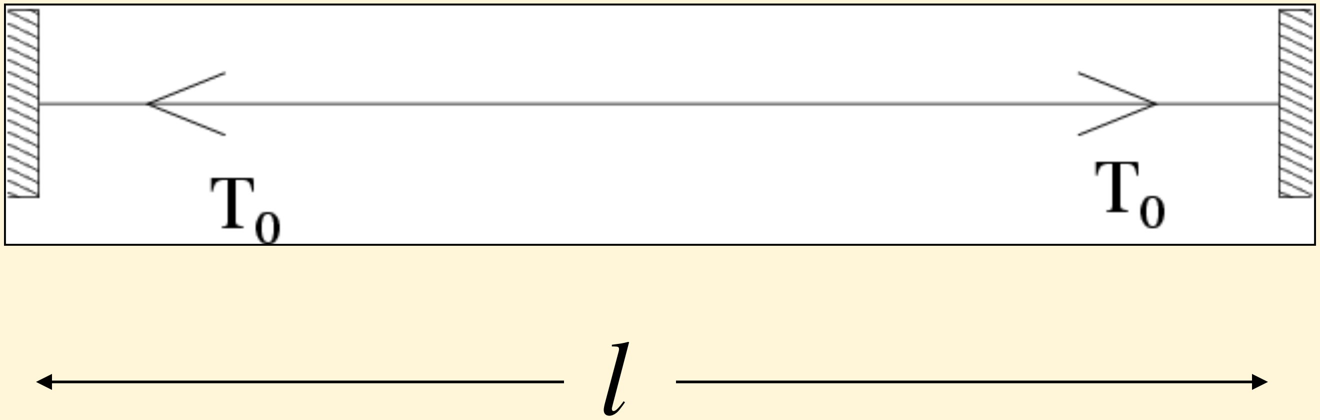
Phase velocity and Group velocity

Boundary conditions for a stretched string



Boundary conditions

$$x = 0, \psi = 0$$

$$x = l, \psi = 0$$

$$\psi = A \exp i(\omega t - kx) + B \exp i(\omega t + kx)$$

$$x = 0, \psi = 0$$

$$A + B = 0$$

$$\psi = A \exp(i\omega t)(\exp(-ikx) - \exp(ikx))$$

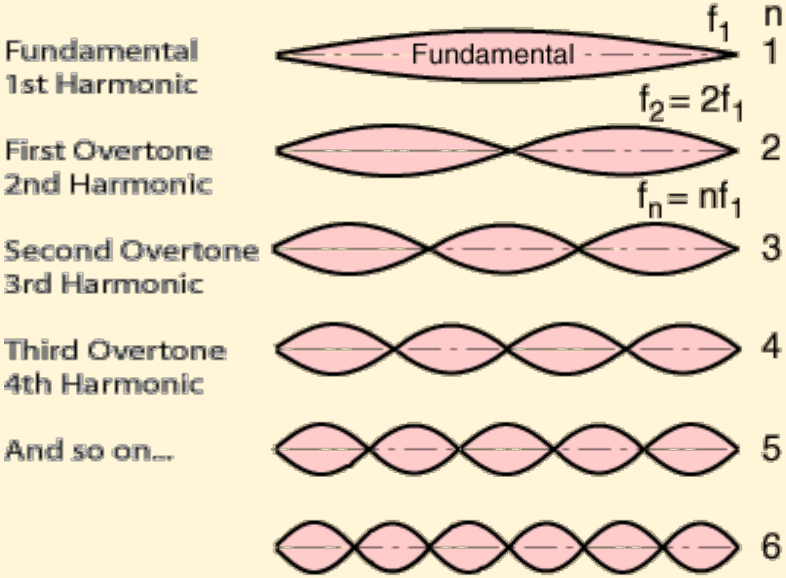
$$\psi = -2i \exp(i\omega t) \sin kx$$

$$\omega^2 = c^2 k^2$$

$$x = l, \psi = 0$$

$$\sin kl = 0$$

$$kl = n\pi$$



(n-1) nodes between boundaries

Progressive waves

$$\xi(x,t) = A\sin(kx - \omega t)$$

$$\psi(z,t) = B\cos(kz - \omega t))$$

$$E_x(y,t) = E_x\exp(i(ky - \omega t))$$

Standing wave

$$\psi = -2i\exp\left(i\omega t\right)\sin kx$$

Wave or Phase velocity

The phase -

$$\phi(x,t) = kx - \omega t$$

$$\phi = 0, \qquad x = 0,\ t = 0$$

To find new position of $\phi = 0,$ at Δt

$$k\Delta x - \omega \Delta t = 0$$

$$\Delta x = \frac{\omega}{k}\Delta t$$

Phase velocity

$$\frac{\Delta x}{\Delta t} = c = \frac{\omega}{k}$$

Wave groups and Group velocity

Superposition of Two Waves of Almost Equal Frequencies

Frequencies

$$\omega_1 = \omega + \Delta\omega$$

$$\omega_2 = \omega - \Delta\omega$$

Wavevectors

$$k_1 = k + \Delta k$$

$$k_2 = k - \Delta k$$

$$\psi(x,t) = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t)$$

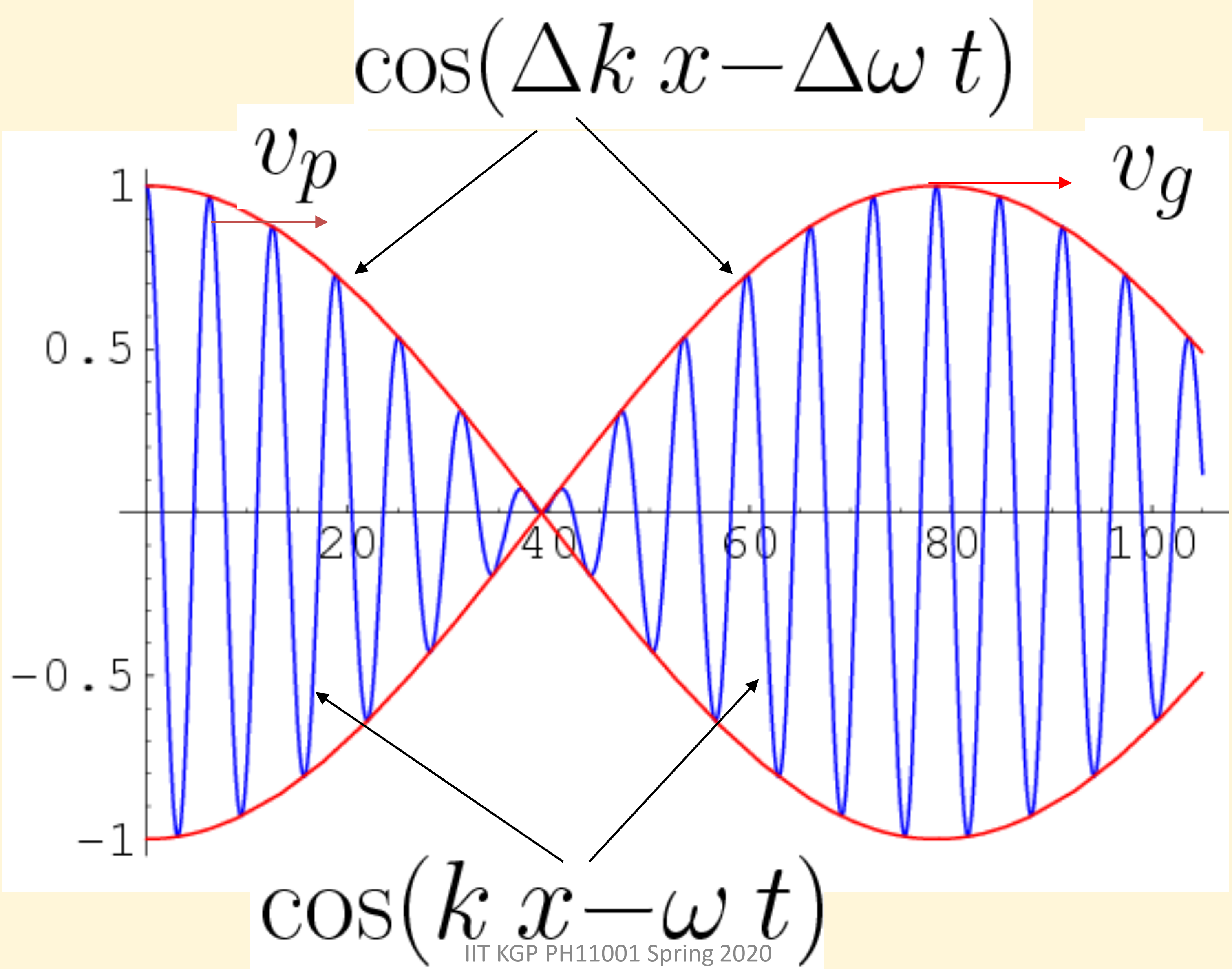
$$\psi(x,t) = 2A\cos(k\,x - \omega\,t)\cdot\cos(\Delta k\,x - \Delta\omega\,t)$$

Phase velocity $$v_p = \frac{\omega}{k}$$

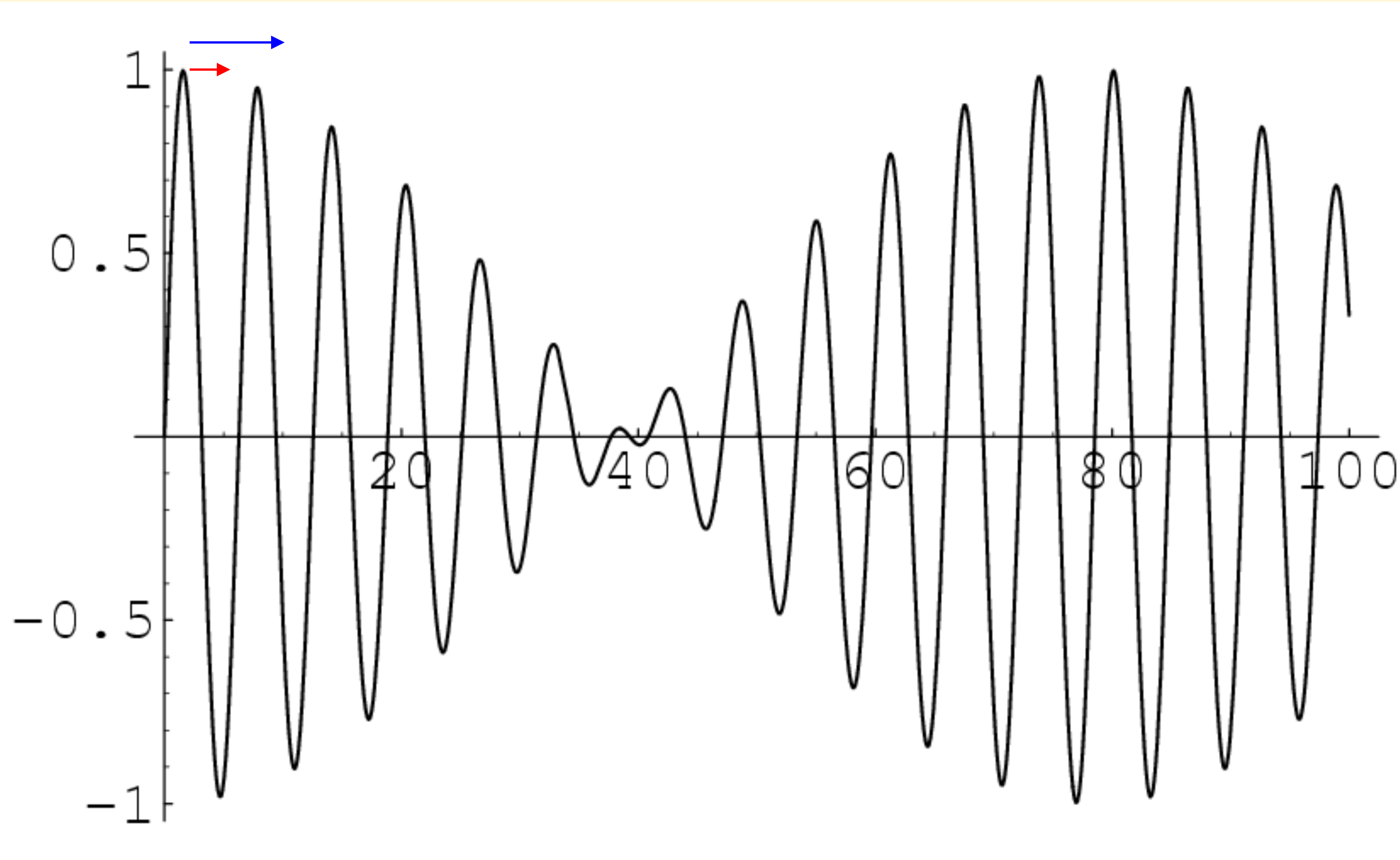
Group velocity $$v_g = \frac{\Delta\omega}{\Delta k} \rightarrow \frac{\partial\omega}{\partial k}$$

$$\Delta k \rightarrow 0$$

$$\boxed{\psi(x,t) = 2A\cos(k\,x - \omega\,t)\cdot\cos(\Delta k\,x - \Delta\omega\,t)}$$



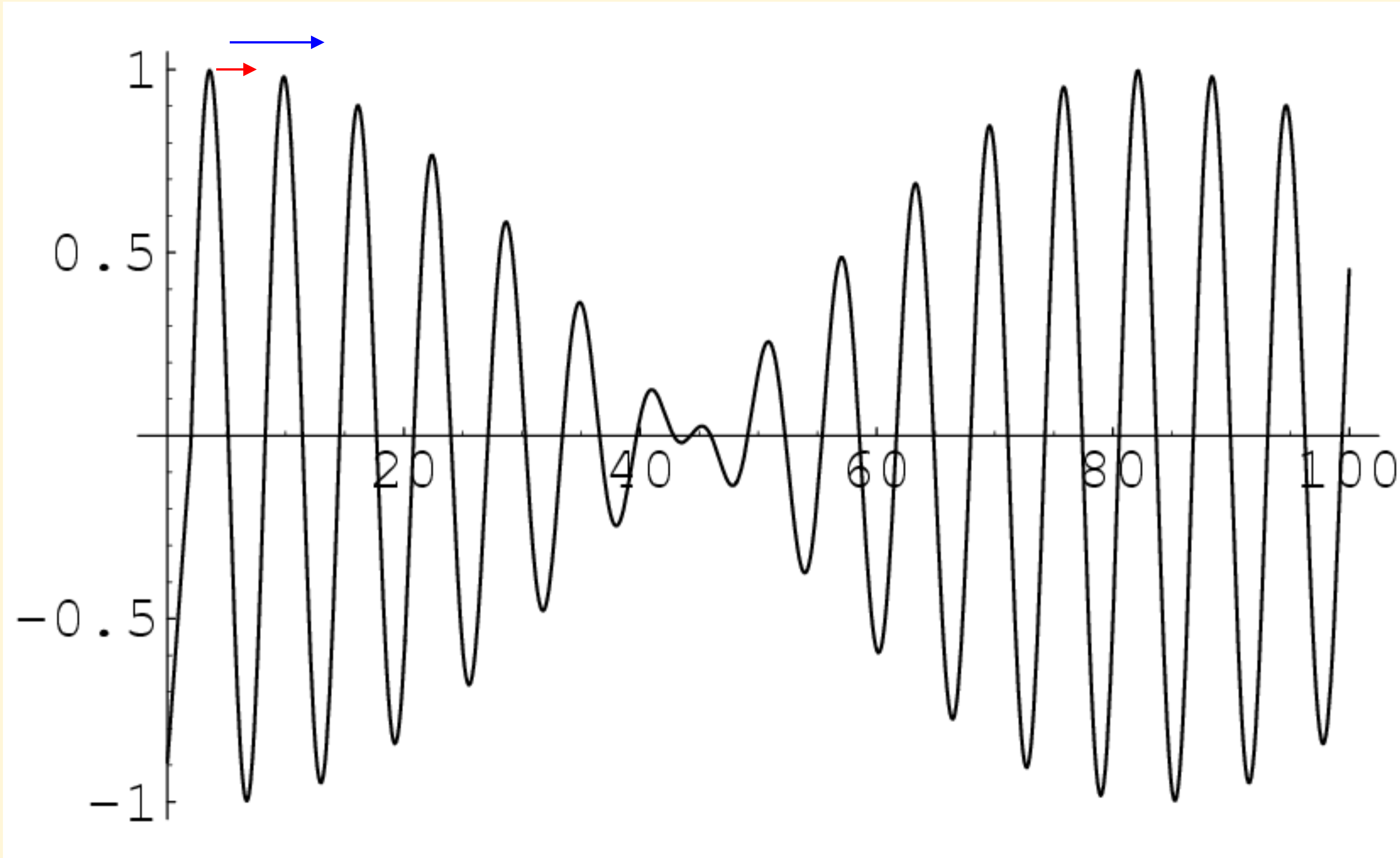
$V_p < V_g$



t = 0

$$\sin(1.00\,x - 2.0\,t)\cos(0.04\,x - 0.2\,t)$$

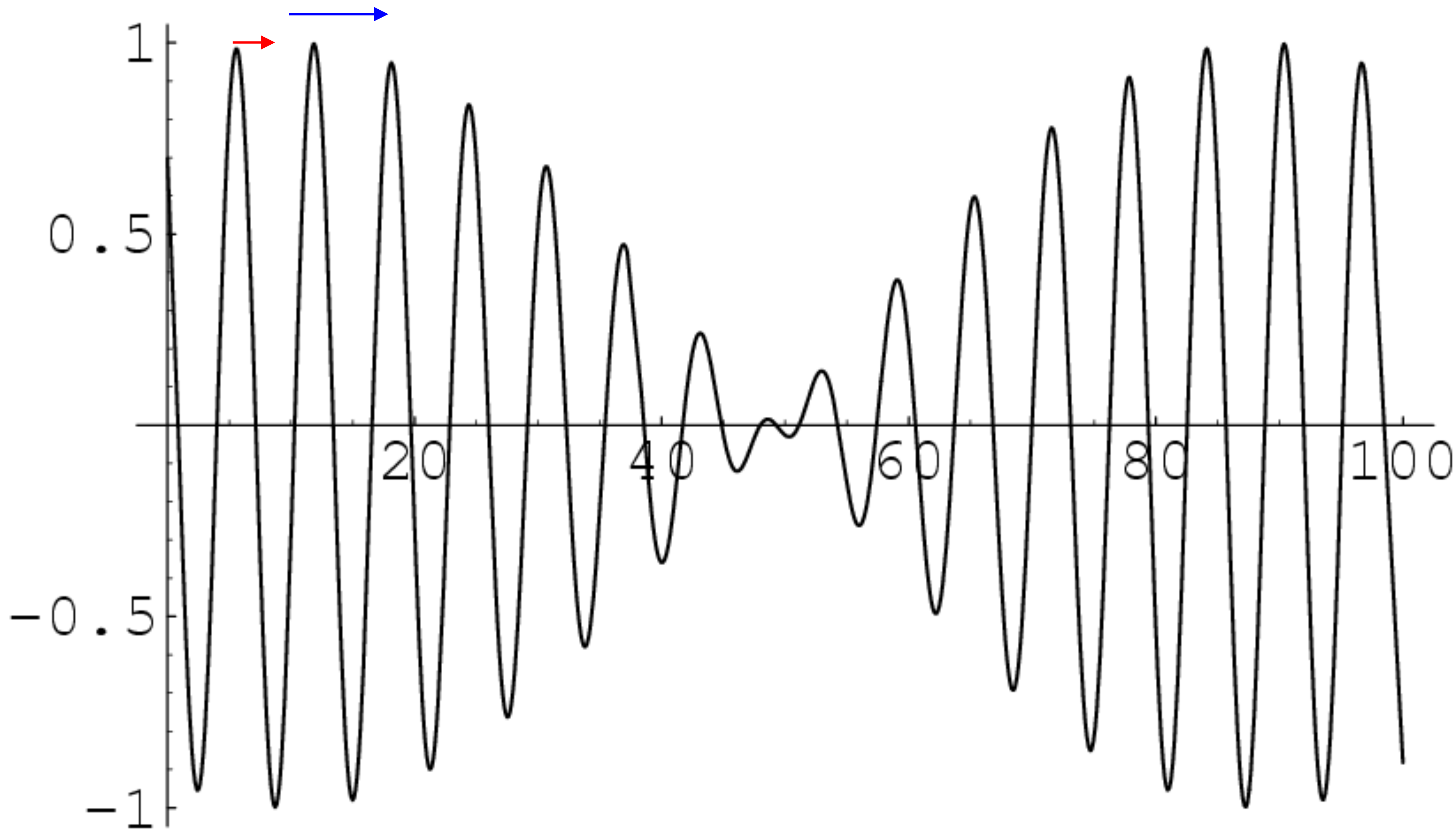
$V_p < V_g$



t = 1

$$\sin(1.00\,x - 2.0\,t)\cos(0.04\,x - 0.2\,t)$$

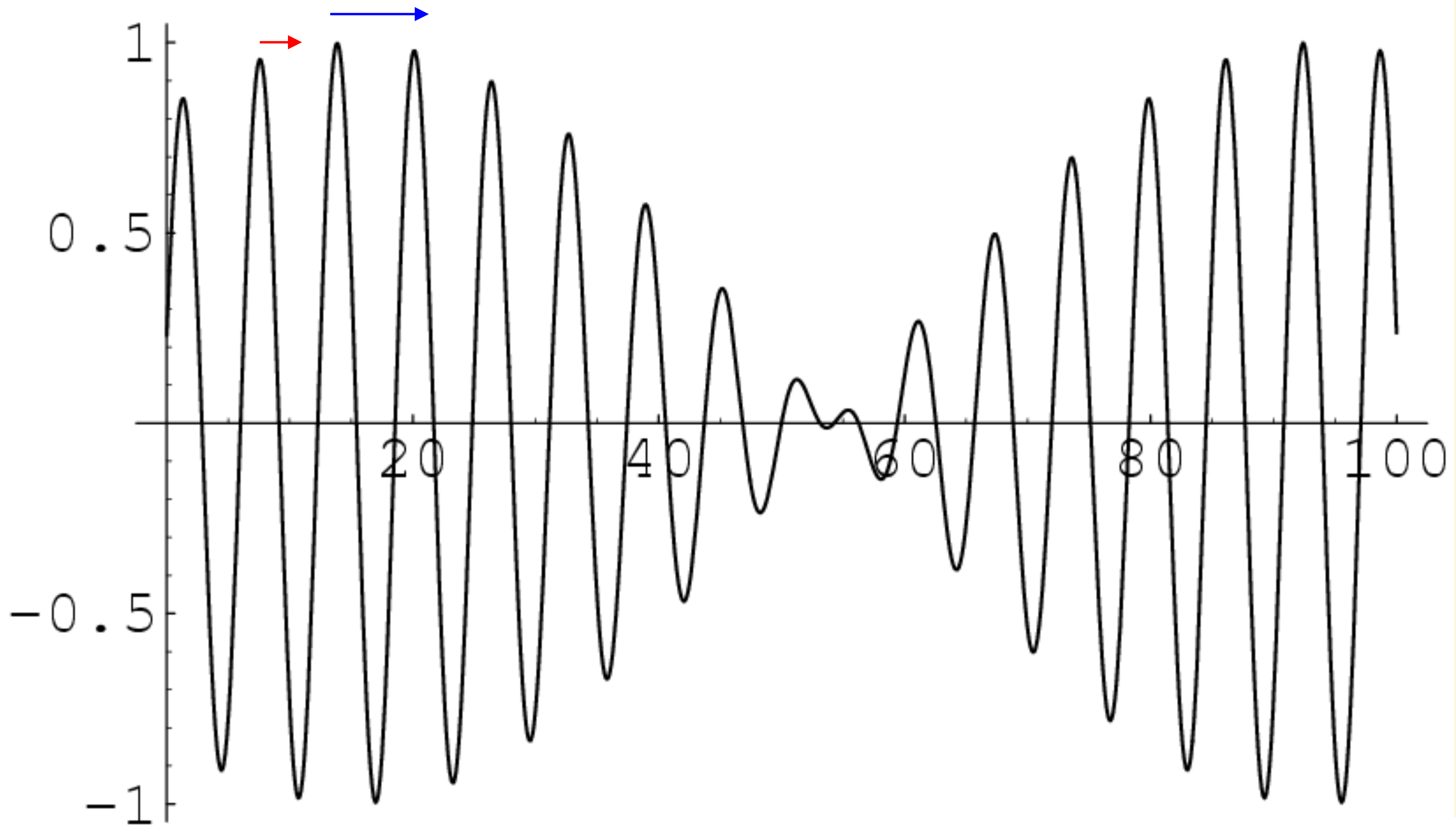
$V_p < V_g$



$t = 2$

$$\sin(1.00\,x - 2.0\,t)\cos(0.04\,x - 0.2\,t)$$

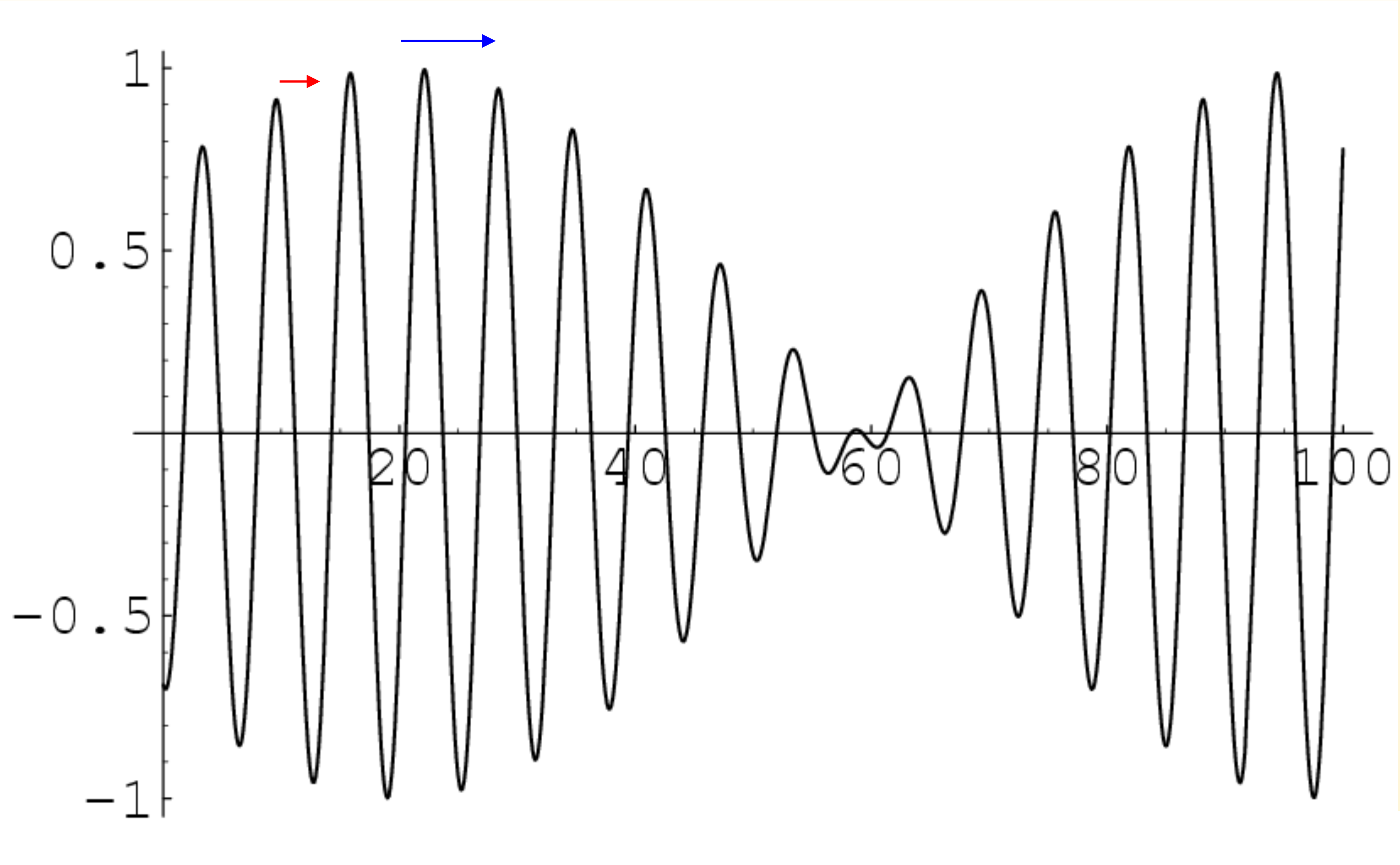
$V_p < V_g$



t = 3

$$\sin(1.00\,x - 2.0\,t)\cos(0.04\,x - 0.2\,t)$$

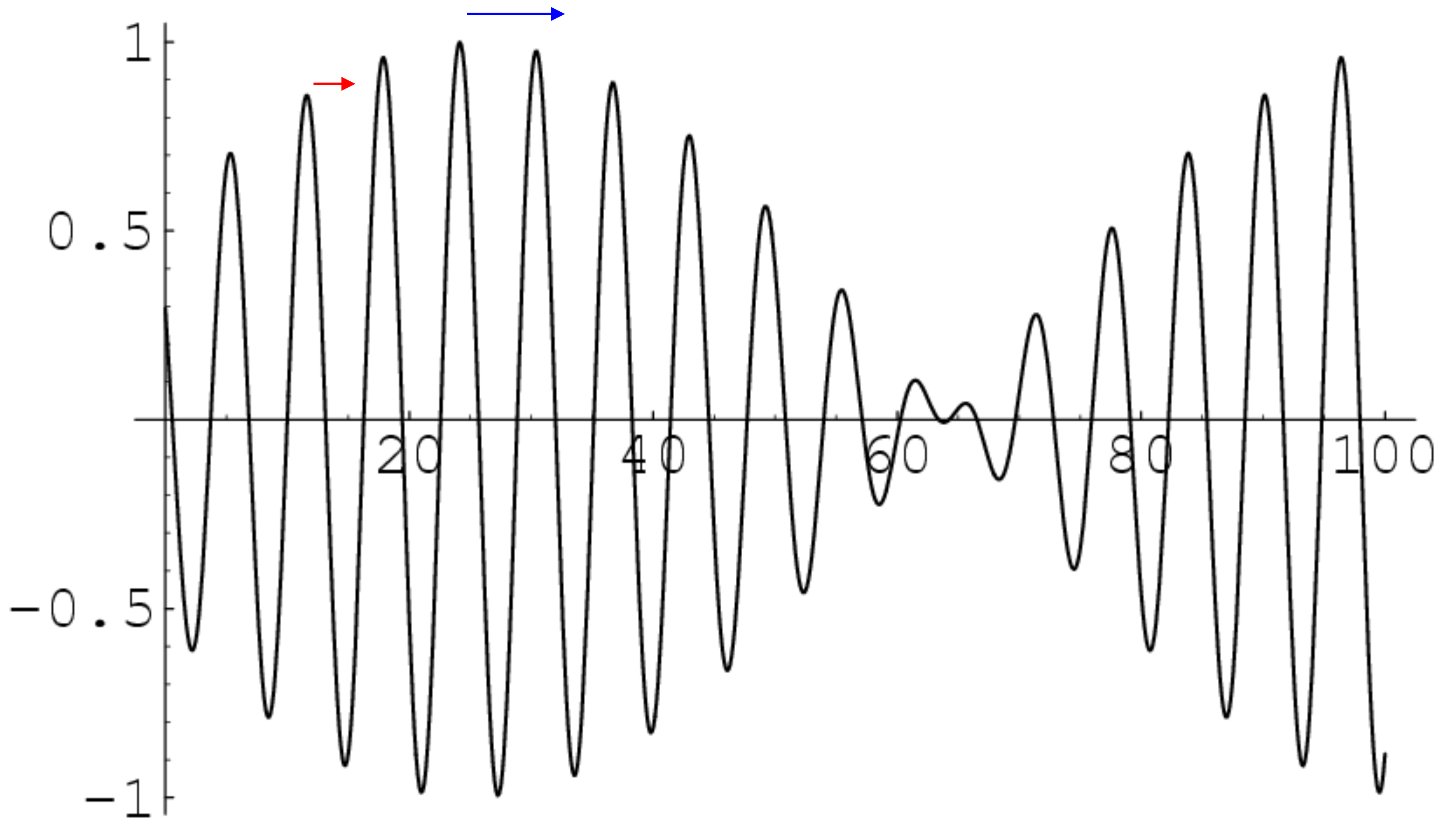
$V_p < V_g$



t = 4

$$\sin(1.00\,x - 2.0\,t)\cos(0.04\,x - 0.2\,t)$$

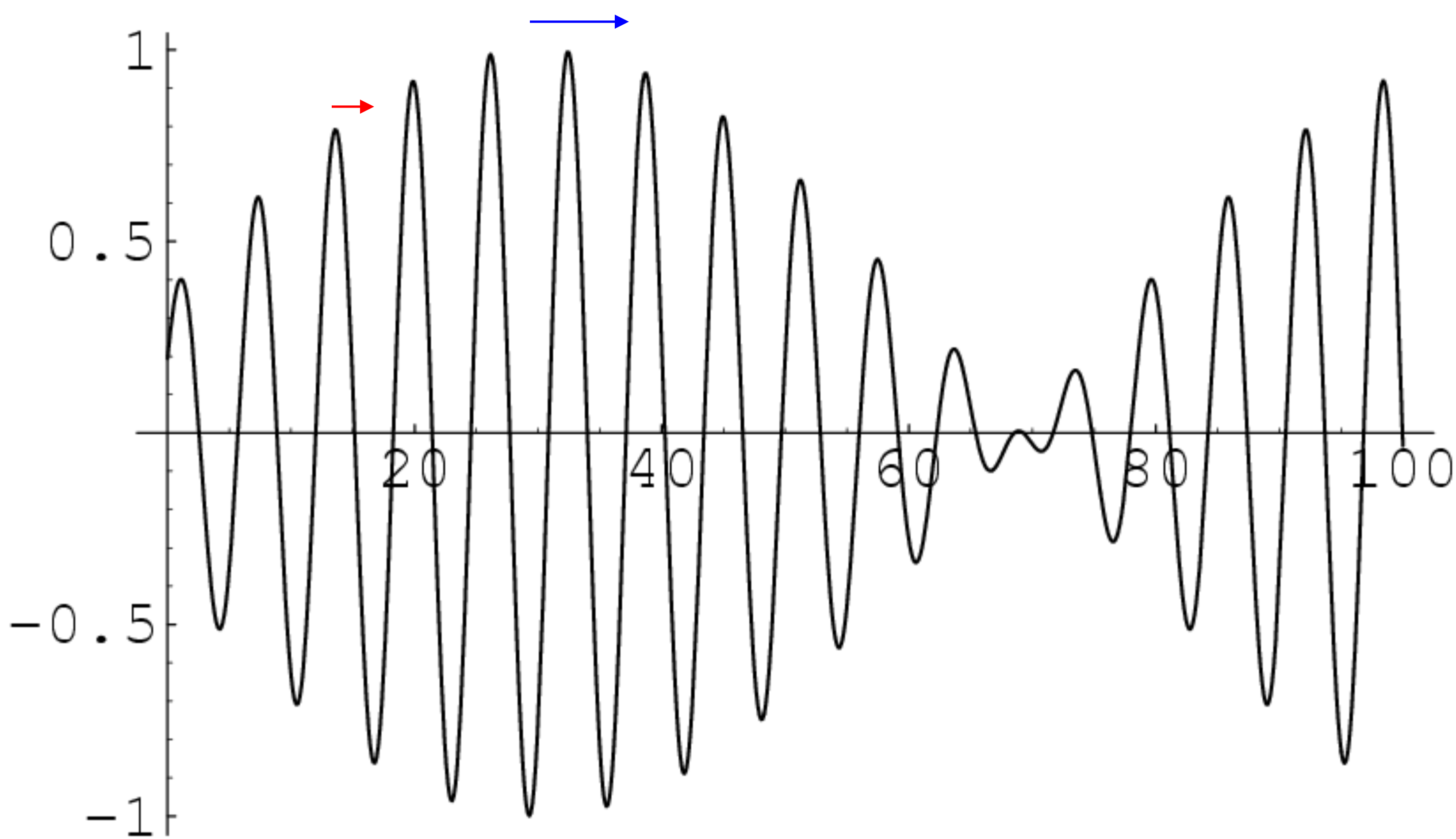
$V_p < V_g$



t = 5

$$\sin(1.00\ x - 2.0\ t)\cos(0.04\ x - 0.2\ t)$$

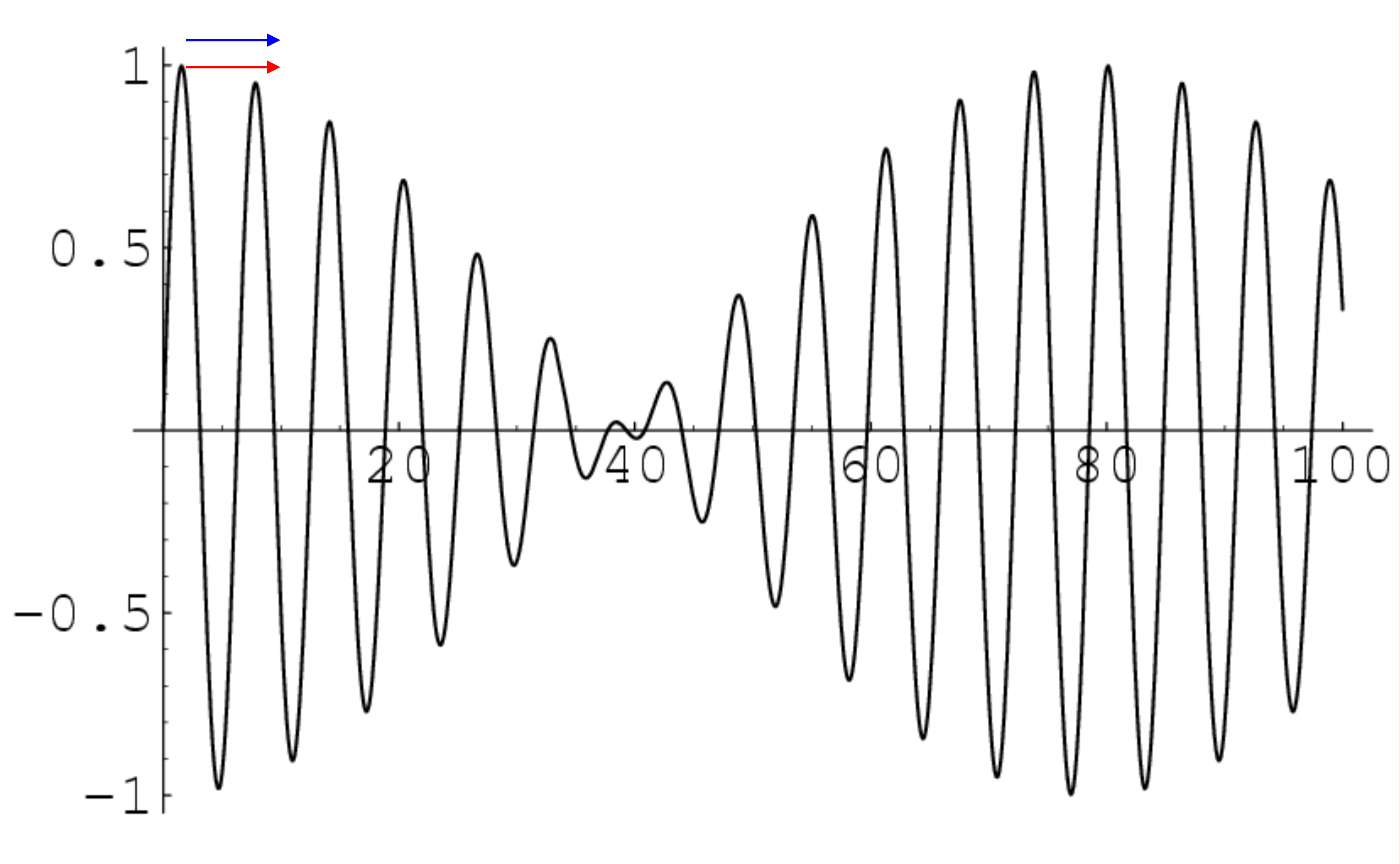
$V_p < V_g$



t = 6

$$\sin(1.00\,x - 2.0\,t)\cos(0.04\,x - 0.2\,t)$$

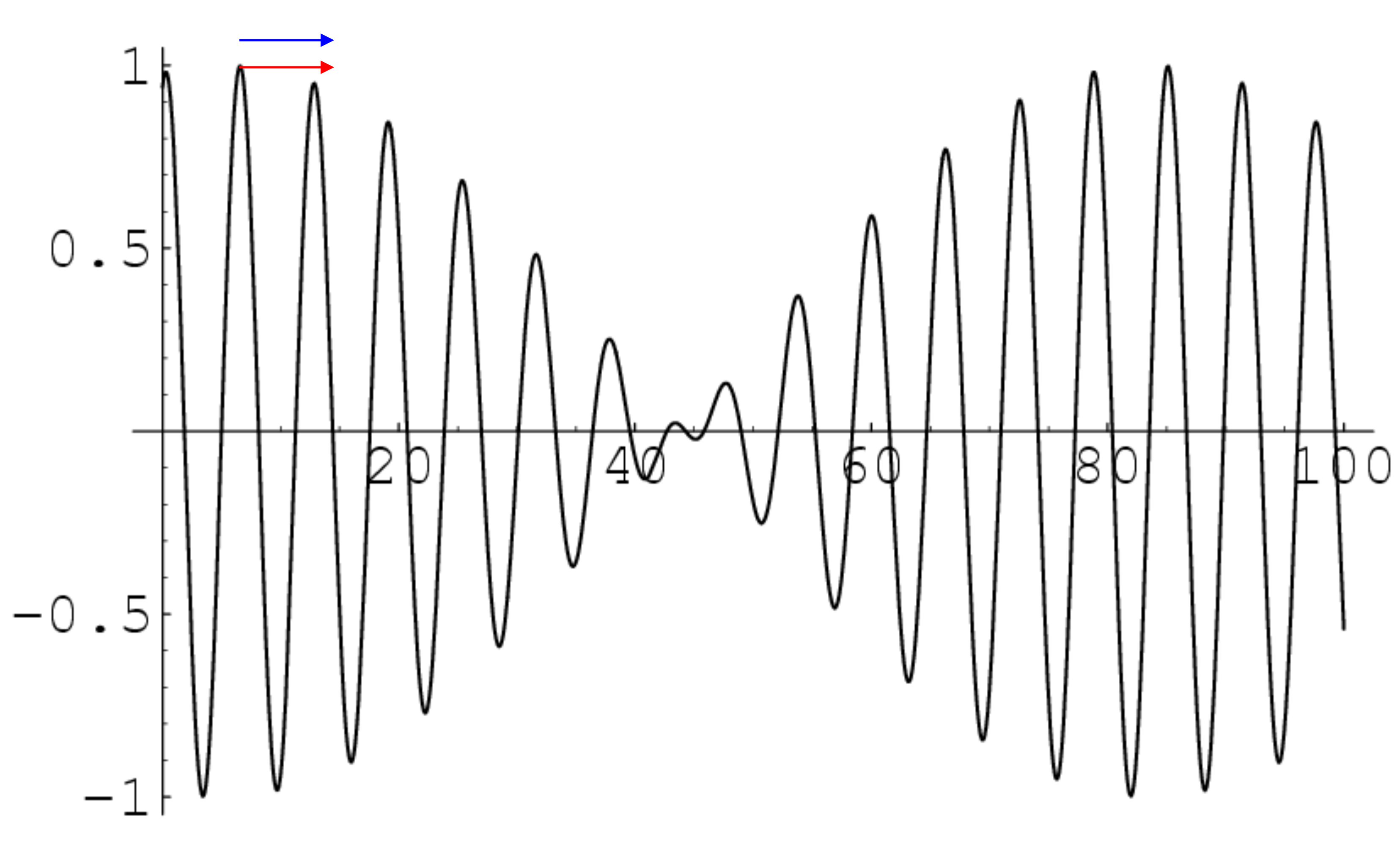
$V_p = V_g$



t = 0

$$\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$$

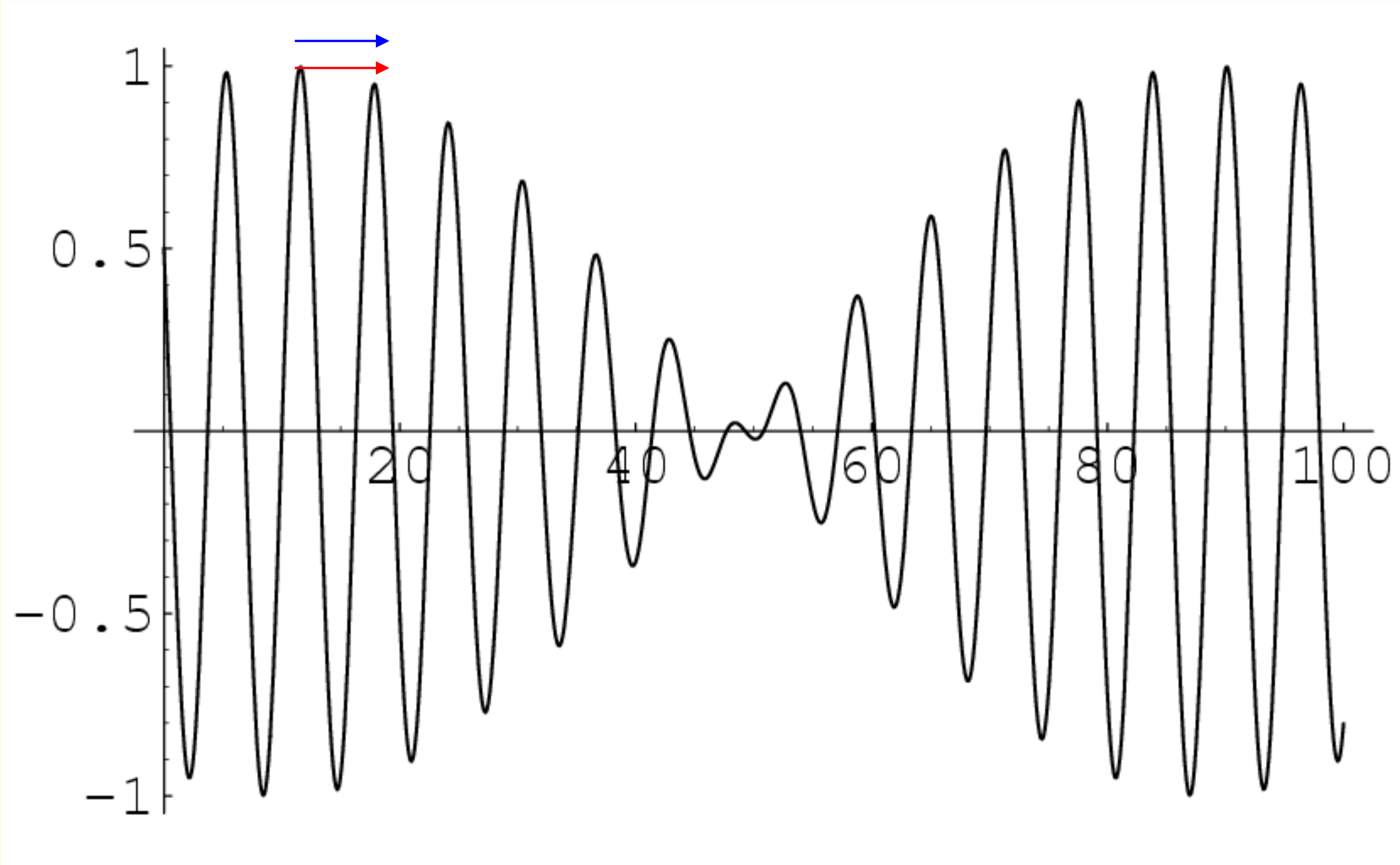
$V_p = V_g$



t = 1

$$\sin(1.00\,x - 5.0\,t)\cos(0.04\,x - 0.2\,t)$$

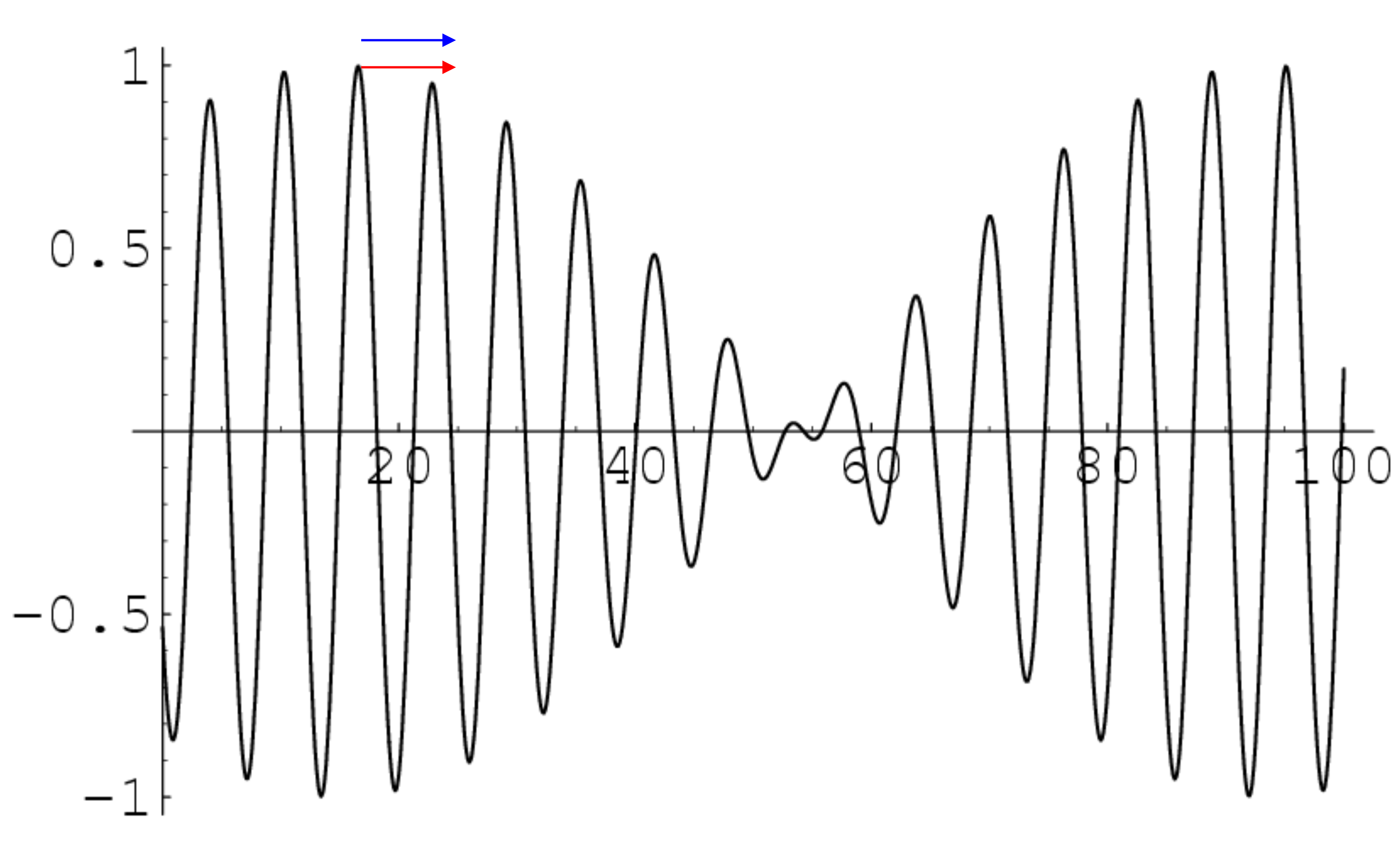
$V_p = V_g$



t = 2

$$\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$$

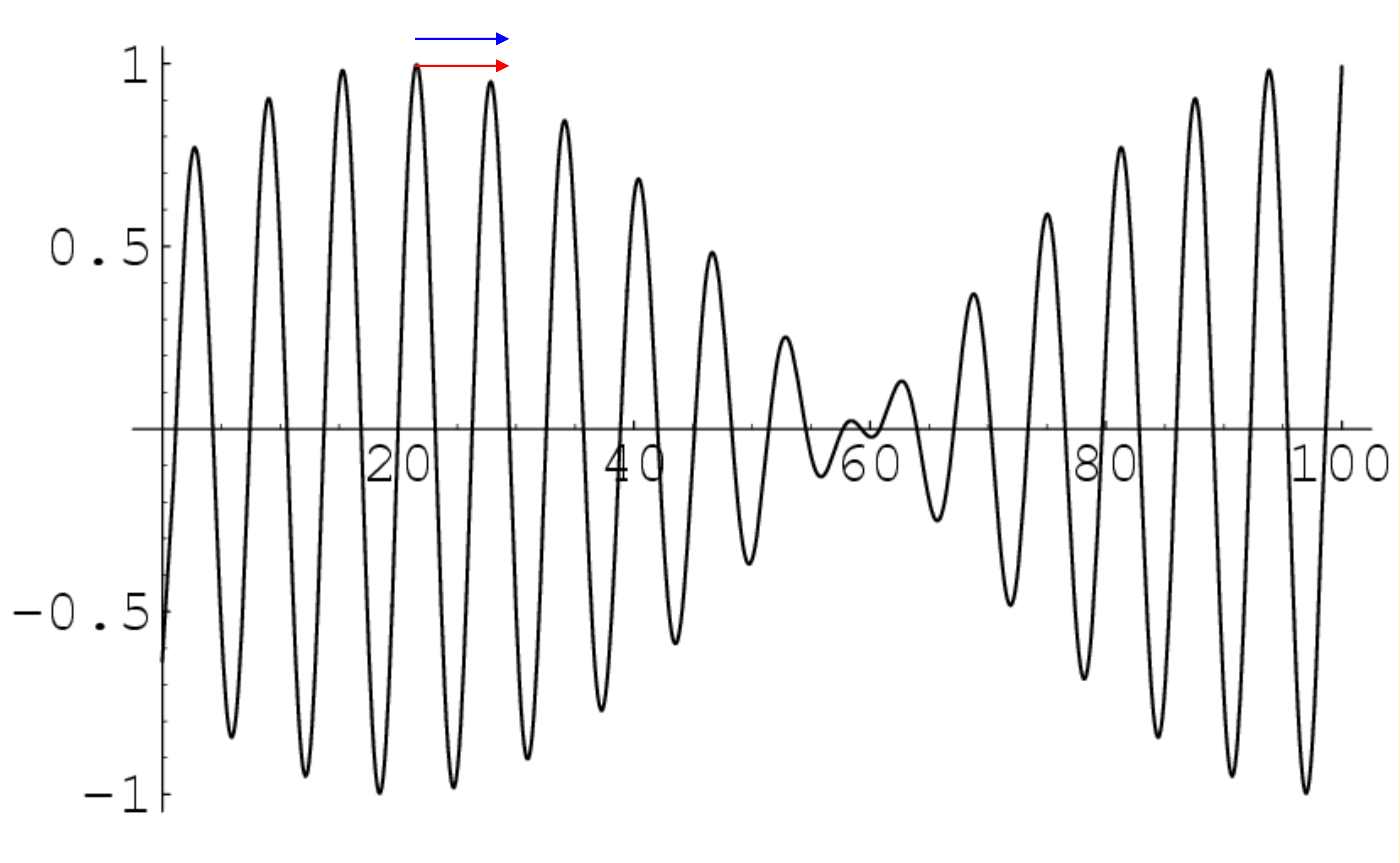
$V_p = V_g$



t = 3

$$\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$$

$V_p = V_g$



t = 4

$$\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$$

Wave Dispersion

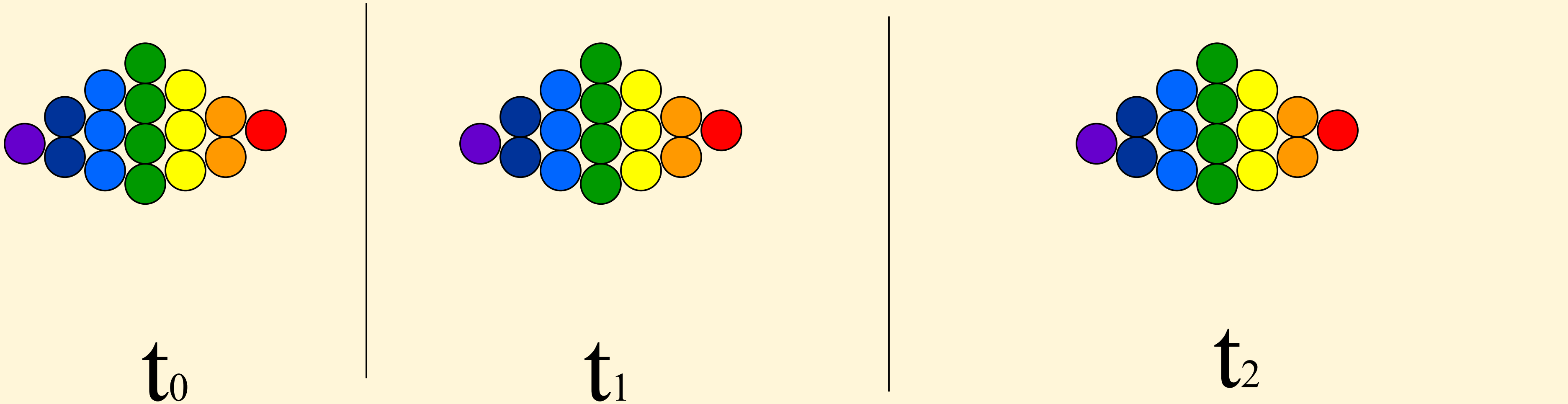
- Simply stated, a dispersion relation is the function ***ω(k)*** for an harmonic wave.

- From these relations the phase velocity and group velocity of the wave can be found and thereby refractive index of the medium can be determine.

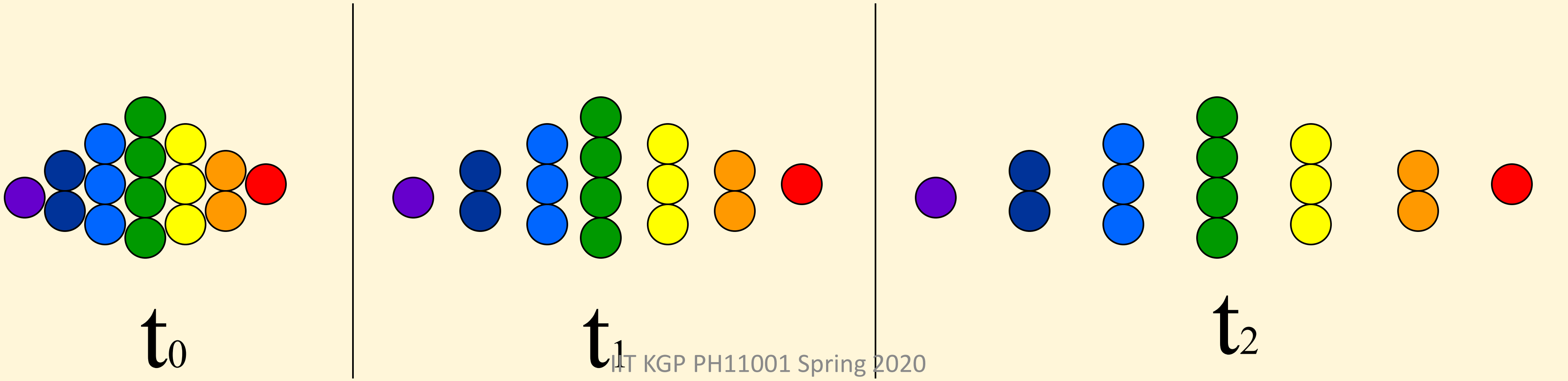
- *A* medium in which phase velocity is frequency dependent is known as a dispersive medium, and a dispersion relation expresses the variation of ***ω*** as a function of ***k***.

$$\boxed{\frac{\Delta\omega}{\Delta k} \neq \frac{\omega_1}{k_1} \neq \frac{\omega_2}{k_2}}$$

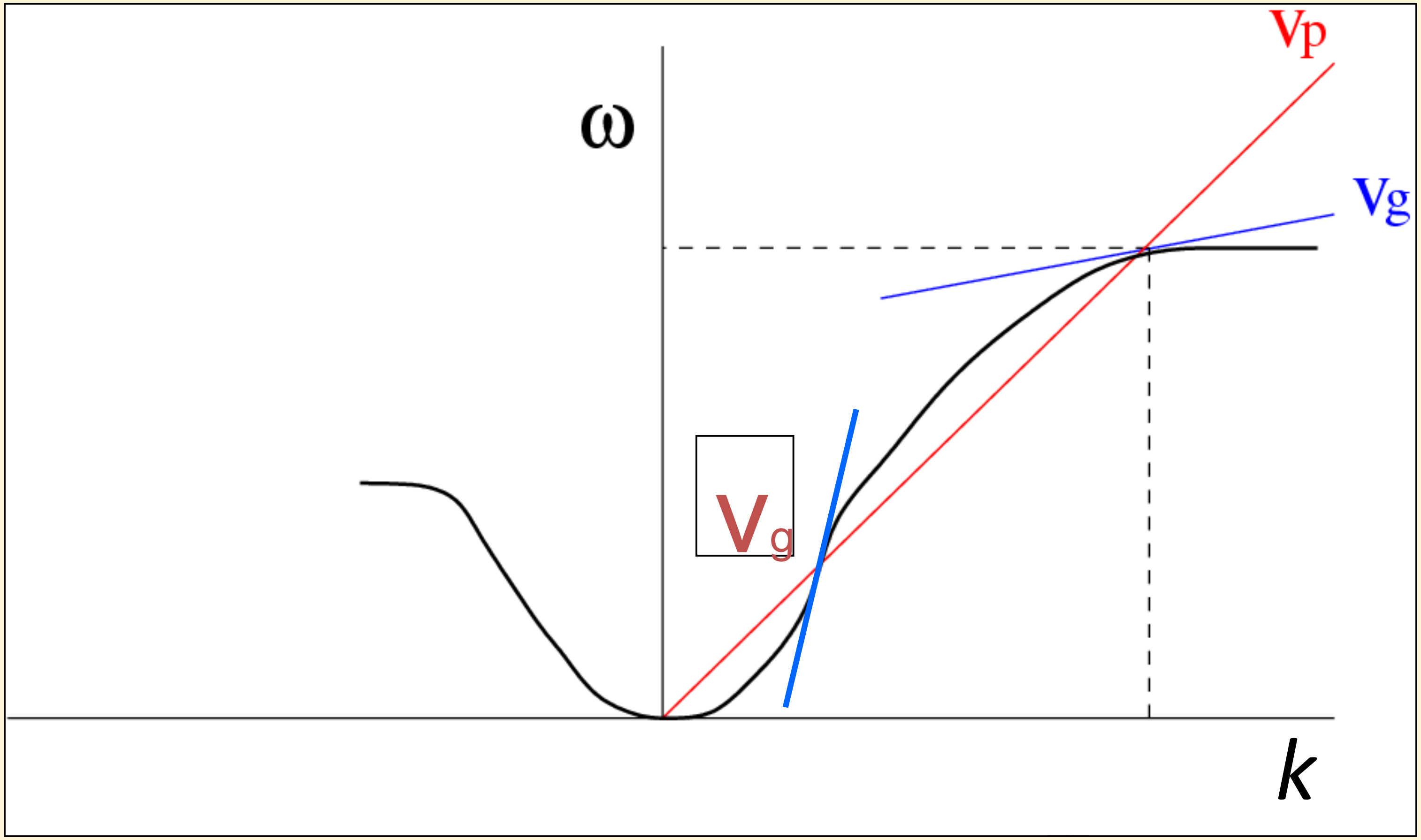
Non dispersive: All colours moving with same speed



Dispersive: Red moving faster than blue



Phase and Group velocity



$v_g < v_p$ →	**Normal Dispersion**
$v_g > v_p$ →	**Anomalous Dispersion**
$v_g = v_p$ →	**Non-dispersive medium**