Solutions to Problem Set-0

Course: PH11001
Spring 2019-20

Problem 1: Scalar (dot) product

Consider the following vector in two dimensions

$$\vec{v} = \cos(\omega t)\ \hat{i} + \sin(\omega t)\ \hat{j},$$

where $\hat{i}$ and $\hat{j}$ are mutually orthogonal unit vectors along the cartesian coordinate axes x and y respectively. Here t is a parameter, which you can physically think of as 'time', and ω is some constant. Clearly, the x and y components of the given vector $\vec{v}$ changes with 'time'.

(a) How does the magnitude and direction of the vector vary with 'time' t?

(b) What is the angle θ made by the vector $\vec{v}$ with the following vector

$$\vec{u} = \sin(\omega t)\ \hat{i} + \cos(\omega t)\ \hat{j}.$$

(c) Plot this angle $\theta(t)$ as a function of t. You may consider $\omega = 1$ for making this plot.

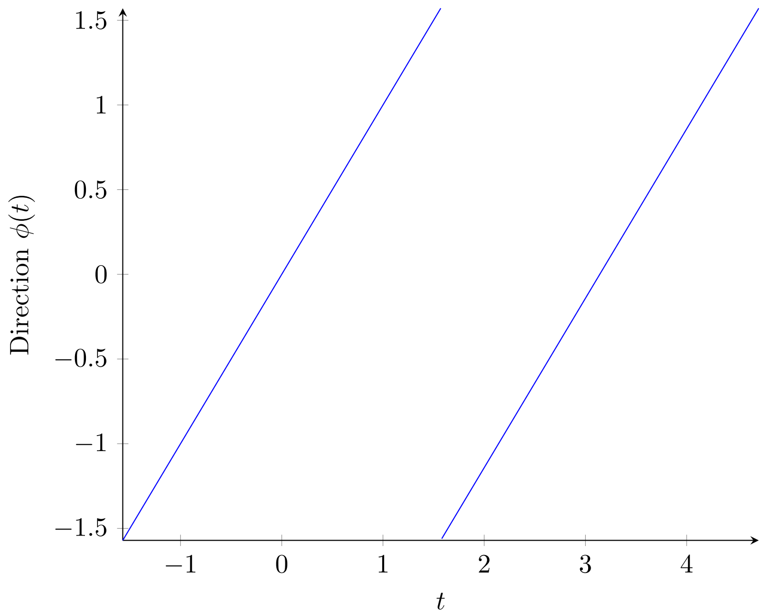
Solution

(a) Magnitude $|\vec{v}| = \sqrt{\cos^2(\omega t) + \sin^2(\omega t)} = 1$ (independent wrt t).

Direction, i.e., angle ϕ made with x-axis $= \tan^{-1}\frac{\sin(\omega t)}{\cos(\omega t)} = \tan^{-1}(\tan\omega t)$

$$= n\pi + \omega t, n \in Z$$

Here, we take n and the sign of the term in brackets such that the conventional range of the arctangent function, $(-\pi/2, \pi/2)$ is respected.

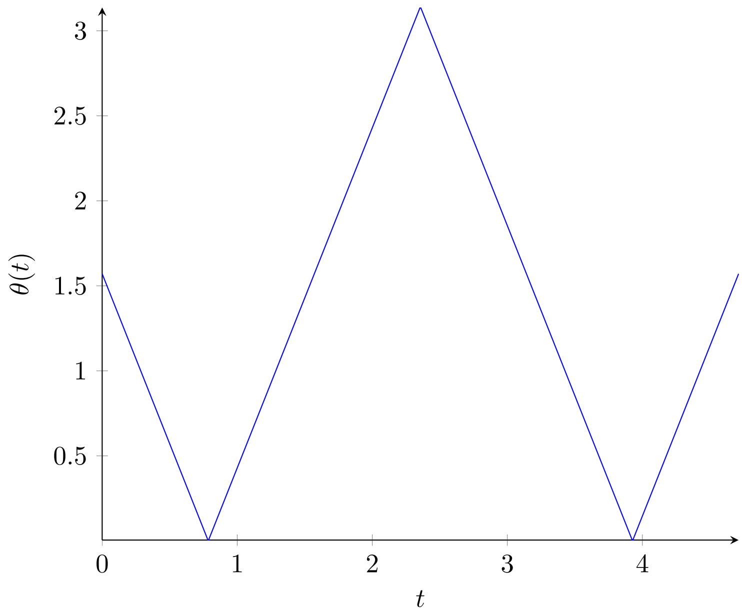


(b) $\cos\theta(t) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = 2\cos\omega t\sin\omega t = \sin 2\omega t = \cos(\pi/2 - 2\omega t)$

$$\implies \theta(t) = 2n\pi \pm (\pi/2 - 2\omega t), n \in Z$$

Here, we take n and the sign of the term in brackets such that the conventional range of the arccosine function, $[0, \pi]$ is respected.

(c) $\vec{v}$ starts aligned along the x-axis, and rotates anti-clockwise, whereas $\vec{u}$ starts along the y-axis and rotates clockwise. Note that $\theta(t)$ becomes 0 at $t = \pi/4$ sec.



Problem 2: Vector (cross) product

Consider the set of following three non-co-planar vectors

$$\vec{a} = 2\,\hat{\imath},\ \vec{b} = \hat{\jmath} + \hat{k},\ \vec{c} = \hat{\imath} + \hat{k},$$

where again $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are mutually orthogonal unit vectors along the cartesian coordinate axes x, y and z respectively.

(a) Explicitly evaluate the following vectors

$$\vec{A} = \frac{\vec{b}\times\vec{c}}{\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)},\ \vec{B} = \frac{\vec{c}\times\vec{a}}{\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)},\ \vec{C} = \frac{\vec{a}\times\vec{b}}{\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)}.$$

(b) Find the volume of the parallelepiped spanned by the vectors $\{\vec{a}, \vec{b}, \vec{c}\}$.

(c) Find the volume of the parallelepiped spanned by the vectors $\{\vec{A}, \vec{B}, \vec{C}\}$. Is this related in any way to your answer in part (b)?

Solution

(a)

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \hat{\imath} + \hat{\jmath} - \hat{k}$$

$$\implies \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = 2$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -2\hat{\jmath} + 2\hat{k}$$

$$\vec{c}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 2\hat{j}$$

$$\therefore \vec{A} = \left(\hat{i}+\hat{j}-\hat{k}\right)/2,\ \vec{B}=\hat{j},\ \vec{C} = -\hat{j}+\hat{k}$$

(b) Volume of parallelepiped $= \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = 2.$

(c) Volume of parallelepiped $= \vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \frac{1}{2}\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \frac{1}{2}$. This is the reciprocal of the answer in part (b).

Problem 3: Taylor Series

(a) Expand $f(x) = \cos x$ in a Taylor series about $x = 0$, and write down the first *three* non-vanishing terms.

(b) Using the first *two* non vanishing terms in the expansion about $x = 0$, evaluate the value of $\cos x$ at $x = \frac{1}{2}$.

(c) Compare your result in part (b) with the exact value of $\cos\frac{1}{2}$ and estimate the error associated with approximating this function with the first two terms of the Taylor series.

(d) Do you expect this error to depend on the value of x (which is $x = \frac{1}{2}$ here). For example, if we evaluated the first two terms at $x = \frac{1}{3}$ instead of $x = \frac{1}{2}$, do you expect the error to increase or decrease. Explain your answer.

(e) How does this error change if you keep all the first three (instead of two) non-vanishing terms of part (a) to evaluate this function at $x = \frac{1}{2}$?

Solution

(a) Taylor series expansion about x=0:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k,$$

$$\begin{aligned}\therefore cos(x) &= \cos(0) + x(-\sin(0)) + \frac{x^2}{2}(-\cos(0)) + \frac{x^3}{6}\sin(0) + \frac{x^4}{24}\cos(0) + \cdots \\ &\approx 1 - \frac{x^2}{2} + \frac{x^4}{24}\end{aligned}$$

(keeping first three non-vanishing terms.)

(b) Using the first two non-vanishing terms to evaluate $cos(\frac{1}{2})$,

$$\cos(\frac{1}{2}) \approx 1 - \frac{1}{2}(\frac{1}{2})^2 = 1 - \frac{1}{8} = \frac{7}{8} = 0.875$$

(c) The exact value of $cos(\frac{1}{2})$ is 0.87758. $\therefore$ Absolute error is 0.00258, and percentage error is 0.29%.

(d) The error does indeed depend on the value of x. The Taylor expansion was calculated around $x = 0$; the further away one evaluates $\cos(x)$ from that point using that particular expansion, the larger the error will be. At $x = 0$ there is no error. At $x = \frac{1}{3}$, one would expect the error to be less than $x = \frac{1}{2}$. Checking:

$$\cos(\frac{1}{3}) \approx 1 - \frac{1}{2}(\frac{1}{3})^2 = 1 - \frac{1}{18} = \frac{17}{18} = 0.94444$$

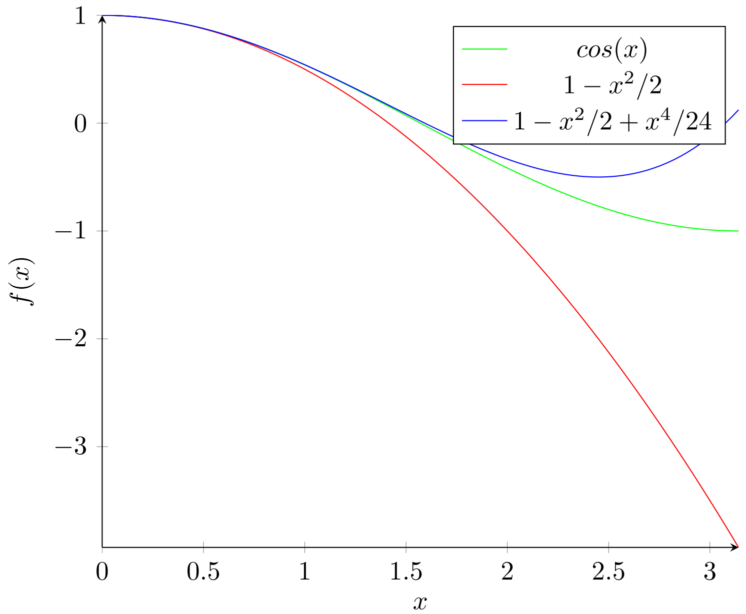
Exact value of $\cos(\frac{1}{3})$ is 0.94496. The percentage error here is 0.054%, significantly less than for $x = \frac{1}{2}$.

(e) Keeping an additional term from the expansion will also reduce the error. Checking:

$$\cos(\frac{1}{2}) \approx 1 - \frac{1}{2}(\frac{1}{2})^2 + \frac{1}{24}(\frac{1}{2})^4 = 1 - \frac{1}{8} + \frac{1}{384} = 0.87760.$$

The percentage error becomes 0.0025%.

The graphs below illustrates how well the two approximations agree with the actual function, and how the error increases as we go further away from x=0.



Problem 4: Second order differential equations

Consider the differential equation

$$\frac{d^2x(t)}{dt^2} + \beta\, x(t) = 0. \tag{1}$$

Solve this equation using the following two distinct methods.

1. *Method 1:*

    (i) Assume $x(t) = \sum_{n=0}^{\infty} a_{(n)} t^n$ and derive a recursion relation between $a_{(n+2)}$ and $a_{(n)}$.

    (ii) Use this to get the series solution to the differential equation in terms of the unknown constants $a_{(0)}$ and $a_{(1)}$.

    (iii) Using the substitutions $a_{(0)} \to A\cos\phi$ and $a_{(1)}/\beta \to -A\sin\phi$ show that the following is a solution to equation (1)

$$x(t) = A\cos(\beta t + \phi)\,.$$

2. *Method 2:*

   (i) Multiply equation (1) with $2\frac{dx(t)}{dt}$ and write it as a total time derivative. Integrate the resultant equation to obtain

$$\left(\frac{dx(t)}{dt}\right)^2 + \beta^2\ x(t)^2 = k^2.$$

   (ii) Rearrange this to the integral form

$$\int dt = \int \frac{dx}{\sqrt{k^2 - \beta^2 x^2}}.$$

   Hence, perform the integral to obtain the most general solution of the differential equation.

Solution

Given Differential equation,

$$\frac{d^2x(t)}{dt^2} + \beta\ x(t) = 0$$

Method-1

(i) Assuming $x(t) = \sum_{n=0}^{\infty} a_{(n)} t^n$ as the soln of the equation, we get,

$$\frac{dx}{dt} = \sum_{n=0}^{\infty} a_{(n)} n t^{n-1}$$
$$\frac{d^2x}{d^2t} = \sum_{n=0}^{\infty} a_{(n)} n(n-1) t^{n-2}$$

Now substituting them back in the given differential equation, we get

$$\sum_{n=0}^{\infty} a_{(n)} n(n-1) t^{n-2} + \beta^2 \sum_{n=0}^{\infty} a_{(n)} t^n = 0$$

We choose a_0 as the coefficient of the lowest non-vanishing term of the series, and so contribution to the coefficient of t^n comes from the term containing a_{n+2} from the first summation and from that with a_n in the second. So for vanishing of each coefficient of t^n yields in,

$$a_{n+2}(n+2)(n+1) + \beta^2 a_n = 0$$

Thus we get the recursion relation,

$$a_{n+2} = -a_n \frac{\beta^2}{(n+2)(n+1)} \quad \text{(Answer)}$$

(ii) The recursion relation yields:

$$
\begin{aligned}
a_2 =& - a_0 \frac{\beta^2}{2} \\
a_3 =& - a_1 \frac{\beta^2}{6} \\
a_4 =& - a_2 \frac{\beta^2}{12} = a_0 \frac{\beta^4}{24} \\
a_5 =& - a_3 \frac{\beta^2}{20} = a_1 \frac{\beta^4}{120} \\
&\vdots
\end{aligned}
$$

Thus writing the final soln.

$$
\begin{aligned}
x(t) =& a_0 + a_1 t - \frac{\beta^2}{2} a_0 t^2 - \frac{\beta^2}{6} a_1 t^3 + \frac{\beta^4}{24} a_0 t^4 + \frac{\beta^4}{120} t^5 + \dots \\
x(t) =& a_0 \left(1 - \frac{(\beta t)^2}{2!} + \frac{(\beta t)^4}{4!} + \dots\right) + \frac{a_1}{\beta}\left(\beta t - \frac{(\beta t)^3}{3!} + \frac{(\beta t)^5}{5!} + \dots\right)
\end{aligned}
$$

(iii) Obtained solution in the previous section can be written as *sine* and *cosine* series respectively as,

$$x(t) = a_0 cos(\beta t) + \frac{a_1}{\beta} sin(\beta t)$$

Now substituting $a_0 \to A\cos\phi$ and $a_1/\beta \to -A\sin\phi$ we get,

$$x(t) = Acos\phi cos(\beta t) - Asin\phi sin(\beta t) = Acos(\beta t + \phi) \qquad \text{(Proved)}$$

Method-2

(i) Multiplying $2\dot{x} = 2\frac{dx}{dt}$ in the given equation we get,

$$
\begin{aligned}
& 2\dot{x}\ddot{x} + 2\beta^2 \dot{x} x = 0 \\
\implies & \frac{d}{dt}\left(\dot{x}^2 + \beta^2 x^2\right) = 0 \\
\implies & \dot{x}^2 + \beta^2 x^2 = k^2 \quad [k^2 \text{ is integration const.}]
\end{aligned}
$$

Hence proved.

(ii) Rearranging the above form to integral form we get,

$$
\begin{aligned}
\int dt =& \int \frac{dx}{\sqrt{k^2 - \beta^2 x^2}} \\
t - t_0 =& \frac{1}{\beta} sin^{-1} \frac{\beta x}{k} - \phi_0 \quad [t_0, \phi_0 \text{ int. const.}] \\
x(t) =& \frac{k}{\beta} sin(\beta t + \beta(\phi_0 - t_0))
\end{aligned}
$$

Now choosing $\frac{k}{\beta} \to A$ and $\beta(\phi_0 - t_0) \to (\phi + \frac{\pi}{2})$ the soln turns out to be,

$$x(t) = Acos(\beta t + \phi)$$

same as we obtained in 1st method.

Problem 5 : Performing line integrals:

Consider a projectile which is fired at an angle of 45° with an initial velocity of 10 ms^{-1}, from the surface of the earth. It starts from point A on the surface of the earth, and drops at point B also on the surface of the earth (at the same level).

1. The curve representing the projectile lies on a plane labeled by cartesian coordinates $\{x, y\}$. Write down the equation of the curve in this plane representing the trajectory of the projectile. Choose your x-axis along the surface of the earth, while the y-axis along the direction perpendicular to the surface of earth. Choose your origin in the xy-plane to be the point where the projectile reaches the highest point above the surface of the earth.

2. Find the *actual* distance travelled by the projectile *along its trajectory.* (Note that, here you are not required to find the distance between A and B on the surface of the earth; rather you need to find the distance along the curve representing the trajectory by performing the necessary line integral on plane containing trajectory.)

Solution

(1) We know, equation of displacement of a particle under acceleration g can be written in general form as,

$$s - s_0 = v.t - \frac{1}{2}g.t^2 \tag{2}$$

Now, for a projectile fired at an angle θ with initial velocity v we obtained,

maximum height, $H = \frac{v^2 sin^2\theta}{2g} = 2.5\ m.$
horizontal range, $R = \frac{v^2 sin2\theta}{g} = 10\ m.$
time of flight, $T = \frac{2vsin\theta}{g} = 1.414\ s.$

Choosing the highest point reached by the projectile as origin, the initial point turns out to be $(-\frac{R}{2}, -H)$ as shown in the figure below. Thus we can write down the the particle coordinates at any instant of time, t as,

$$\begin{aligned} y + H =& vsin\theta.t - \frac{1}{2}g.t^2 \\ \text{and, } x + \frac{R}{2} =& vcos\theta.t \quad \text{[no acceleration along horizontal direction]} \end{aligned}$$

Now eliminating t from above equations we got the equation of trajectory to be,

$$\begin{aligned} y =& \left(x + \frac{R}{2}\right) tan\theta - \frac{g\left(x + \frac{R}{2}\right)^2}{2v^2cos^2\theta} - H \\ y =& 2.5 + x - \frac{1}{10}(5 + x)^2 \quad \text{(Answer)} \end{aligned} \tag{3}$$

(2) We got,

$$\begin{aligned} y =& \frac{v}{\sqrt{2}}.t - \frac{1}{2}g.t^2 - H && \implies \frac{dy}{dt} = \frac{v}{\sqrt{2}} - g.t \\ x =& \frac{v}{\sqrt{2}}.t - \frac{R}{2} && \implies \frac{dx}{dt} = \frac{v}{\sqrt{2}} \end{aligned}$$

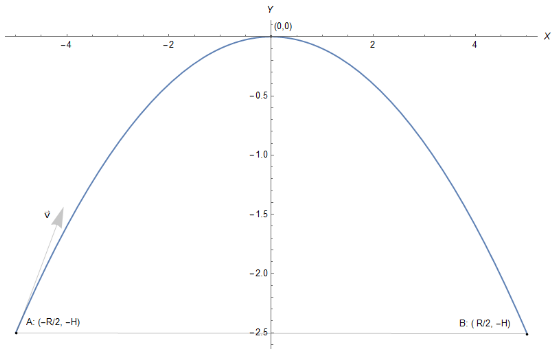


Figure 1: Trajectory of projectile

Now to find the actual distance traveled by the projectile along its trajectory, we need to perform line integral over the curve, now if ds be the small arc length of the curve then we can write down,

$$\begin{aligned}
\frac{ds}{dt} =& \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \\
\implies \int_0^S ds =& \int_0^T \sqrt{\left(gt - \frac{v}{\sqrt{2}}\right)^2 + \frac{v^2}{2}} \quad dt \\
S =& \frac{\sqrt{2}v^2}{2g} + \frac{v^2}{4g} log \left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \right| \\
=& 11.4779 \ m. \quad \text{(Answer)}
\end{aligned}$$

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