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MA 106 : Linear Algebra

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Tutorial Solutions Booklet

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Contents

1 Tutorial 1 (on Lectures 1 and 2)

1.1 Let $\mathbf{A}$ be a square matrix. Show that there is a symmetric matrix $\mathbf{B}$ and there is a skew-symmetric matrix $\mathbf{C}$ such that $\mathbf{A} = \mathbf{B} + \mathbf{C}$. Are $\mathbf{B}$ and $\mathbf{C}$ unique?

Given $\mathbf{B}$ should be symmetric and $\mathbf{C}$ should be skew-symmetric such that $\boxed{\mathbf{A} = \mathbf{B} + \mathbf{C}}$. Take transpose on both sides of this equation. This gives us $\mathbf{A^T} = \mathbf{B^T} + \mathbf{C^T} \Rightarrow \boxed{\mathbf{A^T} = \mathbf{B} - \mathbf{C}}$. Solve these two boxed equations simultaneously to get $\mathbf{B} = \frac{\mathbf{A}+\mathbf{A^T}}{2}$ and $\mathbf{C} = \frac{\mathbf{A}-\mathbf{A^T}}{2}$.
Thus we have $\mathbf{A} = \mathbf{B} + \mathbf{C}$ and clearly, $\mathbf{B}$ is symmetric and $\mathbf{C}$ is skew-symmetric.
By our solution, B and C must be unique

1.2 Let $\mathbf{A} := \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Write (i) the second row of $\mathbf{AB}$ as a linear combination of the rows of $\mathbf{B}$ and (ii) the second column of $\mathbf{AB}$ as a linear combination of the columns of $\mathbf{A}$.

(i) $\mathbf{AB}$ is a 3×3 matrix. The elements of the second row of $\mathbf{AB}$ are given by the expression: $AB_{2,j} = \sum_{k=1}^{2} A_{2,k}B_{k,j}$. Thus, the second row can be written as the linear combination of rows of B as follows:

$$3\begin{bmatrix}1 & 2 & 3\end{bmatrix} + 4\begin{bmatrix}4 & 5 & 6\end{bmatrix}$$

(ii) Similarly, the second column of $\mathbf{AB}$ can be written as as the linear combination of columns of $\mathbf{A}$ as follows:

$$2\begin{bmatrix}1\\3\\5\end{bmatrix} + 5\begin{bmatrix}2\\4\\6\end{bmatrix}$$

1.3 Let $\mathbf{A} := \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -24 & 8 & -5 \\ 0 & -7 & 2 & 2 \end{bmatrix}$. Assuming that $\mathbf{A}$ is invertible, find the last column and the last row of $\mathbf{A}^{-1}$.

$\mathbf{AA^{-1}} = \mathbf{I_4}$, Thus we have the following system of equations to get the last column of $\mathbf{A^{-1}}$:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -24 & 8 & -5 \\ 0 & -7 & 2 & 2 \end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \text{ Solve this to get the last column of } \mathbf{A^{-1}}$$

$$\text{We get: } \begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}^T = \begin{bmatrix}2.75 & -0.5 & -2.25 & 1\end{bmatrix}^T$$

Do a similar process to get the last row. Since we already know x_4, now we'll have to solve a system of only 3 equations and 3 unknowns. Last Row of $\mathbf{A^{-1}} = \begin{bmatrix}-1.5 & -0.5 & 0 & 1\end{bmatrix}$

1.4 Show that the product of two upper triangular matrices is upper triangular. Is this true for lower triangular matrices?

Assume $\mathbf{A}$ and $\mathbf{B}$ are two upper triangular matrices. For these upper triangular matrices, A_{ij} and $B_{ij} = 0$ for $i > j$. We have to show that $AB_{ij} = 0$ for $i > j$ also holds true.
We have $AB_{ij} = A_i^T B_j$ where A_i^T is the i$^{\text{th}}$ row of A and B_j^T is the j$^{\text{th}}$ column of B.

$$\text{Thus, } AB_{i,j} = A_i^T B_j = \sum_{k=1}^{n} A_{ik} B_{kj}$$

$$= \sum_{k=1}^{j} A_{ik} B_{kj} + \sum_{k=j+1}^{n} A_{ik} B_{kj}$$

Now given A, B are upper triangular. So $A_{ij} = 0, B_{ij} = 0$ for $i > j$. Here we are only checking AB_{ij} for $i > j$, so we get $\sum_{k=1}^{j} A_{ik} B_{kj} = 0$ since A_{ik} is zero in the summation. $\sum_{k=j+1}^{n} A_{ik} B_{kj} = 0$ since B_{kj} is zero in the summation.
Similarly we can show that product of two lower triangular matrix is also lower triangular but there we would consider $i < j$ in our analysis.

1.5 The **trace** of a square matrix is the sum of its diagonal entries. Show that trace $(\mathbf{A}+\mathbf{B})$ = trace $(\mathbf{A})+$ trace $(\mathbf{B})$ and trace $(\mathbf{AB})$ = trace $(\mathbf{BA})$ for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$.

Part (a) is trivial.

$$trace(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}$$

$$trace(BA) = \sum_{i=1}^{n} (BA)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} B_{ik} A_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} A_{ki} B_{ik}$$

We have just switched the order of summation as the two summations are over independent axes. Thus we see that $trace(AB) = trace(BA)$ as the two expressions are equivalent

1.6 Find all solutions of the linear system $\mathbf{Ax} = \mathbf{b}$, where (i) $\mathbf{A} := \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$, $\mathbf{b} := \begin{bmatrix} 0 & -1 & 6 & 6 \end{bmatrix}^\mathsf{T}$,

(ii) $\mathbf{A} := \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$, $\mathbf{b} := \begin{bmatrix} 5 & -2 & 9 \end{bmatrix}^\mathsf{T}$,

(iii) $\mathbf{A} := \begin{bmatrix} 0 & 2 & -2 & 1 \\ 2 & -8 & 14 & -5 \\ 1 & 3 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 2 & 2 & 8 \end{bmatrix}^\mathsf{T}$

by reducing $\mathbf{A}$ to a row echelon form.

(i) We perform the row operations to the augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]$$

$R_4 := R_4 - 2R_1$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right]$$

$R_2 := R_2 - 2R_1$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right]$$

$R_3 := R_3 + 5R_2$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right]$$

Swap R_3 and R_4

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_3 = R_3 + 4R_2$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The last row of the augmented matrix is inconsistent. So the system has no solution.

(ii) Performing row operations on the augmented matrix,

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

$R_2 := R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

$R_3 := R_3 + R_1$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right]$$

$R_3 := R_3 + R_2$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

So we get $x_3 = 2$. Back-substituting in $8x_2 + 2x_3 = 12$ we get $x_2 = 1$ and back-substituting in $2x_1 + x_2 + x_3 = 5$, we get $x_1 = 1$.
The solution is; $\mathbf{x} := \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}^{\mathsf{T}}$

(iii) Here the augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 2 & -2 & 1 & 2 \\ 2 & -8 & 14 & -5 & 2 \\ 1 & 3 & 0 & 1 & 8 \end{array}\right]$$

Performing the following operations, we get; Swap R_1 and R_3

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 2 & -8 & 14 & -5 & 2 \\ 0 & 2 & -2 & 1 & 2 \end{array}\right]$$

$R_2 := R_2 - 2R_1$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 0 & -14 & 14 & -7 & -14 \\ 0 & 2 & -2 & 1 & 2 \end{array}\right]$$

Then $R_3 := 7R_3 + R_2$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 0 & -14 & 14 & -7 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the last row is 0, there are infinitely many solutions.

2 Tutorial 2 (on Lectures 3, 4 and 5)

2.1 Find the Row Canonical Form of $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$.

Row1 Pivot1 = 1
Swap R_2 and R_3

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_2 := R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Row2 Pivot2 = -1
$R_2 := R_2/(-1)$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_1 := R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Row3 Pivot3= 1
$R_1 := R_1 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_2 := R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Above Matrix is the row canonical form of the given Matrix.

2.2 Let $\mathbf{A} := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Find $\mathbf{A}^{-1}$ by Gauss-Jordan method.

2.3 An $m{\times}m$ matrix $\mathbf{E}$ is called an **elementary matrix** if it is obtained from the identity matrix $\mathbf{I}$ by an elementary row operation. Write down all elementary matrices.

(i) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. If an elementary row operation transforms $\mathbf{A}$ to $\mathbf{A}'$, then show that $\mathbf{A}' = \mathbf{E}\mathbf{A}$, where $\mathbf{E}$ is the corresponding elementary matrix.

(ii) Show that every elementary matrix is invertible, and find its inverse.

(iii) Show that a square matrix $\mathbf{A}$ is invertible if and only if it is a product of finitely many elementary matrices.

Part i

Each row operation is represented by $\mathbf{E_i}$ matrices. Let's take $\mathbf{E_1}, \mathbf{E_2}, ...\mathbf{E_k}$ be elementary row transformation matrix such that $\mathbf{E} = \mathbf{E_1}\mathbf{E_2}.....\mathbf{E_k}\mathbf{I}$ so we get

$$\mathbf{A'} = \mathbf{E_1}\mathbf{E_2}.....\mathbf{E_k}\mathbf{A}$$

Finally

$$\mathbf{A'} = \mathbf{E}\mathbf{A}$$

Part ii

Earlier we got to know that $\mathbf{E} = \mathbf{E_1}\mathbf{E_2}.....\mathbf{E_k}\mathbf{I}$, here we can see that E_i are elementary matrices which are invertible and hence the product of all such $\mathbf{E_i}$ are invertible. We can get the inverse by

$$\mathbf{E^{-1}} = (\mathbf{E_1}\mathbf{E_2}.....\mathbf{E_k})^{\mathbf{-1}}$$

$$\mathbf{E^{-1}} = \mathbf{E_k^{-1}}\mathbf{E_{k-1}^{-1}}.....\mathbf{E_1^{-1}}$$

Think how can you prove part3 on the basis of first part and second part

Part iii

A square matrix A is invertible if and only if you can row reduce A to an identity matrix I

Let's take the forward case so we have been given matrix is invertible .So on performing k row operations we obtain I

$$\mathbf{E_1}\mathbf{E_2}.....\mathbf{E_k}\mathbf{A} = \mathbf{I}$$

$$\mathbf{A} = \mathbf{E_k^{-1}}\mathbf{E_{k-1}^{-1}}.....\mathbf{E_1^{-1}}$$

Hence its proved

2.4 Let S and T be subsets of $\mathbb{R}^{n\times 1}$ such that $S \subset T$. Show that if S is linearly dependent then so is T, and if T is linearly independent then so is S. Does the converse hold?

Let $S = [v_1, v_2, ...v_s]$. Since $S \subset T$ let $T = [v_1, v_2, ...v_s, u_1, u_2, ..u_t]$. Now suppose if S is **Linearly dependant** then $\exists \alpha_1, \alpha_2...\alpha_s$ such that $\alpha_1 v_1 + \alpha_2 v_2... + \alpha_s v_s = 0$ and not all α_i are zero. Now let $\beta_1 v_1 + \beta_2 v_2 + .. + \beta_s v_s + \beta_{s+1} u_1 + \beta_{s+2} u_2 + ...\beta_{s+t} u_t = 0$. Put $\beta_{s+i} = 0$ where $i \geq 1$ and $\beta_i = \alpha_i$ for $i \leq s$. So this tuple value of β isnt zero hence T is **Linearly dependant**.

If T is Linearly independant then the only solution for $\beta_1 v_1 + \beta_2 v_2 + .. + \beta_s v_s + \beta_{s+1} u_1 + \beta_{s+2} u_2 + ...\beta_{s+t} u_t = 0$ is $\beta_i = 0$. Suppose if S is **Linearly dependant** then it means $\exists \alpha_1, \alpha_2...\alpha_s$ such that $\alpha_1 v_1 + \alpha_2 v_2... + \alpha_s v_s = 0$. Sp put $\beta_i = \alpha_i$ for $i \leq s$ and $\beta_{s+i} = 0$. This tuple satisfies the above equation yet $\beta \neq 0$. So this contradicts that T is Linearly independant. Hence S is **Linearly independant**

2.5 Are the following sets linearly independent?

(i) $\{[1 \quad -1 \quad 1], [3 \quad 5 \quad 2], [1 \quad 2 \quad 1], [1 \quad 1 \quad 1]\} \subset \mathbb{R}^{1\times 3}$,

(ii) $\{[1 \quad 9 \quad 9 \quad 8], [2 \quad 0 \quad 0 \quad 3], [2 \quad 0 \quad 0 \quad 8]\} \subset \mathbb{R}^{1\times 4}$,

(iii) $\{[1 \quad -1 \quad 0]^{\mathsf{T}}, [3 \quad -5 \quad 2]^{\mathsf{T}}, [1 \quad -2 \quad 1]^{\mathsf{T}}\} \subset \mathbb{R}^{3\times 1}$.

2.6 Given a set of s linearly independent row vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_i, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_s\}$ in $\mathbb{R}^{1\times n}$ and $\alpha \in \mathbb{R}$, show that the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_i + \alpha\mathbf{a}_j, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_s\}$ is linearly independent.

$c_1a_1 + c_2a_2 + ...c_ia_i + ..c_ja_j... + c_sa_s = 0$.
Since these vectors are linearly independant, $\forall_k \; c_k = 0$.
Now consider $\beta_1a_1 + \beta_2a_2 + ...\beta_i(a_i + \alpha a_j) + ..\beta_ja_j... + \beta_sa_s = 0$.
So $\beta_1a_1 + \beta_2a_2 + ...\beta_ia_i + ..(\beta_j + \beta_i\alpha)a_j... + \beta_sa_s = 0$.
So $\beta_1 = \beta_2 = ..\beta i.. = \beta_s = 0, \beta_j + \alpha\beta_i = 0$.
Hence $\forall_k\beta_k = 0$. So this set of vectors is also linearly independant.

2.7 Find the ranks of the following matrices.

(i) $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$, (ii) $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$.

2.8 Are the following subsets of $\mathbb{R}^{3\times 1}$ subspaces?

(i) $\{[x_1 \quad x_2 \quad x_3]^{\mathsf{T}} : x_1, x_2, x_3 \in \mathbb{R}, x_1 + x_2 + x_3 = 0\}$,

(ii) $\{[x_1 + x_2 + x_3 \quad x_2 + x_3 \quad x_3]^{\mathsf{T}} : x_1, x_2, x_3 \in \mathbb{R}\}$,

(iii) $\{[x_1 \quad x_2 \quad x_3]^{\mathsf{T}} : x_1, x_2, x_3 \in \mathbb{R}, x_1x_2x_3 = 0\}$

(iv) $\{[x_1 \quad x_2 \quad x_3]^{\mathsf{T}} : x_1, x_2, x_3 \in \mathbb{R}, |x_1|, |x_2|, |x_3| \leq 1\}$.

If so, find a basis for each, and also its dimension.

2.9 Describe all subspaces of $\mathbb{R}$, $\mathbb{R}^{2\times 1}$, $\mathbb{R}^{3\times 1}$ and $\mathbb{R}^{4\times 1}$. Can you visualise them geometrically?

3 Tutorial 3 (on Lectures 6 and 7)

3.1 Let V be a subspace of $\mathbb{R}^{n\times 1}$ with $\dim V = r$, and let S be a finite subset of V such that $\operatorname{span} S = V$. Suppose S has s elements. Show that (i) $s \geq r$, (ii) if $s = r$, then S is a basis for V, (iii) if $s > r$, then S contains basis for V.

3.2 Let $\mathbf{A}' \in \mathbb{R}^{m\times n}$ be in a REF. Show that the pivotal columns of $\mathbf{A}'$ form a basis for the column space $\mathcal{C}(\mathbf{A}')$.

3.3 Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. The set $\mathcal{R}(\mathbf{A})$ consisting of all linear combinations of the rows of $\mathbf{A}$ is called the **row space** of $\mathbf{A}$. Show that $\mathcal{R}(\mathbf{A})$ is a subspace of $\mathbb{R}^{1\times n}$. If $\mathbf{A}'$ is obtained from $\mathbf{A}$ by EROs, then prove that $\mathcal{R}(\mathbf{A}') = \mathcal{R}(\mathbf{A})$. Further, show that the dimension of $\mathcal{R}(\mathbf{A})$ is equal to the rank of $\mathbf{A}$.

3.4 Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$. Show that $\operatorname{rank} \mathbf{AB} \leq \min\{\operatorname{rank} \mathbf{A}, \operatorname{rank} \mathbf{B}\}$.

3.5 Let $\mathbf{A} := \begin{bmatrix} 0 & 0 & 0 & -2 & 1 \\ 0 & 2 & -2 & 14 & -1 \\ 0 & 2 & 3 & 13 & 1 \end{bmatrix}$. Find the rank and the nullity of $\mathbf{A}$. What is the dimension of the solution space of the homogeneous equation $\mathbf{Ax} = \mathbf{0}$? If $\mathbf{b} := \begin{bmatrix} 2 & 2 & 3 \end{bmatrix}^{\mathsf{T}}$, find the general solution of $\mathbf{Ax} = \mathbf{b}$.

3.6 Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$, where $a, b, c \in \mathbb{R}$. Also, prove an analogous formula for a determinant of order n, known as the **Vandermonde determinant**.

$$\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

Use $det(A) = det(A^T)$ and perform $R_k = R_k - R_1$ $\forall$ k=2 to 3

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} = (b-a)(c-a)(c-b)$$

Part 2

To prove general result use induction for n=2 we have

$$\det \begin{bmatrix} 1 & 1 \\ a_1 & a_2 \end{bmatrix} = (a_2 - a_1)$$

Now assume it to be true for n-1 order matrix and if we are able to prove n order matrix from the n-1 order matrix we are done

$$\det\begin{bmatrix} 1 & 1 & 1 & \dots & \dots & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & \dots & \dots & a_n \\ & & & & . & & \\ & & & . & & & \\ & & . & & & & \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \dots & \dots & \dots & a_n^{n-1} \end{bmatrix} = \prod_{1\le i<j\le n}(a_j - a_i)$$

$$det(A) = det(A^T)$$

$$\det\begin{bmatrix} 1 & a_1 & a_1^2 & \dots & \dots & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & \dots & \dots & a_2^{n-1} \\ & & & & . & & \\ & & & . & & & \\ & & . & & & & \\ 1 & a_n & a_n^2 & \dots & \dots & \dots & a_n^{n-1} \end{bmatrix} = \prod_{1\le i<j\le n}(a_j - a_i)$$

$R_k = R_k - R_1$ $\forall$ k=2 to n

$$\det\begin{bmatrix} 1 & a_1 & a_1^2 & \dots & \dots & \dots & a_1^{n-1} \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 & \dots & \dots & \dots & a_2^{n-1} - a_n^{n-1} \\ & & & & . & & \\ & & & . & & & \\ & & . & & & & \\ 0 & a_n - a_1 & a_n^2 - a_1^2 & \dots & \dots & \dots & a_n^{n-1} - a_1^{n-1} \end{bmatrix} -> eqn(I)$$

$$\prod_{1\le j\le n}(a_j - a_1)\det\begin{bmatrix} 1 & a_2 + a_1 & \dots & \dots & \sum_0^{n-1} a_2^{n-2-i} a_1^i \\ & & & . & \\ & & . & & \\ & . & & & \\ . & & & & \\ 1 & a_n + a_1 & \dots & \dots & \sum_0^{n-1} a_n^{n-2-i} a_1^i \end{bmatrix}$$

Now keep on splitting the det by column wise starting from col(2) to col(n) and see only one non zero det would surive and others would vanish

$$\prod_{1\le j\le n}(a_j - a_1)\det\begin{bmatrix} 1 & a_1 & a_1^2 & \dots & \dots & \dots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \dots & \dots & \dots & a_2^{n-2} \\ & & & & . & & \\ & & & . & & & \\ & & . & & & & \\ 1 & a_{n-1} & a_{n-1}^2 & \dots & \dots & \dots & a_{n-1}^{n-2} \end{bmatrix}$$

$$\prod_{1\le i<j\le n} (a_j - a_1) * \prod_{2\le j\le n} (a_j - a_i)$$

$$\prod_{1\le j\le n} (a_j - a_i)$$

Other method

Look at eqn(I) matrix

Use $det(A) = det(A^T)$ and consecutively perform $R_k = R_k - R_{k-1} * a_1$ $\forall$ k=2 to n Try out

3.7 For $n \in \mathbb{N}$, prove that

$$\det \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ & & & & . & & \\ & & & . & & & \\ & & . & & & & \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} = (-1)^{n(n-1)/2}.$$

Use induction Method:

For n=1 we have,

$$\det \begin{bmatrix} 1 \end{bmatrix} = (-1)^{1(1-1)/2} = 1$$

Now assume it to be true for n-1 order matrix and if we are able to prove n order matrix from the n-1 order matrix we are done

$$\text{To prove:: } \det \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ & & & & . & & \\ & & & . & & & \\ & & . & & & & \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} = (-1)^{n(n-1)/2}$$

$$\det \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ & & & & . & & \\ & & & . & & & \\ & & . & & & & \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

Now if we expand via the first row to find det and use result of $det(A)_{n-1}$, we get

$$(-1)^{n+1} \det \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ & & & & \cdot & & \\ & & & \cdot & & & \\ & & \cdot & & & & \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}_{n-1}$$

$$(-1)^{n+1} * (-1)^{(n-1)(n-2)/2} = (-1)^{n(n-1)/2}$$

3.8 For $n \in \mathbb{N}$, prove that

$$\det \begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 2 & 3 & \dots & n-1 & n \\ 3 & 3 & 3 & \dots & n-1 & n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n-1 & n-1 & \dots & n-1 & n \\ n & n & n & \dots & n & n \end{bmatrix} = (-1)^{n+1} n.$$

$$R_n \mapsto \frac{1}{n} R_n$$

$R_i \mapsto R_i - iR_n$ for all $i \in \{1, \dots, n-1\}$.

For example, in the case of $n = 4$, you should have arrived at the following conclusion:

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} = 4 \det \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Write the general case.

Now, expand along the first column. This is simple to do as it has only one non-zero entry.

(Note that you'll get a $(-1)^n$.)

Thus, you get that the original determinant equals the following expression:

$$(-1)^n n \det \begin{bmatrix} 1 & 2 & \cdots & n-1 \\ 0 & 1 & \cdots & n-2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Note that the determinant written above is just 1 as it's a triangular matrix with all diagonal entries 1.

Thus, the answer is $(-1)^n n$.

3.9 Find rank $\mathbf{A}$ using determinants, where $\mathbf{A}$ is

$$\text{(i)}\ \begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix}, \ \text{(ii)}\ \begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix}.$$

Verify by transforming $\mathbf{A}$ to a REF.

4 Tutorial 4 (on Lectures 8, 9 and 10)

4.1 Find the value(s) of α for which Cramer's rule is applicable. For the remaining value(s) of α, find the number of solutions, if any.

$$\begin{array}{ccccccc} \mathrm{x} & + & 2\mathrm{y} & + & 3\mathrm{z} & = & 20 \\ \mathrm{x} & + & 3\mathrm{y} & + & \mathrm{z} & = & 13 \\ \mathrm{x} & + & 6\mathrm{y} & + & \alpha\mathrm{z} & = & \alpha. \end{array}$$

4.2 Find the cofactor matrix $\mathbf{C}$ of the matrix $\mathbf{A}$, and verify $\mathbf{C}^{\mathsf{T}}\mathbf{A} = (\det \mathbf{A})\mathbf{I} = \mathbf{A}\mathbf{C}^{\mathsf{T}}$. If $\det \mathbf{A} \neq 0$, find $\mathbf{A}^{-1}$, where $\mathbf{A}$ is

$$\text{(i)} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{(ii)} \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \text{(iii)} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

4.3 Find the matrix of the linear transformation $T : \mathbb{R}^{3\times 1} \to \mathbb{R}^{4\times 1}$ defined by $T\left(\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathsf{T}}\right) := \begin{bmatrix} x_1 + x_2 & x_2 + x_3 & x_3 + x_1 & x_1 + x_2 + x_3 \end{bmatrix}^{\mathsf{T}}$ with respect to the ordered bases (i) $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^{3\times 1}$ and $F = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ of $\mathbb{R}^{4\times 1}$,

(ii) $E' = (\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_1)$ of $\mathbb{R}^{3\times 1}$ and $F' = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_1, \mathbf{e}_4 + \mathbf{e}_1 + \mathbf{e}_2)$ of $\mathbb{R}^{4\times 1}$, first showing that E' is a basis for $\mathbb{R}^{3\times 1}$ and F' is a basis for $\mathbb{R}^{4\times 1}$.

Part(i)
We have the basis set $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^{3\times 1}$ and $F = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ of $\mathbb{R}^{4\times 1}$,

$\mathrm{T}(\mathbf{e}_1) = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^{\mathsf{T}} = 1\mathbf{e}_1 + 0\mathbf{e}_2 + 1\mathbf{e}_3 + 1\mathbf{e}_4$

$\mathrm{T}(\mathbf{e}_2) = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^{\mathsf{T}} = 1\mathbf{e}_1 + 1\mathbf{e}_2 + 0\mathbf{e}_3 + 1\mathbf{e}_4$

$\mathrm{T}(\mathbf{e}_3) = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}^{\mathsf{T}} = 0\mathbf{e}_1 + 1\mathbf{e}_2 + 1\mathbf{e}_3 + 1\mathbf{e}_4$

$$\mathbf{M}_F^E(T) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Part(ii)
Check whether the set E' and set F' forms a basis set? Indeed yes they form (Try it out)

$\mathrm{T}(\mathbf{e}_1 + \mathbf{e}_2) = \begin{bmatrix} 2 & 1 & 1 & 2 \end{bmatrix}^{\mathsf{T}} = 0(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) + 0(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) + 1(\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_1) + 1(\mathbf{e}_4 + \mathbf{e}_1 + \mathbf{e}_2)$
$=$

$\mathrm{T}(\mathbf{e}_2 + \mathbf{e}_3) = \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}^{\mathsf{T}} = 0(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) + 1(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) + 0(\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_1) + 1(\mathbf{e}_4 + \mathbf{e}_1 + \mathbf{e}_2)$
$=$

$\mathrm{T}(\mathbf{e}_3 + \mathbf{e}_1) = \begin{bmatrix} 1 & 1 & 2 & 2 \end{bmatrix}^{\mathsf{T}} = 0(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) + 1(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) + 1(\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_1) + 0(\mathbf{e}_4 + \mathbf{e}_1 + \mathbf{e}_2)$
$=$

$$\mathbf{M}_{F'}^{E'}(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

4.4 Let $\mathbf{A} \in \mathbb{R}^{4\times 4}$. Let $\mathbf{P} := \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Show that $\mathbf{P}$ is invertible. Find an ordered bases E of $\mathbb{R}^{4\times 1}$

such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{M}_E^E(T_{\mathbf{A}})$.

Using the theorem we get $\mathbf{E} = \{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4\}$

4.5 Let $\lambda \in \mathbb{K}$. Find the geometric multiplicity of the eigenvalue λ of each of the following matrices:

$$\mathbf{A} := \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \mathbf{B} := \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \mathbf{C} := \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Also, find the eigenspace associated with λ in each case.

For $|\mathbf{A} - \mu\mathbf{I}| = 0 = (\mu - \lambda)^3$ its true for all vector $\mathbf{x} = (x_1, x_2, x_3)$ and hence eigen space is $\mathbb{R}^3$

For $|\mathbf{B} - \mu\mathbf{I}| = 0 = (\mu - \lambda)^3$ and for corresponding eigen vector $\mathbf{x} = (x_1, x_2, x_3)$
Solve $(\mathbf{B} - \lambda\mathbf{I})\mathbf{x} = 0 \implies \mathrm{x}_2 = 0$ and hence eigen space is $\mathbb{R}^2$

For $|\mathbf{C} - \mu\mathbf{I}| = 0 = (\mu - \lambda)^3$ and for corresponding eigen vector $\mathbf{x} = (x_1, x_2, x_3)$
Solve $(\mathbf{B} - \lambda\mathbf{I})\mathbf{x} = 0 \implies \mathrm{x}_2 = 0,\ x_3 = 0$ and hence eigen space is $\mathbb{R}$

4.6 Let $\mathbf{A} := \begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix}$. Show that 3 is an eigenvalue of $\mathbf{A}$, and find all eigenvectors of $\mathbf{A}$ corresponding to it. Also, show that $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^{\mathsf{T}}$ is an eigenvector of $\mathbf{A}$, and find the corresponding eigenvalue of $\mathbf{A}$.

Check $|\mathbf{A} - 3\mathbf{I}| = 0$, we get $\det \begin{bmatrix} 0 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 1 & 2 \end{bmatrix} = 0$

$$\mathbf{A}\mathbf{x} = 3\mathbf{x}$$

,

$$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We get $x_1 = 0$ and $x_2 + 2x_3 = 0$. So all eigen vectors $\mathbf{x} = x_3(0, -2, 1)$ where $x_3 \in \mathbb{R}$
To prove $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^{\mathsf{T}}$ is an eigenvector of $\mathbf{A}$

$$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We get the eigen value to be 6.

4.7 Let $\theta \in (-\pi, \pi]$, $\mathbf{A} := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $\mathbb{K} = \mathbb{C}$. Show that $\cos\theta \pm i\sin\theta$ are eigenvalues of $\mathbf{A}$. Find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix, and check your answer.

For $|\mathbf{A}-\mu\mathbf{I}|=0$,

$$\det\left(\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} - \mu\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}\cos\theta-\mu & -\sin\theta\\ \sin\theta & \cos\theta-\mu\end{bmatrix}\right) = 0$$

$$\mu^2 - 2\mu\cos\theta + 1 = 0 \implies \mu = \cos\theta \pm i\sin\theta$$

$$\mathbf{x} = (x_1, x_2) \text{ where } x_1, x_2 \in \mathbb{C}$$

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \mu\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

We get $\cos\theta x_1 - \sin\theta x_2 = (\cos\theta - i\sin\theta)x_1 \implies x_2 = ix_1$

We get $\mathbf{x} = x_1(1, i)$ where $x_1 \in \mathbb{C}$

For other eigen value $\cos\theta x_1 + \sin\theta x_2 = (\cos + i\sin\theta)x_1 \implies x_2 = -ix_1$

We get $\mathbf{x} = x_1(1, -i)$ where $x_1 \in \mathbb{C}$

$\mathbf{P} := \begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$ and Check it $\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix}\cos\theta - i\sin\theta & 0\\ 0 & \cos + i\sin\theta\end{bmatrix}$

4.8 Let $n \geq 2$ and $\mathbf{A} := \begin{bmatrix}1 & \cdots & 1\\ \vdots & \vdots & \vdots\\ 1 & \cdots & 1\end{bmatrix} \in \mathbb{R}^{n\times n}$, that is, $a_{jk} = 1$ for all $j, k = 1, \ldots, n$. Find rank $\mathbf{A}$ and nullity $\mathbf{A}$. Find an eigenvector of $\mathbf{A}$ corresponding to a nonzero eigenvalue by inspection. Find two distinct eigenvalues of $\mathbf{A}$ along with their geometric multiplicities, and find bases for the eigenspaces. Show that $\mathbf{A}$ is diagonalizable, and find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

$Rank\mathbf{A} = 1$, $Nullity\mathbf{A} = n-1$

Eigen vector = $\begin{bmatrix}1 & 1 & ... & 1\end{bmatrix}^T$ for eigen value=n

To find $|\mathbf{A}-\mu\mathbf{I}| = 0$, Swap all rows inititially and perform $R_1 \mapsto \sum_{i=1}^n R_i$ and take (n-μ) common and then $R_k \mapsto R_k - R_1 \forall$ k=2 to n and then expand via last column

we get $\mu^{n-1}(\mu - n) = 0 \implies \mu = 0$ GM is n-1 ,$\mu =$ n GM is 1

Now find eigen vectors corresponding to all eigen values $(\mathbf{A}-\mu\mathbf{I})\mathbf{x} = 0$ we get

For $\mu = 0$,$v = \{\ \mathbf{x} : \sum_{i=1}^n x_i = 0\}$

$For \mu = n$ we get $\mathbf{v} = x_1(1,1,1,1....)^T \forall x_1 \in \mathbb{R}$ $\mathbf{P} := \begin{bmatrix}-1 & -1 & -1 & . & . & -1 & 1\\ 1 & 0 & 0 & . & . & 0 & 1\\ 0 & 1 & 0 & . & . & 0 & 1\\ 0 & 0 & 1 & . & . & 0 & 1\\ . & . & . & . & . & . & .\\ . & . & . & . & . & . & .\\ 0 & 0 & 0 & . & . & 1 & 1\end{bmatrix}$

Perform $\mathbf{P}^{-1}\mathbf{AP}$ to get to a diagonal matrix

5 Tutorial 5 (on Lectures 11, 12 and 13)

5.1 Find all eigenvalues, and their geometric as well as algebraic multiplicities of the following matrices. Are they diagonalizable? If so, find invertible $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

$$\text{(i) } \mathbf{A} := \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}, \quad \text{(ii) } \mathbf{A} := \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, \quad \text{(iii) } \mathbf{A} := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Similar to exercise 4.7 and 4.8

5.2 Let $\mathbf{A} := \begin{bmatrix} 2 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}$. Find a necessary and sufficient condition on a, b, c for $\mathbf{A}$ to be diagonalizable.

You can easily see eigen values are 2,1,2
Just you need to check for nullspace $(\mathbf{A} - \mu\mathbf{I})\mathbf{x} = 0$ or find nullity for $\mu = 2$

$$\begin{bmatrix} 2-\mu & a & b \\ 0 & 1-\mu & c \\ 0 & 0 & 2-\mu \end{bmatrix} \mapsto \begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix}$$

So for nullity equal to 2 we need rank =1 hence R_2 must to be a scalar multiple of R_1
$\frac{a}{-1} = \frac{b}{c} \implies$ b=-ac

5.3 Let $k \in \mathbb{N}$ and

$$\mathbf{A} := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & -1 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{K}^{2k\times 2k},$$

that is, $\mathbf{A}$ has all diagonal entries 0, the subdiagonal entries are $1, 0, 1, 0 \ldots, 1, 0$, and the superdiagonal entries are $-1, 0, -1, 0, \ldots, -1, 0$. Find the characteristic polynomial of $\mathbf{A}$, all eigenvalues of $\mathbf{A}$, and their algebraic as well as geometric multiplicities.

Take $(\mathbf{A} - \mu\mathbf{I})$ and perform $R_{2i} \mapsto R_{2i} + R_{2i-1}/\mu$ $\forall$ i=1 to k
There was a catch that $\mu \neq 0$ (how would you prove that). Hint (find nullity of A)
It's a Upper triangular matrix and whose det is product of diagonal entries
$\mu^k(\mu + 1/\mu)^k = 0 \implies (\mu^2+1)^k = 0 \implies \mu = \pm i$
Find Nullity of $(\mathbf{A} - i\mathbf{I})$ by performing $R_{2i} \mapsto R_{2i} - iR_{2i-1}$ $\forall$ i=1 to k
Characteristic polynomial is $(\mathbf{A^2}+1)^k = 0$

5.4 Let $\lambda \in \mathbb{K}$. Show that λ is an eigenvalue of $\mathbf{A}$ if and only if $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}^*$, but their eigenvectors can be very different.

For forward part,

$$\lambda\|\mathbf{x}\|^2 = \lambda < x, x >=< x, \lambda x >=< x, Ax >$$

Transformation property: $< Ax, y >= (Ax)^*y = x^*(A * y) < x, A^*y >$

$$\lambda\|\mathbf{x}\|^2 =< A^*x, x >$$

Take conjugate on both sides

$$\overline{\lambda}\|\mathbf{x}\|^2 = \overline{< A^*x, x >}$$

$$\overline{\lambda}\|\mathbf{x}\|^2 =< x, A^*x >$$

Similarly prove the backward part (Try it)

Other method:
$|\mathbf{A} - \lambda\mathbf{I}| = 0$. Choose $\mathbf{B} = \mathbf{A} - \lambda\mathbf{I}$ and we get det(B) $= 0$
We can claim that det(B*)=0. So $\mathbf{B}^* = \mathbf{A}^* - \overline{\lambda}\mathbf{I}$.
Now $|\mathbf{A}^* - \overline{\lambda}\mathbf{I}| = 0$ hence $\overline{\lambda}$ is an eigen value of $\mathbf{A}*$

5.5 Let $\mathbf{A} \in \mathbb{K}^{n\times n}$. Show that 0 is an eigenvalue of $\mathbf{A}$ if and only if 0 is an eigenvalue of $\mathbf{A}^*\mathbf{A}$, and its geometric multiplicity is the same. Deduce rank $\mathbf{A}^*\mathbf{A} = $ rank $\mathbf{A}$.

$\mathbf{A}x = 0 \implies A^*Ax = 0 \implies x \in N(A^*A)$
$N(A) \subseteq N(A^*A)$
Now consider $A * Ax = 0 \implies x^*A^*Ax = 0 \implies (Ax)^*Ax = 0 \implies Ax = 0 \implies x \in N(A)$
$N(A^*A) \subseteq N(A)$
Hence $N(A) = N(A^*A)$
All part follows from this because geometric multiplicity of 0 is nullity of the matrix.

5.6 Let $\mathbf{A} := \begin{bmatrix} 2 & i & 1+i \\ -i & 3 & 1 \\ 1-i & -1 & 8 \end{bmatrix}$. Show that no eigenvalue of $\mathbf{A}$ is away from one of the diagonal entries of $\mathbf{A}$ by more than $1 + \sqrt{2}$.

By the Gerschgorin Theorem we know $|\lambda - a_{jj}| \leq \sum_{j\neq k} |a_{jk}|$
Lets calculate $\sum_{j\neq k} |a_{jk}|$ for j=1 it's $1 + \sqrt{2}$
For j=2 it's 2 , For j=3 it's $1 + \sqrt{2}$

5.7 A square matrix $\mathbf{A} := [a_{jk}]$ is called **strictly diagonally dominant** if $|a_{jj}| > \sum_{k\neq j} |a_{jk}|$ for each $j = 1, \ldots, n$. If $\mathbf{A}$ strictly diagonally dominant, show that $\mathbf{A}$ is invertible.

By the Gerschgorin Theorem we know $|\lambda - a_{jj}| \leq \sum_{j\neq k} |a_{jk}|$
We have $\lambda - a_{jj} > -\sum_j |a_{jk}| \mapsto$ I
We already have that $|a_{jj}| > \sum_{k\neq j} |a_{jk}| \implies a_{jj} - \sum_{k\neq j} |a_{jk}| > 0 \mapsto$ II
From I and II we get $\lambda > 0$ hence the matrix is invertible

5.8 Let $\mathbf{A} \in \mathbb{K}^{n\times n}$. Define $\alpha_2 := \max\{\|\mathbf{Ax}\| : \|\mathbf{x}\| = 1\}$, $\alpha_\infty := \max\left\{ \sum_{k=1}^n |a_{jk}| : j = 1, \ldots, n\right\}$ and $\alpha_1 := \max\left\{ \sum_{j=1}^n |a_{jk}| : k = 1, \ldots, n\right\}$, where $\mathbf{A} := [a_{jk}]$. Show that $|\lambda| \leq \min\{\alpha_2, \alpha_\infty, \alpha_1\}$ for every eigenvalue λ.

consider λ to be max of all eigen value
$\alpha_2 \geq \|\mathbf{Ax}\| = \|\lambda\mathbf{x}\| = |\lambda|$

By the Gerschgorin Theorem we know $|\lambda - a_{jj}| \leq \sum_{j\neq k} |a_{jk}|$
$||\lambda| - |a_{jj}|| \leq |\lambda - a_{jj}| \leq \sum_{j\neq k} |a_{jk}|$
$||\lambda| - |a_{jj}|| \leq \sum_{j\neq k} |a_{jk}| \implies |\lambda| - |a_{jj}| \leq \sum_{j\neq k} |a_{jk}|$
$|\lambda| = |a_{jj}| + \sum_{j\neq k} |a_{jk}| \leq \alpha_\infty$

Eigen values of $\mathbf{A}$ and $\mathbf{A}^T$ are same and performing same operations as we did above we can say $|\lambda| \leq \alpha_1$

Other method (An Important General result):
Let $(\lambda, \mathbf{x})$ be eigen pair s.t $\rho(\mathbf{A}) = max|\lambda|$
Find $\mathbf{y} \neq 0$ s.t $\mathbf{xy}^*$ is a non zero matrix , $\|.\|$ is a matrix norm
$\lambda\mathbf{x} = \mathbf{Ax} \implies \lambda\mathbf{xy}^* = \mathbf{Axy}^* \implies |\lambda|\|\mathbf{xy}^*\| = \|\mathbf{Axy}^*\| \leq \|\mathbf{A}\|\|\mathbf{xy}^*\| \implies \rho(\mathbf{A}) \leq \|\mathbf{A}\|$

5.9 Let $\mathbf{x}, \mathbf{y} \in \mathbb{K}^{n\times 1}$. Prove the **parallelogram law**: $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$. In case $\mathbf{x}$ and $\mathbf{y}$ are both nonzero, prove the **cosine law**, which says that $\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$, where the angle $\theta \in [0, \pi]$ between nonzero $\mathbf{x}$ and $\mathbf{y}$ is defined to be $\cos^{-1}\big(\Re\langle \mathbf{x}, \mathbf{y}\rangle / \|\mathbf{x}\|\|\mathbf{y}\|\big)$.

Part.a) You need to use $\|\mathbf{x}+\mathbf{y}\|^2 =< \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} >=$
$< \mathbf{x}, \mathbf{x}+\mathbf{y} > + < \mathbf{y}, \mathbf{x}+\mathbf{y} >=< \mathbf{x}, \mathbf{x} > + < \mathbf{x}, \mathbf{y} > + < \mathbf{y}, \mathbf{x} > + < \mathbf{y}, \mathbf{y} >$ and
Similarly for the other term $\|\mathbf{x}-\mathbf{y}\|^2 =< \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} >=$
$< \mathbf{x}, \mathbf{x}-\mathbf{y} > + < -\mathbf{y}, \mathbf{x}-\mathbf{y} >=< \mathbf{x}, \mathbf{x} > - < \mathbf{x}, \mathbf{y} > - < \mathbf{y}, \mathbf{x} > + < \mathbf{y}, \mathbf{y} >$
Part.b) $(\Re < x, y >) = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$ where $\theta \in [0, \pi]$

6 Tutorial 6 (on Lectures 14, 15 and 16)

6.1 Orthonormalize the following ordered subsets of $\mathbb{K}^{4\times 1}$.

(i) $(\mathbf{e}_1,\ \mathbf{e}_1+\mathbf{e}_2,\ \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\ \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)$

(ii) $\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4,\ -\mathbf{e}_1+\mathbf{e}_2,\ -\mathbf{e}_1+\mathbf{e}_3,\ -\mathbf{e}_1+\mathbf{e}_4)$.

6.2 Use the Gram-Schmidt Orthogonalization Process to orthonormalize the ordered subset

$$\left(\begin{bmatrix}1 & -1 & 2 & 0\end{bmatrix}^{\mathsf{T}}, \begin{bmatrix}1 & 1 & 2 & 0\end{bmatrix}^{\mathsf{T}}, \begin{bmatrix}3 & 0 & 0 & 1\end{bmatrix}^{\mathsf{T}}\right)$$

and obtain an ordered orthonormal set $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. Also, find $\mathbf{u}_4$ such that $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ is an ordered orthonormal basis for $\mathbb{K}^{4\times 1}$. Express the vector $\begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}^{\mathsf{T}}$ as a linear combination of these four basis vectors.

> Let W be the subspace of $\mathbb{K}^{4\times 1}$ spanned by the vectors $\mathbf{x}_1 := \begin{bmatrix}1 & -1 & 2 & 0\end{bmatrix}^{\mathsf{T}}$,
> $\mathbf{x}_2 := \begin{bmatrix}1 & 1 & 2 & 0\end{bmatrix}^{\mathsf{T}}$ and $\mathbf{x}_3 := \begin{bmatrix}3 & 0 & 0 & 1\end{bmatrix}^{\mathsf{T}}$ Let us apply the G-S OP.
>
> Let $u_1 := \frac{x_1}{\|x_1\|} = \frac{\begin{bmatrix}1 & -1 & 2 & 0\end{bmatrix}^{\mathsf{T}}}{\sqrt{6}}$
>
> , $u_2 := \frac{x_2 - P_{u_1}(x_2)}{\|x_2 - P_{u_1}(x_2)\|} = \frac{\begin{bmatrix}1 & 5 & 2 & 0\end{bmatrix}^{\mathsf{T}}}{\sqrt{30}}$
>
> $u_3 := \frac{x3 - P_{u_1}(x3) - P_{u_2}(x3)}{\|x3 - P_{u_1}(x3) - P_{u_2}(x3)\|} = \frac{\begin{bmatrix}12 & 0 & -6 & 5\end{bmatrix}^{\mathsf{T}}}{\sqrt{205}}$
>
> You can check for yourself that $\{u_1, u_2, u_3\}$ is an orthonormal basis
> To extend $\{u_1, u_2, u_3\}$ to an orthonormal basis for V $:= \mathbb{K}^{4\times 1}$, we look for $u_4 := \begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix}^{\mathsf{T}}$
> which is orthogonal to the set $\{x1, x2, x3\}$. Try on your own

6.3 Show that $\mathbf{A} \in \mathbb{K}^{n\times n}$ is unitary if and only if its rows form an orthonormal subset of $\mathbb{K}^{1\times n}$.

6.4 Let $E := (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be the standard basis for $\mathbb{K}^{n\times 1}$, and let $F := (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be an orthonormal basis for $\mathbb{K}^{n\times 1}$. If I denotes the identity map from $\mathbb{K}^{n\times 1}$ to $\mathbb{K}^{n\times 1}$, then show that the matrix $\mathbf{M}_E^F(I)$ is unitary.

6.5 Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and let λ be an eigenvalue of $\mathbf{A}$. Show that $p(\lambda)$ is an eigenvalue of $p(\mathbf{A})$ for every polynomial $p(t)$.

6.6 Suppose $\mathbf{A} \in \mathbb{C}^{3\times 3}$ satisfies $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} = 6\mathbf{I}$.

If $5 \le \det \mathbf{A} \le 7$, determine the eigenvalues of $\mathbf{A}$.

Is $\mathbf{A}$ diagonalizable?

6.7 Let $\mathbf{A} \in \mathbb{K}^{n\times n}$, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$ with a corresponding orthonormal set of eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Show that $\mathbf{A} = \lambda_1 \mathbf{u}_1\mathbf{u}_1^* + \cdots + \lambda_n \mathbf{u}_n\mathbf{u}_n^*$. ($\mathbf{x}\mathbf{y}^*$= outer product of $\mathbf{x}, \mathbf{y}$)

6.8 Let $\mathbf{A} \in \mathbb{K}^{n\times n}$, and $\lambda \in \mathbb{K}$.

(i) Show that λ is an eigenvalue of $\mathbf{A}$ if and only $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}^*$.

(ii) Let $\mathbf{A}$ be unitary. Show that $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$. If λ is an eigenvalue of $\mathbf{A}$, then show that $|\lambda| = 1$.

(iii) Let $\mathbb{K} = \mathbb{C}$ and let $\mathbf{A}$ skew self-adjoint. If λ is an eigenvalue of $\mathbf{A}$, then show that $i\lambda \in \mathbb{R}$.

6.9 Let $\mathbf{A} := [a_{jk}] \in \mathbb{C}^{n\times n}$, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$, counting algebraic multiplicities. Show that $\mathbf{A}$ is normal $\iff \sum_{1\le j,k\le n} |a_{jk}|^2 = \sum_{j=1}^{n} |\lambda_j|^2$.

6.10 A matrix $\mathbf{A} \in \mathbb{K}^{n\times n}$ is called **nilpotent** if there is $m \in \mathbb{N}$ such that $\mathbf{A}^m = \mathbf{O}$. If $\mathbf{A}$ is upper triangular with all diagonal entries equal to 0, then show that $\mathbf{A}$ is nilpotent. Further, if $\mathbf{A} \in \mathbb{C}^{n\times n}$, then show that $\mathbf{A}$ is nilpotent if and only if 0 is the only eigenvalue of $\mathbf{A}$.

7 Tutorial 7 (on Lectures 17, 18 and 19)

7.1 Let $\mathbf{A} \in \mathbb{C}^{n\times n}$. Show that $\mathbf{A}$ is self-adjoint if and only if $\mathbf{A}$ is normal and all eigenvalues of $\mathbf{A}$ are real.

7.2 State and prove a spectral theorem for skew self-adjoint matrices with complex entries.

7.3 Find an orthonormal basis for $\mathbb{K}^{4\times 1}$ consisting of eigenvectors of

$$\mathbf{A} := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}.$$

Write down a spectral representation of $\mathbf{A}$, and find $\mathbf{A}^7\mathbf{x}$, where $\mathbf{x} := \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^{\mathsf{T}}$

7.4 A self adjoint matrix $\mathbf{A}$ is called **positive definite** if $\langle \mathbf{A}\,\mathbf{x},\, \mathbf{x}\rangle > 0$ for all nonzero $\mathbf{x} \in \mathbb{K}^{n\times 1}$. Show that a self-adjoint matrix is positive definite if and only if all eigenvalues of $\mathbf{A}$ are positive.

7.5 Real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are placed on the 4 corners of a square in clockwise order initially. In the next step,

α_1 is replaced by $\beta_1 := (\alpha_2 + \alpha_4)/2$,

α_2 is replaced by $\beta_2 := (\alpha_3 + \alpha_1)/2$,

α_3 is replaced by $\beta_3 := (\alpha_4 + \alpha_2)/2$ and

α_4 is replaced by $\beta_4 := (\alpha_1 + \alpha_3)/2$.

Find the numbers placed on the corners of the square after k such steps. (Hint: Find a set of 4 orthonormal eigenvectors of the matrix $\mathbf{A} := \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$ and use the spectral theorem for $\mathbf{A}$.)

7.6 Let Q be a real quadratic form, and let $\mathbf{A}$ denote the associated real symmetric matrix. Let $g(\mathbf{x}) = \|\mathbf{x}\|^2 - 1$. If $\mathbb{Q}$ has a local extremum at a vector $\mathbf{x}_0$ subject to the constraint $g(\mathbf{x}) = 0$, then show that $\mathbf{x}_0$ is a unit eigenvector of $\mathbf{A}$, and the corresponding eigenvalue λ_0 is the corresponding Lagrange multiplier and equals $Q(\mathbf{x}_0)$.

In particular, the largest eigenvalue of $\mathbf{A}$ is the constrained maximum and the smaller eigenvalue of $\mathbf{A}$ is the constrained minimum of Q.

7.7 Which quadric surface does the equation $7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy - 36 = 0$ describe? Explain by reducing the quadratic form involved to a diagonal form. Express x, y, z in terms of the new coordinates u, v, w.

$\mathbf{Q}(x) = 7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy - 36$ to a diagonal form.

Here $\mathbf{A} := \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix}$ is the associated matrix.

Hence the equation of the given quadric surface becomes

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix}^T - 36 = 0$$

Now find eigen value and corresponding eigen vector and then using GSOP find $\{u_1, u_2, u_3\}$
Change of variable from $\begin{bmatrix} x & y & z \end{bmatrix}^T = \mathbf{C} \begin{bmatrix} u & v & w \end{bmatrix}^T$,where $\mathbf{C} = [\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3]$
Characteristic polynomial is $\lambda^3 - 12\lambda - 180\lambda + 1296 = 0$
Eigen values are {18,-12,6}
Eigen vectors are $\{\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T\}$
By GSOP Orthonormal eigen vectors are $\{\frac{\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T}{\sqrt{3}}, \frac{\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^T}{\sqrt{6}}, \frac{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T}{\sqrt{2}}\}$
$\mathbf{Q}_D(u, v, w) = 18u^2 - 12v^2 + 6w^2$
The quadric surface reduces to $18u^2 - 12v^2 + 6w^2 = 36$
Since eigen values two positive, one negative its **1 sheeted hyperboloid**

$$\begin{bmatrix} x & y & z \end{bmatrix}^T = \mathbf{C} \begin{bmatrix} u & v & w \end{bmatrix}^T$$

$$\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix} \begin{bmatrix} u & v & w \end{bmatrix}^T$$

$x = \frac{-1}{\sqrt{3}}u + \frac{1}{\sqrt{6}}v + \frac{1}{\sqrt{2}}w$, $y = \frac{1}{\sqrt{3}}u + \frac{-1}{\sqrt{6}}v + \frac{1}{\sqrt{2}}w$, $z = \frac{1}{\sqrt{3}}u + \frac{2}{\sqrt{6}}v + 0w$

7.8 Let Y be a subspace of $\mathbb{K}^{n\times 1}$. Show that $(Y^\perp)^\perp = Y$.

Let $\{u_1, u_2, ...u_k\}$ and $\{w_1, w_2, ...w_l\}$ be an orthonormal basis for subspace respectively Y and $Y^\perp$
Every vector $\mathbf{s} \in (Y^\perp)^\perp$ will be perpendicular to $w_j \forall$ j=1 to l
Any vector can be represented in the form of $\mathbf{s} = \mathbf{x} + \mathbf{y}$ where $x \in Y$ and $y \in Y^\perp$

$$\langle s,\, w_j \rangle = 0 \forall j$$

$$\langle x + y,\, \sum \alpha_j w_j \rangle = 0 \forall j$$

Since $\langle x,\, w_j \rangle = 0$ amd $y \in Y^\perp$ $\exists$ some α_j s.t. $y = \sum \alpha_j w_j$

$$\langle \sum \alpha_j w_j,\, \sum \alpha_j w_j \rangle = 0 \forall j$$

It gives us all $\alpha_i' s$ are zero, so y=0 , then $s \in Y$
Hence every vector in $(Y^\perp)^\perp$ lies in Y, i.e $(Y^\perp)^\perp \subseteq Y$

Now let $\mathbf{x} \in Y$ then $x = \sum \alpha_j u_j$

$$\langle x,\, w_i \rangle = \langle \sum \alpha_j u_j,\, w_i \rangle = 0$$

So $x \in W^\perp \implies x \in (Y^\perp)^\perp \implies Y \subseteq (Y^\perp)^\perp$
Hence $Y = (Y^\perp)^\perp$

7.9 Let $\mathbf{A}$ be a self-adjoint matrix. If $\langle \mathbf{A}\mathbf{x},\, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then show that $\mathbf{A} = \mathbf{O}$. Deduce that if $\|\mathbf{A}^*\mathbf{x}\| = \|\mathbf{A}\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then $\mathbf{A}$ is a normal matrix, and if $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then $\mathbf{A}$ is a unitary matrix.

Part i
Self adjoint $\mathbf{A}^* = \mathbf{A}$ and $\langle \mathbf{A}\mathbf{x},\, \mathbf{x}\rangle = \mathbf{x}^*\mathbf{A}^*\mathbf{x} = \mathbf{x}^*\mathbf{A}\mathbf{x} = 0,\ \forall\ \mathbf{x} \in \mathbb{K}^{n\times 1}$
Choose $\mathbf{x} = \mathbf{e}_k$ you get $a_{kk} = 0\ \forall$ k = 1 to n
Choose $\mathbf{x} = \mathbf{e}_k + \mathbf{e}_j$ and we get $a_{kj} + a_{jk} = 0\ \forall$ k,j = 1 to n and $k \neq j$
Choose $\mathbf{x} = \mathbf{e}_k - i\mathbf{e}_j$ and we get $a_{kj} - a_{jk} = 0\ \forall$ k,j = 1 to n and $k \neq j$
Hence $\mathbf{A} = \mathbf{O}$
Part ii) Choose $\mathbf{B} = \mathbf{A}\mathbf{A}^* - \mathbf{A}^*\mathbf{A}$, $\mathbf{B} = \mathbf{B}^*$

$$\|\mathbf{A}^*\mathbf{x}\| = \|\mathbf{A}\mathbf{x}\|$$

Square on both sides

$$\|\mathbf{A}^*\mathbf{x}\|^2 = \|\mathbf{A}\mathbf{x}\|^2 \implies \langle \mathbf{A}^*\mathbf{x},\, \mathbf{A}^*\mathbf{x}\rangle = \langle \mathbf{A}\mathbf{x},\, \mathbf{A}\mathbf{x}\rangle$$
$$(\mathbf{A}^*\mathbf{x})^*\mathbf{A}^*\mathbf{x} = \langle \mathbf{x},\, \mathbf{A}^*\mathbf{A}\mathbf{x}\rangle \implies \mathbf{x}^*\mathbf{A}\mathbf{A}^*\mathbf{x} = \mathbf{x}^*\mathbf{A}^*\mathbf{A}\mathbf{x}$$

We get $\langle \mathbf{B}\mathbf{x},\, \mathbf{x}\rangle = 0$

Hence $\mathbf{A}$ is normal
Part iii) Choose $\mathbf{B} = \mathbf{A}\mathbf{A}^* - \mathbf{I}$, $\mathbf{B} = \mathbf{B}^*$

$$\|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|$$

Square on both sides

$$\|\mathbf{A}\mathbf{x}\|^2 = \|\mathbf{x}\|^2 \implies \langle \mathbf{A}\mathbf{x},\, \mathbf{A}\mathbf{x}\rangle = \langle \mathbf{x},\, \mathbf{x}\rangle$$
$$\langle \mathbf{x},\, \mathbf{A}^*\mathbf{A}\mathbf{x}\rangle = \langle \mathbf{x},\, \mathbf{x}\rangle \implies \mathbf{x}^*\mathbf{A}^*\mathbf{A}\mathbf{x} = \mathbf{x}^*\mathbf{x}$$

We get $\langle \mathbf{B}\mathbf{x},\, \mathbf{x}\rangle = 0$

Hence $\mathbf{A}$ is unitary

7.10 Let E be a nonempty subset of $\mathbb{K}^{n\times 1}$.

(i) If E is not closed, then show that there is $\mathbf{x} \in \mathbb{K}^{n\times 1}$ such that no best approximation to $\mathbf{x}$ exists from E.

(ii) If E is convex, then show that for every $\mathbf{x} \in \mathbb{K}^{n\times 1}$, there is at most one best approximation to $\mathbf{x}$ from E.

Part i
Definition: A non empty subset E of $\mathbb{K}^{n\times 1}$ is not closed, then $\exists$ $\mathbf{x} \in \mathbb{K}^{n\times 1}$ and a sequence (x_n) of points of E s.t $x_n \mapsto x$, but x $\notin$ E
Suppose x had a best approximation from E, say y then

$$||x - y|| \le ||x - u|| \forall u \in E$$

$$||x - y|| \le ||x - x_n|| \forall n \in N$$

Now by passing limit we get $||x - y|| \le 0 \implies ||x - y|| = 0 \implies x = y$
But it is a contradiction since $x \notin E$ and $y \in E$

Part ii
Definition: A set E is convex if u,v $\in E \iff$ (1-$\lambda)u + \lambda v \in E \; \forall \lambda \in [0,1]$
Suppose there are u_1 and u_2 two best approximations from E to $\mathbf{x}$ s.t $||\mathbf{x} - u_i|| = \lambda$
Since E is convex the line joining u_1 and u_2 lies in E
$||\mathbf{x} - \frac{u_1+u_2}{2}|| = ||\frac{\mathbf{x}-u_1}{2} + \frac{\mathbf{x}-u_2}{2}|| \le ||\frac{\mathbf{x}-u_1}{2}|| + ||\frac{\mathbf{x}-u_2}{2}|| = \lambda$
But then it contradicts the definition of best approximation
Hence atmost one approximation

7.11 Find $\mathbf{x} := [x_1, x_2]^\mathsf{T} \in \mathbb{R}^{2\times 1}$ such that the straight line $t = x_1 + x_2 s$ fits the data points $(-1, 2)$, $(0, 0)$, $(1, -3)$ and $(2, -5)$ best in the 'least squares' sense.

The data points are (s,t) = $(-1, 2)$, $(0, 0)$,$(1, -3)$ and $(2, -5)$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \implies \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}$$

To minimise, we need to find the best approximation to the vector $\mathbf{b}$ from the column space $\mathbf{C}(\mathbf{A})$

$\mathbf{A} = [\mathbf{y}_1\mathbf{y}_2]$ and $\mathbf{u}_1 = \frac{y_1}{||y_1||} = \frac{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T}{\sqrt{4}}$ and $\mathbf{u}_2 = \frac{\begin{bmatrix} -1 & 0 & 1 & 2 \end{bmatrix}^T}{\sqrt{6}}$
Best approximation is $\langle \mathbf{u}_1, b \rangle \mathbf{u}_1 + \langle \mathbf{u}_2, b \rangle \mathbf{u}_2 = \begin{bmatrix} 1 & -1.5 & -4 & -6.5 \end{bmatrix}^T$
Now solve $x_1 - x_2 = -1$ and $x_1 + x_2 = -4$ gives $x_1 = -2.5, x_2 = -1.5$

7.12. Let $Q(x_1, \ldots, x_n) := \sum_{j=1}^n \sum_{k=1}^n \alpha_{jk} x_k \overline{x}_j$, where $\alpha_{jk} \in \mathbb{C}$, be a **complex quadratic form**. Show that there is a unique self-adjoint matrix $\mathbf{A}$ such that

$$Q(x_1, \ldots, x_n) = \mathbf{x}^* \mathbf{A}\, \mathbf{x} \quad \text{for all } \mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T} \in \mathbb{C}^{n\times 1}.$$

$Q(x_1, \ldots, x_n) = \sum_{j=1}^n \sum_{k=1}^n \alpha_{jk} x_k \overline{x}_j = \overline{Q} = \sum_{j=1}^n \sum_{k=1}^n \overline{\alpha_{jk} x_k} x_j$
The variable j,k are dummy variable for the summation
$Q = \sum_{j=1}^n \sum_{k=1}^n \overline{\alpha_{kj} x_j} x_k \implies \alpha_{jk} = \overline{\alpha_{kj}}$
To prove uniqueness:
Suppose $Q = \sum_{j=1}^n \sum_{k=1}^n \alpha_{jk} x_k \overline{x}_j = \sum_{j=1}^n \sum_{k=1}^n \beta_{jk} x_k \overline{x}_j$

Choose $\mathbf{x} = \mathbf{e}_k$ you get $\alpha_{kk} = \beta_{jj}$ $\forall$ k = 1 to n where k=j
Choose $\mathbf{x} = \mathbf{e}_k + \mathbf{e}_j$ and we get $\alpha_{kj} + \alpha_{jk} = \beta_{kj} + \beta_{jk}$ $\forall$ k,j = 1 to n and $k \neq j$
Choose $\mathbf{x} = \mathbf{e}_k - i\mathbf{e}_j$ and we get $\alpha_{kj} - \alpha_{jk} = \beta_{kj} - \beta_{jk}$ $\forall$ k,j = 1 to n and $k \neq j$

Hence unique

7.13. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be normal, and let $\mu_1, \dots, \mu_k$ be the distinct eigenvalues of $\mathbf{A}$. Let $Y_j := \mathcal{N}(\mathbf{A} - \mu_j \mathbf{I})$ for $j = 1, \dots, k$. Show that $\mathbb{C}^{n\times 1} = Y_1 \oplus \cdots \oplus Y_k$. Also, if P_j is the orthogonal projection onto Y_j, then show that $P_1 + \cdots + P_k = I$, $P_i P_j = O$ if $i \neq j$ and $\mathbf{Ax} = \mu_1 P_1(\mathbf{x}) + \cdots + \mu_k P_k(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{C}^{n\times 1}$.

Since $\mathbf{A}$ is normal , it is unitarily diagonalizable. So $\mathbb{C}^n$ has a basis of eigen vectors of $\mathbf{A}$
The form would be $\{u_{11}, .., u_{1g_1}, ..., u_{k1}, u_{k2}, ..., u_{kg_k}\}$ where g_j = geometric multiplicity of $\mu_j = dim\mathcal{N}(\mathbf{A} - \mu_j \mathbf{I})$ and μ_{j1}, ...,μ_{jg_j} are eigen vectors of eigen value μ_j for j=1,2,..,k.
We know $g_1 + g_2 .... + g_k = n$ and since $\mathbf{A}$ is diagonalizable. So given any x $\in \mathbb{C}^n$ we can write

$$x = \sum_{j=1}^{k} \sum_{l=1}^{g_j} \alpha_{jl} u_{jl} = y_1 + y_2 + ... + y_k$$

where $y_j = \sum_{l=1}^{g_j} \alpha_{jl} u_{jl} \in Y_j = \mathcal{N}(\mathbf{A} - \mu_j \mathbf{I})$. Thus $\mathbb{C}^n = Y_1 + \cdots + Y_k$
Since coeffiecients α_{jl} are uniquely determined by x, $\alpha_{jl} = \langle u_{jl}, x\rangle$, hence the decomposition is unique and we get $\mathbb{C}^n = Y_1 \oplus \cdots \oplus Y_k$

The orthogonal projection map is defined by $P_j(x) = y_j$ $(1 \leq j \leq k)$ and it is clear that $x = P_1(x) + ... + P_k(x)$ $\forall x \in \mathbb{C}^n$ So $P_1 + ...P_k = I$
Also $P_i P_j = P_i(y_j) = 0$ if $i \neq j$ .Thus $P_i P_j = 0$ if $i \neq j$
Finally since $y_j \in \mathcal{N}(\mathbf{A} - \mu_j \mathbf{I})$, we get $\mathbf{A} y_j = \mu_j y_j$

$$\mathbf{Ax} = \mathbf{Ay}_1 + ...\mathbf{Ay}_k$$

$$\mathbf{Ax} = \mu_1 \mathbf{y}_1 + ...\mu_k \mathbf{y}_k$$

$$\mathbf{Ax} = \mu_1 P_1(\mathbf{x}) + \cdots + \mu_k P_k(\mathbf{x}) \forall x \in \mathbb{C}^n$$

8 Tutorial 8 (on Lectures 20 and 21)

8.1 State why the following sets are not subspaces:

(i) All $m \times n$ matrices with nonnegative entries.

(ii) All solutions of the differential equation $xy' + y = 3x^2$.

(iii) All solutions of the differential equation $y' + y^2 = 0$.

(iv) All invertible $n \times n$ matrices.

(a) $\alpha\mathbf{M}$ if $\alpha < 0$ then it doesn't lie in subspace

(b) $xy_1' + y_1 = 3x^2$ and $xy_2' + y_2 = 3x^2$ and $x(y_1 + y_2)' + y_1 + y_2 - 3x^2 = 3x^2 \neq 0$ it doesn't lie in subspace

(c) $y_1' + y_1^2 = 0$ and $y_2' + y_2^2 = 0$ and $(y_1 + y_2)' + (y_1 + y_2)^2 = 2y_1y_2 \neq 0$ it doesn't lie in subspace

(d) $det(\mathbf{A}), det(\mathbf{B}) \neq 0$ but det(A+B) can be zero if det(A) = -det(B) its not invertible and hence doesnt lie

8.2 Let V denote the vector space of all polynomial functions on $\mathbb{R}$ of degree at most n. Are the following subsets of V in fact subspaces of V? (i) $W_1 := \{p \in V : p(0) = 0\}$,

(ii) $W_2 := \{p \in V : p'(0) = 0 = p''(0)\}$,

(iii) $W_3 := \{p \in V : p \text{ is an odd function}\}$.

If so, find a spanning set for each.

8.3 Let $V := C([-\pi, \pi])$. Show that $S_1 := \{1, \cos, \sin\}$ is a linearly independent subset of V, while $S_2 := \{1, \cos^2, \sin^2\}$ is a linearly dependent subset of V.

8.4 Let $V := \mathbb{R}^{1\times 2}$, and let $v_1 := [1\ \ 0]$, $v_2 := [1\ \ 1]$, $v_3 := [1\ -1]$. Explain why $(24, 12)$ can be written as a linear combination of v_1, v_2, v_3 in two different ways, namely, $4v_1 + 16v_2 + 4v_3$ and $6v_1 + 15v_2 + 3v_3$.

8.5 Let $n \in \mathbb{N}$. Let W_1, W_2, W_3, W_4 denote the subspaces of $n \times n$ real matrices which are diagonal, upper triangular, symmetric and skew-symmetric. Find their dimensions.

8.6 Let V and W be vector spaces over $\mathbb{K}$. Show that $V \times W := \{(v, w) : v \in V \text{ and } w \in W\}$ is a vector space over $\mathbb{K}$ with componentwise addition and scalar multiplication. If $\dim V = n$ and $\dim W = m$, find $\dim V \times W$.

8.7 Let $\mathbf{A} := [a_{jk}] \in \mathbb{K}^{4\times 4}$. Define $T : \mathbb{K}^{2\times 2} \to \mathbb{K}^{2\times 2}$ by

$$T\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix},$$

where $\begin{bmatrix} y_{11} & y_{12} & y_{21} & y_{22} \end{bmatrix}^{\mathsf{T}} := \mathbf{A} \begin{bmatrix} x_{11} & x_{12} & x_{21} & x_{22} \end{bmatrix}^{\mathsf{T}}$. Show that T is linear, and find the matrix of T with respect to the ordered basis $\big(\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{21}, \mathbf{E}_{22}\big)$ of $\mathbb{K}^{2\times 2}$.

8.8 Define $T : \mathcal{P}_2 \to \mathbb{K}^{2\times 1}$ by

$$T(\alpha_0 + \alpha_1 t + \alpha_2 t^2) := \begin{bmatrix} \alpha_0 + \alpha_1 & \alpha_1 + \alpha_2 \end{bmatrix}^{\mathsf{T}}$$

for $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$. If $E := (1, t, t^2)$ and $F := (\mathbf{e}_1, \mathbf{e}_2)$, then find $\mathbf{M}_F^E$. Also, if $E' := \big(1, 1 + t, (1 + t)^2\big)$ and $F' := (\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2)$, then find $\mathbf{M}_{F'}^{E'}$.

8.9 (**Parallelogram law**) Let V be an inner product space. Prove that the norm on V induced by the inner product satisfies $\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$ for all $v, w \in V$.

(Conversely, if there is a norm $\|\cdot\|$ on a vector space V which satisfies the parallelogram law, then it can be shown that there is an inner product $\langle \cdot\,,\,\cdot \rangle$ on V such that $\langle v,\, v\rangle = \|v\|^2$ for all $v \in V$.)

8.10 For $\mathbf{A}, \mathbf{B} \in \mathbb{K}^{m\times n}$, define $\langle \mathbf{A},\, \mathbf{B}\rangle := \operatorname{tr} \mathbf{A}^*\mathbf{B}$. Show that $\langle \cdot\,,\,\cdot \rangle$ is an inner product on $\mathbb{K}^{m\times n}$.

8.11 Show that

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\sin t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \dots\right\}$$

is an orthonormal subset of $C([-\pi, \pi])$.

(This is the beginning of the theory of Fourier Series.)

8.12 Let T be a Hermitian operator on a finite dimensional inner product space V over $\mathbb{K}$. Prove the following.

(i) $\langle T(v),\, v\rangle \in \mathbb{R}$ for every $v \in V$.

(ii) Every eigenvalue of T is real.

(iii) If $\lambda \neq \mu$ are eigenvalues of T with v and w corresponding eigenvectors of T, then $v \perp w$.

(iv) Let W be a subspace of V such that $T(W) \subset W$. Then $T(W^\perp) \subset W^\perp$.