Solution of Tutorial-1 for PH11001 course
Spring 2020, IIT Kharapur
January 15, 2020

Question 1.
A liquid is kept ina U-shaped tube. If h is the equilibrium height of liquid of both arms and $2d$ is the inner separation between the two arms of the U-tube then find the equation of motion for the vertical displacement x of the liquid surface. Also find the frequency of oscillation from the equation of motion.

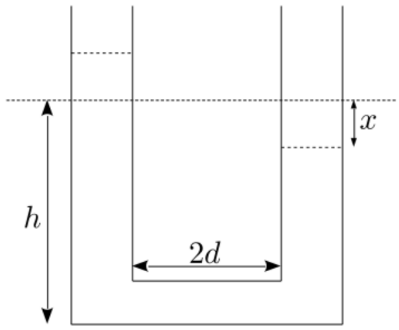

Figure 1: Liquid in U-tube

Solution.
The weight of the displaced volume of the liquid will act as the restoring force on the total mass of the liquid to obtain the simple harmonic motion. Assuming the area A to be uniform over the total length of the u-tube and the density of the liquid to be ρ, the restoring force can be written as

$$F_{res} = mg = (A \times 2x)\rho g = 2A\rho g x.$$

The total mass of the liquid is thus given by

$$M = A(2h + 2d)\rho = 2A\rho(h + d).$$

Therefore, the equation of the SHM is obtained by equating the restoring force with the acceleration of the liquid as follows

$$\begin{aligned} M\frac{d^2x}{dt^2} &= -F_{res} \\ \Rightarrow 2A\rho(h+d)\ddot{x} &= -2A\rho g x \\ \therefore \ddot{x} &= -\frac{g}{h+d}x. \end{aligned} \tag{1}$$

Comparing Eqn. (1) with the equation of SHM given by $\ddot{x} + \omega^2 x = 0$, the frequency of oscillation is given by

$$\omega = \sqrt{\frac{g}{h+d}}.$$

Question 2.
Consider the potential energy of a point mass(m) given by $V(x) = a - bx - cx^2$. Find out :

1. the sign of the coefficient c such that the particle undergoes harmonic oscillation
2. the origin around which the particle oscillates
3. the oscillation frequency

Solution.
2(a). For a particle to oscillate harmonically, force $F = -\frac{dV(x)}{dx} = -Ax$, where A is some constant. The $-ve$ sign indicates the force to be restoring in nature resulting the particle to undergo SHM. The force due to the given potential is

$$F = -\frac{dV(x)}{dx} = 2cx + b = 2c\left(x + \frac{b}{2c}\right).$$

The second term in the expression of the force is due to a shift in the origin by an amount of $-b/(2c)$ from zero. Therefore, the sign of the constant c has to negative $(-ve)$ so that the force is restoring in nature.

2(b). The potential $V(x)$ can be written as

$$V(x) = a - bx - cx^2 = (a + b^2/4c) - c(x + b/2c)^2,$$

which suggests that the potential (parabola) is having a minima at $x_0 = -b/(2c)$. The terms in the first parenthesis is an overall constant which has no effect on the force (*and we shall not bother with it !*), as the quadratic part describes the simple harmonic harmonic motion about the point $-\frac{b}{2c}$. It can also be verified from the expression of the force.

2(c). Assuming the mass of the particle is m, the definition of the restoring force of the particle undergoing simple harmonic motion is $F = -m\omega^2 x$ or $F = -m\omega^2(x - a_0)$ where ω is the frequency of oscillation. Comparing with the current problem we get

$$\begin{aligned} m\omega^2 &= 2c \\ \therefore \quad \omega &= \sqrt{\frac{2c}{m}} \quad . \end{aligned}$$

Note : For any general quadratic potential, the position of the minima determines the origin about which the particle oscillates and it can be found by solving the equation $\left(\frac{dV}{dx}\right)_{x=x_0} = 0$. In case of a minimum, one must have $\left(\frac{d^2V}{dx^2}\right)_{x=x_0} > 0$ which fixes the sign of c in order to get a simple harmonic motion.

Question 3.

A point performs damped oscillations according to the law: $x = a_0 e^{-\beta t}\sin(\omega t)$, find

1. the oscillation amplitude and the velocity of the point at the moment $t = 0$
2. the moments of time at which the point reaches the extreme positions.

Solution.

3(a). The amplitude for the damped motion will decrease with time given by

$$A(t) = a_0 e^{-\beta t}.$$

At $t = 0$ it will be maximum and will fall exponentially with time

$$A(t = 0) = a_0.$$

The velocity will be given by

$$v(t) = \frac{dx(t)}{dt} = a_0(-\beta\sin(\omega t) + \omega\cos(\omega t))e^{-\beta t}$$

At $t = 0$ the velocity is given by

$$v(t = 0) = a_0\omega.$$

3(b). At the extreme position the velocity will be zero, thus

$$\begin{aligned} \beta\sin(\omega t) &= \omega\cos(\omega t)e^{-\beta t} \\ \Rightarrow \tan(\omega t) &= \frac{\omega}{\beta} \\ \therefore \quad t &= \frac{1}{\omega}\tan^{-1}\frac{\omega}{\beta} + n\pi \quad \text{where} \quad n = 0, 1, 2\ldots \end{aligned}$$

Question 4.

A simple pendulum oscillates in a medium for which the logarithmic decrement is equal to $\lambda_0 = 1.5$. What will be the logarithmic decrement if the resistance of the medium increases by a factor $n = 2$? By what factor has the resistance of the medium to be increased for the oscillation to become impossible?

Solution.
From the definition of the logarithmic decrement we get

$$\lambda_0 = \beta T = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}}.$$

Rearranging the equation we get

$$\frac{\beta}{\omega_0} = \frac{1}{\sqrt{1 + \left(\frac{2\pi}{\lambda_0}\right)^2}}.$$

When the resistanc increses by a factor of n, then

$$\begin{aligned} \lambda &= \frac{2\pi n\beta}{\sqrt{\omega_0^2 - n^2\beta^2}} \\ \Rightarrow \lambda &= \frac{2\pi n}{\sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - n^2}} \\ \therefore \quad \lambda &= \frac{n\lambda_0}{\sqrt{1 + (1-n^2)\left(\frac{\lambda_0}{2\pi}\right)^2}} \end{aligned} \tag{2}$$

Putting $n = 2$ and $\lambda_0 = 1.5$ in Eqn. (2) er get the value of logarithmic decreament given by

$$\lambda = 3.3$$

It is clearly evident that λ blows up as the denominator approches zero $n \to \frac{\omega_0}{\beta}$ or it becomes imaginary as n exceed the limiting value. This is equivalent to say that in order to have finite real values of λ we must have

$$\begin{aligned} 1 + (1-n^2)\left(\frac{\lambda_0}{2\pi}\right)^2 &\geq 0 \\ \Rightarrow \quad n^2 - 1 &\leq \left(\frac{2\pi}{\lambda_0}\right)^2 \\ \therefore \quad n &\leq \sqrt{1 + \left(\frac{2\pi}{\lambda_0}\right)^2} \end{aligned}$$

Plugging the numerical value of λ_0, we get the maximum allowed value of n to have a damped simple harmonic motion is $n = 4.3$.

Question 5.
A spring mass system has an underdamped time period $T_0 = 2\pi$ seconds. It is then subjected to critical damping. The mass is pulled to one side and released from rest at $t = 0$. Find the time τ in seconds, $(0 < \tau < \infty)$, at which the damping force exactly balances the spring force.

Solution.
The equation of a damped simple harmonic motion is given by

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = 0$$

The condition for critical damping is given by

$$\beta = \omega_0 = \frac{2\pi}{T_0} = 1$$

The solution is given by

$$x(t) = (c_1 + c_2 t)e^{-\beta t}$$

Using the initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = 0$, the displacement $x(t)$ can be written as

$$x(t) = x_0(1+\beta t)e^{-\beta t} = x_0(1+t)e^{-t}$$

Method 1.
When the damping force is equal to the restoring force, the total acceleration is zero. Thus we have

$$\begin{aligned} \ddot{x}(\tau) &= 0 \\ \Rightarrow \quad -e^{-t}(1-t)|_{t=\tau} &= 0 \\ \therefore \quad \tau &= 1 \end{aligned}$$

Method 2.
When the spring comes back towards the origin from the extreme position, the direction of the damping force is opposite to the direction of the restoring force. Taking care of the direction of these two forces we can write

$$\begin{aligned} F_{\text{damping}} &= -F_{\text{restoring}} \\ \Rightarrow \quad -2\beta\dot{x}(\tau) &= -(-\omega_0^2 x(\tau)) \\ \Rightarrow \quad -2(-\tau e^{-\tau}) &= (1+\tau)e^{-\tau} \\ \therefore \quad \tau &= 1 \end{aligned}$$

So the damping force will be exactly equal to the spring force after $\tau = 1$ sec.