1. The equation of the undamped oscillation :

$$m\frac{d^2x}{dt^2} + kx = 0$$

At $t = 0$ we have $x = 0$ and at this moment as the mass is in equilibrium, the velocity $\frac{dx}{dt} = 0$. Now with this initial conditions we apply a constant force F. The new EOM is

$$m\frac{d^2x}{dt^2} + kx = F$$

Solving this we can get the path of the particle

$$x(t) = \frac{F}{k} + A\cos\omega_0 t + B\sin\omega_0 t$$

, where A and B are constant with time and $\omega_0 = \sqrt{k/m}$ is the fundamental frequency of the undamped oscillator.
Using the above initial conditions, $A = -\frac{F}{k}$ and $B = 0$. So,

$$x(t) = \frac{F}{k}(1 - \cos\omega_0 t) = \frac{2F}{k}\sin^2\frac{\omega_0 t}{2}$$

So At $t = \tau$ the position and velocity of the particle are $x(\tau) = \frac{2F}{k}\sin^2\frac{\omega_0\tau}{2}$ and $\dot{x}(\tau) = \frac{2F\omega_0}{k}\sin\frac{\omega_0\tau}{2}\cos\frac{\omega_0\tau}{2}$.
After $t = \tau$, In absense of the external force, the eom $m\frac{d^2x}{dt^2} + kx = 0$ with the solution $x(t) = a\cos\omega(t-\tau) + b\sin\omega(t-\tau)$. Now at $t = \tau$ the state of the paricle is already obtained, using that, we can find $a = \frac{2F}{k}\sin^2\frac{\omega_0\tau}{2}$ and $b = \frac{2F}{k}\sin\frac{\omega_0\tau}{2}\cos\frac{\omega_0\tau}{2}$.
So the amplitude of the oscillation

$$\text{Amp} = \sqrt{a^2 + b^2} = \frac{2F}{k}|\sin\frac{\omega_0\tau}{2}|$$

2. The eom of the undamped forced oscillation

$$\ddot{x}(t) + \omega_0^2 x(t) = f\cos\omega t$$

with the initial conditions $x(t = 0) = \dot{x}(t = 0) = 0$. The total solution of this differential equation $x = x_h + x_p$. The homogeneous solution (without the force) $x_h = A\cos\omega_0 t + B\sin\omega_0 t$. The particular solution $x_p = \frac{f}{\omega_0^2 - \omega^2}\cos\omega t$. So the total solution

$$x = A\cos\omega_0 t + B\sin\omega_0 t + \frac{f}{\omega_0^2 - \omega^2}\cos\omega t$$

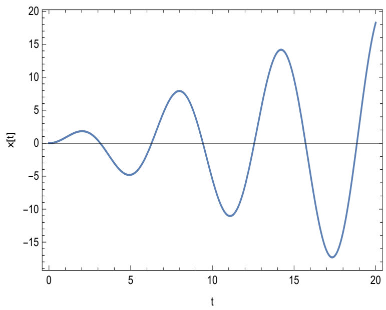
Using the initial condition, $A = -\frac{f}{\omega_0^2 - \omega^2}$ and $B = 0$.

$$x(t) = \frac{f}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t)$$

Near the resonance ($\omega = \omega_0$), we take $\omega = \omega_0 - \Delta\omega$

$$\begin{aligned} x(t) &= \frac{f}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t) \\ &= \frac{f}{\omega_0^2 - (\omega_0 - \Delta\omega)^2}(\cos((\omega_0 - \Delta\omega)t) - \cos\omega_0 t) \\ &= \frac{f}{\omega_0^2 - \omega_0^2 + 2\omega_0\Delta\omega}(\cos\omega_0 t + t\Delta\omega\sin\omega_0 t - \cos\omega_0 t) \\ &= \frac{f}{2\omega_0} t\sin\omega_0 t \end{aligned} \tag{1}$$

To plot this amplitude near the resonance, we take $f/2 = \omega_0 = 1$.



3. Using the equation for amplitude in the case of a damped oscillator with a driven force $F(t) = F_0\cos\omega t$

$$a = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \tag{2}$$

Given $F(t) = 10\cos\omega t$ Where, $F_0 = 10$, $\omega_0 \approx 1$MHz and quality factor, $Q = 1100$ Using these values $\beta = \frac{\omega_0}{2Q} = 454.5$

(a) To find the spring constant we take $\omega = 1$KHz, where $\omega_0 \gg \omega$ and $a = 8.26$mm. Using low frequency limit in (2)

$$a = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

this gives the spring constant $k = 1210.65$N/m.

(b) Take $\omega = 100$MHz, so $\omega_0 \ll \omega$ and $a = 1\mu$m. Here we take high frequency limit

$$a = \frac{F_0}{m\omega^2}$$

this gives mass $m = 10^{-9}$Kg.
Note $\omega_0 = \sqrt{k/m} = 1.1$MHz which is closed to the resonant frequency.

(c) FWHM $= 2\beta = \frac{\omega_0}{Q} = 1000$Hz

(d) The phase difference between the force and the oscillation at $\omega = \omega_0 + \frac{\text{FWHM}}{2} =$ 1100.5KHz is

$$\phi = \tan^{-1}\frac{-2\beta\omega}{(\omega_0^2 - \omega^2)} = 42.2^o$$

4. In this system the solutions are

$$x_0(t) = a_0 \cos\frac{\omega_0+\omega}{2}t \cos\frac{\omega_0-\omega}{2}t = \frac{a_0}{2}\left(\cos\omega t + \cos\omega_0 t\right)$$

and

$$x_1(t) = a_0 \sin\frac{\omega_0+\omega}{2}t \sin\frac{\omega_0-\omega}{2}t = \frac{a_0}{2}\left(\cos\omega t - \cos\omega_0 t\right)$$

, where $a_0 = 40$cm. The frequency modes are $\omega_0 = \sqrt{k/m} = \sqrt{10} = 3.162$per sec and $\omega = \sqrt{(k+2k')/m} = \sqrt{70} = 8.366$per sec.

(a) the faster normal mode's frequency is 8.366Sec^{-1}.

(b)

$$\dot{x}_1(t) = -\frac{a_0}{2}\left(\omega\sin\omega t - \omega_0 \sin\omega_0 t\right)$$

The kinetic energy at time t
$KE(t) = \frac{1}{2}m\dot{x}_1^2 = m\frac{a_0^2}{8}\left(\omega_0^2\sin^2\omega_0 t + \omega^2\sin^2\omega t - \omega_0\omega\cos(\omega+\omega_0)t + \omega_0\omega\cos(\omega_0-\omega)t\right)$
Now, $\langle\sin^2\theta\rangle = \frac{1}{2}$ and $\langle\cos\theta\rangle = 0$. So average kinetic energy is

$$\langle KE(t)\rangle = m\frac{a_0^2}{16}(\omega^2+\omega_0^2) = m\frac{a_0^2}{16}(\frac{k}{m} + \frac{k+2k'}{m}) = \frac{a_0^2}{8}(k+k') = 0.8\text{Joule}$$

(c) As ω_0^2 and ω^2 are proportional to $1/m$, the kinetic energy does not depends on m, mass of the particle.

5. From the previous problem,

$$x_A(t) = x_0 \cos\frac{\omega_0+\omega}{2}t \cos\frac{\omega_0-\omega}{2}t$$

and

$$x_B(t) = a_0 \sin\frac{\omega_0+\omega}{2}t \sin\frac{\omega_0-\omega}{2}t$$

Here the mass A is pulled by small distance x_0 and released from rest, whereas the B is released from the rest at its equilibrium position. Now $\omega_0 = \sqrt{k_A/m}$ and $\omega = \sqrt{(k_A + 2k_{middle})/m}$. $k_{middle} = k_A/100 = \eta k_A$ $[\eta = 0.01]$. So,

$$\omega = \omega_0\sqrt{1+2\eta} \approx \omega_0(1+\eta)$$

The solutions can be approximated as

$$x_A(t) = x_0 \cos\frac{\eta\omega_0}{2}t\cos\omega_0 t$$

and

$$x_B(t) = -a_0 \sin\frac{\eta\omega_0}{2}t\sin\omega_0 t$$

So two mass interchange the state of oscillation with an angular frequency $\eta\omega_0/2$, therefore lifetime of each state is $T_L = \frac{2\pi}{\eta\omega_0}$. In this state the mass oscillates with an angular frequency ω_0, i.e. frequency $\frac{\omega_0}{2\pi}$
Here we are starting A from a maxima, so for its first die down required time is $T_L/2 = \frac{\pi}{\eta\omega_0}$. During this state the total number of oscillations completed by the mass A is $\frac{\omega_0}{2\pi} \times \frac{\pi}{\eta\omega_0} = \frac{1}{2\eta} = 50$