Interactive Session / Tut 8

22nd April 2021

<u>Recall</u>: A <u>subspace</u> of a vector space V is a nonempty subset W which is closed under addition and scalar multiplication, i.e.,

$$u, v \in W \implies u + v \in W$$

$$\alpha \in \mathbb{K},\ v \in W \implies \alpha v \in W$$

Note that every subspace must contain the zero vector of V.

$$(W \neq \phi \implies \exists\ v \in W$$
$$\implies 0 \cdot v \in W$$
$$\implies 0 \in W.)$$

<u>8.1</u> State why the following sets are <u>not</u> subspaces

(i) $\{A \in \mathbb{R}^{m\times n} : a_{jk} \geq 0 \ \forall j,k\}$

— not a subspace since it is not closed under scalar mult^n.

(ii) $\{y = \underset{C^1(\mathbb{R})}{y(x)} : xy' + y = 3x^2\}$

— not a subspace because it doesn't contain $y=0$ or also because it is not closed under addition. (also not closed under scalar mult^n)

(iii) $\{y = y(x) : y' + y^2 = 0\}$

— not a subspace because it is not closed under addition.

[To be precise, one can consider

$\{y \in C^1[a,b] : y' + y^2\} = 0$

and note that it contains nonzero functions, e.g. $y = \dfrac{1}{x-(a/2)}$ satisfies $y' + y^2 = 0$.

but $2y$ doesn't.]

[Aside: In general solutions of homogeneous linear differential equations will constitute a subspace

$$\{ y : P_n(x) y^{(n)} + P_{n-1}(x) y^{(n-1)} + \dots + P_1(x) y = 0 \}.]$$

(e.g. $x^2 y'' + x y' + y = 0$ is a homogen. linear diff. eq^n.)

(iv) all invertible $n \times n$ matrices

$$\{ A \in \mathbb{K}^{n \times n} : A^{-1} \text{ exists} \}$$

– not a subspace since it doesn't contain 0. Also not closed under addition,

e.g. $I + (-I) = 0$ is not invertible, but I and $-I$ are invertible.

8.2 $V = \mathcal{P}_n$ $\quad \mathbb{K} = \mathbb{R}$

(i) $W_1 = \{ p \in V : p(0) = 0 \}$

$p, q \in W_1 \Rightarrow (p+q)(0) = p(0) + q(0) = 0$ and $(\alpha p)(0) = \alpha p(0) = 0 \Rightarrow p+q \in W_1$ and $\alpha p \in W_1$

Also $0 \in W_1$, So this is a subspace.

A spanning set is given by

$$\{x, x^2, x^3, \ldots, x^n\}$$

(ii) $W_2 = \{p \in V : p'(0) = 0 = p''(0)\}$.

— yes, W_2 is a subspace of V.

Note: $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in V$

is in W_2

$$\Longleftrightarrow \quad \begin{aligned} a_1 &= p'(0) = 0 \\ 2a_2 &= p''(0) = 0 \end{aligned}$$

$$\Longleftrightarrow \quad a_1 = 0 = a_2 .$$

Thus a spanning set is given by

$$\{1, x^3, x^4, \ldots, x^n\}$$

Any subset of W_2 which contains this set is also a spanning set, e.g.,

$$\{1, x^3, x^4, \ldots, x^n, 1 + x^3 + x^4\}$$

is a spanning set.

(iii) $W_3 = \{ p \in V : p$ is an odd fn. $\}$
i.e., $p(-x) = -p(x)$

- W_3 is a subspace of V.

A spanning set is given by

$S = \{ x, x^3, x^5, \ldots x^k \}$

where k is the largest odd integer $\leq n$,

i.e., $k = \begin{cases} n & \text{if } n \text{ is odd} \\ -n & \text{if } n \text{ is even} \end{cases}$

(In fact, S is a basis of W_3

since S is also lin. indep.)

<u>8.3</u> $V = C[-\pi, \pi]$

<u>To show:</u> $S_1 = \{ 1, \cos, \sin \}$ is lin. indep.

But $S_2 = \{ 1, \cos^2, \sin^2 \}$ is lin. dependent.

Since $\cos^2 x + \sin^2 x = 1 \quad \forall x \in [-\pi, \pi]$,

we see that S_2 is linearly dependent

$\left[\quad f_0 - f_1 - f_2 = 0 \quad \text{where } \begin{array}{l} f_0 = 1 \\ f_1 = \cos^2 \\ f_2 = \sin^2 \end{array} \right]$

To show that S_1 is lin. indep., suppose we have

$$\alpha_0 \cdot 1 + \alpha_1 \cos + \alpha_2 \sin = 0 \text{ in } V,$$

i.e.,

$$\alpha_0 + \alpha_1 \cos x + \alpha_2 \sin x = 0 \quad \forall x \in [-\pi, \pi]. \qquad \text{--- } (*)$$

In particular, putting $x = 0$ gives

$$\alpha_0 + \alpha_1 = 0 \qquad \Rightarrow \alpha_1 = -\alpha_0$$

putting $x = \pi/2$ gives

$$\alpha_0 + \alpha_2 = 0 \qquad \Rightarrow \alpha_2 = -\alpha_0$$

and putting $x = \pi/4$ gives

$$\alpha_0 + \frac{\alpha_1}{\sqrt{2}} + \frac{\alpha_2}{\sqrt{2}} = 0$$

$$\Rightarrow \alpha_0 = \frac{-(\alpha_1 + \alpha_2)}{\sqrt{2}} = \frac{2\alpha_0}{\sqrt{2}}$$

$$\Rightarrow \alpha_0 = 0$$

$$\Rightarrow \alpha_1 = 0 = \alpha_2$$

This shows that S_1 is lin. indep.

[Aliter : We can differentiate $(*)$ twice & put $x=0$ or integrate & then substitute suitable values.]

8.4 $v_1 = [1 \;\; 0]$

$v_2 = [1 \;\; 1]$

$v_3 = [1 \;\; -1]$

$$[24 \;\; 12] = 4v_1 + 16v_2 + 4v_3$$
$$= 6v_1 + 15v_2 + 3v_3$$

This happens because v_1, v_2, v_3 are linearly dependent, e.g. $v_1 = \frac{1}{2}v_2 + \frac{1}{2}v_3$

8.5 $W_1 = \{\text{diagonal matrices}\}$

$$= \left\{ \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{pmatrix} : d_1, \dots, d_n \in \mathbb{K} \right\}$$

$\dim W_1 = n$ since $\{E_{11}, E_{22}, \dots, E_{nn}\}$ is a basis of W_1

because it is linearly independent and any $D = \operatorname{diag}(d_1, \dots, d_n)$ can be written as $D = \sum_{j=1}^{n} d_j E_{jj}$'s
i.e. $\{E_{11}, \dots, E_{nn}\}$ spans W_1.

$W_2 = \{$ upper triangular matrices $\}$

$$= \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ 0 & a_{22} \cdots & a_{2n} \\ & \ddots & \vdots \\ \bigcirc & & a_{nn} \end{pmatrix} : a_{jk} \in \mathbb{K} \text{ for } j \le k \right\}$$

A basis for W_2 is given by

$$\{ E_{ij} : 1 \le i \le j \le n \}$$

and this has

$$n + (n-1) + \cdots + 2 + 1 = \frac{n(n+1)}{2}$$

elements. So

$$\dim W_2 = \frac{n(n+1)}{2}.$$

$$W_3 = \left\{ \begin{pmatrix} a_{11} & & a_{ij} \\ & \ddots & \\ a_{ij} & & a_{nn} \end{pmatrix} : a_{ij} = a_{ji} \right\}$$

A basis

$$S = \{ E_{ij} + E_{ji} : 1 \le i < j \le n \} \cup \{ E_{ii} : i = 1, \dots, n \}$$

is a basis of W_3. So $\dim W_3 = \frac{n(n+1)}{2}$.

[$A = (a_{ij})$ symmetric

$\implies a_{ij} = a_{ji} \quad \forall\, i, j.$

and $$A = \sum_{1 \le i,j \le n} a_{ij} E_{ij}$$

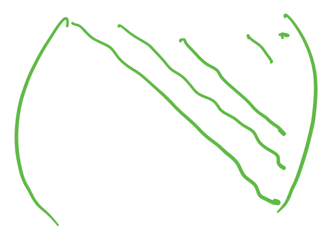
$$= \sum_{i=1}^{n} a_{ii} E_{ii} + \sum_{1 \le i < j \le n} a_{ij} (E_{ij} + E_{ji})$$

This implies that S spans W_3 and also that S is lin. indep.

$\dim W_3 = \#$ of elements in S

$$= \#\{i : 1 \le i \le n\} + \#\{(i,j) : 1 \le i < j \le n\}$$

$$= n + \sum_{i=1}^{n} \sum_{j=i+1}^{n} 1$$



$$= n + \sum_{i=1}^{n} (n - i)$$

$$= n + [(n-1) + (n-2) + \dots + 1 + 0]$$

$$= n(n+1)/2.$$

$W_4 = \{$ skew-symmetric $n \times n$ matrices $\}$

$$= \left\{ \begin{pmatrix} 0 & & a_{ij} \\ & \ddots & \\ -a_{ij} & & 0 \end{pmatrix} \right\}$$

$$= \{ A = (a_{ij}) : \underbrace{a_{ij} = -a_{ji}}_{\Rightarrow\, a_{ii} = 0 \ \forall i} \ \forall i, j \}$$

A basis for this is given by

$$\{ \underbrace{(E_{ij} - E_{ji})} : \underline{1 \le i < j \le n} \}$$

$\Big[$ since $A = \sum_{1 \le i < j \le n} a_{ij} (E_{ij} - E_{ji})$ if $A = [a_{ij}]$ is skew-symm. $\Big]$

So $\dim W_4 = \dfrac{n(n-1)}{2}$.

8.6 $V \times W = \{ (v, w) : v \in V, w \in W \}$

is a vector space w.r.t. componentwise addition and scalar mult^n, i.e.,

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$

and $\alpha(v, w) = (\alpha v, \alpha w)$ for $\alpha \in \mathbb{K}$

$v, v_1, v_2 \in V$

$w, w_1, w_2 \in W$.

since V and W are vector spaces over $\mathbb{K}$, we can easily check that $V \times W$ satisfies the axioms for it to be a vector space over $\mathbb{K}$ w.r.t. addition and scalar mult^n as defined above. (verify!)

Note that zero element of $V \times W$ is $(0, 0)$, the first coordinate being the zero elt of V and the second the zero elt. of W.

If $\dim V = n$ and $\dim W = m$

then $\dim V \times W = m + n$

Let $\{v_1, \cdots, v_n\}$ be a basis of V

and $\{w_1, \cdots, w_m\}$ be " W.

Then $S = \{(v_1, 0), \cdots, (v_n, 0), (0, w_1), (0, w_2), \cdots, (0, w_m)\}$ is a basis of $V \times W$. To see this let $(v, w) \in V \times W$. Then

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$$
$$w = \beta_1 w_1 + \cdots + \beta_m w_m$$

for some $\alpha_j, \beta_k \in \mathbb{K}$. Now

$$(v, w) = (v, 0) + (0, w)$$
$$= \sum_{j=1}^{n} \alpha_j (v_j, 0) + \sum_{k=1}^{m} \beta_k (0, w_k)$$

Thus S spans $V \times W$. Also

$$\sum_{j=1}^{n} \alpha_j (v_j, 0) + \sum_{k=1}^{m} \beta_k (0, w_k) = (0,0)$$

$$\Longrightarrow \sum_{j=1}^{n} \alpha_j v_j = 0 = \sum_{k=1}^{m} \beta_k w_k$$

$$\Longrightarrow \alpha_j = 0 \ \forall j = 1, \ldots, n \ \& \ \beta_k = 0 \ \forall k = 1, \ldots, m.$$

So S is lin. indep.

[**Side Remark:** There is a notion of tensor product $V \otimes W$ of V and W which has the property that
$$\dim(V \otimes W) = (\dim V)(\dim W).$$]

8.7 $\quad A = [a_{jk}] \in \mathbb{K}^{4 \times 4}$

$$T : \mathbb{K}^{2\times 2} \longrightarrow \mathbb{K}^{2 \times 2}$$

$$T\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

where
$$\begin{bmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{bmatrix} = A \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix}$$

Show that : T is linear, i.e., to show

$$T(X + X') = T(X) + T(X')$$
$$T(\alpha X) = \alpha T(X)$$

But this is clear from the defn. and the fact that $A(u+u') = Au + Au'$ & $A(\alpha u) = \alpha A u$ $\forall \alpha \in \mathbb{K}, \forall u, u' \in \mathbb{K}^4$.

matrix of T w.r.t. $\{E_{11}, E_{12}, E_{13}, E_{14}\}$

Since $E_{11} \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ & $A \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}$

we see that $\underline{T(E_{11})} = \begin{bmatrix} a_{11} & a_{21} \\ a_{31} & a_{41} \end{bmatrix}$

$$= a_{11} E_{11} + a_{21} E_{12} + a_{31} E_{21} + a_{41} E_{22}$$

Similar

$E_{12} \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ & $A \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix}$

and $T(E_{12}) = a_{12} E_{11} + a_{22} E_{12} + a_{32} E_{21} + a_{42} E_{22}$

and so on.

Thus

matrix of T w.r.t. $\{E_{11}, E_{12}, E_{13}, E_{14}\}$ $= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

$T(E_{11}) \quad T(E_{12})$

$= A$

8.8 $\quad T : P_2 \longrightarrow \mathbb{K}^{2 \times 1}$

$$T(\alpha_0 + \alpha_1 t + \alpha_2 t^2) = \begin{bmatrix} \alpha_0 + \alpha_1 \\ \alpha_1 + \alpha_2 \end{bmatrix}$$

Note: T is linear (check!)

$$T((\alpha_0 + \alpha_1 t + \alpha_2 t^2) + (\beta_0 + \beta_1 t + \beta_2 t^2)$$

$$= T(\alpha_0 + \beta_0 + (\alpha_1 + \beta_1) t + (\alpha_2 + \beta_2) t^2)$$

$$= \begin{bmatrix} (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1) \\ (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) \end{bmatrix} = \begin{bmatrix} \alpha_0 + \alpha_1 \\ \alpha_1 + \alpha_2 \end{bmatrix} + \begin{bmatrix} \beta_0 + \beta_1 \\ \beta_1 + \beta_2 \end{bmatrix}$$

$$= T(\quad) + T(\quad).$$

$$E = \{1, t, t^2\}, \quad F = \{e_1, e_2\}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$M_F^E(T) = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3} \\ \;\; T(1) \;\; T(t) \;\; T(t^2) \end{array}$$

$$E' = \{1, 1+t, (1+t)^2\}, \quad F' = \{e_1, e_1+e_2\}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_1+e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

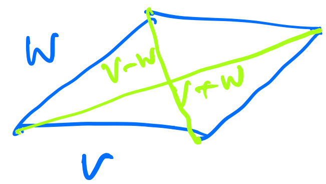
$$M_{F'}^{E'}(T) = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}_{2\times 3} \\ \;\; T(1) \;\; T(1+t) \;\; T((1+t)^2) \end{array}$$

$$T(1+t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$T((1+t)^2) = T(1+2t+t^2) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

8.9 Parallelogram Law

V ips, $v, w \in V$



To show:

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

$$\text{LHS} = \langle v+w, v+w\rangle + \langle v-w, v-w\rangle$$

$$= \langle v+w, v\rangle + \langle v+w, w\rangle + \langle v-w, v\rangle - \langle v-w, w\rangle$$

$$= \underbrace{\langle v, v\rangle} + \cancel{\langle w, v\rangle} + \cancel{\langle v, w\rangle} + \underline{\underline{\langle w, w\rangle}}$$

$$+ \underbrace{\langle v, v\rangle} - \cancel{\langle w, v\rangle} - \cancel{\langle v, w\rangle} + \underline{\underline{\langle w, w\rangle}}$$

$$= 2\|v\|^2 + 2\|w\|^2$$

$$= \text{RHS}.$$

(For a pf. of remark, see B.V. Limaye, Functional Analysis)

8.10

$A, B \in \mathbb{K}^{m\times n}$. To show: $\langle \, , \, \rangle$ defined by

$$\langle A, B \rangle = \operatorname{trace}(\underset{n\times m}{A^*}\underset{m\times n}{B})$$

is an inner product on $\mathbb{K}^{m\times n}$.

i) $\langle A, A \rangle = \operatorname{trace}(A^* A)$

$$= \sum_{j=1}^{n} (A^*A)_{jj}$$

$A = [a_{jk}]$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} \underbrace{a^*_{jk}}_{(j,k)^{th} \text{ entry of } A^*} a_{kj}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} \underbrace{\overline{a_{kj}} \, a_{kj}}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} |a_{kj}|^2$$

Thus $\langle A, A \rangle \geq 0$ and

$$\langle A, A\rangle = 0 \iff |a_{kj}| = 0 \quad \forall k, j$$
$$\iff a_{kj} = 0 \quad \forall k, j$$
$$\iff A = 0.$$

Thus $\langle\ ,\ \rangle$ is positive definite.

(ii) $\langle A, \alpha B + \beta C\rangle$

$$= \operatorname{trace}\left(A^*(\alpha B + \beta C)\right)$$
$$= \operatorname{trace}\left(\alpha A^* B + \beta A^* C\right)$$
$$= \alpha \operatorname{trace}(A^* B) + \beta \operatorname{trace}(A^* C)$$
$$= \alpha \langle A, B\rangle + \beta \langle A, C\rangle.$$

Thus $\langle\ ,\ \rangle$ is linear in the second variable.

(iii) $\langle A, B\rangle = \operatorname{trace}(A^* B)$

$$= \operatorname{trace}\left((B^* A)^*\right)$$
$$= \overline{\operatorname{trace}(B^* A)} = \overline{\langle B, A\rangle}.$$

Thus $\langle , \rangle$ is conjugate symmetric.

By (i), (ii), (iii), we see that $\langle , \rangle$ is an inner product on $\mathbb{K}^{m\times n}$.

8.11 Show that

$$S = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\sin t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \ldots\ldots \right\}$$

is an orthonormal subset of $C[-\pi, \pi]$.

Need to show: (1) every element is a unit vector.

i.e. $$\int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}}\, dt = 1,$$

$$\int_{-\pi}^{\pi} \frac{\cos^2 nt}{\pi}\, dt = 1,$$

$$\int_{-\pi}^{\pi} \frac{\sin^2 nt}{\pi}\, dt = 1,$$

and (2) any two distinct elements are orthogonal, i.e., to show

$$\frac{1}{\sqrt{2}\,\pi}\int_{-\pi}^{\pi} \cos nt \, dt = 0$$

$$\frac{1}{\sqrt{2}\,\pi}\int_{-\pi}^{\pi} \sin nt \, dt = 0$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos mt \cos nt \, dt = 0 \quad \text{if } m \neq n,$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin mt \sin nt \, dt = 0 \quad \text{if } m \neq n,$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos mt \sin nt \, dt = 0 \quad \forall\, m, n$$

These are easily verified.

Thus S is an orthonormal subset of $C[-\pi, \pi]$.

8.12 T Hermitian operator on a finite-dim'l ips V over $\mathbb{K}$

Then $T^* = T$ (by defn. of Hermitian operator)

We have, in general.

$$\langle Tv, w \rangle = \langle v, T^* w \rangle \quad \forall v, w \in V.$$

In this case, since $T = T^*$, we obtain

$$\boxed{(*) \qquad \langle Tv, w \rangle = \langle v, Tw \rangle \quad \forall v, w \in V.}$$

(i) $\langle Tv, v \rangle = \langle v, Tv \rangle$

$= \overline{\langle Tv, v \rangle}$ $\quad \forall v \in V$

(by $*$)

$\Rightarrow \langle Tv, v \rangle \in \mathbb{R}$

[Tv is a shorthand for $T(v)$]

(ii) Let λ be an eigenvalue of T.

Then $Tv = \lambda v$ for some $v \neq 0$.

$(v \in V)$

Now

$$\langle Tv, v\rangle = \langle \lambda v, v\rangle$$
$$= \underline{\bar{\lambda}\langle v, v\rangle}$$

and by (*)

$$\langle Tv, v\rangle = \langle v, Tv\rangle$$
$$= \langle v, \lambda v\rangle$$
$$= \underline{\lambda\langle v, v\rangle}$$

Thus

$$\lambda \|v\|^2 = \bar{\lambda}\|v\|^2$$

since $v \neq 0$, it follows that

$$\lambda = \bar{\lambda}, \text{ i.e., } \lambda \in \mathbb{R}.$$

(iii) If $\lambda \neq \mu$ are eigenvalues of T, with v, w corres. eigenvectors, then

<u>to show</u>: $v \perp w$, i.e., $\langle v, w\rangle = 0$.

We have $Tv = \lambda v$ and

$$Tw = \mu w$$

$$\langle Tv, w\rangle = \langle \lambda v, w\rangle$$

$$= \bar{\lambda}\langle v, w\rangle$$

$$= \lambda\langle v, w\rangle$$

(since $\lambda \in \mathbb{R}$, by (ii))

whereas by (*)

$$\langle Tv, w\rangle = \langle v, Tw\rangle$$

$$= \langle v, \mu w\rangle$$

$$= \mu\langle v, w\rangle.$$

So

$$\lambda\langle v, w\rangle = \mu\langle v, w\rangle$$

But since $\lambda \neq \mu$, we must have

$$\langle v, w\rangle = 0$$

i.e., $v \perp w$.

(iv) W subspace of V,

$T(W) \subset W$.

To show: $T(W^\perp) \subset W^\perp$.

Let $v \in W^\perp$. Then we should show that $T(v) \in W^\perp$, i.e.,

to show: $\langle T(v), w \rangle = 0 \quad \forall w \in W$.

But since $T = T^*$, by (*) we have,

$$\langle Tv, w \rangle = \langle v, Tw \rangle$$

Now $w \in W \Rightarrow Tw \in T(W) \subset W$

$\Rightarrow Tw \in W$

Now since $v \in W^\perp$, we see that

$$\langle v, Tw \rangle = 0$$

This shows that $\langle Tv, w \rangle = 0 \ \forall w \in W$, and so $T(W^\perp) \subset W^\perp$.

About the Exam

Final Exam on 27th Apr (Tue)
at 3 pm. to be given on SAFE.

[Actual exam starts at 3.30 pm
and ends at 5.30 pm.]

Exam will be for 30 marks.

Syllabus : all the course !

Response to a question not directly related to the course:

$f : [-\pi, \pi] \longrightarrow \mathbb{R}$ "nice" function

(or equivalently $f : \mathbb{R} \longrightarrow \mathbb{R}$ "nice" periodic function of period 2π)

Question of Fourier Series:

Can we express f in terms of basic (periodic) functions

$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$

This can be interpreted in many ways.

One way, is to ask whether we can write

$$(*) \quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\forall x \in [-\pi, \pi]$$

for a suitable choice of constants $a_0, a_n, b_n \ (n \geq 1)$?

The orthogonality relations in 8.11 tell us that if $(*)$ is possible, then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx$$

These are the formulas for "Fourier coefficients"

$(*)$ can be interpreted in many ways,

e.g. – pointwise convergence $s_n(f)(x) \to f(x)$ for each $x \in [-\pi,\pi]$

– uniform convergence $\sup_{x\in[-\pi,\pi]} |s_n(f)(x) - f(x)| \to 0$

"L^2 convergence", which means

$\|f - s_n(f)\| \to 0$, i.e. $\int_{-\pi}^{\pi} |f(x) - s_n(f)(x)|^2 dx \to 0$ as $n \to \infty$

where $s_n(f)$ is n^{th} partial sum:

$$s_n(f)(x) = a_0 + \sum_{m=1}^{n} a_m \cos mx + \sum_{m=1}^{n} b_m \sin mx$$

where a_0, a_m, b_m are the Fourier coeff's.

The Fourier coefficients are defined for any integrable function $f:[-\pi,\pi]\to\mathbb{R}$. However, the question of convergence of the Fourier series is a delicate one, especially for pointwise and uniform convergence. But L^2 convergence is relatively easy to achieve. We can show that if f is square integrable, i.e., $\int_{-\pi}^{\pi}|f(x)|^2dx<\infty$, then $\|S_n(f)-f\|\to 0$ as $n\to\infty$.

For more on Fourier series, you may see:

R. Bhatia, Fourier Series

E. M. Stein and R. Shakarachi, Fourier Analysis

For basics of pointwise and uniform convergence, you may see my book:

S. R. Ghorpade and B. V. Limaye, A Course in Calculus and Real Analysis, 2^{nd} Ed.