Solutions to tutorial 12

April 21, 2020

1. Given that $\psi(x) = (\pi/\alpha)^{-1/4} e^{-\frac{\alpha x^2}{2}}$,

a) calculate $\langle x^n \rangle$ for n even. Why does this vanish for n odd?

b) calculate the positional spread $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$.

Sol.(a) The expectation value of x^n is

$$\langle x^n \rangle = \int_{-\infty}^{\infty} \left(\frac{\pi}{a}\right)^{-1/2} e^{-\alpha x^2} x^n dx. \tag{1}$$

For an even 'n' using the standard integration formula

$$\int_0^{\infty} x^n e^{-ax^b} = \frac{1}{b} a^{\frac{-(n+1)}{b}} \Gamma\Big(\frac{n+1}{2}\Big) \tag{2}$$

we get the value of $\langle x^n \rangle$ as

$$\langle x^n \rangle = \sqrt{\frac{\alpha}{\pi}} \alpha^{\frac{-(n+1)}{2}} \Gamma\Big(\frac{n+1}{2}\Big). \tag{3}$$

If 'n' is odd the $e^{-\alpha x^2} x^n$ becomes an odd function and the integral in eq.(1) vanishes.

(b) Using the expression for expectation value of $\langle x^n \rangle$ from part (a) we get-

$$\langle x^2 \rangle = \sqrt{\frac{\alpha}{\pi}} \alpha^{\frac{-3}{2}} \Gamma\Big(\frac{3}{2}\Big) = \frac{1}{2\alpha} \tag{4}$$

as $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2})$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. So

$$\Delta x = \sqrt{\frac{1}{2\alpha}} \tag{5}$$

as $\langle x \rangle = 0$.

2. Show that the operator relation $e^{iap/\hbar}xe^{-iap/\hbar} = x + a$ (**correction**) holds, using

a) the taylor expansion of the exponential and commutation relations,

b) using the momentum representation.

Sol.(a) We first calculate the commutation relation $[x, e^{iap/\hbar}]$. To calculate this we simply taylor expand the exponential function

$$[x, e^{iap/\hbar}] = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{ia}{\hbar}\Big)^n [x, p^n] \tag{6}$$

We know $[x, p] = i\hbar$ and by the method of induction it can be shown that $[x, p^n] = i\hbar n p^{n-1}$ for $n \geq 1$. Using these results we get-

$$\begin{aligned}[x, e^{iap/\hbar}] = \sum_{n=1}^{\infty} \frac{1}{n!}\Big(\frac{ia}{\hbar}\Big)^n i\hbar n p^{n-1} = \Big(\frac{ia}{\hbar}\Big)(i\hbar)\sum_{n=1}^{\infty} \frac{1}{(n-1)!}\Big(\frac{ia}{\hbar}\Big)^{n-1} p^{n-1} \\ = -ae^{iap/\hbar}\end{aligned} \tag{7}$$

So now we write the commutation relation explicitly-

$$\begin{aligned}[x, e^{iap/\hbar}] &= -ae^{iap/\hbar} \\ \implies xe^{iap/\hbar} - e^{iap/\hbar}x &= -ae^{iap/\hbar}\end{aligned} \tag{8}$$

Operating $e^{-iap/\hbar}$ on both LHS and RHS from right side we get-

$$\begin{aligned}x - e^{iap/\hbar}xe^{-iap/\hbar} &= -a \\ \implies e^{iap/\hbar}xe^{-iap/\hbar} &= x + a.\end{aligned} \tag{9}$$

(b) Operating $e^{iap/\hbar}xe^{-iap/\hbar}$ on a wavefunction ψ and remembering that in momentum representation $x = i\hbar\frac{\partial}{\partial p}$ we get-

$$\begin{aligned}e^{iap/\hbar}xe^{-iap/\hbar}\psi &= e^{iap/\hbar}\Big(i\hbar\frac{\partial}{\partial p}\Big)e^{-iap/\hbar}\psi \\ &= e^{iap/\hbar}(i\hbar)\Big(\frac{-ia}{\hbar}e^{-iap/\hbar}\psi + e^{-iap/\hbar}\frac{\partial\psi}{\partial p}\Big) \\ &= a\psi + i\hbar\frac{\partial}{\partial p}\psi = (x+a)\psi\end{aligned} \tag{10}$$

Hence we can say that $e^{iap/\hbar}xe^{-iap/\hbar} = x + a$.

3. A particle is known to be localized in the left half of a box with sides at $x = \pm a/2$, with the wavefn.

$$\psi(x) = \begin{cases} \sqrt{2/a}\,; \ -a/2 < x < 0 \\ 0\,; \ 0 < x < a/2 \end{cases}$$

(a) Will the particle remain localized at later times?

(b) Calculate the probabilities that an energy measurement yields the ground state energy and the energy of the first excited state.

Sol.(a) Since $\psi(x)$ is not an energy eigenstate, so it will not remain stationary over time and hence the particle will not remain localized over time.

(b) The total wavefn of a system can be written as a superposition of its energy eigen states. So $\psi = \sum_{n=1}^{\infty} a_n u_n(x)$ where $u_n(x)$ are the eigen states. We also know that for a particle in a box with dimension $x = \pm a/2$ the eigen states are given as-

$$\begin{aligned} u_n(x) &= \sqrt{\frac{2}{a}} cos\Big(\frac{n\pi x}{a}\Big)\, for\, odd\, n \\ u_n(x) &= \sqrt{\frac{2}{a}} sin\Big(\frac{n\pi x}{a}\Big)\, for\, even\, n. \end{aligned} \tag{11}$$

Hence the coefficient a_n will be-

$$a_n = \int_{-a/2}^{a/2} u_n^*(x)\psi(x)dx = \sqrt{\frac{2}{a}} \int_{-a/2}^{0} u_n^* dx \tag{12}$$

On integrating the above equation for even and odd case separately we get-

$$a_n = \begin{cases} (-1)^{\frac{n-1}{2}} \frac{2}{n\pi}\, ;\, odd\ n \\ [-1 + (-1)^{\frac{n}{2}}] \frac{2}{n\pi}\, ;\, even\ n. \end{cases} \tag{13}$$

For ground state $n = 1$ so $a_1 = 2/\pi$ and hence the probability of the ground state energy to be measured is $a_1^2 = 4/\pi^2$.
For the first excited state $n = 2$ so $a_2 = -2/\pi$ and hence the probability of the ground state energy to be measured is $a_2^2 = 4/\pi^2$.

4. A particle in free space is initially in a wave packet described by: $\psi(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha x^2}{2}}$

a) What is the probability that its momentum is in the range $(p, p + dp)$?
b) What is the expectation value of the energy? Can you justify the answer using the size of the wavefunction and uncertainty principle??
Sol.(a) The wavefunction in the momentum space is given by

$$\begin{aligned} \psi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-\frac{ipx}{\hbar}} dx \\ &= \left(\frac{1}{\pi\alpha\hbar^2}\right)^{\frac{1}{4}} e^{-\frac{p^2}{2\hbar^2\alpha}} \end{aligned}$$

The probability that its momentum is in the range $(p, p+dp)$ is given by $|\psi(p)|^2 = \left(\frac{1}{\pi\alpha\hbar^2}\right)^{\frac{1}{2}} e^{-\frac{p^2}{\hbar^2\alpha}}$

Sol.(b) The expectation value of the energy is given by :

$$
\begin{aligned}
\langle\psi(p)|\frac{p^2}{2m}|\psi(p)\rangle =& \int_{-\infty}^{\infty}\left(\frac{1}{\pi\alpha\hbar^2}\right)^{\frac{1}{2}} e^{-\frac{p^2}{\hbar^2\alpha}}\frac{p^2}{2m}dp \\
=&\frac{\alpha\hbar^2}{4m}
\end{aligned}
$$

Using the uncertainty relation we can estimate the ground state energy of the particle:

$$
\begin{aligned}
E =&\frac{\hbar^2}{8m(\Delta x)^2} \\
=&\frac{\alpha\hbar^2}{4m}
\end{aligned}
$$

where we have use the result $\Delta x = \sqrt{\frac{1}{2\alpha}}$ obtained in the first problem.

5. Consider the eigenfunctions for particle in a box with sides at $x = \pm a$.

Without working out the integral find the expectation value of the operator $x^2p^3 + 3xp^3x + p^3x^2$ for all eigenfunctions.

Sol The wave function for the symmetric particle in a box is given by :

$$
\psi_n(x) = \begin{cases} \sqrt{\frac{1}{a}}\cos\left(\frac{n\pi}{2a}x\right) \; ; \textit{odd } n \\ \sqrt{\frac{1}{a}}\sin\left(\frac{n\pi}{2a}x\right) \; ; \textit{even } n. \end{cases} \tag{14}
$$

Using the commutation relation $[x, p] = i\hbar$, we can write $px = -i\hbar + xp$. Using this relation the second term and the third term in the operator are given by:

$$
\begin{aligned}
3xp^3x =&3xp^2(-i\hbar + xp) \\
=&3xp(-i\hbar p - i\hbar p + xp^2) \\
=&3x(-2i\hbar p^2 - i\hbar p^2 + xp^3) \\
=& - 9i\hbar xp^2 + 3x^2p^3
\end{aligned}
$$

A similar evaluation of the third term p^3x^2, we obtain $p^3x^2 = -6\hbar^2 p - 6i\hbar xp^2 + x^2p^3$. Using these relations the original operator can be rewritten in the following form: $5x^2p^3 - 15i\hbar xp^2 - 6\hbar^2 p$. In this new form it is evident that $\psi_n(x)(5x^2p^3 - 15i\hbar xp^2 - 6\hbar^2 p)\psi_n(x)$ produces an odd function. Hence when integrated over a symmetric range it becomes zero.