Derivation of the maximum likelihood (ML) solutions

The negative log-likelihood (NLL) cost

The estimated normalized PSD matrix is

$$\hat{\mathbf{\Gamma}}_x = \begin{bmatrix} 1 & \hat{\gamma}_x \\ \hat{\gamma}_x^* & 1 \end{bmatrix} \tag{1}$$

where $\gamma_x(\omega) = \frac{E[X_1(\omega,t)X_2^*(\omega,t)]}{\sqrt{E[|X_1(\omega,t)|^2]E[|X_2(\omega,t)|^2]}}$, and $\hat{\gamma}_x(\omega)$ is an estimate of $\gamma_x(\omega)$ obtained by replacing the expectations with sample averages.

The true PSD matrix is supposed to have the form

$$\mathbf{\Gamma}_x = P_s \begin{bmatrix} 1 & e^{j\phi} \\ e^{-j\phi} & 1 \end{bmatrix} + P_v \begin{bmatrix} 1 & \gamma_v \\ \gamma_v^* & 1 \end{bmatrix} = P_s \mathbf{\Phi} + P_v \mathbf{\Gamma}_v \tag{2}$$

where P_s and P_v are the powers of source signal and noises, respectively, and $\mathbf{\Phi}$ and $\mathbf{\Gamma}_v$ the source signal and noise coherence matrices, respectively. Note that we do *not* assume $P_s + P_v = 1$. The degree of freedoms of this model is three.

Then, the sample size normalized negative logarithm likelihood (NLL) function for the power normalized observations is given by

$$J(\hat{\mathbf{\Gamma}}_x | P_s, P_v, \phi) = \log\det \mathbf{\Gamma}_x + \operatorname{tr}(\mathbf{\Gamma}_x^{-1}\hat{\mathbf{\Gamma}}_x) + 2\log\pi \tag{3}$$

where det and tr denote the determinant and trace of a square matrix, respectively.

The gradients of NLL cost

To derive the ML solution, we need to solve the system of equations $\partial J/\partial P_s = 0$, $\partial J/\partial P_v = 0$ and $\partial J/\partial \phi = 0$. Note that for any invertible matrix $\boldsymbol{A}$, we have

$$\mathbf{0} = d\boldsymbol{I} = d(\boldsymbol{A}\boldsymbol{A}^{-1}) = d\boldsymbol{A}\boldsymbol{A}^{-1} + \boldsymbol{A}d\boldsymbol{A}^{-1}$$

Thus,

$$d\boldsymbol{A}^{-1} = -\boldsymbol{A}^{-1}d\boldsymbol{A}\boldsymbol{A}^{-1} \tag{4}$$

With this fact, we can easily write down the derivative of $\mathbf{\Gamma}_x^{-1}$ with respect to P_s, P_v and ϕ. For example, we have

$$\frac{\partial \mathbf{\Gamma}_x^{-1}}{\partial \phi} = -\mathbf{\Gamma}_x^{-1}\frac{\partial \mathbf{\Gamma}_x}{\partial \phi}\mathbf{\Gamma}_x^{-1} = -\mathbf{\Gamma}_x^{-1}P_s\frac{\partial \begin{bmatrix} 1 & e^{j\phi} \\ e^{-j\phi} & 1 \end{bmatrix}}{\partial \phi}\mathbf{\Gamma}_x^{-1} = -\mathbf{\Gamma}_x^{-1}P_s\begin{bmatrix} 0 & je^{j\phi} \\ -je^{-j\phi} & 0 \end{bmatrix}\mathbf{\Gamma}_x^{-1}$$

Here, we give the gradients in their compact forms as below,

$$
\begin{aligned}
\frac{\partial J}{\partial P_s} &= \operatorname{tr}(\boldsymbol{\Gamma}_x^{-1}\boldsymbol{\Phi}) - \operatorname{tr}(\boldsymbol{\Gamma}_x^{-1}\hat{\boldsymbol{\Gamma}}_x\boldsymbol{\Gamma}_x^{-1}\boldsymbol{\Phi}) \\
\frac{\partial J}{\partial P_v} &= \operatorname{tr}(\boldsymbol{\Gamma}_x^{-1}\boldsymbol{\Gamma}_v) - \operatorname{tr}(\boldsymbol{\Gamma}_x^{-1}\hat{\boldsymbol{\Gamma}}_x\boldsymbol{\Gamma}_x^{-1}\boldsymbol{\Gamma}_v) \\
\frac{1}{P_s}\frac{\partial J}{\partial \phi} &= \operatorname{tr}(\boldsymbol{\Gamma}_x^{-1}\frac{d\boldsymbol{\Phi}}{d\phi}) - \operatorname{tr}(\boldsymbol{\Gamma}_x^{-1}\hat{\boldsymbol{\Gamma}}_x\boldsymbol{\Gamma}_x^{-1}\frac{d\boldsymbol{\Phi}}{d\phi})
\end{aligned} \tag{5}
$$

where $\frac{d\boldsymbol{\Phi}}{d\phi} = \begin{bmatrix} 0 & je^{j\phi} \\ -je^{-j\phi} & 0 \end{bmatrix}$.

Solving the set of equations (5)

Proof for relation $\boldsymbol{\Gamma}_x = \hat{\boldsymbol{\Gamma}}_x$ at a stationary point

This conclusion seems obvious, but the proof itself can be tricky. Let us rewrite the set of equations $\partial J/\partial P_s = 0$, $\partial J/\partial P_v = 0$ and $\partial J/\partial \phi = 0$ as

$$
\operatorname{tr}[(\boldsymbol{I}-\boldsymbol{\Gamma}_x^{-1}\hat{\boldsymbol{\Gamma}}_x)(\boldsymbol{\Gamma}_x^{-1}\boldsymbol{\Phi})] = 0,\ \operatorname{tr}[(\boldsymbol{I}-\boldsymbol{\Gamma}_x^{-1}\hat{\boldsymbol{\Gamma}}_x)(\boldsymbol{\Gamma}_x^{-1}\boldsymbol{\Gamma}_v)] = 0,\ \operatorname{tr}[(\boldsymbol{I}-\boldsymbol{\Gamma}_x^{-1}\hat{\boldsymbol{\Gamma}}_x)(\boldsymbol{\Gamma}_x^{-1}\frac{d\boldsymbol{\Phi}}{d\phi})] = 0
$$

Then, using property $\operatorname{tr}(\boldsymbol{AB}) = \operatorname{tr}(\boldsymbol{BA})$, we can further rewrite them as

$$
\operatorname{tr}[(\boldsymbol{I} - \boldsymbol{\Gamma}_x^{-0.5}\hat{\boldsymbol{\Gamma}}_x\boldsymbol{\Gamma}_x^{-0.5})(\boldsymbol{\Gamma}_x^{-0.5}\boldsymbol{\Phi}\boldsymbol{\Gamma}_x^{-0.5})] = 0
$$

$$
\operatorname{tr}[(\boldsymbol{I} - \boldsymbol{\Gamma}_x^{-0.5}\hat{\boldsymbol{\Gamma}}_x\boldsymbol{\Gamma}_x^{-0.5})(\boldsymbol{\Gamma}_x^{-0.5}\boldsymbol{\Gamma}_v\boldsymbol{\Gamma}_x^{-0.5})] = 0
$$

$$
\operatorname{tr}[(\boldsymbol{I} - \boldsymbol{\Gamma}_x^{-0.5}\hat{\boldsymbol{\Gamma}}_x\boldsymbol{\Gamma}_x^{-0.5})(\boldsymbol{\Gamma}_x^{-0.5}\frac{d\boldsymbol{\Phi}}{d\phi}\boldsymbol{\Gamma}_x^{-0.5})] = 0
$$

Note that we can always rewrite $\operatorname{tr}(\boldsymbol{AB})$ as $[\operatorname{vec}(\boldsymbol{A})]^T\operatorname{vec}(\boldsymbol{B}^T)$, where vec is a vectorization operator. Thus, $\operatorname{vec}(\boldsymbol{I} - \boldsymbol{\Gamma}_x^{-0.5}\hat{\boldsymbol{\Gamma}}_x\boldsymbol{\Gamma}_x^{-0.5})$ can be solved from the set of equations

$$
\begin{bmatrix} [\operatorname{vec}(\boldsymbol{\Gamma}_x^{-0.5}\boldsymbol{\Phi}\boldsymbol{\Gamma}_x^{-0.5})]^T \\ [\operatorname{vec}(\boldsymbol{\Gamma}_x^{-0.5}\boldsymbol{\Gamma}_v\boldsymbol{\Gamma}_x^{-0.5})]^T \\ [\operatorname{vec}(\boldsymbol{\Gamma}_x^{-0.5}\frac{d\boldsymbol{\Phi}}{d\phi}\boldsymbol{\Gamma}_x^{-0.5})]^T \end{bmatrix} \operatorname{vec}(\boldsymbol{I} - \boldsymbol{\Gamma}_x^{-0.5}\hat{\boldsymbol{\Gamma}}_x\boldsymbol{\Gamma}_x^{-0.5})^T = \mathbf{0} \tag{6}
$$

We let

$$
\begin{aligned}
[\operatorname{vec}(\boldsymbol{\Gamma}_x^{-0.5}\boldsymbol{\Phi}\boldsymbol{\Gamma}_x^{-0.5})]^T &= [a, b, b^*, a] \\
[\operatorname{vec}(\boldsymbol{\Gamma}_x^{-0.5}\boldsymbol{\Gamma}_v\boldsymbol{\Gamma}_x^{-0.5})]^T &= [c, d, d^*, c] \\
[\operatorname{vec}(\boldsymbol{\Gamma}_x^{-0.5}\frac{d\boldsymbol{\Phi}}{d\phi}\boldsymbol{\Gamma}_x^{-0.5})]^T &= [e, f, f^*, e] \\
[\operatorname{vec}(\boldsymbol{I} - \boldsymbol{\Gamma}_x^{-0.5}\hat{\boldsymbol{\Gamma}}_x\boldsymbol{\Gamma}_x^{-0.5})]^T &= [x, y; y^*, x]
\end{aligned}
$$

Then (6) can be rewritten as

$$\begin{bmatrix} 2a & b & b^* \\ 2c & d & d^* \\ 2e & f & f^* \end{bmatrix} \begin{bmatrix} x \\ y \\ y^* \end{bmatrix} = \mathbf{0}$$

Then, we need to show that $\det(\begin{bmatrix} 2a & b & b^* \\ 2c & d & d^* \\ 2e & f & f^* \end{bmatrix}) \neq 0$ such the above equation leads to $\begin{bmatrix} x \\ y \\ y^* \end{bmatrix} = \mathbf{0}$ and thus $\mathrm{vec}(\boldsymbol{I} - \boldsymbol{\Gamma}_x^{-0.5}\hat{\boldsymbol{\Gamma}}_x\boldsymbol{\Gamma}_x^{-0.5})$, which eventually leads to $\hat{\boldsymbol{\Gamma}}_x = \boldsymbol{\Gamma}_x$ at a stationary point.

Note that

$$\mathrm{rank}(\begin{bmatrix} 2a & b & b^* \\ 2c & d & d^* \\ 2e & f & f^* \end{bmatrix}) = \mathrm{rank}(\begin{bmatrix} a & b & b^* \\ c & d & d^* \\ e & f & f^* \end{bmatrix}) = \mathrm{rank}(\begin{bmatrix} [\mathrm{vec}(\boldsymbol{\Gamma}_x^{-0.5}\boldsymbol{\Phi}\boldsymbol{\Gamma}_x^{-0.5})]^T \\ [\mathrm{vec}(\boldsymbol{\Gamma}_x^{-0.5}\boldsymbol{\Gamma}_v\boldsymbol{\Gamma}_x^{-0.5})]^T \\ [\mathrm{vec}(\boldsymbol{\Gamma}_x^{-0.5}\frac{d\boldsymbol{\Phi}}{d\phi}\boldsymbol{\Gamma}_x^{-0.5})]^T \end{bmatrix})$$

where $\mathrm{rank}(\cdot)$ denotes the rank of a matrix. Also note that

$$\mathrm{vec}(\boldsymbol{ABC}) = (\boldsymbol{C}^T \otimes \boldsymbol{A})\mathrm{vec}(\boldsymbol{B})$$

Thus

$$\mathrm{rank}(\begin{bmatrix} [\mathrm{vec}(\boldsymbol{\Gamma}_x^{-0.5}\boldsymbol{\Phi}\boldsymbol{\Gamma}_x^{-0.5})]^T \\ [\mathrm{vec}(\boldsymbol{\Gamma}_x^{-0.5}\boldsymbol{\Gamma}_v\boldsymbol{\Gamma}_x^{-0.5})]^T \\ [\mathrm{vec}(\boldsymbol{\Gamma}_x^{-0.5}\frac{d\boldsymbol{\Phi}}{d\phi}\boldsymbol{\Gamma}_x^{-0.5})]^T \end{bmatrix}) = \mathrm{rank}(\begin{bmatrix} [\mathrm{vec}(\boldsymbol{\Phi})]^T \\ [\mathrm{vec}(\boldsymbol{\Gamma}_v)]^T \\ [\mathrm{vec}(\frac{d\boldsymbol{\Phi}}{d\phi})]^T \end{bmatrix})$$

since $\mathrm{vec}(\boldsymbol{A}) \mapsto \mathrm{vec}(\boldsymbol{\Gamma}_x^{-0.5}\boldsymbol{A}\boldsymbol{\Gamma}_x^{-0.5}) = (\boldsymbol{\Gamma}_x^{-0.5} \otimes \boldsymbol{\Gamma}_x^{-0.5})\mathrm{vec}(\boldsymbol{A})$ is an invertible linear transform. Now, we have

$$\mathrm{rank}(\begin{bmatrix} [\mathrm{vec}(\boldsymbol{\Phi})]^T \\ [\mathrm{vec}(\boldsymbol{\Gamma}_v)]^T \\ [\mathrm{vec}(\frac{d\boldsymbol{\Phi}}{d\phi})]^T \end{bmatrix}) = \mathrm{rank}(\begin{bmatrix} 1 & e^{-j\phi} & e^{j\phi} & 1 \\ 1 & \gamma_v^* & \gamma_v & 1 \\ 0 & -je^{-j\phi} & je^{j\phi} & 0 \end{bmatrix}) = \begin{cases} 2, & \text{if } \gamma_v = e^{j\phi} \\ 3, & \text{otherwise} \end{cases}$$

Since we assume $|\gamma_v| < 1$, we must have $\mathrm{rank}(\begin{bmatrix} [\mathrm{vec}(\boldsymbol{\Phi})]^T \\ [\mathrm{vec}(\boldsymbol{\Gamma}_v)]^T \\ [\mathrm{vec}(\frac{d\boldsymbol{\Phi}}{d\phi})]^T \end{bmatrix}) = 3$. Thus, solving (6) must lead to $\mathrm{vec}(\boldsymbol{I} - \boldsymbol{\Gamma}_x^{-0.5}\hat{\boldsymbol{\Gamma}}_x\boldsymbol{\Gamma}_x^{-0.5}) = \mathbf{0}$, i.e., $\boldsymbol{\Gamma}_x = \hat{\boldsymbol{\Gamma}}_x$.

Solving for P_s, P_v and ϕ from relation $\boldsymbol{\Gamma}_x = \hat{\boldsymbol{\Gamma}}_x$

Now, starting from $\boldsymbol{\Gamma}_x = \hat{\boldsymbol{\Gamma}}_x$, we can have

$$P_s + P_v = 1,\ P_s e^{j\phi} + P_v\gamma_v = \hat{\gamma}_x$$

Thus,

$$e^{j\phi} = \frac{\hat{\gamma}_x - P_v\gamma_v}{P_s} = \frac{\hat{\gamma}_x - P_v\gamma_v}{1 - P_v}$$

Since $|e^{j\phi}| = 1$, we have

$$\left| \frac{\hat{\gamma}_x - P_v \gamma_v}{1 - P_v} \right| = 1$$

We can solve for P_v from the above equation. There will be two solutions for P_v. But, the one greater than 1 should be discarded as $P_s = 1 - P_v \geq 0$. Those algebra calculations leading to the final equations in the paper are straightforward, and we do not show them here.

Lastly, we would like to emphasize that we do not assume $P_s + P_v = 1$ for the true values of P_s and P_v, although it turns out that their maximum likelihood estimations satisfies property $P_s + P_v = 1$. Actually, if we are to normalize the empirical powers of observations to another *same* arbitrary number, say 10, it turns out that the maximum likelihood estimations for P_s and P_v will satisfy property $P_s + P_v = 10$. The target here is to estimate phase ϕ. Thus, it is perfect to assume that the empirical powers of observations are normalized to an arbitrary value (must be the same for both observations), say 1 without loss of generality.

The Hessian of NLL

Again, (4) is repeatedly used to derive the Hessian of the NLL cost. No further trick expect for the approximation $\hat{\mathbf{\Gamma}}_x \approx \mathbf{\Gamma}_x$ with sample size $T \to \infty$ is used to derive the Fisher information formula given in the paper. We do not show the process here.