# 数值分析作业# 4

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## problem 1

**a.** 首先，求Lagrange辅助函数

$$\begin{pmatrix} L_{2,0} \\ L_{2,1} \\ L_{2,2} \end{pmatrix} = \begin{pmatrix} \frac{50\left(x-\frac{3}{5}\right)\left(x-\frac{3}{10}\right)}{9} \\ -\frac{100x\left(x-\frac{3}{5}\right)}{9} \\ \frac{50x\left(x-\frac{3}{10}\right)}{9} \end{pmatrix}$$

化简得

$$\begin{pmatrix} L_{2,0} \\ L_{2,1} \\ L_{2,2} \end{pmatrix} = \begin{pmatrix} \frac{50x^2}{9} - 5x + 1 \\ -\frac{100x^2}{9} + \frac{20x}{3} \\ \frac{50x^2}{9} - \frac{5x}{3} \end{pmatrix} \tag{1.1}$$

再求Lagrange多项式，根据$P(x) = \begin{pmatrix} L_{2,0} \\ L_{2,1} \\ L_{2,2} \end{pmatrix} \times \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}^t$得

$$P(x) = \left(\frac{50\cos\left(\frac{9}{5}\right)e^{\frac{6}{5}}}{9} - \frac{100\cos\left(\frac{9}{10}\right)e^{\frac{3}{5}}}{9} + \frac{50}{9}\right)x^2 + \left(\frac{20\cos\left(\frac{9}{10}\right)e^{\frac{3}{5}}}{3} - \frac{5\cos\left(\frac{9}{5}\right)e^{\frac{6}{5}}}{3} - 5\right)x + 1$$

化简得

$$P(x) = -11.22x^2 + 3.808x + 1.0 \tag{1.2}$$

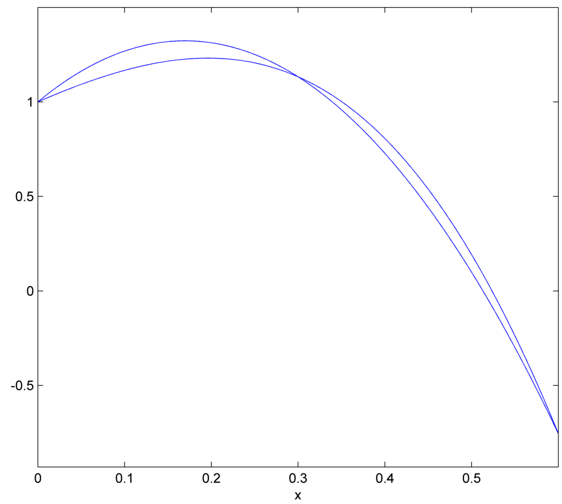


Fig.1.1 Lagrange多项式插值

再求lagrange理论误差界

$$f(x)-P(x)=\left(\frac{23\cos\left(3\xi(x)\right)\mathrm{e}^{2\xi(x)}}{3}+\frac{3\sin\left(3\xi(x)\right)\mathrm{e}^{2\xi(x)}}{2}\right)x\left(x-\frac{3}{5}\right)\left(x-\frac{3}{10}\right)$$

这里$\exists\xi(x)\in(0,0.6)$，很难求得$\xi(x)$的具体值，只能通过最大值估计误差界，为

$$\max_{0\le x\le 0.6}\left|\left(\frac{23\cos\left(3\xi(x)\right)\mathrm{e}^{2\xi(x)}}{3}+\frac{3\sin\left(3\xi(x)\right)\mathrm{e}^{2\xi(x)}}{2}\right)\right|*\max_{0\le x\le 0.6}\left|x\left(x-\frac{3}{5}\right)\left(x-\frac{3}{10}\right)\right|$$

所以，内插的误差上界为0.11371

**b.** 类似地，Lagrange辅助函数：

$$\begin{pmatrix} L_{2,0} \\ L_{2,1} \\ L_{2,2} \end{pmatrix}=\begin{pmatrix} \frac{25\left(x-\frac{12}{5}\right)\left(x-\frac{13}{5}\right)}{6} \\ -\frac{25(x-2)\left(x-\frac{13}{5}\right)}{2} \\ \frac{25(x-2)\left(x-\frac{12}{5}\right)}{3} \end{pmatrix}=\begin{pmatrix} \frac{25x^2}{6}-\frac{125x}{6}+26 \\ -\frac{25x^2}{2}+\frac{115x}{2}-65 \\ \frac{25x^2}{3}-\frac{110x}{3}+40 \end{pmatrix} \tag{1.3}$$

Lagrange多项式插值函数

$$P(x)=\left(\frac{25\sin(\log(2))}{6}-\frac{25\sin\left(\log\left(\frac{12}{5}\right)\right)}{2}+\frac{25\sin\left(\log\left(\frac{13}{5}\right)\right)}{3}\right)x^2+\left(\frac{115\sin\left(\log\left(\frac{12}{5}\right)\right)}{2}-\frac{125\sin(\log(2))}{6}-\frac{110\sin\left(\log\left(\frac{13}{5}\right)\right)}{3}\right)x+\left(26\sin(\log(2))-65\sin\left(\log\left(\frac{12}{5}\right)\right)+40\sin\left(\log\left(\frac{13}{5}\right)\right)\right)$$

化简得，

$$P(x)=-0.1306x^2+0.897x-0.6325 \tag{1.4}$$

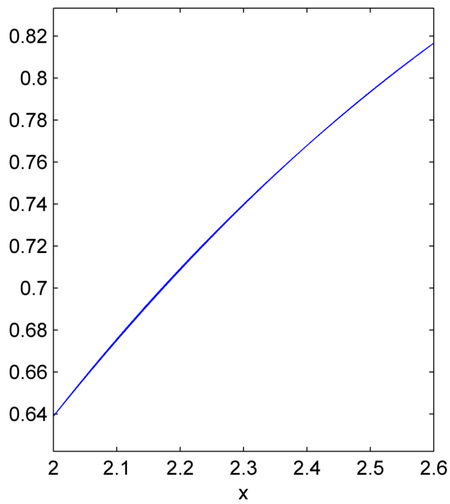


Fig.1.2 内插拟合度较好

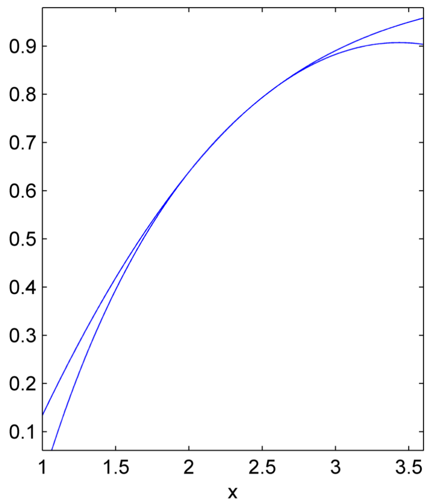

Fig.1.3 外插偏差较大

可以看出Lagrange多项式插值内插拟合得很好，但是外插误差较大。下求内插的误差上界，

$$f(x)-P(x)=\left|\left(\frac{\cos(\log(x))}{6x^3}+\frac{\sin(\log(x))}{2x^3}\right)(x-2)\left(x-\frac{12}{5}\right)\left(x-\frac{13}{5}\right)\right|$$

计算得，内插误差不超过0.00094579

下面是计算Lagrange多项式的m代码，由于用了tex代码生成看起来可能有点复杂，但是实现起来还是蛮方便的。

```
%% problem 1
hdl=fopen('p1.txt','w');
fclose(hdl);
disp('problem 1')
syms f x
f=exp(2*x)*cos(3*x);
xn=[0,0.3,0.6].';
[r,err]=LagInter(f,xn);
%disp(double(coeffs(r)))
disp(vpa(r))
%disp('Visually oriented, it is ') ;pretty(vpa(r))
fprintf('The bound of abosolute error is    ');disp(vpa(err,5))
%clipboard=latex(r);

f=sin(log(x));
xn=[2.0,2.4,2.6].';
[r,err]=LagInter(f,xn);
%disp(double(coeffs(r)))
disp(vpa(r))
%disp('Visually oriented, it is '); pretty(vpa(r))
fprintf('The bound of abosolute error is    ');disp(vpa(err,5))
```

```
function [P,boAbsErr]=LagInter(f,xn)
fhdl=fopen('p1.txt','a');
syms total L_up L_down x
sizen=size(xn,1);
```

```
total=prod(x-xn);
L_up=total./(x-xn);
L_down=subs(L_up,x,xn);L_down=L_down(L_down≠0);
L=L_up./L_down;
fprintf(fhdl,'\\[ %s \\] \n \\[ %s\\] ...
    \n',latex(L),latex(collect(L)));
yn=subs(f,x,xn);
P=L.'*yn;
fprintf(fhdl,'\\[ %s \\] \n \\[ %s \\] \n ...
    ',latex(collect(P)),latex(collect(vpa(P))));
%fprintf('\\[ %s \\] \n  \n',latex(collect(P)));
P=collect(P);%P=coeffs(P);
fprintf(fhdl,'\\[ %s \\] ...
    \n',latex(abs(diff(f,sizen)/factorial(sizen)*total)));
resid1=matlabFunction(abs(diff(f,sizen)/factorial(sizen)));
resid2=matlabFunction(abs(total));
testnum=min(xn):1e-5:max(xn);
boAbsErr=max(resid1(testnum))*max(resid2(testnum));
fclose(fhdl);
```

# problem 2

Lagrange辅助函数：

$$\begin{pmatrix} L_{2,0} \\ L_{2,1} \\ L_{2,2} \end{pmatrix} = \begin{pmatrix} -(x-1)(x-2)\left(x-\frac{1}{2}\right) \\ \frac{8x(x-1)(x-2)}{3} \\ -2x(x-2)\left(x-\frac{1}{2}\right) \\ \frac{x(x-1)\left(x-\frac{1}{2}\right)}{3} \end{pmatrix} = \begin{pmatrix} -1x^3+\frac{7x^2}{2}-\frac{7x}{2}+1 \\ \frac{8x^3}{3}-8x^2+\frac{16x}{3} \\ -2x^3+5x^2-2x \\ \frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{6} \end{pmatrix} \tag{2.1}$$

Lagrange多项式$x^3$项系数为

$$\frac{8}{3}y-6+\frac{2}{3}$$

其值为6，解得：$y=4.25$

# problem 3

**定理 3.1** Neville's method $f$在$x_0, x_1, ...x_n$上有定义，$x_i \neq x_j$, 那么f的第$k+1$阶内插多项式为

$$P_{0,1,\ldots,n} = \frac{(x-x_j)P_{-j}(x)-(x-x_i)P_{-i}(x)}{x_i-x_j} \tag{3.1}$$

其中，$P_{-j}$表示$P_{0,i,\ldots,j,j-1,j+1,\ldots,n}$，为第$k$阶Lagrange多项式

**a** 由$P_{2,3}$和$P_2$得，

$$P_2 = 4$$

**b** 由$P_{0,1}$,得$P_1=3$, $P_0=1$; 由$P_{0,2}$得,$P_2=3$, $P_0=1$
所以$P_{0,1,2}(2.5)=2.25$
又因为$P_{1,2,3}(2.5)=3$,所以

$$P_{0,1,2,3}(2.5)=2.875$$

## problem 4

**定义 4.1** Divided Difference 对于函数$f$，如果$x_0 ... x_n$已知，则定义差分递推式为

$$f[x_\nu] = f(x_\nu) \tag{4.1}$$
$$\text{when } \nu \in \{0, \dots, k\}$$
$$f[x_\nu, \dots, x_{\nu+j}] = \frac{f[x_{\nu+1}, \dots, x_{\nu+j}] - f[x_\nu, \dots, x_{\nu+j-1}]}{x_{\nu+j} - x_\nu} \tag{4.2}$$
$$\text{when } \nu \in \{0, \dots, k-j\},\ j \in \{1, \dots, k\}$$

代入递推式得，

| | | | |
|---|---|---|---|
| $x_0 = 0.0$ | $f[x_0] = 1$ | | |
| | | $f[x_0, x_1] = 5$ | |
| $x_1 = 0.4$ | $f[x_1] = 3$ | | $f[x_0, x_1, x_2] = \frac{50}{7}$ |
| | | $f[x_1, x_2] = 10$ | |
| $x_2 = 0.7$ | $f[x_2] = 6$ | | |

## problem 5

三次样条插值，先假设

$$S_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \text{ for } x \in [x_i, x_{i+1}].$$

需要满足:

1. $S_i(x_i) = y_i, S_i(x_{i+1}) = y_{i+1}$
2. $S'_{i-1}(x_i) = S'_i(x_i)$
3. $S''_{i-1}(x_i) = S''_i(x_i)$
4. $S_0(x)$ 和$S_{n-1}(x)$ 两个边值条件

**a).** 对自然样条插值，还需要满足：$S''_0(x_0) = 0, S''_{n-1}(x_n) = 0$
利用老师的结论可以较快解得**c**：

$$\begin{bmatrix} 1 & 0 & & & & & \\ h_0 & 2(h_0+h_1) & h_1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & & 0 & 1 \end{bmatrix} \mathbf{f}(\tilde{\mathbf{a}}) = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}.$$

代入得，$\vec{c} = \vec{0}$，$\vec{d} = \vec{0}$, $\vec{b} = (0, 1, 0)$, $\vec{a} = (0, 1, 2)$. 所以

$$S(x) = x \qquad \text{for } x \in [0, 2] \tag{5.1}$$

**b).** 对自然样条插值，该题要求满足：$S_0'(x_0)=f'(x_0)=S_{n-1}'(x_n)=f'(x_n)=1$

$$\begin{bmatrix} 2h_0 & h_0 & & & & & \\ h_0 & 2(h_0+h_1) & h_1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & & h_{n-1} & 2h_{n-1} \end{bmatrix} \mathbf{f}'(\tilde{\mathbf{a}}) = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}.$$

解得

$$S(x)=x \qquad \text{for } x\in[0,2] \tag{5.2}$$

说明尽管用了不同的边值条件，但是结果是一样的！自然样条要求两端不能是最大最小，而是拐点；而对于这道题，拐点处斜率恰好为1.

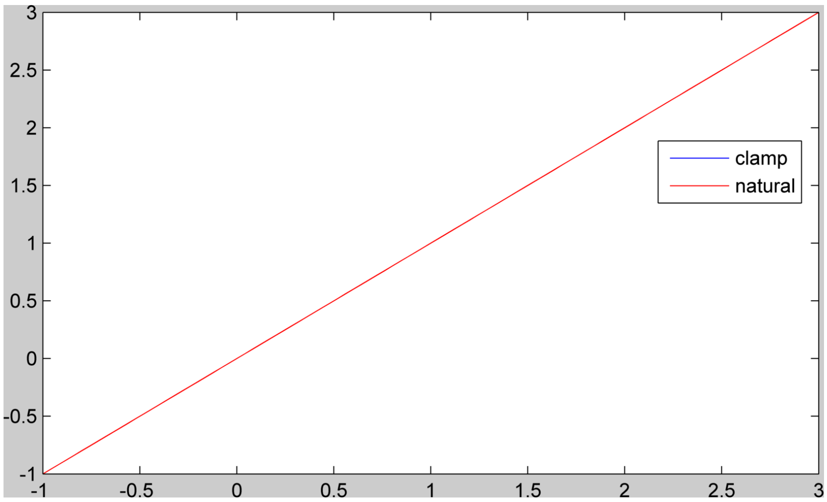


Fig.5.1 两种样条插值

```
%% probelm 5
x=[0,1,2];xx=0:0.01:2;
y=[0,1,2];
ppfunc=csape(x,[0,y,0],'second');
fnplt(ppfunc,[0,2]);
aa=fnbrk(ppfunc,1);
res1=poly2sym(aa.coefs);fprintf('For problem 5 : \npiecewise ...
    function on [0,1] is %s\n and on [1,2] is ',latex(res1));
aaa=fnbrk(ppfunc,2);
syms X
res2=poly2sym(aaa.coefs,X-1);disp(res2)
ppfunc.coefs;
hold on
ppfunc=csape(x,[1,y,1],'clamp');
fnplt(ppfunc,[0,2],'r');
```