Maxwell's Equations

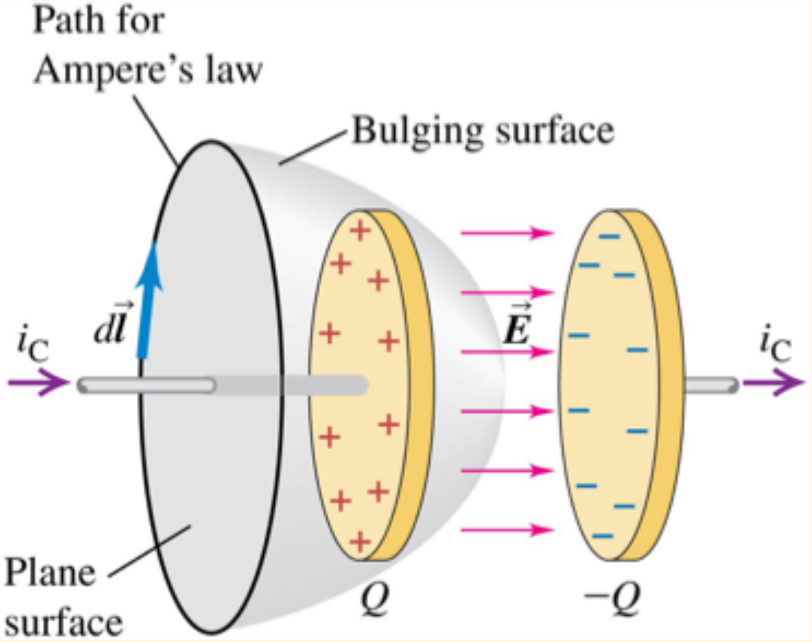
Contradiction with the Ampere's Law

Remember the Amperes Law:

Ampere's law: $$\overbrace{\oint \underbrace{\vec{B} \cdot d\vec{l}}_{\substack{\text{Scalar product of magnetic field}\\ \text{and vector segment of path}}}}^{\text{Line integral around a closed path}} = \underbrace{\mu_0}_{\text{Magnetic constant}} \underbrace{I_{\text{encl}}}_{\text{Net current enclosed by path}}$$

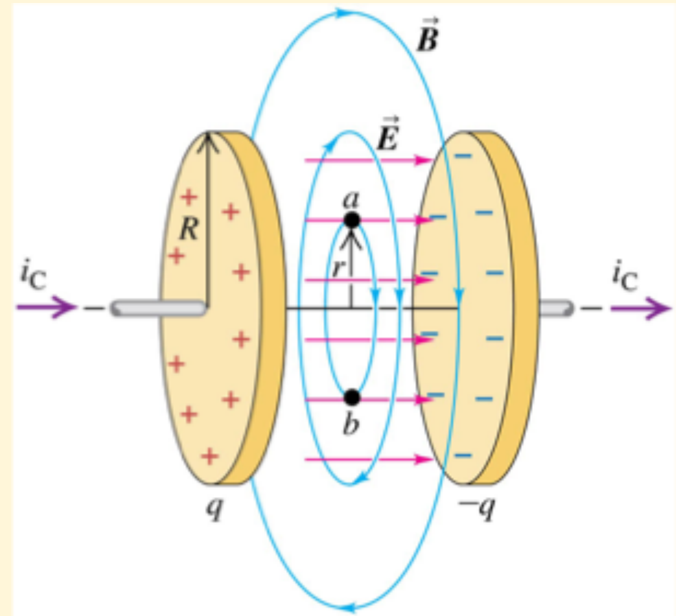
Now consider a charging of a capacitor:

- For the plane circular area bounded by the circle, I_{encl} is the current i_C in the left conductor.
- But the surface that bulges out to the right is bounded by the same circle, and the current through that surface is zero.
- This leads to a contradiction.



Fixing Amperes Law: Displacement Current

- When a capacitor is charging, the electric field is increasing between the plates.
- We can define a fictitious **displacement current** i_D in the region between the plates.
- This can be regarded as the source of the magnetic field between the plates.



Displacement current through an area

$$i_D = \epsilon \frac{d\Phi_E}{dt}$$

Time rate of change of electric flux through area

Permittivity of material in area

Gauss's Law for Electric and Magnetic Fields

Statement of the Coulomb Law:

Gauss's law for $\vec{E}$:

Flux of electric field through a closed surface

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

Charge enclosed by surface

Electric constant

Statement of absence of magnetic monopole:

Gauss's law for $\vec{B}$:

Flux of magnetic field through any closed surface ...

$$\oint \vec{B} \cdot d\vec{A} = 0$$

... equals zero.

Faraday's and Amperes Laws

Faraday's Law of Electromagnetic Induction:

Faraday's law for a stationary integration path:

$$\overbrace{\oint \vec{E}\cdot d\vec{l}}^{\text{Line integral of electric field around path}} = \underbrace{-\frac{d\Phi_B}{dt}}_{\text{Negative of the time rate of change of magnetic flux through path}}$$

Ampere's Law including the Displacement current:

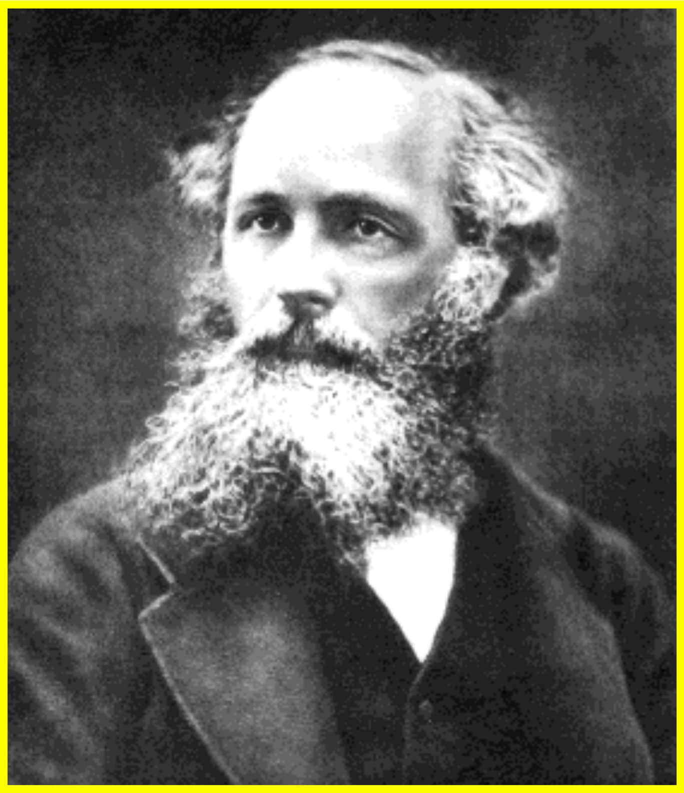
Ampere's law for a stationary integration path:

Electric constant

$$\overbrace{\oint \vec{B}\cdot d\vec{l}}^{\text{Line integral of magnetic field around path}} = \mu_0\left(i_C + \underbrace{\epsilon_0 \overbrace{\frac{d\Phi_E}{dt}}^{\text{Time rate of change of electric flux through path}}}_{\text{Displacement current through path}}\right)_{\text{encl}}$$

Magnetic constant

Conduction current through path



“Let there be electricity and magnetism and there is light”

J.C. Maxwell

Maxwell's Laws in the Integral Form

- There is a remarkable symmetry in Maxwell's equations.
- In empty space where there is no charge, the first two equations are identical in form.
- The third equation says that a changing magnetic flux creates an electric field, and the fourth says that a changing electric flux creates a magnetic field.

In empty space there are no charges, so the fluxes of $\vec{E}$ and $\vec{B}$ through any closed surface are equal to zero.

$$\oint \vec{E} \cdot d\vec{A} = 0$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

In empty space there are no conduction currents, so the line integrals of $\vec{E}$ and $\vec{B}$ around any closed path are related to the rate of change of flux of the other field.

Maxwells Equations in free space

Using Gauss's Law of vector Calculus

The first Maxwells equation in free space:

$$\oint \vec{E} \cdot d\vec{A} = 0$$

$$\oint \vec{E} d\vec{A} = \int \vec{\nabla}.\vec{E} dV \implies \vec{\nabla}.\vec{E} = 0$$

The second Maxwells equation in free space:

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{B} d\vec{A} = \int \vec{\nabla}.\vec{B} dV \implies \vec{\nabla}.\vec{B} = 0$$

Maxwells Equations in free space

Using Stokes Law of vector Calculus

The third Maxwells equation in free space:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{E}.d\vec{l} = \oint (\vec{\nabla} \times \vec{E}).d\vec{A}$$

$$-\frac{d\Phi_B}{dt} = -\oint \frac{d\vec{B}}{dt}.d\vec{A}$$

$$\Longrightarrow \vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

Maxwells Equations in free space

The fourth Maxwells equation in free space:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint \vec{B}.d\vec{l} = \oint (\vec{\nabla} \times \vec{B}).d\vec{A}$$

$$\mu_0 \epsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \epsilon_0 \oint \frac{d\vec{E}}{dt}.d\vec{A}$$

$$\Longrightarrow \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Maxwells Equations in free space

$$\vec{\nabla}.\vec{E} = 0$$

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\vec{\nabla} \times \vec{B} = \mu_0\epsilon_0\frac{d\vec{E}}{dt}$$

Local Conservation of Charge

Maxwell's Fourth Law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

$$\vec{\nabla}. \left(\mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} \right) = \vec{\nabla}. \left(\vec{\nabla} \times \vec{B} \right) = 0$$

Using Maxwell's First Law:

$$\vec{\nabla}.\vec{J} + \frac{d\rho}{dt} \implies -\frac{dQ}{dt} = \oint \vec{J}.d\vec{A}$$

Local conservation of charge says that the rate of decrease of electric charge in a volume is equal to the total charge escaping from the surface of the volume in the form of electric current

Maxwells Equations:
Solutions in presence of charges

The Maxwell equations can be solved exactly in the presence of localized charges.

Important features of the solution:

1. No information can propagate instantaneously
2. The electric field at the time *t* is determined by the position of the charge at an earlier time, when the charge was at *r'*, the retarded position.
3. At large distances the electric and magnetic field falls off as 1/r . The nature of these 1/r terms is what is seen as radiation from a localised source.

Electro-magnetic Radiation

It is also to be noted that only accelerating charges produce radiation.