Problem Set-3

PH11001 (Spring 2019-20)

Mechanical waves and vector calculus

January 25, 2020

1. *Standing waves on a string:*

   Consider a guitar string (say, the 1st-string (E-high): the thinest string). Its tension is adjusted such that when open (string vibrating at full length), its *fundamental frequency* is 330 Hz (E). Keeping the tension same, if we now close the first fret (i.e. put our finger on the first fret), the *fundamental frequency* becomes 350 Hz (F). If the length of the string is about 60 cm, can you estimate the reduction in length of the string when you place your finger on the first fret? Note that this is roughly the distance between the nut (0th-fret) and the first fret.

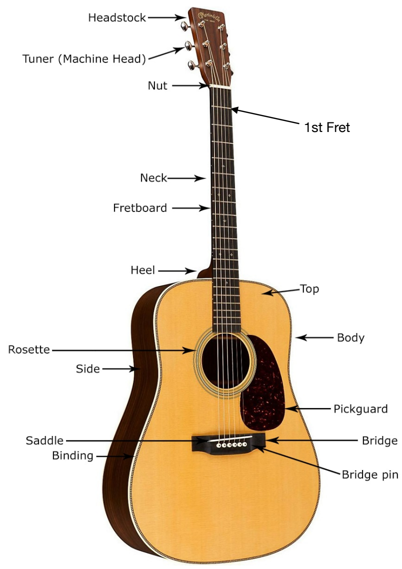

Figure 1: The anatomy of a guitar.

2. *Sound waves in solid:*

   Consider a sound wave travelling in a solid with Young's modulus $Y = 2 \times 10^{11}$kg m^{-1} s^{-2}, and whose mass density is $\rho = 8 \times 10^3$ kg m^{-3}. The wave-solution has the form $\xi(x,t) = A\cos^2(kx - (2\pi \times 10^2)t)$, where x and t are measured in SI units.

   (a) Calculate the velocity of sound in the given solid.

   (b) Compute the wavelength and frequency of the the given wave-form.

3. *Phase velocity and group velocity:*

   Find the phase velocity and group velocity for the following two wave-forms (x and t are measured in SI units)

   (i) $\psi(x,t) = \cos(4t - 2x) + \cos(8t - 4x)$.

   (ii) $\psi(x,t) = \cos(8t - 6x) + \cos(4t - 4x)$.

   Does these waveforms satisfy the wave equation

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

   where v is a *constant* independent of frequency. Explain.

4. *Vector Calculus:*

   (a) Let us denote the unit vectors along the x, y and z-axis to be $\hat{i}$ $\hat{j}$ and $\hat{k}$ respectively. Consider the vector $\vec{v} = yz\hat{i} + xz\hat{j} + yx\hat{k}$. Now consider a cube of side unit length and one of its corners placed at the origin as shown in the fig.2. Now compute the two quantities

$$A = \int_{S_1} \vec{v} \cdot \vec{dS}, \quad B = \int_{S_2} \vec{v} \cdot \vec{dS}.$$

   Here S_1 is the surface which consists of 5 sides of the cube (i)-(v) and S_2 is the surface which consists of the 6th side of the cube (vi) as shown in fig.2. How does A compare with B? If you notice any relation between A and B can you provide a mathematical justification for your observation. For this problem use the direction of the surface elements following the arrows shown in fig.2.

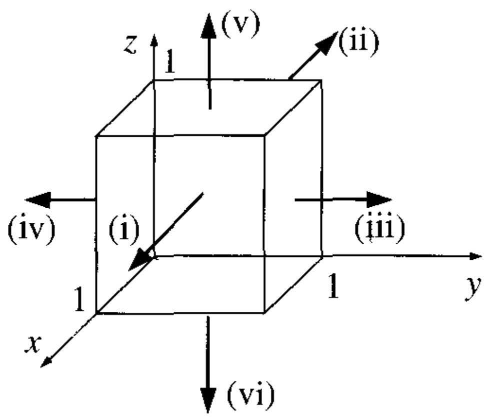

Figure 2: Figure for Problem 4(a).

   (b) Consider the vector $\vec{v} = xy\hat{i} + 2yz\hat{j} + 3zx\hat{k}$. Now compute the two quantities

$$A = \int_S \left(\vec{\nabla} \times \vec{v}\right) \cdot \vec{dS}, \quad B = \oint \vec{v} \cdot \vec{dl}.$$

Here S is the surface of the shaded square as shown in fig.3. While the line integral is to be performed over the boundary line of surface S in the direction as denoted by the arrows in fig.3. Consider the direction of $\vec{dS}$ to be the direction of positive x-axis.

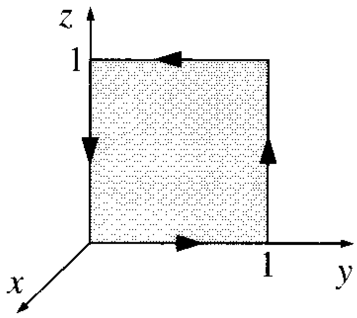

Figure 3: Figure for Problem 4(b).

5. *Vector Calculus:*

   Let us denote the unit vectors along the x, y and z-axis to be $\hat{i}$ $\hat{j}$ and $\hat{k}$ respectively, and the unit vector along the radial direction to be $\hat{r}$. In terms of cartesian coordinates, we can write

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ r = \sqrt{x^2+y^2+z^2},\ \hat{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

   Now consider the multivariable function

$$\phi(x,y,z) = \frac{\hat{r}\cdot\hat{k}}{r^2}$$

   .

   (a) Find $\vec{\nabla}\phi$ (gradient of ϕ), and express your answer completely in terms of the quantities $\hat{k}, \hat{r}, r$.

   (b) Hence evaluate the surface integral

$$\oint \left(\vec{\nabla}\phi\right)\cdot \vec{dS}$$

   where the integration is performed over the (closed) surface of the sphere $x^2+y^2+z^2 = 1$. Consider the outward normal to the surface to be the direction of $\vec{dS}$.