Introduction to the Physics of the Quantum Hall Regime

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Abstract

These lecture notes attempt to explain the main ideas of the theory of the quantum Hall effect. The emphasis is on the localization and interaction physics in the extreme quantum limit which gives rise to the quantum Hall effect. The interaction physics in the extreme quantum limit which is responsible for the fractional quantum Hall effect is discussed at length and from an elementary point of view.

I. WHAT IS THE QUANTUM HALL EFFECT?

The quantum Hall effect occurs in two-dimensional electron systems in the limit of strong perpendicular magnetic fields. Two-dimensional electron gas systems may be realized at the interface between semiconductors and insulators or at the interface between two different semiconductors as illustrated in Fig. [1]. The theory of the quantum Hall effect now takes as its subject all physical properties of two-dimensional electron systems in the limit where the magnetic field is so strong that the mixing of Landau levels by disorder or by electron-electron interactions may be considered as a weak perturbation. I will refer to this limit as the *quantum Hall regime.* Qualitatively [1], physical properties then depend only on the ratio of the disorder potential to the electron-electron interaction potential. These systems have unusual and interesting properties because there is no energy scale, like the band width of a periodic solid or the Fermi energy of an electron liquid in the absence of a magnetic field, which is associated with a simple one-body term in the Hamiltonian and can be used as the basis of a perturbation theory. Disorder and interactions, and in the general case both disorder *and* interactions have to be accounted for using some non-perturbative approach. Neither interactions nor disorder can ever be considered to be weak. The ratio between the energy scales corresponding to these two interactions is, however, important. The limit where the disorder potential is much stronger, is referred to as the *integer* quantum Hall regime whereas the limit where the interaction potential is much stronger, is referred to as the *fractional* quantum Hall regime. (A given sample may be in the integer quantum Hall regime at one magnetic field strength and in the fractional quantum Hall regime at another magnetic field strength.)

One of the challenges in probing the properties of a two-dimensional electronic systems experimentally is in isolating it from its three-dimensional semiconducting or insulating host. This isolation is most easily achieved in measurements of the electrical transport properties and this type of measurement has been the most widely used technique for studying two-dimensional electron systems. The quantum Hall effect was discovered [2] by Klaus von Klitzing, nearly fifteen years ago now, while performing measurements of the electrical transport properties of a two-dimensional electron gas system at the strong magnetic field facility in Grenoble. The experimental setup for these measurements is illustrated schematically in Fig. [2]. At weak magnetic fields the magnetotransport properties of two-dimensional electron gas are well described by the simple Drude theory, outlined in the following section. According to this theory the dissipative resistance R of the two-dimensional electron gas system should be proportional to n^{-1} where n is the areal density of the two-dimensional electron system, while the Hall resistance R_H should be proportional to B/n. What von Klitzing saw was dramatically different. His discovery signaled the occurrence of novel physical phenomena in the quantum Hall regime and engendered a large body of work which has led to a fairly complete understanding of many phenomena which had not even been anticipated prior to his experiments. These notes attempt to review a portion of that new knowledge.

II. DRUDE THEORY OF MAGNETOTRANSPORT

Before discussing what von Klitzing actually observed we briefly review the Drude theory [3] of magnetotransport in two-dimensions. The Drude theory gives results which are qualitatively correct outside of the quantum Hall regime. In the simplest version of this theory it is assumed that the electrons of the two-dimensional electron gas are accelerated by external forces between scattering events to reach a drift velocity,

$$
\begin{aligned}
\vec{v}_D &= \frac{\vec{F}\tau}{m^*} \\
&= -\frac{e\tau}{m^*}\left[\vec{E} + \frac{\vec{v}_D}{c}\,B \times \hat{z}\right]
\end{aligned}
\tag{1}
$$

where $\vec{F}$ is the force experienced by the electrons, m^* is the effective mass of the electrons and τ is the time between scattering events. The magnetic field enters this theory through the appearance of the Lorentz force on the moving electrons. In the Drude theory it is assumed that all electrons drift together so that the current density is related to the electronic drift velocity by

$$
\vec{j} = -ne\,\vec{v}_D.
\tag{2}
$$

Given these equations it is possible to solve for the electric fields present in the two-dimensional electron system given a uniform current density:

$$
\vec{E} = \frac{m^*}{ne^2\tau}\,\vec{j} + \frac{B}{nec}\,\hat{z} \times \vec{j}
\tag{3}
$$

The dissipative resistivity ρ_{xx} (or for a square sample in two-dimensions the resistance R) is given by the ratio of the electric field in the direction of current flow to the current density while the Hall resistivity ρ_{xy} (and the Hall resistance R_H) is given by the ratio of the electric-field component perpendicular to the direction of current flow to the current density. Notice that in the Drude theory ρ_{xx} is independent of magnetic field while ρ_{xy} is proportional to magnetic field. Similar results are obtained with more sophisticated semiclassical theories of electrical transport [3], with the added benefit that explicit expressions capable of quantitative accuracy are obtained for the scattering time, τ.

These theoretical results may be compared with typical experimental results obtained in the quantum Hall regime which are illustrated in Fig. [3] and Fig. [4]. For fields below ~ 0.1Tesla the Drude theory works well. At stronger fields, however, the Hall resistivity becomes nearly constant over certain finite intervals of magnetic field. Over the same magnetic field intervals the dissipative resistivity becomes very small. These features are labeled in Fig. [4] by a set of integers and fractions with odd denominators which we will presently see correspond to Landau level filling factors. In what follows I will attempt to explain why these anomalies in the transport properties of two-dimensional electron systems occur. We'll find that the quantum Hall effect reflects both novel disorder physics and novel interaction physics in the quantum Hall regime.

III. FREE 2D ELECTRON IN A MAGNETIC FIELD

Since the quantum Hall regime is defined by Landau quantization of the kinetic energy, any discussion of the quantum Hall effect must begin with the quantum mechanics of a single free-particle moving in two-dimensions in a perpendicular magnetic field. We'll find it useful to discuss the motion of classical particles in a magnetic field first.

A. Classical Solution

I start with the solution of the classical equations of motion. Here and throughout much of these notes it is convenient to use a complex number notation for two-dimensional vectors. In particular we'll let $z = x + iy$ represent the two-dimensional position vector and the complex number $v = v_x + iv_y$ represent the two-dimensional velocity vector. The classical equations of motion take a compact form when the complex number notation is used:

$$\left. \begin{array}{c} m\ddot{x} = -\frac{eB}{c} \dot{y} \\ m\ddot{y} = \frac{eB}{c} \dot{x} \end{array} \right\} \quad \ddot{z} = i\omega_c \dot{z} \tag{4}$$

Integrating twice we obtain

$$\dot{z} = v_0 e^{i\omega_c t} \tag{5}$$

$$z = C - \frac{iv_0 e^{i\omega_c t}}{\omega_c} \tag{6}$$

where v_0 is the complex number representing the velocity at time $t = 0$. Classical particles moving in two-dimensions in the presence of a perpendicular magnetic field execute circular (cyclotron) motion with an angular frequency $\omega_c = eB/m^*c$. The tangential velocity v_c and the radius for the cyclotron orbits are related by $R_c = v_c/\omega_c$. In Eq. (6) C is a complex integration constant which specifies the position vector for the center of the cyclotron orbit. Cyclotron orbit motion is illustrated in Fig. [5].

B. Quantum Solution

The Hamiltonian for an electron moving in two-dimensions in a perpendicular magnetic field is given by

$$H = \frac{\vec{\pi}^2}{2m^*} \tag{7}$$

where the kinetic momentum is given in a coordinate representation by

$$\vec{\pi} = -i\hbar\vec{\nabla} + \frac{e\vec{A}}{c}. \tag{8}$$

For a uniform magnetic field $\hat{z} \cdot (\vec{\nabla} \times \vec{A}) = B$ so that the vector potential $\vec{A}$ is a linear function of the spatial coordinates. It follows that H is a generalized two-dimensional

harmonic oscillator Hamiltonian which is quadratic in both the spatial coordinates and in the canonical momentum $\vec{p} = -i\hbar\vec{\nabla}$. The eigenstates and eigenvalues of this Hamiltonian may be obtained by a convenient algebraic method, analogous to the 'ladder operator' solution of the one-dimensional harmonic oscillator. The algebraic solution facilitates many calculations we will outline later in these notes. I start from the observation that the x and y components of the kinetic momentum are canonically conjugate coordinates:

$$[\pi_x, \pi_y] = \frac{-i\hbar e}{c}\, \hat{z} \cdot (\vec{\nabla} \times \vec{A}) = \frac{-i\hbar^2}{\ell^2} \tag{9}$$

where $\ell^2 = \hbar c/eB.$ ℓ is known as the magnetic length and is the natural length unit in the quantum Hall regime. In these notes we will always take ℓ as the unit of length although it will frequently be exhibited explicitly for clarity. Note that ℓ is related to the magnetic flux quantum by

$$2\pi\ell^2 B = \Phi_0 \tag{10}$$

I define the first set of ladder operators as follows:

$$a^\dagger \equiv \frac{\ell/\hbar}{\sqrt{2}}(\pi_x + i\pi_y) \tag{11}$$

so that

$$[a, a^\dagger] = 1 \tag{12}$$

and

$$H = \frac{\hbar\omega_c}{2}(aa^\dagger + a^\dagger a). \tag{13}$$

It follows from Eq. (13) that the eigenenergies for the Schrödinger equation of a free particle are $\hbar\omega_c(n + 1/2)$. However, just as the classical kinetic energy is independent of the center coordinate for the cyclotron orbit we might expect that each of these quantum eigenenergies will be degenerate. The degeneracy is revealed by constructing ladder operators from the quantum orbit-center operators:

$$C = z + \frac{i\pi}{m^*\omega_c}. \tag{14}$$

I first note that the x and y components of the cyclotron orbit centers are canonically conjugate coordinates:

$$[C_x, C_y] = i\ell^2. \tag{15}$$

This allows us to define a ladder operator by

$$b \equiv \frac{1}{\sqrt{2}\ell}(C_x + iC_y) \tag{16}$$

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It is readily verified that

$$[b, b^\dagger] = 1 \tag{17}$$

and that

$$[a, b] = [a^\dagger, b] = [H, b] = 0. \tag{18}$$

The cyclotron-orbit-center ladder operators produce a set of degenerate eigenstates of the one-body kinetic energy operator. The set of all eigenstates with a given allowed kinetic energy is called a Landau level. The full set of eigenstates can be generated by using raising operators starting from the bottom of the ladder:

$$|n, m\rangle = \frac{(a^\dagger)^n (b^\dagger)^m}{\sqrt{n!m!}} |0, 0\rangle$$

$$\varepsilon_n = \hbar\omega_c(n + \frac{1}{2}). \tag{19}$$

For many calculations it is useful to choose a specific gauge for the vector potential. In the symmetric gauge,

$$\vec{A} = \frac{B}{2}(-y, x, 0) \tag{20}$$

and the ladder operators may be written in the form,

$$b = \frac{1}{\sqrt{2}}\left(\frac{z}{2\ell} + 2\ell\frac{\partial}{\partial\bar{z}}\right) \tag{21}$$

$$b^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2\ell} - 2\ell\frac{\partial}{\partial z}\right) \tag{22}$$

$$a^\dagger = \frac{i}{\sqrt{2}}\left(\frac{z}{2\ell} - 2\ell\frac{\partial}{\partial\bar{z}}\right) \tag{23}$$

$$a = \frac{-i}{\sqrt{2}}\left(\frac{\bar{z}}{2\ell} + 2\ell\frac{\partial}{\partial z}\right). \tag{24}$$

(Throughout these notes we use an overbar to indicate complex conjugation.) It follows that the orbital wavefunction at the bottom of both the kinetic-momentum and orbit-center ladder operator chains, *i.e.*, the one which is annihilated by both a and b is

$$\psi_{0,0} = \frac{1}{\sqrt{2\pi\ell^2}} e^{-z\bar{z}/4\ell^2}. \tag{25}$$

The set of single-particle orbitals in the lowest Landau level can then be generated by repeated application of $b^\dagger$:

$$\psi_{0,m} = \frac{\bar{z}^m e^{-z\bar{z}/4\ell^2}}{\sqrt{2\pi\ell^2 2^m m!}}. \tag{26}$$

For the symmetric gauge it is convenient to change definitions and drop the factors of i and $-i$ which appear in the expressions for a and $a^\dagger$; this simply changes the phase convention for the higher Landau level wavefunctions.

I will make frequent use of these symmetric gauge wavefunctions and the ladder operator approach we used to obtain them. I start here by deriving an expression for the number of states per unit area in a single Landau level. I first note that the complex coordinates for the position vector can be expressed in terms of ladder operators as follows:

$$z = \sqrt{2}\ell(b + a^\dagger) \tag{27}$$
$$\bar{z} = \sqrt{2}\ell(b^\dagger + a). \tag{28}$$

It follows that

$$\pi\langle n, m|\bar{z}z|n, m\rangle = 2\pi\ell^2(m + n + 1). \tag{29}$$

For large m these orbitals are strongly localized to within $\sim \ell$ of a ring with an radius $r_{m,n} = \ell\sqrt{2(m + n + 1)}$ as can be verified by calculating $\langle n, m|\bar{z}^2z^2|n, m\rangle$ and comparing with the above result. When the lowest Landau level is filled by occupying all single-particle states with $n = 0$ and $m = 0, \cdots, N - 1$ it follows from Eq. (29) that in the thermodynamic limit the area occupied per electron is $2\pi\ell^2$. Each Landau level contains one single-particle state per magnetic flux quantum penetrating the area occupied by the electrons. It is the macroscopic Landau level degeneracy which creates the opportunity for unique physical phenomena in the strong magnetic field limit. Another way to obtain the same result is to note that

$$\sum_{m=0}^{N-1} |\psi_{0,m}(z)|^2 = (2\pi\ell^2)^{-1} \sum_{m=0}^{N-1} \frac{x^m}{m!} \exp(-x) \to (2\pi\ell^2)^{-1} \tag{30}$$

with $x \equiv |z|^2/2\ell^2$.

C. Useful Identities for Symmetric-Gauge Free-Particle Eigenfunctions

I now pause to mention and sketch the derivations of a number of identities involving the symmetric gauge free-particle eigenfunctions which will enter concrete calculations later in these notes. This section of the notes can be skipped on a first reading.

Plane Wave Matrix Elements

The matrix element of a plane wave $\exp(-i\vec{k} \cdot \vec{r})$ can be evaluated by expressing the position in term of ladder operators:

$$\langle n', m'|e^{-i\vec{k}\cdot\vec{r}}|n, m\rangle$$
$$= \langle n', m'|e^{-i\bar{k}(b+a^\dagger)/\sqrt{2}}e^{-ik(b^\dagger+a)/\sqrt{2}}|n, m\rangle$$
$$= e^{-|k|^2/2}\langle n'|e^{-i\bar{k}a^\dagger/\sqrt{2}}e^{-ika/\sqrt{2}}|n\rangle\langle m'|e^{-ikb^\dagger/\sqrt{2}}e^{-i\bar{k}b/\sqrt{2}}|m\rangle \tag{31}$$

The matrix element may be written as the product of a factor associated with the kinetic-momentum ladder operators and a factor associated with the orbit-center ladder operators.

This factorization of operators is frequently possible in the quantum Hall regime and we will find occasion to exploit it. Each factor may be evaluated using the algebra of the ladder operators. For example,

$$\langle m'|e^{-ikb^\dagger/\sqrt{2}}e^{-i\bar{k}b/\sqrt{2}}|m\rangle \equiv G_{m',m}(k)$$
$$= \left(\frac{m!}{m'!}\right)^{1/2}\left(\frac{-ik}{\sqrt{2}}\right)^{m'-m}L_m^{m'-m}\left(\frac{k\bar{k}}{2}\right) \tag{32}$$

where $L_m^{m'-m}$ is a generalized Laguerre polynomial. The second equality in Eq. (32) follows (for $m' > m$) after noting that non-zero matrix elements will arise only when the lowering operator acts no more than m times and the raising operator acts $m' - m$ times more often than the lowering operator. The result is a sum of m terms which can be compared with the definition of the generalized Laguerre polynomials. With this definition we have the result

$$\langle n', m'|e^{-i\vec{k}\cdot\vec{r}}|n, m\rangle = \exp(-|k|^2/2)G_{n',n}(\bar{k})G_{m',m}(k). \tag{33}$$

It is also useful to note from the above definitions that the projection of a plane wave onto the lowest ($n = 0$) Landau level is

$$e^{-i\bar{k}b/\sqrt{2}}e^{-ikb^\dagger/\sqrt{2}} \equiv B(k). \tag{34}$$

Except for the factor of $\exp(-|k|^2/2)$ which comes from interchanging the orders of the raising and lowering factors in Eq. (34) we see that $G_{n,m}(k)$ is the symmetric gauge single-particle eigenstate representation of the projected plane-wave operator.

Inversion

From the explicit form of Eq. (32) it follows that

$$G_{m',m}(-k) = (-)^{m'-m}G_{m',m}(k) \tag{35}$$

Symmetric Gauge Wavefunctions

By letting the two raising operators act repeatedly on $\phi_{0,0}(\vec{r})$ we find that

$$\langle\vec{r}|n, m\rangle \equiv \varphi_{n,m} = \frac{e^{-|z|^2/4}}{\sqrt{2\pi}}G_{m,n}(i\bar{z}) \tag{36}$$

The similarity of the plane-wave matrix elements and the orbital wavefunctions is reminiscent of the position-space, momentum-space duality which occurs for the one-dimensional harmonic oscillator.

Matrix Products We can derive an expression for the product of two $G_{m,m'}(k)$ matrices with different wavevector arguments by using the completeness of the ladder operator eigenstates:

$$\sum_\ell G_{m',\ell}(k_1)G_{\ell,m}(k_2) = \sum_\ell\langle m'|e^{-ik_1b^\dagger/\sqrt{2}}e^{-i\bar{k}_1b/\sqrt{2}}|\ell\rangle$$
$$\times\langle\ell|e^{-ik_2b^\dagger/\sqrt{2}}e^{-i\bar{k}_2b/\sqrt{2}}|m\rangle$$
$$= e^{-\bar{k}_1k_2/2}G_{m',m}(k_1 + k_2). \tag{37}$$

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The additional factor in Eq. (37) comes from interchanging raising and lowering factors after invoking completeness.

Hermitian Conjugate

Taking the complex conjugate of Eq. (32) we find immediately that

$$\bar{G}_{m',m}(k) = G_{m,m'}(-k) = (-)^{m-m'}G_{m,m'}(k) \tag{38}$$

This result is also obvious from the explicit expression for $G_{n,m}(k)$ in term of Laguerre polynomials.

Landau Level Degeneracy

I have previously derived a result for the number of states per unit area in the lowest Landau level. This result can be generalized to arbitrary Landau level using Eq. (37) and Eq. (36):

$$\sum_m |\varphi_{n,m}|^2 = \frac{e^{-|z|^2/2}}{2\pi} \sum_m G_{n,m}(-i\bar{z})G_{m,n}(i\bar{z})$$
$$= \frac{1}{2\pi\ell^2}G_{n,n}(0) = \frac{1}{2\pi\ell^2}. \tag{39}$$

This implies that the number of states per unit area per Landau level is $N_\phi = BA/\Phi_0$, the number of flux quanta which pass through the area of the system.

Full Landau Level Rule

The fact that the charge density of a full Landau level is a constant gives us a result for the trace of the plane-wave matrix elements:

$$\sum_{m=0}^{\infty} G_{m,m}(k) = \int d\vec{r} e^{-i\vec{k}\cdot\vec{r}} e^{|k|^2/2} (2\pi\ell^2)^{-1} = N_\phi \delta_{\vec{k},0}. \tag{40}$$

Orthogonality

We will have occasion to expand quantities of physical interest in terms of the plane-wave matrix elements. The following orthogonality relationship will be useful:

$$\int d^2\vec{k} e^{-|k|^2/2} G_{m',m}(k)G_{n',n}(\bar{k}) = \bar{\varphi}_{n',m'}(0)\varphi_{n,m}(0)$$
$$= \frac{\delta_{m',n'}\delta_{m,n}}{2\pi}. \tag{41}$$

This equation can be understood by recognizing the left-hand side as the Fourier expansion of the matrix element of $\delta(\vec{r})$ between symmetric gauge eigenstates. The second equality follows from Eq. (36) and the observation that $G_{n,m}(0) = 0$ for $n \neq m$.

IV. INCOMPRESSIBILITY

A. What is incompressibility?

In this section of the notes we will argue that incompressibility at zero temperature in the absence of disorder is a necessary condition for the occurrence of the quantum Hall

effect. Before beginning, however, it is useful to state more precisely what we mean by incompressibility. The compressibility (for a two-dimensional system at zero temperature) is defined as the relative area decrease per unit increase in pressure;

$$\kappa \equiv -\frac{1}{V}\frac{\partial V}{\partial P} \tag{42}$$

It is usually more convenient to calculate thermodynamic properties as a function of area rather than as a function of pressure so that the following expression is often more useful:

$$\kappa^{-1} \equiv -V\frac{\partial P}{\partial V} = V\frac{\partial^2 E}{\partial V^2}. \tag{43}$$

For systems of many particles the energy in the thermodynamic limit is an extensive quantity and the energy per particle depends on the area and the particle number only through the particle density, $E = N\epsilon(n)$ where $n = N/A$ is the areal density. This relationship allow us to connect the compressibility with the chemical potential; $\mu = \partial E/\partial N = d(n\epsilon(n))/dn$. Comparing with Eq. (43) we find that

$$\kappa^{-1} = n^2\frac{d\mu}{dn} \tag{44}$$

When we say that the system is incompressible we mean that $\kappa = 0$. This occurs whenever the chemical potential of the system increases discontinuously as a function of density. That is what we really mean by incompressibility. I argue below that the quantum Hall effect can occur whenever the system is incompressible in the absence of disorder at *magnetic field dependent densities.*

Whenever the ground state is incompressible the increase in energy when a particle is added to the system and the decrease in energy when a particle is removed from the system differ even in the thermodynamic limit. It follows that it costs a finite energy to create particle-hole pairs which are not bound to each other and are therefore able to carry current. In this circumstance we say that the system has a 'charge gap'. At this point we might be tempted to conclude that the chemical potential discontinuity can be measured by studying the temperature dependence of activated transport processes in the system. For several reasons this conclusion would be only partially correct. I will return to this point below.

B. Incompressibility Implies Quantization

I now present an argument for the following conclusion: *Incompressibility at T=0 in the absence of disorder is a necessary condition for the occurrence of the quantum Hall effect. Incompressibility at a magnetic-field dependent density always leads to gapless excitations localized at the edges of the system. When such an incompressibility exists in the absence of disorder, the quantum Hall effect will occur if any gapless excitations which occur in the bulk of the system are localized on a length scale small compared to the system size and disorder is not too strong.* Our argument is closely related to the thermodynamic argument

of Widom [4] and related arguments due to Laughlin [5] and Halperin [6]. At least for non-interacting electrons, similar conclusions follow from treatments based on linear response theory [7]. I believe that there are also close connections between the following argument and the beautiful topological quantum number picture of the fractional quantum Hall effect [8], although we don't have the space to pursue these connections here. There are difficulties with the argument which follows, and we will touch on some of these difficulties and how they might be circumvented later. However it is our belief that what follows is the essence of the quantum Hall effect.

I first consider a large but finite two-dimensional electron gas at zero temperature, as illustrated in Fig. [6], at a chemical potential which would fall in a 'charge gap'in the thermodynamic limit. I want to consider the change in the equilibrium local currents, present in the system because of the breaking of time-reversal-invariance by the magnetic field, when we make an infinitesimal change in the chemical potential, $\delta\mu$. I argue that because μ lies in a gap the change in the local current density anywhere in the bulk of the system must be zero. The current density can change, if it does anywhere, only at the edge of the system. It follows from charge conservation that, if there is a change in the current flowing along the edge of the system, it must be constant as we move along the edge. We can relate this change in current to the change in the orbital magnetization:

$$\delta I = \frac{c}{A}\delta M.$$ (45)

Eq. (45) is just the equation for the magnetic moment of a current loop. However,

$$\delta M = \frac{\partial M}{\partial\mu}|_B\delta\mu = \frac{\partial N}{\partial B}|_\mu\delta\mu.$$ (46)

The second equality in Eq. (46) follows from a Maxwell relation. Combining Eq. (45) and Eq. (46) we obtain the following result for the rate at which the equilibrium edge current changes with chemical potential when the chemical potential lies in a charge gap:

$$\frac{\delta I}{\delta\mu} = c\frac{\partial n}{\partial B}|_\mu.$$ (47)

Notice that whenever the charge gap occurs at a density which depends on magnetic field, there *must* be gapless excitations at the edge of the system. (These edge states form a chiral one-dimensional electron system with many interesting properties, especially in the fractional case [9–11]. Unfortunately space does not permit us to review that aspect of the fractional quantum Hall effect theory here.) From a more microscopic point of view, Eq. (47) arises because the edge currents are related to the way in which the spectrum evolves with changes in the the vector potential and hence in the magnetic field [6]. The density at which a charge gap occurs can depend on magnetic field only if states localized at the edge of the system cross the Fermi level as a function of magnetic field.

This property of the edge states is expected to persist even if the chemical potential lies only in a mobility gap and not in a true gap, as illustrated schematically in Fig. [7]. A net current can be carried from source to drain across the system by changing the local chemical potentials only at the edges and having different chemical potentials along the two

edges connecting source and drain. Because of localization the two edges and the bulk are effectively decoupled from each other. Eq. (47) also relates the chemical potential difference between the two edges, equal to eV_H, and the net current carried through the system. There is no voltage drop along an edge since each edge is in local equilibrium. Eq. (47), often called the Streda formula or the Widom-Streda formula, was first derived from the Kubo formula expression for the Hall conductivity by Streda and Smrcka [12] prior to the experimental discovery of the quantum Hall effect. However, its robustness over finite ranges of field when localization occurs was appreciated only after von Klitzing's discovery.

There are difficulties with this argument, most of which are shared with the elegant Landauer-Buttiker [13] edge-state picture of transport in the quantum Hall regime. (The two pictures are essentially equivalent except that our argument removes unnecessary details specific to the case of non-interacting electrons. On the other hand the Landauer-Buttiker picture allows for a natural description of deviations from the quantum Hall effect in finite-size non-interacting electron systems.) The principle difficulty with this explanation is that it would appear to break down when eV_H becomes comparable to the charge gap or the mobility gap. We know that this is not the case, since accurately quantized Hall conductances are seen experimentally when eV_H is hundreds of times larger than $\hbar\omega_c$. Furthermore, the argument appears at first sight to depend on the assumption that all the transport current flows at the edges of the system, an assumption which is certainly not correct in general [14]. These apparent deficiencies can be explained away by appealing to locality properties [15] of the conductivity when the Fermi level lies in a region of localized states.

The fact that it is possible to offer various seemingly different explanations of the quantum Hall effect often creates some confusion. In closing this section, we wish to emphasize that the different explanations are really different points of view on the same physics. For example, the quantization of the Hall conductance is often discussed in terms of the dependence of properties of the system on boundary conditions or on Aharanov-Bohm fluxes in different geometries. Because of localization at edges dependencies on flux are equivalent to dependencies on magnetic field. For states localized at a particular edge of the system dependencies on flux are indistinguishable from changes in the magnetic field strength. A thorough discussion of the connections between the various approaches taken to explain the quantum Hall effect would be a good subject for another set of lectures. However that is not the main subject of these lectures. Instead, we want to emphasize what the occurrence of the quantum Hall effect tells us about the electronic properties of a two-dimensional system in which it occurs. It tells us that the system has an incompressibility at a magnetic-field-dependent density in the absence of disorder and that on the Hall plateau all gapless excitations in the bulk are localized.

V. INTEGER QUANTUM HALL REGIME

The main focus of these notes will be on the fractional Hall regime. I first briefly summarize some highlights of the physics of the integer quantum Hall regime. In the integer quantum Hall regime electron-electron interactions are weak compared to the electron-disorder interaction. Experimentally it appears that interactions do not play an essential role except in the extremely high mobility samples where the fractional quantum Hall effect occurs, so

the integer quantum Hall regime is certainly realizable experimentally. I have considered the non-interacting problem in the absence of disorder in detail in a previous section. We found that the single-particle energies are grouped into macroscopically degenerate Landau levels. The kinetic eigenenergy for states in the n-th Landau level is $\hbar\omega_c(n+1/2)$ and the number of states per unit area in the $n-$th Landau level is $(2\pi\ell^2)^{-1}$. It is usual and customary in the quantum Hall regime to measure the charge density in terms of the charge density of a single Landau level. The Landau level filling factor is defined by

$$\nu \equiv 2\pi\ell^2 n. \tag{48}$$

In the absence of disorder discontinuities occur in the chemical potential of a system of non-interacting electrons at the magnetic-field-dependent densities corresponding to integer values of ν.

I have argued in (IV B) that at zero temperature the Hall conductance will be quantized at $V_H/I \equiv G_H = je^2/h$ and that the dissipative conductance will be zero over the range of filling factors surrounding $\nu = j$ where the localization length is finite. Thus experiments on the integerquantum Hall effect tell us first of all that there is an incompressibility at a magnetic field dependent density in the absence of disorder and from the width of the Hall plateau they tell us the range of filling factors over which states at the Fermi level are localized. The fact that the Hall conductance is finite tells us that in contrast to the zero magnetic field situation not all states are localized. Very early experiments [16] by Paalanen *et. al.* showed that in the integer Hall regime the plateau width approaches the full width of the Landau level as the temperature approaches zero, implying that extended states occur at a single critical energy E_c within each Landau level. Later experiments [17,18] pioneered by Wei and collaborators were able to extract information about the way in which the localization length, ξ, diverges at the critical energy, E_c within each Landau level. The experiments are actually performed in samples with essentially constant density and the position of the Fermi energy within a Landau level is altered by changing the magnetic field so that what is studied is

$$\xi(E_F, B) \sim (B - B_c(E_F))^{-\nu}. \tag{49}$$

(Here ν is *not* the filling factor. If you are bothered by this unfortunate notation you may be consoled by reflecting on the rich weaving of strands in the tapestry of physical theory which it reflects. Since E_c is a linear function of B the same critical exponent applies for the dependence of the localization length on energy at fixed field.) The analysis of these experiments is based on the notion that at finite temperature electron-electron or electron-phonon scattering introduces another relevant length scale L_ϕ which is expected to diverge with a power law as the temperature goes to zero:

$$L_\phi \sim T^{-p/2} \tag{50}$$

As indicated by the notation used L_ϕ is usually thought of as a phase coherence length of diffusing electrons, although this notion is not as precise here as it is at weak magnetic fields where it plays an equally important role in the theory of quantum corrections to Boltzmann transport properties [19]. In this picture the time between inelastic scattering events which

destroy phase coherence diverges as T^{-p} at low temperatures. At finite temperature, dissipation is expected to occur and the Hall conductance is expected to vary with electron density or with field only when $\xi(E_F)$ exceeds L_ϕ. The experiments [17] of Wei *et. al.* showed that the width of field regime where dissipation occurs vanishes like T^κ where $\kappa \approx 0.42$. The result from experiment, then, is that

$$\kappa = p/2\nu \approx 0.42. \tag{51}$$

The principle theoretical problem in the integer Hall regime has been to understand these experimental results. The fact that extended states occurred at a single energy with each Landau level was initially explained [20–23] by considering the limit of disorder potentials which are smooth on the scale of the magnetic length. In this limit the eigenstates are localized along equipotential contours and the localization length is the typical size of a closed equipotential contour. At low energies the equipotential contours will surround minima in the random potential which is assumed to have some finite correlation length. The equipotentials are most easily visualized by considering the energy of interest as a Fermi energy so that the equipotential surrounds regions which have been filled by lower energy electrons. For a given finite system an energy, E_c^- must eventually be reached where the filled region first extends from one side of the sample to the other and the localization length therefore reaches the sample size. We can perform a similar analysis starting from high energies where the equipotentials at the Fermi energy surround areas where the electron states are unoccupied and the localization length is finite. As the Fermi energy is lowered an energy, E_c^+ must eventually be reached where the unoccupied region extends across the sample so that the localization length reaches the sample size. It is clear that $E_c^+ \geq E_c^-$ since the occupied area and the unoccupied area together comprise the entire area of the sample. One or the other must extend from one side of the sample to the other. For $E_c^- < E_F < E_c+$ there exist both regions of unoccupied states and regions of occupied states which extend from one side of the sample to the other. It is clear that this is possible only when the sample size is comparable to the size of typical equipotential contour. In the limit of an infinite sample $E_c^- = E_c^+ = E_c$. Using results from percolation theory Trugman has shown that the size of the typical equipotential contour $\xi \sim |E - E_c|^{-\nu}$ with $\nu = 4/3$.

The diverging localization length at E_c suggests that the localization behavior can be viewed as a quantum ($T = 0$) critical phenomenon and that it should therefore be independent of microscopic details such as the nature of the disorder potential. On the scale of the diverging localization length any disorder potential eventually appears to be rough. This observation has motivated field-theoretical [24] treatments of transport at low temperatures near $\nu = 1/2$. The above results, although they were derived for a smooth disorder potential, should apply for any disorder potential. Numerical calculations have been able to test this hypotheses in considerable detail although serious questions remain. Early work by Ando [25] convincingly established that even for zero correlation length disorder potentials the localization length diverges at a single energy within each Landau level. Paradoxically the numerical calculations have been most convincing for short correlation length disorder potentials rather than for smooth disorder potentials, presumably because the localization length in the latter case exceeds the system sizes at which calculations are possible. Later finite-size-scaling analyses [26,27] were able to convincingly establish that the localization

length diverges with critical exponent $\nu = 7/3$, in disagreement with the percolation theory results. It is possible to explain [28] the discrepancy as a correction due the possibility of tunneling between equipotential contours near saddle points.

VI. INCOMPRESSIBILITY AT FRACTIONAL FILLING FACTORS

We are interested in understanding how electron-electron interactions give rise to incompressibilities at fractional values of the Landau level filling factor ν. We need to learn to treat interactions between electrons which are confined to a single Landau level, usually the lowest ($n = 0$) Landau level. It turns out that some important lessons arise from analyzing the two-body problem in the fractional Hall regime and so we will start there. All of the discussion in this section is based on the symmetric gauge discussed previously. I will assume in these notes that the electrons have been completely spin-polarized by the magnetic field. The physics of spin or other discrete additional degrees of freedom in the fractional quantum Hall regime forms a large and interesting subject [29] which we are not able to review here.

A. Two body problem: Haldane pseudopotentials

We want to find the spectrum of the two-body Hamiltonian. The two-body wavefunctions could be expanded in terms of $N = 2$ Slater determinants formed from antisymmetric products of two single-particle states. If the electrons are confined to a finite area containing N_ϕ units of magnetic flux the number of two-body eigenstates is $\sim N_\phi^2/2$. In the absence of interactions each eigenstate has an eigenvalue $\hbar\omega_c$; since the kinetic energy per particle is trivial constant we usually absorb this constant into the zero of energy and the Hamiltonian then consists of only the interaction term which lifts the degeneracy of all two-body states. To solvethe interacting two-body problem it is useful to transform from the representation of free individual particle wavefunctions to the representation of free center-of-mass and relative eigenstates. Using the complex number representation of two-dimensional coordinates

$$Z \equiv (z_1 + z_2)/2 \tag{52}$$
$$z \equiv z_1 - z_2. \tag{53}$$

Here and below we use the upper case for quantities associated with the center-of-mass coordinate and the lower case for quantities associated with the relative coordinate. I define [30] orbit center ladder operators for the center-of-mass and relative states by

$$b_R^\dagger = \frac{b_1^\dagger - b_2^\dagger}{\sqrt{2}}$$
$$b_r^\dagger = \frac{b_1^\dagger - b_2^\dagger}{\sqrt{2}} \tag{54}$$

so that $[b_r, b_r^\dagger] = [b_R, b_R^\dagger] = 1$ and $[b_r, b_R] = [b_r, b_R^\dagger] = 0$. It is easy to verify from Eq. (24) that b_R involves only Z and that its coordinate space form is identical to that of the individual particle eigenstates except for the replacements $z_i \to Z$ and $\ell \to \ell_R \equiv \ell/\sqrt{2}$. Similarly

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b_r involves only z and the effective magnetic length for the relative motion eigenstates is $\ell_r = \sqrt{2}\ell$. The two-body state in the lowest Landau level with center-of-mass angular momentum M and relative angular momentum m is:

$$|M\rangle_R |m\rangle_r = \frac{(b_R^\dagger)^M (b_r^\dagger)^m}{\sqrt{M!m!}} |0,0\rangle \tag{55}$$

Comparing Eq. (19), Eq. (55) and Eq. (54) it is easy to derive an explicit expression for the unitary transformation relating the two representations for the two-body problem. Including the kinetic energy part, the two-body Hamiltonian may be written as

$$\mathcal{H} = \hbar\omega_c(1 + a_1^\dagger a_1 + a_2^\dagger a_2) + V(\vec{r}_1 - \vec{r}_2)$$
$$\hbar\omega_c(1 + a_R^\dagger a_R + a_r^\dagger a_r) + \sum_m{}' |m\rangle_{rr}\langle m|V|m\rangle_{rr}\langle m|$$
$$\equiv \sum_m{}' V_m P_m^{1,2}. \tag{56}$$

Here $P_m^{1,2}$ projects particles 1 and 2 onto a state of relative angular momentum m. (Note that the electrons are restricted to states of odd relative angular momentum by the antisymmetry requirement on the two-body wavefunction.) The second form for the right-hand side of Eq. (56) follows from the observation that the interaction term in the Hamiltonian acts only on the relative-motion degree-of-freedom. From the assumption that the interactions are isotropic, there is no coupling between states of different total angular momenta. We see that the Hamiltonian is completely specified by a set of numbers V_m which are simply the interaction energies for pairs of particles with relative angular momentum m. This parameterization of the Hamiltonian was first introduced by Haldane [31] and these numbers are known as Haldane pseudopotentials. As we will discuss further below, most of the physics of interacting electrons in the lowest Landau level is controlled by the few smallest pseudopotentials and the fractional quantum Hall effect occurs when we have sufficiently short-ranged repulsive interactions. Thus the problem of interacting electrons depends on a finite number of distinct important energy scales and it is this property which opens up the possibility of chemical potential jumps due to electron-electron interactions. For short-range repulsive interactions $V_m > 0$ and $V_{m'} < V_m$ for $m' > m$ since larger relative angular momenta relative wavefunctions are peaked at larger values of the relative coordinate where the interaction is weaker. We will find it useful in what follows to discuss what for many purposes can be considered as the ideal fractional quantum Hall system, the hard core model, for which $V_1 \neq 0$ and $V_i = 0$ for $i \neq 1$.

With interactions, then, the two-body spectrum consists of a set of eigenvalues $E_i = V_{2i-1}$ with degeneracy $g_i = 1 + 2(N_\phi - i)$ for $i = 1, 2, \cdots, N_\phi$. The degeneracy g_i is determined by the requirement that the sum of the relative angular momentum and the single-particle angular momentum cannot exceed $2N_\phi$. It is readily verified that the total number of states is $\sim N_\phi^2/2$.

B. Some properties of Haldane pseudopotentials

I pause here to establish some useful relations involving Haldane pseudopotentials. Firstly we note that it is possible to derive an explicit expression for the Haldane pseudopotentials in terms of the Fourier transform of the electron-electron interaction.

$$
\begin{aligned}
V_m \equiv {}_r\langle m|V|m\rangle_r &= \int \frac{d^2\vec{q}}{(2\pi)^2} V(\vec{q})_r \langle m|e^{i\vec{q}\cdot\vec{r}}|m\rangle_r \\
&= \int \frac{d^2\vec{q}}{(2\pi)^2} V(\vec{q}) e^{-q^2} L_m(q^2).
\end{aligned}
\tag{57}
$$

The final form for the right-hand side of Eq. (57) follows from Eq. (32) and the observation that the center of mass and relative wavefunctions are identical apart from the replacement $\ell \to \ell_r = \sqrt{2}\ell$. For the case of an ideal two-dimensional electron gas $V(q) = 2\pi e^2/q$ and from Eq. (57)

$$
V_m = \frac{e^2}{\ell}\sqrt{\frac{\pi}{4}}\frac{(2m-1)!!}{2^m m!}.
\tag{58}
$$

(This result is actually most easily obtained by doing the integral for the pseudopotential directly in real space.) Note that for large m the Coulomb V_m approaches $(e^2/\ell)/2\sqrt{m}$ as expected since the relative wavefunction for large m is strongly peaked at a relative separation of $2\ell\sqrt{m}$. Eq. (57) can be inverted to obtain an expression for $V(\vec{q})$ in term of the Haldane pseudopotentials by using a special case of Eq. (41);

$$
V(\vec{q}) = 4\pi \sum_m V_m L_m(q^2).
\tag{59}
$$

This equation is often useful for realizing the hard-core model in numerical calculations.

In these notes we discuss explicitly only the case of the fractional quantum Hall effect in the lowest Landau level of a semiconductor with an isotropic band structure and in a magnetic field which is perpendicular to the two-dimensional layer. It is not, in fact, necessary to be so restrictive. For example the fractional quantum Hall effect can occur when the $n-1$ lowest Landau levels are completely filled and we need to consider the effect of electron-electron interactions among electrons confined to the n-th Landau level. The interaction terms are then completely specified by the two-body matrix elements in the $n-th$ Landau level. Looking at a particular Fourier component of the interaction potential we note that

$$
\begin{aligned}
&\langle n, m'_1 : n, m'_2 | e^{i\vec{q}\cdot(\vec{r}_1-\vec{r}_2)} | n, m_1, : n, m_2 \rangle \\
&= [L_n(q^2/2)]^2 \langle m'_1, m'_2 | e^{i\vec{q}\cdot(\vec{r}_1-\vec{r}_2)} | m_1, , m_2 \rangle
\end{aligned}
\tag{60}
$$

where the $n = 0$ Landau level index is left implicit on the right hand side. Thus the fractional quantum Hall effect in the n-th Landau level with interaction potential $V(\vec{q})$ is equivalent to the fractional quantum Hall effect in the $n = 0$ Landau level with interaction potential $(L_n(q^2/2))^2 V(\vec{q})$. Anisotropy in the band structure [32,33], and the complications associated with the degeneracy [34] at the top of the valence band in $GaAs$ can be accounted

for with similar ease. An interesting example of the quantum Hall effect occurs when the magnetic field is tilted away from the normal to the two-dimensional layer. In this case [35], the system is equivalent to a system with a perpendicular magnetic field but with a non centro-symmetric interaction. The concept of a Haldane pseudopotential must then be generalized to allow matrix elements which are not diagonal in relative angular momentum and one might hope that the physics would be changed in important ways. However, there have been numerous experimental studies [36] of the fractional quantum Hall effect in tilted magnetic fields and nothing very dramatic occurs as long as the spin degree of freedom does not play a role.

C. Incompressibility in the Hard-Core Model and Laughlin Wavefunctions

Now we are in a position to consider many interacting electrons. The Hamiltonian is

$$H = \sum_{i<j} \sum_m V_m P_m^{ij} \tag{61}$$

The occurrence of chemical potential jumps at fractional Landau level filling factors (nearly) follows from the following statement. *In the thermodynamic limit the ground state energy of the hard-core model is zero for $\nu \leq 1/3$ and non-zero for $\nu > 1/3$.* The proof of this statement follows.

Any many-body wavefunction formed completely from electrons in the lowest Landau level must be a sum of products of the lowest Landau level one-body wavefunctions. It follows that many-body wavefunctions must take the form:

$$\Psi[z] = P(z_1, z_2, \cdots, z_N) \prod_k \exp\left(-|z_k|^2/4\right) \tag{62}$$

where $P[z]$ is a polynomial in each of the electronic coordinates. The hard-core model Hamiltonian is non-negative so that its lowest possible eigenenergy is zero. Let's assume that $\Psi[z]$ is an eigenstate of the hard-core model with eigenenergy zero. Then $\Psi[z]$ must be an eigenstate of P_1^{ij} with eigenvalue zero for any pair of particles i and j. The dependence of $\Psi[z]$ on z_i and z_j can be reexpressed in terms of the relative and center-of-mass coordinates for this pair of particles:

$$z_{ij} \equiv z_i - z_j$$
$$Z_{ij} \equiv (z_i + z_j)/2 \tag{63}$$

We can expand

$$P[z] = \sum_m z_{ij}^m F_m \tag{64}$$

where F_m depends on Z_{ij} and on the positions of all the other particles. Only odd values of m appear in the sum. In order for $\Psi[z]$ to be annihilated by the hard-core model F_1 must be identically zero. It follows that z_{ij}^3 is a factor of $P[z]$ so that

$$P[z] \propto (z_1 - z_2)^3 (z_1 - z_3)^3 \cdots (z_{N-1} - z_N)^3 \tag{65}$$

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and hence that the maximum power to which each coordinate appears in $P[z]$ is at least $3(N-1)$. Since the single-particle orbital with angular momentum m is localized along a ring which encloses area $2\pi\ell^2(m+1)$, it follows that the area per electron for the state represented by $\Psi[z]$ for large N is at least $3(2\pi\ell)^2$ or that $\nu^{-1} \geq 3$. This is what we wanted to show. We will have further occasion to relate the area per electron in the system to the degree of $P[z]$. In particular it is useful to note that multiplying polynomials corresponds to adding areas and hence to adding inverse filling factors.

The most spatially compact zero-energy eigenstate of the hard-core model is the one for which the relation sign in Eq. (65) is an equal sign. This wavefunction is known as a Laughlin wavefunction [37]. More precisely, the total angular momentum is a good quantum number for interacting electrons in the symmetric gauge and the total angular momentum is equal to the homogeneous degree of the polynomial $P[z]$. When the relation in Eq. (65) is an equality the degree of $P[z]$ is $M = 3N(N-1)/2$. Incidentally, the area per electron for large N can also be deduced from the total homogeneous degree of a polynomial under the assumption, implicit above in any event, that the electron density is uniform except near the edge of the system. The proof of the relation, $A/N = 2M/N^2$, is left as an exercise. For $M = 3N(N-1)/2$ the Laughlin wavefunction is the only zero-energy eigenstate of the hard-core model. The wavefunction can be generalized to higher powers of $z_i - z_j$:

$$\Psi_m^L[z] = \prod_{i<j}(z_i - z_j)^m \prod_k \exp\left(-|z_k|^2/4\right). \tag{66}$$

These wavefunctions were first suggested as trial wavefunctions for the many-electron ground state at $\nu = 1/m$ by Laughlin [37] and were recognized as exact eigenstates of the hard-core model by Trugman and Kivelson [38].

For $M = 3N(N-1)/2$, it follows from the above that $E_0(N) = E_0(N-1) = 0$ but $E_0(N+1) > 0$. We can define $\mu_N^- \equiv E_0(N) - E_0(N-1)$ and $\mu_N^+ \equiv E_0(N+1) - E_0(N)$ keeping $M = 3N(N-1)/2$. $\lim_{N\to\infty} \mu_N^\pm \equiv \mu^\pm$ gives the chemical potential at Landau level filling factors infinitesimally larger and infinitesimally smaller than $1/3$. Evidently, for the hard-core model $\mu^- = 0$. μ^+ is not known analytically and it may not be immediately evident that it is finite. However all variational approximations which have been explored for $E_0(N+1)$ give finite energies even in the limit $N \to \infty$ and it seems clear that the added electron has a finite probability of being in a state of relative angular momentum 1 with one of the existing electrons. I discuss one such variational wavefunction in the next section. Numerical exact diagonalization calculations [39] have also convincingly supported this conclusion. There is a chemical potential jump of $\sim V_1$ at $\nu = 1/3$ and, as we've stated previously, this leads to the fractional quantum Hall effect.

The hard-core model is not a realistic model for electron-electron interactions in a two-dimensional electron gas. It is useful to think of a mapping between Hamiltonians and vectors whose values are lists of Haldane pseudopotentials. The hard-core model and the model for the real system are two points in this vector space. I have argued above that there is a chemical potential jump at $\nu = 1/3$ for the hard-core model because of the impossibility of avoiding states of relative angular momentum 1 between pairs of electrons for $\nu > 1/3$. The chemical potential jump should be a smooth function of the Haldane pseudopotentials and there should therefore be a finite volume surrounding the hard-core model point in this vector space where the jump remains finite. Experiments showing the fractional quantum

Hall effect at $\nu = 1/3$ can be interpreted as proof that the point in the pseudopotential vector space describing the physical system lies inside this volume. Numerical exact diagonalization calculations also convincingly indicate that this should be the case [40].

VII. LAUGHLIN STATE PROPERTIES

A. Fractionally Charged Quasiparticles

Perhaps the most exotic property of incompressible states at fractional Landau level filling factors, is the fact that they have fractionally charged excitations. Below we give an argument for the occurrence of fractionally charged quasiparticles which is closely related to the one given by Laughlin in his classic paper on the fractional quantum Hall effect [37]. I consider a system with an incompressible ground state at a Landau level filling factor ν; we assume that $\nu = q/p$ is rational since the case of immediate interest is $\nu = 1/3$ and, in any event, incompressibilities can occur only at rational filling factors as far as we know. I imagine piercing the system [41] with an infinitely thin solenoid located at a point we take to be the origin, as illustrated in Fig. [8]. The flux through the solenoid is slowly changed from 0 to $\Phi_0 = hc/e$ at which point it is invisible to the electrons and the solenoid can be removed. The state generated in this way must be an eigenstate of the many electron system. However we can show that this state has charge eq/p added or removed from an area surrounding the origin. According to the Faraday induction law the time dependent flux gives rise to an electric field

$$E_\phi = \frac{1}{c} \frac{d\phi}{dt} \frac{1}{2\pi R} \tag{67}$$

directed azimuthally around the solenoid. Since the ground state has an incompressibility at filling factor ν it has (on a sufficiently long length scale) no dissipative conductance and a Hall conductivity $\sigma_{xy} = \nu e^2/h$. It follows that, if we look along a ring far enough from the origin, the azimuthal electric field gives rise to a radial current,

$$j_r = \nu \frac{e^2}{h} \cdot \frac{1}{c} \frac{d\phi}{dt} \cdot \frac{1}{2\pi R}. \tag{68}$$

Thus when the solenoid is removed the total change in the charge inside the ring is

$$Q = 2\pi R \int j_r \, dt = \nu \frac{e^2}{h} \cdot \frac{1}{c} \cdot \frac{hc}{e} = \nu e. \tag{69}$$

We have generated an excited state with charge νe localized within the microscopic length scale, ℓ, of the origin.

For $\nu = 1/3$ this procedure gives rise to quasiparticles with charges $\pm e/3$ created at fixed total electron number. For a fixed number of electrons the charge comes from the edge of the system, which is removed to infinity in the thermodynamic limit. In general for $\nu = q/p$ this procedure generates localized charge $\pm p/q$; the argument cannot determine the number of fractionally charged quasiparticles created by this procedure. In general it is expected that the procedure generates q quasiparticles of charge $1/p$ located near the origin. For fixed area

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we can generalize this procedure by creating equal numbers of independent quasielectrons and quasiholes. The activation energy observed in transport experiments is the energy to make free quasiparticle-quasihole pairs, Δ. For $\nu = p/q$ this energy will be $1/q$ as large as the chemical potential gap, Δ_μ:

$$\Delta = \Delta_\mu/q \equiv (\mu^+ - \mu^-)/q. \tag{70}$$

Explicit trial wavefunctions for states with a single quasielectron or a single quasihole can be constructed by executing Laughlin's *gedanken* experiment [42]. The quasielectron and quasihole states differ qualitatively, as we can see from the following simple cartoon which contains a considerable degree of truth. Laughlin's *gedanken* experiment for generating quasiparticles has the effect of increasing the angular momentum of each particle by 1, in the case of quasiholes, or in the case of quasielectrons of decreasing the angular momentum of each particle by 1. We can estimate the expectation value for the number operator for each angular momentum in the quasielectron and quasihole states. In the incompressible ground state $\langle n_k \rangle_0 = 1/m$ for all k. In the quasihole state then $\langle n_k \rangle_{qh} = \langle n_{k-1} \rangle_0$ for $k \geq 1$ and $\langle n_0 \rangle_{qh} = 0$. The missing charge is all in the $k = 0$ state. For quasielectrons we have to consider the effect of decreasing the angular momentum by one unit. The $m = 0$ state is then raised to the $n = 1$ Landau level and all the part of the wavefunction for which $m = 0$ was initially occupied will be projected away by the *gedanken* experiment. The wavefunction then needs to be renormalized. The ground state has the property that whenever the $m = 0$ state is occupied the states with $k = 1, \cdots, m - 1$ must be unoccupied. Otherwise an electron at the origin would have a finite probability of having relative angular momentum less than m with one of the other electrons. In our cartoon for the quasielectron state we ignore correlations in $\langle n'_k n_k \rangle_0$ other than those between $k = 0$ and $k = 1, \cdots, m - 1$. Then, since the states with $k = 1, \cdots, m - 1$ are unoccupied when $k = 0$ is occupied and they are occupied with overall probability $1/m$ they must be occupied with probability $1/(m - 1)$ when the $k = 0$ state is empty. It follows that in the quasielectron state $\langle n_k \rangle_{qe} = 1/(m - 1)$ for $k = 0, \cdots, m - 2$ and is $1/m$ otherwise. The excess charge near the origin is $1/m$. Note that the quasielectron is localized over a distance $\sim \sqrt{m}\ell$ whereas the quasihole is localized over a distance $\sim \ell$. Also note that in the quasielectron state $\langle n_1 n_0 \rangle \sim (m - 1)^{-2}$ independent of the number of electrons. An electron at the center of the quasielectron has a finite probability of interacting via a hard-core model so that the chemical potential jump is finite as claimed previously.

In closing we mention another trial wavefunction for the quasihole state which was suggested in Laughlin's original [37] paper:

$$\psi_{qh}^L = \prod_k z_k \prod_{i<j}(z_i - z_j)^m \prod_\ell \exp\left(-|z_\ell|^2/4\right) \tag{71}$$

This trial wavefunction for a quasihole is similar but not identical to the one discussed above. It has the advantage that some of its properties can be evaluated using an analogy with classical Coulomb plasmas which we discuss next.

B. Plasma Analogy

Some properties of the Laughlin wavefunctions for incompressible ground states and for fractionally charged quasihole states can be calculated by exploiting an analogy to classical two-dimensional plasmas. The analogy is based on interpreting the coordinate representation quantum mechanical distribution function, *i.e.*, the square of the many-electron wavefunction, as the canonical ensemble distribution function for a classical system of interacting particles. For example we write

$$|\psi_L|^2 = e^{-U_0/k_B T}. \tag{72}$$

U_0 is defined by this equation. It is convenient (but unnecessary) to choose $k_B T = 1/m$ so that

$$U_0 = -2m^2 \sum_{i<j} \ln|z_i - z_j| + m \sum_k \frac{|z_k|^2}{2}. \tag{73}$$

The first term on the right hand side describes the interaction between particles of charge m in a two-dimensional plasma while the second term describes their interaction with an external electric potential. Noting that the Laplacian of the second term is a constant we see from the Poisson equation that this potential arises from a uniform charge density [43],

$$n_B = \frac{1}{2\pi}. \tag{74}$$

Because of the long range of the plasma interaction, overall charge neutrality is required except within a screening length of the edge of the electron system. Since the electrons have plasma charge m this implies that the electron density is

$$n = \frac{1}{2\pi m} \tag{75}$$

and that the density is uniform, as promised previously. The filling factor $\nu = 1/m$.

A similar argument may be used to deduce the quasihole charge in Laughlin's explicit wavefunction for the quasihole state. The additional factor in the wavefunction gives rise to an additional contribution to the effective potential seen by the plasma particles:

$$U = U_0 + m \sum_k 2 \ln|z_k|. \tag{76}$$

The additional term corresponds to the interaction with an external unit charge located at the origin. Because of the long range of the plasma interaction this charge will be perfectly screened by the charge m plasma particles so that $1/m$ of an electron must be missing from within a plasma screening length of the origin. We have recovered from this explicit wavefunction our previous result for the quasihole charge.

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VIII. CHERN-SIMONS-LANDAU-GINZBURG THEORY

In recent years, issues connected with the quantum statistics of interacting particles in two-dimensions have become important themes in several areas of condensed matter theory, including the theory of the fractional quantum Hall effect. In this section we review the several connected ways in which these issues have arisen in fractional quantum Hall effect theory.

A. Anyons

The Hamiltonian for a system of identical particles commutes with the operator whichpermutes the indices of any pair of particles. It follows that the eigenstates of the many-particle Hamiltonian can be chosen to be eigenstates of the permutation operator. The usual textbook argument notes that since the square of the permutation operator for a particular pair is the identity operator the only possible eigenvalues for the permutation operator are $+1$ or -1 corresponding to Bose and Fermi statistics respectively. In two-dimensions, however, it is physically sensible to allow the permutation operator eigenvalue to be any phase factor:

$$\Psi(z_2, z_1, z_3, \cdots, z_N) = e^{\pm i\theta}\Psi(z_1, z_2, z_3, \cdots, z_N). \tag{77}$$

Crudely, this is true because in two-dimensions exchange paths for a pair electrons can be distinguished by a winding number. (The sign choice for the phase factor in Eq. (77) is specified by the sense of the exchange path.) A continuum of statistics is possible and a given *anyon* system is specified by the statistics angle θ in Eq. (77). (For an introduction to anyon quantum mechanics see the article of Canright and Girvin [44] and work cited therein.) Since elementary particles live in a three-dimensional world they must have Fermi or Bose statistics. However, quasiparticles in purely two-dimensional electronic systems could, at least in principle, have exotic statistics angles. I'll return to this possibility for the fractionally charged quasiparticles of the fractionally quantum Hall effect below.

B. Statistical Transmutation

One interesting application of the theory of anyons has come from the observation [45] that we are free to choose any statistics we like for two-dimensional particles provided that we appropriately modify the Hamiltonian. The following transformation changes a system with statistics angle θ to one with statistic angle $\theta + \pi\alpha$.

$$\Psi' = \prod_{j>k} e^{i\theta_{jk}\alpha}\ \Psi$$

$$H' = \frac{1}{2m}\sum_{j}\left(\vec{p}_j + \frac{e}{c}\vec{A}_j + \frac{e}{c}\vec{a}_j\right)^2 + U \tag{78}$$

where θ_{jk} is $\mathrm{Im}\ln(z_k - z_j)$, $\vec{A}$ is the vector potential corresponding to the external magnetic field, and the 'statistical vector potential' is

$$\vec{a}_j = \frac{-\hbar c}{e} \alpha \sum_{k \neq j} \vec{\nabla}_j \theta_{jk}. \tag{79}$$

Noting that

$$\oint_k \vec{a}_j \cdot d\vec{\ell} = -\Phi_0 \cdot \alpha \tag{80}$$

where the integral is around any closed contour surrounding only particle k we see that the 'statistical vector potential' corresponds to a 'statistical magnetic field' for particle j

$$\vec{B}_j = \vec{\nabla} \times \vec{a}_j = -\Phi_0 \cdot \alpha \cdot \sum_{k \neq j} \delta(\vec{r}_j - \vec{r}_k). \tag{81}$$

This transformation allows us to change the quantum statistics of the particles at the price of attaching 'flux tubes' to each particle as illustrated schematically in Fig. [9] and has become known as statistical transmutation. If each flux tube contains a single magnetic flux quantum and we attach an odd number of flux tubes we change fermions to bosons. The resulting particles are often referred to as composite bosons because of the attached 'flux-tubes'. We will have occasion later to discuss transformations in which even numbers of flux-tubes are attached to each particle, a transformation which does not alter particle statistics. In that case the transformed particles are often referred to as 'composite fermions'. Statistical transmutation is the two-dimensional analog of the familiar Jordan-Wigner transformation in one-dimensional identical particle systems.

C. Quasiparticle Statistics

It has been argued [46,47] that the quasiparticles of the $\nu = 1/m$ incompressible state have fractional statistics with statistics angle $\pm 1/m$. This assertion is motivated by trial wavefunctions for many quasiparticle states. Consider Laughlin's trial wavefunction for a state with a single quasihole located at z_0:

$$\Psi_{z_0}[z] = \prod_i (z_i - z_0) \Phi[z] = |\Psi_{z_0}[z]| e^{i\varphi[z]}. \tag{82}$$

We can easily calculate the total phase change in this wavefunction when the quasiparticle is moved around a closed loop keeping all the particles fixed.

$$\oint dz_0 \frac{d\varphi[z]}{dz_0} = -i \oint \frac{\psi'_{z_0}[z]}{\psi_{z_0}[z]} dz_0 = 2\pi N_0. \tag{83}$$

where N_0 is the number of electrons, and hence the number of zeroes of $\Psi_{z_0}[z]$ (considered as a function of z_0) inside the loop. If a quasiparticle located well in the interior of the loop is introduced in $\Phi[z]$ in Eq. (82) the additional phase change, averaging over all electron coordinates, is $2\pi/m$ since the quasiparticle charge is $1/m$. Arovas *et. al.* have noted [47] that the same phase appears in the wavefunction of a pair of fractional statistics particles where it can be considered as a Berry phase associated with a statistical vector potential and have concluded that the quasiparticles are anyons having statistics angle π/m. There is some

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evidence from finite size exact diagonalization calculations in favor of fractional quasiparticle statistics [48]. There have not been any suggestions of feasible experimental measurements which could establish the quasiparticle statistics. It may seem surprising that the statistics of the quasiparticles does not give rise to qualitative physical effects. I'll explain why this is not the case later in this section.

D. Digression: Lowest Landau Level Density Matrices

I pause here to calculate the ground state one-body density matrix in a coordinate representation for many-body states in the lowest Landau level. It is useful to start by considering its diagonal elements the density, $n(\vec{r})$.

$$
\begin{aligned}
n(\vec{r}) &= \sum_{m',m} \bar{\varphi}_{m'}(z)\varphi_m(z)\langle c_{m'}^\dagger c_m \rangle_0 \\
&= \frac{1}{2\pi} \sum_{m',m} \frac{r^{m+m'} e^{i\theta(m-m')}}{\sqrt{2^{m+m'} m! m'!}} \langle c_{m'}^\dagger c_m \rangle_0 e^{-r^2/2}.
\end{aligned}
\tag{84}
$$

The second form of Eq. (84) follows by explicitly substituting the symmetric gauge lowest Landau level wavefunctions. We see that if the ground state represents an isotropic fluid

$$
\langle c_{m'}^\dagger c_m \rangle_0 \propto \delta_{m',m}
\tag{85}
$$

so that

$$
n(r) = \frac{1}{2\pi} \sum_m \frac{!}{m!} \left(\frac{r^2}{2}\right)^m e^{-r^2/2} \langle n_m \rangle_0.
\tag{86}
$$

If we further assume that the ground state is a constant density fluid at a Landau level filling factor ν, the density is $\nu/2\pi$ and

$$
\langle n_m \rangle_0 = \nu
\tag{87}
$$

independent of m. The second quantized form for the one-body density matrix differs only in the position at which $\bar{\varphi}_m$ is evaluated:

$$
\rho(z, \bar{z}') = \sum_{m',m} \bar{\varphi}_{m'}(z')\varphi_m(z)\langle c_{m'}^\dagger c_m \rangle_0.
\tag{88}
$$

Using the above result for $\langle c_{m'}^\dagger c_m \rangle_0$ we find

$$
\begin{aligned}
n(z, \bar{z}') &= \frac{\nu}{2\pi} \sum_m \frac{1}{m!} \left(\frac{z\bar{z}'}{2}\right)^m e^{-|z|^2/4} e^{-|z'|^2/4} \\
&= \frac{\nu}{2\pi} \exp\left(-\frac{|z-z'|^2}{4}\right) \exp\left((z\bar{z}' - z'\bar{z})/4\right).
\end{aligned}
\tag{89}
$$

The last factor in Eq. (89) is a phase factor. The one-body density matrix has a Gaussian fall off and is completely specified by ν for any isotropic fluid.

E. Boson Off-Diagonal Long-Range Order

Long-range-order in the one-body density matrix, or in two-dimensions quasi-long-range order, is the usual criterion for superfluidity. (For superconductors long range order occurs in the density-matrix of Cooper pairs.) I have just shown that in the fractional Hall regime long range order *does not* occur in the electron one-body density matrix. There is, however, an important connection between superfluidity and the fractional quantum Hall effect which we will now explain. For $\nu = 1/m$ the connection is based on the the statistical transmutation transformation

$$\Psi'[z] = \prod_{j<k} \exp^{-im\theta_{jk}} \Psi[z] \tag{90}$$

which changes fermions into bosons since m is odd. We want to consider the one-body density matrix in this boson representation. In first quantized form the density matrix is:

$$n(z, z') = N \int d^2 z_2 \cdots \int d^2 z_N \psi(z, z_2, \cdots, z_N) \bar{\psi}(z', z_2, \cdots z_N). \tag{91}$$

For Laughlin's wavefunction the effect of the statistical transmutation is to replace the wavefunction by its absolute value. The explicit form of the transformed density matrix is

$$
\begin{aligned}
n'(z, z') &= N \int d^2 z_2 \cdots \int d^2 z_N \psi'(z : [z]) \bar{\psi}'(z'; [z]) \\
&= N \int d^2 z_2 \cdots \int d^2 z_N \prod_{1<k} |z - z_k|^m |z' - z_k|^m e^{-|z|^2/4} \\
&\quad e^{-|z'|^2/4} \prod_{1<\ell<m} |z_\ell - z_m|^{2m} \prod_{1<j} e^{-|z_j|^2/2}.
\end{aligned}
\tag{92}
$$

$n'(z, z')$ clearly drops off much more slowly with $|z - z'|$ than $n(z, z')$ since the integrand in Eq. (92) is real and positive definite. It is possible to show [49] using a plasma analogy argument that

$$n'(z, z') \sim |z - z'|^{-m/2}. \tag{93}$$

(I leave this proof as an exercise for the reader.) Thus in the appropriate boson representation the one-body density matrix of the incompressible ground state has the quasi-long-range order which is associated with a two-dimensional superfluid in the absence of a magnetic field.

This observation has given rise to a useful phenomenology for the fractional quantum Hall effect known as Chern-Simons-Landau-Ginzburg theory [50]. For $\nu = 1/m$, m quanta of magnetic flux pass through the system for each electron. We can choose a statistical transmutation transformation which attaches m flux quanta to each electron whose orientation is opposite to that of the physical magnetic field. (In a Lagrangian formulation these attached flux quanta lead to a Chern-Simons term in the action.) This statistical magnetic field fluctuates in a complicated way in concert with quantum fluctuations in the electronic motion. If the fluctuating magnetic field is treated exactly, something we are not able to do at present, this description of bosons interacting with each other and with a fluctuating

magnetic field is entirely equivalent to the direct fermionic description of the interacting electron system. The simplest approximation is to replace the fluctuating magnetic field by its average, yielding a system of bosons with repulsive interactions in zero magnetic field. Bosons with repulsive interactions are superfluids and therefore share the off-diagonal-long-range-order of the Laughlin state transformed to the boson representation. It has been argued [49] that this long-range-order property occurs in the ground state if and only if the fractional quantum Hall effect occurs, and therefore that the fractional quantum Hall effect is equivalent to superfluidity in the appropriate boson representation. Given that the boson system is a superfluid and that fluctuations in the statistical magnetic field can be treated at a random-phase-approximation level, all essential physical properties associated with the fractional quantum Hall effect can be explained on a phenomenological level [51]. It is important to realize that the validity of the random-phase-approximation treatment of the fluctuating magnetic field cannot be justified on theoretical grounds. It is evidently a poor approximation in the case of non-interacting electrons since nothing like the macroscopic ground state degeneracy of that limit can be recovered. Its apparent validity is evidently due to the same correlations which give rise to the fractional quantum Hall effect.

The polynomial part of the Laughlin state at $\nu = 1/m$ is of degree $m(N-1)$ in each of its coordinates, and hence has $m(N-1)$ zeroes as a function of any of its coordinates. The Laughlin state has the special property that m zeroes are placed at the position of each other particle and they follow these particles as they move about. This is the property of the Laughlin states which makes them have a low energy for short-range repulsive interactions. It is also this property of the Laughlin wavefunctions which makes their boson transformed counterparts purely real and leads to off-diagonal-long-range-order and the order parameter of the Chern-Simons-Landau-Ginzburg theory. A closely related order parameter has been constructed by Read [52] and numerical calculations by Rezayi and Haldane [53] have verified that this order exists if and only if the electron-electron interaction is such that the fractional quantum Hall effect occurs.

IX. COLLECTIVE MODES OF INCOMPRESSIBLE STATES

A. Collective Modes at Zero Magnetic Field

In this section we will discuss the collective modes and the density-density response function at zero temperature for the case where the ground state is incompressible. Our approach is similar to sum rule approaches which are often useful for discussing collective modes and response functions in interacting particle systems. A famous example is Feynman's theory [54] of the collective excitations and response functions of ^{3}He. To orient ourselves to the problem at hand we consider the application of a similar approach to the case of interacting two-dimensional electrons in the absence of a magnetic field. The central quantity in this approach is the dynamic structure factor of the electron system which is defined by

$$s(q, \epsilon) \equiv N^{-1} \sum_n |\langle \Psi_n | \rho_{\vec{q}} | \Psi_0 \rangle|^2 \delta(\epsilon - (E_n - E_0)). \tag{94}$$

Here $\rho_{\vec{q}}$ is the Fourier transform of the one-body density operator;

$$\rho_{\vec{q}} \equiv \int d\vec{r} \exp(-i\vec{q} \cdot \vec{r}) \sum_{i=1}^{N} \delta(\vec{r} - \vec{r}_i) \tag{95}$$

(We use the notation $n(\vec{q}) = \langle \Psi_0 | \rho_{\vec{q}} | \Psi_0 \rangle$ to distinguish the density operator from its ground state expectation value.) Three different moments of the dynamic structure factor have important physical significance:

$$f(q) \equiv \int d\epsilon \, \epsilon s(q, \epsilon) \tag{96}$$

$$s(q) \equiv \int d\epsilon \, s(q, \epsilon) \tag{97}$$

$$\chi(q) = 2n \int d\epsilon \frac{s(q, \epsilon)}{\epsilon}. \tag{98}$$

Both $f(q)$ and $s(q)$ can be expressed in terms of ground state expectation values of appropriate operators:

$$f(q) = N^{-1} \langle \Psi_0 | \rho_{-\vec{q}} [H, \rho_{\vec{q}}] | \Psi_0 \rangle = \frac{\hbar^2 q^2}{2m} \tag{99}$$

and

$$s(q) = N^{-1} \langle \Psi_0 | \rho_{-\vec{q}} \rho_{\vec{q}} | \Psi_0 \rangle. \tag{100}$$

The second equality in Eq. (99) is known as the f-sum rule. The first moment of the dynamic structure factor turns out to be proportional to q^2 and completely independent of the electron electron interaction. It can be shown [55] that $\chi(q)$ is the static density-density response function.

Because of the long range interactions in interacting electron systems, some general statements can be made about the behavior of $\chi(q)$ at small q. This is most easily seen by considering the energy of the electronic system as a functional of the Fourier components of the ground state electronic density. This assumption can be formally justified by appealing to the Hohenberg-Kohn theorems of density-functional theory [56]. If we consider the charge density induced in the uniform system by a weak external potential $V^{ext}(\vec{q})$ we find that

$$\chi(q) \equiv -\frac{n(\vec{q})}{V^{ext}(\vec{q})} = \frac{1}{A} \left(\frac{\delta^2 E^{int}}{\delta n(\vec{q}) \delta n(-\vec{q})} \right)^{-1}. \tag{101}$$

The response function, $\chi(q)$, is inversely proportional to the stiffness of the internal energy of the electron system when density modulation is introduced at wavevector $\vec{q}$. For systems with long range interactions it is customary to separate the contribution to the stiffness from the Hartree energy of the charge density so that for a two-dimensional electron system

$$\chi^{-1}(q) = \frac{2\pi e^2}{q} + \Pi^{-1}(q). \tag{102}$$

We may take the static polarization function, $\Pi(q)$ to be defined by Eq. (102). It can be demonstrated from the above discussion that

$$\lim_{q \to 0} \Pi^{-1}(q) = \frac{d\mu}{dn} \tag{103}$$

where the energy of the system is calculated with a neutralizing background at every density. (This connection between $s(q, \epsilon)$ and $\chi(q)$ is often referred to as the compressibility sum rule. For short range interactions the connection is more direct since the Hartree-interaction need not be separated.) As long as $\lim_{q \to 0} \Pi(q)$ is finite, a result which is believed to apply whenever the two-dimensional electron system has a fluid ground state, we can conclude that

$$\lim_{q \to 0} \chi(q) = \frac{q}{2\pi e^2}. \tag{104}$$

Independent of any assumptions about the long wavelength behavior of $\Pi(q)$ it follows from the above arguments that $\chi(q)$ must vanish at least like q for q going to zero. A two-dimensional electron system cannot change its density on long length scales because of the infinite energy cost of violating overall charge neutrality. This behavior is sometimes referred to as incompressibility since $d\mu/dn$ is infinite unless the neutralizing positive background charge density is changed along with the electron density. We see below that the incompressible states associated with the fractional quantum hall effect are 'much more' incompressible since $\chi(q)$ vanishes like q^4 at small q.

We can use the above results to estimate the elementary excitation energies and the response functions of the two-dimensional electron system. The estimates are based on the assumption that there is a single excited state which contributes to $s(q, \epsilon)$ at each q; this approximation is known as the single-mode approximation. The assumption is well justified in the present case for $q \to 0$ since in this limit $\rho \propto \vec{q} \cdot (\sum_i r_i)$, (up to a constant) which involves only the center of mass coordinate of the electrons. The center-of-mass motion is completely decoupled from the complicated relative motion of the particles of the gas and its excited states are labeled by momentum. For larger q we know that many individual particle-hole pair excitations are possible at a given q and the single-mode-approximation should be regarded critically. Given the single mode approximation we can compare our exact results for $\chi(q)$ and $f(q)$ to obtain an estimate of the energy of the collective excited state:

$$\epsilon_{pl}(q) = \left(\frac{2\pi ne^2 q\hbar^2}{m}\right)^{1/2}. \tag{105}$$

This is, of course, the long-wavelength expression for the plasmon collective excitation of the zero-field two-dimensional electron gas. Given Eq. (105) it follows that for $q \to 0$

$$S(q, \epsilon) = \frac{\hbar^2 q^2}{2m\epsilon_{pl}(q)} \delta(\epsilon - \epsilon_{pl}(q)) \tag{106}$$

so that $S(q) \sim q^{3/2}$ at small q compared to the familiar result that for non-interacting electrons $S(q) \sim q$.

B. Correlation Function Moments in the Fractional Hall Regime

I now want to apply a similar analysis in the fractional Hall regime. We'll find that our conclusions depend on some general properties of uniform fluid state correlation functions which we now establish. Consider the second quantization expression for the two-point distribution function

$$
\begin{aligned}
n^{(2)}(\vec{r}_1, \vec{r}_2) \\
= \sum_{\substack{m_1, m_2 \\ m_1', m_2'}} \bar{\varphi}_{m_1'}(\vec{r}_1) \varphi_{m_1}(\vec{r}_1) \bar{\varphi}_{m_2'}(\vec{r}_2) \varphi_{m_2}(\vec{r}_2) \langle c_{m_1'}^\dagger c_{m_2'}^\dagger c_{m_2} c_{m_1} \rangle_0
\end{aligned}
\tag{107}
$$

Assuming that the ground state is a uniform isotropic fluid, $n^{(2)}(\vec{r}_1, \vec{r}_2) = n^2 g(|\vec{r}_1 - \vec{r}_2|)$ where $g(r)$ is the usual dimensionless pair distribution function normalized so that it goes to one at large r. We can evaluate $g(r)$ by choosing $\vec{r}_1 = 0$ and $\vec{r}_2 = r\hat{x}$. We take advantage of the frequently convenient fact that only the $m = 0$ orbital is non-zero at the origin to find,

$$
g(r) = n^{-2} \sum_{m \neq 0} \left(\frac{r^2}{2}\right)^m \frac{1}{m!} e^{-r^2/2} \langle \hat{n}_m \hat{n}_0 \rangle_0.
\tag{108}
$$

In the following we define $x \equiv r^2/2$. It it possible to derive general expressions for two spatial moments of the dimensionless pair correlation function

$$
\begin{aligned}
h(r) &\equiv g(r) - 1 \\
&= \nu^{-2} \sum_{m=0}^{\infty} \frac{x^m}{m!} e^{-x} \Big((1 - \delta_{m,0}) \langle \hat{n}_m \hat{n}_0 \rangle - \langle \hat{n}_m \rangle \langle \hat{n}_0 \rangle \Big).
\end{aligned}
\tag{109}
$$

The zeroth moment of $h(r)$ is

$$
n \int d^2\vec{r}\, h(r) = \nu^{-1} \Big[\langle \hat{N} \hat{n}_0 \rangle - \langle \hat{n}_0 \hat{n}_0 \rangle - \langle \hat{N} \rangle \langle \hat{n}_0 \rangle \Big] = -1,
\tag{110}
$$

where the the second equality follows from the fact that the total number operator is a good quantum number. The first moment of $h(r)$ is

$$
\begin{aligned}
n \int d^2\vec{r}\, \frac{r^2}{2} h(r) &= -1 + \nu^{-1} \Big[\langle \hat{M} \hat{n}_0 \rangle - \langle \hat{M} \rangle \langle \hat{n}_0 \rangle \Big] \\
&= -1
\end{aligned}
\tag{111}
$$

where $\hat{M} = \sum_m m \hat{n}_m$ and the second equality this time follows from the fact that the total angular momentum is a good quantum number. These two sum rules for the pair correlation function apply to Laughlin's wavefunction. In that special case they can be derived from the plasma analogy using perfect screening properties of the plasma but we see here that they have a more general validity. The fact that these sum rules apply to any many-electron wavefunction in the lowest Landau level which represents an isotropic fluid, suggests that there is a deep connection between the suppression of long wavelength density fluctuations in two-dimensional plasmas by long-range interactions and the suppression of density fluctuations in the fractional quantum Hall regime by projection onto a single Landau level.

C. Projected Static Structure Factor

The pair correlation function and the static structure factor discussed above are closely related. To see this recall that

$$s(k) = \frac{1}{N}\langle \rho_{-\vec{k}}\rho_{\vec{k}}\rangle_0 = \left\langle \frac{1}{N}\sum_{i,j} e^{i\vec{k}\cdot(\vec{r}_i - \vec{r}_j)}\right\rangle_0 . \tag{112}$$

The $i = j$ terms in Eq. (112) sum to 1 and the $i \neq j$ terms are proportional to a Fourier transform of the two-point distribution function:

$$s(k) = 1 + \frac{1}{N}\int d\vec{r}\int d\vec{r'}e^{i\vec{k}\cdot(\vec{r}-\vec{r'})}n^{(2)}(\vec{r},\vec{r'})$$

$$= 1 + N\delta_{\vec{k},0} + n\int d\vec{r}e^{i\vec{k}\cdot\vec{r}}h(r) \equiv 1 + N\delta_{\vec{k},0} + h(k) \tag{113}$$

In Eq. (113) we have used the the first quantization expression for the two-point distribution function:

$$n^{(2)}(\vec{r},\vec{r'}) \equiv \sum_{i\neq j}\left\langle \delta(\vec{r}-\vec{r}_j)\delta(\vec{r'}-\vec{r}_j)\right\rangle_0 \tag{114}$$

and adopted the conventional definition for $h(k)$, the Fourier transform of the pair correlation function. The moments derived above for $h(r)$ imply the following general result for the long-wavelength behavior of $h(k)$.

$$h(k) = n\int d\vec{r}h(r) + \frac{k^2}{2}\left(-n\int dr\frac{r^2}{2}h(r)\right) + \mathcal{O}(k^4)$$

$$= -1 + \frac{k^2}{2} + \cdots . \tag{115}$$

In the fractional Hall regime, we are interested primarily in the low energy elementary excitations of the system which do not involve the promotion of electrons to higher Landau levels. It is explicit in Eq. (31) that $\rho_{\vec{k}}$ maps states in the lowest Landau level partly to higher Landau levels. To generate the excited states of interest it is necessary to project the density operator onto the lowest Landau level:

$$\bar{\rho}_{\vec{k}} \equiv \sum_i {}_i\langle 0|e^{-i\vec{k}\cdot\vec{r}_i}|0\rangle_i = \sum_i B_i(k) \tag{116}$$

where $B_i(k)$ was expressed in terms of the orbit center ladder operators for particle i in Eq. (34). Note that

$$B_i(k_1)B_i(k_2) = e^{k_1\bar{k}_2/2}B_i(k_1 + k_2) \tag{117}$$

so that $B_i(-k)B_i(k) = \exp(-|k|^2/2)$. The projected static structure factor for $\vec{k} \neq 0$ obeys

$$\bar{s}(k) \equiv \frac{1}{N}\langle \bar{\rho}_{-\vec{k}}\bar{\rho}_{\vec{k}}\rangle = \frac{1}{N}\sum_{i\neq j}\langle e^{i\vec{k}\cdot\vec{r}_i}e^{-i\vec{k}\cdot\vec{r}_j}\rangle_0 + \frac{1}{N}\sum_i\langle B_i(-k)B_i(k)\rangle$$

$$= s(k) - 1 + e^{-|k|^2/2} = h(k) + e^{-|k|^2/2} \tag{118}$$

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To obtain this result we've noted that the ground state is entirely in the lowest Landau level so that the projection of $\exp(-i\vec{k}\cdot\vec{r})$ is necessary only when the two particle indices are identical. Combining Eq. (115) with Eq. (118) we obtain the remarkable result that

$$\bar{s}(k) = \left(-1 + \frac{|k|^2}{2} + \mathcal{O}(|k|^4)\right) + \left(1 - \frac{|k|^2}{2} + \mathcal{O}(|k|^4)\right). \tag{119}$$

Inserting the definition of the projected static structure factor we can conclude that there are no dipole matrix elements within the lowest Landau level:

$$\sum_m \left|\langle\psi_m|e^{-i\vec{k}\cdot\vec{r}}|\psi_0\rangle\right|^2 = \mathcal{O}(|k|^4) \tag{120}$$

By dipole matrix elements we mean those that would arise at the lowest order in the expansion of $\exp(-i\vec{k}\cdot\vec{r})$ (if this were valid) and give rise to contributions to $\bar{s}(k) \sim k^2$. (For finite systems dipole matrix elements do occur within the lowest Landau level but only for excitations that are localized at the edge of the system.) Note that since the real part of the conductivity is related to $s(q,\omega)$ by the continuity equation, it follows that $\sigma_{xx}(\omega) \equiv 0$ within the lowest Landau level if no disorder potential is present.

D. Magnetorotons

We are now in a position to discuss the intra-Landau-level collective modes of the fractional quantum Hall effect using the single-mode approximation. The calculation we outline below was first performed by Girvin *et al.* [57] The calculation is based on a version of the single-mode-approximation projected onto the lowest Landau level. These modes have become known as magnetorotons because their existence was first suggested based on a strong-magnetic-field generalization of Feynman's theory of the He^3 phonon-roton excitation spectrum. We assume that $|\psi_{\vec{k}}\rangle = \bar{\rho}_{\vec{k}}|\psi_0\rangle$ is an approximate eigenstate of the many-electron Hamiltonian and attempt to evaluate its energy and the matrix element of the (projected) density operator between this state and the ground state so that we can estimate response functions. The energy is

$$E(\vec{k}) = \frac{\langle\psi_{\vec{k}}|\mathcal{H}|\psi_{\vec{k}}\rangle}{\langle\psi_{\vec{k}}|\psi_{\vec{k}}\rangle} = E_0 + \frac{\langle\psi_0|\bar{\rho}_{-\vec{k}}[\mathcal{H},\bar{\rho}_{\vec{k}}]|\psi_0\rangle}{\langle\psi_0|\bar{\rho}_{-\vec{k}}\bar{\rho}_{\vec{k}}|\psi_0\rangle} \equiv E_0 + \frac{\bar{f}(\vec{k})}{\bar{s}(\vec{k})}. \tag{121}$$

In Eq. (121) $\bar{f}(k)$ is the projected version of the usual f-sum rule.

$$\bar{f}(k) \equiv \frac{1}{N}\sum_m |\langle\psi_m|\bar{\rho}_{\vec{k}}|\psi_0\rangle|^2(E_m - E_0) = \bar{f}(-k). \tag{122}$$

The second equality follows from inversion symmetry in the isotropic fluid and it allows us to write

$$\bar{f}(k) = \frac{1}{2N}\langle\psi_0|[\bar{\rho}_{-k},[\mathcal{H},\rho_k]]|\psi_0\rangle. \tag{123}$$

The Hamiltonian consists of the interaction term alone which can be written in the form

$$\mathcal{H} = \frac{1}{2} \int \frac{d^2\vec{q}}{(2\pi)^2} v(q) \bar{\rho}_{-q} \bar{\rho}_q + \text{ constant.} \tag{124}$$

It is amusing to note that for the usual unprojected f-sum rule the entire contribution comes from the kinetic energy term in the Hamiltonian. Here however there is a contribution from the interaction term because the projected density operators do not commute,

$$[\bar{\rho}_{k_1}, \bar{\rho}_{k_2}] = \sum_i [B_i(k_1), B_i(k_2)] = (e^{k_1 \bar{k}_2/2} - e^{k_2 \bar{k}_1/2}) \bar{\rho}_{k_1+k_2} \tag{125}$$

After a little patient work using Eq. (125), the expression for $\bar{f}(k)$ can be expressed in terms of the projected static structure factor.

$$\bar{f}(k) = \int \frac{d^2\vec{q}}{(2\pi)^2} v(q)(1 - \cos{(\hat{z} \cdot (\vec{q} \times \vec{k}))})(\tilde{\bar{s}}(k+q) - \tilde{\bar{s}}(q)) e^{-|q|^2/2} \tag{126}$$

where

$$\tilde{\bar{s}}(q) \equiv e^{|q|^2/2} \bar{s}(q). \tag{127}$$

For an isotropic fluid it follows from Eq. (126) that $\bar{f}(k) \sim k^4$ at small k. Since $\bar{s}(k) \sim k^4$ the single-mode approximation gives collective excitation energies which remain finite in the long wavelength limit, unlike the two-dimensional plasmon excitations at zero magnetic field. For incompressible ground states there is therefore a gap for both neutral and charged excitations. The single-mode-approximation magnetoroton excitation energies calculated by Girvin *et. al.* are shown in Fig. [10]. The excitation energies are expressed in terms of the natural energy unit in the strong magnetic field limit for $1/r$ interactions, e^2/ℓ. Because of the excitation gap, $\chi(k)$ also vanishes like k^4, much faster than the linear in k behavior required by long-range interactions at zero magnetic field. (Actually there is a quadratic contribution to $\chi(k)$ which comes from higher Landau level excitations but it is proportional to B^{-1} in the strong magnetic field limit.)

Because $\bar{s}(k) \sim k^4$ at long wavelengths the magnetoroton is not purely a center-of-mass motion excitation. We cannot offer a compelling argument in favor of the accuracy of the single-mode approximation even in the long wavelength limit, however we do get some indication that is is likely to be accurate for incompressible ground states by comparing with the $\nu = 1$ case. The following discussion follows a line of thought first advanced by Kallin and Halperin [58]. At $\nu = 1$ the ground state in the strong magnetic field limit is the single Slater determinant which has all the single-particle orbitals in the lowest Landau level occupied. The ground state is incompressible and, for non-interacting electrons, the lowest energy neutral excitations are the N_ϕ^2 particle-hole pairs all of which have excitation energy $\hbar\omega_c$. This degeneracy is lifted by electron-electron interactions but the excitation energies can still be evaluated exactly since they are labeled by wavevector and there is precisely one state at each wavevector. The excited state is

$$|\Psi_{\vec{k}}\rangle \equiv \sum_i |n = 1\rangle_i \langle n = 0| B_i(k) |\Psi_0\rangle. \tag{128}$$

The energy of this state can easily be evaluated exactly; the calculation is similar to that used above to calculate magnetoroton energies [59]. For large $|k|$ it turns out that

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$$E(k) = \Delta\mu - e^2/k\ell^2 \tag{129}$$

where $\Delta\mu = \hbar\omega_c + \sqrt{\pi/2}(e^2/\ell)$ is the chemical potential discontinuity which is shifted upward by electron-electron interactions. The electron and hole in a magnetic field moving parallel to each other with a velocity just sufficient so that their oppositely directed Lorentz forces cancel their attractive Coulombic forces. Requiring the group velocity in Eq. (129) to give canceling Lorentz and Coulomb forces implies that the electron and hole in $|\Psi_{\vec{k}}\rangle$ to be separated by $\hat{z} \times \vec{k}\ell^2$; this is consistent with the excitonic correction to the particle-hole excitation energy in Eq. (129). In fact this property of $|\Psi_{\vec{k}}\rangle$ can be established explicitly. We should expect this picture to break down when $|k|\ell \sim 1$ since the shortest possible localization length which is $\sim \ell$ becomes comparable to the separation between the particle and the hole. In fact it turns out that for small $|k|$

$$E(k) = \hbar\omega_c + e^2k/2 \tag{130}$$

which is the long wavelength dispersion for the classical magnetoplasmon collective mode at long wavelengths. It seems that the neutral excitations gradually crossover from having a collective character to having a particle-hole character as the wavevector increases. This contrasts with the familiar case at zero magnetic field where collective excitations and particle-hole excitations are clearly distinct, although the fact that particle-hole excitations capture more and more oscillator strength at larger wavevectors is still reflected in the strong field behavior. By analogy with what happens at $\nu = 1$ we expect that the single-mode-approximation magnetoroton dispersion should be accurate for $k\ell$ small. This has been confirmed [60,32] by comparisons between the single-mode-approximation theory and numerical exact diagonalization calculations. Results of some of these calculations are shown in Fig. [10].

X. HIERARCHY STATES

So far we have discussed only the case of the fractional quantum Hall effect at filling factor $\nu = 1/m$ where m is an odd integer. Experimentally the fractional quantum Hall effect occurs at a large number of other filling factors. Up to the present time the fractional quantum Hall effect has been observed, for fully spin polarized electron systems at $\nu = \nu_n \equiv n/(2n+1)$ and for $\nu = 1 - \nu_n$ for $n = 1, 2, 3, 4, 5, 6, 7$ as well as for $\nu = 1/5, 4/5, 2/7, 5/7, 2/9$. The largest charge gaps occur for $\nu = 1/3$ and $\nu = 2/3$. We have argued that these experimental results tell us that the interacting two-dimensional electron system has chemical potential discontinuities at all these filling factors. Presumably there are other chemical potential discontinuities which are still masked by disorder even in the highest quality samples available today and the ones we see have the largest charge gaps. We would like to be able to estimate the charge gaps and to understand why it is at these particular filling factors that the fractional quantum Hall effect occurs; of particular interest is the prominence of the series of filling factors ν_n and $1 - \nu_n$. This series of fractions shows up prominently in numerical exact diagonalization calculations [39] for the hard-core model as shown in Fig. [11]. For reasons which will become clear it is common to refer to the incompressible ground states at filling factors $\nu \neq 1/m$ as hierarchy states. The reader is warned that we are now approaching a

border on the map of knowledge in the fractional Hall regime. *This way lie dragons.* We start this section with some exact results and, with that equipment in hand, venture into more rugged territory.

A. Bosonization

We now discuss a method of bosonization of the many-fermion system in a strong magnetic field which is different from, but not unrelated to, the statistical transmutation transformation discussed earlier. It is useful to start with an observation concerning the many-fermion wavefunction for a full Landau level. For finite N, a full Landau level state is formed by occupying all single-particle states from $m = 0$ to $m = N - 1$. The many-fermion wavefunction is a single Slater determinant of the form $\Psi[z] = P[z] \prod_k \exp\left(-|z_k|^2/4\right)$ where

$$P[z] = \begin{vmatrix} z_1^0 & \cdots & z_N^0 \\ \vdots & & \vdots \\ z_1^{N-1} & & z_N^{N-1} \end{vmatrix} = \prod_{i<j}(z_i - z_j) \equiv P_V[z]. \tag{131}$$

The equality in Eq. (131) can be understood by noting that $P[z]$ must change sign when any pair of particles are interchanged and therefore must vanish when $z_i = z_j$ for any $i \neq j$. It follows that $z_i - z_j$ is a factor of $P[z]$ for any $i \neq j$. $P_V[z]$ is the lowest degree polynomial with this property and one can easily check that it has the same degree ($M = N(N-1)/2$) as the Slater determinant. This proves the equality up to a constant; we leave the proof that the constant is equal to one as an exercise. More generally any wavefunction in the lowest Landau level must be of the same form and must be antisymmetric. It follows that we can always write

$$P[z] = Q[z] P_V[z] \tag{132}$$

where $Q[z]$ is a homogeneous *symmetric* polynomial and therefore is the polynomial part of a *many-boson* wavefunction. Multiplying a many-boson wavefunctions by $P_V[z]$ increases its homogeneous degree by $N(N-1)/2$. It follows from the fact that multiplying polynomials is equivalent to adding areas that if $Q[z]$ represents a boson fluid with filling factor ν_B then $P[z]$ represents a fermion fluid with filling factor satisfying $\nu_F^{-1} = \nu_B^{-1} + 1$.

This mapping between a many-fermion system in the fractional Hall regime at filling factor ν and a many-boson system at filling factor $\nu_B = \nu/(1-\nu)$ can be made more precise. We consider a many-fermion system with N electrons which we allow to occupy orbitals with $m = 0, \cdots, N_\phi - 1$ so that the filling factor is $\nu = N/N_\phi$. The set of

$$g_{N,N_\phi} = \frac{N_\phi!}{N!(N_\phi - N)!} \tag{133}$$

many-fermion states corresponds to the set of independent antisymmetric polynomials in N coordinates for which the maximum power of any coordinate is $N_\phi - 1$. Each of these antisymmetric polynomials is divisible by $P_V[z]$ with a quotient which is a symmetric polynomial. The maximum degree to which any coordinate can appear in the symmetric polynomial is

$N_\phi - 1 - N \equiv N_\phi^B - 1$, *i.e.*, all quotients belong to the Hilbert space of N bosons in the fractional Hall regime which are in a Landau level containing N_ϕ^B orbitals. (The number of independent symmetric polynomials in N coordinates with maximum power k is equal to the number of independent antisymmetric polynomials with maximum power $k + N$. It is easy to perform the division by $P_V[z]$ explicitly [61] for small numbers of electrons and small maximum powers.) Matrix elements of any operator expressible in terms of fermion coordinates, and in particular of the Hamiltonian, will be preserved if they are transformed according to

$$\hat{O}_B[z] = \bar{P}_V[z] \hat{O}_F P_V[z]. \tag{134}$$

(Note that the many-boson inner product is defined by $\hat{O}_F = 1$.) We will make arguments below based on the assumption that these changes do not change the physics in any essential way.

It is useful togive some simple examples of boson wavefunctions and their fermion counterparts. The lowest order symmetric polynomial has degree $M^B = 1$;

$$Q[z] = \sum_i z_i \tag{135}$$

$Q[z]P_V[z]$ is an antisymmetric polynomial of degree $N(N-1)/2+1$. There is only one such polynomial;

$$P[z] = \begin{vmatrix} z_1^0 & \cdots & z_N^0 \\ \vdots & & \\ z_n^{N-2} & \cdots & z_N^{N-2} \\ z_1^N & & z_N^N \end{vmatrix}. \tag{136}$$

We see that the boson state in which $N - 1$ bosons are in the state with angular momentum 0 and one boson is in the state with angular momentum 1 is equivalent to the fermion state in which a particle-hole excitation has been made at the edge of the full Landau level state by promoting a single electron from angular momentum $N - 1$ to angular momentum N. Another example was already introduced in connection with Laughlin's quasihole state, $Q[z] = \prod_i z_i$. This polynomial has $M_B = N$ and increases the power of every coordinate by one in every term of $P_V[z]$ so that

$$P[z] = Q[z]P_V[z] = \begin{vmatrix} z_1^1 & \cdots & z_N^1 \\ \vdots & & \vdots \\ z_1^N & & z_N^N \end{vmatrix}. \tag{137}$$

The boson state where all N bosons have angular momentum 1 corresponds to a fermion state in which the $m = 0$ state is empty. This is a state with a single integer charge hole in a full Landau level. As we've discussed previously when this $Q[z]$ multiplies the $\nu = 1/m$ Laughlin state it creates a state with a fractionally charged quasihole.

B. Particle-hole symmetry

The two-dimensional electron system in the fractional Hall regime has an exact particle-hole symmetry whose importance in constructing a theory of hierarchy states was first emphasized by Girvin [62]. The existence of this symmetry is most easily established by working in an occupation number representation where the Hamiltonian is

$$H = \frac{1}{2} \sum_{m_1, m_1', m_2, m_2'} e_{m_1'}^{\dagger} e_{m_2'}^{\dagger} e_{m_2} e_{m_1} \langle m_1', m_2' | V_{ee} | m_1, m_2 \rangle. \tag{138}$$

Here $e_m^{\dagger}$ and e_m are the creation and annihilation operators for the state with angular momentum m and we have used a standard notation for two-body matrix elements of the electron electron interaction. We make the particle-hole transformation by defining creation and annihilation operators for holes using,

$$h_m^{\dagger} \equiv e_m \tag{139}$$

$$h_m \equiv e_m^{\dagger}. \tag{140}$$

The holes operators obey the same anticommutation operators as electron operators. The vacuum state for electrons is related to the vacuum state for holes by

$$|0\rangle_e = \prod_m h_m^{\dagger} |0\rangle_h. \tag{141}$$

We wish to show that the Hamiltonian for holes is identical to that for electrons up to a constant which is proportional to the difference between the number of electrons and the number of holes. To see this we normal order the two body piece in the Hamiltonian expressed in terms of hole creation and annihilation operators. This procedure leaves us with some one body terms which we need to evaluate;

$$\begin{aligned}
e_{m_1'}^{\dagger} e_{m_2'}^{\dagger} e_{m_2} e_{m_1} &= h_{m_1}^{\dagger} h_{m_2}^{\dagger} h_{m_2'} h_{m_1'} \\
&\quad - \delta_{m_1, m_2'} h_{m_1'} h_{m_2}^{\dagger} + \delta_{m_1, m_1'} h_{m_2'} h_{m_2}^{\dagger} \\
&\quad - \delta_{m_2', m_2} h_{m_1}^{\dagger} h_{m_1'} + \delta_{m_1', m_2} h_{m_1}^{\dagger} h_{m_2'}. \tag{142}
\end{aligned}$$

The one-body terms turn out to be constants. To see this it is useful to write the two-body matrix elements in terms of the Fourier representation of the electron-electron interaction:

$$\langle m_1', m_2' | V_{ee} | m_1, m_2 \rangle = \int \frac{d^2 \vec{k}}{(2\pi)^2} V_{ee}(\vec{k}) e^{-|k|^2}$$
$$\times G_{m_1', m_1}(k) G_{m_2', m_2}(-k). \tag{143}$$

Eq. (143) follows from the expression for plane-wave matrix elements in the lowest Landau level, Eq. (31). We can now use other identities we derived from the algebra of our ladder operator solution of free particle problem. From Eq. (40) it follows that

$$\sum_{m_1} \langle m_1 m_2' | V_{ee} | m_1, m_2 \rangle = \delta_{m_2', m_2} (2\pi \ell^2)^{-1} V_{ee}(\vec{k} = 0)$$

$$\equiv \delta_{m_2', m_2} 2\epsilon_H \tag{144}$$

where ϵ_H is the Hartree energy per electron of the ground state when the Landau level is full. Similarly from Eq. (37) it follows that

$$\sum_{m_1}\langle m_1' m_1 | V_{ee} | m_1, m_2 \rangle = \delta_{m_1', m_2} \int \frac{d^2\vec{k}}{(2\pi)^2} V_{ee}(\vec{k}) e^{-|k|^2/2}$$

$$\equiv \delta_{m_1', m_2} 2\epsilon_X \tag{145}$$

where $-\epsilon_X$ is the exchange energy per electron of the ground state when the Landau level is full. For the two models featured in these notes ϵ_H can be calculated analytically. For the Coulomb model ϵ_H is zero because of the omnipresent neutralizing background required to get a finite zero of energy. For the hard-core model it follows from Eq. (59) that $\epsilon_H = V_1$. Similarly, exchange energies of the full Landau level are $\sqrt{\pi/8}(e^2/\ell)$ and $-V_1$ for the Coulomb and hard-core models respectively. Combining these results the Hamiltonian can be rewritten in terms of normal ordered hole creation and annihilation operators;

$$H = \frac{1}{2} \sum_{\substack{m_1, m_2 \\ m_1', m_2'}} h_{m_1'}^\dagger h_{m_2'}^\dagger h_{m_2} h_{m_1} \langle m_1, m_2 | V_{ee} | m_1', m_2' \rangle$$

$$+ (\epsilon_H - \epsilon_X)(N - N_h) \tag{146}$$

where $N_h \equiv N_\phi - N$ is the number of holes in the Landau level.

Since the matrix elements of the electron-electron interaction terms are real it follows that the spectrum of H for a given number of electrons in a Landau level is identical to that for a given number of holes in a Landau level apart from a known constant which can be interpreted as the interaction of the holes with its vacuum (which is a full Landau level). In particular, the excitation spectra are identical at filling factors ν and $1 - \nu$. The energy per electron, ϵ, in the ground state satisfies,

$$\nu\epsilon(\nu) = (1 - \nu)\epsilon(1 - \nu) + (2\nu - 1)(\epsilon_H - \epsilon_X) \tag{147}$$

and the chemical potential $\mu(\nu)$ satisfies

$$\mu(\nu) + \mu(1 - \nu) = 2(\epsilon_H - \epsilon_X). \tag{148}$$

The chemical potential discontinuities at filling factors ν and $1 - \nu$ are identical. (I leave it as an exercise for the reader to determine how the correlation functions at filling factors ν and $1 - \nu$ are related.)

Bosonization and particle-hole symmetry allow a given situation to be described in languages which appear to be somewhat different. Consider, for example, the case where the Landau level filling factor is near one. It may be more economical, for example, to use particle-hole symmetry to describe the system as consisting of $N_h = N_\phi - N$ Fermi particles rather than N Fermi particles. In both descriptions the number of single-particle states available to theFermi particles is $N_\phi = N + N_h$. As shown above we can do this without approximation. Now we can apply the bosonization transformation. The number of states available to the Bose particles is then reduced by the number of particles i.e., from $N + N_h$ to N. The bosonization transformation is not as clean since it requires a change in the

Hamiltonian and the inner product which we don't know how to treat exactly. We assume on reasonable physical grounds that this change is unimportant; there is fairly strong evidence [63] to support this assumption from numerical exact diagonalization transformations. Note that the number of states available to these Bose particles is N; it is as if the Bose particles lie in a Landau level created by a magnetic field of one flux quantum for each electron. We'll see this result again soon, coming from a different but related direction.

C. Correlation factors

There is another non-unitary transformation of the many-electron wavefunctions which is physically important:

$$\Psi'[z] = \prod_{i<j}(z_i - z_j)^2 \Psi[z]. \tag{149}$$

According to our angular-momentum-counting arguments this transformation changes the filling factor according to

$$\nu'^{-1} = 2 + \nu^{-1}. \tag{150}$$

I will refer to this as the 'correlation-factor' transformation. For $\nu > 1/3$ this transformation maps the Hilbert space at filling factor ν to the part of the Hilbert space at filling factor ν' which is projected out by letting $V_1 \to \infty$. For sufficiently short-range interactions this transformation should therefore map the many-particle Hilbert space at filling factor ν to the low-energy portion of the many-particle Hilbert space at filling factor ν'. It is common [61] to argue on physical grounds that the changes in the Hamiltonian and the inner product which result from the transformation are sufficiently innocuous that the transformation maps eigenstates to eigenstates, although the numerical evidence is less convincing in this case [64]. However, the weaker conclusion that for sufficiently short-range interactions an incompressibility at filling factor ν, implies an incompressibility at filling factor ν' is well supported both by numerical calculations and by the ultimate jury, experiment.

It is possible to 'derive' the Laughlin states using the correlation-factor transformation. Starting from the full Landau level state at $\nu = 1$ the correlation factor transformation generates a sequence of non-degenerate grounds states at $\nu = 1/m$ for odd m which are precisely the Laughlin states. The transformation also maps states with $\nu < 1$ to states with $\nu < 1/3$. The reader can verify that the state with a single hole in a full Landau level, discussed above, is mapped precisely to Laughlin's approximate wavefunction for a single fractionally charged quasihole in the incompressible $\nu = 1/3$ state. Similarly states with many holes are mapped to states with many fractionally charged quasiholes. Since the number of particles and the dimension of the Hilbert space are preserved by the correlation-function transformation it follows that if the quasiholes are regarded as fermions they have Landau levels with degeneracy $N + N_h$, while if they are regarded as bosons they have Landau levels with degeneracy N. Systems with many holes can form incompressible states. We know, for example, from the particle-hole transformation that when $N_h = N_\phi/3$ the holes form a $\nu = 1/3$ incompressible state. The correlation-factor transformation will generate from this state an incompressible ground state at $\nu = 2/7$. Thus we can explain many of the

incompressibilities which are observed experimentally. However, no combination of the above transformations will generate an incompressibility in the filling factor range $1/3 < \nu < 2/3$. It is precisely in this range of filling factors that the most robust fractional quantum Hall effects occur. To account for these incompressibilities we have a choice of stepping onto one of a number of thinner theoretical limbs.

D. Classical Hierarchies

The original hierarchy schemes were motivated by the indication from early experiments that the fractional quantum Hall effect for spin-polarized electrons could occur at any rational value of $\nu = q/p$ with p odd. Related pictures which could account for an incompressibility at all such filling factors were advanced initially by Laughlin [65], Haldane [31], and Halperin [46]. Attempts to obtain more quantitative estimates of hierarchy state properties were also made [61,66] using similarly motivated approximations These three pictures all start from the observation that the Laughlin incompressible states have fractionally charged quasiparticles and attempt to describe the system at nearby filling factors in terms of quasi-particle degrees of freedom. The three pictures differ in the statistics assumed for the quasiparticles. As recognized by Halperin [46], the three pictures are distinguished only if predictions depend on the statistics representation in which long-range gauge forces, representing fluctuations about an average magnetic field, are absent. It turns out that this distinction does not lead to any difference in the filling factors at which incompressible states are expected.

We describe this classical hierarchy pictures by following Haldane and assuming that the quasiparticles are bosons. We start by considering filling factors $\nu < 1/m$ so that the system contains a dilute gas of charge $1/m$ quasiholes. As we've discussed above, the low-energy part of the Hilbert space is that of N_h bose particles in a Landau level with degeneracy N. The low-energy part of the spectrum will be broadened by interactions between the quasiholes and we expect that the ground state will be incompressible when the quasiholes can form a boson Laughlin state $i.e.$, when $N_h/N = 1/2n$. Since for this case $N_\phi = mN + N_h$ this happens when the electron filling factor is $(m + 1/2n)^{-1}$. We get precisely the same result when we describe the quasiholes as fermions since the particles are then in a Landau level with degeneracy $N + N_h$. So far everything is based on the relatively solid results of the previous subsections. The hierarchy picture is based on the expectation that the same ideas should apply to fractionally charged quasielectron excitations which are created when N_ϕ is decreased at fixed N. In the quasielectron case Bose Laughlin states for the quasielectrons occur when $N_\phi = mN - N/2n$ or at filling factor $(m - 1/2n)^{-1}$. Although we cannot offer compelling analytical arguments in favor of treating the quasielectrons as a Bose gas in a Landau level with degeneracy N, we do have a substantial [67] body of detailed numerical studies which support this notion. The line of argument can be continued by assuming that for filling factors near $(m \pm 1/2n)^{-1}$ we can apply the same argument to the quasiparticles of the Bose Laughlin states formed by the quasiparticles at the first level of the hierarchy. It is possible to demonstrate that all filling factors with an odd denominator can be generated in this way.

E. Neoclassical Hierarchies

A picture of the hierarchy states can be generated by generalizing the Chern-Simons-Landau-Ginzburg picture of the fractional quantum Hall effect [68]. In this theory the quasi-particles are associated with vortices of the Bose superfluid. The hierarchy is constructed by taking advantage of the approximate particle-vortex duality of superfluids. The gas of quasiparticles, which was mapped to a gas of vortices in a superfluid by the Chern-Simons-Landau-Ginzburg theory is mapped back to a gas of Bose particles using the particle-vortex duality of superfluids. These Bose particles can then have a fractional quantum Hall effect along the lines of the original hierarchy.

A separate picture of the hierarchy which is associated with a fermion to fermion statistical transmutation arose from several related lines of investigation. A system of electrons at filling factor $p/(2np \pm 1)$ has $2n \pm 1/p$ flux quanta for each electron. We can apply a statistical transmutation in which $2n$ flux-quanta, directed in opposition to the flux from the physical magnetic field, are attached to each electron. When this flux is treated in a mean-field approximation the result is a system with p electrons for each quantum of net magnetic flux $i.e.,$ the mean field system has p filled Landau levels. If the mean-field approximation is justified, the fractional quantum Hall effect could then be explained as an integer quantum Hall effect of composite fermions [69] consisting of electrons and an even number of attached flux quanta. It can be argued that the neglect of fluctuations is justified by the gap between Landau levels associated which produces the integer quantum Hall effect. This is not sufficient, however, since fluctuations evidently alter the physics completely when there are no interactions between the electrons and the fractional quantum Hall effect does not occur. The simplest application of this approach is to the Laughlin filling factors which are generated from this approach with $p = 1$ where it is the fermion representation equivalent of the Chern-Simons-Landau-Ginzburg theory. Lopez and Fredkin [70] have shown that the Laughlin wavefunctions can be derived in this approach when the fluctuations are treated in a random phase approximation. At this level accounting for the fluctuating magnetic fields associated with $2m$ flux quanta on the particles give rise to the correlation factors, $\prod(z_i - z_j)^{2m}$ whose significance was explained from a microscopic point of view above.

For $p > 1$ this transformation maps the system at the mean field level to a system with more than one filled Landau level. One appealing aspect of this approach is that the sequence of fractions generated with the smallest values of m are precisely the set of fractions for which the most robust fractional quantum Hall effects occur experimentally. At least at a superficial level this provides a rationale for this experimental fact. A difficulty with this approach becomes more apparent when we consider $p > 1$. The mean-field state has components in higher Landau levels which are unphysical in the strong magnetic field limit. As we have mentioned earlier any approach based on statistical transmutation requires that we enlarge our usual Hilbert space to include the high energy states with electrons in higher Landau levels. In the composite fermion approach the gap at the mean-field level is proportional to $\hbar\omega_c$, a quantity which has absolutely no meaning in the fractional Hall regime. If it were possible to treat the fluctuations accurately this scale would be replaced by the interaction scale e^2/ℓ, but to date it has only been possible to due this phenomenologically. Despite this weakness, explicitly trail wavefunctions proposed by Jain [69], in which correlation factors are attached to the mean-field states and a projection onto the lowest Landau level

is performed, have proved to be remarkably accurate for system containing a small number of electrons. For example Jain has proposed the following wavefunction,

$$\Psi_{2/5}^{J}[z] = \overline{\prod_{i<j}(z_i - z_j)^2 \Psi_{\nu=2}[z]} \tag{151}$$

where the overbar implies projection onto the lowest Landau level, as a trial wavefunction for the incompressible ground state at $\nu = 2/5$.

F. Overview on Hierarchy States

We have seen that there are several approaches for understanding hierarchy states which appear on the surface to be quite different. All are phenomenological in that they are unable to predict quantitative values for measurable quantities, such as the charge gaps at particular filling factors. They are therefore distinguishable on the basis of qualitative predictions and here, as shown by Read [71], there are no differences. Every approach makes identical predictions for the excitation structure at every hierarchy state filling factor. All approaches can be generalized so that they can generate fractional quantum Hall effects at all filling factors with odd denominators. There are, however, some mild differences in the weak statements that can be made about the values of ν at which the fractional quantum Hall effect occurs. In one common interpretation, the classical hierarchies predict that if, in a given sample, the fractional quantum Hall effect does not occur at a particular filling factor, then it cannot occur at any filling factor which evolves from that filling factor at a later generation in the hierarchy. I am not aware of violations of this prediction in experiment. On the other hand the composite fermion approach has the advantage that the filling factors in the 'main sequence' ($\nu = \nu_n$ or $\nu_n = 1 = \nu_n$) appear on a more equal footing.

The composite fermion approach also makes interesting predictions for the properties of the system at filling factors for which the ground state is not incompressible and the fractional quantum Hall effect does not occur. For $\nu = 1/2n$, there exist fermion to fermion statistical transmutation transformations in which the mean-field state is a fermion gas in no magnetic field. In this case, the many electron system should be a Fermi liquid if the fluctuations in the statistical magnetic field are sufficiently innocuous. Halperin, Read, and Lee have recently completed a thorough theoretical study of this scenario [72] which analyses the role of electron-electron interactions in suppressing fluctuations at low order in perturbation theory. On the experimental side several recent studies [73] have uncovered very suggestive evidence of Fermi-liquid-like behavior for ν near $1/2$. Experiment provides even clearer evidence of an important length scale in the system which appears to diverge like the cyclotron orbit radius associated with the mean magnetic field seen by composite fermions as ν approaches $1/2$.

XI. WHAT'S NOT HERE

These notes provide an introduction to physics in the quantum Hall regime. This continues to be a very active area of physics both theoretically and experimentally and there

are many fascinating topics which we have not been able to even touch upon. I conclude by mentioning a few of these.

In the integer quantum Hall regime theory [74,27] predicts that at the energy where the localization length diverges $\sigma_{xx} = \sigma_{xy} = 0.5 e^2/h$. So far it is not very clear that these expectations are consistent with experiment. One possible explanation is that even weak interactions change the value of σ_{xx} at the critical energy. The whole area of the interplay between disorder and interactions has not been thoroughly explored. It appears [75] that interactions create a dip in the tunneling density-of-states even in the integer quantum Hall regime. Weak interactions influence physical properties in the integer quantum Hall regime in a way which interpolates, as the localization length changes, between being similar to interaction effects in insulators and being similar to interaction effects in metals.

Gapless edge excitations are an essential companion of the quantum Hall effect as we have emphasized. The structure of the edge excitation spectrum in the fractional case can be quite intricate [9,10] in reflection of the complicated nature of the hierarchical ground states. The edge system comprises a chiral realization of a one-dimensional electron gas which is predicted [10,11] to have the power law behavior of low-temperature physical properties associated with the breakdown of Fermi liquid theory in interacting one-dimensional electron systems. The unique aspect of fractional Hall systems as one dimensional electron systems is that the exponents may be related exactly to the dimensionless quantized Hall conductance rather than being non-universal quantities dependent on high-energy physics. Experimental study of these effects has recently been initiated. [76]

Finally I would like to mention recent work on double-layer systems [77] which can exhibit a number of highly unusual properties associated with unusual broken symmetry in which phase coherence occurs between electrons in different layers which are isolated apart from having inter-layer Coulomb interactions. Experimental study of these phenomena has also been initiated recently and some interesting phenomena have been observed. [78]

ACKNOWLEDGMENTS

REFERENCES

[1] Some quantitative issues will depend on the precise nature of the disorder potential.

[2] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980). The international research effort which culminated with this discovery has been discussed by G. Landwehr, Metrologia **22**, 118 (1986).

[3] See for example, N.W. Ashcroft and M.D. Mermin, *Introduction to Solid State Physics*, (Saunders, Philadelphia, 1976).

[4] A. Widom, Phys. Rev. B **28**, 4858 (1983); Phys. Lett. **90A**, 474 (1982); A. Widom and T.D. Clark, J. Phys. D **15**, L181 (1982).

[5] R.B. Laughlin, Phys. Rev. B **23**, 5632 (1981); R.E. Prange, Phys. Rev. B **23**, 4802 (1981).

[6] B.I. Halperin, Phys. Rev. B **25**, 2185 (1982); A.H. MacDonald and P. Streda, Phys. Rev. B **29**, 1616 (1984).

[7] D.J. Thouless, J. Phys. C **14**, 3475 (1991); P. Streda, J. Phys. C **15**, L717 (1982); H. Aoki and T. Ando, Solid State Commun. **38**, 1079 (1981).

[8] D.J. Thouless, M. Kohmoto, P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982); J.E. Avron, R. Seiler, and B. Simon, Phys. Rev. Lett. **51**, 51 (1983). For a review see J. E. Avron in this volume.

[9] A.H. MacDonald, Phys. Rev. Lett. **64**, 220 (1990); M.D. Johnson and A.H. MacDonald, Phys. Rev. Lett. **67**, 2060 (1991); Michael Stone, Phys. Rev. B **42**, 8399 (1990); Michael Stone, Ann. Phys. (NY) **207**, 38 (1991).

[10] X.G. Wen, Phys. Rev. Lett. **64**, 2206 (1990); Phys. Rev. B **41**, 12838 (1990); Phys. Rev. Lett. **66**, 802 (1991); Int. J. Mod. Phys. **6**, 1711 (1992).

[11] C.L. Kane and M.P.A. Fisher, Phys. Rev. B **46**, 15233 (1992); C.L. Kane, M.P.A. Fisher, and J. Polchinski, preprint (1994).

[12] P. Streda and L. Smrcka, Phys. Stat. Solidi B **70**, 537 (1975).

[13] M. Buttiker, Phys. Rev. Lett. **57**, 1761 (1986); M. Buttiker, Phys. Rev. B **38**, 9375 (1988).

[14] D.J. Thouless, Phys. Rev. Lett. **71**, 1879 (1991); C. Wexler and D.J. Thouless, Phys. Rev. B **49**, 4815 (1994); D.J. Thouless, J. Phys. C **18**, 6211 (1995); A.H. MacDonald, T.M. Rice and W.F. Brinkman, Phys. Rev. B **28**, 3648 (1983).

[15] Qian Niu and D.J. Thouless, Phys. Rev. B **35**, 2188 (1987).

[16] M.A. Paalanen, D.C. Tsui, and A.C. Gossard, Phys. Rev. B **25**, 2185 (1982).

[17] H.P. Wei, D.C. Tsui, M. Paalanen, and A.M.M. Pruisken, Phys. Rev. Lett. **61**, 1294 (1988); Phys. Rev. B **33**, 1488 (1986).

[18] S. Koch, R. Haug, K. von Klitzing, and K. Ploog, Phys. Rev. Lett. **67**, 883 (1991).

[19] See for example B.A. Altschuler in this volume and work cited therein.

[20] M. Tsukada, J. Phys. Soc. Jpn. **41**, 1466 (1976).

[21] R.F. Kasarinov and Serge Luryi, Phys. Rev. B **25**, 7626 (1982).

[22] R.E. Prange and R. Joynt, Phys. Rev. B **25**, 2943 (1982).

[23] S.V. Iordansky, Solid State Commun. **43**, 1 (1982).

[24] A.M.M. Pruisken, Phys. Rev. B **31**, 416 (1985); A.M.M. Pruisken in *The Quantum Hall Effect*, edited by R.E. Prange and S.M. Girvin (Springer, Berlin, 1986), Chap 5.

[25] Tsuneya Ando, J. Phys. Soc. Jpn. **52**, 1740 (1983); J. Phys. Soc. Jpn. **53**, 3101 (1984); J. Phys. Soc. Jpn. **53**, 3126 (1984).

[26] B. Huckestein and B. Kramer, Phys. Rev. Lett. **64**, 1437 (1990).

[27] J.T. Chalker and P.D. Coddington, J. Phys. C **21**, 2665 (1988).

[28] G.V. Mil'nikov and I.M. Solcolov, JETP Lett. **48**, 536 (1988).

[29] See for example B.I. Halperin, Helv. Phys. Acta **56**, 75 (1983); A.H. MacDonald, Surface Science **229**, 1 (1990); X.G. Wen and A. Zee, Phys. Rev. Lett. **69**, 1811 (1992); Kun Yang, K. Moon, L. Zheng, A.H. MacDonald, S.M. Girvin, D. Yoshioka, and Shou-Cheng Zhang, Phys. Rev. Lett. **72**, 732 (1994).

[30] Corresponding definitions apply for the velocity ladder operators which change the Landau level indices.

[31] F.D.M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).

[32] F.D.M. Haldane, in *The Quantum Hall Effect*, edited by Richard E. Prange and Steven M. Girvin, (Springer-Verlag, New York, 1987).

[33] Lian Zheng and A.H. MacDonald, unpublished.

[34] A.H. MacDonald and U. Ekenberg, Phys. Rev. B **39**, 5959 (1989); S.-R. Eric Yang, A.H. MacDonald, and D. Yoshioka, Phys. Rev. B **41**, 1290 (1990).

[35] J.D. Nickila and A.H. MacDonald, unpublished.

[36] Experimental references on tilted magnetic fields; Haug, Fureaux and collaborators.

[37] R.B. Laughlin, Phys. Rev. Lett. **51**, 605 (1983).

[38] S.A. Trugman and S. Kivelson, Phys. Rev. B **31**, 5280 (1985).

[39] C. Gros and A.H. MacDonald, Phys. Rev. B **42**, 10811 (1990).

[40] Numerical exact diagonalization calculations have played an important role in the theory of the fractional quantum Hall effect which we do not discuss in any depth in these notes. For an introduction to this literature see for example, N. d'Ambrumenil and R. Morf, Phys. Rev. B **40**, 6108 (1989); G. Fano, F. Ortolani and E. Colombo, Phys. Rev. B **34**, 2670 (1986); D. Yoshioka, J. Phys. Soc. Jpn. **55**, 885 (1986); E.H. Rezayi, Phys. Rev. B **36**, 5454 (1987); Tapash Chakraborty and P. Pietiläinen, Phys. Rev. Lett. **59**, 2784 (1987).

[41] A word of caution is in order. The effect of a magnetic flux on the system appears through the kinetic energy term which becomes irrelevant when we project onto a particular Landau level. In order to consider the effect of such a flux we need to relax the projection onto a single Landau level, perform the operation, and if appropriate reapply the projection.

[42] A.H. MacDonald and S.M. Girvin, Phys. Rev. B **33**, 4414 (1986); A.H. MacDonald and S.M. Girvin, Phys. Rev. B **34**, 5639 (1986).

[43] The zero of the potential is arbitrary and linear terms could also in principle be present without contributing to the Laplacian. The fact that linear terms are absent means that the force due to the background potential vanishes at the origin from which we can deduce that the background charge which produces the potential is symmetric around the origin. We can imagine the background charge as a disk of uniform plasma charge density whose radius is larger than the maximum radius at which electrons can be found with significant probability. The electron system will then be centered on the origin.

[44] G.C. Canright and S.M. Girvin, Science **247**, 1197 (1990).

[45] J.M. Leinaas and J. Myhreim, Nuovo Cimento **37B**, 1 (1977); F. Wilkzek, *Phys. Rev.*

Lett. **49**, 957 (1980).

[46] B.I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).

[47] D. Arovas, J.R. Schrieffer, F. Wilczek and A. Zee, Nucl. Phys. **251**, 117 (1985).

[48] P. Beran and R. Morf, Phys. Rev. B **43**, 12654 (1991); M.D. Johnson and G.S. Canright, Phys. Rev. B **49**, 2947 (1994).

[49] S.M. Girvin and A.H. MacDonald, Phys. Rev. Lett. **58**, 1251 (1987).

[50] For a review of this approach see Shou Cheng Zhang, International Journal of Modern Physics B **6**, 25 (1992).

[51] S.C. Zhang, H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989); **62**, 980 (1989); D.H. Lee and S.C. Zhang, Phys. Rev. Lett. **66**, 1220 (1991).

[52] N. Read, Phys. Rev. Lett. **66**, 1220 (1989).

[53] E. Rezayi, and F.D.M. Haldane, Phys. Rev. Lett. **61**, 1985 (1988).

[54] R.P. Feynman, Phys. Rev. **91**, 1291, 1301 (1953); Phys. Rev. **94**, 262 (1954); R.P. Feynman and M. Cohen, Phys. Rev. **102**, 1189 (1956).

[55] See, for example, the thorough discussion of David Pines and Philippe Nozières *The Theory of Quantum Liguids*, (Addison-Wesley, New York, 1989).

[56] P. Hohenberg and W. Kohn papers, Phys. Rev. **136**, B864 (1964).

[57] S.M. Girvin, A.H. MacDonald, and P.M. Platzman, Phys. Rev. Lett. **54**, 581 (1985); Phys. Rev. B **33**, 2481 (1986).

[58] C. Kallin and B.I. Halperin, Phys. Rev. B **30**, 5655 (1984); Phys. Rev. B **31**, 3635 (1985).

[59] A.H. MacDonald, H.C.A. Oji, and S.M. Girvin, Phys. Rev. Lett. **55**, 2208 (1985).

[60] F.D.M. Haldane and E.H. Rezayi, Phys. Rev. Lett. **54**, 237 (1985); F.D.M. Haldane, Phys. Rev. Lett. **55**, 2095 (1985). See also Ref. [40].

[61] A.H. MacDonald and D.M. Murray, Phys. Rev. B **32**, 2707 (1985).

[62] S.M. Girvin, Phys. Rev. B **29**, 6012 (1984).

[63] X.C.Xie, S.He, and S.Das Sarma, Phys. Rev. Lett. **66**,389 (1991).

[64] Rudolf Morf, private communication.

[65] R.B. Laughlin, Surface Science **141**, 11 (1984); Chapter 7 in *The Quantum Hall Effect*, edited by Steven M. Girvin and Richard E. Prange (Springer-Verlag, New York, 1987).

[66] A.H. MacDonald, G.C. Aers, and M.W.C. Dharma-wardana, Phys. Rev. B **31**, 5529 (1985).

[67] N. d'Ambrumenil and R. Morf, Phys. Rev B **40**, 6108 (1989); Song He *et. al.* Phys. Rev. Lett. **68**, 3460 (1990) and work cited therein.

[68] D.H. Lee and M.P.A. Fisher, Phys. Rev. Lett. **63**, 903 (1989).

[69] J.K. Jain, Phys. Rev. Lett. **63**, 199 (1989); Phys. Rev. B **40**, 8079 (1989); **41**, 7653 (1990).

[70] Ana Lopez and Eduardo Fradkin, Phys. Rev. B **47**, 7080 (1993).

[71] N. Read, Phys. Rev. Lett. **65**, 1502 (1990).

[72] B.I. Halperin, P.A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).

[73] R.L. Willett, R.R. Ruel, M.A. Paalanen, K.W. West, and L.N. Pfeiffer, Phys. Rev. B **47**, 7344 (1993); D.R. Leadley, R.J. Nicholas, J.J. Harris, and C.T. Foxon, Phys. Rev. Lett. **72**, 1906 (1994); W. Kang, H.L. Störmer, L.N. Pfeiffer, K.W. Baldwin, and K.W. West, Phys. Rev. Lett. **71**, 3850 (1993); R.L. Willett, R.R. Ruel, K.W. West, and L.N. Pfeiffer, Phys. Rev. Lett. **71**, 3846 (1993); V.J. Goldman, Bo Su and J.K. Jain, Phys.

Rev. Lett. **72**, 2065 (1994).

[74] D.H. Lee, S. Kivelson, and S.C. Zhang, Phys. Rev. Lett. **68**, 2386 (1992); M.P.A. Fisher, Phys. Rev. Lett. **65**, 923 (1990).

[75] S.R.E. Yang and A.H. MacDonald, Phys. Rev. Lett. **70**, 4110 (1993).

[76] F.B. Milliken, C.P. Umbach, and R.A. Webb, preprint (1994).

[77] Kun Yang, K. Moon, L. Zheng, A.H. MacDonald, S.M. Girvin, D. Yoshioka, and Shou-Cheng Zhang, Phys. Rev. Lett. **72**, 732 (1994); X.G. Wen and A. Zee, Phys. Rev. Lett. **69**, 1811 (1992); Z.F. Ezawa and A. Iwazaki, Int. J. Mod. Phys. **19**, 3205 (1992); H.A. Fertig, Phys. Rev. B **40**, 1087 (1989); A.H. MacDonald, P.M. Platzman and Greg Boebinger, Phys. Rev. Lett. **65**, 175 (1990); S. He, S. Das Sarma and X.C. Xie, Phys. Rev. B **47**, 4394 (1993); Rene Côté, Luis Brey and A.H. MacDonald, Phys. Rev. B **46**, 10239 (1992).

[78] S.Q. Murphy, J.P. Eisenstein, G.S. Boebinger, L.N. Pfeiffer, and K.W. West, Phys. Rev. Lett. **72**, 728 (1994); T.S. Lay, Y.W. Suen, H.C. Manoharan, X. Ying, M.B. Santos and M. Shayegan, preprint (1994).

[79] See for example S. Kivelson, D.H. Lee, and S.C. Zhang, Phys. Rev. B **46**, 2223 (1992) and work cited therin.

FIG. 1. Band edge profiles for a typical two-dimensional electron gas system at a $GaAs$ $Al_xGa_{1-x}As$ interface. The electrons are bound to the interface because of charge transfer of electrons from ionized donors in the larger bandgap semiconductor. I will limit my attention here to the case where the two-dimensional electron gas occurs in the conduction band, as illustrated. I take the direction perpendicular to the interface to be the $\hat{z}$ direction. Its single-particle Schrödinger equation has bound states at the interface whose energetic separation may be considered as a large energy. This leads to two-dimensional 'subbands'; we will assume that all electrons are confined to the lowest energy 'subband'. Electrons move freely parallel to the interface but see a random potential which has contributions from remote ionized donors, as well as from ionized acceptors in the $GaAs$ and from imperfections in the interface between the two-semiconductors. The effective mass for the two-dimensional electron system in the lowest subband depends weakly on the details of a particular system but is typically quite close to the bulk $GaAs$ conduction band effective mass, $m^* \approx 0.067 m_e$.

FIG. 2. Hall bar geometry. In this typical six probe measurement a current, I, flows from source to drain. The dissipative resistance, R, is the ratio of the voltage drop along the direction of current flow ($V_A - V_B$ or $V_D - V_C$) to I. The Hall resistance, R_H, is the ratio of the voltage drop across the sample ($V_A - V_D$ or $V_B - V_C$) to I.

FIG. 3. Dissipative and Hall resistivity data for a typical two-dimensional electron gas system in the integer quantum Hall regime. These measurements were made by H.P. Wei and D.C. Tsui.

FIG. 4. Dissipative and Hall resistivity data for a typical two-dimensional electron gas system in the fractional quantum Hall regime. These measurements were made by H.P. Wei and D.C. Tsui.

FIG. 5. Complex number description of a classical cyclotron orbit.

FIG. 6. A large but finite two-dimensional electron gas. In panel (a) the chemical potential lies in a gap and the only low-energy excitations are localized at the edge of the system. In panel (b) the chemical potential lies in a mobility gap so that there are low-energy excitations in the bulk but they are localized away from the edge. In panel (c) a net current is carried from source to drain by having local equilibria at different chemical potentials on upper and lower edges.

FIG. 7. The rate at which the edge current changes with chemical potential in a gap (a) and in a mobility gap (b). This illustration is for the case where the disorder potential varies in one direction only. The properties at the edge do not change as the scale of the disorder is increased so that states localized in the bulk of the system occur at the chemical potential.

FIG. 8. Illustration of the Laughlin's gedanken experiment leading to fractionally charged quasiparticle states when an incompressibility occurs at fractional Landau level filling factors

FIG. 9. Schematic illustration of 'flux-tube' attachment.

FIG. 10. Magnetoroton excitation energies for $\nu = 1/3$, and $\nu = 1/5$ calculated by Girvin, MacDonald, and Platzman.

FIG. 11. Finite size estimates of the dependence of the chemical potential on filling factor for the hard-core model. N_L is the degeneracy of the lowest Landau level on the sphere and the chemical potential is estimated from $\nu \approx \mu_n \equiv E_0(N) - E_0(N-1)$.