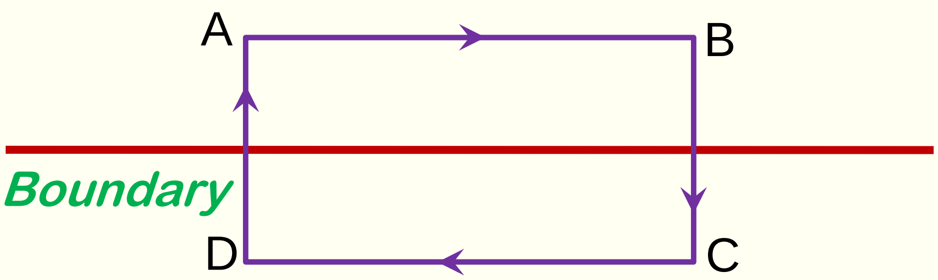
Waves at Interface

Boundary condition for electric field at dielectric interface

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \oint \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B}.d\vec{s}$$



$$\oint \vec{E}.d\vec{l} = \int_{ABCDA} \vec{E}.d\vec{l} = \int_{AB} \vec{E}.d\vec{l} + \int_{BC} \vec{E}.d\vec{l} + \int_{CD} \vec{E}.d\vec{l} + \int_{DA} \vec{E}.d\vec{l}$$

If $BC \to 0,\ DA \to 0$ **The flux will be zero**

$$\int_{AB} \vec{E}.\hat{n}\, dl + \int_{CD} \vec{E}.(-\hat{n})dl = 0 \Rightarrow (E_1)^{\parallel} = (E_2)^{\parallel}$$

Parallel component (with interface) remain conserved

Waves at Interface

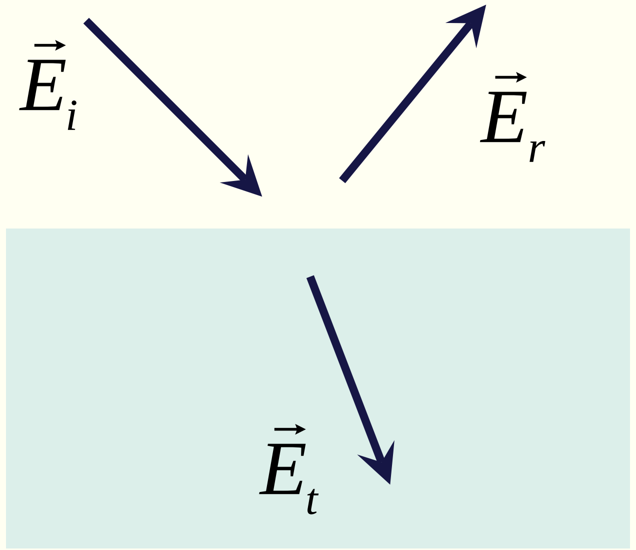
Consider a monochromatic planar light wave incident at an interface:

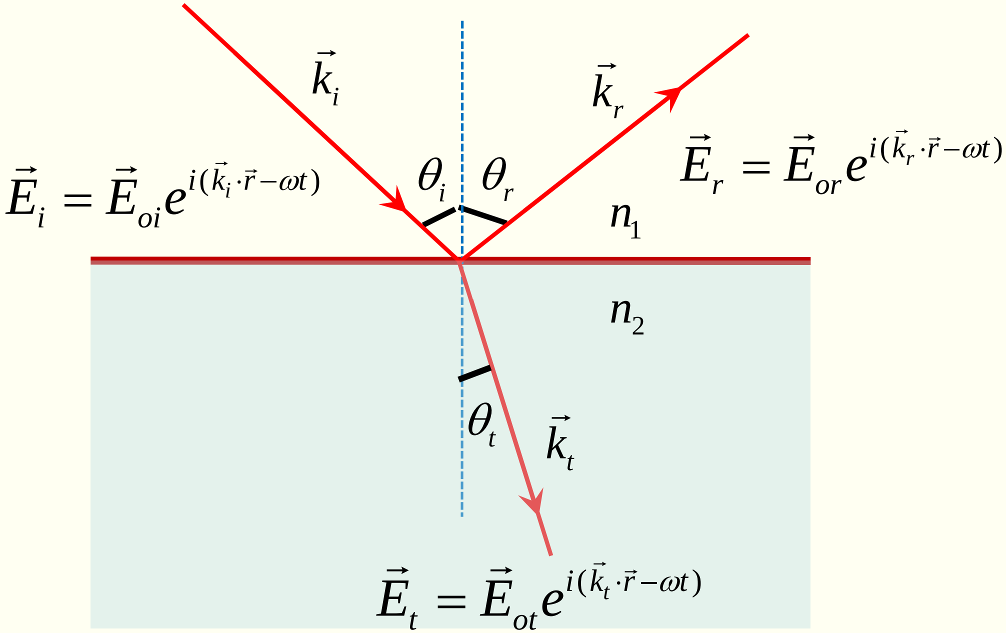
$$\vec{E}_i = \vec{E}_{oi} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$$

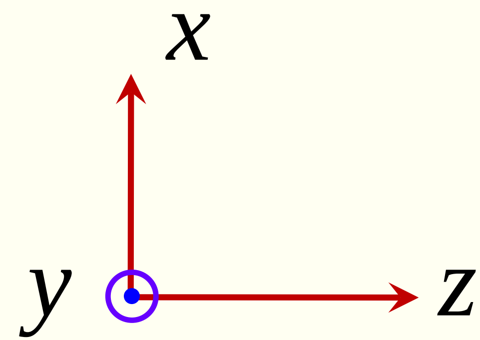
The form of the reflected and transmitted waves

$$\vec{E}_r = \vec{E}_{or} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

$$\vec{E}_t = \vec{E}_{ot} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$







At interface ($x = 0$)

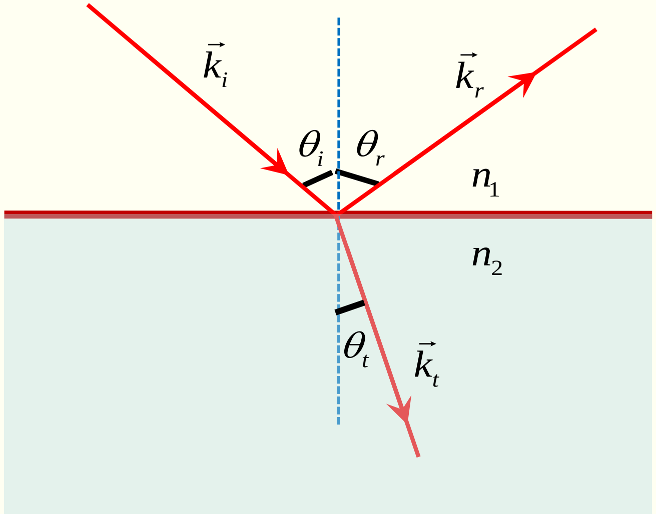
$$(\vec{E}_i)^{\parallel} + (\vec{E}_r)^{\parallel} = (\vec{E}_t)^{\parallel}$$

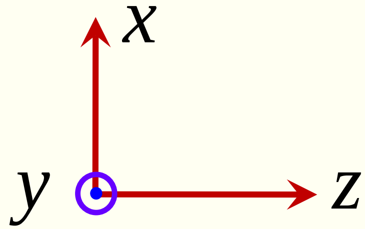
$$\left(\vec{E}_{oi}\right)^{\parallel} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)} + \left(\vec{E}_{or}\right)^{\parallel} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)} = \left(\vec{E}_{ot}\right)^{\parallel} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

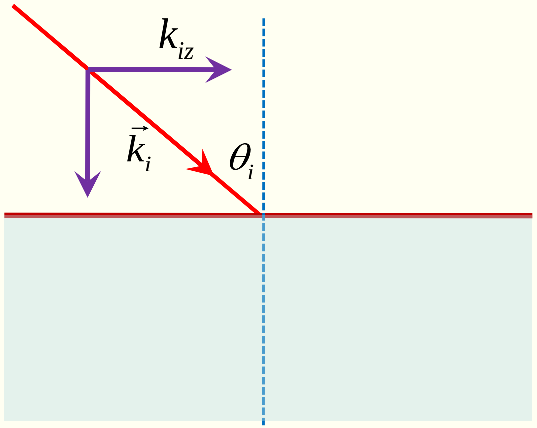
$$\left(\vec{E}_{oi}\right)^{\parallel} e^{i(k_{iy} y + k_{iz} z - \omega t)} + \left(\vec{E}_{or}\right)^{\parallel} e^{i(k_{ry} y + k_{rz} z - \omega t)} = \left(\vec{E}_{ot}\right)^{\parallel} e^{i(k_{ty} y + k_{tz} z - \omega t)}$$

This equation is valid for all values of time and all points on the interface (yz plane).........so we can have

$$k_{iy} = k_{ry} = k_{ty}$$
$$k_{iz} = k_{rz} = k_{tz}$$







$$k_{iz} = k_{rz} \Rightarrow k_i \sin\theta_i = k_r \sin\theta_r$$

$$\Downarrow$$

$$\theta_i = \theta_r$$

Law of Reflection

$$k_i = \frac{\omega}{c} n_1 \qquad k_r = \frac{\omega}{c} n_1$$

$$k_{iz} = k_{tz} \Rightarrow k_i \sin\theta_i = k_t \sin\theta_t$$

$$\Downarrow$$

Snell's Law $$n_1 \sin\theta_i = n_2 \sin\theta_t$$

Law of Refraction

$$k_t = \frac{\omega_t}{c} n_2$$

$$\omega_i = \omega_t$$

Principle of superposition

Principle of superposition

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

(Differential wave equation)

$\Downarrow$ **(Individual solution)**

$$\psi_1(\vec{\mathrm{r}},\mathrm{t}),\ \psi_2(\vec{\mathrm{r}},\mathrm{t}),\ \psi_3(\vec{\mathrm{r}},\mathrm{t}),\ \psi_4(\vec{\mathrm{r}},\mathrm{t})\ldots\ldots\ldots\ldots$$

$\Downarrow$ **(General solution)**

$$\psi(\vec{\mathrm{r}},\mathrm{t})=\sum_{i=1}^{n}C_i\psi_i(\vec{\mathrm{r}},\mathrm{t})$$

$\Rightarrow$ **Linear combination**

Principle of superposition: Resultant disturbance of any point in a medium is the algebraic sum of the separate constituent waves

Simple Example: Addition of wave of same frequency (but different phase)

$$E_1 = E_{01}\sin(\omega t + \alpha_1)$$
$$E_2 = E_{02}\sin(\omega t + \alpha_2)$$

$$\Rightarrow \quad E = E_1 + E_2$$

$$E = E_{01}\left[\sin(\omega\,\mathrm{t})\cos\alpha_1 + \cos(\omega\,\mathrm{t})\sin\alpha_1\right] + E_{02}\left[\sin(\omega\,\mathrm{t})\cos\alpha_2 + \cos(\omega\,\mathrm{t})\sin\alpha_2\right]$$

$$E = \left[E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2\right]\sin(\omega t) + \left[E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2\right]\cos(\omega\,\mathrm{t})$$

$$E_0\cos\alpha = \left[E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2\right]$$
$$E_0\sin\alpha = \left[E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2\right]$$

Final Wave

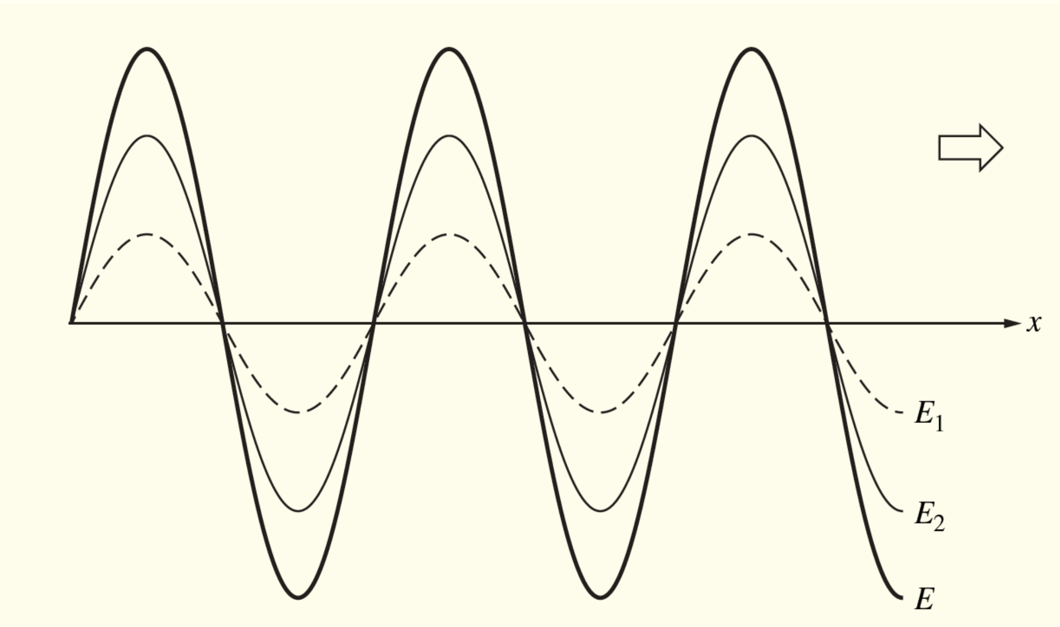
$$\Rightarrow \quad E = E_0\sin(\omega\,\mathrm{t} + \alpha)$$

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_2 - \alpha_1)$$

$$\tan\alpha = \frac{\left[E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2\right]}{\left[E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2\right]}$$

If $(\alpha_2 - \alpha_1)$ is constant over time then E_1, E_2 are **coherent** to each other

In phase $\delta = 0$

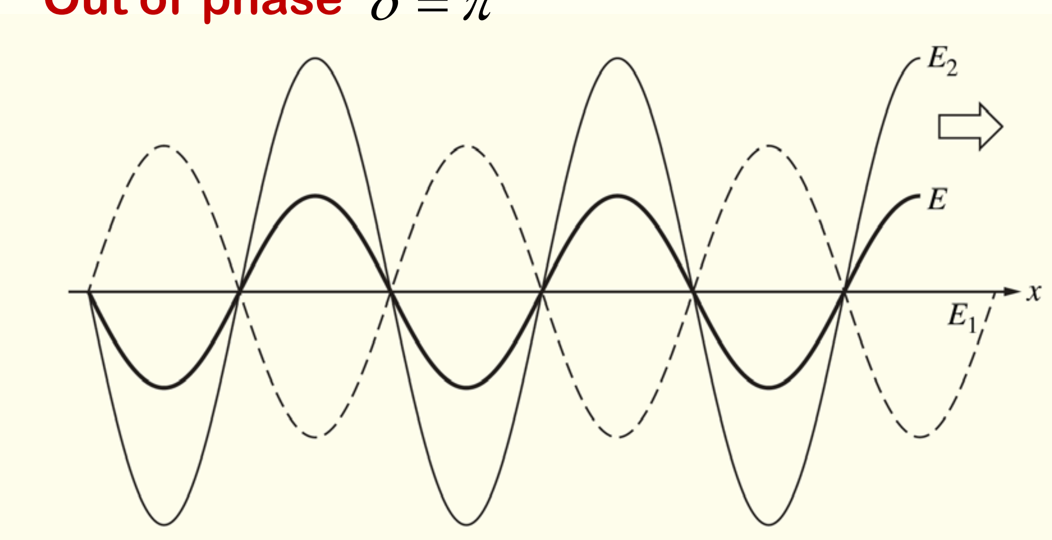


$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos\delta$$

$$\delta = (\alpha_2 - \alpha_1)$$

$$\tan\alpha = \frac{\left[E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2\right]}{\left[E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2\right]}$$

Out of phase $\delta = \pi$



Superposition of many wave

$$E = \sum_{i=1}^{n} E_{0i}\cos(\omega t \pm \alpha_i)$$

$$\Downarrow$$

$$E = E_0\cos(\omega t \pm \alpha)$$

$$E_0^2 = \sum_{i=1}^{n} E_{0i}^2 + 2\sum_{j>i}^{n}\sum_{i=1}^{n} E_{0i}E_{0j}\cos(\alpha_i - \alpha_j)$$

$$\tan\alpha = \frac{\sum_{i=1}^{n} E_{0i}\sin\alpha_i}{\sum_{i=1}^{n} E_{0i}\cos\alpha_i}$$

The Complex Method

$$E_1 = E_{01}\cos(\omega t + \alpha_1) \xrightarrow{\text{Complex form}} \tilde{E}_1 = E_{01}e^{i(\omega t+\alpha_1)}$$

Superposition of N number of waves with same frequency

$$\boxed{E_{01}e^{i(\omega t+\alpha_1)} + E_{02}e^{i(\omega t+\alpha_2)} + E_{03}e^{i(\omega t+\alpha_3)} + \ldots\ldots + E_{0N}e^{i(\omega t+\alpha_N)} = e^{i\omega t}\sum_{j=1}^{N} E_{0i}e^{i\alpha_j}}$$

The superposition of the wave is written as, $\tilde{E} = E_0 e^{i(\omega t+\alpha)}$

$$\tilde{E} = E_0 e^{i(\omega t+\alpha)} = e^{i\omega t}\sum_{j=1}^{N} E_{0j}e^{i\alpha_j} \Longrightarrow E_0 e^{i\alpha} = \sum_{j=1}^{N} E_{0j}e^{i\alpha_j}$$

Complex Amplitude

$$\boxed{E_0^2 = \left(E_0 e^{i\alpha}\right)\left(E_0 e^{i\alpha}\right)^*}$$

For N=2

$$E_0^2 = \left(E_{01}e^{i\alpha_1} + E_{02}e^{i\alpha_2}\right)\left(E_{01}e^{-i\alpha_1} + E_{02}e^{-i\alpha_2}\right)$$

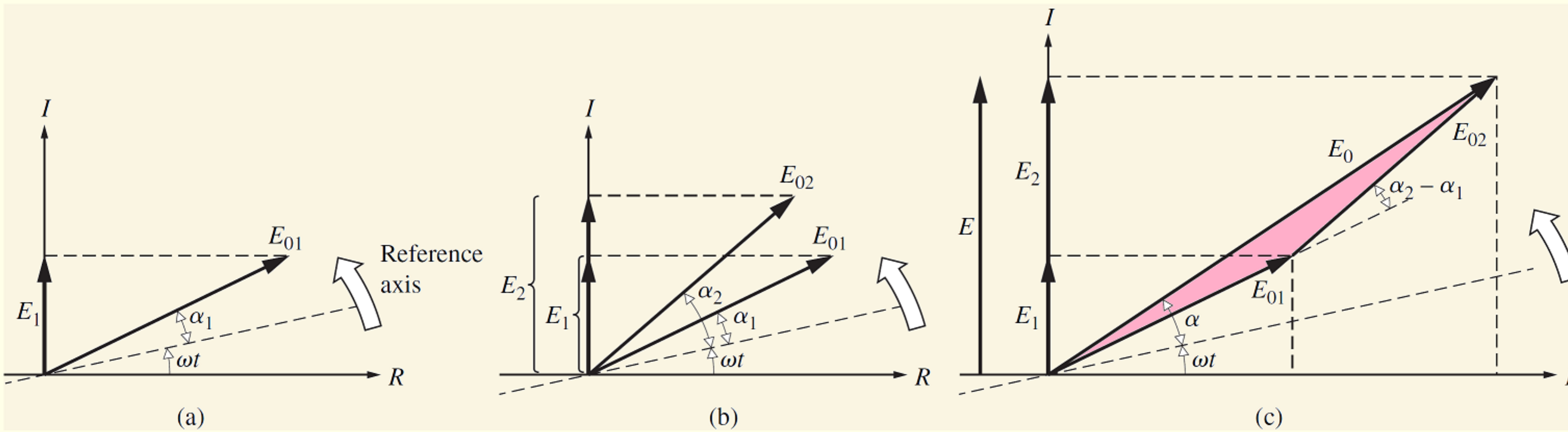
$$= E_{01}^2 + E_{02}^2 + E_{01}E_{02}\left(e^{i(\alpha_1-\alpha_2)} + e^{-i(\alpha_1-\alpha_2)}\right) = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_1-\alpha_2)$$

Phasor Addition

$$E_1 = E_{01} e^{i(\omega t + \alpha_1)}$$

$$E_2 = E_{02} e^{i(\omega t + \alpha_2)}$$

$$\Rightarrow \quad E = E_1 + E_2 = E_0 e^{i(\omega t + \alpha)}$$



$$E_0^2 = \left(E_{01} e^{i\alpha_1} + E_{02} e^{i\alpha_2}\right)\left(E_{01} e^{-i\alpha_1} + E_{02} e^{-i\alpha_2}\right)$$

$$= E_{01}^2 + E_{02}^2 + E_{01}E_{02}\left(e^{i(\alpha_1 - \alpha_2)} + e^{-i(\alpha_1 - \alpha_2)}\right)$$

$$= E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_1 - \alpha_2)$$