Solutions to Problem Set-5

Electrodynamics, Electromagnetic waves

Problem 1: Flux through a given surface

1. Consider a tiny (point) dipole sitting at the origin, pointing towards the z axis. Consider the distance between the positive and negative charge $d \ll 1$ units. The dipole moment of the dipole is given by $\vec{p} = p\hat{k}$. Its Electric field is given by

$$\vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\vec{p}\cdot\hat{r})\hat{r}}{r^3} - \frac{\vec{p}}{r^3}\right)$$

Find the flux of this electric field through the *hemisphere* given by $x^2+y^2+z^2 = 1,\ z \geq 0$.

2. Consider spherical polar coordinates (r,θ,ϕ), which has the usual relation with cartesian coordinates (x,y,z). Also $(\hat{i},\hat{j},\hat{k})$ are unit vectors along x, y and z respectively. Now the magnetic field in a region is given by

$$\vec{B}(r,\theta,\phi) = B_0\left(-\sin\phi\ \hat{i} + \cos\phi\ \hat{j}\right),$$

where B_0 is a constant. Calculate the flux of this magnetic field through the $x = 0$ plane (in the positive x-direction).

Solution

1. Electric field is given by,

$$\vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\vec{p}\cdot\hat{r})\hat{r}}{r^3} - \frac{\vec{p}}{r^3}\right)$$

Now the surface vector of the sphere can be found using gradient operator as,

$$\vec{n} = \vec{\nabla}(x^2+y^2+z^2)$$
$$\Longrightarrow \hat{n} = \frac{x\ \hat{i} + y\ \hat{j} + z\ \hat{k}}{\sqrt{x^2+y^2+z^2}} = \hat{r}$$

Thus we get,

$$\vec{E}\cdot\hat{n} = \frac{1}{4\pi\epsilon_0}\left(\frac{3\ p\ (\hat{k}\cdot\hat{r})(\hat{r}\cdot\hat{r})}{r^3} - \frac{p}{r^3}(\hat{k}\cdot\hat{r})\right)$$

Using $\hat{r} = sin\theta cos\phi\ \hat{i} + sin\theta sin\phi\ \hat{j} + cos\theta\ \hat{k}$, it simplifies to,

$$\vec{E}\cdot\hat{n} = \frac{1}{4\pi\epsilon_0}\frac{2\ p\ cos\theta}{r^3}$$

Now the flux thorough the given hemisphere will be,

$$\int \vec{E}\cdot\hat{n}dS = \frac{1}{4\pi\epsilon_0}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{2\pi}\frac{2\ p\ cos\theta}{r^3}r^2\ sin\theta\ d\theta\ d\phi \quad [\ \because r^2 = x^2+y^2+z^2 = 1\ ]$$
$$= \frac{p}{2\epsilon_0} \quad [\text{ Answer }]$$

2. Given magnetic field,

$$\vec{B}(r,\theta,\phi) = B_0 \left(-\sin\phi\ \hat{i} + \cos\phi\ \hat{j}\right)$$

Now the flux of this magnetic field through the $x = 0$ plane (in the positive x -direction),

$$\begin{aligned}
\int \vec{B} \cdot d\vec{S} &= \int \vec{B} \cdot \hat{i} dy\ dz \\
&= -\ B_0 \int sin\phi\ dy\ dz \\
&= -\ B_0 \left[\int_{r=0}^{R} \int_{\theta=0}^{2\pi} sin\phi\ r\ dr\ d\theta\right]_{\phi=\pi/2,\ -\pi/2} \qquad \text{[in spherical polar coordinates]} \\
&= -\ B_0 \int_{r=0}^{R} rdr \left[\int_{\theta=0}^{\pi} d\theta - \int_{\theta=\pi}^{2\pi} d\theta\right] \quad [\ \because \theta[0,\pi] \to \phi = \frac{\pi}{2};\ \theta[\pi, 2\pi] \to \phi = -\frac{\pi}{2}] \\
&= 0 \quad \text{[Answer]}
\end{aligned}$$

The result can be visualised looking at the magnetic field as well. The magnetic flux lines are basically circular about the z-axis, so the flux lines penetrating $y - z$ plan are opposite to each other for $y < 0$ and $y > 0$ regions, thus resulting zero net flux.

Problem 2: Faraday's law

A current configuration creates a time-dependent electric and magnetic field given, in cylindrical polar coordinates by,

$$\vec{B} = \frac{e^{-t}}{r}\ \hat{k}$$

Now consider a moving loop of wire $C(t)$ in this electric and magnetic field. This loop is a circular loop and it lies in the $z = 0$ plane with its center at the origin, and it has a time varying radius $R(t) = 1 + t$. Find the induced EMF in this loop. Clearly explain the origin(s) of this EMF in the loop.

Solution We have to find the flux of the system (Φ) at any instance of time and then using Faraday's law we can get the induced EMF. EMF generated in the loop is,

$$\begin{aligned}
\mathcal{E}_{emf} &= -\ \frac{d\Phi}{dt} \\
&= -\ \frac{d}{dt} \int \vec{B} \cdot d\vec{S} \\
&= -\ \frac{d}{dt} \int_{r=0}^{R(t)} \int_{\phi=0}^{2\pi} \frac{e^{-t}}{r}\ rdr\ d\theta \\
&= -\ 2\pi \frac{d}{dt} \left[e^{-t} R(t)\right] \\
&= 2\pi t e^{-t}
\end{aligned}$$

Problem 3: Electromagnetic waves

The Electric and magnetic field in a region is given by

$$\vec{E} = E_0\ \hat{i} \cos\left(\omega\left(t - \frac{z}{c}\right)\right),\ \vec{B} = B_0\ \hat{j} \cos\left(\omega\left(t - \frac{z}{c}\right)\right),$$

where E_0, B_0 and ω are constants and c is the speed of light in vacuum.

Calculate the divergence and curl of $\vec{E}$ and $\vec{B}$ in the given region. From your answer, can you predict the charge density and current density in the given region?

Solution Given electric and magnetic fields are,

$$\vec{E} = E_0 \; \hat{i} \cos\left(\omega\left(t - \frac{z}{c}\right)\right), \; \vec{B} = B_0 \; \hat{j} \cos\left(\omega\left(t - \frac{z}{c}\right)\right)$$

Now divergence of the fields,

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

And curl of the fields are,

$$\vec{\nabla} \times \vec{E} = \hat{j}\frac{\partial E}{\partial z} = \frac{\omega E_0}{c} sin\left(\omega\left(t - \frac{z}{c}\right)\right) \; \hat{j}$$

$$\vec{\nabla} \times \vec{B} = -\hat{i}\frac{\partial B}{\partial z} = - \; \frac{\omega B_0}{c} sin\left(\omega\left(t - \frac{z}{c}\right)\right) \; \hat{i}$$

From above results we can predict the <u>charge density</u> to be,

$$\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} = 0$$

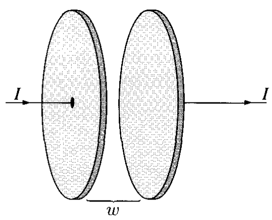
and the <u>current density</u> to be,

$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \left(- \; \frac{\omega B_0}{\mu_0 c} + \epsilon_0 E_0\right) sin\left(\omega\left(t - \frac{z}{c}\right)\right) \; \hat{i}$$

$$=0 \quad [\text{using the relations } B_0 c = E_0 \text{ and } c^2 = \frac{1}{\mu_0 \epsilon_0})]$$

Problem 4: Poynting Vector

1. Consider a steady current I resulting in the charging of a circular parallel plate capacitor as shown below. The charge is zero at time $t = 0$, when the steady current starts flowing and a uniform surface charge density starts developing with the flow of the current. An external agency is responsible to maintain the steady current.



   (a) Find the electric and magnetic fields in the gap, as a function of time. Ignore any edge effects (you can thinking of finding the electric and magnetic fields deep inside the gap away from the edges).
   (b) Calculate the energy density and the Poynting vector inside the gap.

Solution

(a) We know the electric field for a parallel plate capacitor (note: can be derived by using Gauss' law on a rectangular pillbox intersected by an infinite sheet with a charge density σ, and then considering two such sheets with opposite charge put close together):

$$\vec{E} = \frac{\sigma(t)}{\epsilon_0}\hat{z}$$

Here $\hat{z}$ points in the direction of the current. We use cylindrical co-ordinates since they suit the symmetry of the problem. The charge density is given by: (Here we assume the charge spreads evenly on the surface)

$$\sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{It}{\pi a^2}$$

$$\implies \vec{E}(t) = \frac{It}{\pi \epsilon_0 a^2}\hat{z} \text{ [Answer]}$$

Displacement current is given by:

$$i_D = \epsilon \frac{d\Phi_E}{dt}$$

Consider a circular Amperean loop of radius s near the center of the gap, concentric with the circular plates. The flux of the electric field through the loop is:

$$\Phi_E = \pi s^2 E$$

$$\implies i_D = \epsilon_0 \pi s^2 \frac{dE}{dt} = \epsilon_0 \pi s^2 . (\frac{I}{\pi \epsilon_0 a^2}) = I\frac{s^2}{a^2}$$

The magnetic field induced by this current will be along $\hat{\phi}$ (using the right-hand rule). Now, using Ampere's law for the same loop to find the magnetic field:

$$\oint_S \vec{B}.\vec{dl} = \mu_o I_{enc} \implies B.2\pi s = \mu_0 I \frac{s^2}{a^2}$$

$$\implies \vec{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi} \text{ [Answer]}$$

(b) Energy density:

$$u_{em} = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2) = \frac{1}{2}[\epsilon_0(\frac{It}{\pi\epsilon_0 a^2})^2 + \frac{1}{\mu_0}(\frac{\mu_0 I s}{2\pi a^2})^2] = \frac{\mu_0 I^2}{2\pi^2 a^4}(c^2t^2 + s^2/4) \text{ [Answer]}$$

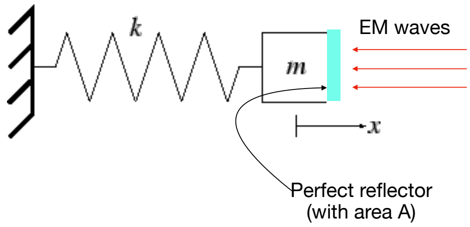
Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0}(\frac{It}{\pi\epsilon_0 a^2})(\frac{\mu_0 I s}{2\pi a^2})(-\hat{s}) = -\frac{I^2 ts}{2\pi^2\epsilon_0 a^4}\hat{s} \text{ [Answer]}$$

Problem 5: Radiation Pressure

1. Consider a situation as shown in the following figure. A perfect reflector is attached to the mass of a spring-mass system; a beam of plane electromagnetic waves is incident on the reflector and reflected away. The beam of plane electromagnetic wave is incident on the mass m at time t=0 and is kept on upto time $t = \tau$. At $t = 0$, the mass m is at rest and is located at the equilibrium position of the spring. The electric field in the beam of light is given by

$$\vec{E} = E_0\ \hat{k}\cos\left(\omega\left(t - \frac{x}{c}\right)\right)$$



Find the amplitude of oscillation after for time $t > \tau$.

Note that we should assume $\omega \gg \sqrt{\frac{k}{m}}$ and the area A is big enough so that it can be considered as a macroscopic object where radiation pressure due to the beam can act.

Solution Radiation pressure on a perfect reflector is given by:

$$P_{rad} = \frac{2I}{c}$$

Now, for a plane wave:

$$I = \frac{1}{2}\epsilon_0 c E^2$$

$$\therefore P_{rad} = \epsilon_0 E_0^2$$

When this pressure acts on the surface of the reflector with area A, the force becomes:

$$F = A\epsilon_0 E_0^2$$

Rest of this problem is exactly like Q1. in Tutorial-2, where the constant force F can be replaced by the force due to radiation pressure.

The equation of the undamped oscillation :

$$m\frac{d^2x}{dt^2} + kx = 0$$

At $t = 0$ we have $x = 0$ and at this moment as the mass is in equilibrium, the velocity $\frac{dx}{dt} = 0$. Now with this initial conditions we apply a constant force F. The new EOM is

$$m\frac{d^2x}{dt^2} + kx = F$$

Solving this we can get the path of the particle

$$x(t) = \frac{F}{k} + A\cos\omega_0 t + B\sin\omega_0 t$$

where A and B are constant with time and $\omega_0 = \sqrt{k/m}$ is the fundamental frequency of the undamped oscillator.
Using the above initial conditions, $A = -\frac{F}{k}$ and $B = 0$. So,

$$x(t) = \frac{F}{k}(1 - \cos\omega_0 t) = \frac{2F}{k}\sin^2\frac{\omega_0 t}{2}$$

So At $t = \tau$ the position and velocity of the particle are $x(\tau) = \frac{2F}{k}\sin^2\frac{\omega_0\tau}{2}$ and $\dot{x}(\tau) = \frac{2F\omega_0}{k}\sin\frac{\omega_0\tau}{2}\cos\frac{\omega_0\tau}{2}$.

After $t = \tau$, In absense of the external force, the eom $m\frac{d^2x}{dt^2} + kx = 0$ with the solution $x(t) = a\cos\omega(t-\tau) + b\sin\omega(t-\tau)$. Now at $t = \tau$ the state of the paricle is already obtained, using that, we can find $a = \frac{2F}{k}\sin^2\frac{\omega_0\tau}{2}$ and $b = \frac{2F}{k}\sin\frac{\omega_0\tau}{2}\cos\frac{\omega_0\tau}{2}$. So the amplitude of the oscillation

$$\text{Amp} = \sqrt{a^2+b^2} = \frac{2F}{k}|\sin\frac{\omega_0\tau}{2}|$$

$$= \frac{2A\epsilon_0 E_0^2}{k}|\sin(\frac{\omega_0\tau}{2})| \text{ [Answer]}$$

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