Tutorial based on Tut Sheet 6 & 7.1–7.5

15th April 2021

6.3 $\quad A \in \mathbb{K}^{n\times n}$

A unitary $\iff$ rows of A form an orthonormal subset of $\mathbb{K}^{1\times n}$.

(Recall : A is unitary if $A^*A = I = AA^*$ or equivalently if the columns of A form an orthonormal set in $\mathbb{K}^{n\times 1}$)

$$(A^*A)^T = A^T(A^*)^T = I^T = I$$

$$(AA^*)^T = (A^*)^T(A^T) = I^T = I$$

Further

$$(A^*)^T = (A^T)^* = \bar{A}$$

So this shows that A^T is unitary.

Hence the columns of A^T form an orthonormal set, which shows that the rows of A form an orthonormal subset of $\mathbb{K}^{1\times n}$.

6.4

$E = (e_1, \dots, e_n)$ std. basis of $\mathbb{K}^{n\times 1}$

$F = (u_1, \dots, u_n)$ an orthonormal basis of $\mathbb{K}^{n\times 1}$.

$$\Longrightarrow \underbrace{M_E^{F}(I)}\ \text{is unitary.}$$

Since

$$I u_k = \underline{u_k} = \begin{bmatrix} u_{1k} \\ \vdots \\ u_{nk} \end{bmatrix} = \underline{u_{1k} e_1 + \cdots + u_{nk} e_n}$$

for $k = 1, \dots, n$

Thus

$$M_E^F(I) = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & & u_{2n} \\ \vdots & \vdots & & \vdots \\ \underbrace{u_{n1}}_{Iu_1} & \underbrace{u_{n2}}_{Iu_2} & & \underbrace{u_{nn}}_{Iu_n} \end{bmatrix}$$

Since columns are orthonormal, the above matrix is unitary.

Recall: (in general) $T : V \longrightarrow W$, with V having basis E and W having basis F.

$E = (v_1, \dots, v_n)$

$F = (w_1, \dots, w_m)$.

$$M_F^E(T) = (a_{jk})_{m\times n} \quad \text{if} \quad T v_k = \sum_{j=1}^{m} a_{jk} w_j$$

for $k = 1, \dots, n$

Note: $T \circ S$

$$M_F^D(T\circ S) = M_F^E(T)\, M_E^D(S)$$

$$\begin{tikzcd} U \arrow[r, "S"] \arrow[rr, bend left] & V \arrow[r, "T"] & W \\[-20pt] D & E & F \end{tikzcd}$$

Note: $\underbrace{M_E^E(I)}_{= I} = M_E^F(I)\, M_F^E(I)$

So $M_E^F(I)$ is invertible and

$M_F^E(I) = M_E^F(I)^{-1}$.

Hence

$M_E^F(I)$ unitary $\Longrightarrow M_F^E(I)$ unitary

6.5 $A \in \mathbb{C}^{n\times n}$, λ eigenvalue of A

$\Longrightarrow p(\lambda)$ e.v. of $p(A)$ for every poly. $p(t)$.

λ eigenvalue of A

$\Longrightarrow \quad AX = \lambda X$ for some $X \neq 0$

$\Longrightarrow \quad A^k X = \lambda^k X \quad \underline{\forall k \geq 0}$ $\left(A^2X = A(\lambda X) = \lambda AX = \lambda^2 X \right)$

$\Longrightarrow \quad p(A)X = p(\lambda)X$

$\Longrightarrow \quad p(\lambda)$ is an eigenvalue of $p(A)$.

6.6 $A \in \mathbb{C}^{3\times 3}$ $\quad A^3 - 6A^2 + 11A = 6I$

Thus A satisfies $p(A) = 0$, where

$p(t) = t^3 - 6t^2 + 11t - 6$

$= (t-1)(t-2)(t-3)$

Thus λ eigenvalue of A

$$\Longrightarrow p(\lambda) = 0$$

$$\Longrightarrow \lambda = 1 \text{ or } 2 \text{ or } 3.$$

So the possibilities for the eigenvalues of A are

			det		
1 1 1	1 1 2	1 1 3	1	2	3
2 2 2	2 2 1	2 2 3	8	4	12
3 3 3	3 3 1	3 3 2	27	9	18
1 2 3			6		

We are also given that

$$5 \leq \det A \leq 7$$

This is possible only when the eigenvalues are 1, 2 and 3, each with alg. mult $= 1$. (and also geom-mult $= 1$ for each of them)

Hence A is diagonalisable.

6.7 $A \in \mathbb{K}^{n \times n}$ $\lambda_1, \dots, \lambda_n$ eigenvalues of A with an orthonormal set $u_1, \dots, u_n$ of eigenvectors $\in \mathbb{K}^{n \times 1}$

To show: $A = \lambda_1 u_1 u_1^* + \dots + \lambda_n u_n u_n^*$.

$$A u_j = \lambda_j u_j$$

$$\Longrightarrow \sum_{j=1}^{n} A u_j u_j^* = \sum_{j=1}^{n} \lambda_j u_j u_j^*$$

$$\Rightarrow A\underbrace{\left(\sum_{j=1}^{n} u_j u_j^*\right)}_{UU^*} = \sum_{j=1}^{n} \lambda_j (u_j u_j^*)$$

$$\Rightarrow A = \sum_{j=1}^{n} \lambda_j (u_j u_j^*)$$

since $UU^* = I$ because U is unitary, having columns orthonormal.

Aliter: $$x = \sum_{k=1}^{n} \langle u_k, x\rangle u_k$$

$$\Rightarrow Ax = \sum_{k=1}^{n} \lambda_k \underbrace{\langle u_k\, x\rangle}_{\|} u_k$$

$$\Rightarrow Ax = \sum_{k=1}^{n} \lambda_k \underbrace{(u_k^* x)}\, u_k$$

$$= \sum_{k=1}^{n} \lambda_k u_k (u_k^* x)$$

$$= \left(\sum_{k=1}^{n} \lambda_k u_k u_k^*\right) x$$

Now taking $x = e_j$ for $j = 1, \cdots, n$, we obtain

j^{th} col. of A = j^{th} col. of $\sum_{k=1}^{n} \lambda_k u_k u_k^*$ $\forall j = 1, \cdots, n$

So $$A = \sum_{k=1}^{n} \lambda_k u_k u_k^*.$$

<u>Aliter:</u> $\quad U^* A U = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$\Longrightarrow \quad A = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^*$$

$$= \sum_{j=1}^{n} \lambda_j u_j u_j^* . \qquad \text{(check!)}$$

<u>6.8</u> $\quad A \in \mathbb{K}^{n \times n}$

(i) λ ev of $A \iff \bar{\lambda}$ ev of A^*

$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$

$\det(A - \lambda I) = 0 \iff \det(\underbrace{A^* - \bar{\lambda} I}) = 0$

$\qquad\qquad\qquad\qquad\qquad \det(A - \lambda I)^* = 0$

(ii) A unitary $\Longrightarrow \|Ax\| = \|x\|$ since

$\|Ax\|^2 = \langle Ax, Ax \rangle = (Ax)^* A x = x^* \underbrace{A^* A} x = x^* x = \|x\|^2$

Next, $\left(\underline{\lambda \text{ ev of } A \Longrightarrow |\lambda| = 1} \right)$ follows since

$\|x\| = \|Ax\| = \|\lambda x\|$ for $\underbrace{\text{some } x \neq 0}$

$\qquad = |\lambda| \, \|x\|$

$\Longrightarrow |\lambda| = 1.$

(iii) $\mathbb{K} = \mathbb{C}$, A skew self-adjoint, i.e., $A^* = -A$.

λ ev of $A \Longrightarrow i\lambda \in \mathbb{R}$.

$\Downarrow$

$\bar{\lambda}$ ev of $A^* = -A$

Use the fact that A is normal and so $\bar{\lambda}$ is an eigenvalue of A^* with eigenvector x if $Ax = \lambda x$, $x \neq 0$. Now $A^* x = \bar{\lambda} x$. But $A^* = -A$.

So $-Ax = \bar{\lambda} x \implies -\lambda x = \bar{\lambda} x$

$$\implies -\lambda = \bar{\lambda} \quad \text{since } x \neq 0$$

$$\implies \lambda = bi \quad \text{for some } b \in \mathbb{R}$$

$$\left(\lambda = a + ib \implies \bar{\lambda} = a - bi, \quad -\lambda = -a - ib \right) \implies \lambda i \in \mathbb{R}.$$

Aliter: λ eigenvalue of A

$$\implies Ax = \lambda x \quad \text{for some } x \neq 0$$

Since $A^* = -A$,

$$\langle Ax, x \rangle = \langle x, A^* x \rangle$$
$$= \langle x, -Ax \rangle$$

$$\implies \langle \lambda x, x \rangle = \langle x, -\lambda x \rangle$$

$$\implies \bar{\lambda} \langle x, x \rangle = -\lambda \langle x, x \rangle$$

$$\implies \bar{\lambda} = -\lambda \quad \text{since } x \neq 0$$

$$\implies \lambda = bi \quad \text{for some } b \in \mathbb{R}$$

$$\implies i\lambda \in \mathbb{R}.$$

6.9: Exer!

6.10 Def: A is nilpotent if $A^m = 0$ for some m

- A upper triangular with 0's on the diagonal $\implies$ A is nilpotent

$$\underbrace{A = \begin{pmatrix} 0 & * & * \cdots * \\ & \ddots & * \\ & & 0 \end{pmatrix}} \implies A^2 = \begin{pmatrix} 0 & 0 & * \\ & \ddots & 0 \\ & & 0 \end{pmatrix} \implies A^3 = \begin{pmatrix} 0 & 0 & 0 & * \\ & \ddots & \ddots & 0 \\ & & \ddots & 0 \\ & & & 0 \end{pmatrix}$$

$\implies A^n = 0.$

$$\left\{ \text{Ex: } \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

Since A is upper triangular with 0's on diagonal we see that.

$$Ae_1 = 0$$
$$Ae_2 \in \text{span}\{e_1\}$$
$$Ae_3 \in \text{span}\{e_1, e_2\}$$
$$\vdots$$
$$Ae_n \in \text{span}\{e_1, \ldots, e_{n-1}\}$$

that is

$$Ae_j \in \text{span}\{e_1, \ldots, e_{j-1}\} \quad \forall j = 1, \ldots, n$$

Hence

$$A^2 e_j \in \text{span}\{e_1, \ldots, e_{j-2}\} \quad \forall j = 1, \ldots, n$$
$$\vdots$$
$$A^{n-1} e_j \in \text{span}\{e_1\} \quad \text{and} \quad A^n e_j = 0 \quad \forall j = 1, \ldots, n$$

so $A^n = 0$. Hence A is nilpotent.

$A \in \mathbb{C}^{n \times n}$:

A nilpotent $\Longleftrightarrow$ 0 is the only eigenvalue of A.

A nilpotent $\Longrightarrow A^m = 0$ for some m

$\Longrightarrow$ A satisfies the poly. $p(t) = t^m$.

$\Longrightarrow$ every eigenvalue λ of A satisfies $p(\lambda) = 0$, i.e., $\lambda^m = 0$

$\Longrightarrow$ every e.v. of A is 0,

Conversely, if

0 is the only eigenvalue of A,

then by Schur's theorem, there is an invertible P s.t.

$P^{-1}AP = B$, B upper triangular

Diagonal entries of B are the eigenvalues of B and hence of A. So B is upper triangular with 0 on the diagonal. Hence $B^n = 0$.

But then $A = PBP^{-1} \Longrightarrow A^n = PB^nP^{-1} = 0$.

So A is nilpotent.

<u>Soln. to 6.9</u> (added later)

$A = (a_{jk}) \in \mathbb{C}^{n \times n}$, $\lambda_1, \ldots, \lambda_n$ eigenvalues of A (counting alg. mult.)

To show:

$$A \text{ normal} \Longleftrightarrow \sum_{1 \le j,k \le n} |a_{jk}|^2 = \sum_{j=1}^{n} |\lambda_j|^2$$

"$\Rightarrow$" Suppose A is normal. Then $AA^* = A^*A$ and by the Spectral Theorem,

$$U^*AU = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$ for some unitary matrix U.

Now since $A^* = \overline{A^T}$,

$(j,j)^{th}$ entry of AA^*

$$= \sum_{k=1}^{n} a_{jk} (A^*)_{kj} = \sum_{k=1}^{n} a_{jk} \overline{a}_{jk} = \sum_{k=1}^{n} |a_{jk}|^2,$$

for $j = 1, \dots, n$.

Thus

$$\sum_{j=1}^{n} \sum_{k=1}^{n} |a_{jk}|^2 = \text{trace}(AA^*).$$

On the other hand, $A = UDU^*$ and so

$$AA^* = (UDU^*)(UDU^*)^* = (UDU^*)(UD^*U^*)$$
$$= U(DD^*)U^*.$$

Thus

$$\text{trace}(AA^*) = \text{trace}(U(DD^*)U^*)$$
$$= \text{trace}(U^*U(DD^*))$$ [$\because$ trace(AB) = trace(BA)]
$$= \text{trace}(DD^*)$$
$$= \text{trace}\begin{pmatrix} \lambda_1\overline{\lambda_1} & & & 0 \\ & \lambda_2\overline{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \lambda_n\overline{\lambda_n} \end{pmatrix} = \sum_{j=1}^{n} |\lambda_j|^2.$$

This shows that

$$A \text{ is normal } \Longrightarrow \sum_{j=1}^{n}\sum_{k=1}^{n} |a_{jk}|^2 = \sum_{j=1}^{n} |\lambda_j|^2$$

Conversely, suppose $A \in \mathbb{C}^{n\times n}$ satisfies the condition $\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{n} |a_{jk}|^2 = \sum_{j=1}^{n} |\lambda_j|^2$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, counting alg. multiplicities.

By Schur's theorem, $U^* A U = B$ for some unitary matrix U and upper triangular matrix B. Let $B = (b_{jk})$. Then as seen before,

$$\begin{aligned}
\sum_{j=1}^{n}\sum_{k=1}^{n} |b_{jk}|^2 &= \operatorname{trace}(BB^*) \\
&= \operatorname{trace}(U^*AU(U^*AU)^*) \\
&= \operatorname{trace}(U^*AA^*U) \\
&= \operatorname{trace}(UU^*AA^*) \\
&= \operatorname{trace}(AA^*) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} |a_{jk}|^2 \\
&= \sum_{j=1}^{n} |\lambda_j|^2 \quad \text{by hypothesis.}
\end{aligned}$$

But $\lambda_1, \dots, \lambda_n$ are eigenvalues of A and hence they are also the eigenvalues of B (since $B \sim A$). Since B is upper triangular, its eigenvalues are its diagonal elements $b_{11}, b_{22}, \dots, b_{nn}$. Thus

$$\sum_{j=1}^{n} |\lambda_j|^2 = \sum_{j=1}^{n} |b_{jj}|^2.$$

It follows that

$$\sum_{j=1}^{n} \sum_{k=1}^{n} |b_{jk}|^2 = \sum_{j=1}^{n} |b_{jj}|^2$$

$$\Rightarrow \sum_{\substack{1 \le j,k \le n \\ j \ne k}} |b_{jk}|^2 = 0$$

$$\Rightarrow b_{jk} = 0 \quad \text{for all } j,k = 1, \dots, n \text{ with } j \ne k.$$

So B is a diagonal matrix. Now since $B = U^* A U$, we see that A is unitarily diagonalizable. Hence A is normal.

This completes the proof.