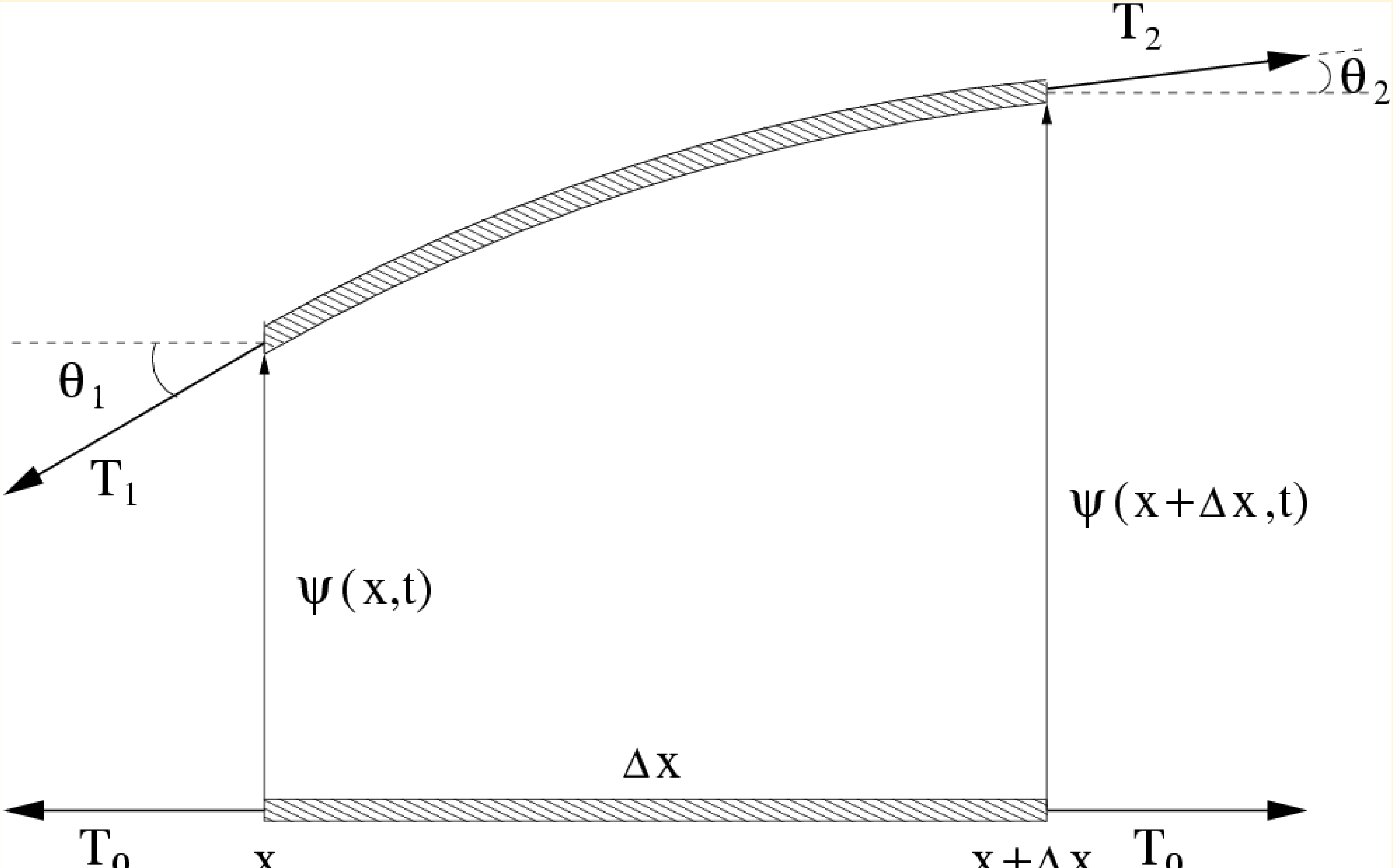
Transverse waves

Transverse vibrations in strings



Horizontal components of forces

$$F_x = T_2 \cos\theta_2 - T_1 \cos\theta_1 = 0$$

$$T_2 \cos\theta_2 = T_1 \cos\theta_1 = T_0$$

Vertical components of forces

$$F_y = T_2 \sin\theta_2 - T_1 \sin\theta_1$$

$$F_y = T_2 \cos\theta_2 \tan\theta_2 - T_1 \cos\theta_1 \tan\theta_1$$

$$F_y = T_0 \tan\theta_2 - T_0 \tan\theta_1$$

$$F_y = T_0\left(\frac{\partial\psi(x+\Delta x,t)}{\partial x}\right) - \quad T_0\left(\frac{\partial\psi(x,t)}{\partial x}\right)$$

$$F_y = T_0\frac{\partial}{\partial x}\left(\psi(x+\Delta x,t) - \psi(x,t)\right)$$

$$F_y = T_0\left(\frac{\partial^2\psi}{\partial x^2}\right)\Delta x$$

$$= \mu \Delta x \left(\frac{\partial^2 \psi}{\partial t^2} \right)$$

μ = mass per unit length

$$\left(\frac{\partial^2 \psi}{\partial x^2} \right) = \frac{\mu}{T_0} \left(\frac{\partial^2 \psi}{\partial t^2} \right)$$

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0}$$

$$c^2 = \frac{T_0}{\mu}$$

Solution to the Wave Equation

Wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

$y = f_1(ct - x)$ or $y = f_2(ct + x)$

will satisfy the above equation

General Solution

$$y = f_1(ct - x) + f_2(ct + x)$$

We have -

$$\frac{\partial y}{\partial x} = -f_1'(ct - x)$$ **and** $$\frac{\partial^2 y}{\partial x^2} = f_1''(ct - x)$$

Similarly-

$$\frac{\partial y}{\partial t} = cf_1'(ct - x)$$ **and** $$\frac{\partial^2 y}{\partial t^2} = c^2 f_1''(ct - x)$$

Thus, y is the solution of wave equation -

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Physical significance of y = f(ct+x)

"y" is the simple harmonic displacement of an oscillator at position "x" and time "t", thus it can be expressed as -

$$y = a\sin\left(\omega t - \phi\right)$$

The bracket (ct -x) in the expression $y = f\left(ct - x\right)$ has the dimension of length and for the function to be a sine or cosine, its argument must have the dimensions of radians, so –

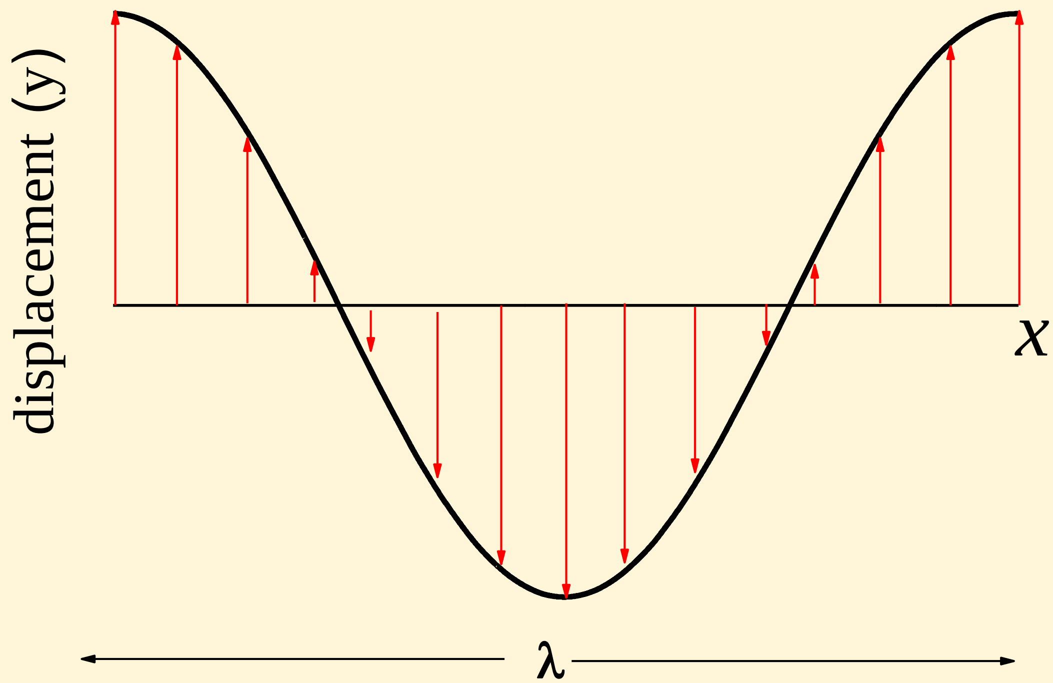
$$y = a\sin\left(\omega t - \phi\right) = a\sin\frac{2\pi}{\lambda}\left(ct - x\right)$$

is a solution of the wave equation $2\pi c/\lambda = \omega = 2\pi\nu$ where

ν is the oscillation frequency and $\phi = 2\pi x/\lambda$

$$y = a\sin(\omega t - \phi) = a\sin\frac{2\pi}{\lambda}(ct - x)$$

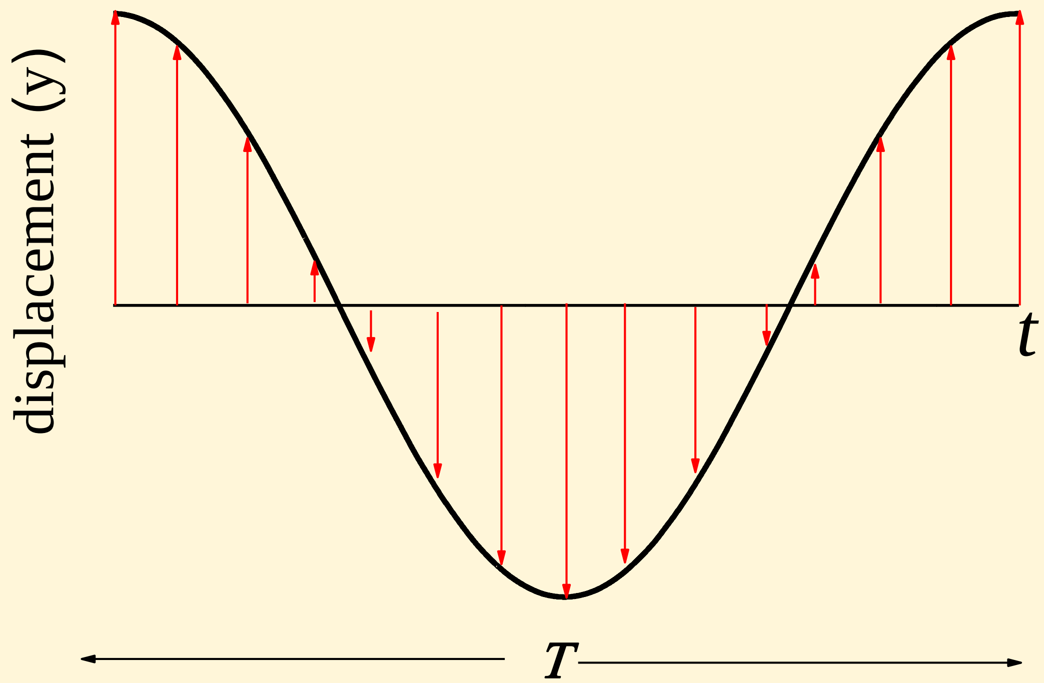
The locus of oscillator displacements in a continuous medium at some fixed time (like a photograph)



λ is the distance (in x) between any two oscillators having a phase difference of 2π radians

$$y = a\sin(\omega t - \phi) = a\sin\frac{2\pi}{\lambda}(ct - x)$$

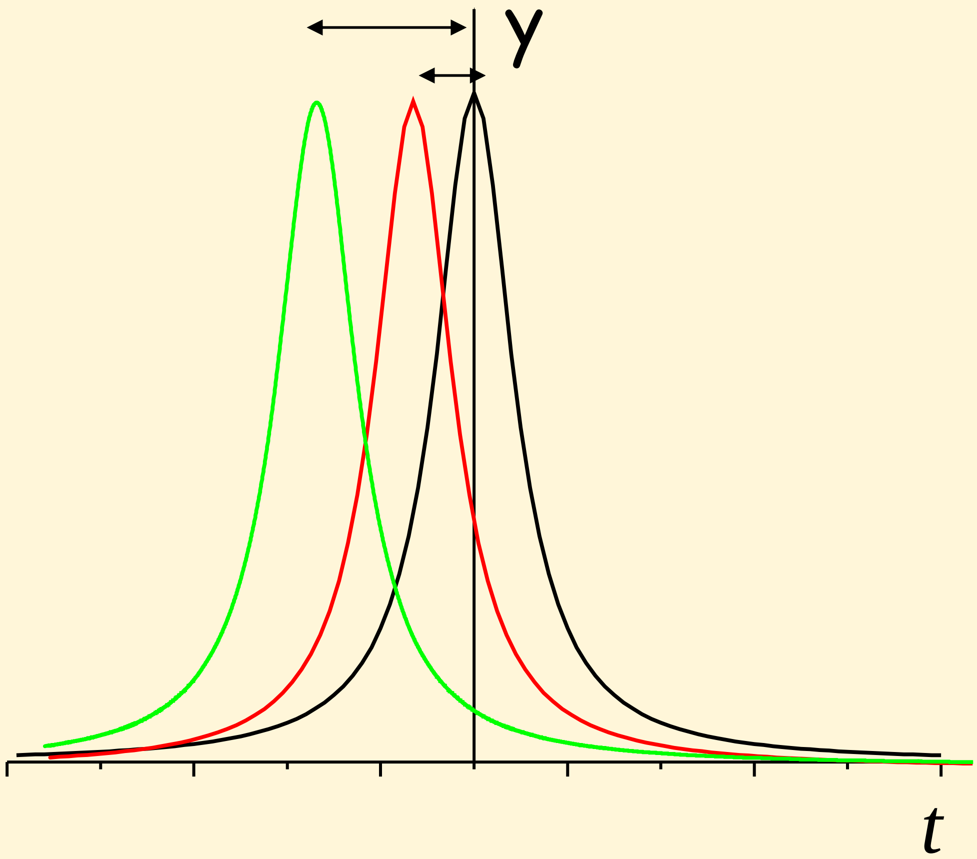
The locus of oscillator displacements in a continuous medium at some fixed position as a function of time



T is the time interval after which the oscillator undergoes a phase difference of 2π radians ($\omega = \frac{2\pi}{T}$)

Significance of $y = f(ct+x)$

wave moving to the left



Equivalent expressions which are equally valid

$$y = a \sin \frac{2\pi}{\lambda}(ct - x)$$

$$y = a \sin 2\pi\left(\nu t - \frac{x}{\lambda}\right)$$

$$y = a \sin \omega\left(t - \frac{x}{c}\right)$$

$$y = a \sin(\omega t - kx)$$

Here $\boxed{k = 2\pi/\lambda = \omega/c}$ is the wave-number

For disturbance propagating in all directions

$$\boxed{\frac{\partial^2}{\partial x^2} \rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \nabla^2 \quad \text{(Laplacian operator)}}$$

$$\boxed{\nabla^2 \xi(\bar{r}, t) = \frac{1}{c_s^2} \frac{\partial^2 \xi}{\partial t^2}}$$