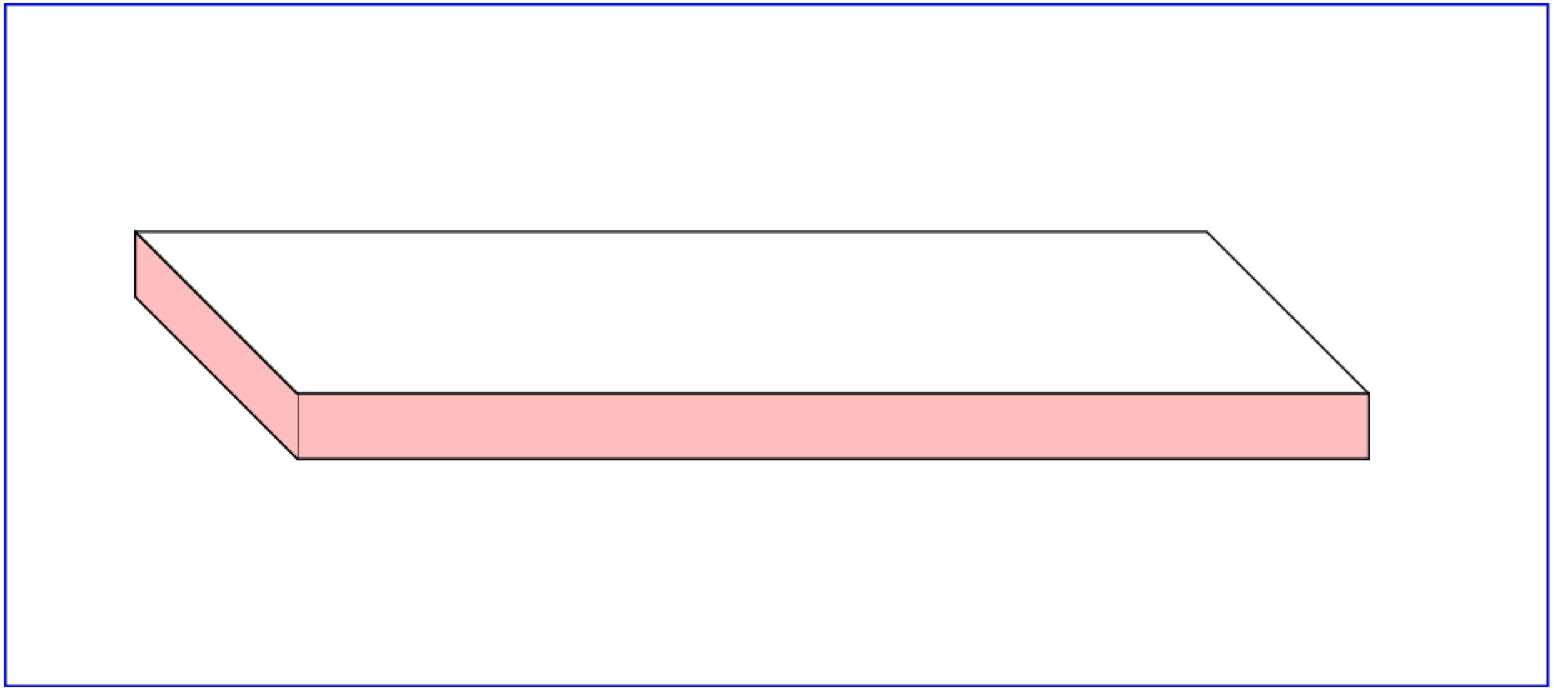
Wave & Wave Equation

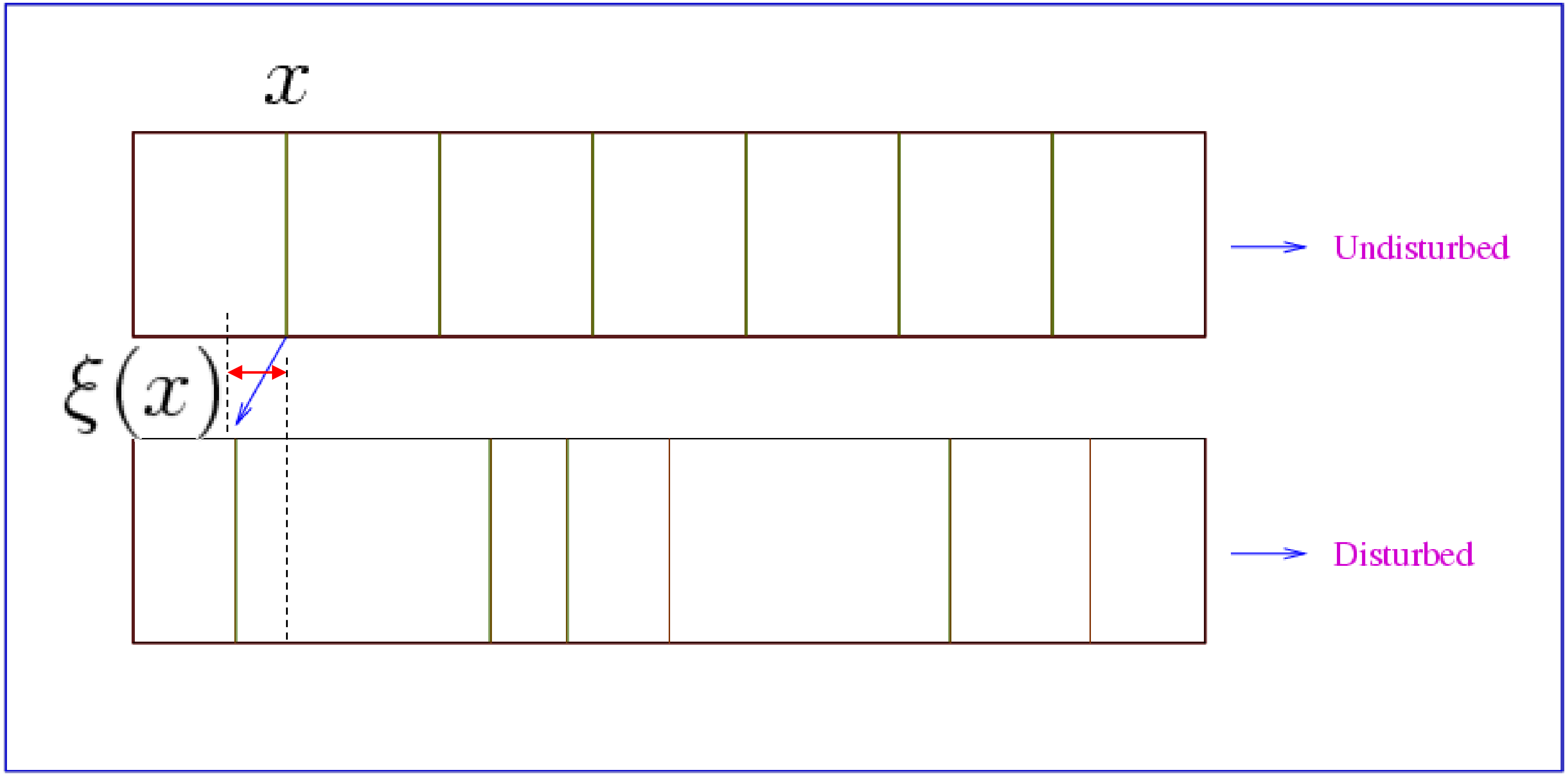
Wave Equation

Longitudinal waves

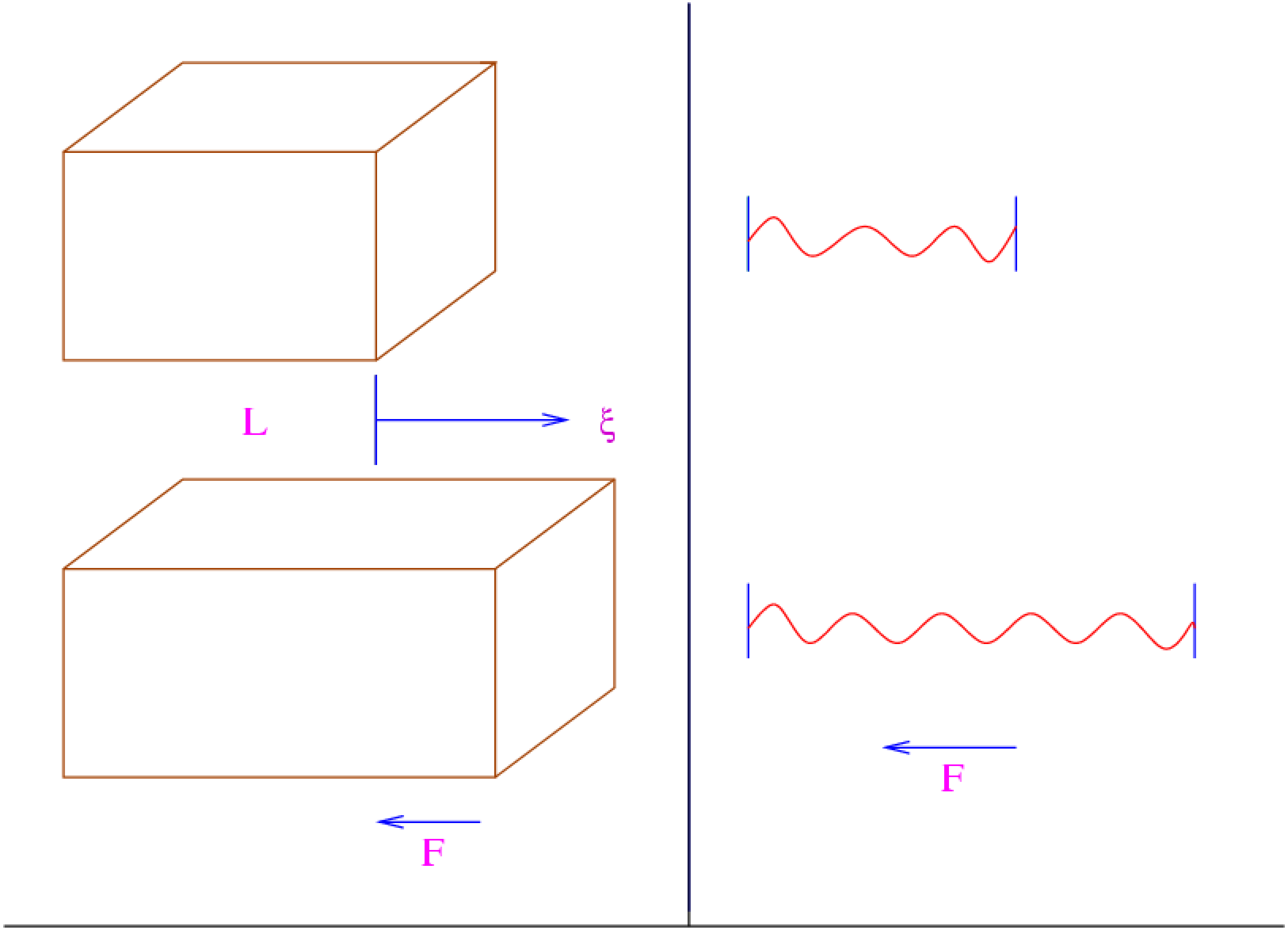
Longitudinal waves in an elastic medium

Shown below is a long rod of cross sectional area A





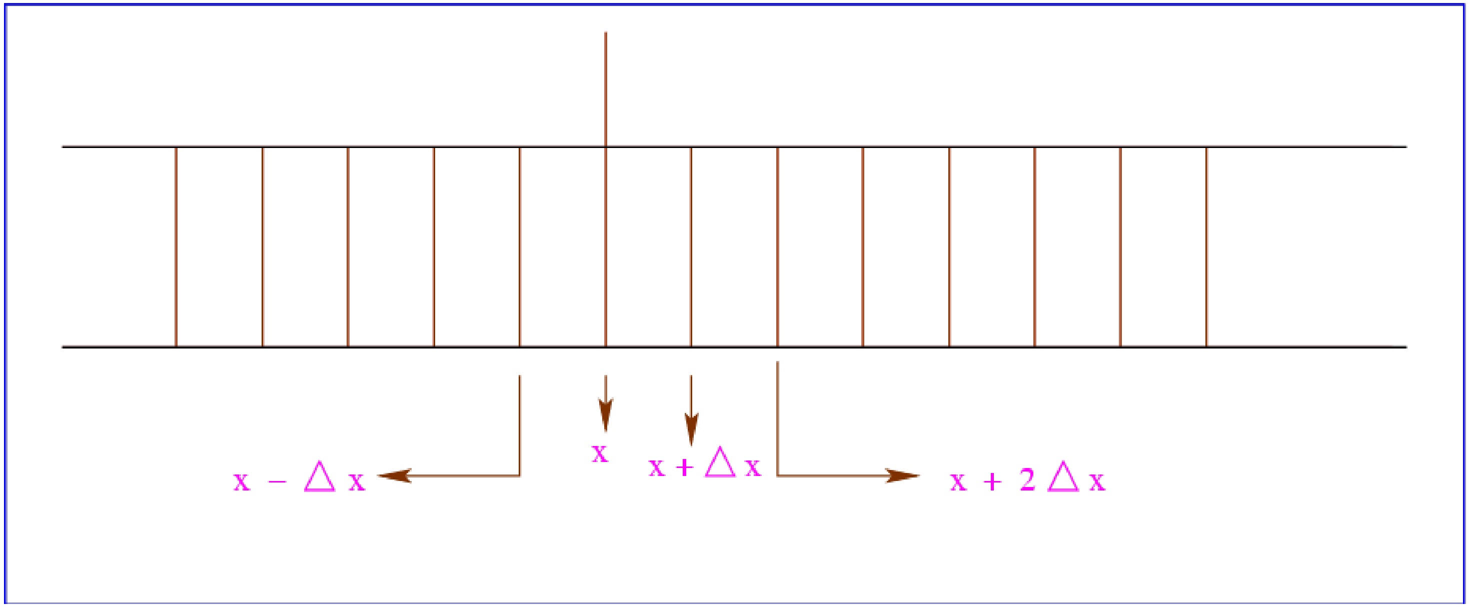
"$\xi(x)$" denotes the horizontal displacement of a point on the rod originally at "x"

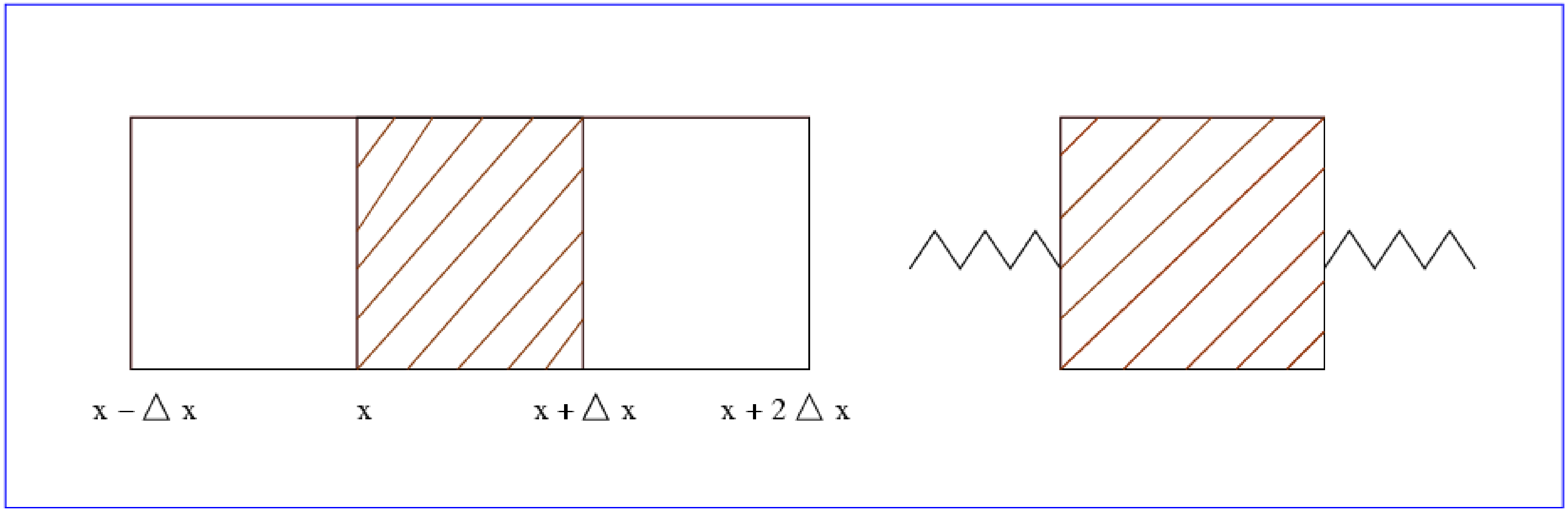

L
ξ
F
F

What happens to an elastic solid when it is compressed or stretched?

$$\text{Stress} = F/A$$
$$\text{Strain} = \frac{\xi}{L}$$

$$Y = \frac{Stress}{Strain} \quad \text{(Youngs modulus)}$$
$$F = \left(\frac{YA}{L}\right)\xi$$
$$F = kx \rightarrow \text{Spring}$$


x − △ x
x
x + △ x
x + 2 △ x


x − △ x
x
x + △ x
x + 2 △ x

Equation of motion

Force equation for slab x

$$\Delta x \ \varrho A \frac{\partial^2 \xi(x,t)}{\partial t^2} = F$$

compression of the slab on the right is

$$\xi(x+\Delta x) - \xi(x+2\Delta x)$$

Stress = F/A = Y × Strain

Force from the right on the slab

$$-\left(\frac{YA}{\Delta x}\right)\left[\xi(x+\Delta x,t)-\xi(x+2\Delta x,t)\right]$$

$$\approx YA\frac{\partial\xi}{\partial x}(x+\Delta x,t)$$

Similarly the force from the left is

$$-\left(\frac{YA}{\Delta x}\right)\left[\xi(x,t)-\xi(x-\Delta x,t)\right]$$

$$= -YA\frac{\partial}{\partial x}\xi(x,t)$$

Total Force

$$= \quad YA\left[\frac{\partial}{\partial x}\xi(x+\Delta x,t)-\frac{\partial}{\partial x}\xi(x,t)\right]$$

$$= \quad YA \frac{\partial}{\partial x}\left[\xi(x+\Delta x, t) - \xi(x,t)\right]$$

$$= \quad YA\Delta x \frac{\partial}{\partial x}\left[\frac{1}{\Delta x}(\xi(x+\Delta x, t) - \xi(x,t))\right]$$

$$= YA\Delta x \frac{\partial^2 \xi}{\partial x^2}$$

Force equation

$$\varrho A \Delta x \frac{\partial^2 \xi}{\partial t^2} = Y A \Delta x \frac{\partial^2 \xi}{\partial x^2}$$

$$\frac{\partial^2 \xi}{\partial x^2} - \left(\frac{\varrho}{Y}\right) \frac{\partial^2 \xi}{\partial t^2} = 0$$

Wave equation

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} = 0$$

Speed (check the dimension) of the wave

$$c = \sqrt{\frac{Y}{\varrho}}$$