Problem Set-0

PH11001 (Spring 2019-20)

Mathematical prerequisites

December 27, 2019

1. *Vector algebra: scalar (dot) product:*

   Consider the following vector in two dimensions

   $$\vec{v} = \cos(\omega t)\ \hat{\imath} + \sin(\omega t)\ \hat{\jmath},$$

   where $\hat{\imath}$ and $\hat{\jmath}$ are mutually orthogonal unit vectors along the cartesian coordinate axes x and y respectively. Here t is a parameter, which you can physically think of as 'time', and ω is some constant. Clearly, the x and y components of the given vector $\vec{v}$ changes with 'time'.

   (a) How does the magnitude and direction of the vector vary with 'time' t?

   (b) What is the angle θ made by the vector $\vec{v}$ with the following vector

   $$\vec{u} = \sin(\omega t)\ \hat{\imath} + \cos(\omega t)\ \hat{\jmath}.$$

   (c) Plot this angle $\theta(t)$ as a function of t. You may consider $\omega = 1$ for making this plot.

2. *Vector algebra: vector (cross) product:*

   Consider the set of following three non-co-planner vectors

   $$\vec{a} = 2\ \hat{\imath},\ \vec{b} = \hat{\jmath} + \hat{k},\ \vec{c} = \hat{\imath} + \hat{k},$$

   where again $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are mutually orthogonal unit vectors along the cartesian coordinate axes x, y and z respectively.

   (a) Explicitly evaluate the following vectors

   $$\vec{A} = \frac{\vec{b} \times \vec{c}}{\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)},\ \vec{B} = \frac{\vec{c} \times \vec{a}}{\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)},\ \vec{C} = \frac{\vec{a} \times \vec{b}}{\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)}.$$

   (b) Find the volume of the parallelepiped spanned by the vectors $\{\vec{a}, \vec{b}, \vec{c}\}$.

   (c) Find the volume of the parallelepiped spanned by the vectors $\{\vec{A}, \vec{B}, \vec{C}\}$. Is this related in any way to your answer in part (b)?

3. *Taylor series:*

   (a) Expand $f(x) = \cos x$ in a Taylor series about $x = 0$, and write down the first *three* non-vanishing terms.

(b) Using the first *two* non vanishing terms in the expansion about $x = 0$, evaluate the value of $\cos x$ at $x = \frac{1}{2}$.

(c) Compare your result in part (b) with the exact value of $\cos\frac{1}{2}$ and estimate the error associated with approximating this function with the first two terms of the Taylor series.

(d) Do you expect this error to depend on the value of x (which is $x = \frac{1}{2}$ here). For example, if we evaluated the first two terms at $x = \frac{1}{3}$ instead of $x = \frac{1}{2}$, do you expect the error to increase or decrease. Explain your answer.

(e) How does this error change if you keep all the first three (instead of two) non-vanishing terms of part (a) to evaluate this function at $x = \frac{1}{2}$?

4. *Solving second order ordinary differential equations by the series method:*

Consider the differential equation

$$\frac{d^2x(t)}{dt^2} + \beta\; x(t) = 0. \tag{1}$$

Solve this equation using the following two distinct methods.

(a) *Method 1:*

(i) Assume $x(t) = \sum_{n=0}^{\infty} a_{(n)} t^n$ and derive a recursion relation between $a_{(n+2)}$ and $a_{(n)}$.

(ii) Use this to get the series solution to the differential equation in terms of the unknown constants $a_{(0)}$ and $a_{(1)}$.

(iii) Using the substitutions $a_{(0)} \to A\cos\phi$ and $a_{(1)}/\beta \to -A\sin\phi$ show that the following is a solution to equation (1)

$$x(t) = A\cos\left(\beta t + \phi\right).$$

(b) *Method 2:*

(i) Multiply equation (1) with $2\frac{dx(t)}{dt}$ and write it as a total time derivative. Integrate the resultant equation to obtain

$$\left(\frac{dx(t)}{dt}\right)^2 + \beta^2\; x(t)^2 = k^2.$$

(ii) Rearrange this to the integral form

$$\int dt = \int \frac{dx}{\sqrt{k^2 - \beta^2 x^2}}.$$

Hence, perform the integral to obtain the most general solution of the differential equation.

5. *Performing line integrals:*

Consider a projectile which is fired at an angle of $45°$ with an initial velocity of 10 ms^{-1}, from the surface of the earth. It starts from point A on the surface of the earth, and drops at point B also on the surface of the earth (at the same level).

(a) The curve representing the projectile lies on a plane labeled by cartesian coordinates $\{x, y\}$. Write down the equation of the curve in this plane representing the trajectory of the projectile. Choose your x-axis along the surface of the earth, while the y-axis along the direction perpendicular to the surface of earth. Choose your origin in the xy-plane to be the point where the projectile reaches the highest point above the surface of the earth.

(b) Find the *actual* distance travelled by the projectile *along its trajectory.* (Note that, here you are not required to find the distance between A and B on the surface of the earth; rather you need to find the distance along the curve representing the trajectory by performing the necessary line integral on plane containing trajectory.)