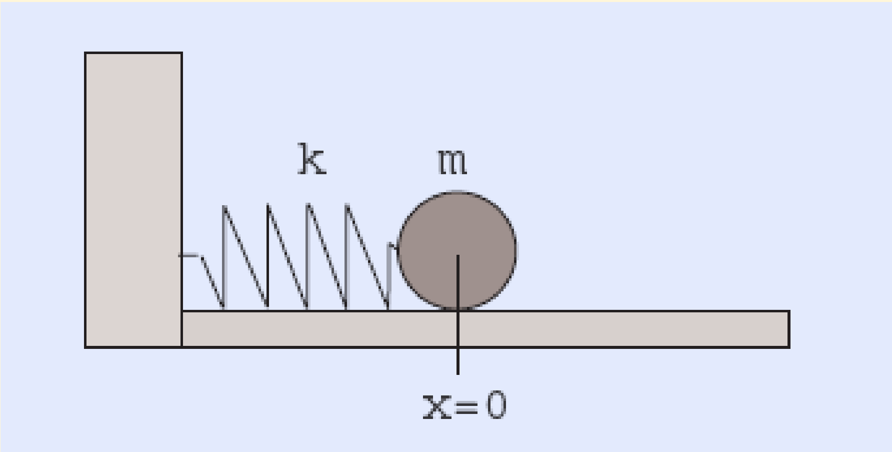
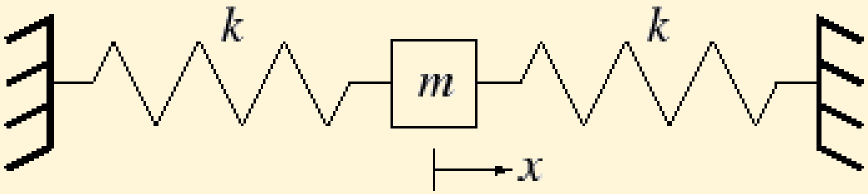
A single spring-mass system



$$m\frac{d^2\mathrm{x}}{dt^2} = -k\mathrm{x}$$

$$\mathrm{x} = \mathrm{A}\cos(\omega \mathrm{t} + \phi)$$

One mass + Two springs

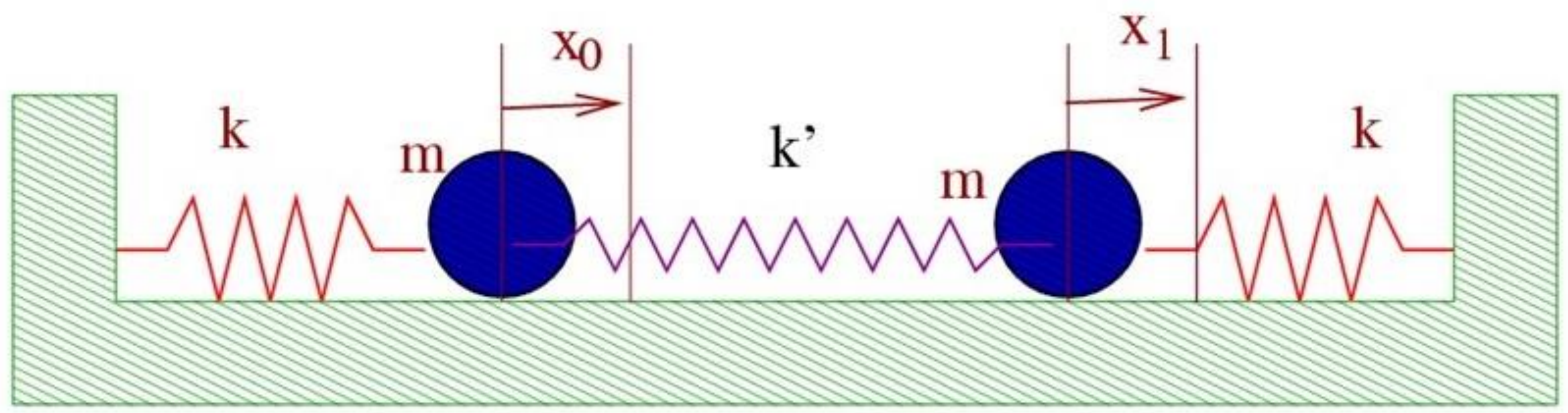


$$m\ddot{x} = k(-x) + k(-x)$$

$$= -2kx$$

$$\omega = \sqrt{\frac{2k}{m}}$$

Coupled oscillators

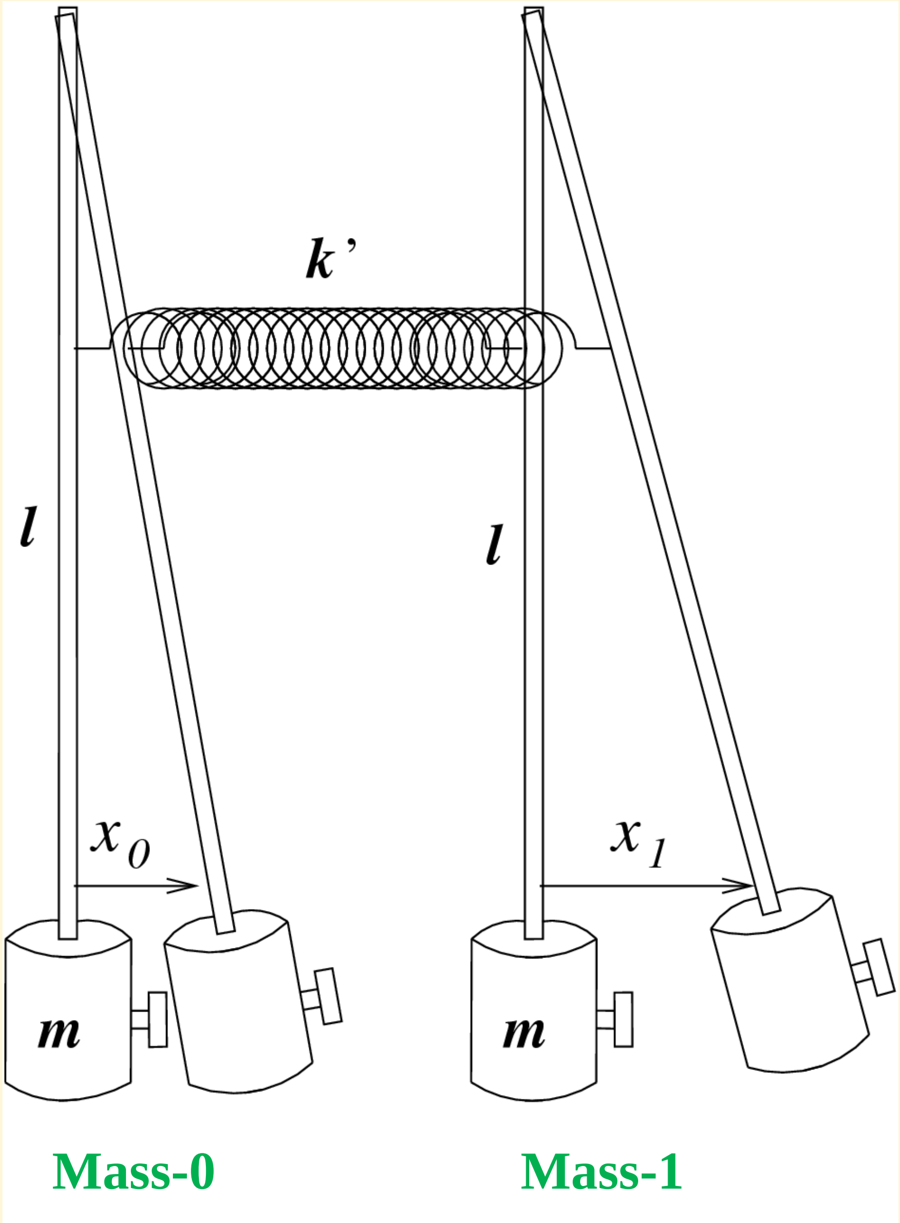


$$m\frac{d^2\,x_0}{dt^2} = -kx_0 - k'(x_0 - x_1)$$

$$m\frac{d^2\,x_1}{dt^2} = -kx_1 - k'(x_1 - x_0)$$

SHM term

Coupling term



Considering small-angle approximation

$$\sin\theta_1 = \frac{x_1}{l}; \sin\theta_o = \frac{x_o}{l}$$

Equation of motion

Total force on Mass-1

$$m\ddot{x}_1 = -mg\frac{x_1}{l} - k(x_1 - x_0)$$

Total force on Mass-0

$$m\ddot{x}_0 = -mg\frac{x_0}{l} - k(x_0 - x_1)$$

SHM term

Coupling term

Let $\omega_0^2 = \frac{g}{l}$ Natural freq. of each pendulum

$$\ddot{x}_0 + \omega_0^2 x_0 = -\frac{k}{m}\left(x_0 - x_1\right)$$

$$\ddot{x}_1 + \omega_0^2 x_1 = -\frac{k}{m}\left(x_1 - x_0\right)$$

Adding: $\ddot{x}_1 + \ddot{x}_0 + \omega_0^2\left(x_1 + x_0\right) = 0$

Subtracting: $\ddot{x}_1 - \ddot{x}_0 + \left(\omega_0^2 + \frac{2k}{m}\right)\left(x_1 - x_0\right) = 0$

Normal Co-ordinates

$$x_1 + x_0 = q_1$$

$$x_1 - x_0 = q_2$$

Which gives a set of linear differential equations with constant coefficients in which each equation contains only one dependent variable (our Simple Harmonic equations in q_1 and q_2 only)

Normal modes

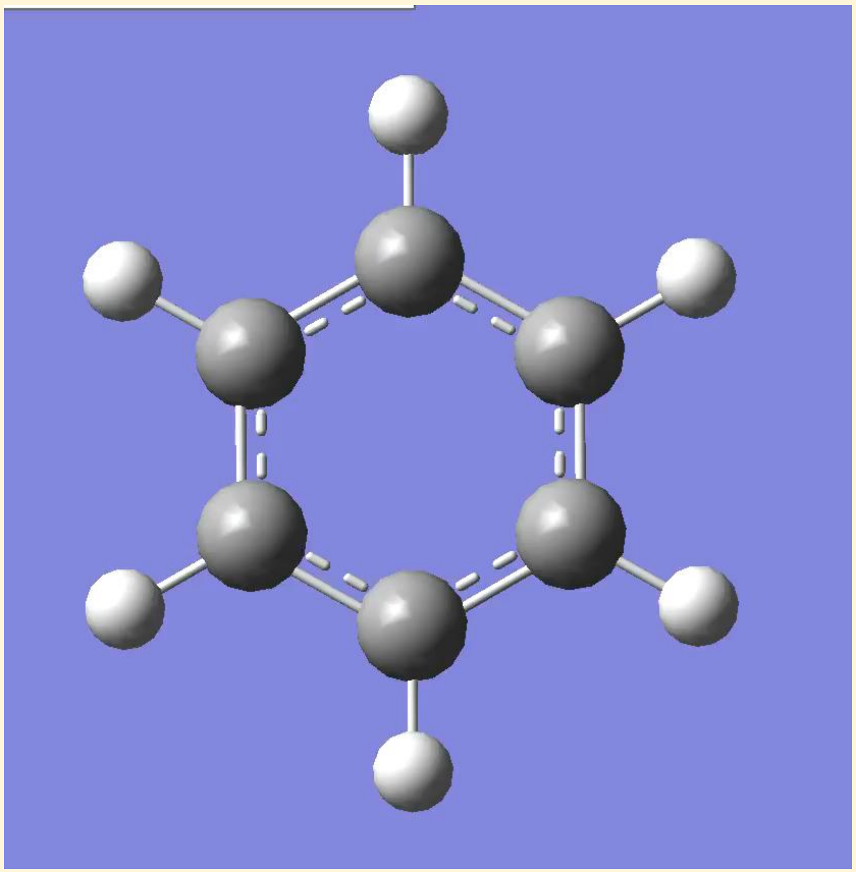
$$\ddot{q}_1 + \omega_0^2 q_1 = 0$$

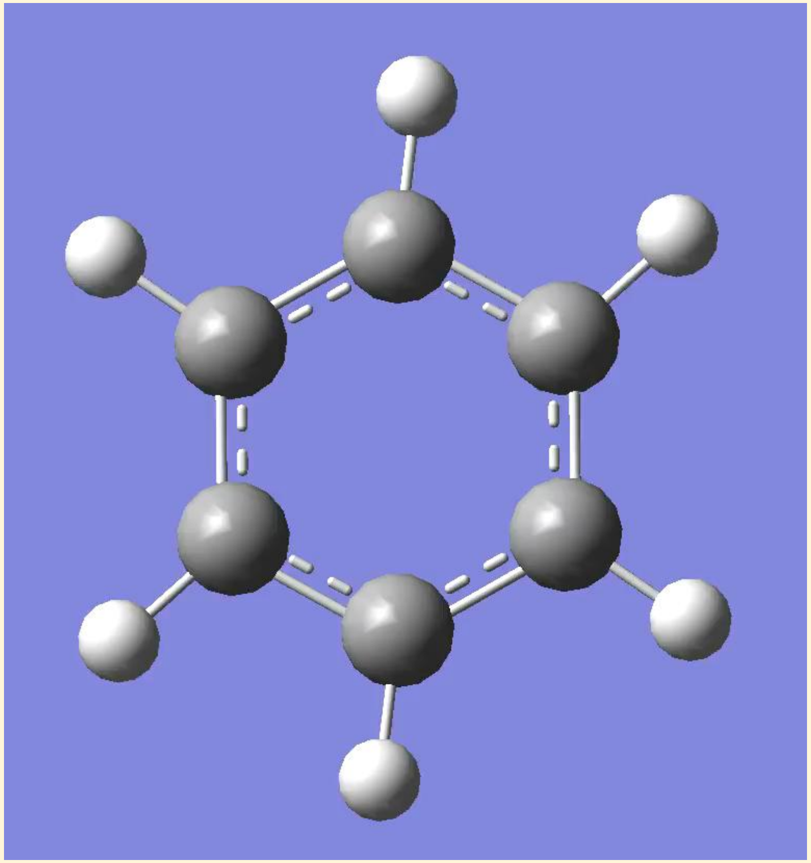
$$\ddot{q}_2 + \left(\omega_0^2 + \frac{2k}{m} \right) q_2 = 0$$

A vibration involving only one dependent variable is called a **normal mode** of vibration and has its own **normal frequency**.
The importance of the normal modes of vibration is that they are entirely independent of each other

Normal mode: A way in which the system can move in a steady state, in which all parts of the system move with the same frequency. The parts may have different (zero or negative) amplitudes

Normal modes of Benzene





Normal frequencies

Slow mode $\omega_1 = \omega_0$

Fast mode $\omega_2 = \left(\omega_1^2 + \frac{2k}{m}\right)^{1/2}$

Solutions

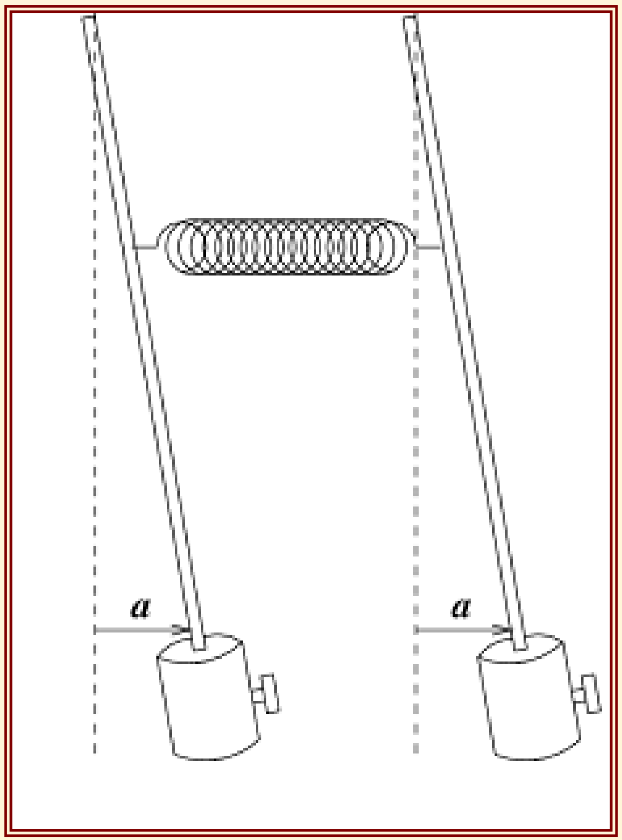
$$\boxed{\begin{aligned} q_{\;1} &= x_1 + x_0 = q_{10}\cos\left(\omega_1 t + \phi_1\right) \\ q_{\;2} &= x_1 - x_0 = q_{20}\cos\left(\omega_2 t + \phi_2\right) \end{aligned}}$$

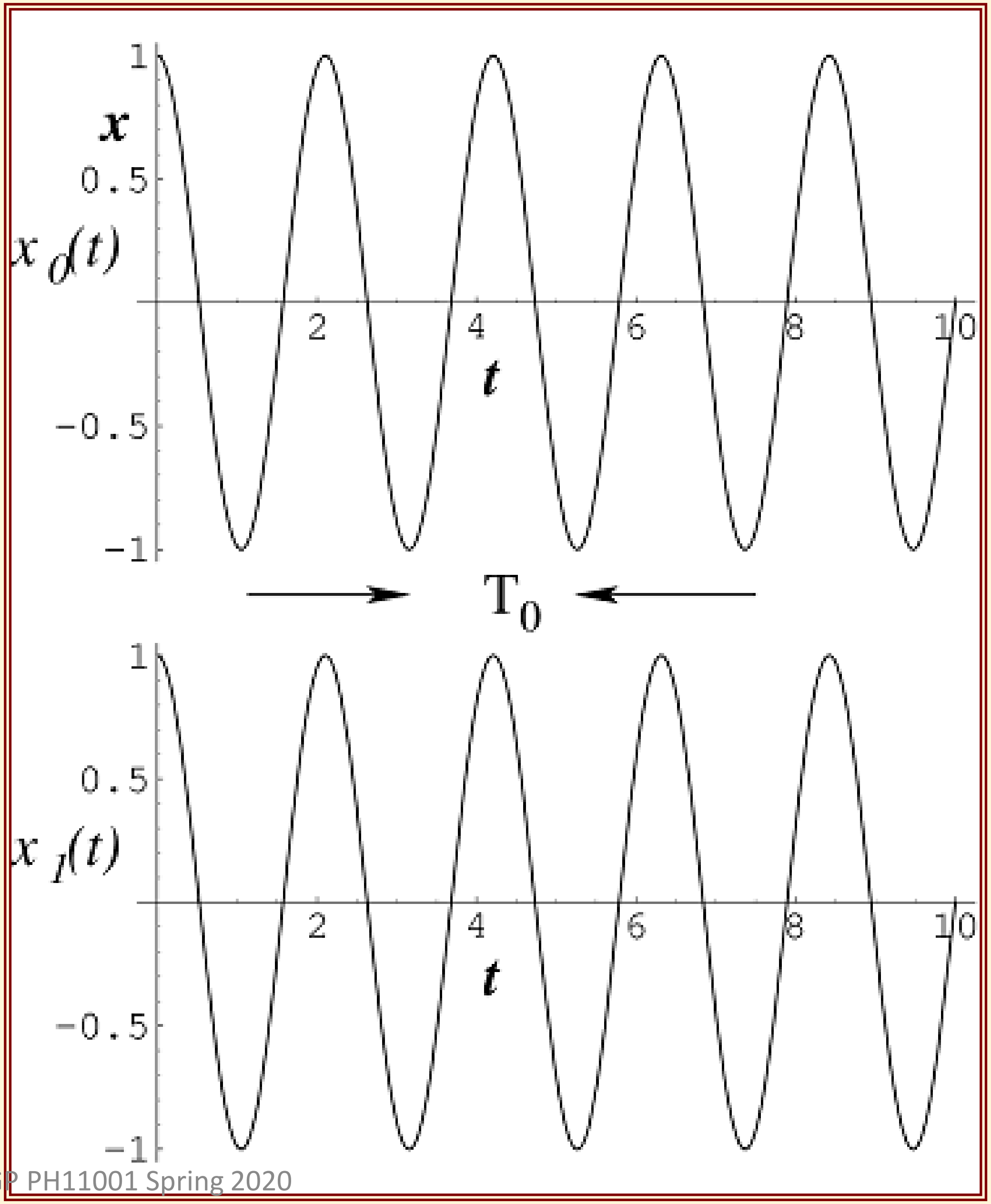
Normal mode amplitudes : q_{10} and q_{20}

In-phase vibration (Pendulum mode)

$$q_2 = 0(x_0 = x_1)$$

$$\ddot{q}_1 + \omega_1^2 q_1 = 0$$

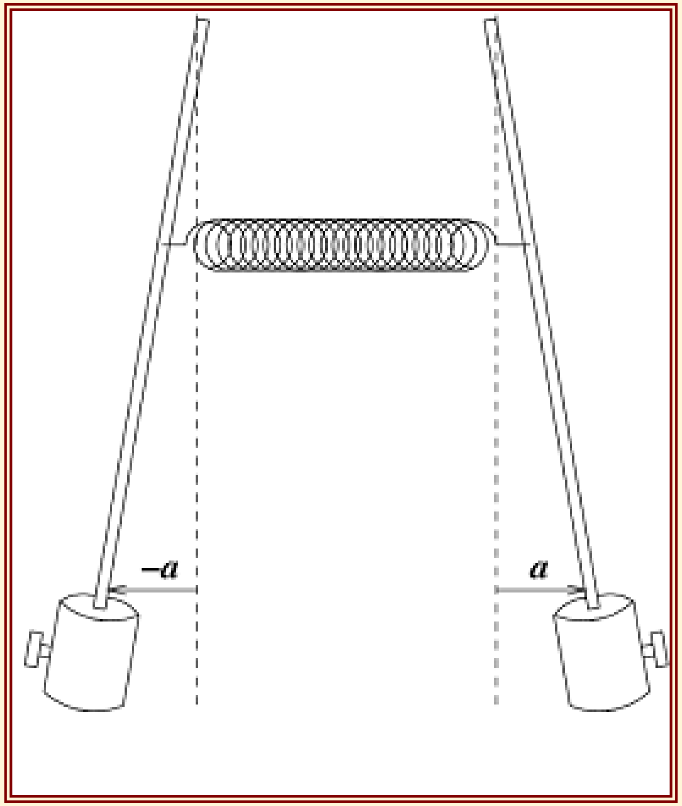


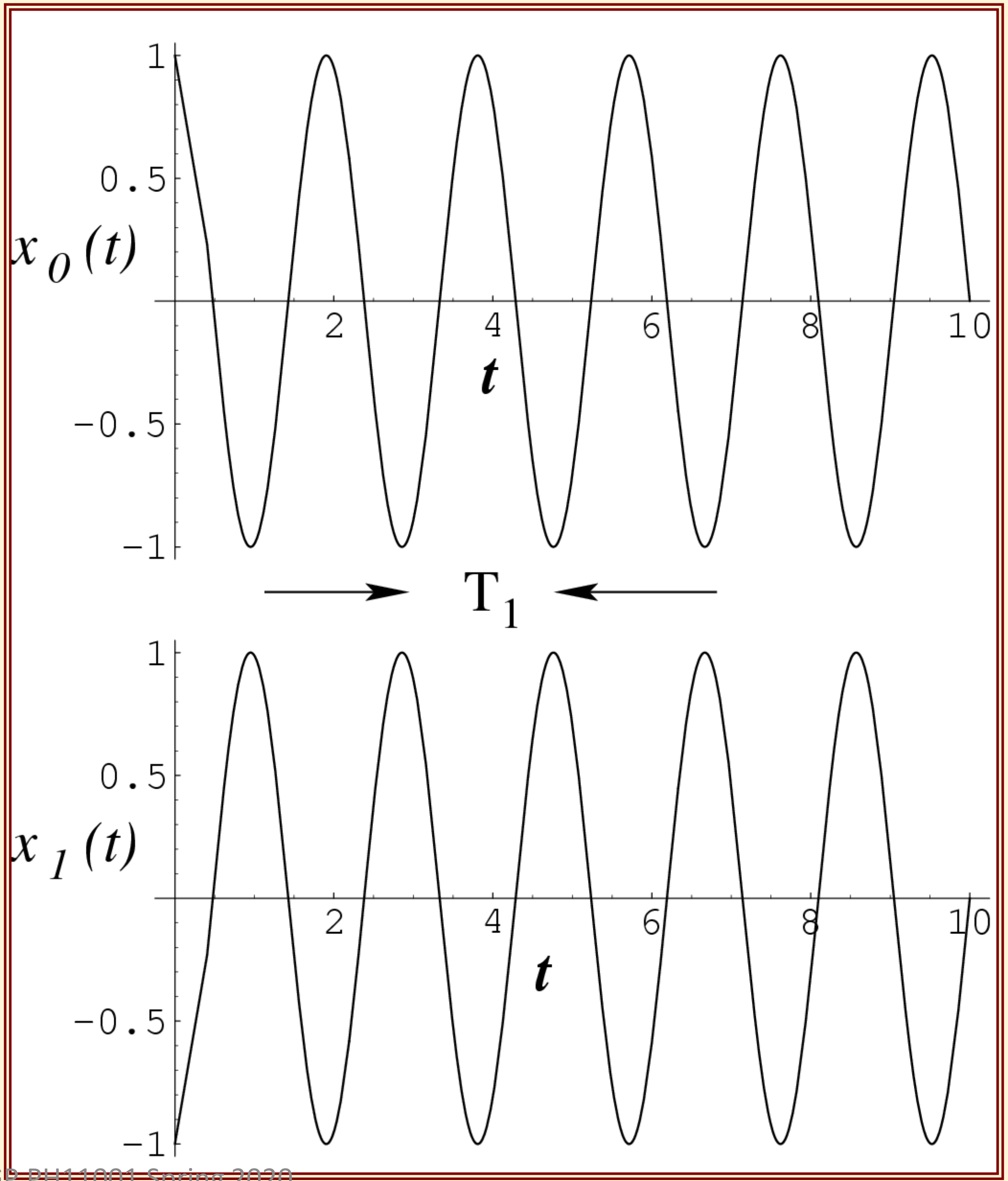


Out-of-phase vibration (Breathing mode)

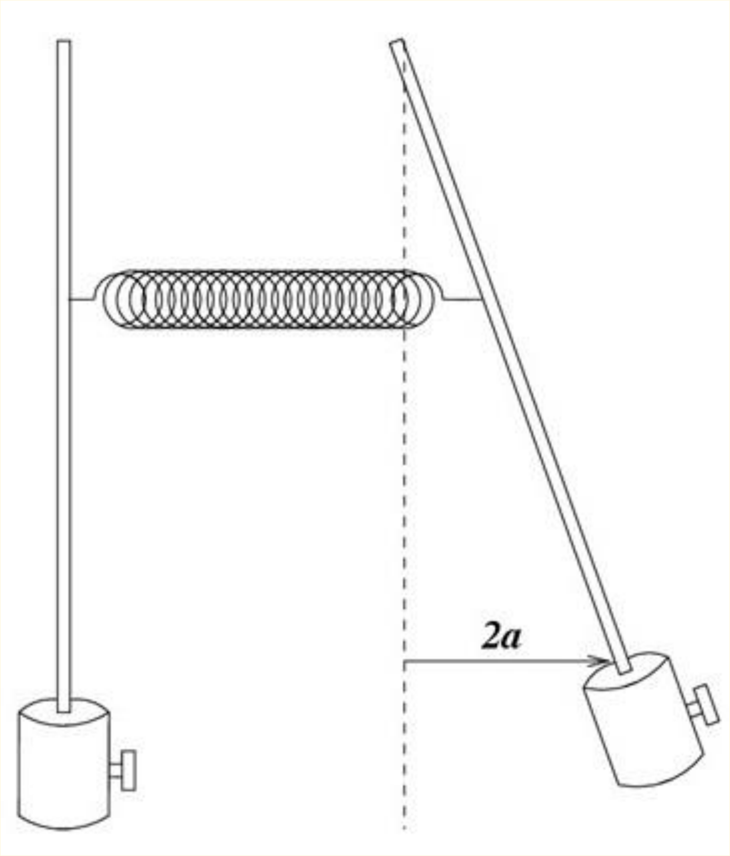
$$q_1 = 0 \quad (x_0 = -x_1)$$

$$\ddot{q}_2 + \left(\omega_1^2 + \frac{2k}{m} \right) q_2 = 0$$





Let us choose the following example



$$x_1 = 2a \quad \& \quad x_0 = 0 \quad \text{at} \quad t = 0$$

How will the system evolve with time?

$$q_1 = x_1 + x_0 = q_{10}\cos\left(\omega_1 t + \phi_1\right)$$

$$q_2 = x_1 - x_0 = q_{20}\cos\left(\omega_2 t + \phi_2\right)$$

Following the initial conditions:

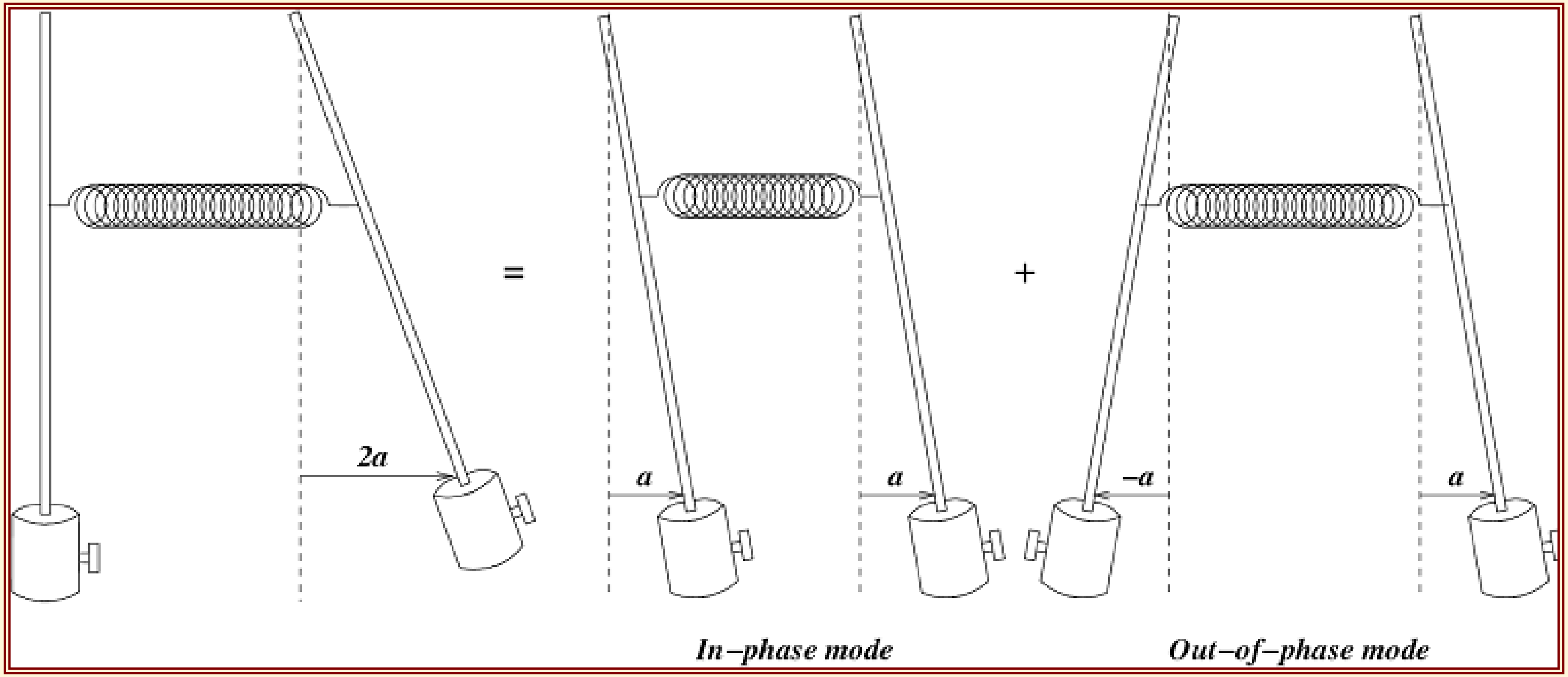
$$q_{10} = q_{20} = 2a \quad \& \quad \phi_1 = \phi_2 = o$$

Pendulum displacements

$$\boxed{\begin{aligned} x_1 &= \frac{1}{2}\left(q_1 + q_2\right) = a\left[\cos\omega_1 t + \cos\omega_2 t\right] \\ &= 2a\cos\frac{\left(\omega_2 - \omega_1\right)t}{2}\cos\frac{\left(\omega_1 + \omega_2\right)t}{2} \end{aligned}}$$

$$\boxed{\begin{aligned} x_0 &= \frac{1}{2}\left(q_1 - q_2\right) = a\left[\cos\omega_1 t - \cos\omega_2 t\right] \\ &= 2a\sin\frac{\left(\omega_2 - \omega_1\right)t}{2}\sin\frac{\left(\omega_1 + \omega_2\right)t}{2} \end{aligned}}$$

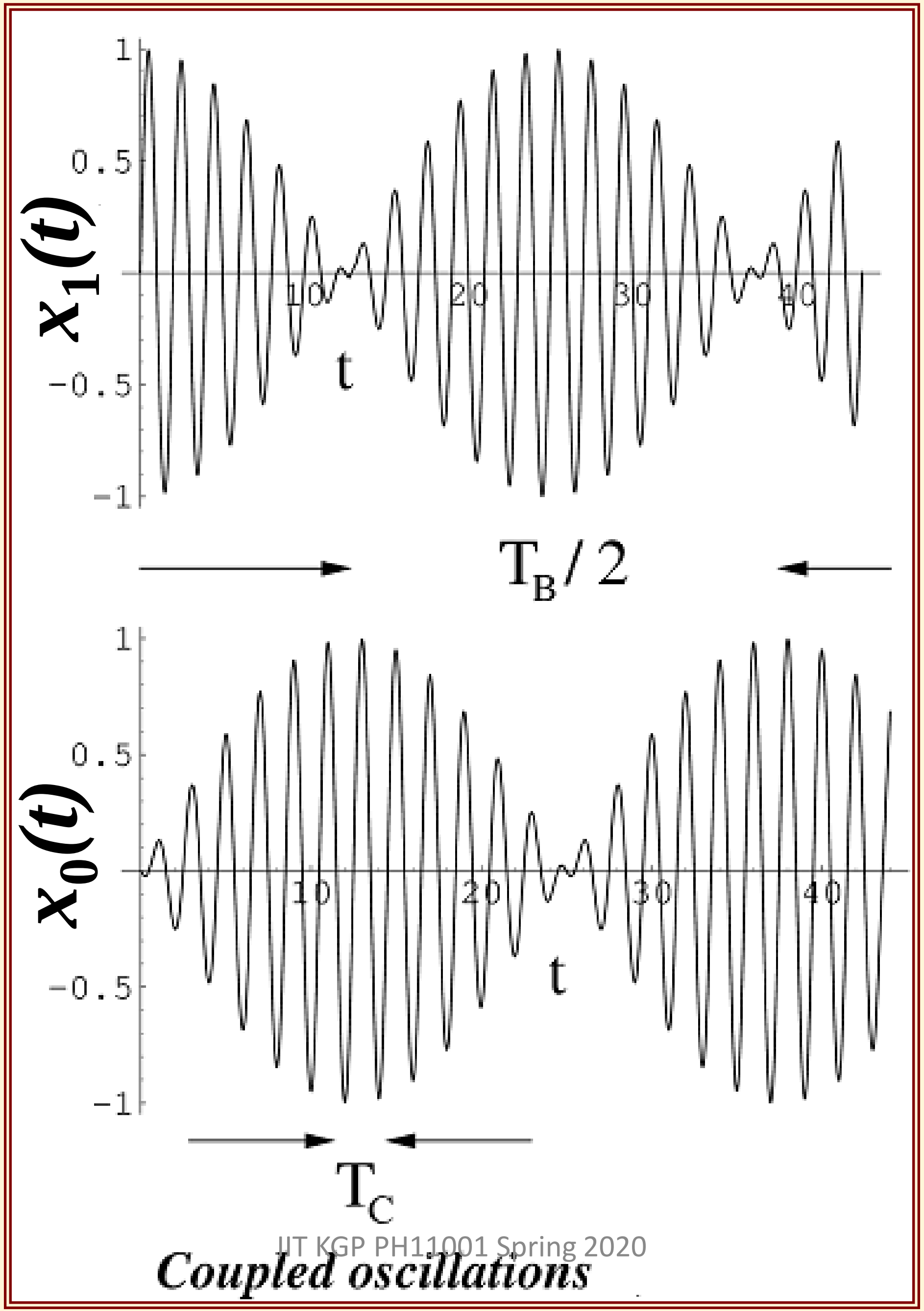
Superposition of Normal Modes



$$x_1 = \frac{1}{2}(q_1 + q_2)$$

$$x_0 = \frac{1}{2}(q_1 - q_2)$$

Evolution with time for individual pendulum



Condition for complete energy exchange

The masses M_0 and M_1 have to be equal and-

For $(\omega_2 - \omega_1)t = n\pi$ (n is odd integer), $x_1 = 0$

$$x_0 = 2a\,\mathrm{Sin}\,\frac{(\omega_2 + \omega_1)\times n\pi}{(\omega_2 - \omega_1)\times 2}$$

$$\boxed{\frac{(\omega_2 + \omega_1)}{(\omega_2 - \omega_1)} = \frac{m}{n}}$$

m is odd integer

Else, neither of the two pendulums will ever be stationary