DS 203

Assignment 1

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1. A product is manufactured by two factories A and B. 80% of the product is manufactured in company A and rest in company B. 30% of the product manufactured by company A are defective while 10% of the product manufactured by company B are defective. A sample of the product is randomly selected from the market. Then,
   1. What is the probability that the sample is defective.
   2. What is the probability that a defective sample in the market is manufactured at company A.

Solution: E = event that sample is defective
F_A = Event that sample is from Factory A.
F_B = Event that sample is from Factory B.

$$\begin{aligned} P(E \mid F_A) &= 0.3 \\ P(E \mid F_B) &= 0.1 \\ P(F_A) &= 0.8 \\ P(F_B) &= 1 - P(F_A) = 1 - 0.8 = 0.2 \end{aligned}$$

(a) By Total probability theorem we get ,

$$\begin{aligned} P(E) &= P(E \mid F_A) * P(F_A) + P(E \mid F_B) * P(F_B) \\ &= 0.3 * 0.8 + 0.1 * 0.2 \\ &= 0.26 \end{aligned}$$

(b) By Baye's Theorem we get,

$$\begin{aligned} P(F_A \mid E) &= \frac{P(E \mid F_A) * P(F_A)}{P(E \mid F_A) * P(F_A) + P(E \mid F_B) * P(F_B)} \\ &= \frac{0.3 * 0.8}{0.3 * 0.8 + 0.1 * 0.2} \\ &= \frac{0.24}{0.26} \\ &= 0.923 \end{aligned}$$

2. A particular webserver may be working or not working. If the webserver is not working, any attempt to access it fails. Even if the webserver is working, an attempt to access it can fail due to network congestion beyond the control of the webserver. Suppose that the a priori probability that the server is working is 0.8. Suppose that if the server is working, then each access attempt is successful with probability 0.9, independently of other access attempts. Find the following quantities.
   1. P(first access attempt fails)
   2. P(server is working | first access attempt fails)
   3. P(second access attempt fails | first access attempt fails)
   4. P(server is working | first and second access attempts fail).

Solution: S_W = Event where server is working.
S_{NW}= Event where server is not working.
E_i = Event where i_{th} access attempt gets failed.

$$\begin{aligned} P(S_W) &= 0.8 \\ P(S_{NW}) &= 0.2 \\ P(E_i \mid S_W) &= 0.1 \\ P(E_i \mid S_{NW} &= 1 \end{aligned}$$

(a) Using Total Probability Theorem

$$\begin{aligned} P(E_1) &= P(E_1 \mid S_W) * P(S_W) + P(E_1 \mid S_{NW}) * P(S_{NW}) \\ &= 0.1 * 0.8 + 1 * 0.2 \\ &= 0.28 \end{aligned}$$

(b) Using Baye's Theorem

$$\begin{aligned} P(S_W \mid E_1) &= \frac{P(E_1 \mid S_W) * P(S_W)}{P(E_1 \mid S_W) * P(S_W) + P(E_1 \mid S_{NW}) * P(S_{NW})} \\ &= \frac{0.1 * 0.8}{0.1 * 0.8 + 1 * 0.2} \\ &= \frac{0.08}{0.28} \\ &= 0.2857 \end{aligned}$$

(c) E_1 and E_2 are independent event when the server is in working case
Using conditional probability

$$\begin{aligned} P(E_2 \mid E_1) &= \frac{P(E_2 \cap E_1)}{P(E_1)} \\ &= \frac{P(E_2 \cap E_1 \mid S_W) * P(S_W) + P(E_2 \cap E_1 \mid S_{NW}) * P(S_{NW})}{P(E_1)} \\ &= \frac{(P(E_2 \mid S_W) * P(E_1 \mid S_W)) * P(S_W) + P(E_2 \cap E_1 \mid S_{NW}) * P(S_{NW})}{P(E_1)} \\ &= \frac{0.1 * 0.1 * 0.8 + 1 * 0.2}{0.28} \\ &= \frac{0.208}{0.28} \\ &= 0.7429 \end{aligned}$$

(d) Here by Baye's theorem,

$$
\begin{aligned}
P(S_W \mid E_1 \cap E_2) &= \frac{P(E_2 \cap E_1 \mid S_W) * P(S_W)}{P(E_1 \cap E_2)} \\
&= \frac{(P(E_2 \mid S_W) * P(E_1 \mid S_W))}{P(E_1 \cap E_2)} \\
&= \frac{0.1 * 0.1 * 0.8}{0.208} \\
&= \frac{0.008}{0.208} \\
&= 0.03846
\end{aligned}
$$

3. Two dice are rolled. What is the probability that at least one is a six? If the two faces are different, what is the probability that at least one is a six?

Solution: $E_1 = 6$ appears on first dice
$E_2 = 6$ appears on second dice
They are clearly an independent event

(a)

$$
\begin{aligned}
P(\text{atleast one six}) = P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \\
&= \frac{1}{6} + \frac{1}{6} - \frac{1}{36} \\
&= \frac{11}{36}
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(\text{atleast one six} \mid \text{ different faces}) &= \frac{P(\text{atleast one six} \cap \text{ different faces})}{P(differentfaces} \\
&= \frac{\frac{11-1}{36}}{\frac{30}{36}} \\
&= 0.3333
\end{aligned}
$$

4. Suppose that 5 percent of men and 1 percent of women are color-blind. A color-blind person is chosen at random. What is the probability of this person being male? Assume that there are an equal number of males and females.

Solution: M = Male
F = Female
E = event of being color blind

$$
\begin{aligned}
P(M) = P(F) &= 0.5 \\
P(E|M) &= 0.05 \\
P(E|F) &= 0.01
\end{aligned}
$$

Using Baye's Theorem we can get

$$
\begin{aligned}
P(M|E) &= \frac{P(E|M) * P(M)}{P(E|M) * P(M) + P(E|F) * P(F)} \\
&= \frac{0.05 * 0.5}{0.05 * 0.5 + 0.01 * 0.5} \\
&= \frac{0.05}{0.06} \\
&= 0.8333
\end{aligned}
$$

5. (a) Suppose that an event E is independent of itself. Show that either P(E) = 0 or P(E) = 1.
(b) Events A and B have probabilities P(A) = 0.3 and P(B) = 0.4. What is P($A \cup B$) if A and B are independent? What is P($A \cup B$) if A and B are mutually exclusive?
(c) Now suppose that P(A) = 0.6 and P(B) = 0.8. In this case, could the events A and B be independent? Could they be mutually exclusive?

Solution:

(a) In general for independent events, P($A \cap B) = P(A)P(B)$
We have A= B =E

$$
\begin{aligned}
P(E \cap E) &= P(E)P(E) \\
P(E) &= P(E)^2 \\
\implies P(E) &= 0, 1
\end{aligned}
$$

(b) Since A and B are independent , P($A \cap B) = P(A)P(B) = 0.3 * 0.4 = 0.12$

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= 0.3 + 0.4 - 0.12 \\
&= 0.58
\end{aligned}
$$

If A and B are mutually exclusive then P($A \cap B) = 0$

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= 0.3 + 0.4 - 0 \\
&= 0.7
\end{aligned}
$$

(c) If A and B are independent, then P($A \cap B$) = P(A)*P(B) = 0.6*0.8 = 0.48

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= 0.6 + 0.8 - 0.48 \\
&= 0.92
\end{aligned}
$$

So it is possible because P $(A \cup B) \leq 1$
If A and B are mutually exclusive then P$(A \cap B) = 0$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.6 + 0.8 - 0 \\ &= 1.4 \end{aligned}$$

So it is not possible because P $(A \cup B) > 1$

6. Which of the following are valid CDF's? For each that is not valid, state at least one reason why. For each that is valid, find P$(X^2 > 5)$.

Solution:

(a) $F(x) = \begin{cases} e^{-x^2}/4 & \text{if } x < 0 \\ (1 - e^{-x^2})/4 & \text{if } x \geq 0 \end{cases}$

Since F(x) is non decreasing function for all x $\in \mathbb{R}$
$lim_{x \to \infty}$ F(x) $= 1$
$lim_{x \to -\infty}$ F(x) $= 0$
and F(x) is right continuous.
Hence F(x) can be a CDF.

$$\begin{aligned} P(X^2 > 5) &= P(X < -\sqrt{5} \cap X > \sqrt{5}) \\ &= P(X < -\sqrt{5}) + P(X > \sqrt{5}) \\ &= P(X < -\sqrt{5}) + (1 - P(X \leq \sqrt{5})) \\ &= lim_{h \to 0^-} F(-\sqrt{5} + h) + 1 - F(\sqrt{5}) \\ &= lim_{h \to 0^-} (e^{(-\sqrt{5}+h)^2}/4) + 1 - (1 - e^{-\sqrt{5}^2})/4 \\ &= e^{-5}/4 + e^{-5}/4 \\ &= 0.00337 \end{aligned}$$

(b) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 + e^{-x} & \text{if } 0 \leq x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$

It can't be a CDF because our F(x) must be non decreasing throughout but F(x) is decreasing in $0 \leq x < 3$

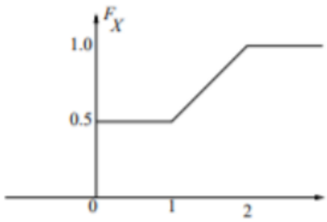
(c) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 + x/20 & \text{if } 0 \leq x < 10 \\ 1 & \text{if } x \geq 10 \end{cases}$

Since F(x) is non decreasing function for all x $\in \mathbb{R}$
$lim_{x \to \infty}$ F(x) $= 1$
$lim_{x \to -\infty}$ F(x) $= 0$
and F(x) is right continuous.

Hence F(x) can be a CDF.

$$\begin{aligned} P(X^2 > 5) &= P(X < -\sqrt{5} \cap X > \sqrt{5}) \\ &= P(X < -\sqrt{5}) + P(X > \sqrt{5}) \\ &= P(X < -\sqrt{5}) + (1 - P(X \leq \sqrt{5})) \\ &= lim_{h \to 0^-} F(-\sqrt{5} + h) + 1 - F(\sqrt{5}) \\ &= lim_{h \to 0^-} 0 + 1 - (0.5 + \sqrt{5}/20) \\ &= 0 + 0.5 - \sqrt{5}/20 \\ &= 0.3882 \end{aligned}$$

7. Let X have the CDF shown.



Solution: $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 & \text{if } 0 \leq x < 1 \\ x/2 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$

(a)

$$P(X \leq 0.8) = F(0.8) = 0.5$$

(b) Using f(x) $= \frac{dF(x)}{dx}$

$$f(x) = \begin{cases} \delta(x)/2 & \text{if } x < 1 \\ 1/2 & \text{if } 1 \leq x < 2 \\ 0 & \text{if } x \geq 2 \end{cases}$$

here $\delta(x)$ is dirac delta fn

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^{1} \frac{x\delta(x)}{2} dx + \int_{1}^{2} \frac{x}{2} dx + 0 \\ &= 0 + 3/4 + 0 \\ &= 0.75 \end{aligned}$$

(c)

$$\begin{aligned}
Var(X) &= \int_{-\infty}^{\infty} (x - E(X))^2 f(x)dx \\
&= \int_{-\infty}^{1} (x - 0.75)^2 \frac{\delta(x)}{2} dx + \int_{1}^{2} (x - 0.75)^2 \frac{1}{2} dx + 0 \\
&= \frac{0.75^2}{2} + \frac{(1.25)^3 - (0.25)^3}{6} + 0 \\
&= 0.6042
\end{aligned}$$

8. If the density function of X equals

$$f(x) = \begin{cases} ce^{-2x} & \text{if } 0 \leq x < \infty \\ 0 & \text{if } x < 0 \end{cases}$$

find c. What is the value of P($X > 2$) ?

Solution: It would satisfy the below equation

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= 1 \\
\int_{0}^{\infty} ce^{-2x} + \int_{-\infty}^{0} 0dx &= 1 \\
\implies c &= 2
\end{aligned}$$

$$\begin{aligned}
P(X > 2) &= \int_{2}^{\infty} f(x)dx \\
&= \int_{2}^{\infty} 2e^{-2x}dx \\
&= e^{-4} = 0.0183
\end{aligned}$$

9. Suppose a coin having probability 0.7 of coming up heads is tossed three times. Let X denote the number of heads that appear in the three tosses. Determine the probability mass function of X.

Solution: P(H) = 0.7 = p and P(T) = 0.3 = q and n=3
Its a Binomial distribution

$$\begin{aligned}
P(X = 0) &= \tbinom{3}{0} p^0 q^3 = 0.3^3 = 0.027 \\
P(X = 1) &= \tbinom{3}{1} p^1 q^2 = 3 * 0.7 * 0.3^2 = 0.189 \\
P(X = 0) &= \tbinom{3}{2} p^0 q^3 = 3 * 0.7^2 0.3^1 = 0.441 \\
P(X = 0) &= \tbinom{3}{3} p^0 q^3 = 0.7^3 = 0.343
\end{aligned}$$

So PMF = { P(X=i) ,i = 0,1,2,3 }

10. Let X is a random variable with probability density function

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } otherwise \end{cases}$$

Find P($X \geq 0.4 \mid X \leq 0.8$).

Solution:

$$\begin{aligned}
P(X \geq 0.4 \mid X \leq 0.8) &= \frac{P(X \geq 0.4 \cap X \leq 0.8)}{P(X \leq 0.8)} \\
&= \frac{P(0.4 \geq X \leq 0.8)}{P(X \leq 0.8)} \\
&= \frac{\int_{0.4}^{0.8} f(x)dx}{\int_{-\infty}^{0.8} f(x)dx} \\
&= \frac{\int_{0.4}^{0.8} 2xdx}{\int_{0}^{0.8} 2xdx} \\
&= \frac{0.48}{0.64} \\
&= 0.75
\end{aligned}$$

11. Let X is an exponentially distributed random variable with parameter λ. For any a, b $>$ 0, find P($X > a + b | X > a$).

Solution:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } 0 \leq x \\ 0 & \text{if } otherwise \end{cases}$$

$$\begin{aligned}
P(X > a + b | X > a) &= \frac{P(X > a + b \cap X > a)}{P(X > a)} \\
&= \frac{P(X > a + b)}{P(X > a)} \\
&= \frac{\int_{a+b}^{\infty} \lambda e^{-\lambda x} dx}{\int_{a}^{\infty} \lambda e^{-\lambda x} dx} \\
&= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} \\
&= e^{-\lambda b}
\end{aligned}$$

12. Suppose five fair coins are tossed. Let E be the event that all coins land heads. Define a random variable I_E

$$I_E = \begin{cases} 1 & \text{if E occurs} \\ 0 & \text{if } E^c \text{ occurs} \end{cases}$$

For what outcomes in the original sample space does I_E equals 1 ? what is P$\{I_E = 1\}$

Solution: E_i be the event where i_{th} coin lands head and they are independent since probability of landing head on other coin doesn't depend

$$\begin{aligned} P(I_E = 1) &= P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5) \\ &= P(E_1)P(E_2)P(E_3)P(E_4)P(E_5) \\ &= \frac{1}{2^5} \\ &= 0.03125 \end{aligned}$$

13. Suppose the distribution function of X is given by

$$F(b) = \begin{cases} 0 & \text{if } 0 < b \\ 0.5 & \text{if } 0 \leq b < 1 \\ 1 & \text{if } 1 \leq b \end{cases}$$

What is the probability mass function of X?

Solution:

$$\begin{aligned} P(X = 0) &= 0.5 \\ P(X = 1) &= 0.5 \\ PMF &= \{P(X = i), i = 0, 1\} \end{aligned}$$

14. A ball is drawn from an urn containing three white and three black balls. After the ball is drawn, it is then replaced and another ball is drawn. This goes on indefinitely. What is the probability that of the first four balls drawn, exactly two are white

Solution: We can take it to be a binomial distribution as it is related to getting r=2 white ones in n=4 trials. P(W)= P(B) = p = q= 0.5

$$\begin{aligned} P(X = 2) &= \binom{n}{r} p^r * q^{n-r} \\ &= \binom{4}{2} 0.5^2 * 0.5^2 \\ &= 0.375 \end{aligned}$$

15. A coin having probability p of coming up heads is successively flipped until the r_{th} head appears. Argue that X, the number of flips required, will be n, n $\geq$ r, with probability

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, n \geq r$$

Solution: We want to have total r head and n flips and also r_{th} head would be on n_{th} flips. So to take care of that in n-1 flips we needs r-1 head randomly which can be related to binomial distribution

$$P\{X = (r-1)\} = \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r}$$

Now to take the n_{th} flip into consideration & E is the event described in question

$$P(E) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

16. Suppose that the number of typographical errors on a single page of this book has a Poisson distribution with parameter $\lambda = 1$. Calculate the probability that there is at least one error on this page.

Solution: X = no. of typographical errors
For poisson distribution, P(X=i) = $\frac{\lambda^i e^{-\lambda}}{i!}$

$$\begin{aligned} P(X \geq 1) &= \sum_{i=1}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} \\ &= e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} \\ &= e^{-\lambda}(e^{\lambda} - 1) \\ &= 1 - e^{-1} \\ &= 0.6321 \end{aligned}$$