Damped Forced Oscillation

$$\begin{aligned} m\ddot{x} + \; r\dot{x} + kx &= F(t) \\ \ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= F(t)/m \end{aligned}$$

$F(t) =$ Time dependent function

$$F(t) = F_0 \cos \omega t$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos \omega t = f_0 \cos \omega t$$

Transient and Steady-state solutions

The transient term (Complimentary function) dies away with time and is the solution to the equation discussed earlier:

$$m\ddot{x} + r\dot{x} + kx = 0$$

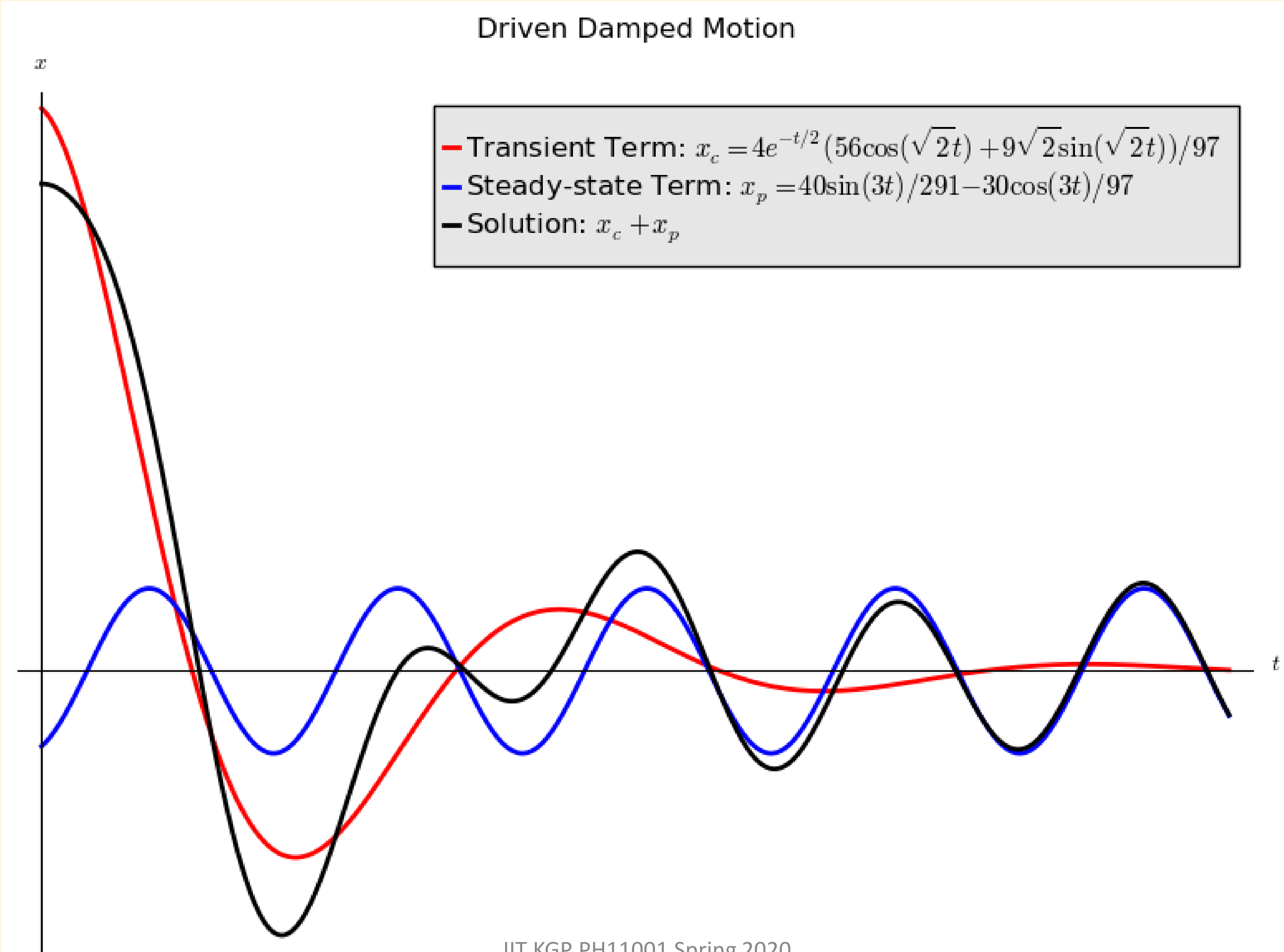
This contributes the term: $x = Ce^{pt}$

$$p_{1,2} = -\frac{r}{2m} \pm \sqrt{\left(\frac{r}{2m}\right)^2 - \frac{k}{m}}$$

$$x(t) = \exp(-\beta t)[A_1 \exp(-i\sqrt{\omega_0^2 - \beta^2}\ t) + A_2 \exp(i\sqrt{\omega_0^2 - \beta^2}\ t)]$$

The steady state term (Particular solution) describes the behaviour of the oscillator after the transient term has died away

Transient and Steady-state solutions



Companion equation:

$$F(t) = F_0 \sin \omega t,$$

$$\ddot{y} + 2\beta \dot{y} + \omega_0^2 y = \frac{F_0}{m} \sin \omega t = f_0 \sin \omega t$$

$$z = x + iy$$

General equation

$$\boxed{\ddot{z} + 2\beta \dot{z} + \omega_0^2 z = \frac{F_0}{m} \exp(i\omega t) = f_0 \exp(i\omega t)}$$

Try steady state solution (*Particular solution*)

$$z(t) = z_0 \exp(i\omega t)$$

where $$z_0 = |z_0| \exp(i\phi)$$

$$z(t) = |z_0| \exp(i(\omega t + \phi))$$
$$= |z_0|(\cos(\omega t + \phi) + i \sin(\omega t + \phi))$$

Depending on the case, one can then get
$$\begin{cases} x(t) = |z_0| \cos(\omega t + \phi) \\ y(t) = |z_0| \sin(\omega t + \phi) \end{cases}$$

Plugging *z* into the general equation

$$\boxed{\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = \frac{F_0}{m}\exp(i\omega t) = f_0 \exp(i\omega t)}$$

$$z_0(-\omega^2 + 2i\beta\omega + \omega_0^2) = f_0$$

$$z(t) = z_0 \exp(i\omega t)$$

$$z_0 = |z_0| \exp(i\phi)$$

$$|z_0| = \sqrt{z_0 z_0^*} = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\phi = \tan^{-1}\left(\frac{-2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

Writing the amplitude in a different form

$$A = \frac{1}{\omega} \frac{f_0}{\sqrt{\frac{(\omega_o^2 - \omega^2)^2}{\omega^2} + (2\beta)^2}}$$

It may appear that the max amplitude appears at $\omega = \omega_o$. However, the term is multiplied by the factor $1/\omega$, which increase as ω decreases, shifting the peak to a slightly lower value of ω_o.

This maximum amplitude is the resonance.

We see that the response amplitude at high frequencies approaches zero

In the opposite limit $\omega \to 0,$

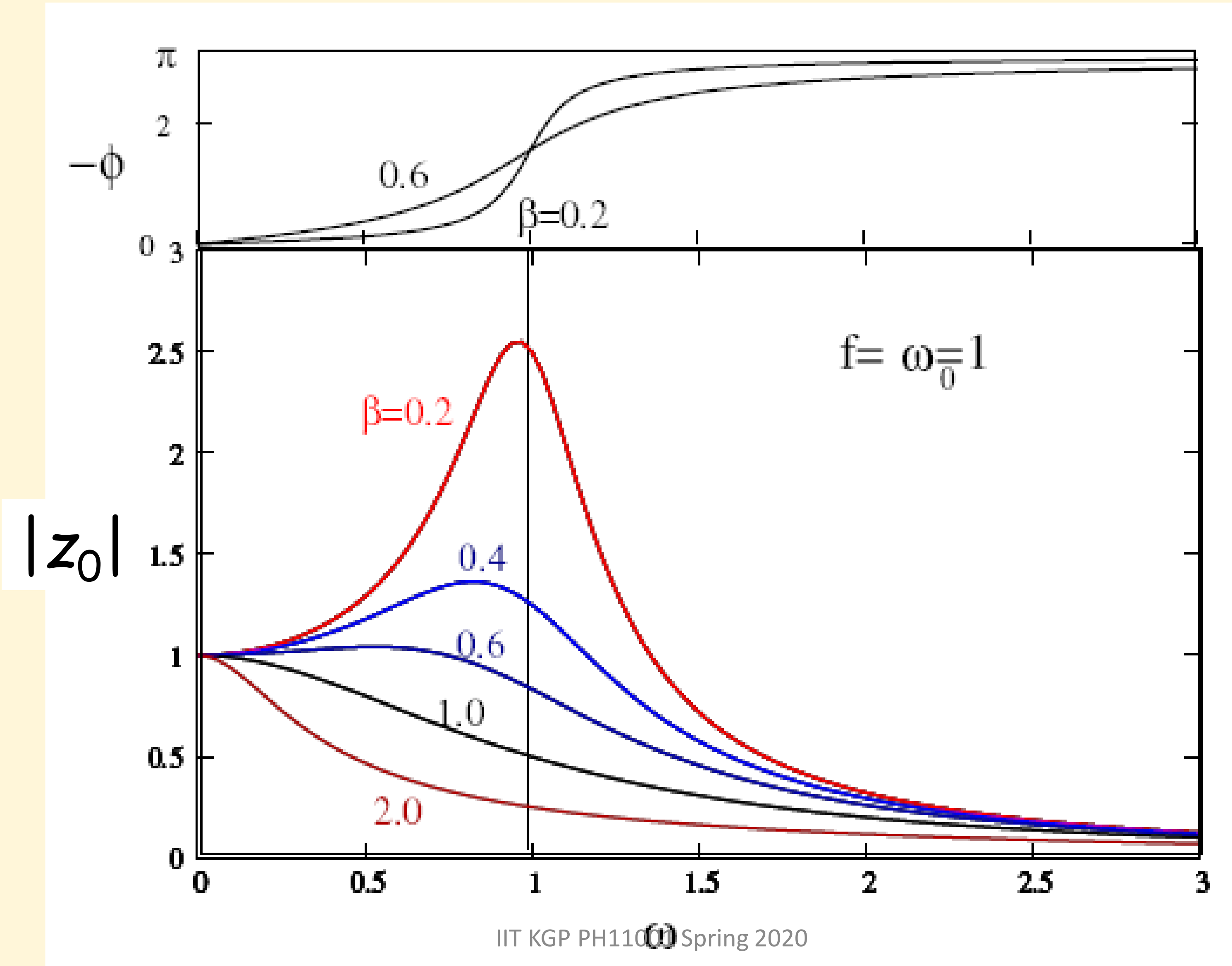
$$A = A(\omega \to 0) = \frac{f_0}{\omega_o^2}$$

For the phase -

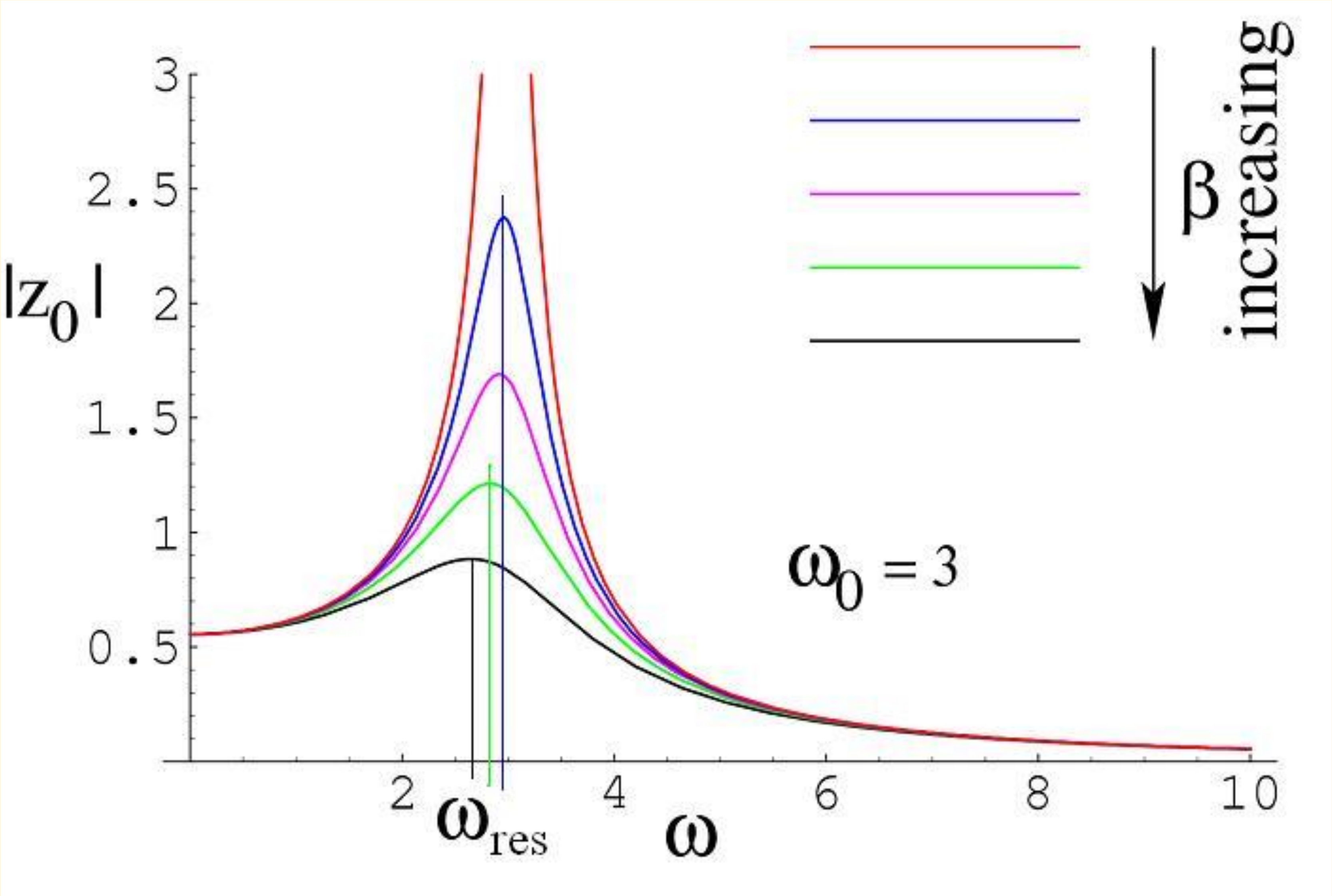
$$\phi = \tan^{-1}\left(\frac{-2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

At $\omega = \omega_0$, the phase ϕ is $\frac{\pi}{2}$

Amplitude resonance at - $\omega = \sqrt{\omega_0^2 - 2\beta^2}$



$$\omega = \sqrt{\omega_0^2 - 2\beta^2} \quad \& \quad |z_0| = \sqrt{z_0 z_0^*} = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$



For mild damping (β << ω_o), ω = ω_o (approximately)

Observations from Mexico Earthquake 1985

Tall and small stay up; medium fall: Mexico, 1985—10,000 die

Figure 16 Collapse of upper stories of steel building

After the earthquake

Interestingly, the short and tall buildings remained standing. Medium-height buildings were the most vulnerable structures in the September 19 earthquake. Of the buildings that either collapsed or incurred serious damage, about 60% were in the 6-15 story range. The resonance frequency of such buildings coincided with the frequency range amplified most frequently in the subsoils.

Sources: *Eng. Struct.* **9**, 74 (1987); www.iris.edu

Power drawn from the external force

Instantaneous power - $P(t) = F(t)\dot{x}(t)$

$$P(t) = [F\cos(\omega t)][- \mid \tilde{x} \mid \omega \sin(\omega t + \phi)]$$

Average power - $\langle P \rangle(\omega) = -\frac{1}{2}\omega F \mid \tilde{x} \mid \sin\ \phi$

$$< \cos^2 \omega t > \equiv \frac{1}{T}\int_0^T dt\ \ \cos^2 \omega t$$

$$= \frac{1}{2T}\int_0^T dt(1 + \cos 2\omega t) = \frac{1}{2}$$

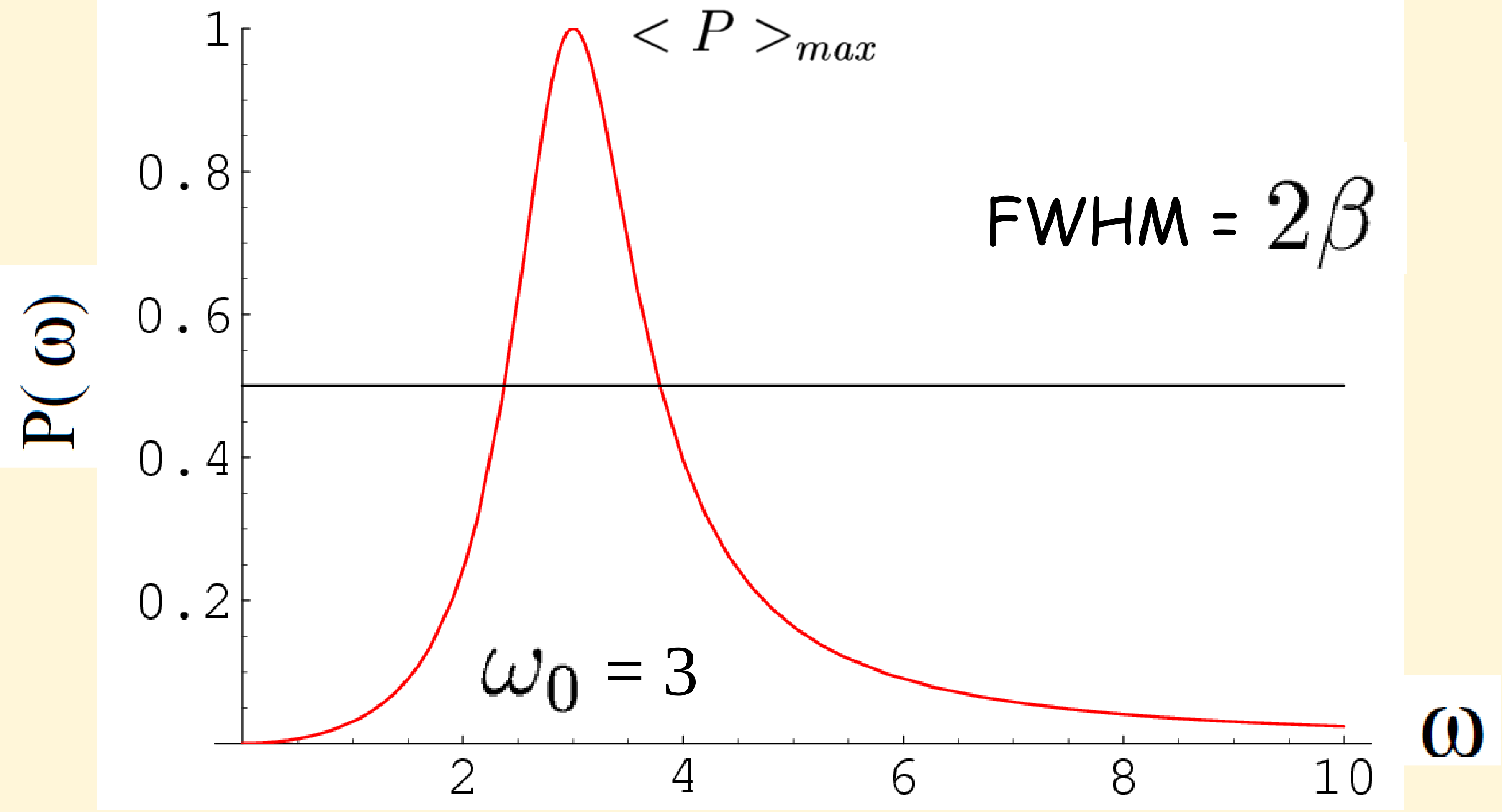
$$< \sin \omega t \cos \omega t >= 0$$

Using - $\mid \tilde{x} \mid \sin\phi = \dfrac{-2\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}\left(\dfrac{F}{m}\right)$

$$\boxed{\langle P \rangle(\omega) = \frac{\beta\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}\left(\frac{F^2}{m}\right)}$$

Power Resonance

$$\langle P \rangle(\omega) = \frac{\beta\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}\left(\frac{F^2}{m}\right)$$



$$Q_0 = \frac{\omega_0}{2\beta} = \frac{\textbf{Central frequency}}{\textbf{FWHM}}$$

Tacoma Narrows Bridge

The **Tacoma Narrows Bridge** is a mile-long (1600 meter) suspension bridge with a main span of 850 m (the third-largest in the world when it was first built) that carries Washington State Route 16 across the
Tacoma Narrows of Puget Sound from Tacoma to Gig Harbor, Washington (collapsed in wind, 1940)

https://www.youtube.com/watch?v=mXTSnZgrfxM