Linear homogeneous second-order differential equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

a, b are real constants.

Let m_1, m_2 be the roots of $m^2 + am + b = 0$. Then there are 3 cases.

Case 1. m_1, m_2 real and distinct:

$$y = c_1e^{m_1x} + c_2e^{m_2x}$$

Case 2. m_1, m_2 real and equal:

$$y = c_1e^{m_1x} + c_2xe^{m_1x}$$

Case 3. $m_1 = p + qi,\ m_2 = p - qi$:

$$y = e^{px}(c_1 \cos qx + c_2 \sin qx)$$

where $p = -a/2,\ q = \sqrt{b - a^2/4}$.

Undamped Simple Harmonic Oscillations

Let's find the general solution...

The equation of motion -

$$\frac{d^2x}{dt^2}+\omega^2 x=0 \quad where \quad \omega^2=\frac{k}{m}$$

This is a second order linear homogeneous equation with constant coefficients.

The general solution is given by:

$$x= c_1 e^{i\omega_0 t}+c_2 e^{-i\omega_0 t}$$

Constants c_1 and c_2 to be determined by the initial conditions.

Special cases

- The mass is pulled to one side and released from rest at t=0

$$x = a_0, \text{ and } \frac{dx}{dt} = 0 \text{ at } t = 0 \qquad \boxed{x = a_0 \cos \omega_0 t}$$

- The mass is hit and is given a speed v_0 at its equilibrium position at t=0

$$x = 0, \, and \quad \frac{dx}{dt} = v_0 \quad \text{at } t = 0 \qquad \boxed{x = \frac{v_0}{\omega_0} \sin \omega_0 t}$$

- The mass is given a speed ***v*** at a displacement ***a*** at t=0

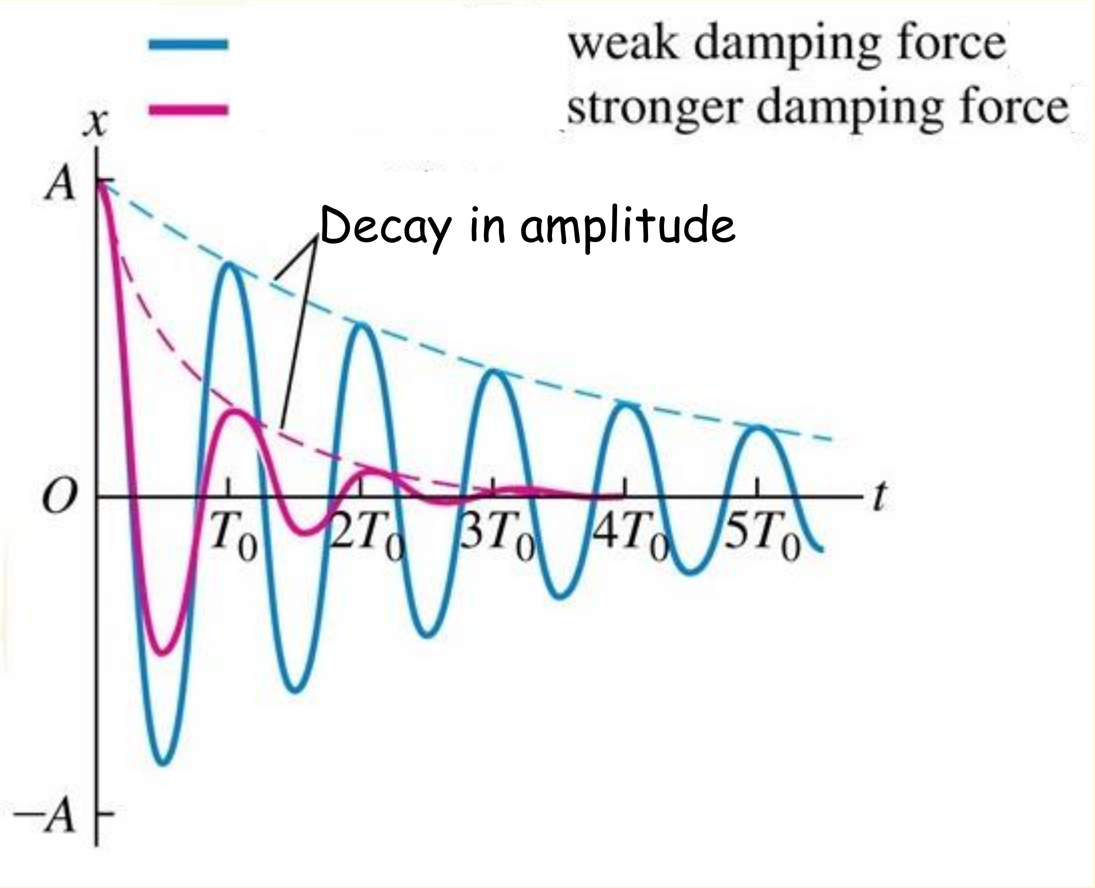
$$x = a, and \quad \frac{dx}{dt} = v \quad at \ t = 0$$

$$x = a_0 \cos(\omega_0 t + \alpha)$$

$$where \ a_0 = \sqrt{a^2 + \left(\frac{v}{\omega_0}\right)^2} \quad and \quad \alpha = \arctan\left(-\frac{v}{\omega_0 a}\right)$$

Damped Free Oscillations

- Real-world systems have some dissipative forces that decrease the amplitude
- The decrease in amplitude is called **damping** and the motion is called **damped oscillation**



Damped Free Oscillations

Resistive force is proportional to velocity

$$F = m\ddot{x} = -rv - kx$$

$$m\ddot{x} + rv + kx = 0$$

$$\ddot{x} + \frac{r}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$$

$$\beta = \frac{r}{2m}$$

$$\omega_0^2 = \frac{k}{m}$$

Solution

- The equation is a second order linear homogeneous equation with constant coefficients.
- Solution can be found which has the form: $\boldsymbol{x = Ce^{pt}}$ where C has the dimensions of x, and p has the dimensions of T^{-1}.

$$\dot{x} = pCe^{pt}; \ddot{x} = p^2Ce^{pt}$$

$$m\ddot{x} + r\dot{x} + kx = 0 \Rightarrow Ce^{pt}(mp^2 + rp + k) = 0$$

$$x = Ce^{pt} = 0 \quad \textit{Trivial solution}$$

$$mp^2 + rp + k = 0$$

- Solving the quadratic equations gives us the two roots:

$$p_{1,2} = -\frac{r}{2m} \pm \sqrt{\left(\frac{r}{2m}\right)^2 - \frac{k}{m}}$$

$$p_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_o^2}$$

$$\beta = \frac{r}{2m}$$

$$\omega_0^2 = \frac{k}{m}$$

- The general solution takes the form:

$$x = x_1 + x_2 = A_1e^{p_1t} + A_2e^{p_2t}$$

Case I: Overdamped $(\beta^2 > \omega_0^2)$

(Heavy damping)

$\beta = \frac{r}{2m}$

$\omega_0^2 = \frac{k}{m}$

The square root term is +ve: The damping resistance term dominates the stiffness term.

$$x(t) = A_1 \exp(-\beta t + \sqrt{\beta^2 - \omega_0^2}\, t) + A_2 \exp(-\beta t - \sqrt{\beta^2 - \omega_0^2}\, t)$$

Let: $q = \sqrt{\left(\frac{r}{2m}\right)^2 - \frac{k}{m}}$ $\quad x = e^{-\beta t}(A_1 e^{qt} + A_2 e^{-qt})$

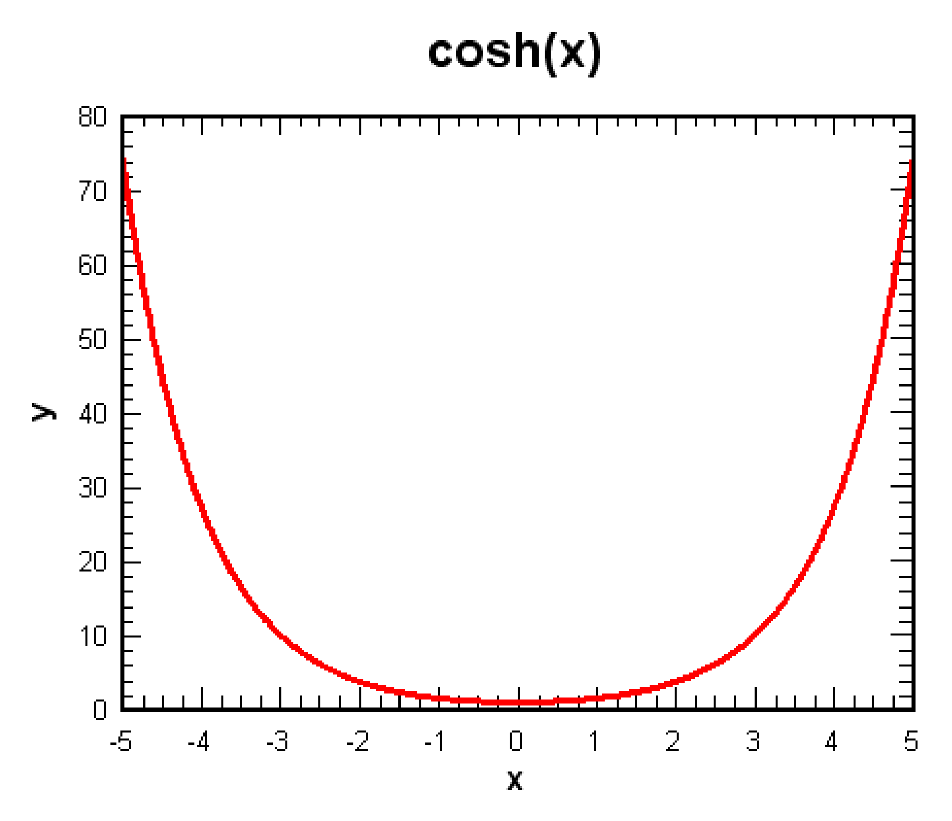
Now, if: $A_1 = (A+B)/2$ $\quad A_2 = (A-B)/2$

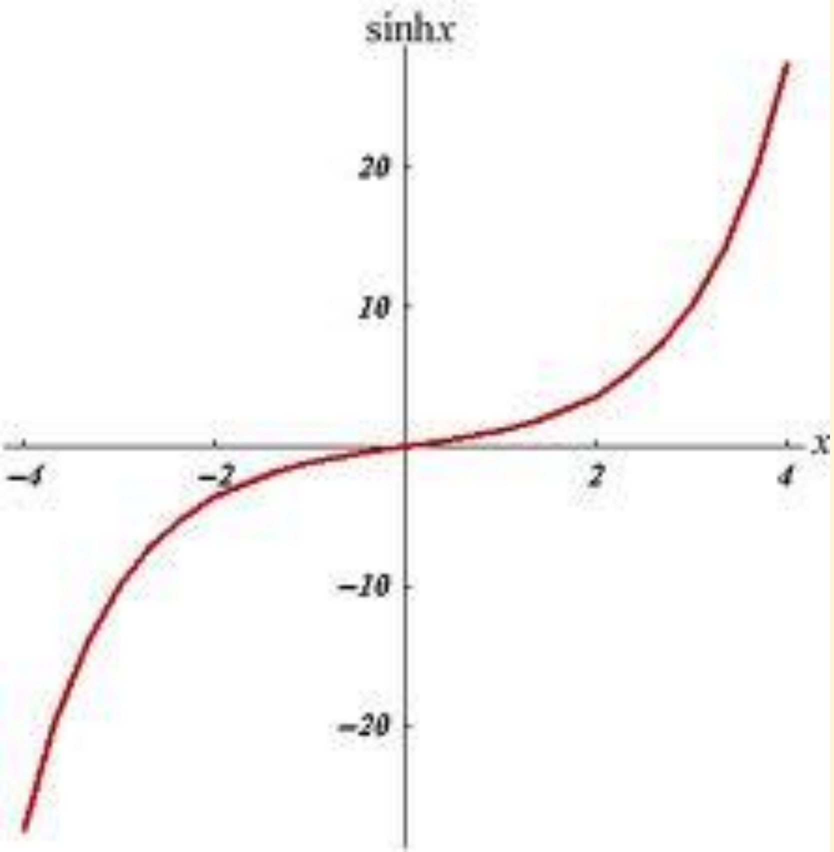
Then displacement is: $x = e^{-\beta t}\left(\frac{A}{2}(e^{qt} + e^{-qt}) + \frac{B}{2}(e^{qt} - e^{-qt})\right)$

$$x = e^{-\beta t}\left(A(\cosh(qt)) + B(\sinh(qt))\right)$$

Hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$





$$\frac{d\cosh x}{dx} = \sinh x \qquad \frac{d\sinh x}{dx} = \cosh x$$

$$x = e^{-\beta t}\left(A(\cosh(qt)) + B(\sinh(qt))\right)$$

$$x(t) = A\exp(-\beta t)\cosh(\sqrt{\beta^2 - \omega_0^2}\,t)$$
$$+B\exp(-\beta t)\sinh(\sqrt{\beta^2 - \omega_0^2}\,t)$$

$$A_1 = (A + B)/2 \qquad A_2 = (A - B)/2$$

- **Non-oscillatory behavior can be observed.**
- **But, the actual displacement will depend upon the boundary conditions**

$$x(0) = A_1 + A_2 = A$$

$$\dot{x}(t) = A_1(-\beta + \sqrt{\beta^2 - \omega_0^2}) \exp(-\beta t + \sqrt{\beta^2 - \omega_0^2}\, t)$$

$$+A_2(-\beta - \sqrt{\beta^2 - \omega_0^2}) \exp(-\beta t - \sqrt{\beta^2 - \omega_0^2}\, t)$$

$$\dot{x}(0) = -\beta(A_1 + A_2) + \sqrt{\beta^2 - \omega_0^2}\,(A_1 - A_2)$$

$$\frac{\dot{x}(0) + \beta x(0)}{\sqrt{\beta^2 - \omega_0^2}} = (A_1 - A_2) = B$$

CaseII: Critical damping $(\beta^2 = \omega_0^2)$

$$\beta = \frac{r}{2m}$$

$$\omega_0^2 = \frac{k}{m}$$

- The damping resistance term and the stiffness terms are balanced.
- When *r* reaches a critical value, the system will not oscillate and quickly comes back to equilibrium.

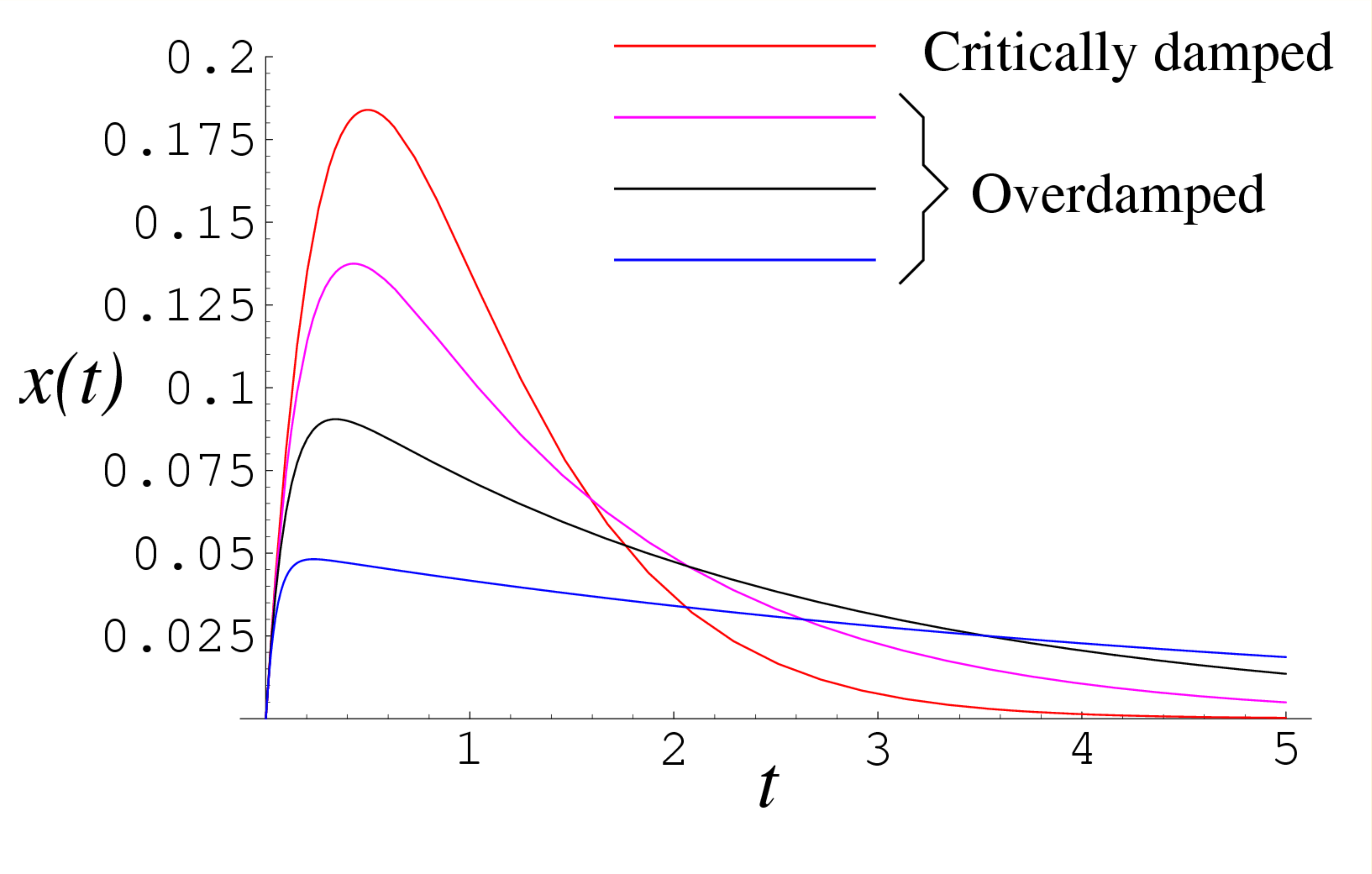
The quadratic equation in *p* has equal roots, which, in a differential equation solution demands that *C* must be written as (*A*+*Bt*).

$$A \exp(-\beta t).$$

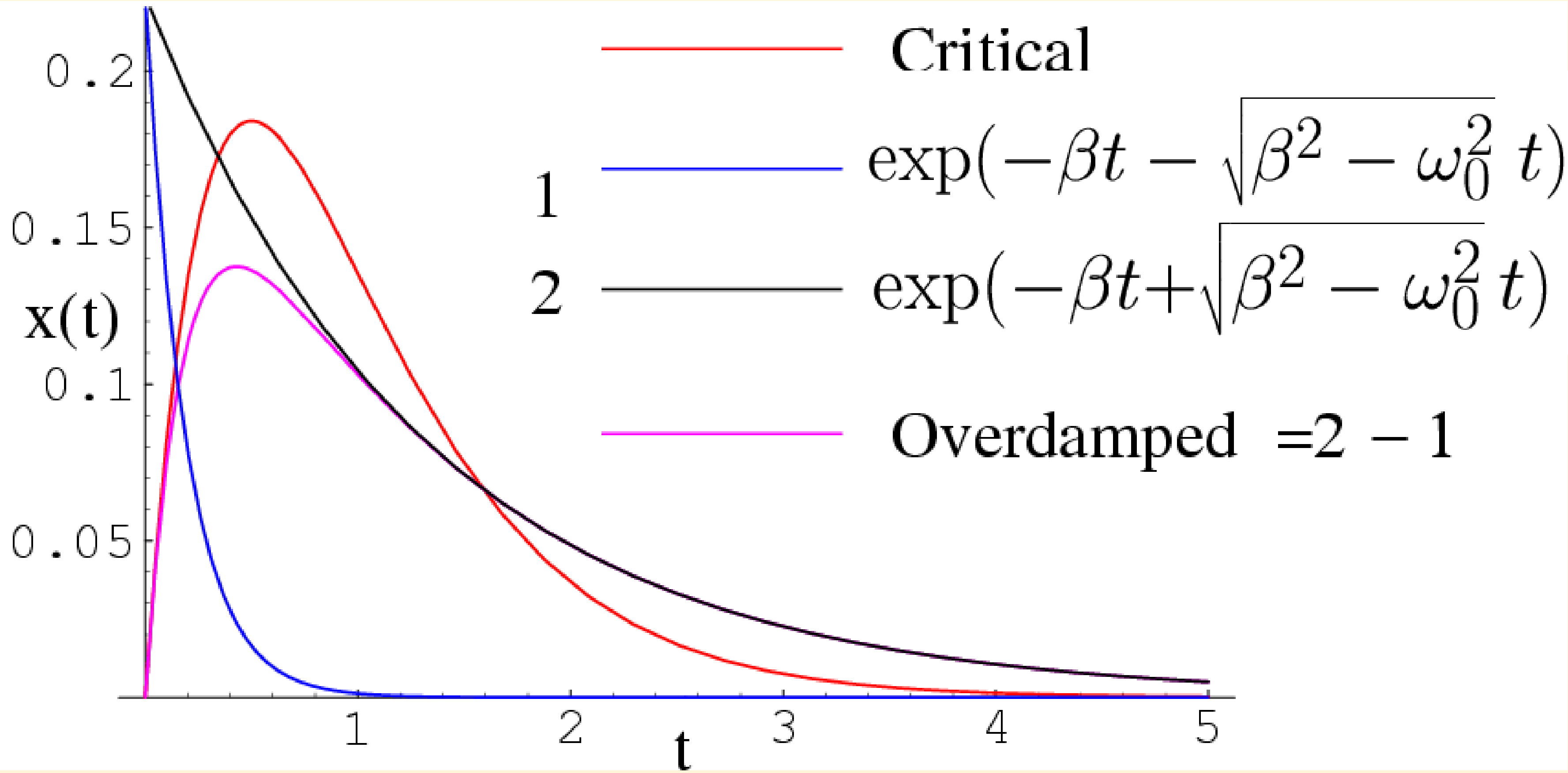
$$Bt \exp(-\beta t)$$

$$\boxed{x(t) = (A + Bt) \exp(-\beta t)}$$

$A=0 \quad B=2$


Critically damped
Overdamped
0.2
0.175
0.15
0.125
0.1
0.075
0.05
0.025
x(t)
1
2
3
4
5
t

$A=0 \quad B=2$



Critical
1
$\exp(-\beta t - \sqrt{\beta^2 - \omega_0^2}\, t)$
2
$\exp(-\beta t + \sqrt{\beta^2 - \omega_0^2}\, t)$
Overdamped =2 − 1
x(t)
t
0.2
0.15
0.1
0.05
1
2
3
4
5