MA 106
Tutorial 8 Solutions

D1 T5

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QUESTION 7.7

QUESTION 7.7

$\mathbf{Q}(x) = 7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy - 36$ to a diagonal form.

Here $\mathbf{A} := \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix}$ is the associated matrix.

Hence the equation of the given quadric surface becomes

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix}^T - 36 = 0$$

Now find eigen value and corresponding eigen vector and then using GSOP find $\{u_1, u_2, u_3\}$

Change of variable from $\begin{bmatrix} x & y & z \end{bmatrix}^T = \mathbf{C} \begin{bmatrix} u & v & w \end{bmatrix}^T$,where $\mathbf{C} = [\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3]$

QUESTION 7.7

Characteristic polynomial is $\lambda^3 - 12\lambda - 180\lambda + 1296 = 0$

Eigen values are {18,-12,6}

Eigen vectors are $\{\begin{bmatrix}-1 & 1 & 1\end{bmatrix}^T, \begin{bmatrix}1 & -1 & 2\end{bmatrix}^T, \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T\}$

By GSOP Orthonormal eigen vectors are $\{\frac{\begin{bmatrix}-1 & 1 & 1\end{bmatrix}^T}{\sqrt{3}}, \frac{\begin{bmatrix}1 & -1 & 2\end{bmatrix}^T}{\sqrt{6}}, \frac{\begin{bmatrix}1 & 1 & 0\end{bmatrix}^T}{\sqrt{2}}\}$

$Q_D(u, v, w) = 18u^2 - 12v^2 + 6w^2$

The quadric surface reduces to $18u^2 - 12v^2 + 6w^2 = 36$

Since eigen values two positive, one negative its **1 sheeted hyperboloid**

QUESTION 7.7

$$\begin{bmatrix} x & y & z \end{bmatrix}^T = \mathbf{C} \begin{bmatrix} u & v & w \end{bmatrix}^T$$

$$\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix} \begin{bmatrix} u & v & w \end{bmatrix}^T$$

$$x = \frac{-1}{\sqrt{3}}u + \frac{1}{\sqrt{6}}v + \frac{1}{\sqrt{2}}w, \; y = \frac{1}{\sqrt{3}}u + \frac{-1}{\sqrt{6}}v + \frac{1}{\sqrt{2}}w, \; z = \frac{1}{\sqrt{3}}u + \frac{2}{\sqrt{6}}v + 0w$$

QUESTION 7.8

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Let $\{u_1, u_2, ...u_k\}$ and $\{w_1, w_2, ...w_l\}$ be an orthonormal basis for subspace respectively Y and $Y^\perp$

Every vector $\mathbf{s} \in (Y^\perp)^\perp$ will be perpendicular to $w_j \forall$ j=1 to l

Any vector can be represented in the form of $\mathbf{s} = \mathbf{x} + \mathbf{y}$ where $x \in Y$ and $y \in Y^\perp$

$$\langle s, w_j \rangle = 0 \forall j$$

$$\langle x + y, \sum \alpha_j w_j \rangle = 0 \forall j$$

Since $\langle x, w_j \rangle = 0$ amd $y \in Y^\perp$ $\exists$ some α_j s.t. $y = \sum \alpha_j w_j$

$$\langle \sum \alpha_j w_j, \sum \alpha_j w_j \rangle = 0 \forall j$$

QUESTION 7.8

It gives us all $\alpha_i's$ are zero, so y=0 , then $s \in Y$

Hence every vector in $(Y^{\perp})^{\perp}$ lies in Y, i.e $(Y^{\perp})^{\perp} \subseteq Y$

Now let **x** $\in Y$ then $x = \sum \alpha_j u_j$

$$\langle x, w_i \rangle = \langle \sum \alpha_j u_j, w_i \rangle = 0$$

So $x \in W^{\perp} \implies x \in (Y^{\perp})^{\perp} \implies Y \subseteq (Y^{\perp})^{\perp}$

Hence $Y = (Y^{\perp})^{\perp}$

QUESTION 7.9

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Part i

Self adjoint $\mathbf{A}^* = \mathbf{A}$ and $\langle \mathbf{Ax}, \mathbf{x} \rangle = \mathbf{x}^*\mathbf{A}^*\mathbf{x} = \mathbf{x}^*\mathbf{Ax} = 0, \forall \mathbf{x} \in 1$

Choose $\mathbf{x} = \mathbf{e}_k$ you get $a_{kk} = 0$ ∀ k = 1 to n

Choose $\mathbf{x} = \mathbf{e}_k + \mathbf{e}_j$ and we get $a_{kj} + a_{jk} = 0$ ∀ k,j = 1 to n and $k \neq j$

Choose $\mathbf{x} = \mathbf{e}_k - i\mathbf{e}_j$ and we get $a_{kj} - a_{jk} = 0$ ∀ k,j = 1 to n and $k \neq j$

Hence $\mathbf{A} = \mathbf{O}$

QUESTION 7.9

Part ii) Choose $B = AA^* - A^*A$, $B = B^*$

$$\|Ax\| = \|Ax\|$$

Square on both sides

$$\|Ax\|^2 = \|Ax\|^2 \implies \langle A^*x, A^*x\rangle = \langle Ax, Ax\rangle$$

$$(A^*x)^*A^*x = \langle x, A^*Ax\rangle \implies x^*AA^*x = x^*A^*Ax$$

We get $\langle Bx, x\rangle = 0$

Hence A is normal

Part iii) Choose $B = AA^* - I$, $B = B^*$

$$\|Ax\| = \|x\|$$

Square on both sides

$$\|Ax\|^2 = \|x\|^2 \implies \langle Ax, Ax\rangle = \langle x, x\rangle$$

$$\langle x, A^*Ax\rangle = \langle x, x\rangle \implies x^*A^*Ax = x^*x$$

We get $\langle Bx, x\rangle = 0$

Hence A is unitary

QUESTION 7.10

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Part i)

Definition: A non empty subset E of $\mathbb{K}^n$ is not closed, then $\exists$ **x** $\in \mathbb{K}^n$ and a sequence (x_n) of points of E s.t $x_n \mapsto x$, but x $\notin$ E

Suppose x had a best approximation from E, say y then

$$||x-y|| \leq ||x-u|| \forall u \in E$$

$$||x-y|| \leq ||x-x_n|| \forall n \in N$$

Now by passing limit we get $||x-y|| \leq 0 \implies ||x-y|| = 0 \implies x = y$

But it is a contradiction since $x \notin E$ and $y \in E$

QUESTION 7.10

Part ii)

Definition: A set E is convex if u,v $\in E \iff$ (1-λ)$u + \lambda v \in E \; \forall \lambda \in [0,1]$

Suppose there are u_1 and u_2 two best approximations from E to $\mathbf{x}$ s.t $||\mathbf{x} - u_i|| = \lambda$

Since E is convex the line joining u_1 and u_2 lies in E

$||\mathbf{x} - \frac{u_1+u_2}{2}|| = ||\frac{\mathbf{x}-u_1}{2} + \frac{\mathbf{x}-u_2}{2}|| \leq ||\frac{\mathbf{x}-u_1}{2}|| + ||\frac{\mathbf{x}-u_2}{2}|| = \lambda$

But then it contradicts the definition of best approximation

Hence atmost one approximation

QUESTION 7.11

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The data points are (s,t) = $(-1, 2)$, $(0, 0)$,$(1, -3)$ and $(2, -5)$

$$\mathbf{Ax} = \mathbf{b} \implies \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}$$

To minimise, we need to find the best approximation to the vector **b** from the column space **C(A)**

$\mathbf{A} = [\mathbf{y}_1\mathbf{y}_2]$ and $\mathbf{u}_1 = \frac{y_1}{||y_1||} = \frac{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T}{\sqrt{4}}$ and $\mathbf{u}_2 = \frac{\begin{bmatrix} -1 & 0 & 1 & 2 \end{bmatrix}^T}{\sqrt{6}}$

Best approximation is $\langle \mathbf{u}_1, b \rangle \mathbf{u}_1 + \langle \mathbf{u}_2, b \rangle \mathbf{u}_2$ = $\begin{bmatrix} 1 & -1.5 & -4 & -6.5 \end{bmatrix}^T$

Now solve $x_1 - x_2 = -1$ and $x_1 + x_2 = -4$ gives $x_1 = -2.5, x_2 = -1.5$

QUESTION 7.12

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$Q(x_1, \dots, x_n) = \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} x_k \overline{x}_j = \overline{Q} = \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{\alpha_{jk}} \overline{x_k} x_j$

The variable j,k are dummy variable for the summation

$Q = \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{\alpha_{kj}} \overline{x_j} x_k \implies \alpha_{jk} = \overline{\alpha_{kj}}$

To prove uniqueness:

Suppose $Q = \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} x_k \overline{x}_j = \sum_{j=1}^{n} \sum_{k=1}^{n} \beta_{jk} x_k \overline{x}_j$

Choose $\mathbf{x} = \mathbf{e}_k$ you get $\alpha_{kk} = \beta_{jj}$ $\forall$ k = 1 to n where k=j

Choose $\mathbf{x} = \mathbf{e}_k + \mathbf{e}_j$ and we get $\alpha_{kj} + \alpha_{jk} = \beta_{kj} + \beta_{jk}$ $\forall$ k,j = 1 to n and $k \neq j$

Choose $\mathbf{x} = \mathbf{e}_k - i\mathbf{e}_j$ and we get $\alpha_{kj} - \alpha_{jk} = \beta_{kj} - \beta_{jk}$ $\forall$ k,j = 1 to n and $k \neq j$

Hence unique

QUESTION 7.13

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Since **A** is normal , it is unitarily diagonalizable.
So $\mathbb{C}^n$ has a basis of eigen vectors of **A**
The form would be $\{u_{11}, .., u_{1g_1}, ..., u_{k1}, u_{k2}, ..., u_{kg_k}\}$ where g_j = geometric multiplicity of $\mu_j = dim\mathcal{N}(\mathbf{A} - \mu_j\mathbf{I})$ and μ_{j1}, ...,μ_{jg_j} are eigen vectors of eigen value μ_j for j=1,2,..,k.
We know $g_1 + g_2.... + g_k = n$ and since **A** is diagonalizable. So given any x $\in \mathbb{C}^n$ we can write

$$x = \sum_{j=1}^{k} \sum_{l=1}^{g_j} \alpha_{jl} u_{jl} = y_1 + y_2 + ... + y_k$$

where $y_j = \sum_{l=1}^{g_j} \alpha_{jl} u_{jl} \in Y_j = \mathcal{N}(\mathbf{A} - \mu_j \mathbf{I})$. Thus $\mathbb{C}^n = Y_1 + \cdots + Y_k$

Since coeffiecients α_{jl} are uniquely determined by x, $\alpha_{jl} = \langle u_{jl}, x \rangle$, hence the decomposition is unique and we get $\mathbb{C}^n = Y_1 \oplus \cdots \oplus Y_k$

The orthogonal projection map is defined by $P_j(x) = y_j$ $(1 \leq j \leq k)$ and it is clear that $x = P_1(x) + ... + P_k(x)$ $\forall x \in \mathbb{C}^n$ So $P_1 + ...P_k = I$

Also $P_iP_j = P_i(y_j) = 0$ if $i \neq j$ .Thus $P_iP_j = 0$ if $i \neq j$

Finally since $y_j \in \mathcal{N}(\mathbf{A} - \mu_j\mathbf{I})$, we get $\mathbf{A}y_j = \mu_j y_j$

$$\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y}_1 + ...\mathbf{A}\mathbf{y}_k$$

$$\mathbf{A}\mathbf{x} = \mu_1\mathbf{y}_1 + ...\mu_k\mathbf{y}_k$$

$$\mathbf{A}\mathbf{x} = \mu_1 P_1(\mathbf{x}) + \cdots + \mu_k P_k(\mathbf{x}) \forall x \in \mathbb{C}^n$$

QUESTION 8.1

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1. αM if $\alpha < 0$ then it doesn't lie in subspace
2. $xy_1' + y_1 = 3x^2$ and $xy_2' + y_2 = 3x^2$ and $x(y_1 + y_2)' + y_1 + y_2 - 3x^2 = 3x^2 \neq 0$ it doesn't lie in subspace
3. $y_1' + y_1^2 = 0$ and $y_2' + y_2^2 = 0$ and $(y_1 + y_2)' + (y_1 + y_2)^2 = 2y_1y_2 \neq 0$ it doesn't lie in subspace
4. $det(\mathsf{A}), det(\mathsf{B}) \neq 0$ but det(A+B) can be zero if det(A) = -det(B) its not invertible and hence doesnt lie

QUESTIONS?

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THANK YOU