Interactive Session / Extra Tutorial

16th April 2021

Tut Sheet No. 7

7.1 $A \in \mathbb{C}^{n\times n}$. Show that

A self-adjoint $\iff$ A normal and all eigenvalues of A are real.

"$\Rightarrow$" If A is self-adjoint, i.e., $A^* = A$, then clearly A is normal (i.e. $A^*A = AA^*$) and we have seen in the class that all the eigenvalues of A are real. (This either follows from the spectral thm for self-adjoint matrices, or it can be seen independently as follows: If $Ax = \lambda x$ for some $x \neq 0$, then

$$\bar{\lambda}\,\|x\|^2 = \langle Ax, x\rangle = \langle x, A^*x\rangle = \langle x, Ax\rangle$$
$$= \langle x, \lambda x\rangle$$
$$= \lambda \langle x, x\rangle$$
$$= \lambda \|x\|^2$$

and hence $\bar{\lambda} = \lambda \Rightarrow \lambda \in \mathbb{R}$.

"$\Leftarrow$" Conversely, suppose A is normal and all the eigenvalues of A are real. Then by the spectral thm. for normal matrices,

$A = UDU^*$ for some unitary matrix U and diagonal matrix D.

Moreover, diagonal entries of D are the eigenvalues of A (since $A \sim D$) and hence real. So, in particular, $D = D^*$. Hence

$$A^* = (UDU^*)^* = UD^*U^*$$
$$= UDU^* = A.$$

So A is self-adjoint.

7.2 Spectral Theorem for skew self-adjoint matrices over $\mathbb{C}$

Let $A \in \mathbb{C}^{n\times n}$. Then

A is skew self-adjoint if and only if A is unitarily diagonalizable and all the eigenvalues of A are purely imaginary.

Proof: Suppose A is skew self-adjoint. Then A is normal and hence A is unitarily diagonalizable. Now if $U^*AU = D$ for some unitary matrix U and diagonal matrix D, then the diagonal entries of D are the eigenvalues of A and moreover,

$$D^* = (U^*AU)^* = U^*A^*U = -U^*AU = -D.$$

Thus if $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, then $D^* = \begin{pmatrix} \bar{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\lambda}_n \end{pmatrix}$

and so $\overline{\lambda_j} = -\lambda_j \quad \forall j = 1, \dots, n$. This implies that λ_j is purely imaginary $\forall j = 1, \dots, n$. ($\overline{a+ib} = -a-ib$, $a - ib$ for $a, b \in \mathbb{R}$ $\implies a = 0$)

Conversely, if A is unitarily diagonalizable and every eigenvalue of A is purely imaginary, then $U^*AU = D$ for some unitary matrix U and diagonal matrix D. Moreover $D^* = -D$.

So
$$A^* = (UDU^*)^* = UD^*U^* = -UDU^* = -A.$$

Hence A is skew self-adjoint.

Aliter: A skew self-adjoint (i.e., $A^* = -A$)

$\iff B = iA$ self-adjoint ($B^* = \bar{i} A^*$
$= -i(-A)$
$= iA = B$)

So we can apply Spectral Thm. for self-adjoint matrices to B to obtain the desired result.

7.3 Find an orthonormal basis for $\mathbb{K}^{4\times 1}$ consisting of eigenvectors of

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}$$

Write down a spectral repn. of A & find $A^7 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

Soln: Note that A is real symmetric and so all the eigenvalues of A are real. Also

$$p_A(t) = \det(A - tI) = \begin{vmatrix} -1-t & 0 & 0 & 0 \\ 0 & -1-t & 0 & 0 \\ 0 & 0 & -1-t & -4 \\ 0 & 0 & -4 & -1-t \end{vmatrix}$$

$$= (-1-t)^2 \left[(-1-t)^2 - 16 \right]$$

$$= (t+1)^2 (t^2 + 2t - 15)$$

$$= (t+1)^2 (t+5)(t-3)$$

So the eigenvalues are

-1 with alg. mult. $= 2$

3 " $= 1$

-5 " $= 1$

Basis for $N(A+I)$, the eigenspace corres. to $\lambda=-1$

$(A+I)x = 0$

$$\Longleftrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underline{0}$$

$$\Longleftrightarrow \quad -4x_3 = 0$$
$$-4x_4 = 0$$

$$\Longleftrightarrow \quad x_3 = x_4 = 0.$$

so a basis for $N(A+I)$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \{e_1, e_2\}$$

which is already orthonormal

Basis for $N(A-3I)$ $\Longleftrightarrow$ $\lambda = 3$

$(A-3I)x = 0$

$$\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & -4 & -4 \end{bmatrix} \underline{x} = \underline{0}$$

$$\xrightarrow{R_4 - R_3} \begin{bmatrix} -4 & 0 & 0 & 0^{*} \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underline{0}$$

$$\Longleftrightarrow \quad -4x_3 - 4x_4 = 0 \Longrightarrow x_3 = -x_4$$
$$x_2 = 0, \quad x_1 = 0$$

So a basis for $N(A-3I)$ is given by

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

and an orthonormal basis is given by

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

Basis for $N(A+5I)$ $\longleftrightarrow$ $\lambda = -5$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & -4 & 4 \end{bmatrix} \underline{x} = \underline{0}$$

$$\xrightarrow{R_4+R_3} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{x} = 0$$

$$\longleftrightarrow \begin{aligned} 4x_1 &= 0 \\ 4x_2 &= 0 \\ 4x_3 - 4x_4 &= 0 \implies x_3 = x_4 \end{aligned}$$

So a basis for $N(A+5I)$ is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and an o.n. basis } \left\{ \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}.$$

Putting these together, we obtain an orthonormal basis of $\mathbb{K}^{4\times 1}$ consisting of eigenvectors of A as follows

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\} = \left\{ e_1, e_2, \frac{-e_3+e_4}{\sqrt{2}}, \frac{e_3+e_4}{\sqrt{2}} \right\}.$$

Aside: If x, y are eigenvectors of A corres. to eigenvalues λ, μ with $\lambda \neq \mu$ and A is normal, then

$$\begin{aligned} \langle Ax, y \rangle &= \langle \lambda x, y \rangle = \bar{\lambda} \langle x, y \rangle \\ \langle Ax, y \rangle &= \langle x, A^* y \rangle \\ &= \langle x, \bar{\mu} y \rangle \\ &= \bar{\mu} \langle x, y \rangle \end{aligned}$$

(since A is normal, $Ay = \mu y \Rightarrow A^* y = \bar{\mu} y$)

So

$$\bar{\mu} \langle x, y \rangle = \bar{\lambda} \langle x, y \rangle$$

if $\langle x, y \rangle \neq 0$, then we'd get $\bar{\mu} = \bar{\lambda}$, i.e., $\mu = \lambda$, a contradiction

so $\langle x, y \rangle = 0$. Thus eigenvectors corres. to distinct eigenvalues of a normal matrix A are orthogonal

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Spectral Representation of A:

$$Ax = \sum_{j=1}^{n} \lambda_j \langle u_j, x \rangle u_j \qquad \text{---} (*)$$

where $\{u_1, \dots, u_n\}$ is an orthonormal basis of eigenvectors of A corres. to ev's $\lambda_1, \dots, \lambda_n$.

In this case,

$$Ax = -\langle e_1, x\rangle e_1 - \langle e_2, x\rangle e_2$$
$$+ 3\left\langle \frac{-e_3+e_4}{\sqrt{2}}, x\right\rangle \frac{-e_3+e_4}{\sqrt{2}} - 5\left\langle \frac{e_3+e_4}{\sqrt{2}}, x\right\rangle \frac{e_3+e_4}{\sqrt{2}}$$

We know in general that

$$(*) \Longrightarrow A^k x = \sum_{j=1}^{n} \lambda_j^k \langle u_j, x\rangle u_j$$

So in this case,

$$A^7 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = (-1)^7 \cdot 1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-1)^7 \cdot 2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{3^7}{2}(-3+4)\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$
$$+ \frac{(-5)^7}{2}(3+4)\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ -2 \\ (-3^7 - 7\cdot 5^7)/2 \\ (3^7 - 7\cdot 5^7)/2 \end{bmatrix}.$$

<u>7.4</u> A self-adjoint matrix A is called <u>positive definite</u> if $\langle Ax, x\rangle > 0$ $\forall x \in \mathbb{K}^n, x \neq \underline{0}$.

<u>To show:</u> Let A be a self-adjoint matrix. Then A positive definite $\Longleftrightarrow$ all eigenvalues of A are positive.

"$\Longrightarrow$" Suppose A is positive definite. Let λ be an eigenvalue of A. Then $Ax = \lambda x$ for some $x \neq 0$.

Also
$$\langle Ax, x \rangle > 0 \quad \text{since } x \neq 0.$$

So
$$\langle \lambda x, x \rangle > 0$$

$$\Longrightarrow \quad \bar{\lambda} \langle x, x \rangle > 0$$

$$\Longrightarrow \quad \lambda \langle x, x \rangle > 0$$

since $\lambda = \bar{\lambda}$ as the ev's of A are real, because A is self-adjoint

$$\Longrightarrow \quad \lambda \|x\|^2 > 0$$

$$\Longrightarrow \quad \lambda > 0.$$

Thus every eigenvalue of A is positive.

"$\Longleftarrow$" Conversely, suppose every eigenvalue of A is positive. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A, counting alg. mult's.

Since A is self-adjoint, we can find an orthonormal basis $(u_1, \dots, u_n)$ of $\mathbb{K}^n$ s.t. $u_1, \dots, u_n$ are eigenvectors corres. to $\lambda_1, \dots, \lambda_n$, resp. Note that

$$\langle Au_i, u_j \rangle = \langle \lambda_i u_i, u_j \rangle = \lambda_i \langle u_i, u_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \lambda_i & \text{if } i = j. \end{cases}$$

Given any $x \in \mathbb{K}^n$, we can write

$$x = \sum_{j=1}^{n} \alpha_j u_j \quad \text{for unique } \alpha_j \in \mathbb{K}$$
(in fact $\alpha_j = \langle u_j, x \rangle$)

and then
$$Ax = \sum_{j=1}^{n} \alpha_j A u_j = \sum_{j=1}^{n} \lambda_j \alpha_j u_j$$

So

$$\langle Ax, x\rangle = \left\langle \sum_{j=1}^{n} \lambda_j \alpha_j u_j \, , \sum_{k=1}^{n} \alpha_k u_k \right\rangle$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_j \overline{\alpha_j} \alpha_k \underbrace{\langle u_j, u_k\rangle}_{= \delta_{jk}}$$

$$= \sum_{j=1}^{n} \lambda_j \overline{\alpha_j} \alpha_j$$

$$= \sum_{j=1}^{n} \underbrace{\lambda_j}_{>0} \underbrace{|\alpha_j|^2}_{\geq 0}$$

Thus if $x \neq 0$, then $\alpha_j \neq 0$ for some j and therefore $\langle Ax, x\rangle > 0$ (since $\lambda_j > 0 \; \forall j$)

Alter: $U^* A U = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ $\lambda_j > 0 \; \forall j$

$$\Downarrow$$

$$A = U D U^*$$

$$\langle Ax, x\rangle = (Ax)^* x$$

$$= x^* A^* x$$

$$= x^* U D^* U^* x$$

$$= y^* D y \quad \text{where } y = U^* x = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \sum_{j=1}^{n} \lambda_j |y_j|^2$$

and $x \neq 0 \Rightarrow y \neq 0$ (since U^* is invertible)

$$\Rightarrow \langle Ax, x\rangle = \sum_{j=1}^{n} \lambda_j |y_j|^2 > 0.$$

Question: If two matrices have the same eigenvalues and same eigenvectors, are they equal?

Answer: No, in general. Consider, for example,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

The only eigenvalue of A as well as B is 0. Also,

$$AX = 0 \Rightarrow X_2 = 0, \text{ and } BX = 0 \Rightarrow X_2 = 0, \text{ So } e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is an eigenvector of A as well as B corres. to the eigenvalue $\lambda = 0$. But of course $A \neq B$.

[note: $Ae_1 = \underline{0} = 0 \cdot e_1$.]

However if A and B are diagonalizable and have same eigenvalues and the corres. eigenspaces are the same as well, then $A = B$.

— this is since we can find a basis $(u_1, u_2, \dots, u_n)$ of $\mathbb{K}^n$ consisting of eigenvectors of A (& hence of B)

Now

$$Au_j = \lambda_j u_j = Bu_j \quad \forall j = 1, \dots, n.$$

Hence

$$Ax = Bx \quad \forall x \in \mathbb{K}^n.$$

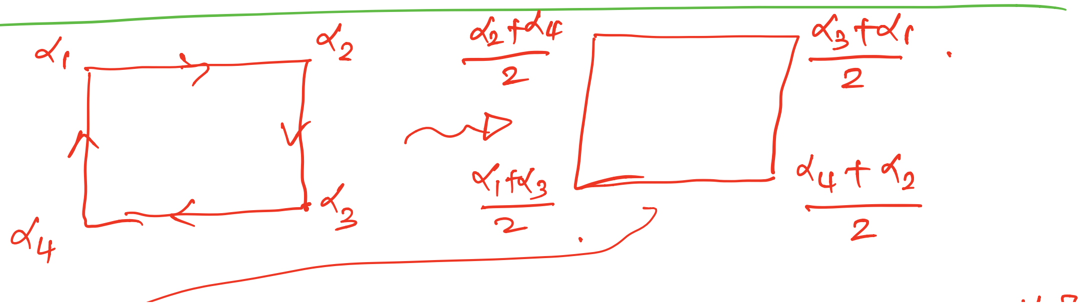
In particular,

$$j^{th} \text{ col. of } A = Ae_j = Be_j = j^{th} \text{ col. of } B \quad \forall j = 1, \dots, n.$$

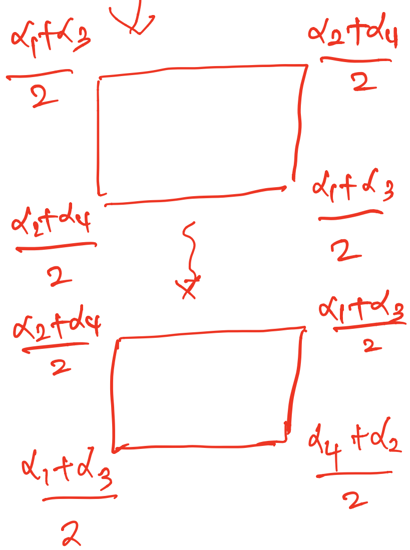
and so

$$A = B.$$

7.5



$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{\alpha_2+\alpha_4}{2} \\ \frac{\alpha_3+\alpha_1}{2} \\ (\alpha_4+\alpha_2)/2 \\ (\alpha_1+\alpha_3)/2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$



$$\alpha \quad A\alpha \quad A^2\alpha \quad A\alpha \quad A^2\alpha \quad \ldots$$

$$A^3 = A. \qquad A^4 = A^2 \qquad A^5 = A$$

After k steps $(k \geq 1)$

$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$ changes to $A^k \alpha$

which is

$$\begin{cases} A\alpha & \text{if } k \text{ odd} \\ A^2\alpha & \text{if } k \text{ even} \end{cases}$$

Solving using the "Hint"

$$p_A(t) = \det(A - tI) = \begin{vmatrix} -t & 1/2 & 0 & 1/2 \\ 1/2 & -t & 1/2 & 0 \\ 0 & 1/2 & -t & 1/2 \\ 1/2 & 0 & 1/2 & -t \end{vmatrix}$$

$$\xrightarrow[R_4 - R_2]{R_1 + 2t R_2} \begin{vmatrix} 0 & \frac{1}{2} - 2t^2 & t & 1/2 \\ 1/2 & -t & 1/2 & 0 \\ 0 & 1/2 & -t & 1/2 \\ 0 & +t & 0 & -t \end{vmatrix}$$

$$= -\frac{1}{2}\begin{vmatrix} \frac{1}{2}-2t^2 & t & 1/2 \\ \frac{1}{2} & -t & 1/2 \\ t & 0 & -t \end{vmatrix}$$

$$= \frac{-1}{2}\begin{vmatrix} 1-2t^2 & t & 1/2 \\ 1 & -t & 1/2 \\ 0 & 0 & -t \end{vmatrix} \quad \text{by } C_1+C_3$$

$$= \frac{t}{2}\left[t(2t^2-1)-t\right]$$

$$= \frac{t}{2}\left[2t^3-2t\right]$$

$$= t^2(t-1)(t+1)$$

Eigenvalues: 0 with alg. mult 2, $1, -1$ each with mult. 1

$\underline{N(A) = N(A-0I)}$

$$\longleftrightarrow \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} X = 0$$

$R_4 - R_2 \longrightarrow R_3 - R_1$

$$\frac{x_1+x_3}{2} = 0 \implies x_1 = -x_3$$

$$\frac{x_2+x_4}{2} = 0 \implies x_2 = -x_4$$

So a basis for $N(A)$ is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and an o.n. basis is

$$\left\{ \frac{-e_1+e_3}{\sqrt{2}}, \frac{-e_2+e_4}{\sqrt{2}} \right\}$$

<u>Basis for $N(A-I)$</u>

$$\begin{bmatrix} -1 & 1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 & 0 \\ 0 & 1/2 & -1 & 1/2 \\ 1/2 & 0 & 1/2 & -1 \end{bmatrix} X = 0$$

is satisfied by $X = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

So an orthonormal basis is given by $\dfrac{e_1+e_2+e_3+e_4}{2}$

(Here the space is 1-dim'l since alg. mult $=1 \Rightarrow$ geom. mult $=1$)

<u>Basis for $N(A+I)$</u>

$$\begin{bmatrix} 1 & 1/2 & 0 & 1/2 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 1/2 & 0 & 1/2 & 1 \end{bmatrix} X = 0$$

Consider $X = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. It is nonzero and satisfies the above eq^n. Since alg. mult is 1,

an orthonormal basis of $N(A+I)$ is $\left\{ \dfrac{e_1-e_2+e_3-e_4}{2} \right\}$

So an orthonormal basis for $\mathbb{K}^4$ consisting of eigenvectors of A is given by

$$\left\{ \frac{-e_1+e_3}{\sqrt{2}}, \frac{-e_2+e_4}{\sqrt{2}}, \frac{e_1+e_2+e_3+e_4}{2}, \frac{e_1-e_2+e_3-e_4}{2} \right\}.$$

Denote this by

$$\{u_1, u_2, u_3, u_4\}.$$

Then the spectral representation gives

$$A^k x = \sum_{j=1}^{4} \lambda_j^k \langle u_j, x \rangle u_j$$

$$= \langle u_3, x \rangle u_3 + (-1)^k \langle u_4, x \rangle u_4 .$$

So if $x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$, then

$$A^k x = \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + (-1)^k \frac{(\alpha_1-\alpha_2+\alpha_3-\alpha_4)}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{cases} \begin{bmatrix} \frac{\alpha_1+\alpha_3}{2} & \frac{\alpha_2+\alpha_4}{2} & \frac{\alpha_1+\alpha_3}{2} & \frac{\alpha_2+\alpha_4}{2} \end{bmatrix}^T & \text{if } k \text{ even} \\ \begin{bmatrix} \frac{\alpha_2+\alpha_4}{2} & \frac{\alpha_1+\alpha_3}{2} & \frac{\alpha_2+\alpha_4}{2} & \frac{\alpha_1+\alpha_3}{2} \end{bmatrix}^T & \text{if } k \text{ odd.} \end{cases}$$