MA 106
Tutorial 5 Solutions

D1 T5

GYANDEV GUPTA

April 07, 2021

IIT BOMBAY

QUESTION 5.1

QUESTION 5.2

QUESTION 5.3

QUESTION 5.4

QUESTION 5.5

QUESTION 5.6

QUESTION 5.7

QUESTION 5.8

QUESTION 5.9

QUESTION 6.1

QUESTION 6.2

QUESTION 5.1

QUESTION 5.1

Similar to exercise 4.7 and 4.8

QUESTION 5.2

QUESTION 5.2

Theorem

Let $\mathbf{A} \in \mathbb{K}^{n \times n}$, and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $\mathbf{A}$. Let g_j be the geometric multiplicity of λ_j for $j = 1, \ldots, k$. Then $g_1 + \cdots + g_k \leq n$. Further, $\mathbf{A}$ is diagonalizable if and only if $g_1 + \cdots + g_k = n$.

You can easily see eigen values are 2,1,2

Just you need to check for nullspace $(\mathbf{A} - \mu \mathbf{I})\mathbf{x} = 0$ or find nullity for $\mu = 2$

$$\begin{bmatrix} 2-\mu & a & b \\ 0 & 1-\mu & c \\ 0 & 0 & 2-\mu \end{bmatrix} \mapsto \begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix}$$

For nullity equal to 2 we need rank=1 hence R_2 must to be a scalar multiple of R_1

$\frac{a}{-1} = \frac{b}{c} \implies$ b=-ac

QUESTION 5.3

Take $(\mathsf{A} - \mu\mathsf{I})$ and perform $R_{2i} \mapsto R_{2i} + R_{2i-1}/\mu \; \forall$ i=1 to k
There was a catch that $\mu \neq 0$ (how would you prove that). Hint (find nullity of A)
It's a Upper triangular matrix and whose det is product of diagonal entries
$\mu^k(\mu + 1/\mu)^k = 0 \implies (\mu^2 + 1)^k = 0 \implies \mu = \pm i$
We know that GM $\leq$ AM hence GM is k for both eigen value

OR

Find Nullity of $(\mathsf{A} - i\mathsf{I})$ by performing $R_{2i} \mapsto R_{2i} - iR_{2i-1} \; \forall$ i=1 to k . we easily get GM=k for both
Characteristic polynomial is $(\mathsf{A}^2 + 1)^k = 0$

QUESTION 5.4

QUESTION 5.4

For forward part,

$$\lambda \|\mathbf{x}\|^2 = \lambda < x, x >=< x, \lambda x >=< x, Ax >$$

Transformation property: $< Ax, y >= y^* Ax = (A^* y)^* x =< x, A^* y >$

$$\lambda \|\mathbf{x}\|^2 =< A^* x, x >$$

Take conjugate on both sides

$$\overline{\lambda} \|\mathbf{x}\|^2 = \overline{< A^* x, x >}$$

$$\overline{\lambda} \|\mathbf{x}\|^2 =< x, A^* x >$$

Similarly prove the backward part (Try it)

QUESTION 5.4

Other method:

$|\mathbf{A} - \lambda \mathbf{I}| = 0$. Choose $\mathbf{B} = \mathbf{A} - \lambda \mathbf{I}$ and we get det(B) = 0

We can claim that det(B*)=0. So $\mathbf{B}^* = \mathbf{A}^* - \overline{\lambda}\mathbf{I}$.

Now $|\mathbf{A}^* - \overline{\lambda}\mathbf{I}| = 0$ hence $\overline{\lambda}$ is an eigen value of $\mathbf{A}*$

QUESTION 5.5

QUESTION 5.5

$\mathbf{A}x = 0 \implies A^*Ax = 0 \implies x \in N(A^*A)$

$N(A) \subseteq N(A^*A)$

Now consider

$A * Ax = 0 \implies x^*A^*Ax = 0 \implies (Ax)^*Ax = 0 \implies Ax = 0 \implies x \in N(A)$

$N(A^*A) \subseteq N(A)$

Hence $N(A) = N(A^*A)$

All part follows from this because geometric multiplicity of 0 is nullity of the matrix.

QUESTION 5.6

QUESTION 5.6

By the Gerschgorin Theorem we know $|\lambda - a_{jj}| \leq \sum_{j \neq k} |a_{jk}|$

Lets calculate $\sum_{j \neq k} |a_{jk}|$

For j=1 it's 1 + $\sqrt{2}$

For j=2 it's 2

For j=3 it's 1 + $\sqrt{2}$

QUESTION 5.7

QUESTION 5.7

By the Gerschgorin Theorem we know $|\lambda - a_{jj}| \leq \sum_{j \neq k} |a_{jk}|$
We have $\lambda - a_{jj} > -\sum_{j} |a_{jk}| \mapsto \text{I}$
We already have that $|a_{jj}| > \sum_{k \neq j} |a_{jk}| \implies a_{jj} - \sum_{k \neq j} |a_{jk}| > 0 \mapsto \text{II}$
From I and II we get $\lambda > 0$ hence the matrix is invertible

QUESTION 5.8

QUESTION 5.8

consider λ to be max of all eigen value

$\alpha_2 \geq \|\mathbf{A}\mathbf{x}\| = \|\lambda \mathbf{x}\| = |\lambda|$

By the Gerschgorin Theorem we know $|\lambda - a_{jj}| \leq \sum_{j \neq k} |a_{jk}|$

$||\lambda| - |a_{jj}|| \leq |\lambda - a_{jj}| \leq \sum_{j \neq k} |a_{jk}|$

$||\lambda| - |a_{jj}|| \leq \sum_{j \neq k} |a_{jk}| \implies |\lambda| - |a_{jj}| \leq \sum_{j \neq k} |a_{jk}|$

$|\lambda| = |a_{jj}| + \sum_{j \neq k} |a_{jk}| \leq \alpha_\infty$

Eigen values of $\mathbf{A}$ and $\mathbf{A}^T$ are same and performing same operations as we did above we can say $|\lambda| \leq \alpha_1$

QESTION 5.8

You can leave this slide since matrix norm isn't covered or won't be covered

Other method (An Important General result):

Let $(\lambda, \mathbf{x})$ be eigen pair s.t $\rho(\mathbf{A}) = max|\lambda|$

Find $\mathbf{y} \neq 0$ s.t $\mathbf{xy}^*$ is a non zero matrix , $\|.\|$ is a matrix norm

$\lambda\mathbf{x} = \mathbf{Ax} \implies \lambda\mathbf{xy}^* = \mathbf{Axy}^* \implies |\lambda|\|\mathbf{xy}^*\| = \|\mathbf{Axy}^*\| \leq \|\mathbf{A}\|\|\mathbf{xy}^*\| \implies \rho(\mathbf{A}) \leq \|\mathbf{A}\|$

QUESTION 5.9

QUESTION 5.9

Part.a) You need to use $\|\mathbf{x}+\mathbf{y}\|^2 =< \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} >=$

<x,x+y>+<y,x+y> = <x,x> + <x,y> + <y,x> + <y,y>and

Similarly for the other term $\|\mathbf{x}-\mathbf{y}\|^2 =< \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} >=$

<x,x-y>+<-y,x-y> = <x,x> - <x,y> - <y,x> + <y,y>

Part.b) $(\text{Re} < x, y >) = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$ where $\theta \in [0, \pi]$

QUESTION 6.1

QUESTION 6.1

QUESTION 6.2

Let W be the subspace of $\mathbb{K}^{4\times 1}$ spanned by the vectors $\mathbf{x}_1 := \begin{bmatrix} 1 & -1 & 2 & 0 \end{bmatrix}^T$, $\mathrm{x}_2 := \begin{bmatrix} 1 & 1 & 2 & 0 \end{bmatrix}^T$ and $\mathbf{x}_3 := \begin{bmatrix} 3 & 0 & 0 & 1 \end{bmatrix}^T$ Let us apply the G-S OP.

Let $u_1 := \frac{x_1}{||x_1||} = \frac{\begin{bmatrix} 1 & -1 & 2 & 0 \end{bmatrix}^T}{\sqrt{6}}$

, $u_2 := \frac{x_2 - P_{u_1}(x_2)}{||x_2 - P_{u_1}(x_2)||} = \frac{\begin{bmatrix} 1 & 5 & 2 & 0 \end{bmatrix}^T}{\sqrt{30}}$

$u_3 := \frac{x3 - P_{u_1}(x3) - P_{u_2}(x3)}{||x3 - P_{u_1}(x3) - P_{u_2}(x3)||} = \frac{\begin{bmatrix} 12 & 0 & -6 & 5 \end{bmatrix}^T}{\sqrt{205}}$

QUESTION 6.2

You can check for yourself that $\{u_1, u_2, u_3\}$ is an orthonormal basis

To extend $\{u_1, u_2, u_3\}$ to an orthonormal basis for V := $\mathbb{K}^{4\times1}$, we look for $u_4 := \begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix}^T$ which is orthogonal to the set $\{x1, x2, x3\}$ where $\|u_4\| = 1$. Try on your own

Questions?

Contact me via 190100051@iitb.ac.in

Thank You