# Quantum Tunneling Phenomenon

# Quantum Mechanics - Brief Recap

The wavefunction $\psi(x,t)$ (considering 1D) is governed by Schrödinger's equation -

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi(x,t) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x,t)\psi}$$

For time independent potential $V(x)$, we can use separation of variables -

$$\psi(x,t) = X(x)T(t)$$

Substituting back in Schrödinger's equation -

$$Xi\hbar\frac{dT}{dt} = \frac{-\hbar^2}{2m}T\frac{d^2X}{dx^2} + V(x)XT$$

# Quantum Mechanics Brief Recap – Contd.

Dividing by $\psi(x,t)$ and setting both sides equal to constant E -

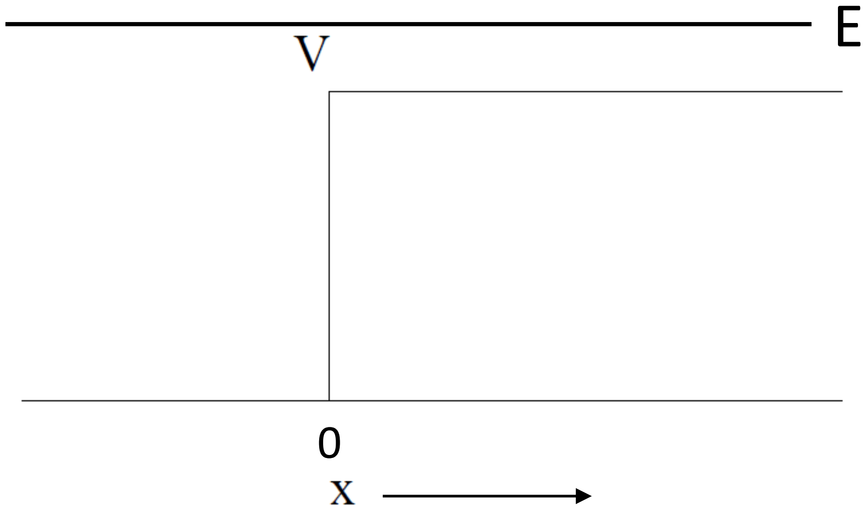
$$i\hbar \frac{1}{T}\frac{dT}{dt} = \frac{-\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} + V(x) = E$$

$$\boxed{\begin{aligned} T(t) &= Ae^{-iEt/\hbar} \\ \psi(x,t) &= Ae^{-iEt/\hbar}X(x) \end{aligned}}$$

To obtain $X(x)$, we need to solve -

$$\frac{\hbar^2}{2m}\frac{d^2X}{dx^2} = -[E - V(x)]X$$

# Step Potential: Case I - (Energy of particle E) > V




We have to solve for - $$\frac{d^2X}{dx^2} = -\frac{2m}{\hbar^2}(E-V)X$$

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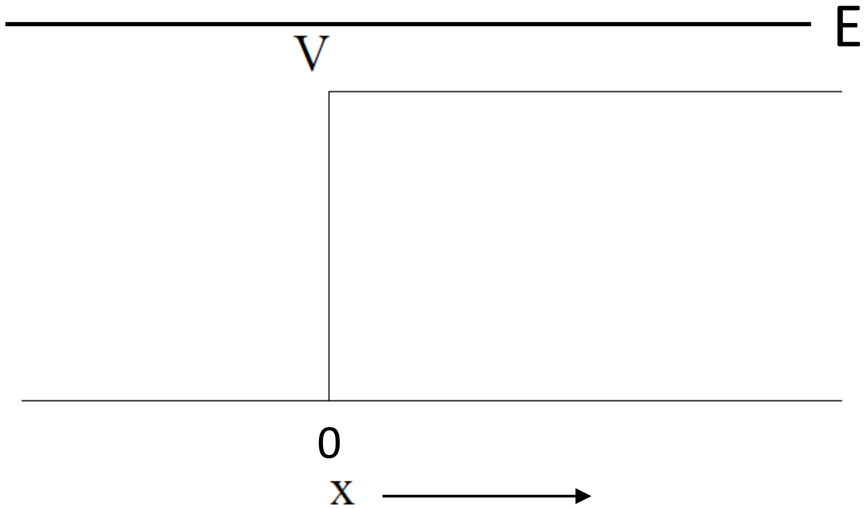


We have to solve for - $$\frac{d^2X}{dx^2} = -\frac{2m}{\hbar^2}(E-V)X$$

For right-side, the momentum is - $p' = \sqrt{2m(E-V)}$

$$\frac{d^2X}{dx^2} = \frac{-p'^2}{\hbar^2}X$$

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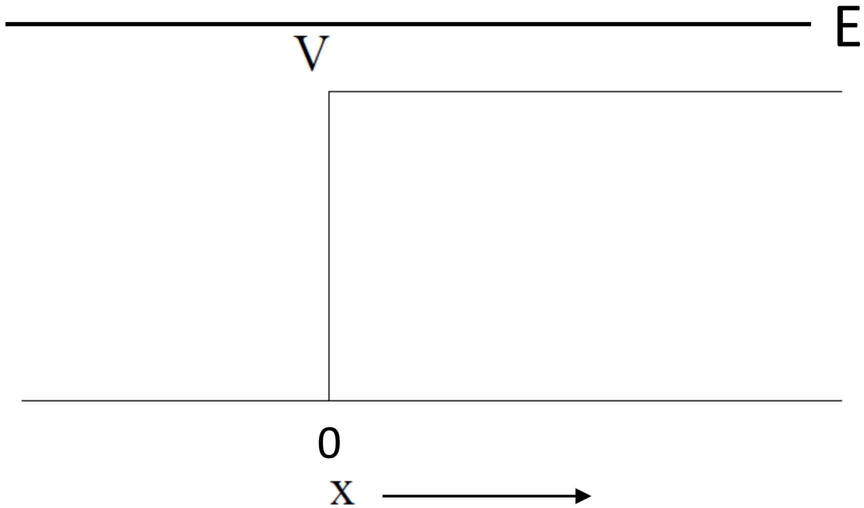


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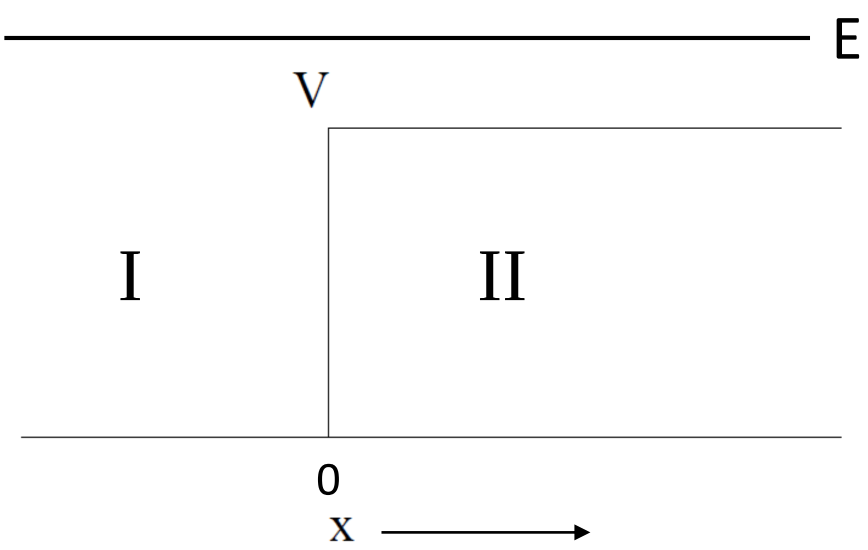
$$X(x) = A_1' e^{ip'x/\hbar} + A_2' e^{-ip'x/\hbar}$$

# Step Potential: Case I - (Energy of particle E) > V

Solution : $\psi_{II}(x,t) = e^{-iEt/\hbar}\left[A_1^{'} e^{ip'x/\hbar} + A_2^{'} e^{-ip'x/\hbar}\right]$

$$\psi_{I}(x,t) = e^{-iEt/\hbar}\left[A_1 e^{ipx/\hbar} + A_2 e^{-ipx/\hbar}\right]$$

$$p = \sqrt{2mE} \qquad p' = \sqrt{2m(E-V)}$$



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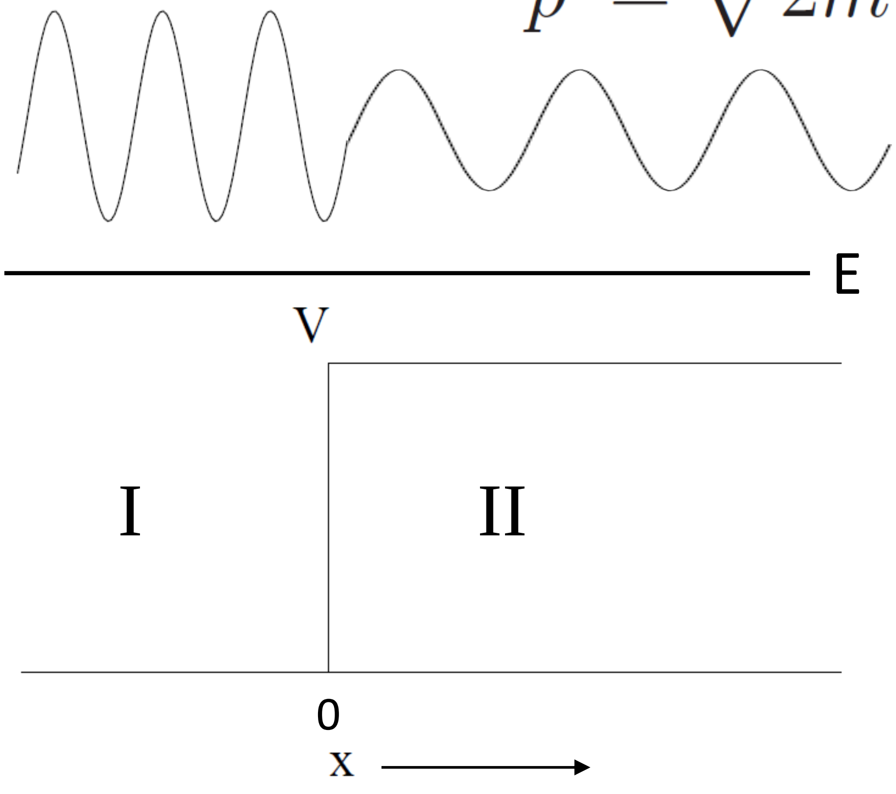
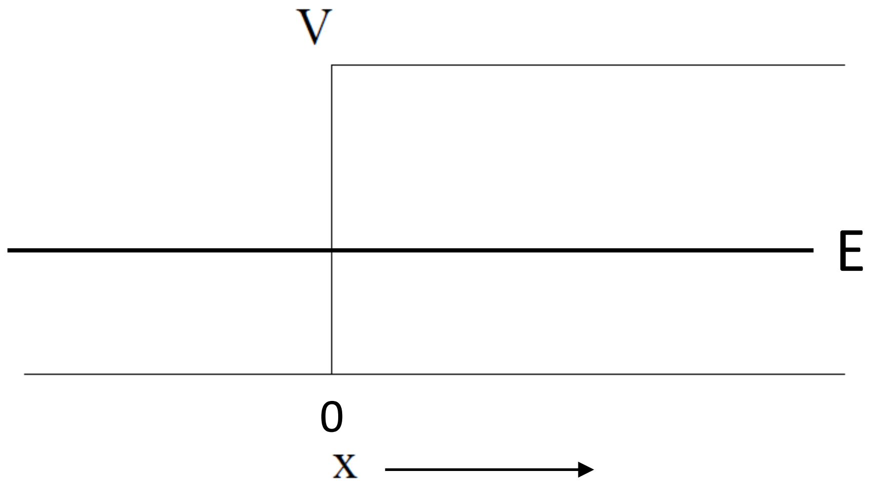
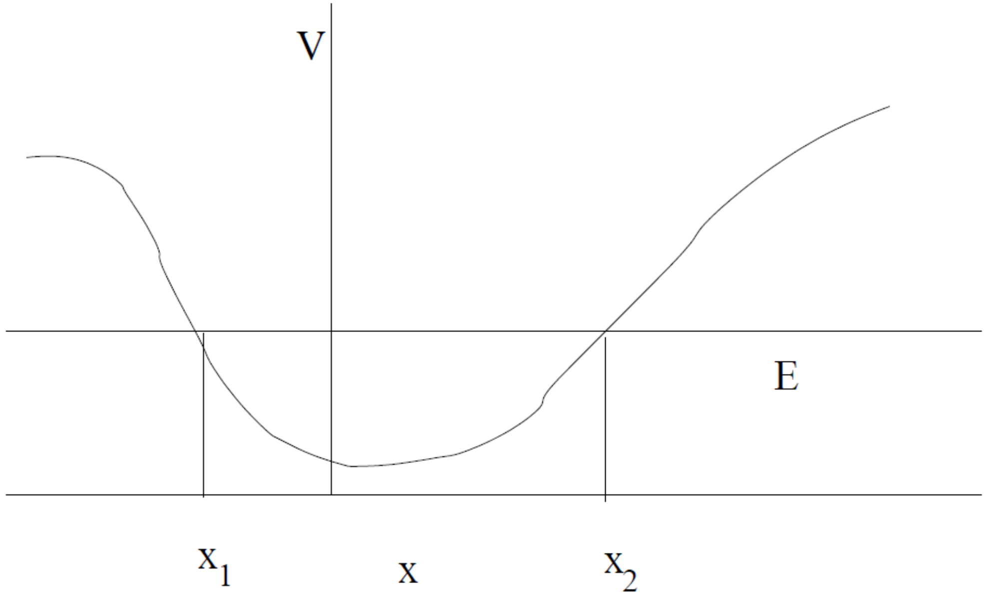


# Step Potential: Case II - (Energy of particle E) < V



# In Classical Mechanics



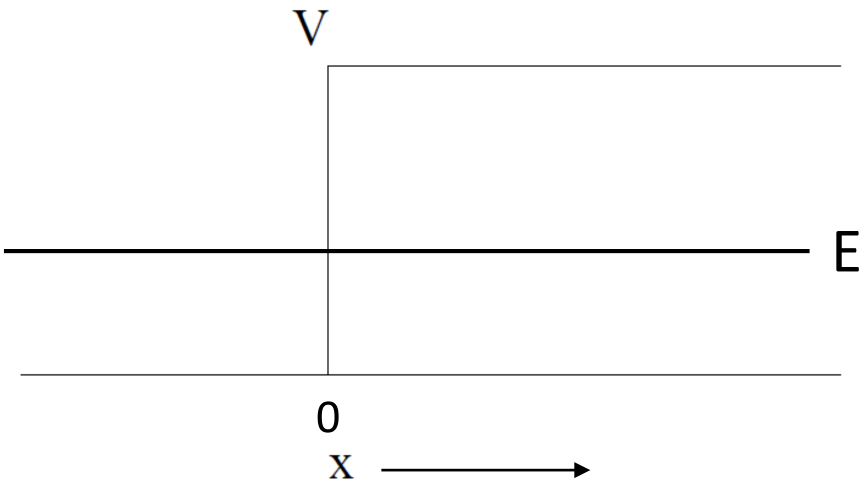

$$\boxed{\frac{p^2}{2m} + V(x) = E}$$

$$p = \pm\sqrt{2m(E - V(x))}$$

In Classical Mechanics, the particle's motion is restricted between $x_1$ and $x_2$, as otherwise $p$ becomes imaginary

What happens in Quantum Mechanics?

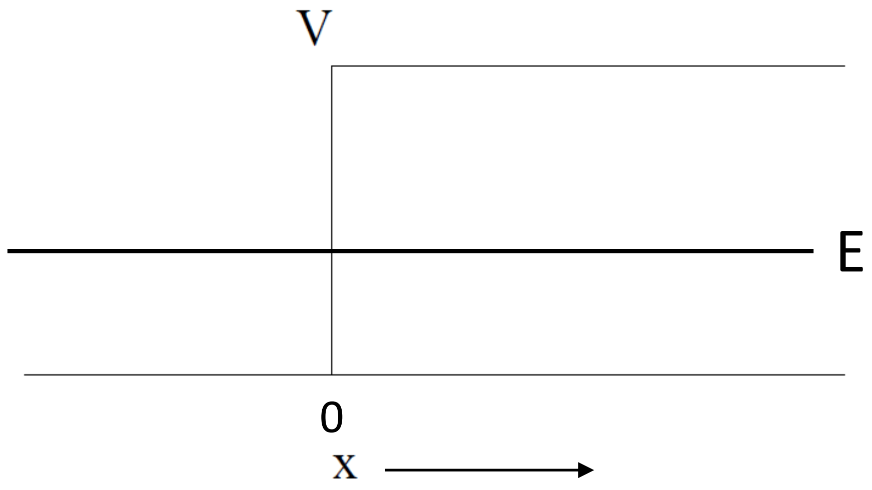
# Step Potential: Case II - (Energy of particle E) < V




Defining - $\sqrt{2m(E-V)} = \sqrt{-1}\sqrt{2m(V-E)} = iq$

$$\frac{d^2X}{dx^2} = \frac{q^2}{\hbar^2}X$$

# Step Potential: Case II - (Energy of particle E) < V

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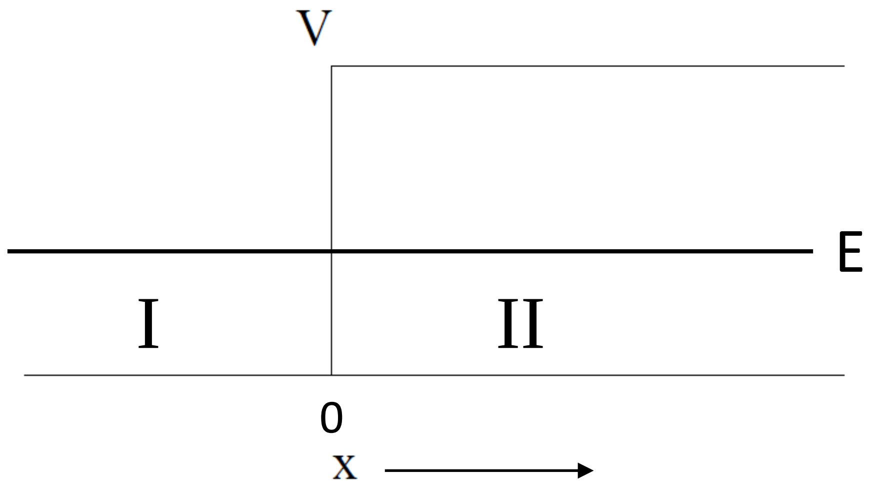
$$X(x) = A_1 e^{-qx/\hbar} + A_2 e^{qx/\hbar}$$

# Step Potential: Case II - (Energy of particle E) < V




$$X(x) = A_1 e^{-qx/\hbar} + A_2 e^{qx/\hbar}$$

To ensure wavefunction remains finite as $x \to +\infty$, $A_2 = 0$

$$X_{II}(x) = A_1 e^{-qx/\hbar}$$

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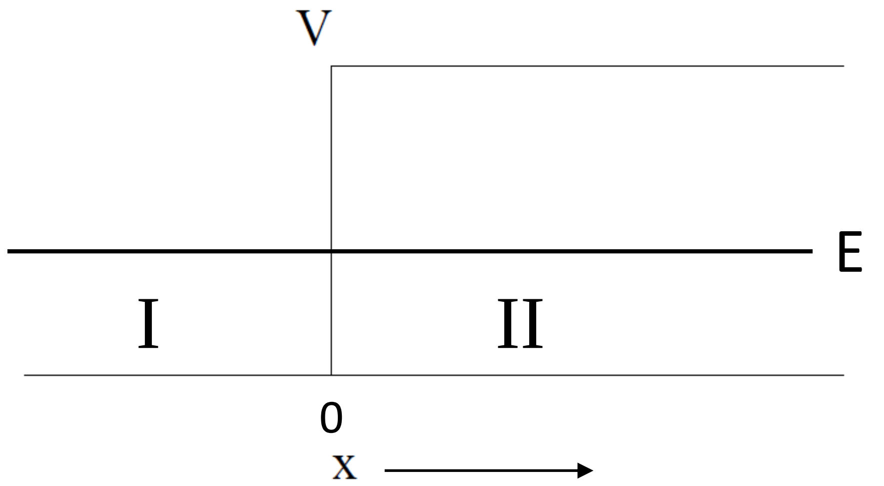



$$X(x) = A_1 e^{-qx/\hbar} + A_2 e^{qx/\hbar}$$

To ensure wavefunction remains finite as $x \to +\infty$, $A_2 = 0$

$$X_{II}(x) = A_1 e^{-qx/\hbar}$$

Total solution for within the potential step -

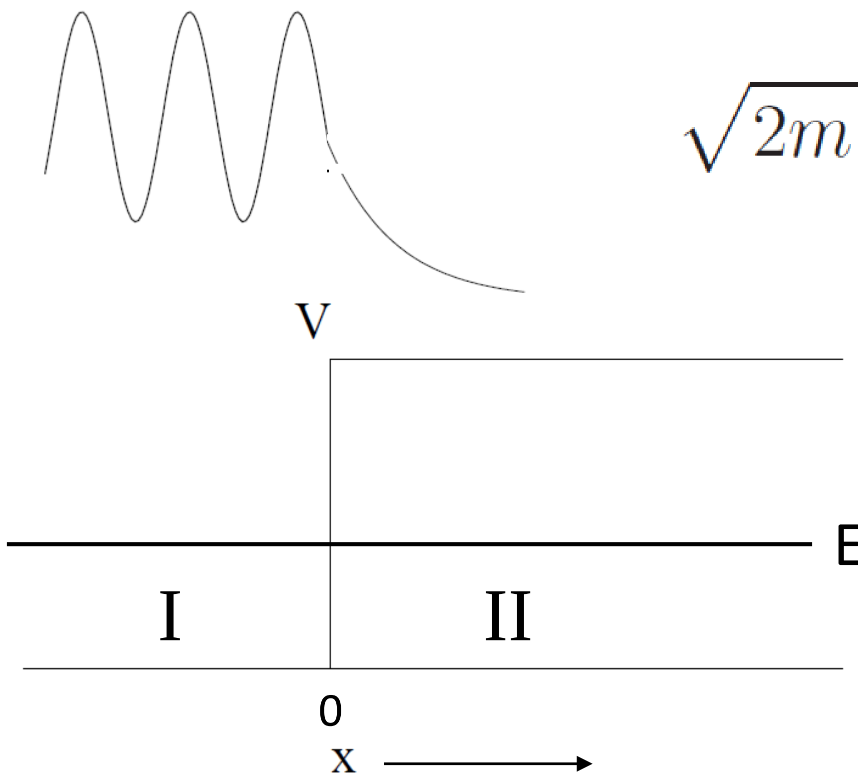
$$\psi_{II}(x,t) = A_1 e^{-iEt/\hbar} e^{-qx/\hbar}$$

# Step Potential: Case II - (Energy of particle E) < V

$$\psi_{II}(x,t) = A_1 e^{-iEt/\hbar} e^{-qx/\hbar}$$

$$\psi_{I}(x,t) = e^{-iEt/\hbar}\left[A_1 e^{ipx/\hbar} + A_2 e^{-ipx/\hbar}\right]$$

$$p = \sqrt{2mE}$$




$$\sqrt{2m(V-E)} = q$$

# Step Potential: Case II - (Energy of particle E) < V

$$\psi_{II}(x,t) = A_1 e^{-iEt/\hbar} e^{-qx/\hbar}$$

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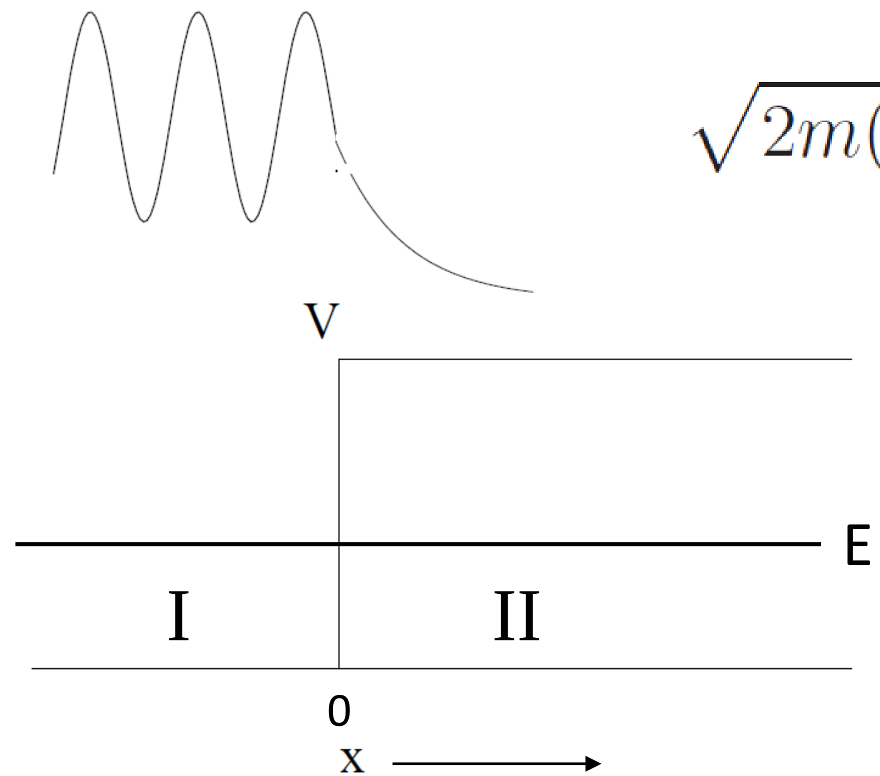
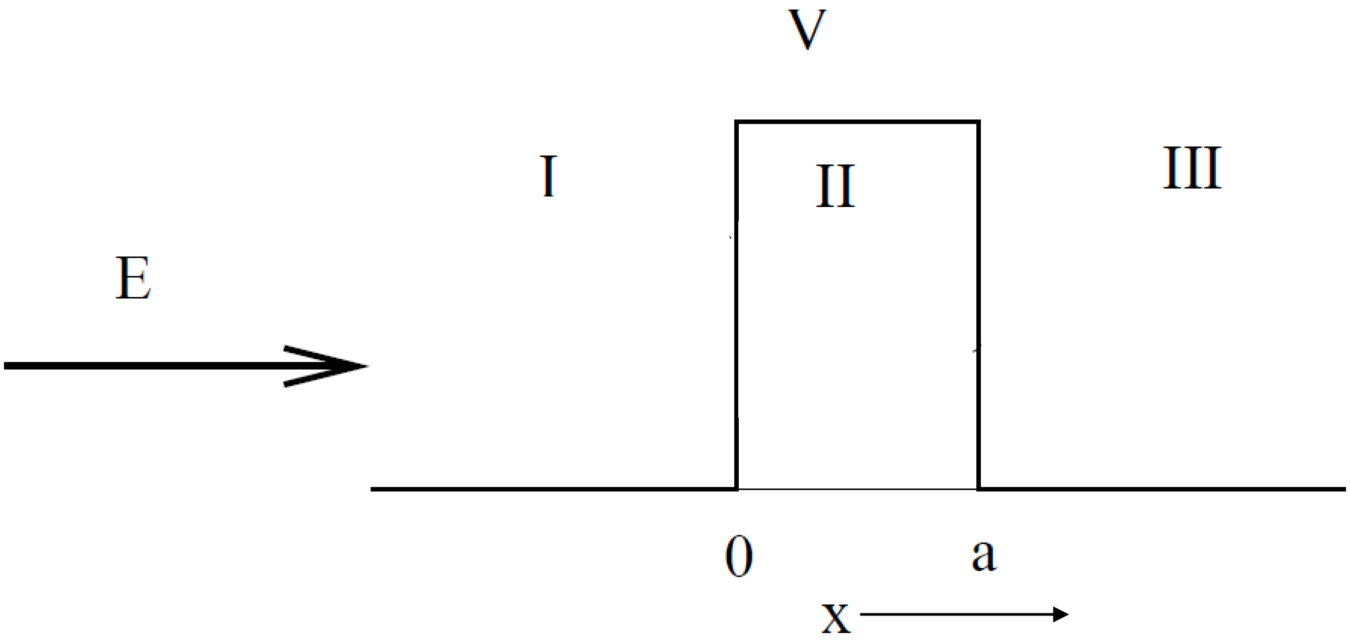



So finite probability of finding particle within barrier for E<V !

# Quantum Tunneling



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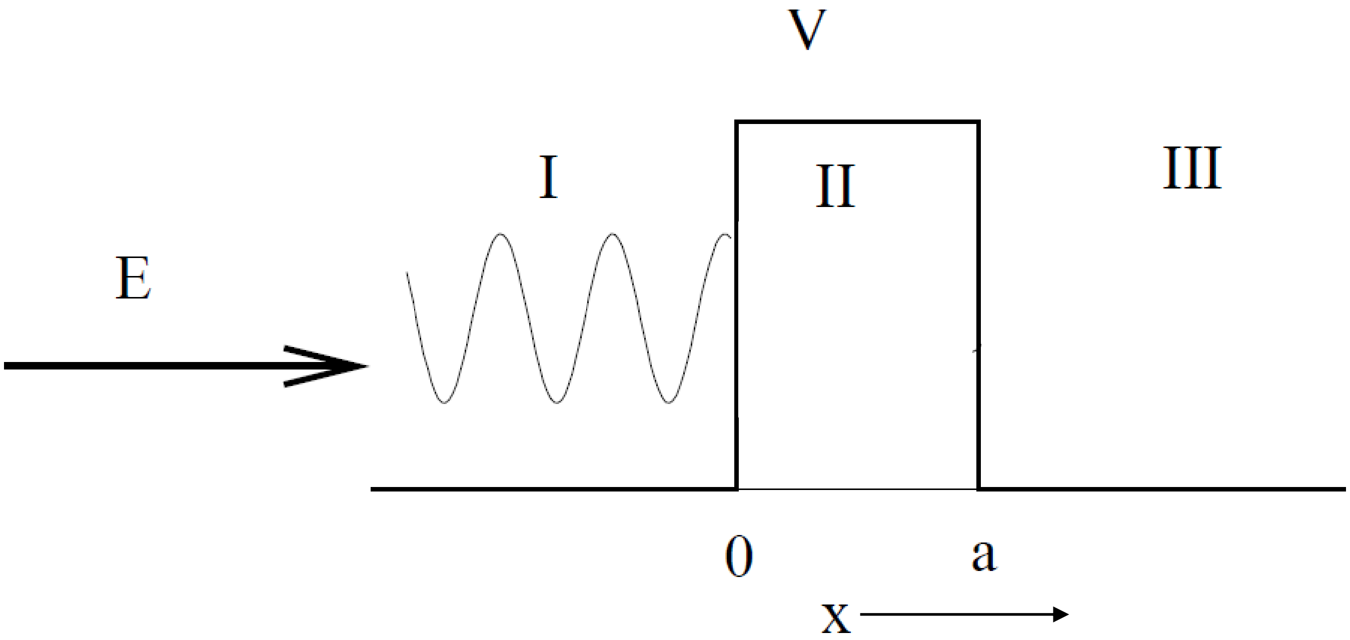
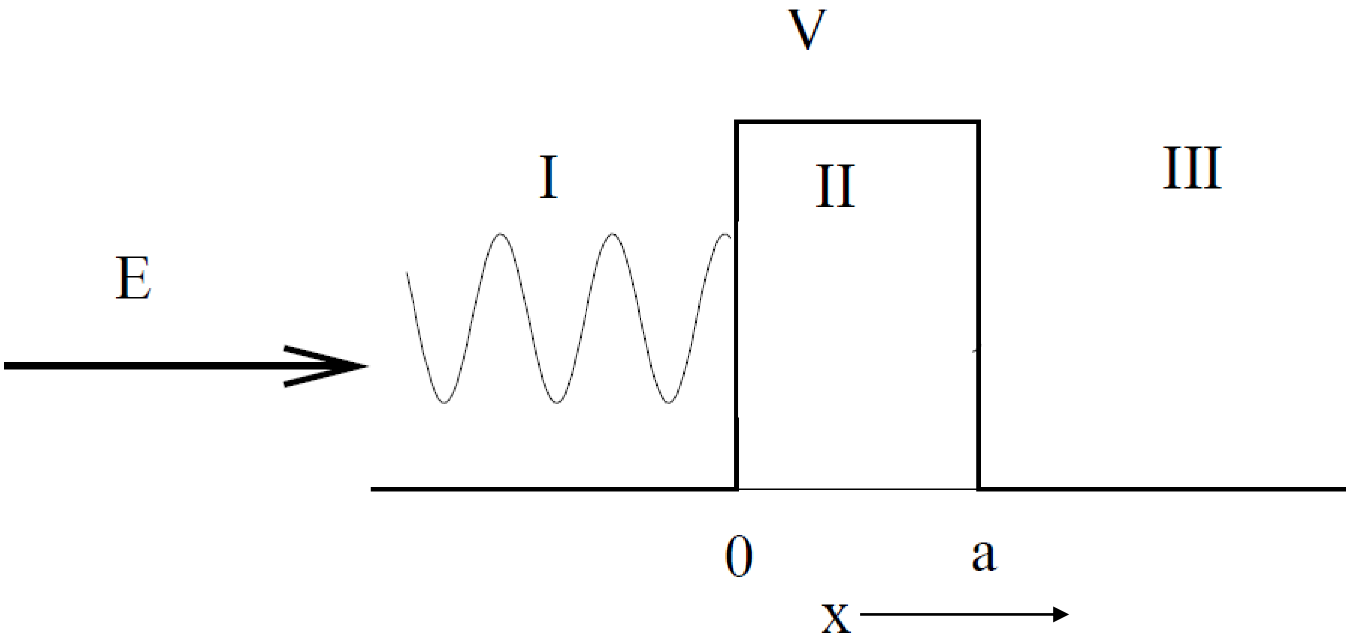



$$\psi_I(x,t) = e^{-iEt//\hbar}\left[A_I e^{ipx/\hbar} + B_I e^{-ipx/\hbar}\right]$$

# Quantum Tunneling

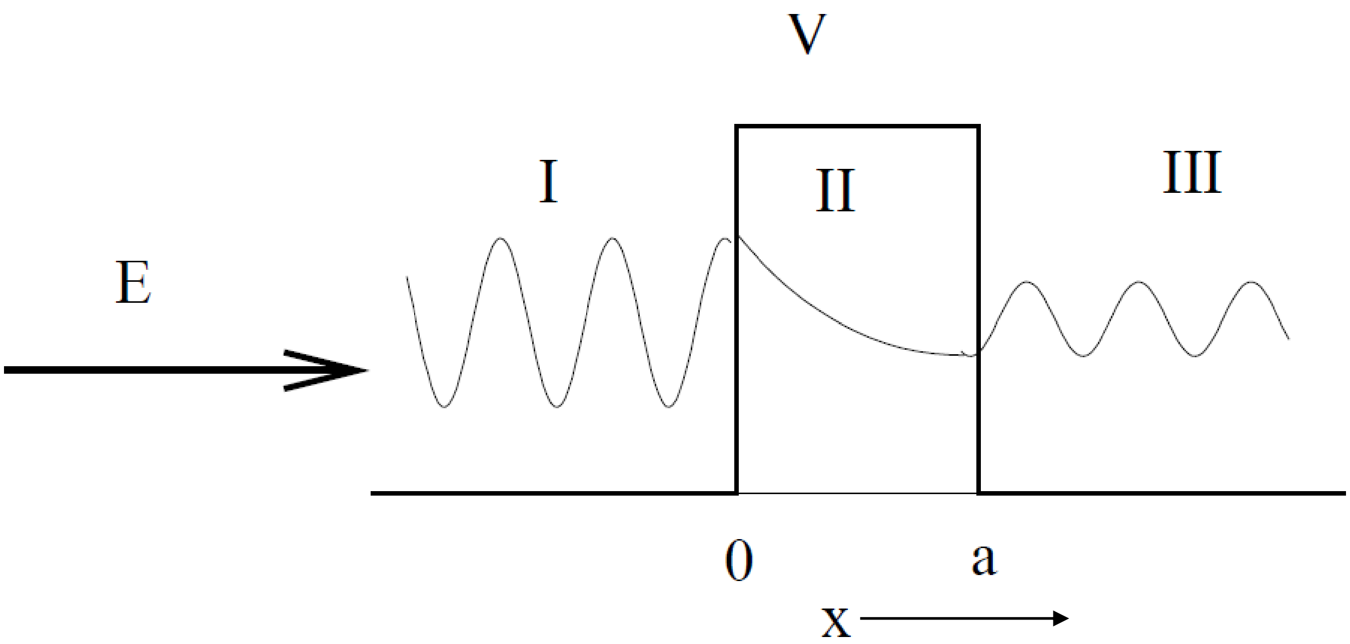



$$\psi_I(x,t) = e^{-iEt//\hbar}\left[\underbrace{A_I e^{ipx/\hbar}} + \underbrace{B_I e^{-ipx/\hbar}}\right]$$

Incident particle travelling along $+x$ direction

Reflected particle travelling along $-x$ direction

# Quantum Tunneling

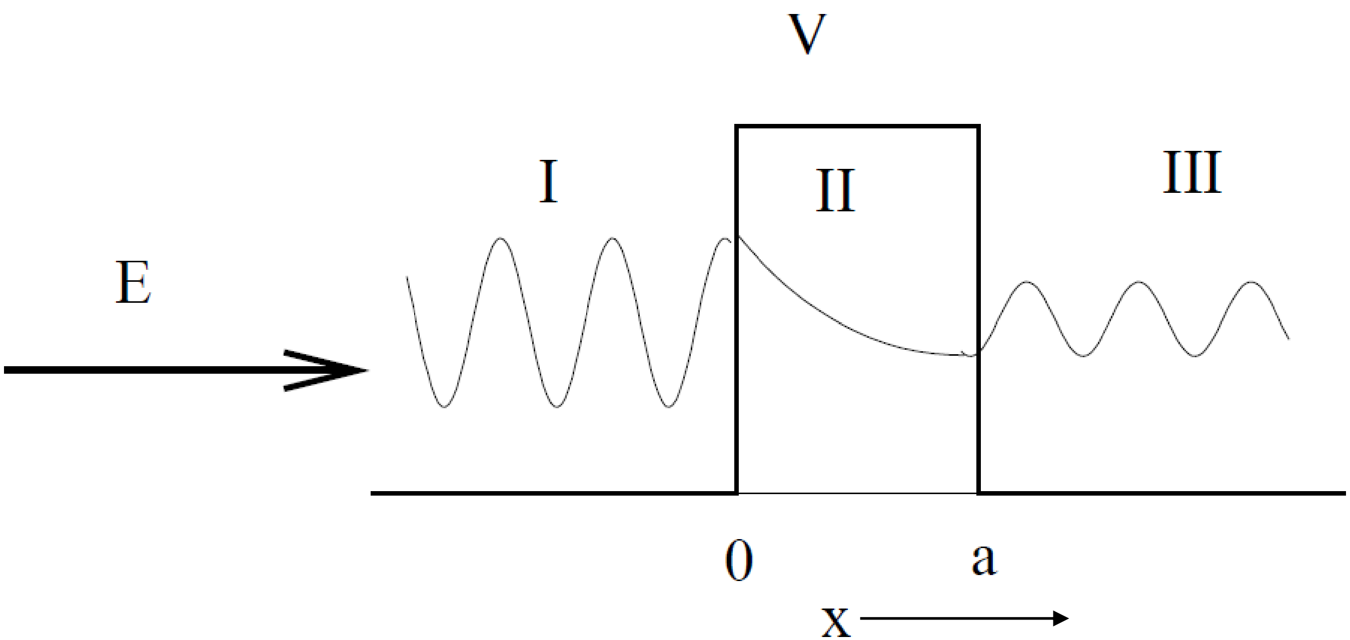



$$\psi_I(x,t) = e^{-iEt//\hbar}\left[A_I e^{ipx/\hbar} + B_I e^{-ipx/\hbar}\right]$$

$$\psi_{II}(x,t) = e^{-iEt/\hbar}\left[A_{II} e^{-qx/\hbar} + B_{II} e^{qx/\hbar}\right]$$

$$\psi_{III}(x,t) = e^{-iEt/\hbar}\left[A_{III} e^{ipx/\hbar} + B_{III} e^{-ipx/\hbar}\right]$$

# Quantum Tunneling

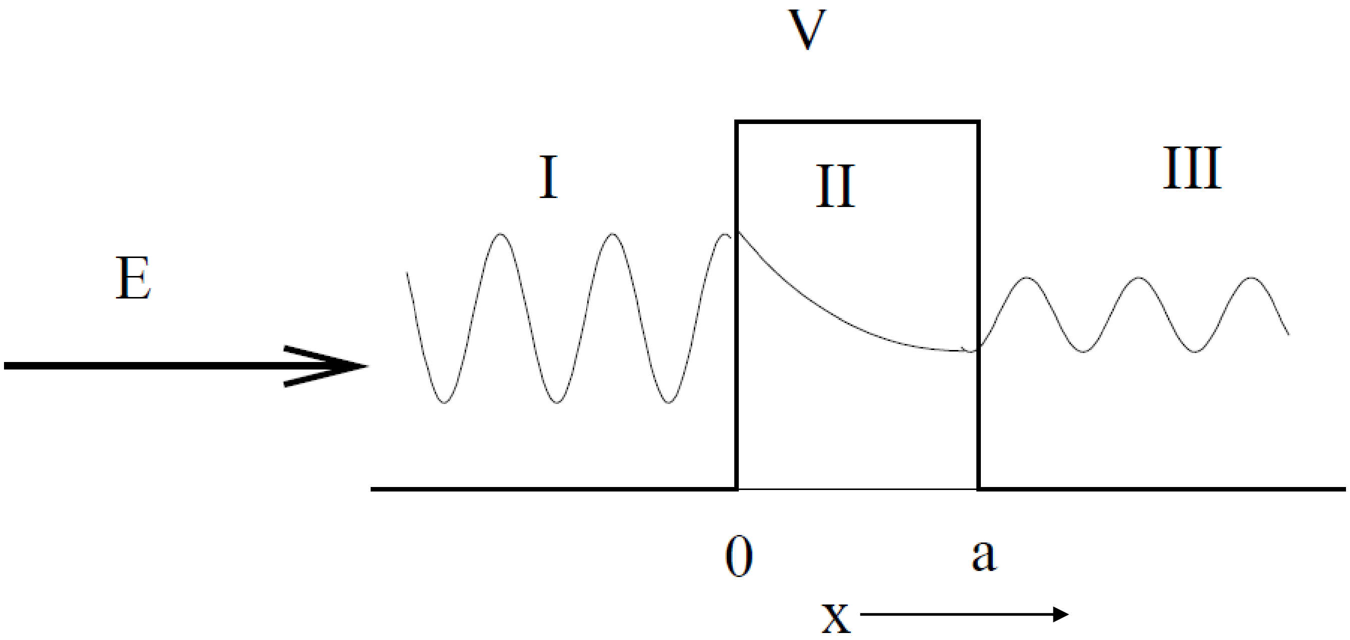



$$\psi_{III}(x,t) = e^{-iEt/\hbar}\left[\underbrace{A_{III}e^{ipx/\hbar}} + \underbrace{B_{III}e^{-ipx/\hbar}}\right]$$

Particle travelling through the barrier and along $+x$ direction

Particle incident from the right along $-x$ direction. Here $B_{III} = 0$

# Quantum Tunneling

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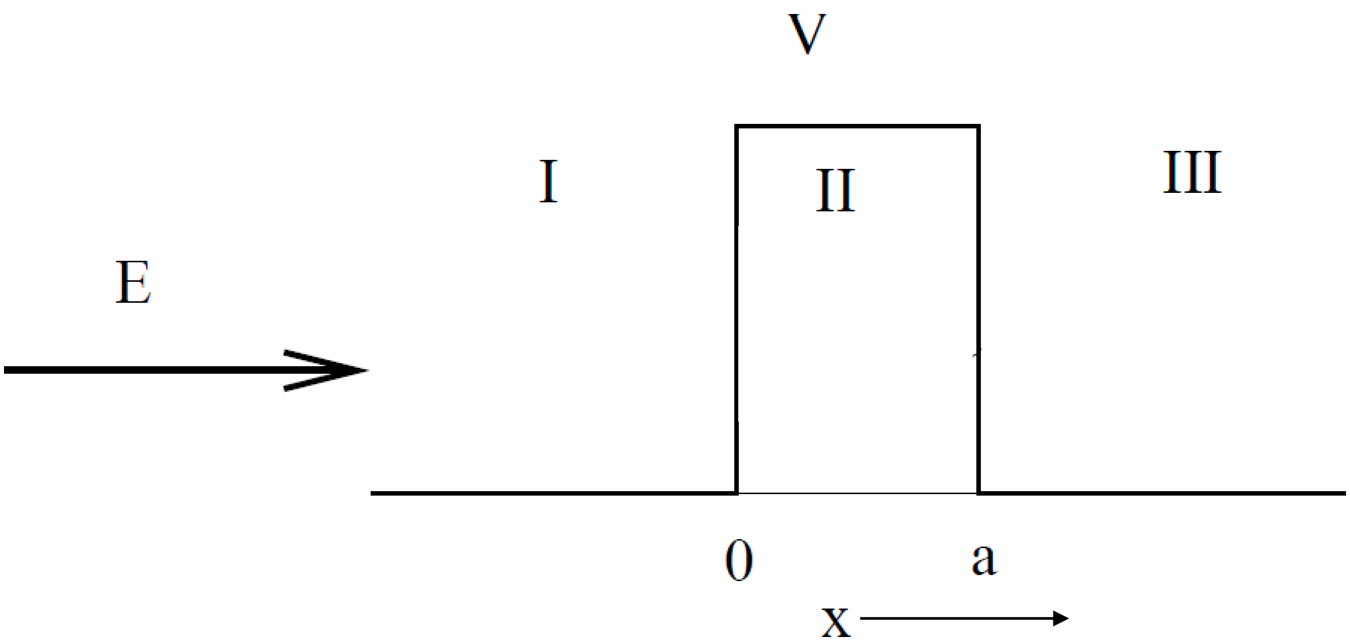
$$\psi_{III}(x,t) = e^{-iEt/\hbar}\left[A_{III} e^{ipx/\hbar}\right]$$

# Quantum Tunneling




Boundary conditions to be satisfied at $x = 0$ –

$$\psi_I(0,t) = \psi_{II}(0,t)$$

$$\left(\frac{\partial \psi_I}{\partial x}\right)_{x=0} = \left(\frac{\partial \psi_{II}}{\partial x}\right)_{x=0}$$

# Origin of boundary conditions

The wavefunction $\psi(x,t)$ (considering 1D) is governed by Schrödinger's equation -

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi(x,t) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x,t)\psi}$$

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## Further assumption for ease of calculation

Let us assume that the potential step is very high, i.e. $V \gg E$

$$q = \sqrt{2m(V-E)} \approx \sqrt{2mV}$$

$$p = \sqrt{2mE} \qquad \frac{p}{q} = \sqrt{\frac{E}{V}} \ll 1$$

# Quantum Tunneling

$$\psi_I(x,t) = e^{-iEt//\hbar}\left[A_I e^{ipx/\hbar} + B_I e^{-ipx/\hbar}\right]$$

$$\psi_{II}(x,t) = e^{-iEt/\hbar}\left[A_{II} e^{-qx/\hbar} + B_{II} e^{qx/\hbar}\right]$$

Applying boundary conditions at $x = 0$

$$\psi_I(0,t) = \psi_{II}(0,t) \Rightarrow A_I + B_I = A_{II} + B_{II}$$

$$\left(\frac{\partial \psi_I}{\partial x}\right)_{x=0} = \left(\frac{\partial \psi_{II}}{\partial x}\right)_{x=0} \Rightarrow ip\left(A_I - B_I\right) = -q\left(A_{II} - B_{II}\right)$$

$$A_I - B_I = \frac{iq}{p}\left(A_{II} - B_{II}\right)$$

# Quantum Tunneling

$$\psi_{II}(x,t) = e^{-iEt/\hbar}\left[A_{II}e^{-qx/\hbar} + B_{II}e^{qx/\hbar}\right]$$

$$\psi_{III}(x,t) = e^{-iEt/\hbar}\left[A_{III}e^{ipx/\hbar}\right]$$

## Applying boundary conditions at $x = a$

Wavefunction continuity -

$$A_{II}e^{-qa/\hbar} + B_{II}e^{qa/\hbar} = A_{III}e^{ipa/\hbar}$$

First deriavative continuity -

$$-q\left[A_{II}e^{-qa/\hbar} - B_{II}e^{qa/\hbar}\right] = ipA_{III}e^{ipa/\hbar}$$

$$A_{II}e^{-qa/\hbar} - B_{II}e^{qa/\hbar} = \frac{-ip}{q}A_{III}e^{ipa/\hbar}$$

# Quantum Tunneling

Applying boundary conditions at $x = a$

First deriavative continuity -

$$A_{II}e^{-qa/\hbar} - B_{II}e^{qa/\hbar} = \frac{-ip}{q}A_{III}e^{ipa/\hbar}$$

Since - $p/q \ll 1$

$$A_{II}e^{-qa/\hbar} - B_{II}e^{qa/\hbar} = 0$$

$$B_{II} = e^{-2qa/\hbar}A_{II} \ll A_{II}$$

Small

$$\psi_{II}(x,t) = e^{-iEt/\hbar}\left[A_{II}e^{-qx/\hbar} + \cancel{B_{II}}e^{qx/\hbar}\right]$$

# Quantum Tunneling

Wavefunction continuity at $x = a$

$$A_{II}e^{-qa/\hbar} + B_{II}e^{qa/\hbar} = A_{III}e^{ipa/\hbar}$$

Also from last slide - $$A_{II}e^{-qa/\hbar} - B_{II}e^{qa/\hbar} = 0$$

$$A_{III} = 2e^{-ipa/\hbar}e^{-qa/\hbar}A_{II}$$

# Quantum Tunneling

Wavefunction continuity at $x = a$

$$A_{II}e^{-qa/\hbar} + B_{II}e^{qa/\hbar} = A_{III}e^{ipa/\hbar}$$

Also from last slide - $A_{II}e^{-qa/\hbar} - B_{II}e^{qa/\hbar} = 0$

$$A_{III} = 2e^{-ipa/\hbar}e^{-qa/\hbar}A_{II}$$

Recalling boundary conditions at $x = 0$

$$A_I + B_I = A_{II} + \cancel{B_{II}} \quad \& \quad A_I - B_I = \frac{iq}{p}(A_{II} - \cancel{B_{II}})$$

$$\left(1 + \frac{iq}{p}\right)A_{II} = 2A_I$$

# Quantum Tunneling

$$\left(1 + \frac{iq}{p}\right) A_{II} = 2A_I$$

Since $q/p \gg 1$

$$A_{II} = -\frac{ip}{q} 2A_I$$

# Quantum Tunneling

$$\left(1+\frac{iq}{p}\right)A_{II} = 2A_I$$

Since $q/p \gg 1$

$$A_{II} = -\frac{ip}{q}2A_I$$

Also from last slide -

$$A_{III} = 2e^{-ipa/\hbar}e^{-qa/\hbar}A_{II}$$

$$A_{III} = -4i\frac{p}{q}\,e^{-ipa/\hbar}\,e^{-qa/\hbar}\,A_I$$

# Quantum Tunneling

$$A_{III} = -4i\frac{p}{q}\, e^{-ipa/\hbar}\, e^{-qa/\hbar}\, A_I$$

The transmission coefficient *T* is -

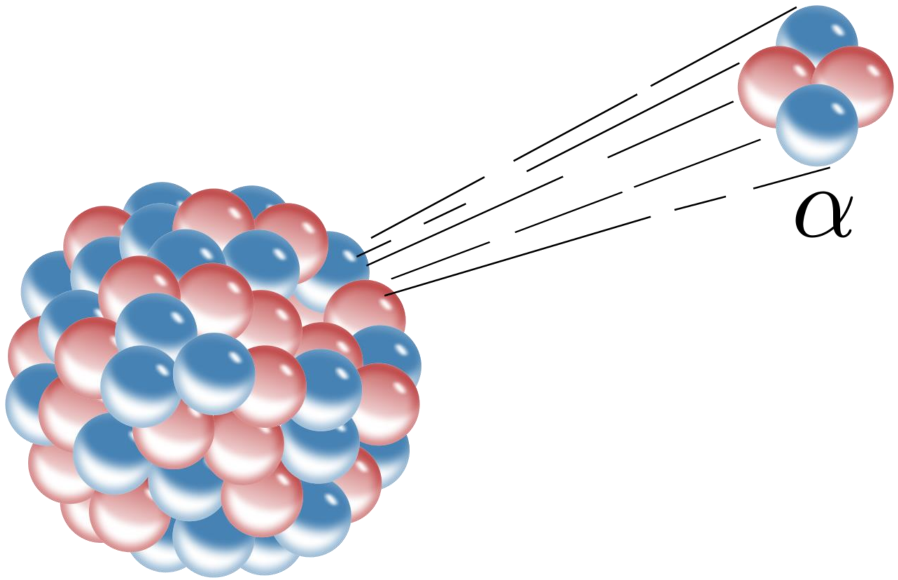
$$T = \frac{|A_{III}|^2}{|A_I|^2} = 16\frac{p^2}{q^2}e^{-2qa/\hbar}$$

Since - $\frac{p}{q} = \sqrt{\frac{E}{V}} \ll 1$

$$\boxed{T = 16\frac{E}{V}e^{-2a\sqrt{2mV}/\hbar}}$$

# Some applications of Quantum Tunneling effect

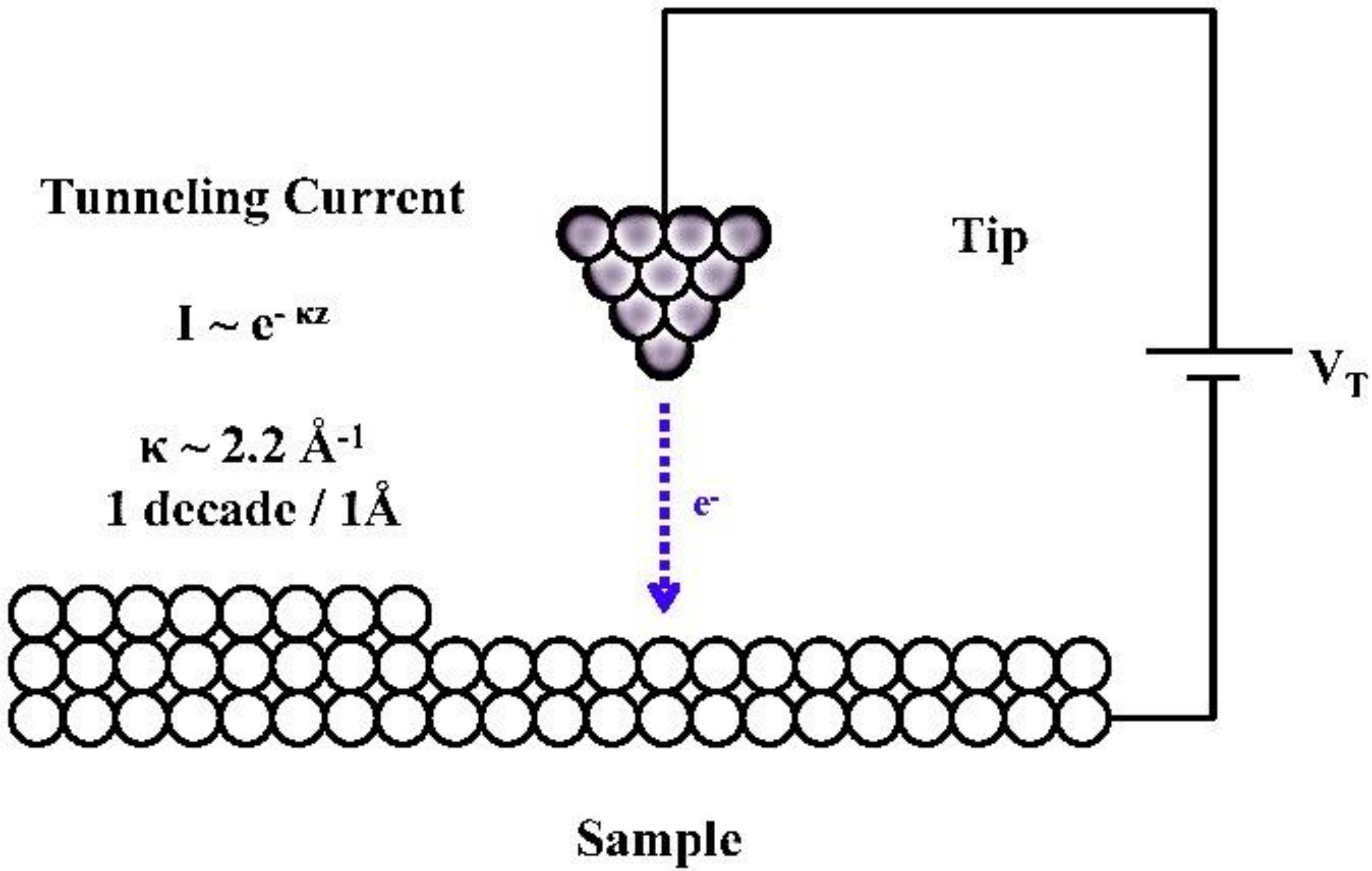
# Alpha Decay




**Alpha particle (a Helium-4 nucleus) is held within nucleus by nuclear forces.**
**It experiences a potential barrier to come outside the nucleus.**
**Still Alpha particles can come outside nucleus because of tunneling**

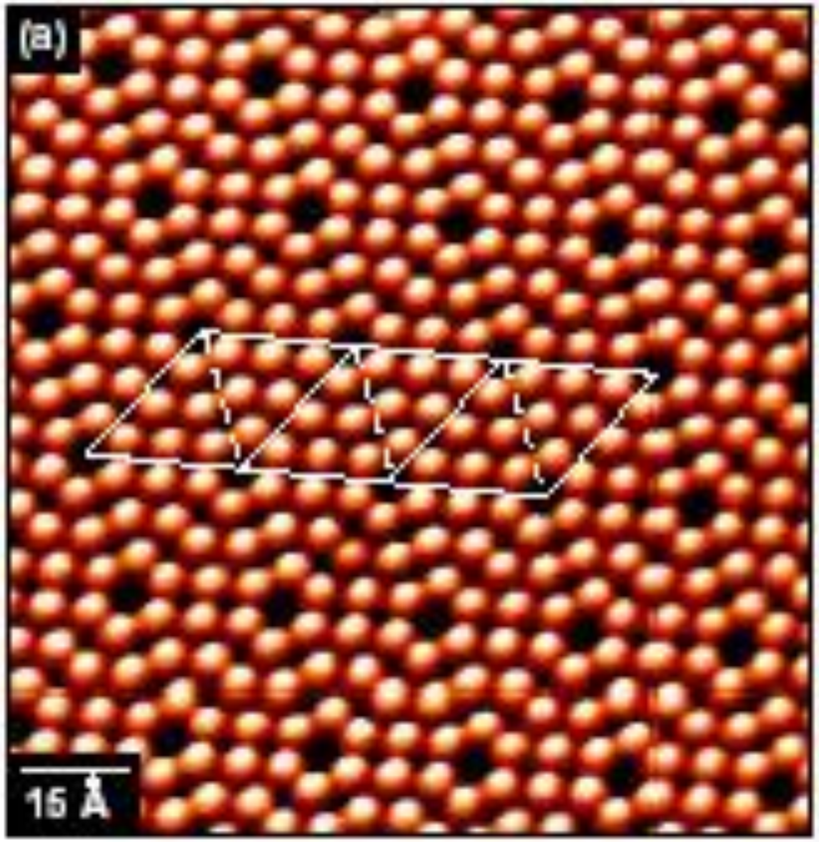
Source of image: Wikipedia

# Scanning Tunneling Microscope




Electron tunneling across the air gap

# Scanning Tunneling Microscope




STM scan of a Si sample
Each bright spot indicates a Si atom