Problem Set-4

PH11001 (Spring 2019-20)

Vector Calculus, Electrostatics, Magnetistatics

February 3, 2020

1. *Divergence and curl of vector fields:*

   Consider a force (vector) field given by

   $$\vec{F} = (x^2 + y^2 + z^2)^n (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).$$

   Find

   (a) $\int_V \left(\vec{\nabla} \cdot F\right) dV$, where V is the volume of the sphere of radius R.

   (b) $\nabla \times \vec{F}$

   (c) a scalar field $\phi(x, y, z)$ such that $\vec{F} = -\nabla\phi$.

   (d) For what value of the exponent n does the scalar field diverge at both the origin as well as infinity?

2. *Divergence theorem*

   (a) If ϕ is a any scalar field and the surface integral is performed over the closed surface S which is the boundary of volume V, then using 'divergence theorem' show that

   $$\int_V \vec{\nabla}\phi = \oint_S \phi \, d\vec{S}$$

   [Hint: Take the vector in 'divergence theorem' to be of the special form $\vec{A} = \vec{C}\phi$, where $\vec{C}$ is a constant, but arbitrary vector. Note that $\vec{A}$ and ϕ are vector and scalar fields respectively.]

   (b) Using 'divergence theorem' show that $\oint d\vec{S} = 0$ for any closed surface. Now, if $\oint \hat{n} \cdot d\vec{S}$ is the total surface area of the closes surface what should be $\hat{n}$?

3. *Stokes theorem theorem*

   Using stokes theorem prove the following identities

   (a) If ϕ is a any scalar field and the line integral is performed over the closed line C which is the boundary of surface S, then show that

   $$\int_S d\vec{S} \times \vec{\nabla}\phi = \oint_C \phi \, d\vec{\ell}$$

   (Same hint as problem 2(a) can be useful here as well.)

(b) Using stokes theorem argue that, if $\vec{B} = \vec{\nabla} \times \vec{A}$, then $\oint_S \vec{B} \cdot d\vec{S} = 0$, for any closed surface S. Can you arrive at the same conclusion using the divergence theorem?

4. *Electrostatics*

(a) Imagine a situation where our world is 2 dimensional instead of 3 dimensional (in addition there is ofcourse time in both the cases), and the local form of Gauss law in electrostatics remains the same, i.e. we still have

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}.$$

where ρ is the charge density (charge per unit area, since in 2 dimensions 'area' is like 'volume'). Make a prediction on the nature of the modified 'Coulomb's law' in such an imaginary world?

(b) Use Gauss Law, to obtain the electric field (everywhere) due to a static uniform charge density ρ, occupying the spherical shell with inner radius a and outer radius b (i.e. the region $a \leq r \leq b$, r being the radial distance from the origin). Make a plot of magnitude of the electric field as a function of the radial coordinate r.

5. *Magnetostatics*

Consider a wire segment carrying a steady current I as shown in the figure. Compute the magnetic field created due to this steady current at the point P, which is shown in the figure.