1. Consider the following function in momentum space:

$$\begin{aligned} g(k) &= g_0 \, (-k_0 \leq k \leq k_0) \\ &= 0 \, \text{otherwise} \end{aligned}$$

Find the corresponding function in the position space, and also evaluate $\Delta x \Delta k$.

<u>Solution</u>: The wavefunction in the position space is

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx} dk \\ &= \frac{g_0}{\sqrt{2\pi}} \int_{-k_0}^{k_0} e^{ikx} dk \\ &= g_0 \sqrt{\frac{2}{\pi}} \frac{\sin(k_0 x)}{x} \end{aligned}$$

Now

$$\begin{aligned} \langle f|f\rangle &= \int_{-\infty}^{\infty} dx f(x) f^*(x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' g(k) g^*(k') e^{i(k-k')x} \\ &= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' g(k) g^*(k') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{i(k-k')x} \right) \\ &= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' g(k) g^*(k') \delta(k-k') \\ &= \int_{-\infty}^{\infty} dk g(k) g^*(k) \\ &= 2k_0 g_0^2 \end{aligned}$$

The normalized function in position space $f(x) = \frac{1}{\sqrt{\pi k_0}} \frac{\sin(k_0 x)}{x}$.

The spread in k-space (i.e., the distance between two zeros) is $\Delta k = 2k_0$.

Taking $x \neq 0$ but $\sin(k_0 x) = 0 = \sin(n\pi)$ (where $n = \pm 1, \pm 2, \cdots$), the spread in x-space is $\Delta x = 2\pi/k_0$.

Therefore, the product $\Delta x \Delta k = 4\pi$.

2. Consider the following thought experiment (proposed by Dicke- Wittke, see the figure at the end of the tutorial-11). Suppose there is a cylindrical bird cage with regular spacing $a = \frac{2\pi R}{N}$ between the bars, where N is the number of bars and R is the radius of cylinder. If radiation is emitted from the axis of the cylinder, the bars can act as a diffraction grating.

a) If the beam emerges from the cage at an angle θ with the original direction, find the condition for maximum intensity which related θ and the wavelength λ.

b) The intensity peak can also be interpreted by assuming that the particles are scattered through θ off the bars of the cage. Considering the momentum transferred to the cage, find the angular momentum transferred to the cage.

c) Using the above and the De Broglie relation, show that angular momentum is quantized.

Solution: a) Here the angle of diffraction is θ and grating spacing is a, so according to the theory of diffraction the condition for the maximum intensity at that angle is $a \sin\theta = k\lambda$, where $k = 1, 2, 3, \cdots$. So,

$$\sin\theta = k\frac{N\lambda}{2\pi R}$$

b) Now as the radiation (particles according to duality) emerges from the axis of the cylinder radially, it has the linear momentum only directed radially outward, no angular momentum around the axis. After diffraction, the beam makes an angle θ with the radial direction. Thus after diffraction, the momentum has two components – radially outward and in the direction perpendicular to the radial direction (i.e. tangential to the circular cross-section of the cage). This tangential component gives the angular momentum around the axis of cylinder. So the transferred angular momentum is

$$L = pR\sin\theta = k\frac{N\lambda}{2\pi}p$$

where p is the linear momentum of the beam. [the tangential velocity $v_t = \frac{p}{m}\sin\theta$, the angular momentum $L = mv_tR$]

c) According to the De Broglie wave-particle duality, the momentum of the beam

$$p = \frac{h}{\lambda}$$

, where h is Plank constant. Therefore the transferred angular momentum

$$L = kN\hbar$$

is quantized.

3. Use the uncertainty relation to estimate the ground state energy of:
a) a harmonic oscillator, the energy being given by: $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$
b) a particle in the potential $V = \alpha x^4$, $\alpha =$ const.

Solution: Assume Δx and Δp are position and momentum uncertainties respectively. So if uncertainties are minimum $\Delta x \Delta p = \frac{\hbar}{2}$. Now at ground state, (the energy is minimum) the position and momentum have minimum alowed values, $x = \Delta x$ and $p = \Delta p = \frac{\hbar}{2\Delta x}$.

(a) Thus the energy can be written as

$$E = \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2$$

. To find the minimum value w.r.t. Δx, we set

$$-\frac{\hbar^2}{4m(\Delta x)^3} + m\omega^2(\Delta x) = 0$$

which gives $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$. Therefore the ground state energy is $\frac{1}{2}\hbar\omega$.

(b) Here the total energy of the particle can be written as

$$E = \frac{\hbar^2}{8m(\Delta x)^2} + \alpha(\Delta x)^4$$

By setting the derivative w.r.t. Δx equal to zero

$$-\frac{\hbar^2}{4m(\Delta x)^3} + 4\alpha(\Delta x)^3 = 0$$

we get

$$\Delta x = \left[\frac{\hbar^2}{16m\alpha}\right]^{1/6}$$

So the ground state energy is

$$\begin{aligned} E &= \frac{\hbar^2}{8m}\left[\frac{16m\alpha}{\hbar^2}\right]^{1/3} + \alpha\left[\frac{\hbar^2}{16m\alpha}\right]^{2/3} \\ &= 2\alpha\left[\frac{\hbar^2}{16m\alpha}\right]^{2/3} + \alpha\left[\frac{\hbar^2}{16m\alpha}\right]^{2/3} \\ &= 3\alpha\left[\frac{\hbar^2}{16m\alpha}\right]^{2/3} \end{aligned}$$

4. Monochromatic light with $\lambda = 6000$ angstrom passes through a fast shutter that opens for 10^{-9} sec. What will be the spread in the wavelengths in the light that is no longer monochromatic after passing through the shutter?

Solution: The energy-time uncertainty principle gives

$$\Delta E \Delta t = \frac{\hbar}{2} \Rightarrow \Delta\lambda\Delta t = \frac{\lambda^2}{4\pi c} \Rightarrow \Delta\lambda = \frac{\lambda^2}{4\pi c \Delta t}$$

Here the wavelength of the monochromatic light $\lambda = 6.0 \times 10^{-7}$m, velocity $c = 3 \times 10^8$m/sec and $\Delta t = 10^{-9}$sec. So the width of the wavelength band is

$$\Delta\lambda = \frac{36.0 \times 10^{-14}}{4\pi \times 3 \times 10^8 \times 10^{-9}} \times 10^{10} \mathring{A} = 0.9554 \times 10^{-3} \mathring{A}$$

5. The smallest separation resolved by a microscope is of the order of magnitude of the wavelength used. What energy would one need for the electrons to have in order to resolve separation of 5 angstrom in an electron microscope?

Solution: To achieve the specified resolution of $5\mathring{A}$ the electron beam need to have the energy so that it's corresponding De Broglie wavelength is of the order of $5\mathring{A}$, i.e. $\lambda = 5 \times 10^{-10}$m. So it's momentum $p = \hbar k = \frac{h}{\lambda}$.
Therefore the required energy

$$E = \frac{p^2}{2m_e} = \frac{h^2}{2m_e\lambda^2} = \frac{43.96 \times 10^{-68}}{2 \times 9.11 \times 10^{-31} \times 25 \times 10^{-20}} = 0.0965 \times 10^{-17}\text{J} = 6.03\text{eV}$$

[where $h = 6.63 \times 10^{-34}$J.s, $m_e = 9.11 \times 10^{-31}$kg and 1eV$= 1.6 \times 10^{-19}$J]