MA 106
Tutorial 3 Solutions

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QUESTION 3.6

QUESTION 3.7

QUESTION 3.8

QUESTION 3.9

QUESTION 4.1

QUESTION 4.2

QUESTION 3.6

QUESTION 3.6

$$\det\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

Use $det(A) = det(A^T)$ and perform $R_k = R_k - R_1$ $\forall$ k=2 to 3

$(b-a)(c^2-a^2) - (c-a)(b^2-a^2)$

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} = (b-a)(c-a)(c-b)$$

$c+a-b-a$

QUESTION 3.6

Use induction: for n=2 we have (Why we didn't take n=1 ?Think)

$$\det \begin{bmatrix} 1 & 1 \\ a_1 & a_2 \end{bmatrix} = (a_2 - a_1)$$

Now assume it to be true for n-1 order matrix and if we are able to prove n order matrix from the n-1 order matrix we are done

$$\text{To prove: } \det \begin{bmatrix} 1 & 1 & 1 & \dots & \dots & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & \dots & \dots & a_n \\ . & . & . & . & . & . & . \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \dots & \dots & \dots & a_n^{n-1} \end{bmatrix} = \prod_{1 \le i < j \le n} (a_j - a_i)$$

QUESTION 3.6

$$det(A) = det(A^T)$$

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \dots & \dots & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & \dots & \dots & a_2^{n-1} \\ & & & & \cdot & & \\ & & & \cdot & & & \\ & & \cdot & & & & \\ 1 & a_n & a_n^2 & \dots & \dots & \dots & a_n^{n-1} \end{bmatrix} = \prod_{1 \le i < j \le n} (a_j - a_i)$$

$R_k = R_k - R_1$ $\forall$ k=2 to n

QUESTION 3.6

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \dots & \dots & \dots & a_1^{n-1} \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 & \dots & \dots & \dots & a_2^{n-1} - a_n^{n-1} \\ & & & & \cdot & & \\ & & & \cdot & & & \\ & & \cdot & & & & \\ 0 & a_n - a_1 & a_n^2 - a_1^2 & \dots & \dots & \dots & a_n^{n-1} - a_1^{n-1} \end{bmatrix} - > eqn(I)$$

QUESTION 3.6

$a^n - b^n = (a^{n-1}b^0 + a^{n-2}b^1 + \dots\dots + a^0 b^{n-1})(a-b)$

$\det[c_1\ c_2\ c_3 \dots c_n]$

$\Rightarrow \det[c_1\ a_1\ c_3\ c_4 \dots c_n]$

$+ \det[c_1\ a_1\ c_3\ c_4 \dots c_n]$

$a_2^2 + a_1 a_2 + a_1^2$

$$\prod_{2\leq j\leq n}(a_j - a_1)\,\det\begin{bmatrix} 1 & a_2+a_1 & \dots & \dots & \sum_0^{n-1} a_2^{n-2-i}a_1^i \\ & & & \cdot & \\ & & \cdot & & \\ & \cdot & & & \\ \cdot & & & & \\ 1 & a_n+a_1 & \dots & \dots & \sum_0^{n-1} a_n^{n-2-i}a_1^i \end{bmatrix}$$

Now keep on splitting the det by column wise starting from col(2) to col(n) and see only one non zero det would surive and others would vanish

QUESTION 3.6

$$\prod_{2\leq j\leq n}(a_j - a_1)\,\mathrm{det}\begin{bmatrix} 1 & a_2 & a_2^2 & \dots & \dots & \dots & a_2^{n-2} \\ 1 & a_3 & a_3^2 & \dots & \dots & \dots & a_3^{n-2} \\ & & & & . & & \\ & & & . & & & \\ & & . & & & & \\ 1 & a_n & a_n^2 & \dots & \dots & \dots & a_n^{n-2} \end{bmatrix}$$

$$\prod_{2\leq i<j\leq n}(a_j - a_i) * \prod_{2\leq j\leq n}(a_j - a_1)$$

$$\prod_{1\leq i<j\leq n}(a_j - a_i)$$

$$= \det \begin{bmatrix} 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} \\ \vdots & & & & \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-2} \end{bmatrix} \times \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} \\ 0 & 1 & a_1 & \cdots & a_1^{n-3} \\ & & 1 & a_1 \cdots & a_1^{n-4} \\ & & & \ddots & \vdots \\ 0 & 0 & & & 1 \end{bmatrix}$$

Other method

Look at eqn(I) matrix

Use $det(A) = det(A^T)$ and consecutively perform $\boxed{R_k = R_k - R_{k-1} * a_1}$ $\forall$ k=2 to n Try out

QUESTION 3.7

Use induction Method:

For n=1 we have,

$$\det\begin{bmatrix} 1 \end{bmatrix} = (-1)^{1(1-1)/2} = 1$$

Now assume it to be true for n-1 order matrix and if we are able to prove n order matrix from the n-1 order matrix we are done

To prove:: $$\det\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ & & & & . & & \\ & & & . & & & \\ & & . & & & & \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} = (-1)^{n(n-1)/2}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det(A) = a_{ij}(-1)^{i+j} M$$

$$1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} -2$$

$$\det \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ & & & & . & & \\ & & & . & & & \\ & & . & & & & \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$1(-1)^{(1+n)} \det A_{n-1}$$

Now if we expand via the first row to find det and use result of $det(A)_{n-1}$, we get

QUESTION 3.7

$$(-1)^{n+1} \det \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ & & & & \cdot & & \\ & & & \cdot & & & \\ & & \cdot & & & & \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}_{n-1}$$

$$(-1)^{n+1} * (-1)^{(n-1)(n-2)/2} = (-1)^{n(n-1)/2}$$

QUESTION 3.8

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$R_n \mapsto \frac{1}{n} R_n$
$R_i \mapsto R_i - iR_n$ for all $i \in \{1, \ldots, n-1\}$.
For example, in the case of $n = 4$, you should have arrived at the following conclusion:

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} = 4 \det \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Write the general case.
Now, expand along the first column. This is simple to do as it has only one non-zero entry. (Note that you'll get a $(-1)^{n+1}$.)

QUESTION 3.8

Thus, you get that the original determinant equals the following expression:

$$(-1)^{n+1} n \det \begin{bmatrix} 1 & 2 & \cdots & n-1 \\ 0 & 1 & \cdots & n-2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Note that the determinant written above is just 1 as it's a triangular matrix with all diagonal entries 1.

Thus, the answer is $(-1)^{n+1}n$.

One can also use induction to prove this (Try out)

QUESTION 3.9

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QUESTION 4.1

QUESTION 4.1

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 6 & \alpha \end{vmatrix} = (1)(-7) - 6(-2) + \alpha \neq 0$$

$$-7 + 12 + \alpha \neq 0 \Rightarrow \boxed{\alpha \neq -5}$$

$\boxed{-5 \neq 0}$

$\boxed{\alpha = -5}$

$x + 2y + 3z = 20$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 6 & -5 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 4 & -8 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = $$

QUESTION 4.2

QUESTION 4.2

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

$$C^T A$$

$$C = \begin{bmatrix} 1/240 & -1/60 & 1/72 \\ -1/60 & 1/45 & -1/12 \\ 1/72 & -1/12 & 1/12 \end{bmatrix}$$

$$1\left(\frac{1}{240}\right) - \frac{1}{2}\left(\frac{1}{60}\right) + \frac{1}{3}\left(\frac{1}{72}\right)$$

$$\left(\frac{1}{240} + \frac{1}{120} + \frac{1}{216}\right)$$

QUESTIONS?

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THANK YOU