Solutions to tutorial 3

January 30, 2020

Q 1. Consider a guitar string (say, the 1st-string (E-high): the thinest string). Its tension is adjusted such that when open (string vibrating at full length), its *fundamental frequency* is 330 Hz (E). Keeping the tension same, if we now close the first fret (i.e. put our finger on the first fret), the *fundamental frequency* becomes 350 Hz (F). If the length of the string is about 60 cm, can you estimate the reduction in length of the string when you place your finger on the first fret? Note that this is roughly the distance between the nut (0th-fret) and the first fret.

Ans 1. The fundamental frequency for 60 cm length of the string is 330 Hz. When the length is changed, the fundamental frequency becomes 350 Hz. Hence, applying the formula:

$$f = \frac{1}{2l}\sqrt{\frac{T}{m}} \tag{1}$$

for two different lengths l_1 and l_2 corresponding to frequencies f_1 and f_2 we get,

$$\frac{f_1}{f_2} = \frac{l_2}{l_1} \implies l_2 = 60\Big(\frac{330}{350}\Big) = 56.57cm. \tag{2}$$

Hence the change in length is 60-56.57 = 3.42 cm.

Q 2. Consider a sound wave travelling in a solid with Young's modulus $Y = 2 \times 10^{11}$kg m^{-1} s^{-2}, and whose mass density is $\rho = 8 \times 10^3$ kg m^{-3}. The wave-solution has the form $\xi(x,t) = A\cos^2(kx - (2\pi \times 10^2)t)$, where x and t are measured in SI units.

1. Calculate the velocity of sound in the given solid.
2. Compute the wavelength and frequency of the the given wave-form.

Ans 2. Given: Young's modulus = $2{\times}10^{11}$ kg m^{-1} s^{-1}, density $\rho = 8 \times 10^3$ kg m^{-3} with wave solution as $\zeta(x,t) = Acos^2(kx - (2\pi \times 10^2)t)$.

(a) The velocity of the wave is given by:

$$c = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{2 \times 10^{11}}{8 \times 10^3}} = 5,000m/s. \tag{3}$$

(b) The solution ζ(x,t) can be re-written as

$$\begin{aligned} \zeta(x,t) &= \frac{A}{2}\Big(1 + cos2(kx - (2\pi\times 10^2)t)\Big) \\ \implies \zeta'(x,t) = \zeta(x,t) - \frac{A}{2} &= \frac{A}{2}cos2(kx - (2\pi\times 10^2)t) \\ \implies \zeta'(x,t) &= \frac{A}{2}cos(k'x - \omega t) \end{aligned} \tag{4}$$

where $\zeta'(x,t)$ also satisfies the same wave equation as a constant does not affect the differential equation. So $\omega = 4\pi \times 10^2$ per sec and the frequency therefore is $\nu = \omega/2\pi = 200$Hz. As the velocity is already known so the wavelength is $\lambda = c/\nu = 25$m.

Q 3. Find the phase velocity and group velocity for the following two wave-forms (x and t are measured in SI units)

(i) $\psi(x,t) = \cos(4t - 2x) + \cos(8t - 4x)$.

(ii) $\psi(x,t) = \cos(8t - 6x) + \cos(4t - 4x)$.

Does these waveforms satisfy the wave equation

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

where v is a *constant* independent of frequency. Explain.

3 (i). The wave function which is a superposition of two waves is given as ψ(x,t) = $cos(4t - 2x)$ +$cos(8t - 4x)$. This can be re-written as:

$$\psi(x,t) = 2cos\Big(\frac{4t - 2x + 8t - 4x}{2}\Big)cos\Big(\frac{4t - 2x - 8t + 4x}{2}\Big) = 2cos(6t{-}3x)cos(2t{-}x). \tag{5}$$

Comparing with the equation of a wave packet we find that the smaller frequency will act as $\Delta\omega = 2$ (frequency of the envelope), $\Delta k = 1$ while the higher frequency will be the frequency of the inner wave $\omega = 6$, $k = 3$.
Hence the phase velocity is $\omega/k = 2$ while the group velocity is $\Delta\omega/\Delta k = 2$ in proper units. Since the two velocities are equal so the medium is **non-dispersive**.

(ii) The waveform is- ψ(x,t) = $cos(8t - 6x)$ +$cos(4t - 4x)$. Proceeding similarly as done above we get

$$\psi(x,t) = 2cos(6t - 5x)cos(4t - 2x) \tag{6}$$

So, the phase velocity is 6/5 while the group velocity is 2 in proper units. Here the velocities differ so the medium is **dispersive**.

The wave form in (i) will satisfy the wave equation with v = phase velocity = 2 which is independent of frequency as is true for any non-dispersive medium. On the contrary, the wave form in (ii) **does not** satisfy the wave equation as now the velocity of the wave will change with frequency owing to the dispersive nature of the medium.

Q 4.

(a) Let us denote the unit vectors along the x, y and z-axis to be $\hat{i}$ $\hat{j}$ and $\hat{k}$ respectively. Consider the vector $\vec{v} = yz\hat{i} + xz\hat{j} + yx\hat{k}$. Now consider a cube of side unit length and one of its corners placed at the origin as shown in the fig.1. Now compute the two quantities

$$A = \int_{S_1} \vec{v} \cdot d\vec{S}, \quad B = \int_{S_2} \vec{v} \cdot d\vec{S}.$$

Here S_1 is the surface which consists of 5 sides of the cube (i)-(v) and S_2 is the surface which consists of the 6th side of the cube (vi) as shown in fig.1. How does A compare with B? If you notice any relation between A and B can you provide a mathematical justification for your observation. For this problem use the direction of the surface elements following the arrows shown in fig.1.

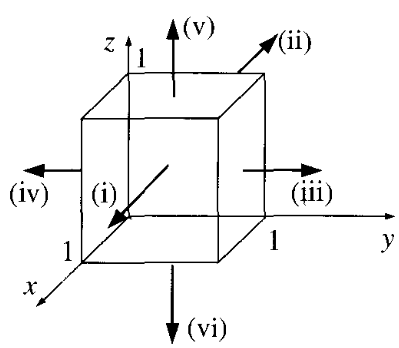

Figure 1: Figure for Problem 4(a).

(b) Consider the vector $\vec{v} = xy\hat{i} + 2yz\hat{j} + 3zx\hat{k}$. Now compute the two quantities

$$A = \int_S \left(\vec{\nabla} \times \vec{v}\right) \cdot d\vec{S}, \quad B = \oint \vec{v} \cdot d\vec{l}.$$

Here S is the surface of the shaded square as shown in fig.2. While the line integral is to be performed over the boundary line of surface S in the

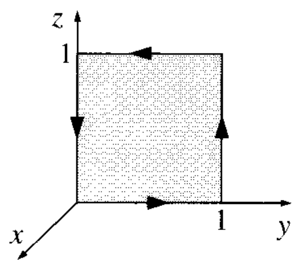

Figure 2: Figure for Problem 4(b).

direction as denoted by the arrows in fig.2. Consider the direction of $\vec{dS}$ to be the direction of positive x-axis.

Ans 4 . (a) The given vector is $v = yz\hat{i} + zx\hat{j} + xy\hat{k}$.

Along surface 1 : Surface element : $dydz\ \hat{i}$.
The integral is given by : $\int yzdydz = \frac{1}{4}$

Along surface 2: surface element : $-dydz\ \hat{i}$.
The integral is given by: $-\int yzdydz = -\frac{1}{4}$

Along surface 3 : Surface element : $dzdx\ \hat{j}$.
The integral is given by : $\int xzdxdz = \frac{1}{4}$

Along surface 4: surface element : $-dxdz\ \hat{j}$.
The integral is given by: $-\int xzdxdz = -\frac{1}{4}$

Along surface 5: surface element : $-dxdy\ \hat{k}$.
The integral is given by : $\int xydxdy = \frac{1}{4}$

Along surface 6: surface element : $-dxdy\ \hat{k}$.
The integral is given by : $\int xydxdy = -\frac{1}{4}$

Hence the total integral $A = \int_{S_1} \vec{v}.\vec{dS} = \frac{1}{4}$

According to Gauss divergence theorem : $\int_V (\vec{\nabla}.\vec{v})dV = \int_S \vec{v}.\vec{dS}$

For the given vector the divergence is zero. Hence $\int_S \vec{v}.\vec{dS} = A + B = 0$ and $A = -B$.

(b) The vector is given by $v = xy\hat{i} + 2yz\hat{j} + 3zx\hat{k}$. Computing A:

$$\nabla \times v = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{bmatrix} = -2y\hat{i} - 3z\hat{j} - x\hat{k}. \tag{7}$$

The integral $A = \int_S (\nabla \times v) dS = \int_S -2ydydz = -1$.

Now, computing B: Line element along y axis and $z = 0$: The vector component is zero hence the integral is also zero.

Line element along z axis and $y = 1$: The vector component is zero as we are in $x = 0$ plane.

Line element along the y axis and $z = 1$: the vector component is given by $2y\hat{j}$. The integral is given by $-\int 2ydy = -1$

Line element along z axis and $y = 0$: The vector component is zero as we are in $x = 0$ plane.
Hence $B = \int \vec{v}.\vec{dl} = -1$

Q 5. Let us denote the unit vectors along the x, y and z-axis to be $\hat{i}$ $\hat{j}$ and $\hat{k}$ respectively, and the unit vector along the radial direction to be $\hat{r}$. In terms of cartesian coordinates, we can write

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \ r = \sqrt{x^2 + y^2 + z^2}, \ \hat{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Now consider the multivariable function

$$\phi(x, y, z) = \frac{\hat{r} \cdot \hat{k}}{r^2}$$

.

(a) Find $\vec{\nabla}\phi$ (gradient of ϕ), and express your answer completely in terms of the quantities $\hat{k}, \hat{r}, r$.

(b) Hence evaluate the surface integral

$$\oint \left(\vec{\nabla}\phi\right) \cdot d\vec{S}$$

where the integration is performed over the (closed) surface of the sphere $x^2 + y^2 + z^2 = 1$. Consider the outward normal to the surface to be the direction of $d\vec{S}$.

Ans 5. **(a)** The scalar is given as $\phi(x,y,z) = \frac{\hat{r}.\hat{k}}{r^2} = \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$.

The gradient of this scalar is given by:

$$\nabla\phi = \frac{-3z}{(x^2+y^2+z^2)^{\frac{5}{2}}}(x\hat{i}+y\hat{j}+z\hat{k}) + \frac{\hat{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = -\frac{3(\vec{r}.\hat{k})\vec{r}}{r^5} + \frac{\hat{k}}{r^3}. \quad (8)$$

(b) Now the gradient obtained in the previous problem can be expressed in spherical polar coordinates when we substitute $\hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$.

We obtain $\nabla\phi = -\frac{2\cos\theta}{r^3}\hat{r} - \frac{\sin\theta}{r^3}\hat{\theta}$.

The integral with the normal along $\hat{r}$ on the surface of the sphere then becomes $\int(\vec{\nabla}\phi).\vec{dS} = \int -2\cos\theta\sin\theta d\theta d\phi = 0$. So the flux passing through the surface is 0.