Quantum Mechanics

Lectrue 4

The story so far

- Several experimental observations suggests that both light and matter has a particle nature and a wave nature simultaneously.
- Such observations, including those of the YDS experiment with electrons, requires us to introduce a concept of probability amplitude $\psi(x)$ (wave-function) associated with each quantum particle.
- This $\psi(x)$ is a complex valued function, and $|\psi(x)|^2$ represents probability density.
- The idea of wave-packets can be used to describe localized particles.

Classical determinism is to be replaced with probabilistic dynamics in the quantum paradigm.

Probability distributions and probability densities

The chance (or probability) of getting "5" in one throw of a single die is 1/6.

The operational meaning of this statement is this: If one casts the same die 6,000 times, one expects that in very nearly 1,000 cases the die will come to rest with number "5" face up. ($6000 \times 1/6 = 1000$)

If one throws 6000 dice once, one would obtain the same result

The result of any individual throw cannot be predicted, but the total number of successes in a given large number of operations can be predicted with considerable accuracy.

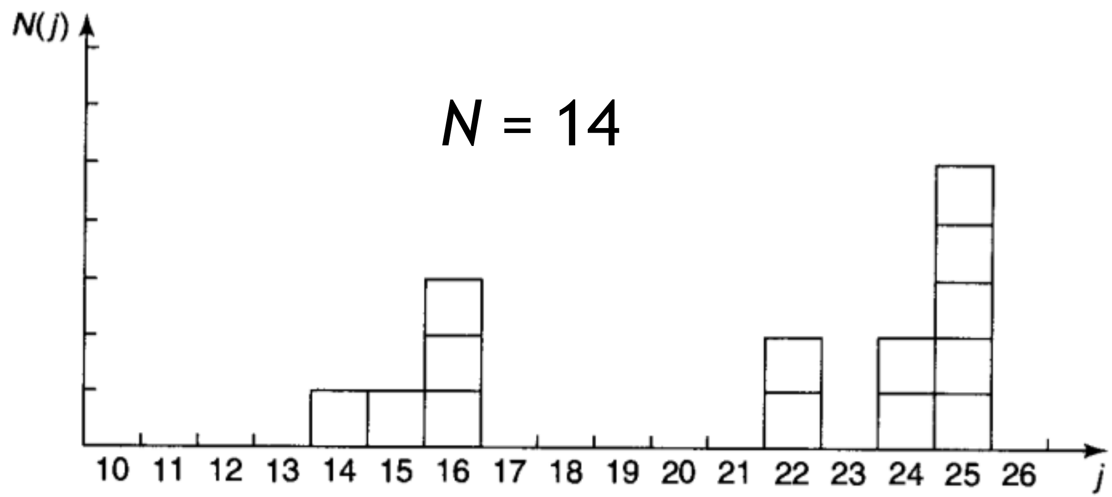
Think of 14 people of different ages in a group

1 aged 14
1 aged 15
3 aged 16
2 aged 22
2 aged 24
5 aged 25

If we represent the number of people of age *j* by *N(j)* then N(14) =1, N(15) =1, N(16) = 3, [N(17) = N(18) = N(19) = N(20) = N(21) = 0], N(22) = 2, N(24) = 2, N(25) = 5.

The total no. of people in the room is

$$N = \sum_{j=0}^{\infty} N(j).$$



We can now ask several questions about this distribution and seek answers

Question 1: What is the probability that a person's age is 15?

Answer: 1/14, since there are 14 people in all, and one of these 14 people has the age of 15.

If $P(j)$ is the probability of getting age j, then $P(14) = 1/14$, $P(15) = 1/14$, $P(16) = 3/14$, $P(17) = 0$, and so on. In general,

$$\boxed{P(j) = \frac{N(j)}{N}.}$$

In particular the sum of all probabilities is 1:

$$\boxed{\sum_{j=1}^{\infty} P(j) = 1.}$$

Question 2: What is the average (or mean) age?

Answer: $$\frac{(14)+(15)+3(16)+2(22)+2(24)+5(25)}{14}=\frac{294}{14}=21.$$

In general, the average value of j is given by:

$$\langle j\rangle=\frac{\sum jN(j)}{N}=\sum_{i=0}^{\infty} jP(j).$$

Question 3: What is the average of the squares of the ages?

Answer: One can get it from $14^2 = 196$, with probability 1/14, $15^2 = 225$, with probability 1/14, $16^2 = 256$, with probability 3/14, and so on. The average then is

$$\langle j^2 \rangle = \sum_{j=0}^{\infty} j^2 P(j).$$

In general, the average of some function of j is given by

$$\langle f(j) \rangle = \sum_{j=0}^{\infty} f(j) P(j).$$

Note: The average of squares ($<j^2>$) is in general **not** equal to the square of the average ($<j>^2$).

We need a numerical measure of the amount of "spread" in a distribution. We define how far each individual deviates from the average,

$$\Delta j = j - \langle j \rangle,$$

Then

$$\langle \Delta j \rangle = \sum (j - \langle j \rangle) P(j) = \sum j P(j) - \langle j \rangle \sum P(j)$$
$$= \langle j \rangle - \langle j \rangle = 0.$$

This is not very useful

We rather work with $(\Delta j)^2$ and its average:

$$\sigma^2 \equiv \langle (\Delta j)^2 \rangle.$$

$$\sigma^2 = \langle (\Delta j)^2 \rangle = \sum (\Delta j)^2 P(j) = \sum (j - \langle j \rangle)^2 P(j)$$

$$= \sum (j^2 - 2j\langle j \rangle + \langle j \rangle^2) P(j)$$

$$= \sum j^2 P(j) - 2\langle j \rangle \sum j P(j) + \langle j \rangle^2 \sum P(j)$$

$$= \langle j^2 \rangle - 2\langle j \rangle \langle j \rangle + \langle j \rangle^2,$$

$$\sigma^2 = \langle j^2 \rangle - \langle j \rangle^2.$$

σ^2 is called the variance and
σ is called the standard deviation.

So far we have been dealing with a discrete variable. For a continuous variable x, we replace $P(j)$ by $\rho(x)$ and shift from summation to integration.

The parameters like average, standard deviation etc. can be defined in terms of a probability density, $\rho(x)$. The probability that x lies between a and b (a finite interval) is given by the integral of $\rho(x)$:

$$\boxed{P_{ab} = \int_a^b \rho(x)\,dx,}$$

Normalization $$\int_{-\infty}^{+\infty} \rho(x)\,dx = 1,$$

Average value of x $$\langle x \rangle = \int_{-\infty}^{+\infty} x\rho(x)\,dx,$$

Average value of a function f(x) $$\langle f(x) \rangle = \int_{-\infty}^{+\infty} f(x)\rho(x)\,dx,$$

Variance $$\sigma^2 \equiv \langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2.$$

The wave-function

- The wave-function (probability amplitude) associated with a quantum particle is a complex valued function of position and time: $\psi(x,t)$.

- The associated probability density

$$\rho(x,t) = |\psi(x,t)|^2 = \psi^*(x,t)\psi(x,t)$$

represents the probability distribution. The probability of finding the particle between x and $x + dx$ is $\rho(x,t)dx = |\psi(x,t)|^2 dx$.

The wave-function

- The probability of finding the particle between any two points a and b separated by a finite distance is given by

$$P_{ab} = \int_a^b dx\ \psi^*(x,t)\psi(x,t)$$

- The particle must be somewhere within $-\infty < x < \infty$, so the total probability of finding the particle within this whole range must be unity

$$\boxed{\int_{-\infty}^{\infty} dx\ \psi^*(x,t)\psi(x,t) = 1, \quad \text{(at any } t\text{).}}$$

Normalization of the wave-function

- If a given wave-function is not properly *normalized* then we may find
$$\int_{-\infty}^{\infty} dx\ \psi^*(x,t)\psi(x,t) = N,$$
where N is a **finite** number not equal to 1.

- Then it is possible to define
$$\tilde{\psi}(x,t) = \frac{\psi(x,t)}{\sqrt{N}}, \quad \text{such that} \quad \int_{-\infty}^{\infty} dx\ \tilde{\psi}^*(x,t)\tilde{\psi}(x,t) = 1.$$

- Here, $\tilde{\psi}(x,t)$ is referred to as the normalized wave-function, which has a physical interpretation as probability amplitude.

- Wave-function must fall-off rapidly for large x and it should be finite for all finite values of x.

Wave-functions must be square integrable functions.

An example :

Find the appropriate normalization of the wave-function $\psi(x) = \exp\left(-\frac{x^2}{2a^2}\right)$.

The integration throughout space for this wave-function is given by

$$\int_{-\infty}^{\infty} dx\ \psi^*(x)\psi(x) = \int_{-\infty}^{\infty} dx\ \exp\left(-\frac{x^2}{a^2}\right) \equiv N, \text{ say.}$$

Then,

$$N^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\ \exp\left(-\frac{x^2+y^2}{a^2}\right) = \int_0^{2\pi}\int_0^{\infty} d\phi r dr \exp\left(-\frac{r^2}{a^2}\right)$$
$$= 2\pi \int_0^{\infty} r dr \exp\left(-\frac{r^2}{a^2}\right), \text{ substitue } \rho = \frac{r^2}{a^2} \Rightarrow a^2 d\rho = 2r dr.$$
$$= \pi a^2 \int_0^{\infty} d\rho \exp(-\rho) = \left(a\sqrt{\pi}\right)^2$$

Therefore, the appropriately normalized wave-function is given by

$$\tilde{\psi}(x) = \frac{\psi(x)}{\sqrt{N}} = \frac{1}{\pi^{\frac{1}{4}}\sqrt{a}} \exp\left(-\frac{x^2}{2a^2}\right).$$

Exercise: Find the average, variance and standard deviation for this probability distribution.

3D generalization

In 3 dimensions, the wave-function is a complex function all three spatial coordinates, in addition to time

$$\psi = \psi(x, y, z, t).$$

So the normalization condition is given by

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dxdydz \; \psi^*(x, y, z, t)\psi(x, y, z, t) = 1$$

The phase of the wave-function

- Two wave-function should be linearly superposed to obtain the combined effect

$$\psi = \psi_1 + \psi_2 = |\psi_1|e^{i\theta_1} + |\psi_2|e^{i\theta_2}.$$

- The total probability is therefore, NOT simply the sum of the individual probabilities:

$$\begin{aligned} |\psi|^2 &= (|\psi_1|e^{i\theta_1} + |\psi_2|e^{i\theta_2})(|\psi_1|e^{-i\theta_1} + |\psi_2|e^{-i\theta_2}) \\ &= |\psi_1|^2 + |\psi_2|^2 + 2|\psi_1||\psi_2|\cos(\theta_1 - \theta_2) \end{aligned}$$

- The cross-term in this probability density is responsible for interference and diffraction effects.

The bra and the ket

- The wave-function represents a state of the system.

- The *space* of all states describing a quantum particle consists of all complex square integrable functions. Such a space has a *vector space structure*. This is like vectors in 3 D space which you are familiar with, only now we have an *infinite dimension space* to deal with.

- The infinite dimensional nature of the state space may be understood as follows. You have seen how a function may be written in terms of sin and cos functions (Fourier Series). Continuing the analogy with a vector in 3 dimensions, you may think of the sin and cos functions to be like the unit vectors along x, y and z directions.

$$f(x) = \frac{a_0}{2} + \sum_{\mathbf{n=1}}^{\infty} \left(a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right) \rightarrow \vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}.$$

- In general, infinite number of of such sin and cos functions are necessary, in this sense the space of such functions is infinite dimensional.

- The wave-function $\psi(x,t)$ thus represent a vector in this abstract *state space* $\mathscr{E}$, which Dirac represented by a **bra** : $|\psi(t)\rangle$

- For every **bra** there is a **ket** : $\langle\psi(t)|$

Inner Product

- The names **bra** and **ket** were given purposefully, because you can put them together to form a *braket*

$$\langle \psi(t)|\psi(t)\rangle = \int_{-\infty}^{\infty} dx\ \psi^*(x,t)\psi(x,t)$$

- We can do a similar operation with two different states $|\psi(t)\rangle$ and $|\phi(t)\rangle$

$$\langle \phi(t)|\psi(t)\rangle = \int_{-\infty}^{\infty} dx\ \phi^*(x,t)\psi(x,t) = \langle \psi(t)|\phi(t)\rangle^*$$

- Inner product : $\langle \phi(t)|\psi(t)\rangle \rightarrow$ a generalization of the idea of a dot product.

- For normalized states $\langle \psi(t)|\psi(t)\rangle = 1$.

- The **bra** is analogous to a column matrix, while the **ket** is analogous to a row matrix.

Physical observables as operators

- All physical observables are represented as *operators* acting on states in the state space $\mathscr{E}$.

- Classical dynamical variables such as position $\hat{X}$, momentum $\hat{P}$ and energy $\widehat{\mathcal{H}}$ are now operators.

- On the vectors of $\mathscr{E}$ operators can be thought of as square matrices acting on column matrices. In terms of the wave-functions, operators will be represented by differential operators *.

- The expectation value (average) of an observable $\hat{O}$ when a particle is in a given quantum state $|\psi(t)\rangle$ (or wave-function $\psi(x,t)$) is given by

$$\langle \hat{O} \rangle = \langle \psi(t)|\hat{O}|\psi(t)\rangle = \int_{-\infty}^{\infty} dx \; \psi^*(x,t) \;\; \left(\hat{O} \; \psi(x,t)\right).$$

*Differential operators may also be thought of as analogues to square matrices.

Hermitian Operators

- An operator has a set of eigenvalues and eigenstates in $\mathscr{E}$: $\widehat{O}|\phi_n\rangle = \lambda_n|\phi_n\rangle$

- A Hermitian operator is equal to its own hermitian conjugate. In our language, a Hermitian operator is one which satisfy

$$\langle\phi|\widehat{O}|\psi\rangle = \langle\psi|\widehat{O}|\phi\rangle^* \Rightarrow \int_{-\infty}^{\infty} dx\ \phi^*(\widehat{O}\psi) = \left(\int_{-\infty}^{\infty} dx\ \psi^*(\widehat{O}\phi)\right)^* = \int_{-\infty}^{\infty} dx\ \psi(\widehat{O}\phi)^*$$

- **The eigenvalues of Hermitian operators must be real**

$$\langle\phi_n|\widehat{O}|\phi_n\rangle = \lambda_n\langle\phi_n|\phi_n\rangle = \lambda_n, \quad \langle\phi_n|\widehat{O}|\phi_n\rangle^* = \lambda_n^*$$

Hermitian Operator $\Rightarrow \langle\phi_n|\widehat{O}|\phi_n\rangle = \langle\phi_n|\widehat{O}|\phi_n\rangle^* \Rightarrow \lambda_n = \lambda_n^*$.

- **The eigenvector of Hermitian operators forms a basis in $\mathscr{E}$.**

Physical observables must be Hermitian operators.

Position and Momentum operators

- Consider the eigenstates of the position and momentum operators

$$\hat{X}|x\rangle = x|x\rangle, \quad \hat{P}|p\rangle = p|p\rangle.$$

- The the state $|\psi(t)\rangle$ in the **position representation** is given by

$$\langle x|\psi(t)\rangle = \psi(x,t)$$

  This $\psi(x,t)$ is what we have referred to as the wave-function earlier.

- Similarly, we can write the **same** state $|\psi(t)\rangle$ in the **momentum representation**

$$\langle p|\psi(t)\rangle = \bar{\psi}(p,t)$$

- Here $\psi(x,t)$ is related to $\bar{\psi}(p,t)$ by the relation

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp \quad \bar{\psi}(p,t) \exp\left(\frac{i}{\hbar}px\right)$$

Momentum Operator in position representation

- The momentum operator can be represented as a differential operator in position representation †

$$\langle x|\hat{P}|\psi(t)\rangle = -i\hbar\frac{\partial}{\partial x}\psi(x,t) \Rightarrow \boxed{\hat{P} = -i\hbar\frac{\partial}{\partial x}}$$

- The expectation value of the momentum operator in a state $|\psi(t)\rangle$ can be computed in the position representation in the following way

$$\langle\psi(t)|\hat{P}|\psi(t)\rangle = \int_{-\infty}^{+\infty} dx \quad \psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x,t).$$

- The position operator in position representation is simply

$$\langle x|\hat{X}|\psi(t)\rangle = x\ \psi(x,t) \Rightarrow \boxed{\hat{X} = x}$$

†It is easy to deduce this from the discussion of the previous slide, but it requires a few additional ideas such as completeness relations of basis vectors. For simplicity, let us assume that this form of the operator is given.

$\hat{X}$ and $\hat{P}$ are hermitian operators

- Position operator is hermitian

$$\langle\phi|\hat{X}|\psi\rangle = \int_{-\infty}^{\infty} dx\ \phi^*(x)(x\psi(x)) = \left(\int_{-\infty}^{\infty} dx\ \psi^*(x)(x\phi(x))\right)^* = \langle\psi|\hat{X}|\phi\rangle^*$$

- Momentum operator is hermitian

$$\begin{aligned}
\langle\phi|\hat{P}|\psi\rangle &= \int_{-\infty}^{\infty} dx\ \phi^*(x)(-i\hbar\frac{\partial}{\partial x}\psi(x)) \\
&= -i\hbar\left(\Big[\phi^*(x)\psi(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\ \psi(x)\frac{\partial\phi^*(x)}{\partial x}\right) \\
&= i\hbar\int_{-\infty}^{\infty} dx\ \psi(x)\frac{\partial\phi^*(x)}{\partial x} \\
&= \left(\mathbf{-i}\hbar\int_{-\infty}^{\infty} dx\ \psi^*(x)\frac{\partial\phi(x)}{\partial x}\right)^* \\
&= \left(\int_{-\infty}^{\infty} dx\ \psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\phi(x)\right)^* \\
&= \langle\psi|\hat{P}|\phi\rangle^*
\end{aligned}$$

$\widehat{X}$ and $\widehat{P}$ do not commute

- In general operators may not commute, just like square matrices may not commute under multiplication.

$$[\widehat{A}, \widehat{B}] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A} \neq 0, \quad \text{in general.}$$

- Let us find $[\widehat{X}, \widehat{P}]$ using their position representations

$$\begin{aligned}
\langle\phi|[\widehat{X}, \widehat{P}]|\psi\rangle &= \int_{-\infty}^{\infty} dx\ \phi^*(x)[\widehat{X}, \widehat{P}]\psi(x) \\
&= \int_{-\infty}^{\infty} dx\ \phi^*(x)\left(x\left(-i\hbar\frac{\partial}{\partial x}\right) - \left(-i\hbar\frac{\partial}{\partial x}\right)x\right)\psi(x) \\
&= \int_{-\infty}^{\infty} dx\ \phi^*(x)\left(x\left(-i\hbar\frac{\partial}{\partial x}\right) - x\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\psi(x) \\
&\quad + \int_{-\infty}^{\infty} dx\ \phi^*(x)(i\hbar)\psi(x)
\end{aligned}$$

$$\int_{-\infty}^{\infty} dx\ \phi^*(x)[\widehat{X}, \widehat{P}]\psi(x) = \int_{-\infty}^{\infty} dx\ \phi^*(x)(i\hbar)\psi(x) \Rightarrow \boxed{[\widehat{X}, \widehat{P}] = i\hbar\mathbb{1}}$$

Non-Commuting conjugate observables and Uncertainty principle

- Let us consider two non-commuting conjugate variables like $[\hat{X}, \hat{P}] = i\hbar\mathbb{1}$.

- In any given state $|\psi\rangle$, the variance in the corresponding probability distribution, for these observables are given by

$$\sigma_X^2 = \left\langle \left(\hat{X} - \langle\psi|\hat{X}|\psi\rangle\right)^2 \right\rangle, \quad \sigma_P^2 = \left\langle \left(\hat{P} - \langle\psi|\hat{P}|\psi\rangle\right)^2 \right\rangle$$

- Then we can argue

$$\boxed{\sigma_X \sigma_P \geq \frac{\hbar}{2}}$$

The uncertainty principle applies to non-commuting observables.

The Postulates of Quantum Mechanics

Postulate 1: At a given time t_0 the state of a physical quantum mechanical system is defined by a vector $|\psi\rangle$, which belongs to the state space $\mathscr{E}$.

The principle of linear superposition is implied by this postulate since $\mathscr{E}$ is a linear vector space.

Postulate 2: Every measurable quantity $\mathcal{A}$ is described by an Hermitian operator $\hat{\mathcal{A}}$ acting in $\mathscr{E}$.

$$\boxed{\text{Observables are Hermitian operators}}$$

Postulate 3: The only possible result of measurement of a physics quantity $\mathcal{A}$ are the eigenvalues of the corresponding operator $\widehat{\mathcal{A}}$.

Postulate 4: Consider a physical quantity, represented by operator $\widehat{\mathcal{A}}$, has a set of (discrete non-degenerate [‡]) eigenvalues $\{a_n\}$ with corresponding **normalized** eigenvectors $|\phi_n\rangle$. Let the system be in some arbitrary **normalized** state $|\psi\rangle$. Then the probability of obtaining the eigenvalue a_n, when we measure $\mathcal{A}$ is given by

$$\mathcal{P}(a_n) = |\langle\phi_n|\psi\rangle|^2 .$$

[‡] A similar postulate also exists for when we have continuous and degenerate eigenvalues, we refrain from discussing that to keep it simple.

Postulate 5: The state of the system immediately after the measurement of $\mathcal{A}$, is always an eigenvector of $\hat{\mathcal{A}}$ with eigenvalue a_n i.e. $|\phi_n\rangle$, where a_n is the outcome of the measurement.

On measurement, the quantum state collapses to an eigenstate of the observable being measured.

Postulate 6: The time evolution of the state $|\psi(t)\rangle$, is governed by the **Schrödinger equation**

$$\boxed{i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \widehat{\mathcal{H}} |\psi(t)\rangle,}$$

where $\widehat{\mathcal{H}}$ **is called the Hamiltonian operator** and it is associated with the total energy of the system. For a particle experiencing a force due to a potential $V(x, y, z)$, the Hamiltonian operator is given by

$$\widehat{\mathcal{H}} = \frac{\widehat{p}^2}{2m} + V(\widehat{x}, \widehat{y}, \widehat{z}).$$

In the position representation this operator can be represented as

$$\widehat{\mathcal{H}}(x,y,z) = \frac{-\hbar^2\nabla^2}{2m} + V(x,y,z).$$

Also in the position representation the state $|\psi(t)\rangle$ will be represented by the function $\psi(x,y,z,t)$.

This function should then satisfy the differential equation

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi(x,y,z,t) = \frac{-\hbar^2}{2m}\nabla^2\psi(x,y,z,t) + V(x,y,z)\psi(x,y,z,t).}$$

In the simpler case of a one-dimensional problem, we simply have the equation

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi(x,t) = \frac{-\hbar^2}{2m}\;\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t).}$$

Schrödinger equation is a linear differential equation which is consistent with the linear superposition principle of the wave-function.

The time-independent Schrödinger equation

- The general time-dependent Schrödinger equation is given by

$$i\hbar\frac{\partial}{\partial t}\psi(x,y,z,t) = \widehat{\mathcal{H}}\psi(x,y,z,t)$$

  Let us assume that the Hamiltonian operator is time independent, for instance the Hamiltonian operator of a single particle under the influence of a potential

$$\widehat{\mathcal{H}} = \frac{\widehat{p}^2}{2m} + V(\widehat{x},\widehat{y},\widehat{z}).$$

- Let us see what happens to the eigen-states of the hamiltonian

$$\widehat{\mathcal{H}}\ \psi_E(x,y,z,t) = E\ \psi_E(x,y,z,t)$$

  where E is a specific eigen-value of the Hamiltonian operator.

- Now let us also do a separation of variables $\psi_E(x,y,z,t) = \tilde{\psi}_E(x,y,z)\phi(t)$. Then, the Schrödinger equation will reduce to

$$i\hbar\frac{\partial}{\partial t}\tilde{\psi}_E(x,y,z)\phi(t) = E\tilde{\psi}_E(x,y,z)\phi(t) \Rightarrow i\hbar\frac{d}{dt}\phi(t) = E\phi(t) \Rightarrow \phi(t) = e^{-i\frac{Et}{\hbar}}.$$

- Thus such energy eingen-states must always have the form

$$\boxed{\psi_E(x,y,z,t) = e^{-i\frac{Et}{\hbar}}\tilde{\psi}_E(x,y,z),}$$

where $\psi_E(x,y,z)$ must be a solution to

$$\begin{aligned}
&\widehat{\mathcal{H}}\ \tilde{\psi}_E(x,y,z)\phi(t) = E\ \tilde{\psi}_E(x,y,z)\phi(t)\\
\Rightarrow &\left(\frac{\widehat{p}^2}{2m} + V(\widehat{x},\widehat{y},\widehat{z})\right)\ \tilde{\psi}_E(x,y,z)\phi(t) = E\ \tilde{\psi}_E(x,y,z)\phi(t)\\
\Rightarrow &\left(-\frac{\hbar^2}{2m}\nabla^2\tilde{\psi}_E(x,y,z) + V(x,y,z)\tilde{\psi}_E(x,y,z)\right) = E\ \tilde{\psi}_E(x,y,z)
\end{aligned}$$

- **Note that, since such an energy eigenstate always has the form**

$$\boxed{\psi_E(x,y,z,t) = e^{-i\frac{Et}{\hbar}}\tilde{\psi}_E(x,y,z),}$$

the probability density $\rho = |\psi_E(x,y,z,t)|^2 = |\tilde{\psi}_E(x,y,z)|^2$ **is independent of time.**

- Therefore the eigenstates of the time-independent Hamiltonian are called **stationary states** and the equation

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\tilde{\psi}_E(x,y,z) + V(x,y,z)\tilde{\psi}_E(x,y,z) = E\ \tilde{\psi}_E(x,y,z)}$$

is called the **time-independent Schrödinger equation**.

- In one dimension, we have

$$\boxed{-\frac{\hbar^2}{2m}\ \frac{d^2\tilde{\psi}_E(x)}{dx^2} + V(x)\tilde{\psi}_E(x) = E\ \tilde{\psi}_E(x)}.$$