Case III: Underdamped $(\beta^2 < \omega_0^2)$

$$\beta = \frac{r}{2m}$$

$$\omega_0^2 = \frac{k}{m}$$

- The square root term is -ve:
  The stiffness term dominates the damping resistance term.
- The system is lightly damped and gives oscillatory damped simple harmonic motion.

$$x(t) = \exp(-\beta t)[A_1 \exp(-i\sqrt{\omega_0^2 - \beta^2}\ t)$$

$$+ A_2 \exp(i\sqrt{\omega_0^2 - \beta^2}\ t)]$$

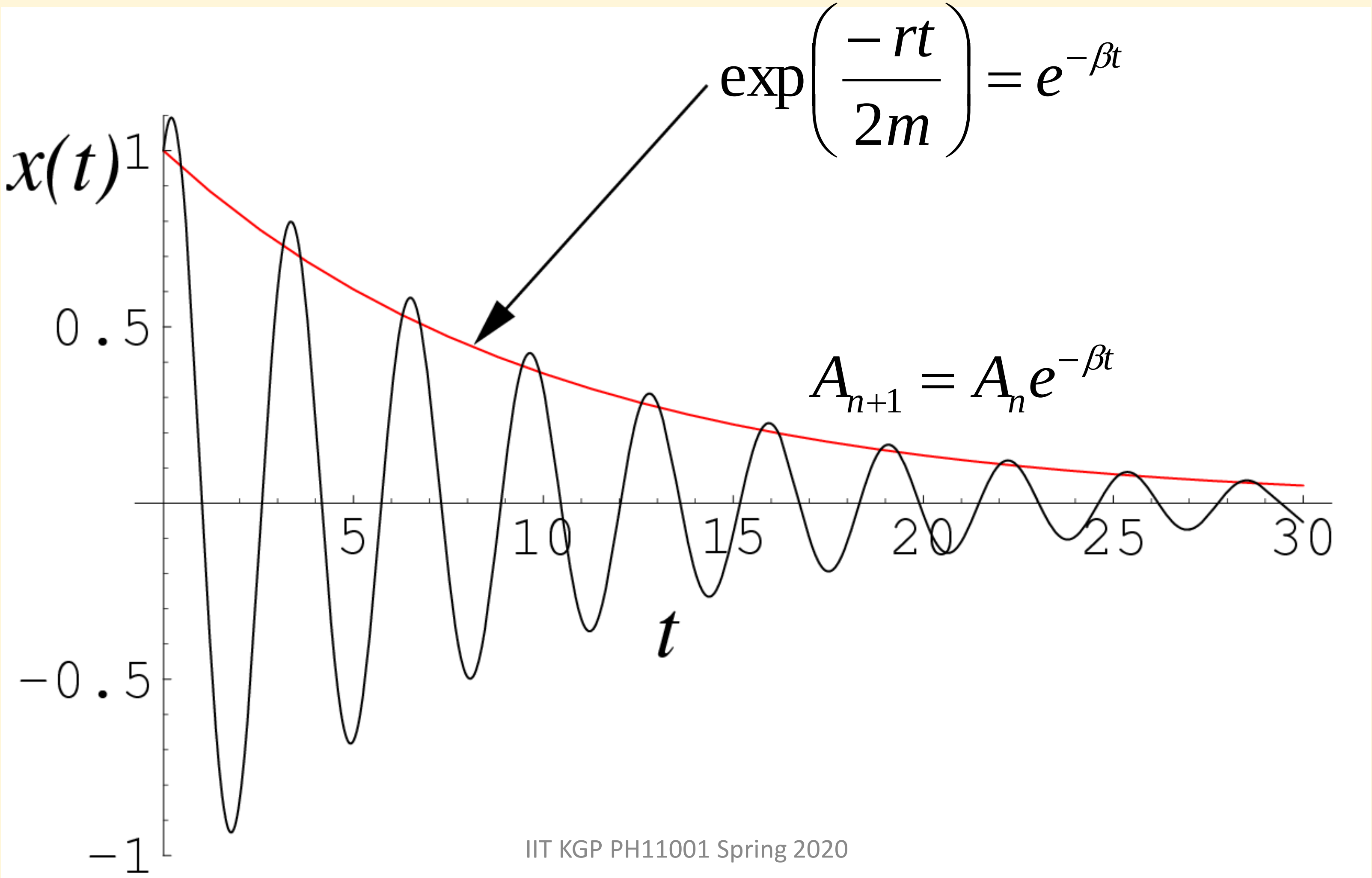
$$A_1 = A_2 = A/2$$

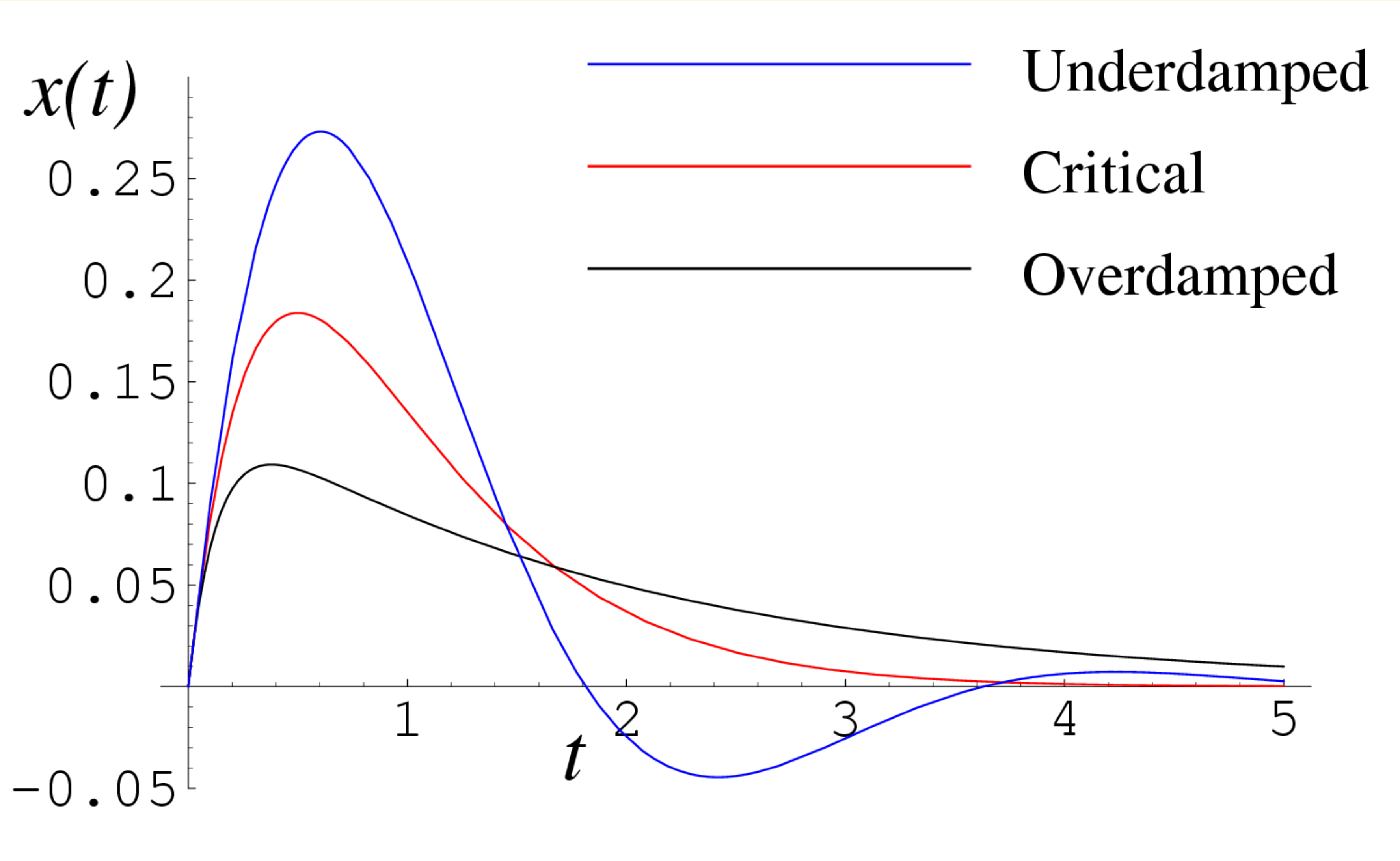
$$\boxed{x(t) = A\exp(-\beta t)\cos(\omega' t)}$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{r^2}{4m^2}} = \sqrt{\omega_0^2 - \beta^2}$$

- Its frequency is reduced: $\omega < \omega_0$ - (which means that the time period is increased)

- Its amplitude decays exponentially

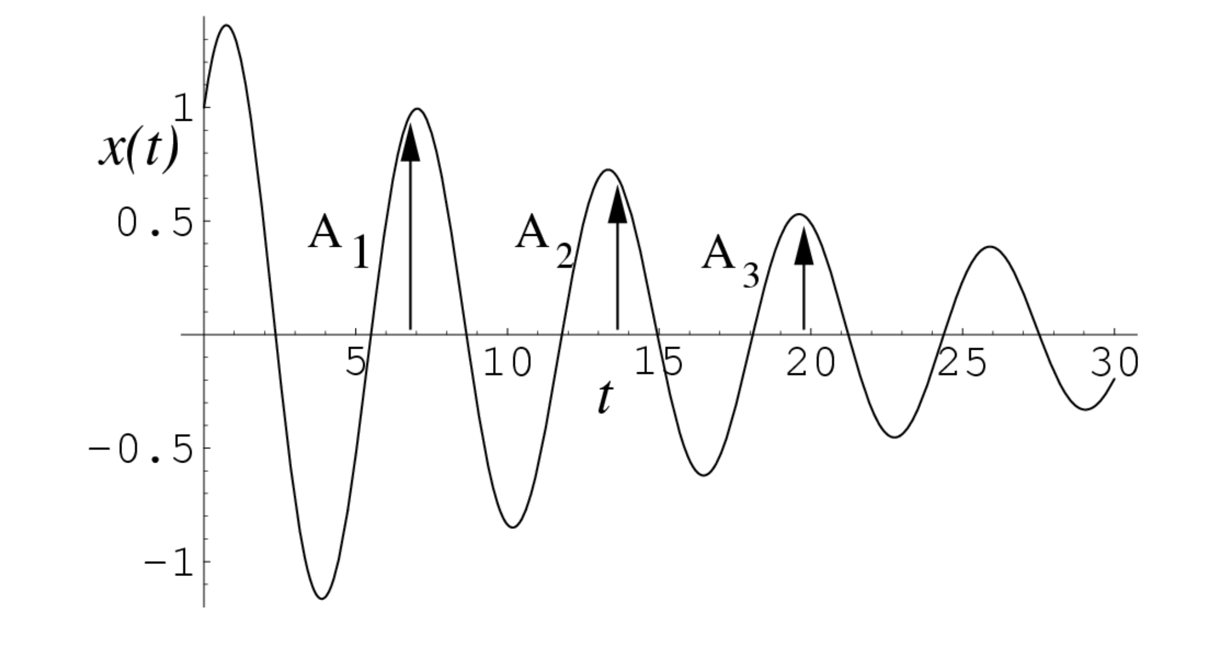




x(t)
0.25
0.2
0.15
0.1
0.05
-0.05
1
2
3
4
5
t
Underdamped
Critical
Overdamped

Logarithmic decrement

$$\lambda = \ln\left(\frac{A_n}{A_{n+1}}\right) = \beta t$$



Relaxation Time

Relaxation time is the time taken for the amplitude to decay to *1/e* of its original value

When *t* = relaxation time $\quad A_t = A_o e^{-1}$

which implies relaxation time, $t = \dfrac{2m}{r}$

Energy Dissipation for damped SHM

The total energy remains the sum of kinetic and potential energies:

$$E = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} s x^2$$

$\dfrac{dE}{dt}$ is not zero, but negative because energy is lost-

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} s x^2\right) = \dot{x}(m\ddot{x} + sx) \\ &= \dot{x}(-r\dot{x}) \quad \text{for} \quad m\dot{x} + r\dot{x} + sx = 0\end{aligned}$$

The rate of energy loss is equal to the rate of doing work against the frictional force.

The Quality Factor or Q-value of a damped SHO

Q-factor measures the rate at which the energy decays

Since the amplitude is represented by - $A = A_0\, \mathrm{e}^{-rt/2m}$

Since energy (E) varies as (Amplitude)^2

$$E = E_0\, \mathrm{e}^{(-r/m)t}$$

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The time for the energy to decay to $E_0 e^{-1}$ is given by $t = \frac{m}{r}$ sec.

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If r (damping) is small-

$$Q = \frac{\omega_0 m}{r}$$

Also-

$$\frac{Q}{2\pi} = \frac{\text{Energy stored in the system}}{\text{Energy lost per cycle}}$$

Undamped Forced Oscillation

The equation of motion is

$$\begin{aligned} m\ddot{x} + kx &= F(t) \\ \ddot{x} + \omega_0^2 x &= F(t)/m \end{aligned}$$

Linear inhomogeneous differential equation of order n=2

Particular solution - $P(t)$

$$\frac{d^2P(t)}{dt^2} + \omega_0^2 P(t) = F(t)/m$$

Complementary function - $C(t)$

$$\frac{d^2C(t)}{dt^2} + \omega_0^2 C(t) = 0$$

$$\frac{d^2(P(t) + C(t))}{dt^2} + \omega_0^2 (P(t) + C(t)) = F(t)/m$$

General solution:

$$x(t) = P(t) + C(t)$$

As the complimentary function has been discussed earlier, we shall focus on the Particular solution $P(t)$ in the following

Driving force

$$F = F_0 \cos(\omega t + \psi)$$

$$= \text{Re of } \; F_0 e^{i(\omega t + \psi)}$$

$$F = \hat{F} e^{i\omega t} \quad \text{where} \quad \hat{F} = F_0 e^{+i\psi}$$

Why this specific functional form for external force?

Since, for any arbitrary time varying force

$$F(t) = \sum_{n=1,\dots}^{\infty} F_n \cos(\omega_n t + \psi_n)$$

We have to solve first x_n corresponding to F_n, and later get x

$$m\ddot{x}_n + kx_n = F_n \cos(\omega_n t + \psi_n)$$

$$x(t) = \sum_n x_n(t)$$

The equation of motion is

$$\frac{d^2x}{dt^2}+\frac{k}{m}x=\frac{\hat{F}e^{i\omega t}}{m}$$

$$x = x_r + ix_i$$

Taking trial solution: $x = \hat{A}e^{i\omega t}$

$$(i\omega)^2\hat{A}+\frac{k}{m}\hat{A}=\frac{\hat{F}}{m}$$

$$\hat{A}=\frac{\hat{F}/m}{\left(\frac{k}{m}-\omega^2\right)}=\frac{\hat{F}}{m\left(\omega_o^2-\omega^2\right)}$$

Amplitude, Relative Phase

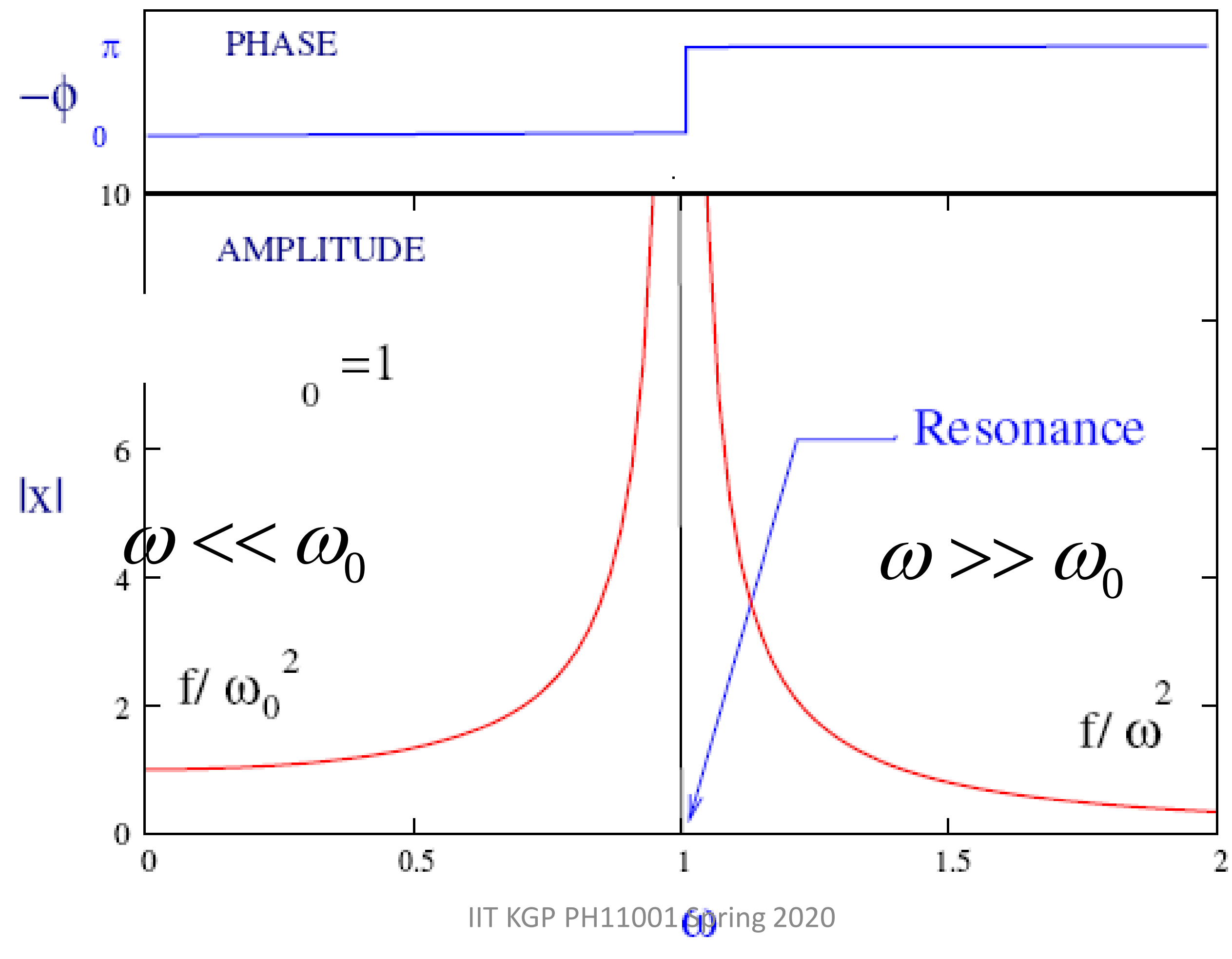
$$\hat{A} = \frac{\hat{F}}{m\left(\omega_o^2 - \omega^2\right)}$$

$$\text{for } \ \omega \langle \omega_0 \Rightarrow \phi = 0$$

$$\text{for } \ \omega \rangle \omega_0 \Rightarrow \phi = -\pi$$

Amplitude and Phase

$$\hat{A} = \frac{f}{\left(\omega_o^2 - \omega^2\right)}$$



$$\hat{A} = \frac{\hat{F}}{m\left(\omega_o^2 - \omega^2\right)}$$

Low Frequency Response - $\omega << \omega_0$

$$x(t) = \frac{\hat{F}}{m\omega_0^2} e^{i\omega t} = \frac{F}{k} e^{i(\omega t + \psi)}$$

Stiffness Controlled Regime

High Frequency Response - $\omega >> \omega_0$

$$x(t) = -\frac{F}{m\omega^2} e^{i(\omega t + \psi)}$$

Mass Controlled Regime