Solutions to tutorial 4

February 4, 2020

1. *Divergence and curl of vector fields:*

   Consider a force (vector) field given by

$$\vec{F} = (x^2 + y^2 + z^2)^n (x\hat{i} + y\hat{j} + z\hat{k}).$$

   Find

   (a) $\int_V \left(\vec{\nabla} \cdot F\right) dV$, where V is the volume of the sphere of radius R.

   Ans: The divergence of F is given by $\vec{\nabla} \cdot F = (3 + 2n)(x^2 + y^2 + z^2)^n = (3 + 2n)r^{2n}$.

$$\begin{aligned}\int \vec{\nabla} \cdot F dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} ((3 + 2n)r^{2n})r^2 \sin\theta dr d\theta d\phi \\ &= 4\pi R^{2n+3}\end{aligned}$$

   (b) $\vec{\nabla} \times \vec{F}$

   Ans:

$$\begin{aligned}\vec{\nabla} \times F &= \begin{bmatrix} \hat{j} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix} \\ &= 0\end{aligned}$$

   (c) a scalar field $\phi(x, y, z)$ such that $\vec{F} = -\vec{\nabla}\phi$.

   Ans: We have $\vec{F} = -\vec{\nabla}\phi = r^{2n+1}\hat{r}$
   Gradient can be expressed in terms of Spherical Polar Coordinates as:

$$\vec{\nabla}\phi = \hat{r}\frac{\partial \phi}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial \phi}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta}\frac{\partial \phi}{\partial \phi}$$

   Since $\vec{F}$ is independent of θ and ϕ so we can write gradient as:

$$\vec{\nabla}\phi = \hat{r}\frac{\partial \phi}{\partial r} = r^{2n+1}\hat{r}$$

   Integrating this we get

$$\phi = -\frac{r^{2n+2}}{2n + 2} + C$$

   (d) For what value of the exponent n does the scalar field diverge at both the origin as well as infinity?

   Ans: Special case : When $n = -1$,

$$\phi = -\log r + C$$

   Here we can see that ϕ diverges at both zero and infinity because of the property of log

2. *Divergence theorem*

   (a) If ϕ is a any scalar field and the surface integral is performed over the closed surface S which is the boundary of volume V, then using 'divergence theorem' show that

$$\int_V \vec{\nabla}\phi = \oint_S \phi \, d\vec{S}$$

   [Hint: Take the vector in 'divergence theorem' to be of the special form $\vec{A} = \vec{C}\phi$, where $\vec{C}$ is a constant, but arbitrary vector. Note that $\vec{A}$ and ϕ are vector and scalar fields respectively.]

   Ans: Let $\vec{A} = \vec{C}\phi$. Then $\int_V \vec{\nabla}(\vec{C}\phi)dV = \oint_S \vec{C}\phi.\hat{n}d\vec{S}$. Also note that $\vec{\nabla}\cdot\phi\vec{C} = \nabla\phi\cdot\vec{C} = \vec{C}\cdot\nabla\phi$.

$$\vec{C}\cdot\int_V(\vec{\nabla}\phi)dV = \vec{C}\cdot\int\phi\hat{n}dS$$
$$\Longrightarrow \int_V \vec{\nabla}\phi dV = \oint_S \phi d\vec{S}$$

   (b) Using 'divergence theorem' show that $\oint d\vec{S} = 0$ for any closed surface. Now, if $\oint \hat{n}\cdot d\vec{S}$ is the total surface area of the closes surface what should be $\hat{n}$?

   Ans: It can be shown easily by taking $\phi = 1$ or any constant in the above expression which gives

$$\int_V \vec{\nabla}K dV = \oint_S K d\vec{S} = 0$$

   The second part can be solved as follows: $\oint_s \hat{n}\cdot d\vec{S} = \oint_S \hat{n}\cdot\hat{n}dS = \oint_s dS$ which represents the total surface area. Here $\hat{n}$ should be unit vector parallel to $d\vec{S}$, ie the normal to the surface.

3. *Stokes theorem*

   Using stokes theorem prove the following identities

   (a) If ϕ is a any scalar field and the line integral is performed over the closed line C which is the boundary of surface S, then show that

$$\int_S d\vec{S}\times\vec{\nabla}\phi = \oint_C \phi \, d\vec{\ell}$$

   (Same hint as problem 2(a) can be useful here as well.)

   Ans: Let $\vec{A} = \vec{C}\phi$, where $\vec{C}$ is a constant. Stoke's Theorem gives

$$\oint_s \left(\vec{\nabla}\times\vec{A}\right).\hat{n}dS = \oint_C \vec{A}.d\vec{l}$$
$$\Longrightarrow \oint_s \left(\vec{\nabla}\times\vec{C}\phi\right).\hat{n}dS = \oint_C \vec{A}.d\vec{l}$$
$$\Longrightarrow \oint_s \phi\left(\vec{\nabla}\times\vec{C}\right).\hat{n}dS - \oint_s \left(\vec{C}\times\vec{\nabla}\phi\right).\hat{n}dS = \oint_C \vec{A}.d\vec{l}$$
$$\Longrightarrow \oint_s \left(\vec{C}\times\vec{\nabla}\phi\right).\hat{n}dS = \oint_C \vec{A}.d\vec{l} \quad \left[\vec{\nabla}\times\vec{C} = 0\right]$$
$$\Longrightarrow \oint_s \vec{C}.\left(\hat{n}dS\times\vec{\nabla}\phi\right) = \oint_C \vec{C}\phi.d\vec{l}$$
$$\Longrightarrow \oint_s \left(\hat{n}dS\times\vec{\nabla}\phi\right) = \oint_C \phi.d\vec{l}$$

   Hence proved.

(b) Using stokes theorem argue that, if $\vec{B} = \vec{\nabla} \times \vec{A}$, then $\oint_S \vec{B} \cdot d\vec{S} = 0$, for any closed surface S. Can you arrive at the same conclusion using the divergence theorem?

Ans: Divergence theorem gives

$$\int_V \left(\vec{\nabla}.\vec{B}\right) dV = \oint_S \vec{B}.d\vec{S}$$

which gives after putting $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\int_V \vec{\nabla}.\left(\vec{\nabla} \times \vec{A}\right) dV = \oint_S \vec{B}.d\vec{S} = 0$$

since divergence of a curl is zero.

Same result can be seen using Stokes theorem. This can be shown as follows:
According to stokes theorem,

$$\oint_s \left(\vec{\nabla} \times \vec{A}\right).\hat{n}dS = \oint_C \vec{A}.d\vec{l} = 0$$

The above line integral is zero as here the surface is a closed surface which has no boundary. So the line integral which represents the boundary of the surface becomes zero.

4. *Electrostatics*

(a) Imagine a situation where our world is 2 dimensional instead of 3 dimensional (in addition there is ofcourse time in both the cases), and the local form of Gauss law in electrostatics remains the same, i.e. we still have

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}.$$

where ρ is the charge density (charge per unit area, since in 2 dimensions 'area' is like 'volume'). Make a prediction on the nature of the modified 'Coulomb's law' in such an imaginary world?

Ans: We know from Guass's law, $\int_V \left(\vec{\nabla}.\vec{E}\right) dV = \frac{Q}{\epsilon_0}$. Applying divergence theorem in 2D,

$$\int_v \left(\vec{\nabla}.\vec{E}\right) dV = \int_C \vec{E}.\hat{n}dl$$

, where C is the boundary of the surface and $\hat{n}$ is the normal unit vector to the boundary. Hence,

$$\int_C \vec{E}.\hat{n}dl = \frac{Q}{\epsilon_0}$$

This yields

$$\begin{aligned} E.2\pi r &= \frac{Q}{\epsilon_0} \\ \implies E &= \frac{Q}{2\pi\epsilon_0 r}\hat{r} \\ \implies \vec{F} = q\vec{E} &= \frac{1}{2\pi\epsilon_0}\frac{qQ}{r} \end{aligned}$$

This is the well-known coulomb's law.

(b) Use Gauss Law, to obtain the electric field (everywhere) due to a static uniform charge density ρ, occupying the spherical shell with inner radius a and outer radius b (i.e. the region $a \leq r \leq b$, r being the radial distance from the origin). Make a plot of magnitude of the electric field as a function of the radial coordinate r.

Ans: At $r < 0$, $Q_{enc} = 0$ implies $E = 0$. Now, we know
$\oint_s \vec{E}.d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$, for $a \leq r \leq b$. This yields

$$
\begin{aligned}
|\vec{E}|4\pi r^2 &= \frac{1}{\epsilon_0}\int \rho dV \\
\implies |\vec{E}|4\pi r^2 &= \frac{1}{\epsilon_0}\int \rho r^2 \sin\theta d\theta d\phi dr \\
\implies |\vec{E}|4\pi r^2 &= \frac{4\pi}{\epsilon_0}\int_a^r \rho r^2 dr \\
\implies |\vec{E}| &= \frac{\rho}{3\epsilon_0}\frac{r^3 - a^3}{r^2}
\end{aligned}
$$

, for $r \leq a$ and for $a \leq r \leq b$ and outside the shell,

$$
\begin{aligned}
|\vec{E}|4\pi r^2 &= \frac{4\pi}{\epsilon_0}\int_a^b \rho r^2 dr \\
\implies |\vec{E}| &= \frac{\rho}{3\epsilon_0}\frac{b^3 - a^3}{r^2}
\end{aligned}
$$

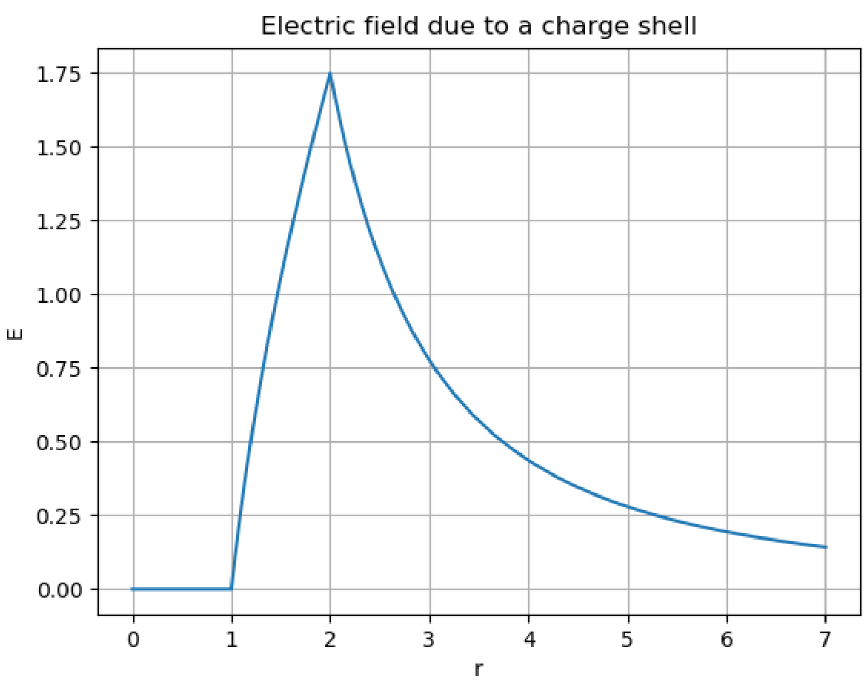

Figure 1: $a = 1$ and $b = 2$

5. *Magnetostatics*

Consider a wire segment carrying a steady current I as shown in the figure. Compute the magnetic field created due to this steady current at the point P, which is shown in the figure.

Ans: Consider a infinitesimal length element dx at a distance x from origin and subtending an angle θ with respect to y axis. The magnetic field prdouced at point P due to this infinitsmal element is given by :

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{x} \times \vec{r}}{r^3}.$$

Now $d\vec{x} \times \vec{r} = r dx \cos\theta \hat{z}$. Hence $d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{dx \cos\theta \hat{z}}{r^2}$. Now, as $\tan\theta = \frac{x}{s} \implies x = s\tan\theta$. Substituting dx and $r = \frac{s}{\cos\theta}$ and integrating over the full range we get:

$$\begin{aligned} \vec{B} &= \int_{\theta_1}^{\theta_2} \frac{\mu_0 I}{4\pi s} \cos\theta d\theta \hat{z} \\ &= \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)\hat{z} \end{aligned}$$

Tutorial-4 is solved by Tara Singha and Priyadarshini Pandit