Solution of Tutorial-6 for PH11001 course
Spring 2020, IIT Kharapur
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Question 1.
Assume we have two string joining at the origin ($x = 0$). The velocity of the wave in the string on the left ($x < 0$) is $v_1 = 20\ m/s$ whereas the velocity of the wave at the right ($x > 0$) is $v_2 = 10\ m/s$. The string on the left has a wave with an amplitude of 3 cm and a wavelength of 1 m moving towards the junction.
a) What are the amplitude of the reflected and transmitted waves and the wavelength of the transmitted wave?
b) Calculate the ratio of the power transmitted to the power reflected.

Solution.
a) On transmission the frequency of the waves doesn't change. Thus we have

$$\frac{v_1}{\lambda_1} = \frac{v_2}{\lambda_2} \Rightarrow \frac{\lambda_2}{\lambda_1} = \frac{v_2}{v_1}$$

So, the wavelength of the transmitted wave $\lambda_2 = 0.5$ m (wavelength of reflected wav remains unchanged)
Now, if the amplitudes of the incident wave and the transmitted wave are A and C respectively then

$$\begin{aligned} C &= \left(\frac{2v_2}{v_2 + v_1}\right) A \\ \therefore C &= \frac{2 \times 3}{1+2} = 2 \quad \text{cm.} \end{aligned}$$

Thus, the amplitude of the reflected ray is given by

$$B = C - A = -1 \quad \text{cm} \quad \text{(the negative sign suggests, on reflection it suffers a } \pi \text{ phase change)}$$

The relation between μ (mass per unit length) and v is $v = \sqrt{\frac{T}{\mu}}$ (T is tension on the string)
If we consider tension to remain constant then

$$v_1^2 \mu_1 = v_2^2 \mu_2 \implies \frac{\mu_2}{\mu_1} = 4$$

b)

$$\frac{\text{Average power of transmitted wave}}{\text{Average power of reflected wave}} = \frac{\frac{1}{2}\mu_2 v_2 C^2}{\frac{1}{2}\mu_1 v_1 B^2} = 8$$

Question 2.
A beam of light enters a glass prism at an angle α and emerges into the air at an angle β. After passing through the prism, the beam is deviated from the original direction by an angle γ. Find the angle of the prism ϕ and refractive index of the material which it is made of.

Solution.

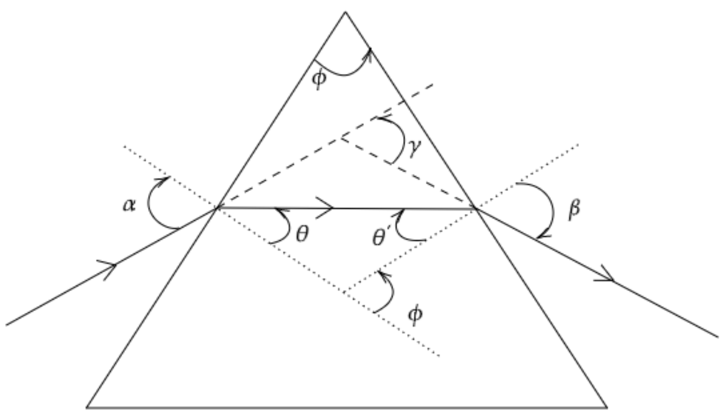


Figure 1: Ray diagram for refraction in a prism

From the Figure it is clearly evident that

- Refraction from first surface $n\sin\theta = \sin\alpha$
- Refraction from second surface $n\sin\theta' = \sin\beta$
- $\phi = \theta + \theta'$
- $\gamma = \alpha - \theta + \beta - \theta'$

Simplifying the last two conditions we get angle of prism to be

$$\boxed{\phi = \alpha + \beta - \gamma}$$

From the third expression we get

$$\begin{aligned}
\phi - \sin^{-1}\left(\frac{1}{n}\sin\alpha\right) &= \sin^{-1}\left(\frac{1}{n}\sin\beta\right) \\
\Rightarrow \sin\phi\sqrt{1-\left(\frac{1}{n}\sin\alpha\right)^2} - \cos\phi\left(\frac{1}{n}\sin\alpha\right) &= \left(\frac{1}{n}\sin\beta\right) \\
\Rightarrow \sin^2\phi 1 - \left(\frac{1}{n}\sin\alpha\right)^2 &= \frac{1}{n^2}(\sin\beta + \cos\phi\sin\alpha)^2 \\
\Rightarrow n^2 - \sin^2\alpha &= \frac{1}{\sin^2\phi}(\sin\beta + \cos\phi\sin\alpha)^2
\end{aligned}$$

So the refractive index can be written as

$$\boxed{n = \sqrt{\sin^2\alpha + \frac{1}{\sin^2\phi}(\sin\beta + \cos\phi\sin\alpha)^2}}$$

Alternatively for small angles we approximate $\sin\theta = \theta$ and $\sin^{-1}x = x$. After some algebric manupulation we get

$$\begin{aligned}
\beta &= \sin^{-1}(n\sin\theta') \\
\Rightarrow \beta &= \sin^{-1}(n\sin(\phi - \theta)) \\
\Rightarrow \beta &= \sin^{-1}(n\sin(\phi - \sin^{-1}\left(\frac{1}{n}\sin\alpha\right))) \\
\Rightarrow \beta &= n\sin\left(\phi - \frac{\alpha}{n}\right) \\
\Rightarrow \beta &= n\phi - \alpha \\
\therefore n &= \frac{\alpha + \beta}{\phi}
\end{aligned}$$

So the refractive index can be written as

$$\boxed{n = \frac{\alpha + \beta}{\alpha + \beta - \gamma}}$$

Question 3.
Consider 3 waves, $\psi_1 = 2\sin(\omega t + \pi/3)$; $\psi_2 = 3\cos(\omega t + \pi/4)$; $\psi_3 = 5\sin(\omega t + \pi/5)$. Find out the resultant amplitude and phase when they superpose.

Solution.
The resultant wave is given by

$$\begin{aligned}
\phi &= \phi_1 + \phi_2 + \phi_3 \\
\Rightarrow \phi &= (2\cos\pi/3 - 3\sin\pi/4 + 5\cos\pi/5)\sin\omega t + (2\sin\pi/3 + 3\cos\pi/4 + 5\sin\pi/5)\cos\omega t \\
\Rightarrow \phi &= \phi_0\cos\alpha\sin\omega t + \phi_0\sin\alpha\cos\omega t \\
\therefore \phi &= \phi_0\sin(\omega t + \alpha)
\end{aligned}$$

where the amplitude is given by

$$\begin{aligned}
\phi_0 &= ((\phi_0\cos\alpha)^2 + (\phi_0\sin\alpha)^2)^{1/2} \\
\Rightarrow \phi_0 &= \left(2^2+3^2+5^2+12\sin(\frac{\pi}{3}-\frac{\pi}{4})+20\cos(\frac{\pi}{3}-\frac{\pi}{5})+30\sin(\frac{\pi}{5}-\frac{\pi}{4})\right)^{1/2} \\
\therefore \phi_0 &= 7.39 \quad \text{unit}
\end{aligned}$$

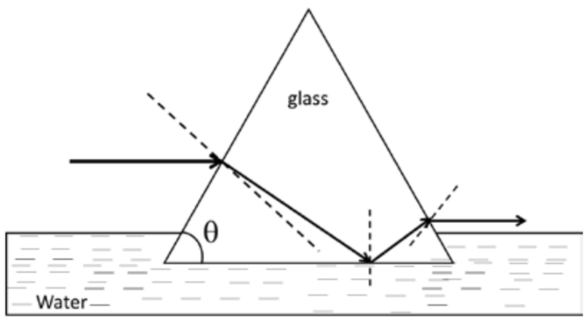
and the angle is given by

$$\begin{aligned}
\alpha &= \frac{\phi_0\sin\alpha}{\phi_0\cos\alpha} \\
\Rightarrow \alpha &= \tan^{-1}\left(\frac{(2\sin\pi/3+3\cos\pi/4+5\sin\pi/5)}{(2\cos\pi/3-3\sin\pi/4+5\cos\pi/5)}\right) \\
\therefore \alpha &= 67.74\,^{o}
\end{aligned}$$

Question 4.

a) Draw a graph of θ_t (angle of transmittance) vs θ_i (angle of incidence) for an air-glass boundary where refractive index for glass is $n_g = 1.5$.

b) A glass prism whose cross section is an isosceles triangle stands with its (horizontal base in water; the angles which its two equal sides make with the base are each equal to θ as shown in the figure 1. An incident ray of light, above and parallel to the water surface and perpendicular to the prism axis, is internally reflected at the glass-water interface and subsequently re-emerges into the air. Taking the refractive indices of glass and water to be 3/2 and 4/3 respectively find the angle θ.



Solution.

a) The graph of θ_t vs θ_i for an air-glass boundary is given in Figure

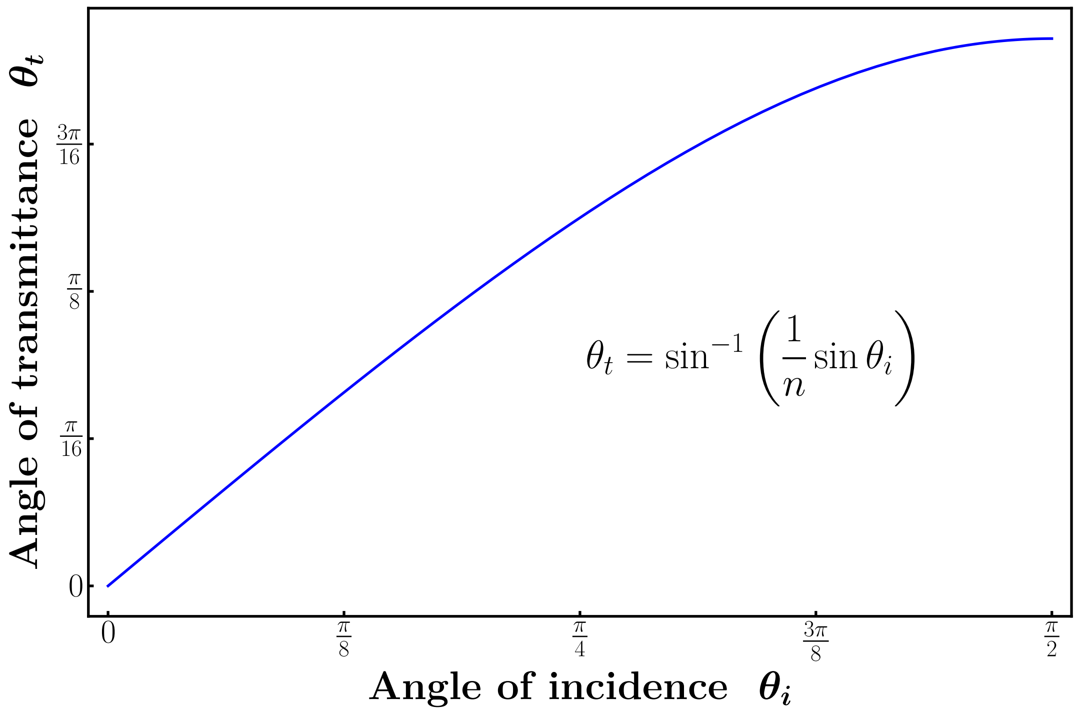


Figure 2: Variation of θ_t w.r.t θ_i

b) From the figure it is clear that the incident angle at the first surface and the emergence angle at the second surface is $90° - \theta$. Suppose the angle of refraction is θ_r. Using Snell's law of refraction at the first surface

$$\sin\theta_r = \frac{2}{3}\cos\theta$$

The incident angle at the glass water interface can be found from geometry to be $(\theta + \theta_r)$. The critical angle for the glass water interface is found to be

$$\sin\theta_c = \frac{4/3}{3/2} = \frac{8}{9}$$

So, for total internal reflection one must have

$$\begin{aligned}
(\theta + \theta_r) &> \theta_c \\
\Rightarrow \sin\theta \times \sqrt{1 - \frac{4}{9}\cos^2\theta} + \cos\theta \times \frac{2}{3}\cos\theta &> \frac{8}{9} \quad \text{(taking sin on both sides)} \\
\cos\theta &< \sqrt{\frac{17}{21}} \\
\therefore \theta &> \cos^{-1}\sqrt{\frac{17}{21}}
\end{aligned}$$

So the minimum allowed angle is given by $\theta = \cos^{-1}\sqrt{\frac{17}{21}} = 25.88°$.

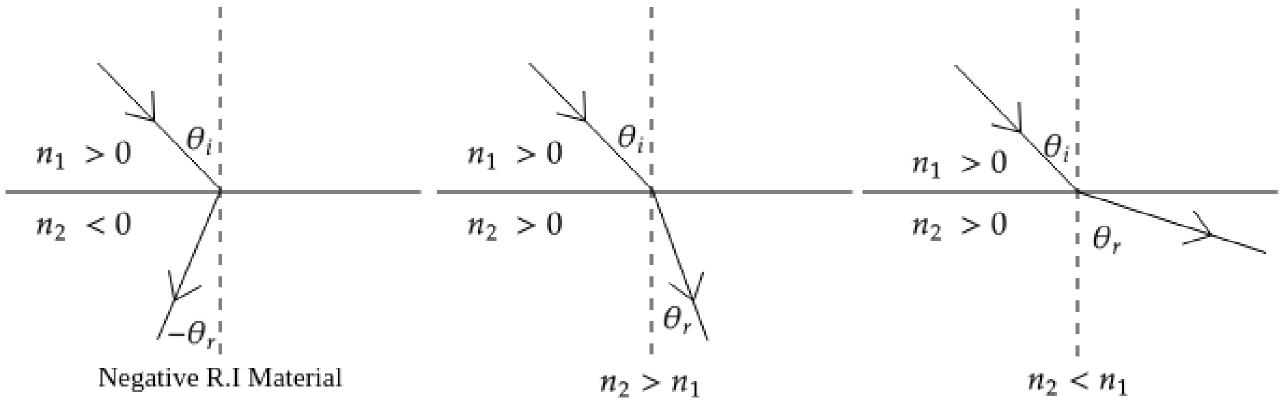
Question 5.

a) Consider Snell's law of refraction. If the medium of incidence has an index $n_1 > 0$ and the other medium has an index $n_2 < 0$, then draw the refracted ray by assuming some angle of incidence θ_i. What difference do you notice in comparison to the case when $n_1, n_2 > 0$?

b) Recall that the refractive index can be written as $n = \sqrt{\epsilon_r \mu_r}$ where $\epsilon_r = \frac{\epsilon}{\epsilon_0}$ and $\mu_r = \frac{\mu}{\mu_0}$. . It is known that ϵ and μ are complex quantities and also functions of ω, with their real and imaginary parts related to physical quantities. Show, with an example , that using $\epsilon < 0$ and $\mu < 0$ it is possible to have the refractive index $n < 0$.

Solution.

a) The refraction for all the cases are depicted in the following figure



b) In general the permitivitty and the permiability of a material can be a complex quantity i.e $\epsilon_r = |\epsilon_r|e^{i\phi_\epsilon}$ and $\mu_r = |\mu_r|e^{i\phi_\mu}$. So the refractive index can be ritten as

$$n = \sqrt{\epsilon_r \mu_r} = \sqrt{|\epsilon_r||\mu_r|}e^{i\dfrac{\phi_\epsilon + \phi_\mu}{2}}$$

When ϵ_r and μ_r are both real and negative then we have $\phi_\epsilon = \phi_\mu = \pi$. So, the refractive index is given by

$$\begin{aligned}
n &= \sqrt{|\epsilon_r||\mu_r|}e^{i\pi} \\
\therefore \quad n &= -\sqrt{|\epsilon_r||\mu_r|}
\end{aligned}$$