Electromagnetic Wave Equation: Solutions

Classes of solution to wave equation

A plane wave satisfies wave equation in Cartesian coordinates

A spherical wave satisfies wave equation in spherical polar coordinates

A cylindrical wave satisfies wave equation in cylindrical coordinates

Solution of 3D wave equation: Plane Wave

In Cartesian coordinates

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Separation of variables

$$\psi\left(x, y, z, t\right) = X(x)Y\left(y\right)Z(z)T(t)$$

Substituting for ψ we obtain

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \frac{1}{c^2}\left(\frac{1}{T}\frac{\partial^2 T}{\partial t^2}\right)$$

Variables are separated out
Each variable-term independent
And must be a constant

So we may write

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2; \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2;$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k_z^2; \left(\frac{1}{T}\frac{\partial^2 T}{\partial t^2}\right) = -\omega^2$$

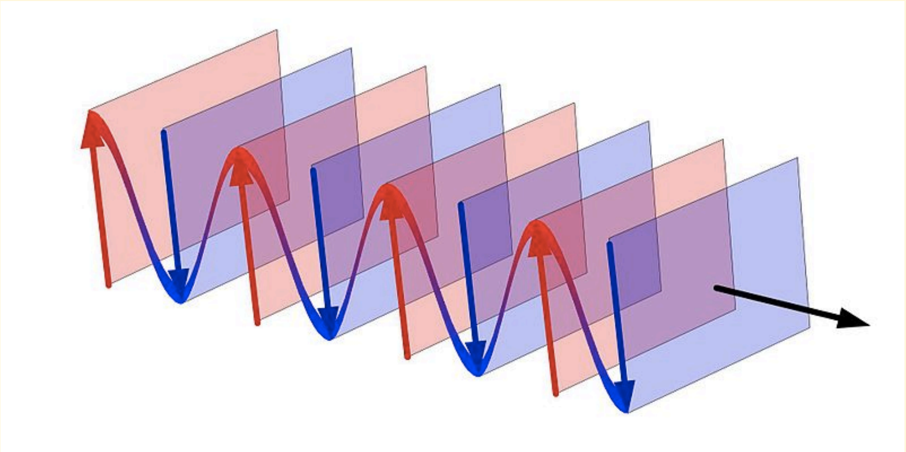
where we use

$$\omega^2/c^2 = k_x^2 + k_y^2 + k_z^2 = k^2$$

Solutions are then

$$X(x) = e^{\pm ik_x x};\; Y(y) = e^{\pm ik_y y};$$

$$Z(z) = e^{\pm ik_z z};\; T(t) = e^{\pm i\omega t}$$



Total Solution is

$$\begin{aligned}\psi(x,y,z) &= X(x)Y(y)Z(z)T(t) \\ &= Ae^{i[\omega t \pm (k_x x + k_y y + k_z z)]} \\ &= Ae^{i\left[\omega t \pm \vec{k}.\vec{r}\right]}\end{aligned}$$

Properties of Plane EM waves

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}\exp(i(\mathbf{k}\cdot\mathbf{r} - \omega t))$$

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}\exp(i(k_x x + k_y y + k_z z - \omega t))$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}\exp(i(k_x x + k_y y + k_z z - \omega t))$$

$$\mathbf{k} \cdot \mathbf{k} = k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 / c^2$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \mathbf{k} \cdot \mathbf{E} = 0$$

Wave vector **k** is perpendicular to **E**

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = \mathbf{k} \cdot \mathbf{B} = 0$$

Wave vector **k** is perpendicular to **B**

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}$$

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$$

$$\hat{\mathbf{k}} \times \mathbf{E} = \frac{\omega}{k}\mathbf{B} = c\mathbf{B}$$

B is perpendicular to **E**

$$\nabla \times \mathbf{B}(\mathbf{r},t) = \mu_0\epsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}$$

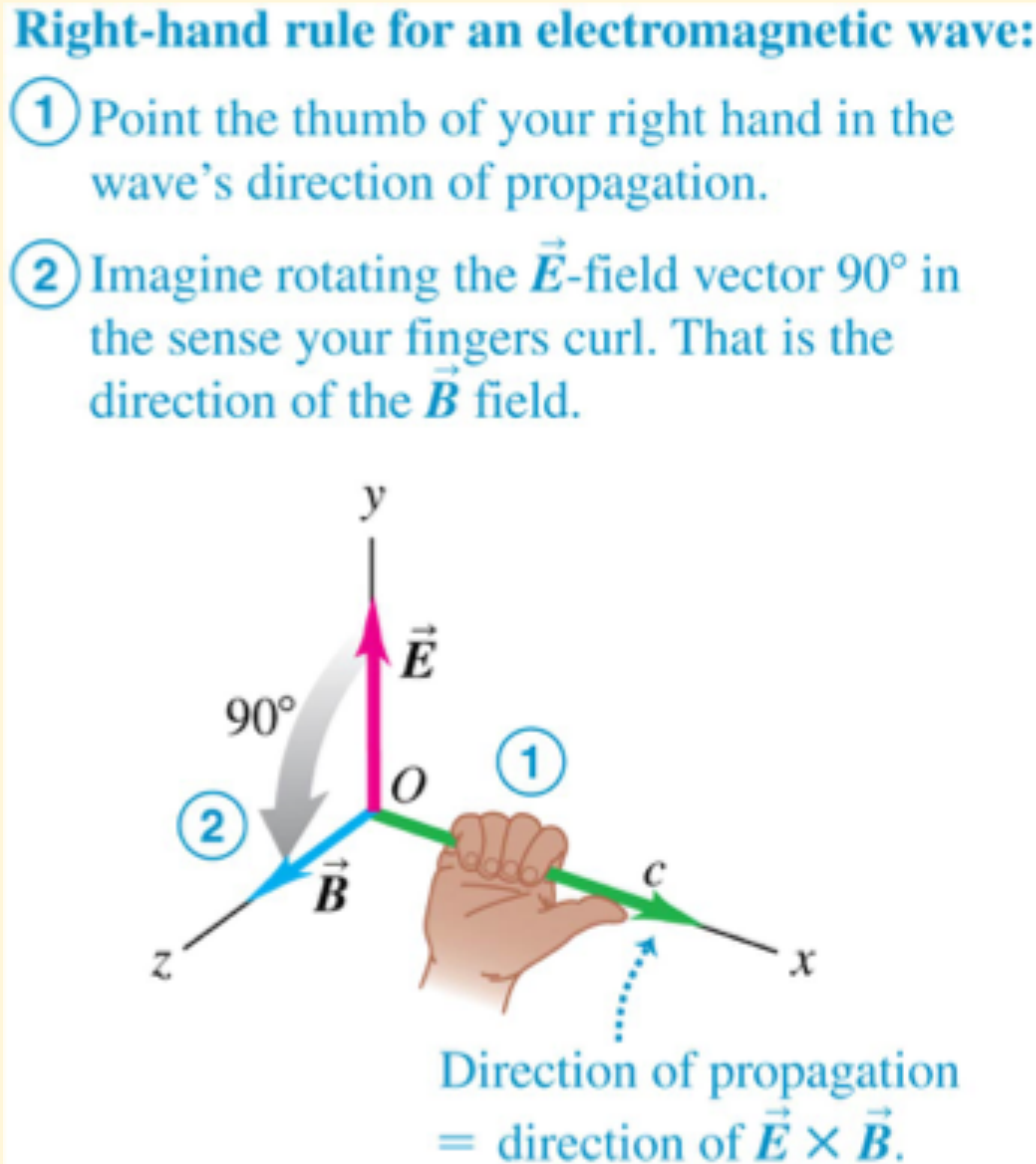
$$\mathbf{k} \times \mathbf{B} = -\frac{\omega}{c^2}\mathbf{E}$$

$$\mathbf{B} \times \hat{\mathbf{k}} = \frac{\omega}{kc^2}\mathbf{E} = \frac{1}{c}\mathbf{E}$$

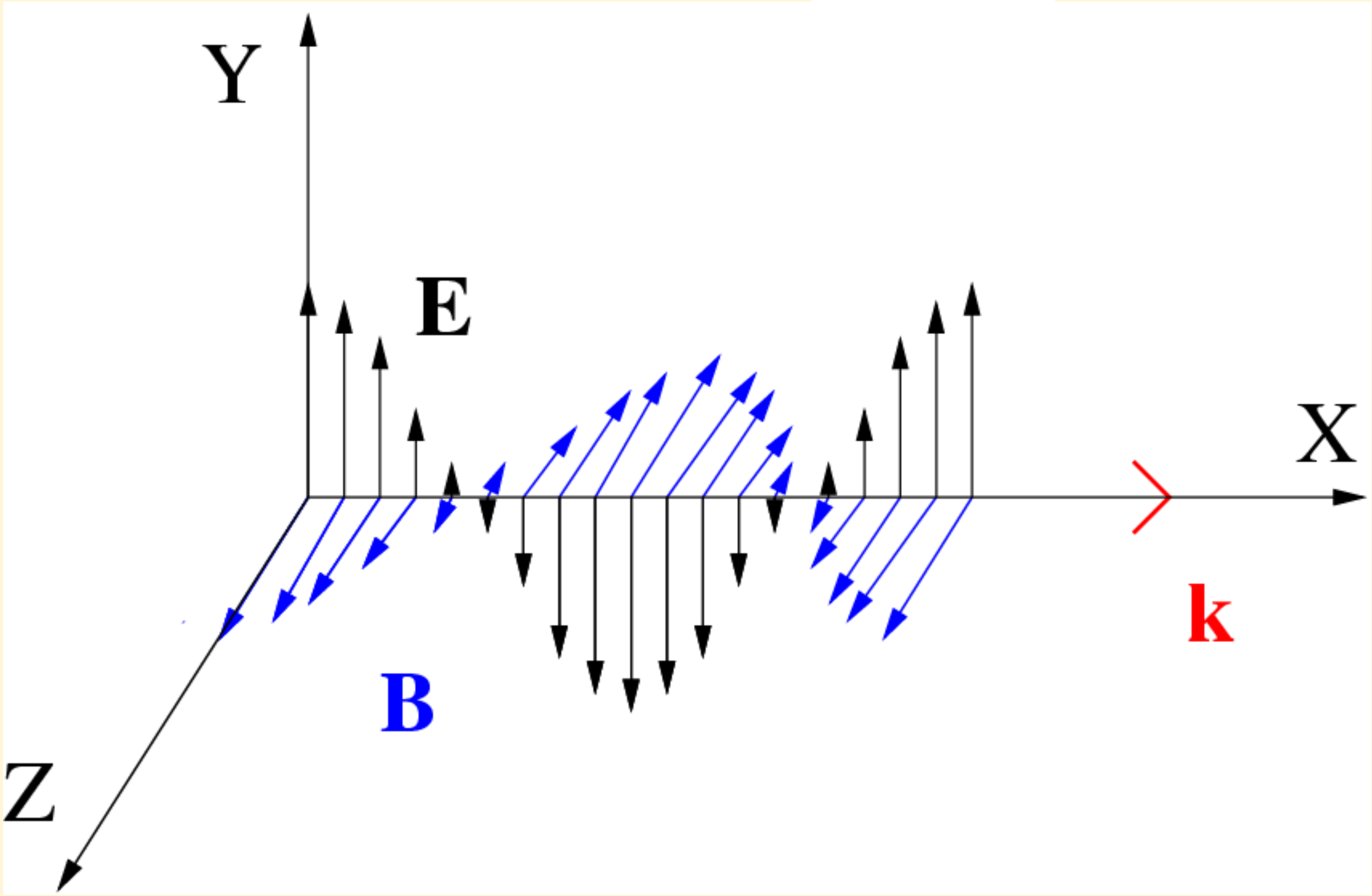
$$c\mathbf{B} \times \hat{\mathbf{k}} = \mathbf{E}$$

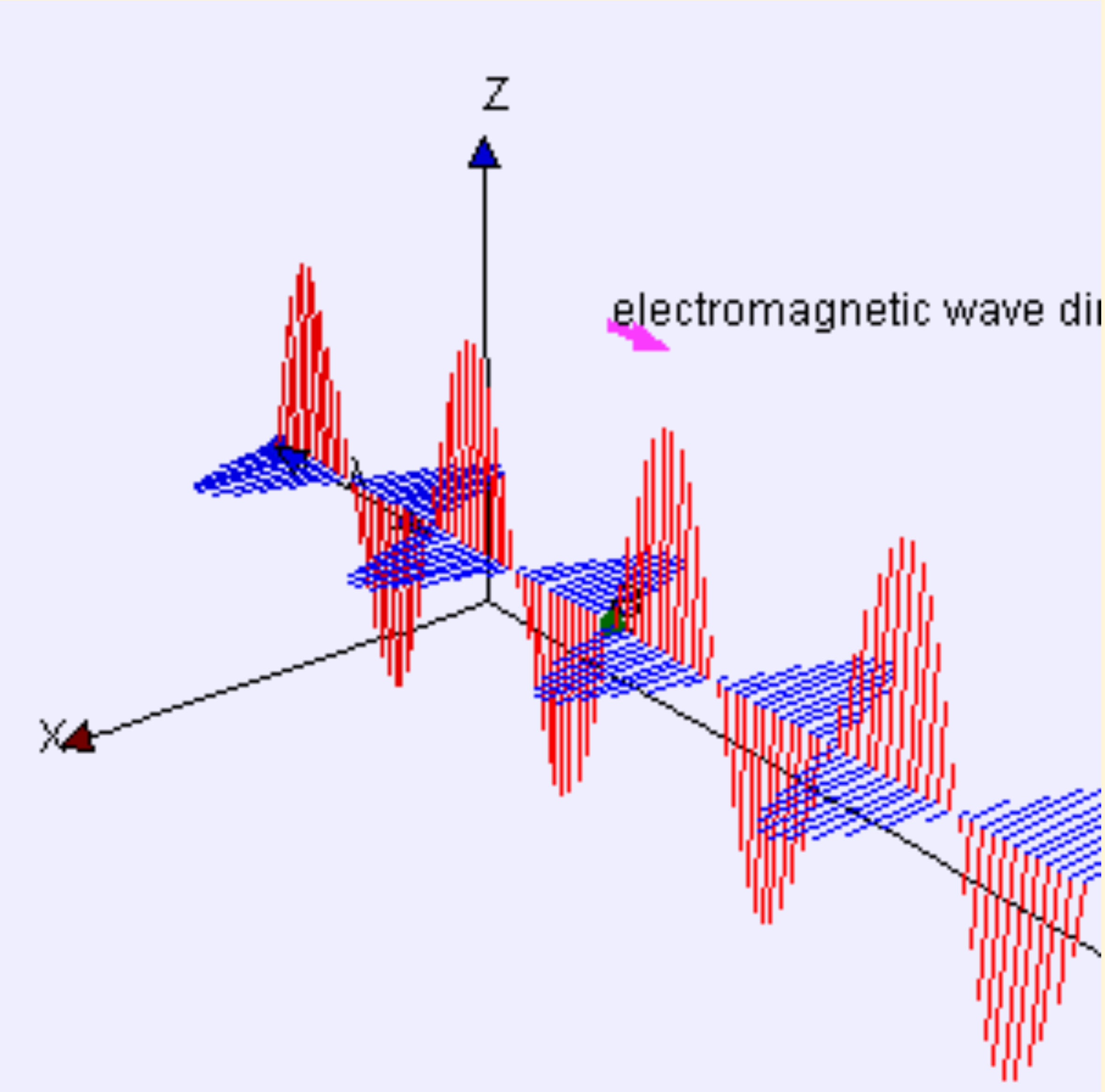
B, k and **E** make a right handed Cartesian co-ordinate system

Electromagnetic waves: orientation of fields



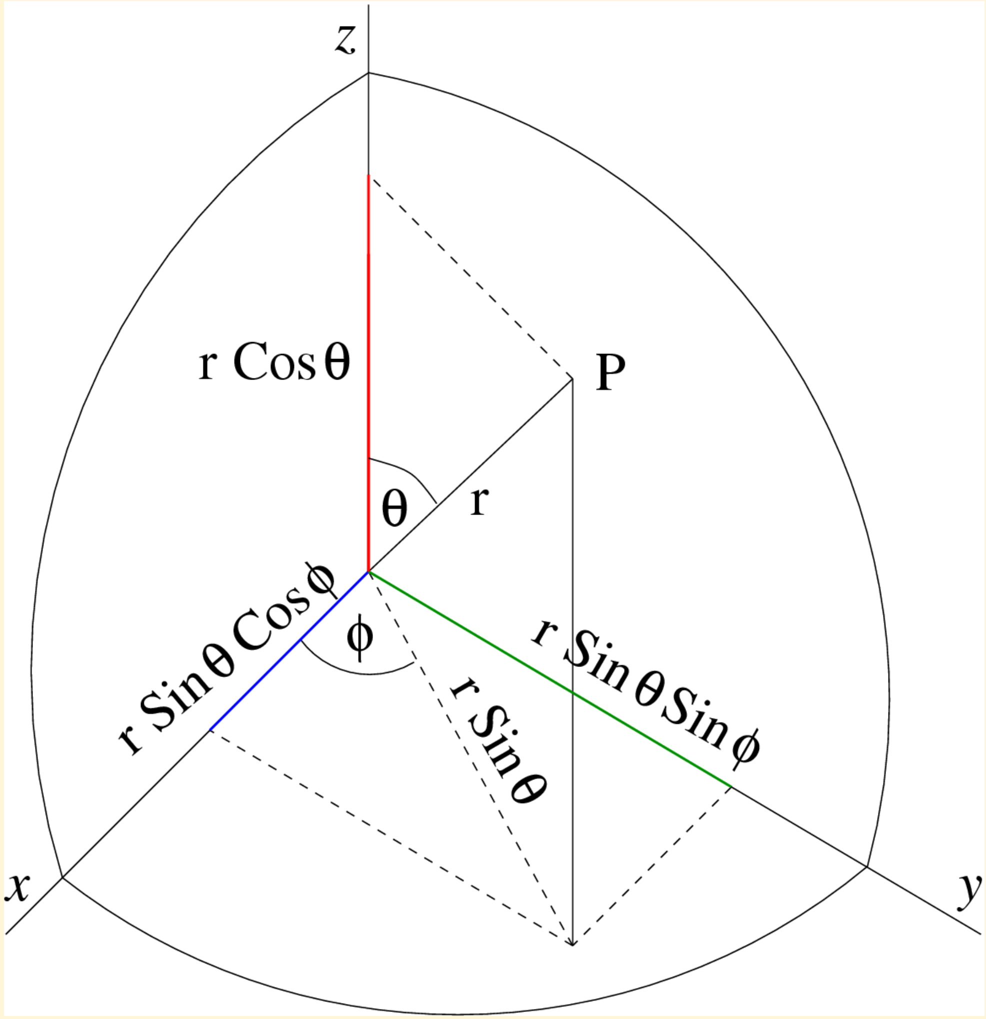
Plane EM waves in vacuum




Z
electromagnetic wave di
X

Solution of 3D wave equation: Spherical waves

Spherical coordinates (r, θ, φ):
radial distance r,
polar angle θ (theta), and
azimuthal angle φ (phi)

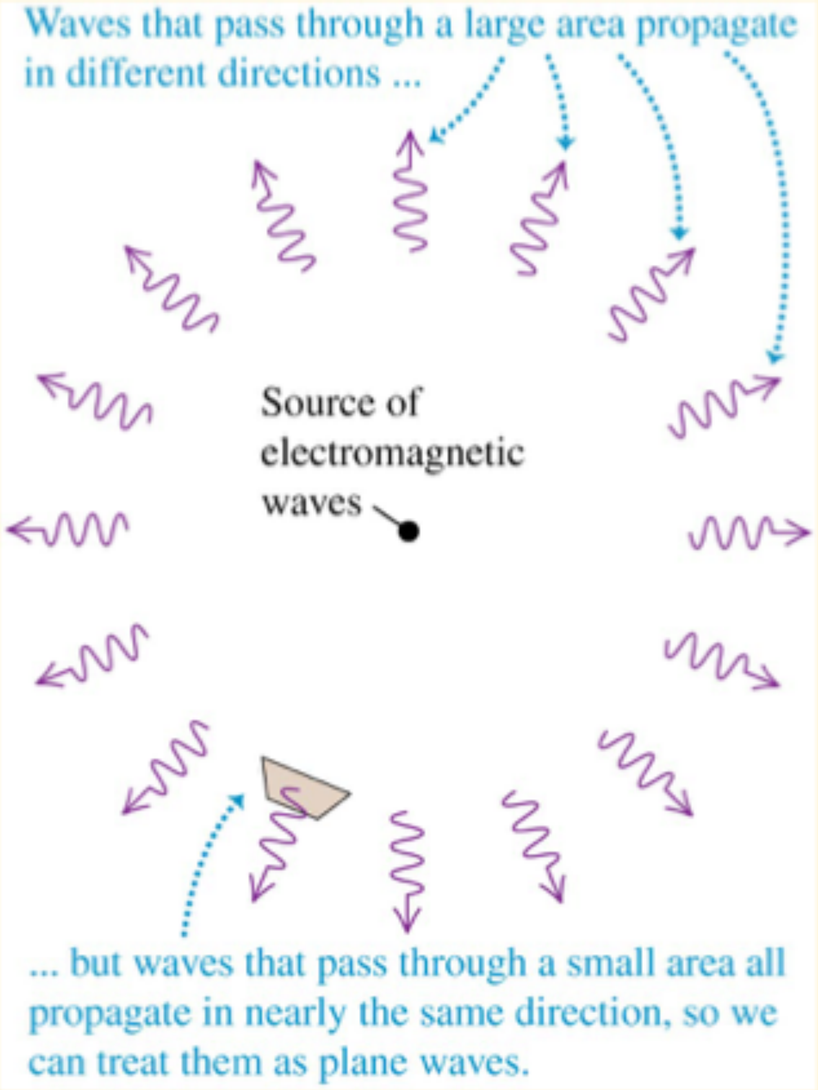


$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

Spherical waves: Origin



$$\psi(\mathbf{r}) = \psi(r, \theta, \phi) = \psi(r)$$

$$\nabla^2 \psi(r) = \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

Spherical waves Eqn: Solution

Spherical Wave Eqn — Substitute — Reduced Spherical Wave Eqn

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \rightarrow \psi(r)=\frac{u(r)}{r} \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)=\frac{1}{r}\frac{\partial^2 u}{\partial r^2}$$

Separation of variables

$$u(r,t)=R(r)T(t)$$

Which follows that

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2}=\frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2}=-k^2$$

Solutions are

$$R(r)=e^{\pm ikr};T(t)=e^{\pm i\omega t}\omega=kc$$

Total solution is

$$u(r)=e^{i\left(\omega t\pm kr\right)}$$

$$\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(r\psi)$$

$$r\psi(r) = A\exp(ik(r - vt))$$

$$\psi(r) = \frac{A}{r}\exp(ik(r - vt))$$

Final form of solution

$$\psi(r) = \frac{1}{r} e^{i\left(\omega t \pm \mathrm{kr}\right)}$$

General solution

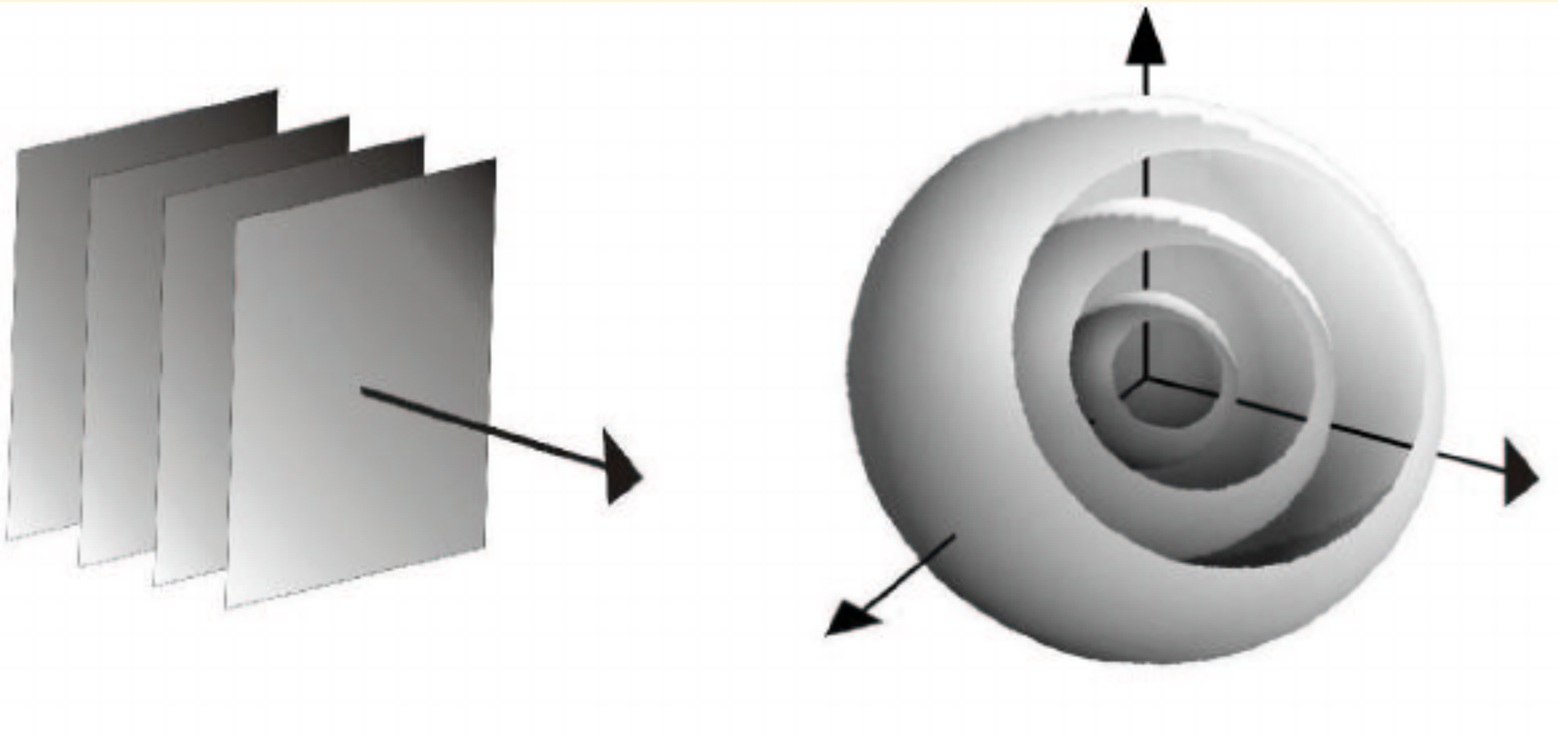
$$\psi(r) = \frac{1}{r} e^{i\left(\omega t - \mathrm{kr}\right)} + \frac{1}{r} e^{i\left(\omega t + \mathrm{kr}\right)}$$

outgoing waves

incoming waves

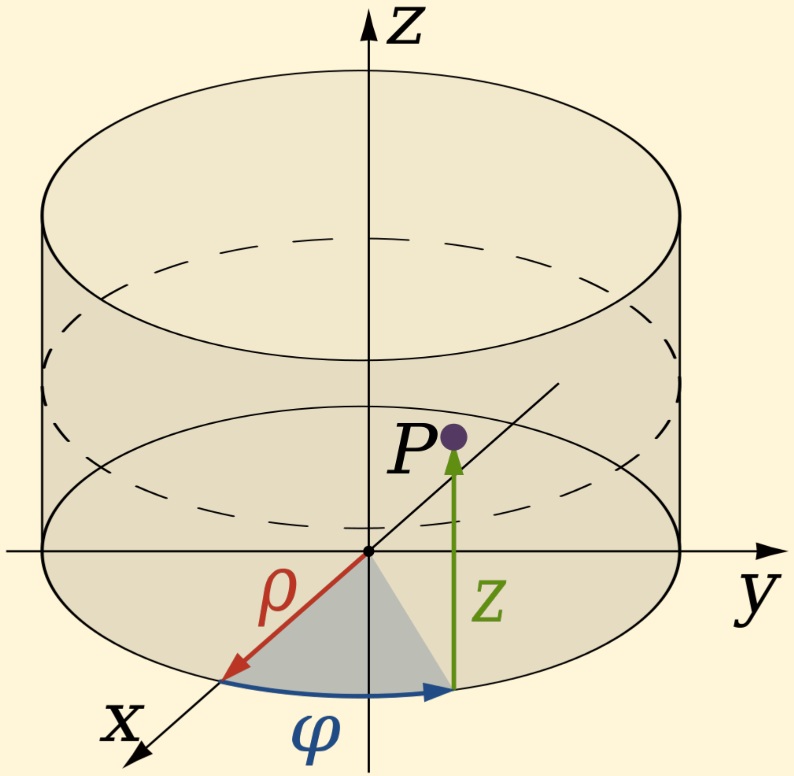
Plane Vs Spherical Wave

The surfaces of equal phases



Cylindrical waves

Cylindrical Coordinate Surfaces(ρ, φ, z). The three orthogonal components, ρ (red), φ (blue), and z (green), each increasing at a constant rate. The point is at the intersection between the three coloured surfaces.



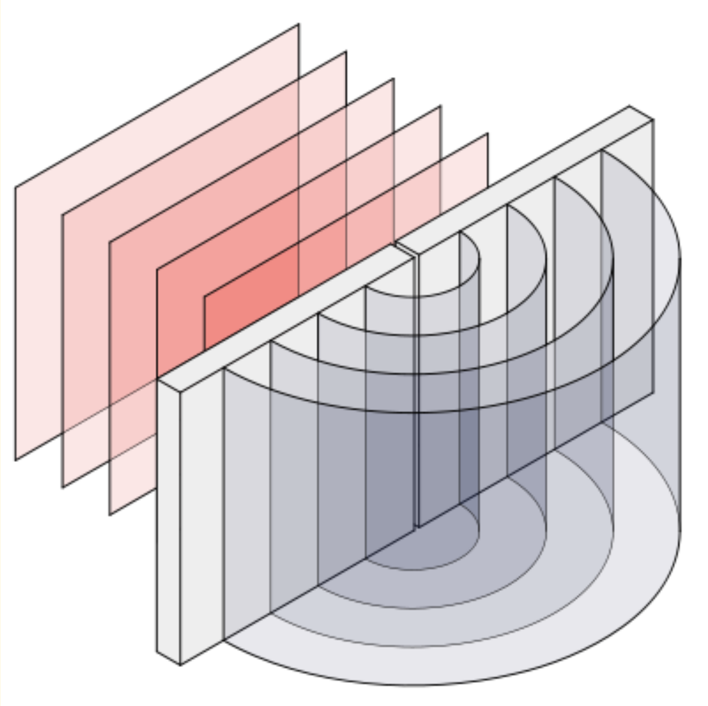
$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Where,

$$\nabla^2 \equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

With angular and azimuthal symmetry, the Laplacian simplifies and the wave equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

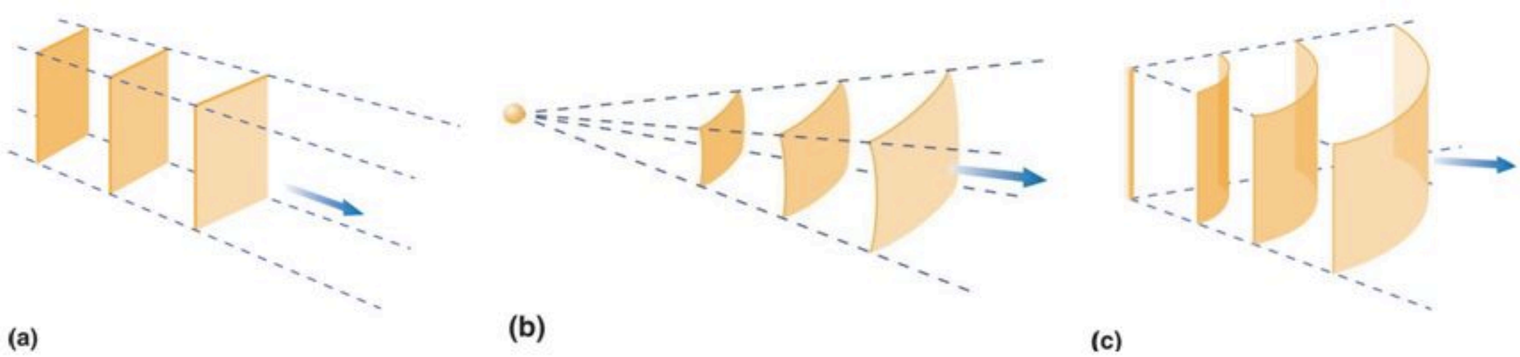


The solutions are Bessel functions.
For large *r*, they are approximated as

$$\psi(\rho,t)\sim\frac{A}{\sqrt{r}}\cos(kr\pm\omega t)$$

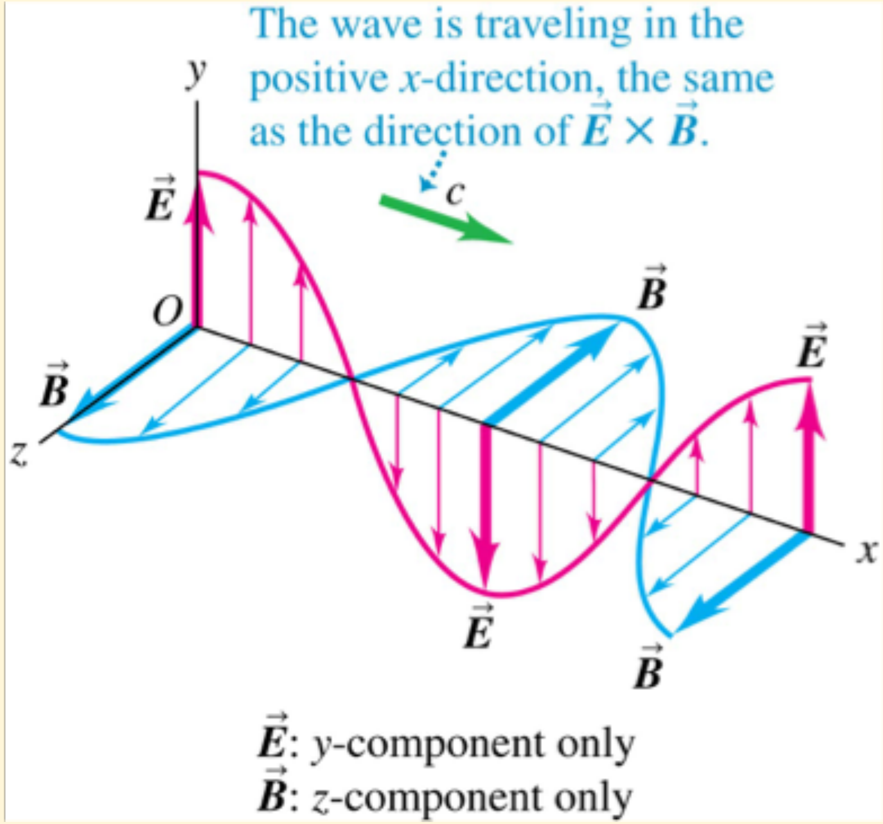
Nature of the wavefront depend on the source:

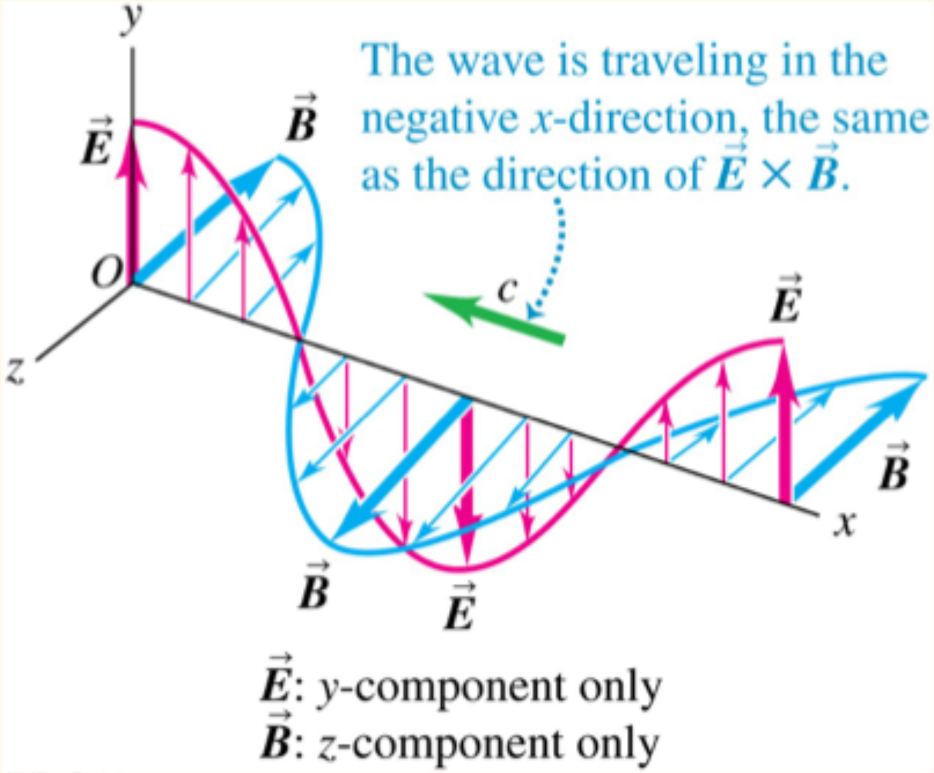
Example of waves



a) Plane waves
b) spherical waves
c) cylindrical waves

Polarisation in Plane EM waves





Shown is a linearly polarized sinusoidal electromagnetic wave traveling in the ***+x*** direction.

Shown is a linearly polarized sinusoidal electromagnetic wave traveling in the ***-x*** direction.

EM waves in medium

- Electromagnetic waves can travel in certain types of matter, such as air, water, or glass.
- When electromagnetic waves travel in nonconducting materials—that is, dielectrics—the speed *v* of the waves depends on the dielectric constant of the material.

Speed of electromagnetic waves in a dielectric

Permeability

Speed of light in vacuum

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{KK_{\mathrm{m}}}}\frac{1}{\sqrt{\epsilon_0\mu_0}} = \frac{c}{\sqrt{KK_{\mathrm{m}}}}$$

Permittivity

Dielectric constant

Relative permeability

Electric constant

Magnetic constant

- The ratio of the speed *c* in vacuum to the speed *v* in a material is known in optics as the **index of refraction** *n* of the material.

$$\frac{c}{v} = n = \sqrt{KK_{\mathrm{m}}} \cong \sqrt{K}$$

Energy in a EM waves

The strength and propagation direction of electromagnetic radiation is given by the Poynting vector, S. The magnitude of the Poynting vector represents the energy flux of the wave, or the power that the wave delivers per unit area perpendicular to the wave's travel direction. As can be seen in the equation below, for light traveling in vacuum, the **Poynting vector** at a given time and position is proportional to the cross product of the electric and magnetic fields composing the light wave at that same time and position:

$$\vec{S} \equiv \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

For EM waves:

Intensity of a sinusoidal electromagnetic wave in vacuum

Electric-field amplitude   Magnetic-field amplitude   Electric constant

$$I = S_{\mathrm{av}} = \frac{E_{\max}B_{\max}}{2\mu_0} = \frac{E_{\max}{}^2}{2\mu_0 c} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}E_{\max}{}^2 = \frac{1}{2}\epsilon_0 c E_{\max}{}^2$$

Magnitude of average Poynting vector   Magnetic constant   Speed of light in vacuum

Harnessing Energy from EM waves

These rooftop solar panels
are tilted to be face-on to the
sun so that the panels can
absorb the maximum amount of wave energy.

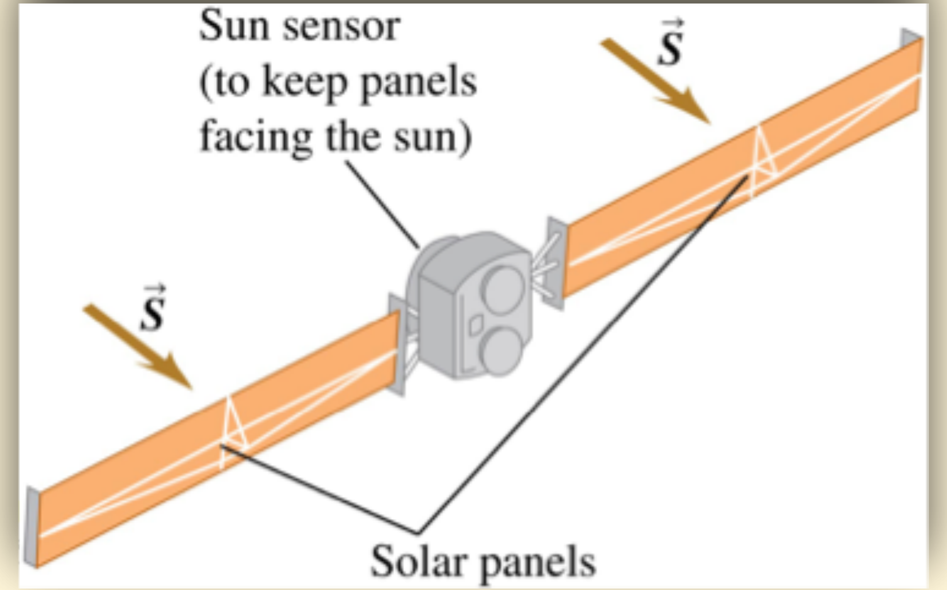
Electromagnetic radiation pressure

Electromagnetic waves carry momentum and can therefore exert **radiation pressure** on a surface:

$$p_{\mathrm{rad}} = \frac{S_{\mathrm{av}}}{c} = \frac{I}{c} \qquad \text{(radiation pressure, wave totally absorbed)}$$

$$p_{\mathrm{rad}} = \frac{2S_{\mathrm{av}}}{c} = \frac{2I}{c} \qquad \text{(radiation pressure, wave totally reflected)}$$

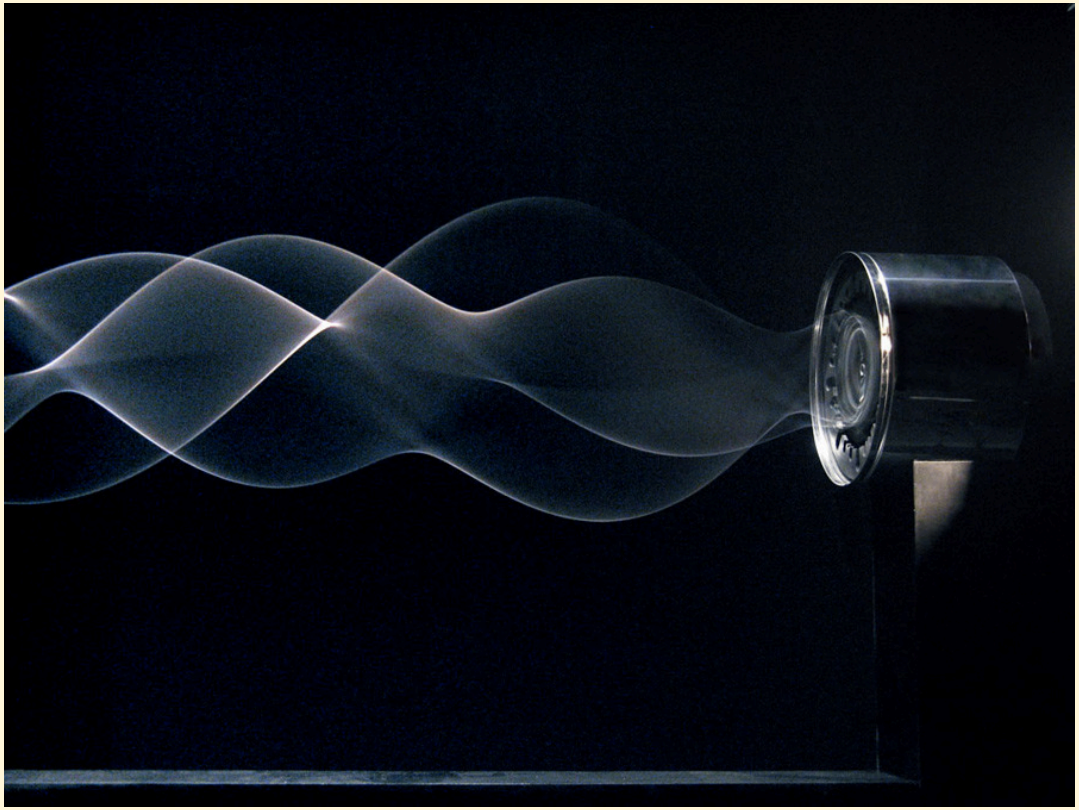
For example, if the solar panels on an earth-orbiting satellite are perpendicular to the sunlight, and the radiation is completely absorbed, the average radiation pressure is $4.7 \times 10^{-6}\ \mathrm{N/m^2}$.



Standing electromagnetic waves

- Electromagnetic waves can be reflected by a conductor or dielectric, which can lead to *standing waves*.

As time elapses, the pattern does not move along the *x*-axis; instead, at every point the electric and magnetic field vectors simply oscillate.



Node formation: an Application

- A typical microwave oven sets up a standing electromagnetic wave with λ = 12.2 cm, a wavelength that is strongly absorbed by the water in food.
- Because the wave has nodes spaced $\lambda/2$ = 6.1 cm apart, the food must be rotated while cooking.
- Otherwise, the portion that lies at a node—where the electric-field amplitude is zero—will remain cold.