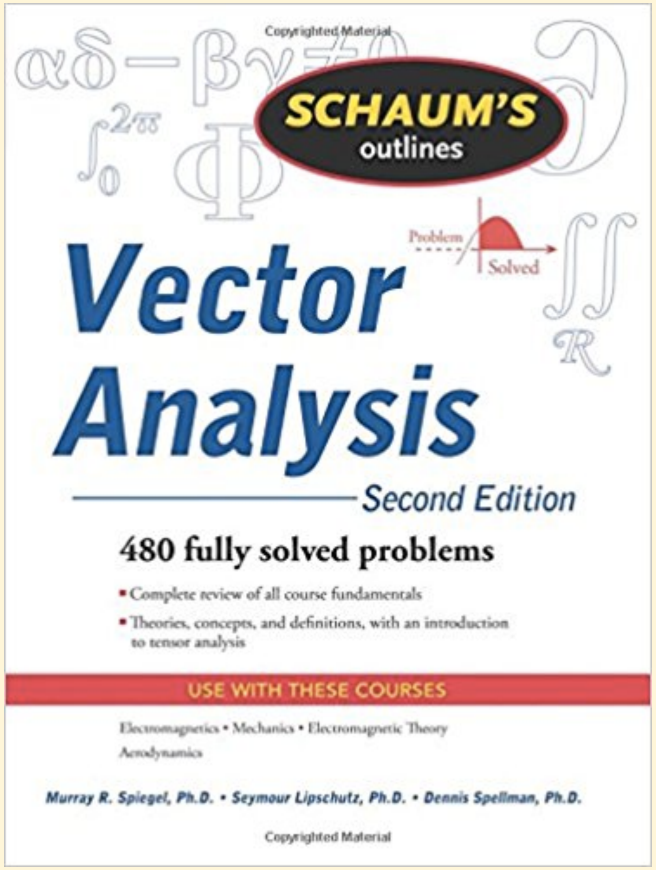
Vector Analysis & Vector Calculus

Reference:
VECTOR ANALYSIS: Schaum's Outlines Series
by Murray Spiegel , Seymour Lipschutz



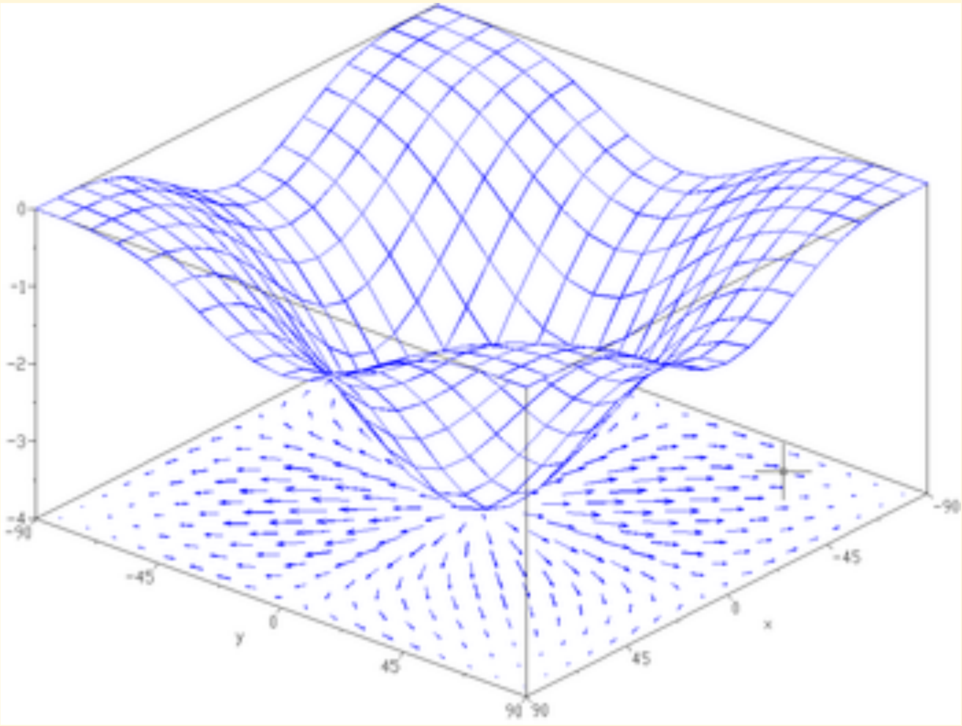
Gradient of a Scalar Field

For a scalar function *W* of three variable W*(x,y,z)*, the gradient of *W* is a vector quantity given by:

$$\vec{\nabla} W(x,y,z) = \frac{\partial W(x,y,z)}{\partial x}\hat{x} + \frac{\partial W(x,y,z)}{\partial y}\hat{y} + \frac{\partial W(x,y,z)}{\partial z}\hat{z}$$

- The gradient points in the direction of the greatest rate of change (increase/decrease) of the function.

- and its magnitude is the slope (rate of change) of the graph in that direction.



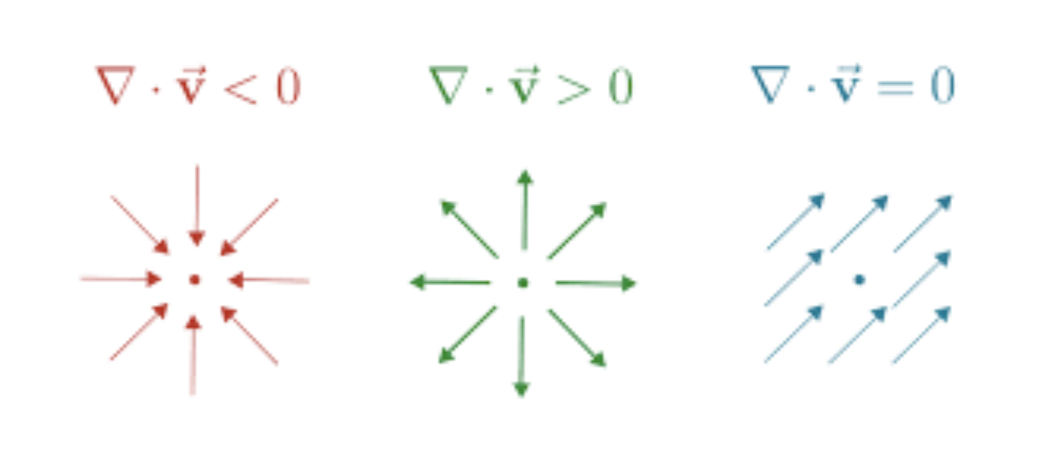
Divergence of a Vector Field

For a vector ***T*** the divergence of ***T*** is given by:

$$\vec{\nabla}.\vec{T} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right).(T_x\hat{x} + T_y\hat{y} + T_z\hat{z})$$

$$= \left(\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + \frac{\partial T_z}{\partial z}\right)$$

It is a measure of how much the vector ***T*** diverges / spreads out from the point in question.



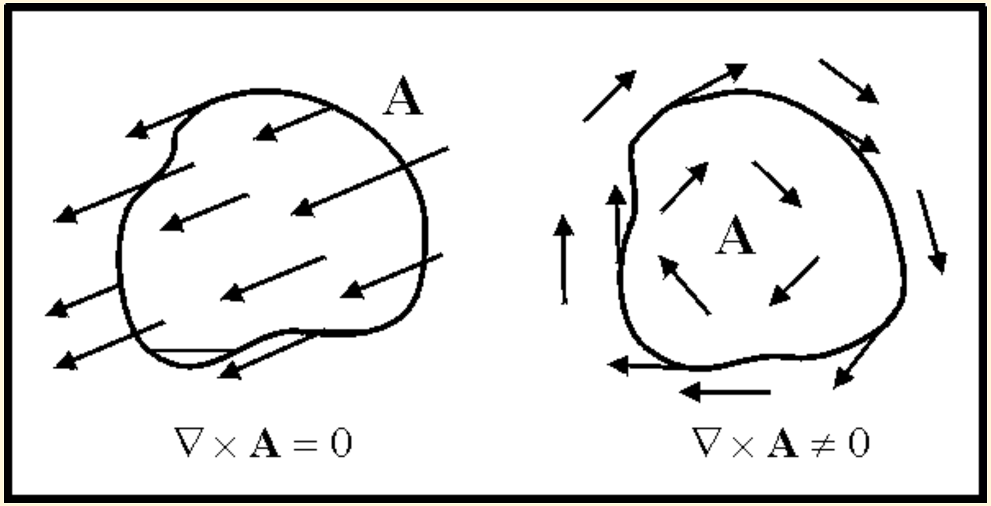
Curl of a Vector Field

For a vector ***T*** the Curl of ***T*** is given by:

$$\nabla \times \vec{T} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \times \left(T_x\hat{x} + T_y\hat{y} + T_z\hat{z}\right)$$

$$= \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ T_x & T_y & T_z \end{pmatrix}$$

It is a measure of how much the vector ***T*** curls around the point in question.



Laplacian

$$\vec{\nabla}.\vec{\nabla} \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Used in: Maxwells equation; Navier Stokes Equation,

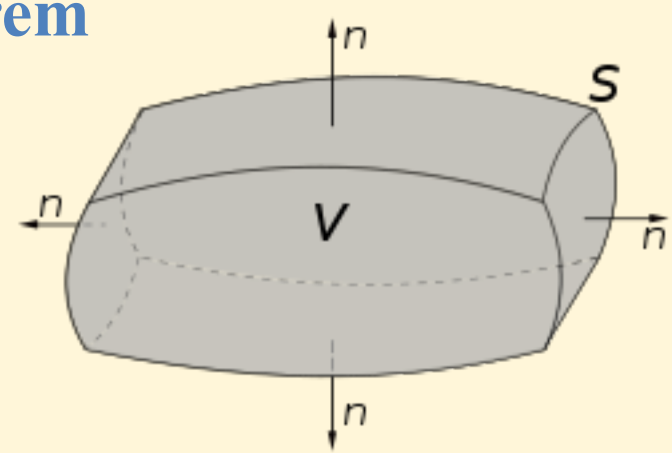
D’Alembertain

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

Used in: Relativistic electrodynamics

DIVERGENCE THEOREM

Gauss's Theorem



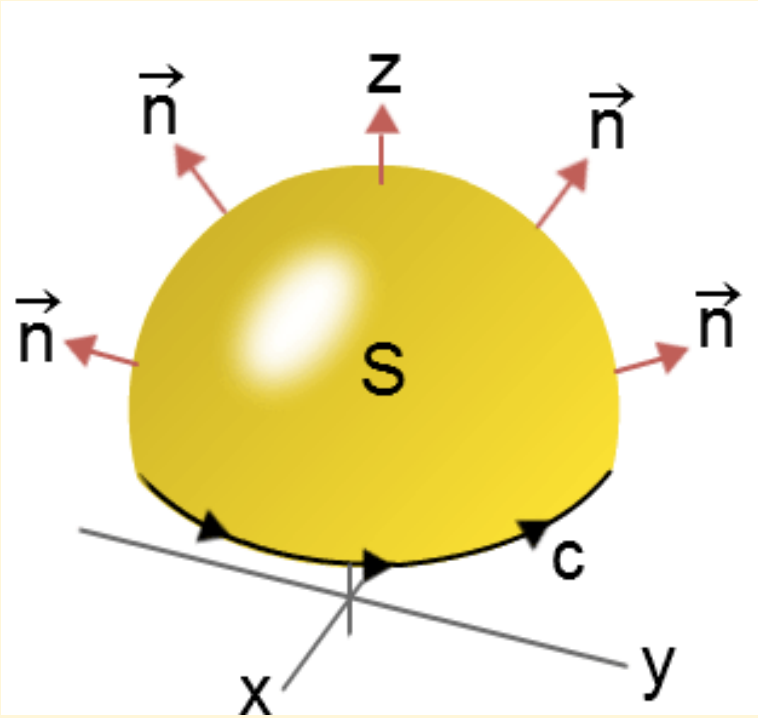
$$\int_V \left(\nabla \cdot \vec{E}\right) d\tau = \oint_S \vec{E} \cdot d\vec{a}$$

Integral of a derivative (in this case the divergence) over a volume is equal to the value of the function at the surface that bounds the volume.

$$\int_V \left(\text{Faucets within the volume}\right) = \oint_S \left(\text{Flow out through the surface}\right)$$

Stokes Theorem

$$\int_S (\vec{\nabla} \times \vec{X}) d\vec{A} = \oint_C \vec{X} . d\vec{l}$$



Where S is an open surface and C is a closed contour around the oped face of the surface

Integral of a deriavative (in this case the curl) over a patch of surface is equal to the value of the function at the boundary (perimeter of the patch).

Some Vector Calculus Identities

(1) Divergence of a Curl is always zero

$$\vec{\nabla}.\left(\vec{\nabla}\times\vec{X}\right)=0$$

(2) Curl of a gradient is always zero

$$\vec{\nabla}\times\vec{\nabla}W=0$$

(3) Curl of a curl

$$\vec{\nabla}\times\vec{\nabla}\times\vec{X}=\vec{\nabla}\left(\vec{\nabla}.\vec{X}\right)-\nabla^2\vec{X}$$