EE 502 P: Analytical Methods for Electrical Engineering

Homework 7: Graphs and Networks, Probabilty and Markov Processes

Due December 12, 2021 by 11:59 PM

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Instructions: Please use this notebook as a template. Answer all questions using well formatted Markdown with embedded LaTeX equations, executable Jupyter cells, or both. Submit your homework solutions as an `.ipynb` file via Canvas.

Although you may discuss the homework with others, you must turn in your own, original work.

Things to remember:

- Use complete sentences. Equations should appear in text as grammatical elements.
- Comment your code.
- Label your axes. Title your plots. Use legends where appropriate.
- Before submitting a notebook, choose Kernel -> Restart and Run All to make sure your notebook runs when the cells are evaluated in order.

0. Warmup (Do not turn in)

- Make sure you download, read, and run the notebook for lectures 8 and 9. Work through the notebook cell by cell and see what happens when you change the expressions, and make up some of your own.
- The material covered in class is intended to be an introductory overview of the incredibly rich and expansive subject of Graph Theory.
- Most of the section on basic graph theory is from chapter one of

  > Bollobas, [Modern Graph Theory](#).

- The spanning tree algorithm by Prim is from

  > Gibbons, [Algorithmic Graph Theory](#).

- Graphs as matrices is covered in

  > Nica, [A Brief Introduction to Spectral Graph Theory](#).

The consensus algorithm is described in [these notes](#) by Richard Murray.

You will also need `networkx` which can be obtained [here](#). If you are on Google CoLab, see [here](#) for how to install packages.

- Most of the section on probability is from chapter of the really, really, really good book:

  > Feller, [An Introduction to Probability Theory and Its Applications, Vol 1](#), 3rd Edition, Wiley, 1968.

- The material on Markov Chains can be found in numerous textbooks. A good online source is Chapter 11 of

> **Grinstead, [Introduction to Probability](#).**

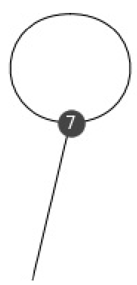
1. Creating a Graph

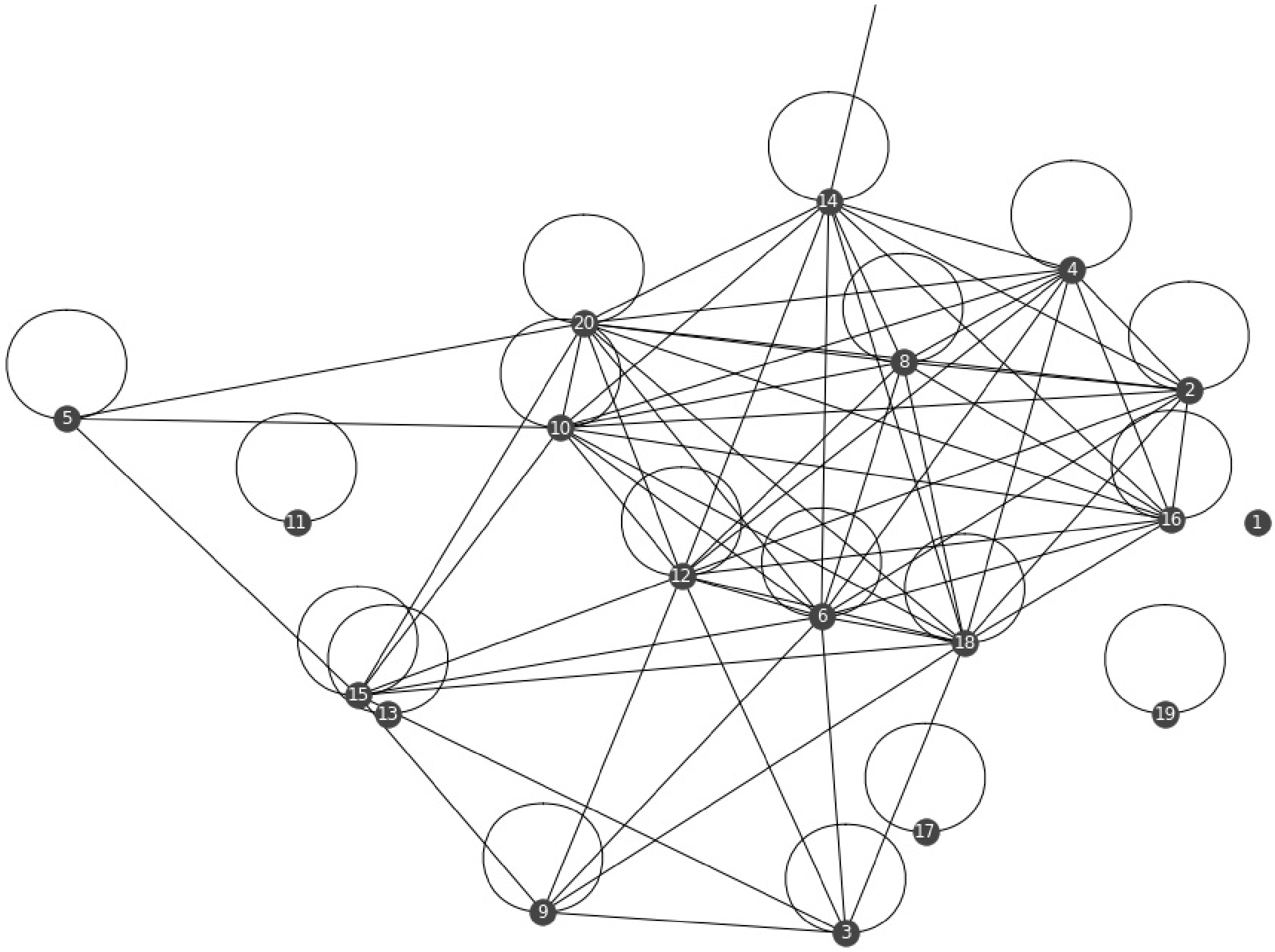
Let $V=\{1,\ldots,20\}$ and say that $uv \in E$ if and only if u and v have a common factor other than 1. Use `networkx` to define this graph and render it. Make sure to label the nodes.

In [132]:

```
import networkx as nx
import matplotlib.pyplot as plt

def hcf(x, y):
   if x > y:
       smaller = y
   else:
       smaller = x
   for i in range(1,smaller + 1):
       if((x % i == 0) and (y % i == 0)):
           hcf = i
   return x,y,hcf

V = [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20]
result_pair = []
result_factor = []

for x in V:
  for y in V:
    a,b,c = hcf(x,y)
    if c > 1 and a <= b:
      result_pair.append((a,b))
      result_factor.append((c))
print(result_pair)

G = nx.Graph()
G.add_nodes_from(V)
G.add_edges_from(result_pair)

basic_graph,ax = plt.subplots(1,1,figsize=(17,17))
nx.draw(G,
        ax=ax,
        pos=nx.kamada_kawai_layout(G),
        with_labels=True,
        node_color='#444444',
        font_color="white")
```

```
[(2, 2), (2, 4), (2, 6), (2, 8), (2, 10), (2, 12), (2, 14), (2, 16), (2, 18), (2, 20), (3
, 3), (3, 6), (3, 9), (3, 12), (3, 15), (3, 18), (4, 4), (4, 6), (4, 8), (4, 10), (4, 12)
, (4, 14), (4, 16), (4, 18), (4, 20), (5, 5), (5, 10), (5, 15), (5, 20), (6, 6), (6, 8),
(6, 9), (6, 10), (6, 12), (6, 14), (6, 15), (6, 16), (6, 18), (6, 20), (7, 7), (7, 14), (
8, 8), (8, 10), (8, 12), (8, 14), (8, 16), (8, 18), (8, 20), (9, 9), (9, 12), (9, 15), (9
, 18), (10, 10), (10, 12), (10, 14), (10, 15), (10, 16), (10, 18), (10, 20), (11, 11), (1
2, 12), (12, 14), (12, 15), (12, 16), (12, 18), (12, 20), (13, 13), (14, 14), (14, 16), (
14, 18), (14, 20), (15, 15), (15, 18), (15, 20), (16, 16), (16, 18), (16, 20), (17, 17),
(18, 18), (18, 20), (19, 19), (20, 20)]
```





2. Bipartite Graphs

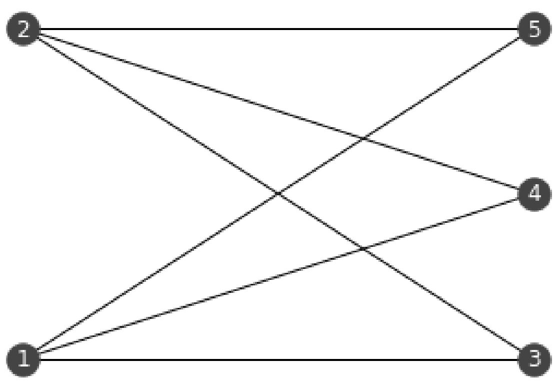
a) What is the greatest number of edges a connected bipartite graph with 5 nodes can have? Draw all five node graphs with this number of edges (up to isomorphism).

b) What is the least number of edges a connected bipartite graph with 5 nodes can have? Draw all five node graphs with this number of edges (up to isomorphism).

a)

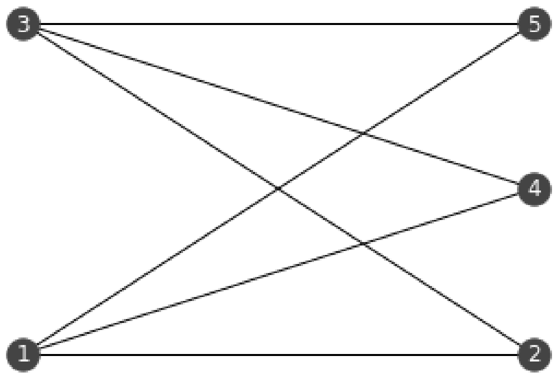
In [133]:

```
G = nx.Graph()
G.add_nodes_from([1,2,3,4,5])
G.add_edges_from([(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)])

bipartite_graph,ax = plt.subplots(1,1)
nx.draw(G, ax=ax, pos=nx.bipartite_layout(G,[1,2]),with_labels=True, node_color='#444444
',font_color="white")
```



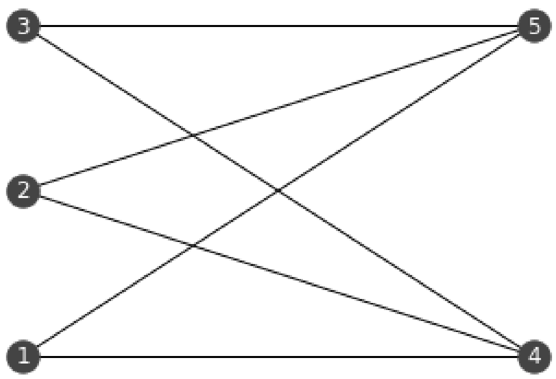
In [134]:

```
G = nx.Graph()
G.add_nodes_from([1,2,3,4,5])
G.add_edges_from([(1,2),(1,4),(1,5),(2,3),(3,4),(3,5)])

bipartite_graph,ax = plt.subplots(1,1)
nx.draw(G, ax=ax, pos=nx.bipartite_layout(G,[1,3]),with_labels=True, node_color='#444444
',font_color="white")
```



In [135]:

```
G = nx.Graph()
G.add_nodes_from([1,2,3,4,5])
G.add_edges_from([(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)])

bipartite_graph,ax = plt.subplots(1,1)
nx.draw(G, ax=ax, pos=nx.bipartite_layout(G,[1,2,3]),with_labels=True, node_color='#4444
44',font_color="white")
```

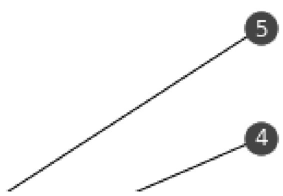


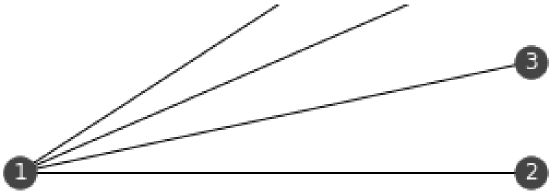
the greatest number of edges a connected bipartite graph with 5 nodes is 6

b)

In [136]:

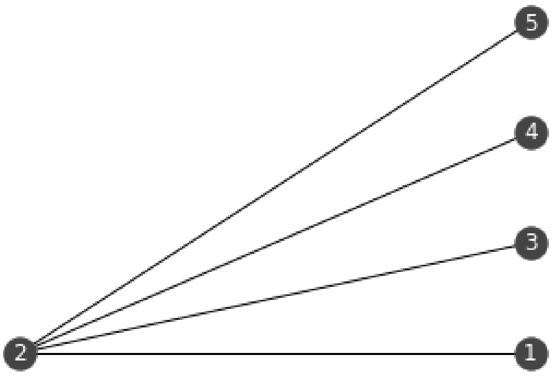
```
G = nx.Graph()
G.add_nodes_from([1,2,3,4,5])
G.add_edges_from([(1,2),(1,3),(1,4),(1,5)])

bipartite_graph,ax = plt.subplots(1,1)
nx.draw(G, ax=ax, pos=nx.bipartite_layout(G,[1]),with_labels=True, node_color='#444444',
font_color="white")
```





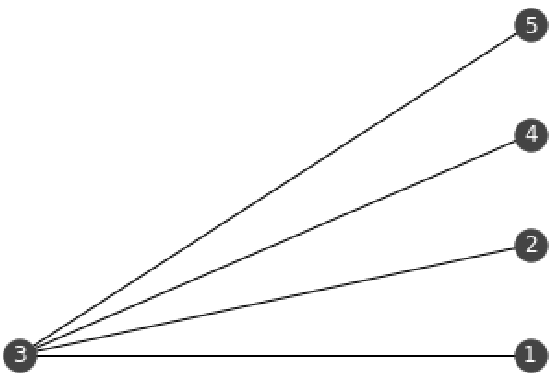
In [137]:

```
G = nx.Graph()
G.add_nodes_from([1,2,3,4,5])
G.add_edges_from([(2,1),(2,3),(2,4),(2,5)])

bipartite_graph,ax = plt.subplots(1,1)
nx.draw(G, ax=ax, pos=nx.bipartite_layout(G,[2]),with_labels=True, node_color='#444444',
font_color="white")
```



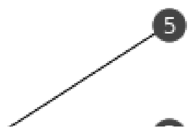
In [138]:

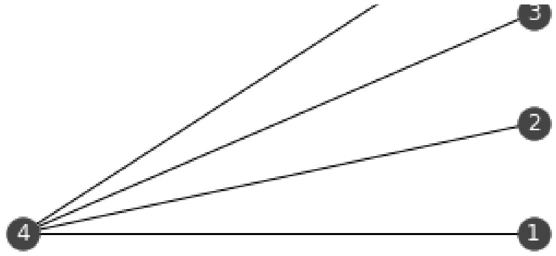
```
G = nx.Graph()
G.add_nodes_from([1,2,3,4,5])
G.add_edges_from([(3,1),(3,2),(3,4),(3,5)])

bipartite_graph,ax = plt.subplots(1,1)
nx.draw(G, ax=ax, pos=nx.bipartite_layout(G,[3]),with_labels=True, node_color='#444444',
font_color="white")
```



In [139]:

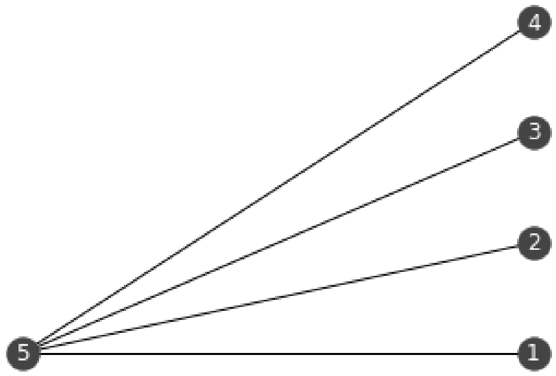
```
G = nx.Graph()
G.add_nodes_from([1,2,3,4,5])
G.add_edges_from([(4,2),(4,3),(1,4),(4,5)])

bipartite_graph,ax = plt.subplots(1,1)
nx.draw(G, ax=ax, pos=nx.bipartite_layout(G,[4]),with_labels=True, node_color='#444444',
font_color="white")
```





In [140]:

```
G = nx.Graph()
G.add_nodes_from([1,2,3,4,5])
G.add_edges_from([(5,2),(5,3),(5,4),(1,5)])

bipartite_graph,ax = plt.subplots(1,1)
nx.draw(G, ax=ax, pos=nx.bipartite_layout(G,[5]),with_labels=True, node_color='#444444',
font_color="white")
```



the least number of edges a connected bipartite graph with 5 nodes is 4

3. The Laplacian

Consider the random graphs returned by `nx.fast_gnp_random_graph(n,p)` where n is the number of nodes and p is the probability of an edge between any two nodes.

Make a list of graphs with 10 nodes with p ranging from 0 to 1 by steps of 0.01. For each graph, compute the eigenvalues of the Laplacian. Then plot all the eigenvalues for against p. For example, above $p = 0.01$ you would have a point for each of the 10 real eigenvalues of the graph you made with $p = 0.01$.

What trends do you see in the eigenvalues?

Based on experimentation, what are the eigenvalues of the Laplacian of a completely connected graph?

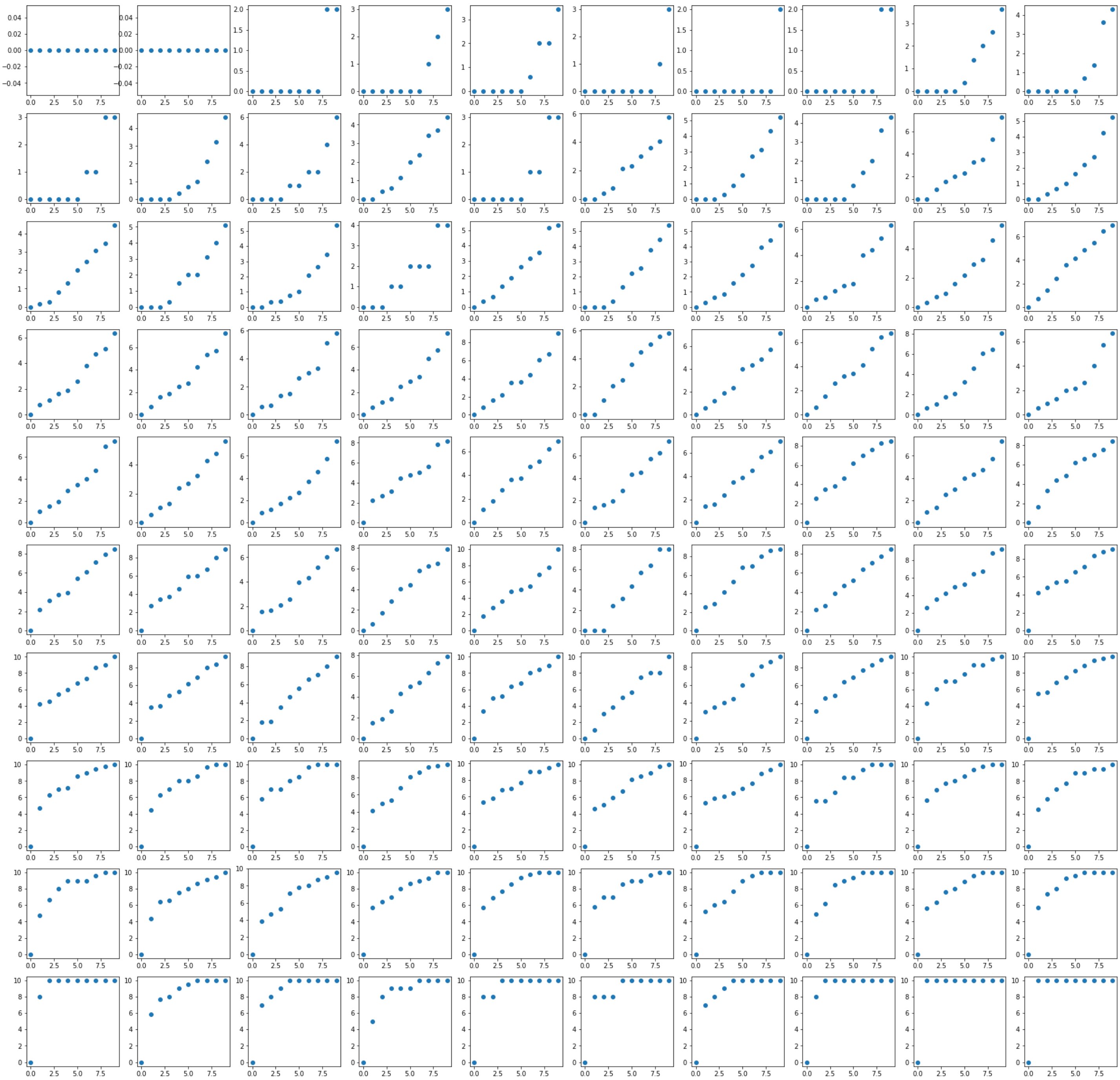
In [141]:

```
import numpy as np

plt.subplots_adjust(hspace=20,wspace=20)
plt.subplots(figsize = (30,30))
for p in range(100):
  G = nx.fast_gnp_random_graph(10,(p+1)/100)
#  print("%d component(s)" % nx.number_connected_components(G))
  L = nx.laplacian_matrix(G).todense()
  eigs = np.linalg.eigvals(L)
#  plt.subplots_adjust(hspace=1, wspace=1)
  plt.subplot(10, 10, (p+1))
  plt.plot(np.sort(eigs),'o')
#  plt.title("Eigenvalues of Laplacian of Random Matrix")
```

```
/usr/local/lib/python3.7/dist-packages/numpy/core/_asarray.py:83: ComplexWarning: Casting
complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)
/usr/local/lib/python3.7/dist-packages/numpy/core/_asarray.py:83: ComplexWarning: Casting
```

```
/usr/local/lib/python3.7/dist-packages/numpy/core/_asarray.py:83: ComplexWarning: Casting
complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)
/usr/local/lib/python3.7/dist-packages/numpy/core/_asarray.py:83: ComplexWarning: Casting
complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)
```

```
<Figure size 432x288 with 0 Axes>
```



4. Dice

Suppose two dice are tossed, but that the dice are weighted so that 1 comes up as twice as likely as any other number for both dice.

a) What is the probability of getting a sum of seven?

b) What is the probability of getting a sum of seven given that the first die comes up 1?

c) What is the probability neither die coming up 1?

d) What is the PDF of the random variable X defined to the the sum of the two dice?

In [142]:

```
d1 = 2/7
```

```
d2 = 1/7
d3 = 1/7
d4 = 1/7
d5 = 1/7
d6 = 1/7
d = [0,2/7,1/7,1/7,1/7,1/7,1/7]
```

In [143]:

```
def fun_p(x):
  p = 0
  for i in range(x):
        if i > 0:
          a = x - i
          if i < 7 and a < 7:
            pp = d[a]*d[i]
            p = p + pp


  return p
```

a) probability is 0.16327

In [144]:

```
#  method 1
#(1,6)
p1 = d1*d6
#(2,5)
p2 = d2*d5
#(3,4)
p3 = d3*d4
#(4,3)
p4 = d4*d3
#(5,2)
p5 = d5*d2
#(6,1)
p6 = d6*d1

p = p1+p2+p3+p4+p5+p6
print(p)

#  method 2
print(fun_p(7))
```

```
0.16326530612244897
0.16326530612244897
```

b) If the first die is 1, so the other die must be 6

since the probability is $\frac{1}{7} \approx 0.14286$

c)probability is $\frac{25}{49} \approx 0.5102$

In [145]:

```
p = 1-(d1*d1+2*d1*(1-d1))
p
```

Out[145]:

```
0.5102040816326531
```

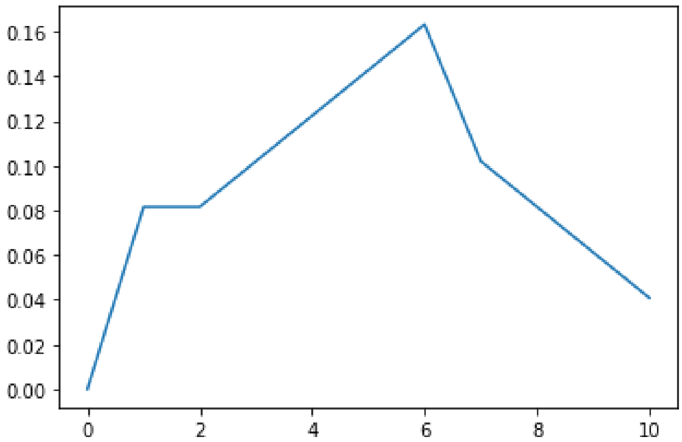
d)

In [146]:

```
L = []
for i in range(12):
  if i > 0:
    a = fun_p(i)
    L.append(a)
plt.plot(L)
```

Out[146]:

```
[<matplotlib.lines.Line2D at 0x7f53b3958610>]
```



5. Functions of a Random Variable

Suppose X is distributed uniformly in the interval $[0,1]$. That is, $f_X(x) = 1$ if $x \in [0,1]$ and $f_X(x) = 0$ otherwise.

a) What is the CDF of X?

b) Let $Y = X^2$. Find and plot F_Y and f_Y.

a) $CDF = \left\{ \begin{matrix} y \\ = 0, x \in (-\infty, 0) \\ y \\ = x, x \in [0,1] \\ y \\ = 1, x \in (1, +\infty) \end{matrix} \right\}$

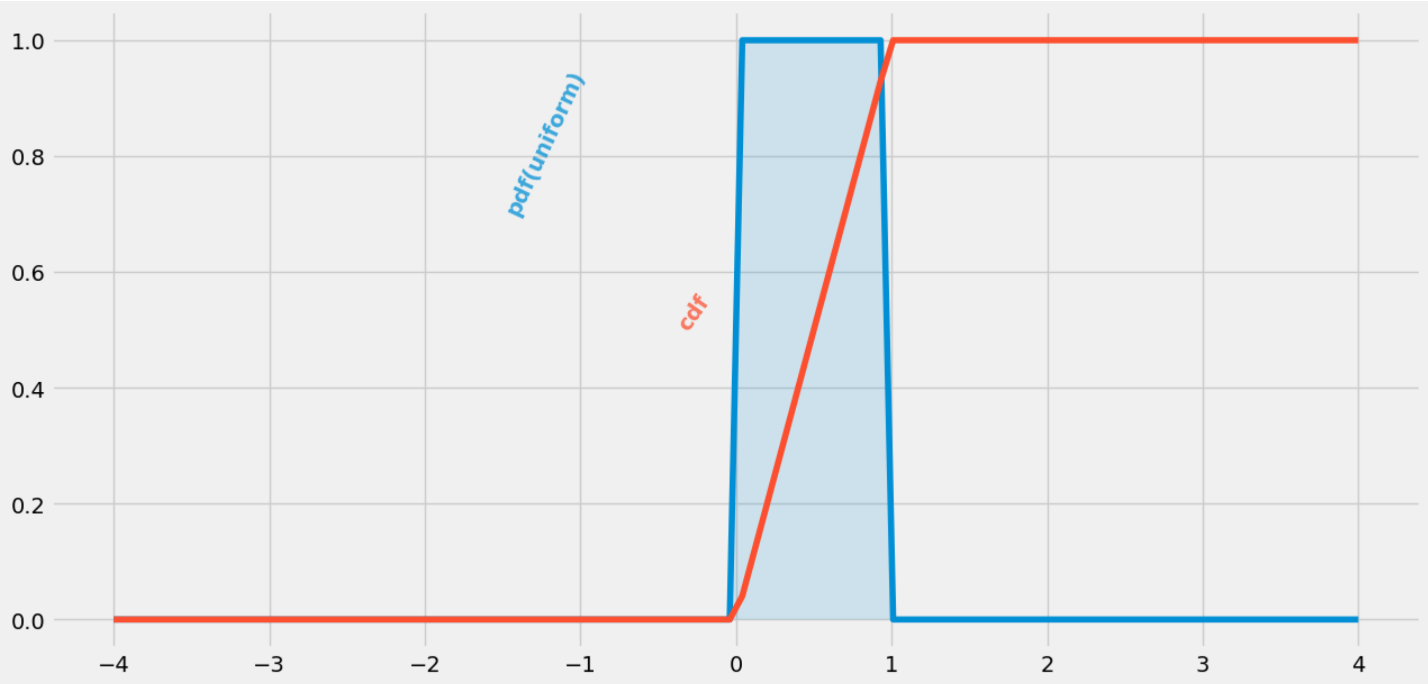
In [147]:

```
import numpy as np
import scipy.stats as stats
import matplotlib.pyplot as plt
import matplotlib.style as style
from IPython.core.display import HTML

#PLOTTING CONFIG
%matplotlib inline
style.use('fivethirtyeight')
plt.rcParams['figure.figsize']=(14,7)
plt.figure(dpi=100)

#PDF
plt.plot(np.linspace(-4,4,100),stats.uniform.pdf(np.linspace(-4,4,100)))
plt.fill_between(np.linspace(-4,4,100),stats.uniform.pdf(np.linspace(-4,4,100)),alpha=0.
15)

#CDF
plt.plot(np.linspace(-4,4,100),stats.uniform.cdf(np.linspace(-4,4,100)))

#LEGEND
plt.text(x=-1.5,y=0.7,s="pdf(uniform)",rotation=65,alpha=0.75,weight="bold",color="#008f
```

```
d5")
plt.text(x=-0.4,y=0.5,s="cdf",rotation=55,alpha=0.75,weight="bold",color="#fc4f30")
```

Out[147]:

```
Text(-0.4, 0.5, 'cdf')
```



b)

$F_Y(x) = P[Y \leq x] = p[X^2 \leq x] = P[-\sqrt{x} \leq X \leq \sqrt{x}] = P[X \leq \sqrt{x}] = F_X(\sqrt{x}) = \sqrt{x}, (x \in [0,1]) f_Y(x) =$

$\therefore F_Y(x)$

$$= \begin{Bmatrix} 0, x \in (-\infty, 0) \\ \sqrt{x}, x \in [0,1] \\ 1, x \in (1, +\infty) \end{Bmatrix}$$

$f_Y(x)$

$=$

$$\begin{cases} y = 0, x \in (-\infty, 0) \cup (0, \infty) \\ \frac{1}{2\sqrt{x}}, x \in [0,1] \end{cases}$$

$\}$

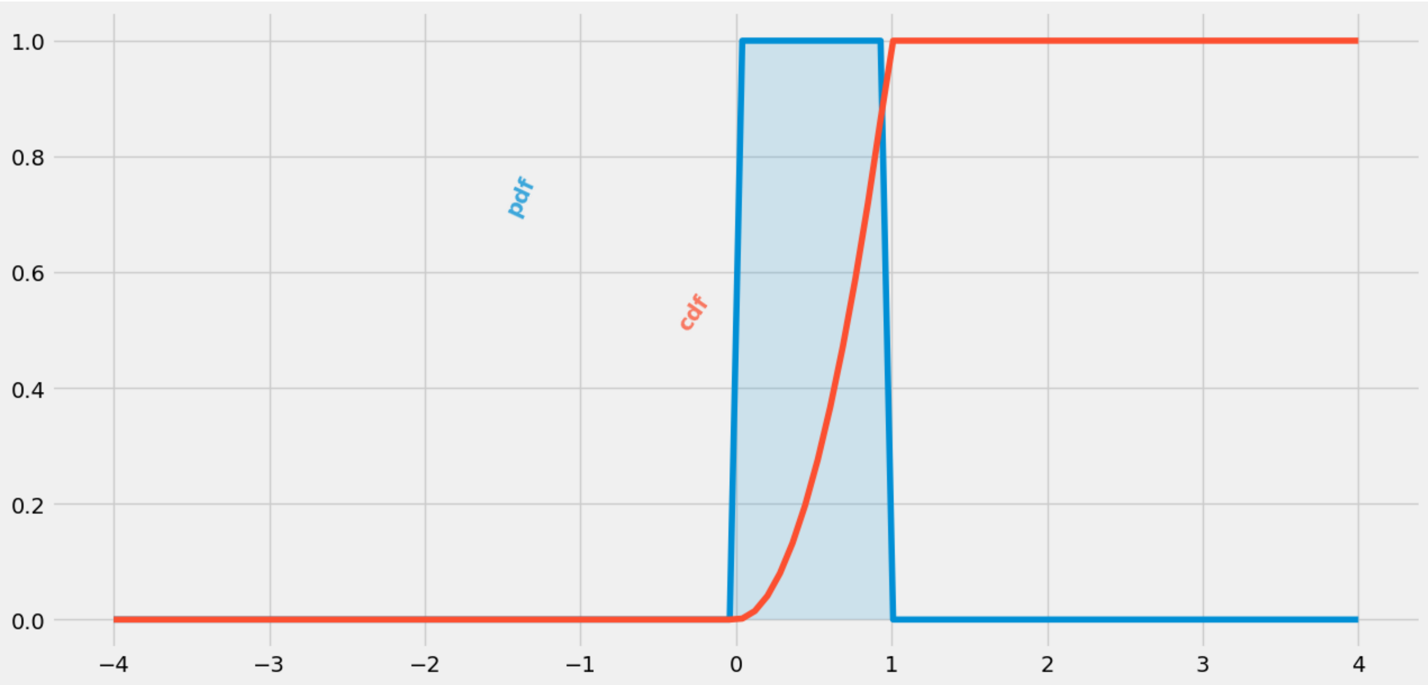
In [148]:

```
import numpy as np
import scipy.stats as stats
import matplotlib.pyplot as plt
import matplotlib.style as style
from IPython.core.display import HTML

#PLOTTING CONFIG
%matplotlib inline
style.use('fivethirtyeight')
plt.rcParams['figure.figsize']=(14,7)
plt.figure(dpi=100)

#PDF
x = np.linspace(-4,4,100)
y_1 = stats.uniform.pdf(x)
y_pdf = y_1 * y_1
plt.plot(x,y_pdf)
```

```
plt.fill_between(x,y_pdf,alpha=0.15)

#CDF
y_2 = stats.uniform.cdf(x)
y_cdf = y_2 * y_2
plt.plot(x,y_cdf)

#LEGEND
plt.text(x=-1.5,y=0.7,s="pdf",rotation=65,alpha=0.75,weight="bold",color="#008fd5")
plt.text(x=-0.4,y=0.5,s="cdf",rotation=55,alpha=0.75,weight="bold",color="#fc4f30")
```

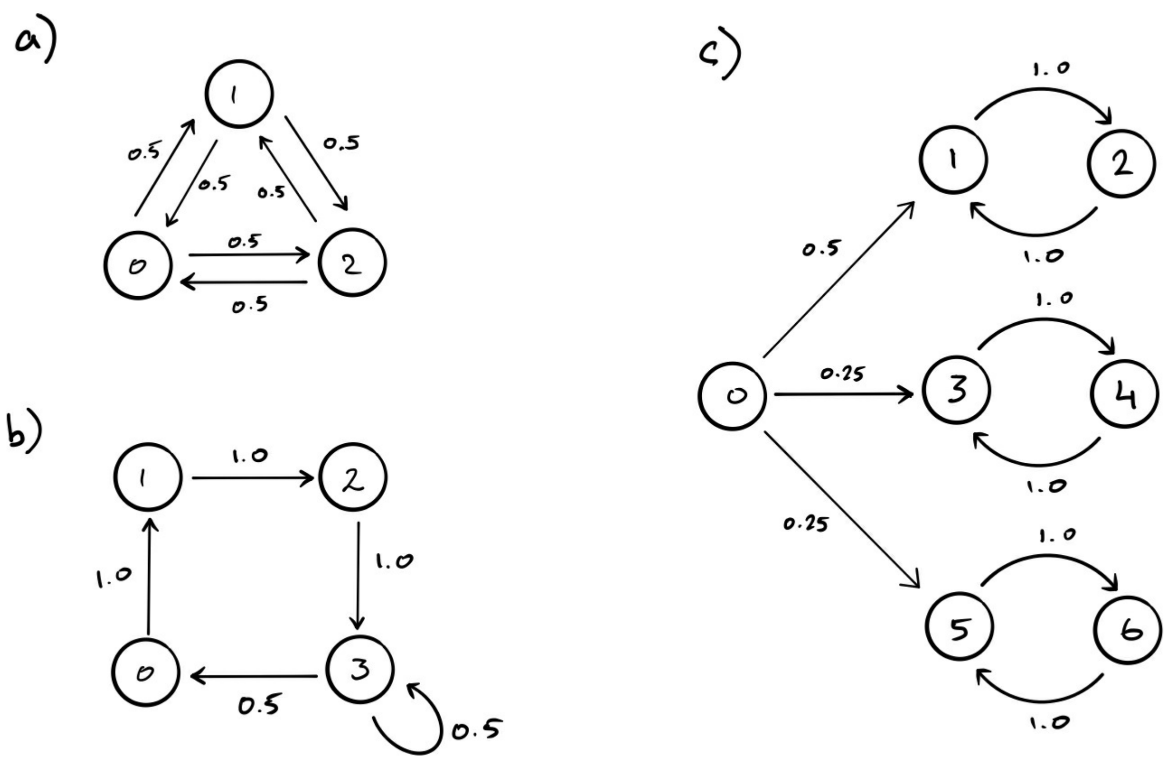
Out[148]:

```
Text(-0.4, 0.5, 'cdf')
```



6. Markov Process Properties

Consider the following three Markov Processes.



For each process,

i) Find the period of each state.

ii) Identify transient states.

iii) Identify absorbing states and absorbing subsets of states.

iv) Identify ergodic states.

a) period: all states are 3

transient states: None

absorbing states: None

ergodic states: None

b) period: all states are 4

transient states: None

absorbing states: None

ergodic states: 3

c) period: states {1,2},{3,4},{5,6}are 2

transient states: 0

absorbing subsets: {1,2},{3,4},{5,6}

ergodic states: None

7. Stationary Distributions

For each Markov Process in Problem 6:

i) Find the update matrix Q.

ii) Simulate the dynamics $p_{k+1} = p_k Q$ starting at the distribution in which $p_0(0) = 1$. Make a plot of the dynamics for each system. To do this, plot a 2D grid where grid point (k, i) is a gray-scale square corresponding to the probability that the process is in state i at time k (the Gambler's Ruin example in the notes does this).

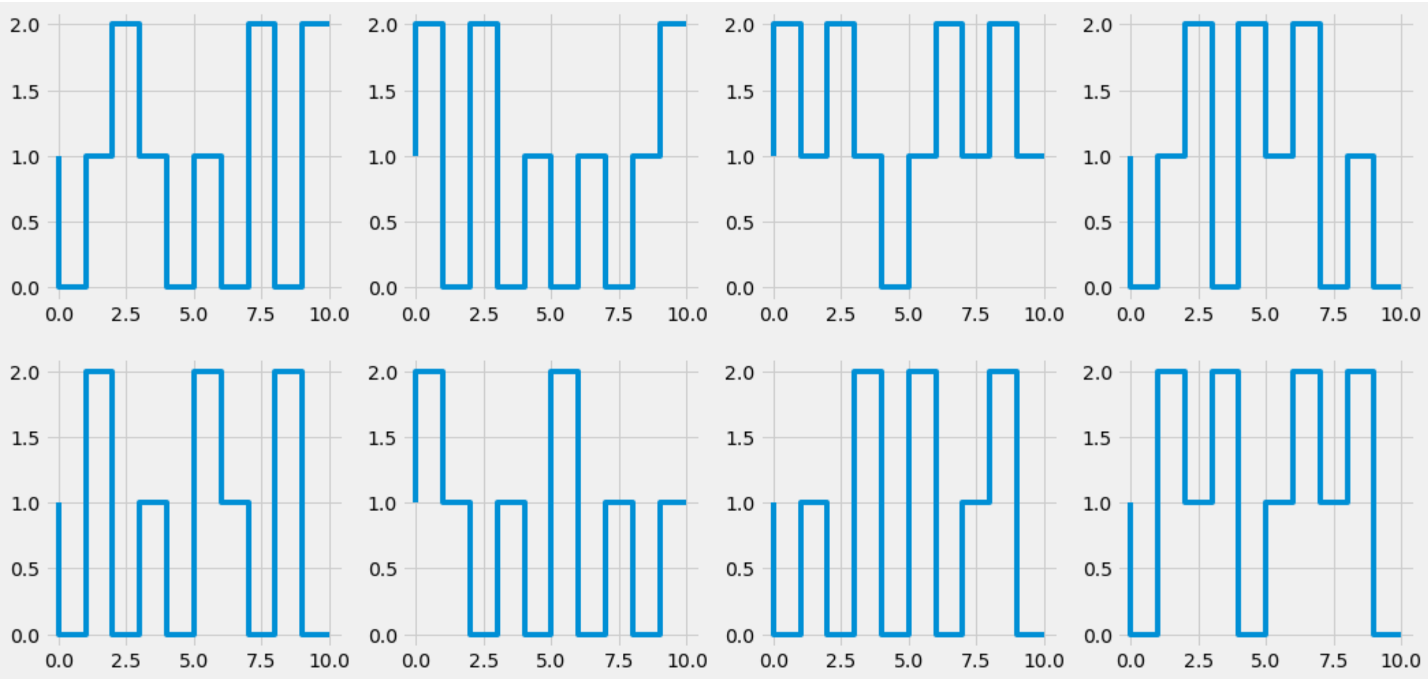
a)

$$Q(0,0) = 0, Q(0,1) = 0.5, Q(0,2) = 0.5Q(1,0) = 0.5, Q(1,1) = 0, Q(1,2) = 0.5Q(2,0) = 0.5, Q(2,1) = 0.5$$

In [149]:

```
import matplotlib.pyplot as plt
import random
def gamble(p,n,k):
    x = 1
    trajectory = [x]
    for i in range(k):
      if x == 0:
        if random.random() > p:
            x = 1
        else:
            x = 2
```

```
            trajectory.append(x)
        elif x == 1:
            if random.random() > p:
                x = 2
            else:
                x = 0
            trajectory.append(x)
        else: #x = 2
            if random.random() > p:
                x = 0
            else:
                x = 1
            trajectory.append(x)
    return trajectory

p = 0.5
n = 10
k = 10
gamble_trajectories,ax = plt.subplots(2,4,figsize=(16,8))

for i in range(8):
    x = gamble(p,n,k)
    t =range(len(x))
    a = ax[int(i/4)][i%4]
    a.step(t,x)
    # if x[-1] == n:
    #     a.set_title("Win :-)")
    # else:
    #     a.set_title("Ruin :-(")
```

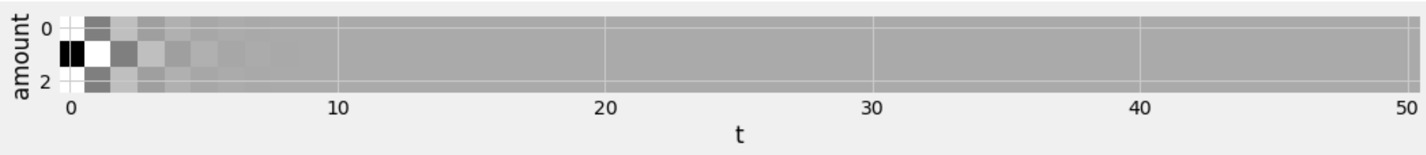


In [150]:

```
n = 2

P = np.zeros((n+1,n+1))
P[0,0] = 0
P[n,n] = 0
P[2,1] = 0.5
P[0,1] = 0.5
P[1,0] = 0.5
P[0,2] = 0.5
P[1,0] = 0.5
P[1,2] = 0.5
P[2,0] = 0.5
print(P)
p = np.zeros(n+1)
p[1] = 1

trajectory = [p]
```

```
for t in range(50):
    p = P.dot(p)
    trajectory.append(p)

trajectory = np.matrix(trajectory).transpose()

fig,ax = plt.subplots(1,1,figsize=(16,5))
ax.imshow(trajectory,cmap="gray_r")
ax.set_xlabel("t")
ax.set_ylabel("amount");
```

```
[[0.  0.5 0.5]
 [0.5 0.  0.5]
 [0.5 0.5 0. ]]
```

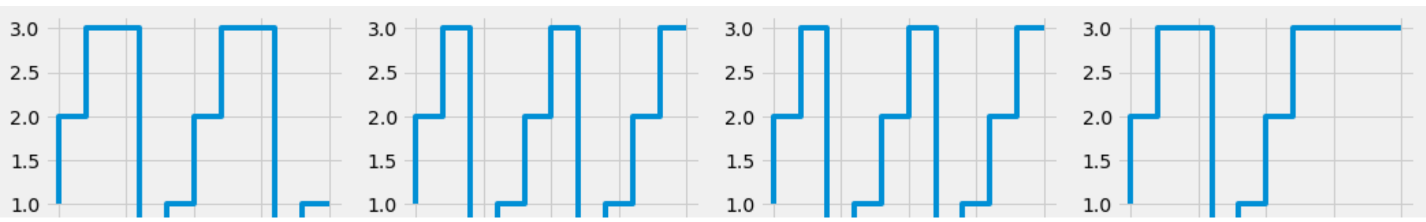


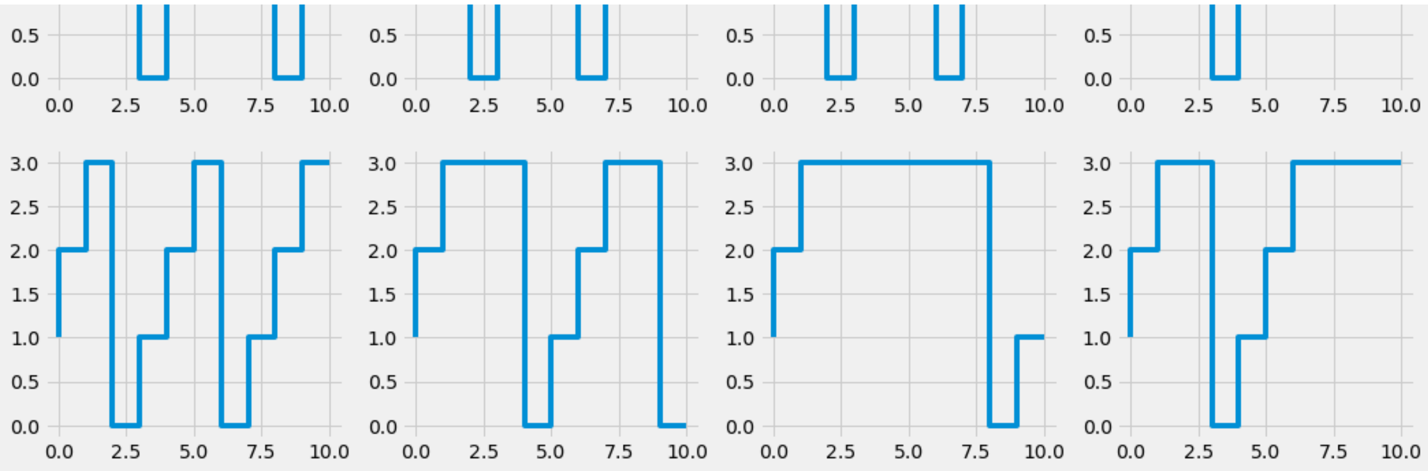
b)Q

$$(0,1) = 1Q(1,2) = 1Q(2,3) = 1Q(3,3) = 0.5Q(3,0) = 0.5Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

In [151]:

```
import matplotlib.pyplot as plt
import random
def gamble(p,n,k):
    x = 1
    trajectory = [x]
    for i in range(k):
      if x == 0:
        x = 1
        trajectory.append(x)
      elif x == 1:
        x = 2
        trajectory.append(x)
      elif x == 2:
        x = 3
        trajectory.append(x)
      else: # x == 3
        if random.random() > p:
          x = 3
        else:
          x = 0
        trajectory.append(x)
    return trajectory

p = 0.5
n = 10
k = 10
gamble_trajectories,ax = plt.subplots(2,4,figsize=(16,8))

for i in range(8):
    x = gamble(p,n,k)
    t =range(len(x))
    a = ax[int(i/4)][i%4]
    a.step(t,x)
```

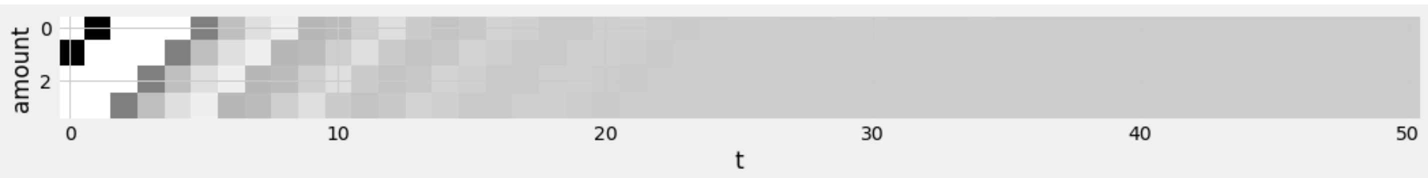




In [152]:

```
n = 3

P = np.zeros((n+1,n+1))
P[0,1] = 1
P[1,2] = 1
P[2,3] = 1
P[3,0] = 0.5
P[3,3] = 0.5
print(P)
p = np.zeros(n+1)
p[1] = 1

trajectory = [p]
for t in range(50):
    p = P.dot(p)
    trajectory.append(p)

trajectory = np.matrix(trajectory).transpose()

fig,ax = plt.subplots(1,1,figsize=(16,5))
ax.imshow(trajectory,cmap="gray_r")
ax.set_xlabel("t")
ax.set_ylabel("amount");
```

```
[[0.  1.  0.  0. ]
 [0.  0.  1.  0. ]
 [0.  0.  0.  1. ]
 [0.5 0.  0.  0.5]]
```



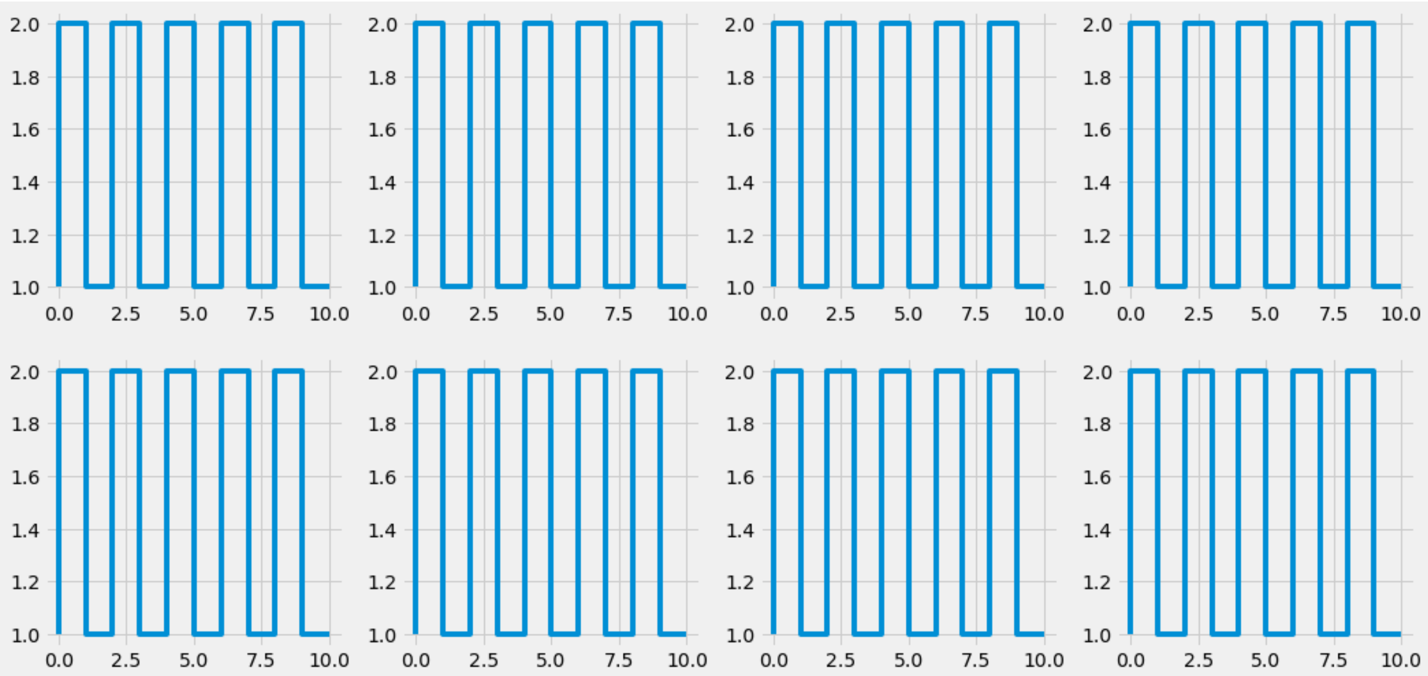
c)

$$Q(0,1) = 0.5Q(0,3) = 0.25Q(0,5) = 0.25Q(1,2) = 1Q(2,1) = 1Q(3,4) = 1Q(4,3) = 1Q(5,6) = 1Q(6,5)$$

In [153]:

```
import matplotlib.pyplot as plt
import random
```

```
def gamble(p,n,k):
    x = 1
    trajectory = [x]
    for i in range(k):
      if x == 1:
        x = 2
        trajectory.append(x)
      else:
        x = 1
        trajectory.append(x)
    return trajectory

p = 1
n = 10
k = 10
gamble_trajectories,ax = plt.subplots(2,4,figsize=(16,8))

for i in range(8):
    x = gamble(p,n,k)
    t =range(len(x))
    a = ax[int(i/4)][i%4]
    a.step(t,x)
```



In [156]:

```
n = 6

P = np.zeros((n+1,n+1))
P[0,1] = 0.5
P[0,3] = 0.25
P[0,5] = 0.25
P[1,2] = 1
P[2,1] = 1
P[3,4] = 1
P[4,3] = 1
P[5,6] = 1
P[6,5] = 1
print(P)
p = np.zeros(n+1)
p[1] = 1

trajectory = [p]
for t in range(50):
    p = P.dot(p)
    trajectory.append(p)

trajectory = np.matrix(trajectory).transpose()
plt.rcParams["axes.grid"] = False
fig,ax = plt.subplots(1,1,figsize=(16,5))
```

```
ax.imshow(trajectory,cmap="gray_r")
ax.set_xlabel("t")
ax.set_ylabel("amount");
```

```
[[0.   0.5  0.   0.25 0.   0.25 0.  ]
 [0.   0.   1.   0.   0.   0.   0.  ]
 [0.   1.   0.   0.   0.   0.   0.  ]
 [0.   0.   0.   0.   1.   0.   0.  ]
 [0.   0.   0.   1.   0.   0.   0.  ]
 [0.   0.   0.   0.   0.   0.   1.  ]
 [0.   0.   0.   0.   0.   1.   0.  ]]
```