



In the figure of the control loop above, we want to find the transfer function, so we solve for:

$$H[z] = \frac{\Phi_O[z]}{\Phi_I[z]}$$

In the time domain representation, we have:

$$\phi_O[n] = \phi_O[n-1] + \alpha[\phi_I[n] - \phi_O[n]] + \sum_{i=0}^n \beta[\phi_I[n-i] - \phi_O[n-i]]$$

The last integral is a first-order filter, which is easily shown to be represented in the z-domain as:

$$\frac{z}{z-1}$$

So using the bilinear z-transform, we convert the discrete time series to:

$$\Phi_O[z] = \Phi_O[z]z^{-1} + \alpha[\Phi_I[z] - \Phi_O[z]] + \beta[\Phi_I[z] - \Phi_O[z]]\frac{z}{z-1}$$

Rearranging, we get:

$$\Phi_O[z] \left[1 - z^{-1} + \alpha + \beta \frac{z}{z-1} \right] = \Phi_I[z] \left[\alpha + \beta \frac{z}{z-1} \right]$$

Multiplying both sides by $\frac{z}{z-1}$ gives us:

$$\Phi_O[z] \left[\frac{z-1}{z-1} + \left[\alpha + \beta \frac{z}{z-1} \right] \frac{z}{z-1} \right] = \Phi_I[z] \left[\alpha + \beta \frac{z}{z-1} \right] \frac{z}{z-1}$$

$$\Phi_O[z] \left[1 + \left[\alpha + \beta \frac{z}{z-1} \right] \frac{z}{z-1} \right] = \Phi_I[z] \left[\alpha + \beta \frac{z}{z-1} \right] \frac{z}{z-1}$$

Therefore,

$$H[z] = \frac{\Phi_O[z]}{\Phi_I[z]} = \frac{\left[\alpha + \beta \frac{z}{z-1} \right] \frac{z}{z-1}}{1 + \left[\alpha + \beta \frac{z}{z-1} \right] \frac{z}{z-1}}$$

We want to reformat this equation into the classical 2nd loop function:

$$H_{REF}[s] = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Using Tustin's method to move $H_{REF}[s]$ to the z-domain:

$$s = \frac{2}{T_s} \frac{Z-1}{Z+1}$$

If we work through the algebra, and substitute $\theta_n = \frac{\omega_n T_s}{2}$, where θ_n is the undamped natural frequency, we get:

$$H_{REF}[z] = \frac{4\theta_n(\zeta + \theta_n)}{1 + 2\zeta\theta_n + \theta_n^2} \frac{z - \frac{\zeta}{\zeta + \theta_n}}{z^2 - 2 \frac{1 + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} z + \frac{1 - 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2}}$$

This looks messy now, but we can find substitutions for this equation in terms of our loop gains, α and β , where α is known as the proportional gain and β is known as the integral gain. Specifically,

$$\frac{1 + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} = 1 - \frac{\alpha + \beta}{2}$$

$$\frac{\alpha + \beta}{2} = 1 - \frac{1 + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} = \frac{1 + 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} - \frac{1 + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} = \frac{2\zeta\theta_n + 2\theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2}$$

$$\alpha + \beta = \frac{4\zeta\theta_n + 4\theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2}$$

Similarly,

$$\frac{1 - 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} = 1 - \alpha$$

$$\alpha = 1 - \frac{1 - 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} = \frac{1 + 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} - \frac{1 - 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} =$$

$$\alpha = \frac{4\zeta\theta_n}{1 + 2\zeta\theta_n + \theta_n^2}$$

Which leaves,

$$\alpha = \frac{4\zeta\theta_n}{1 + 2\zeta\theta_n + \theta_n^2}$$

$$\beta = \frac{4\theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2}$$

$$\theta_n = \frac{\varpi_n T_s}{2}$$