MA 106
Tutorial 4 Solutions

D1 T5

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QUESTION 4.3

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Part(i)

We have the basis set $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^{3\times 1}$ and $F = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ of $\mathbb{R}^{4\times 1}$,

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T = 1\mathbf{e}_1 + 0\mathbf{e}_2 + 1\mathbf{e}_3 + 1\mathbf{e}_4$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^T = 1\mathbf{e}_1 + 1\mathbf{e}_2 + 0\mathbf{e}_3 + 1\mathbf{e}_4$$

$$T(\mathbf{e}_3) = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}^T = 0\mathbf{e}_1 + 1\mathbf{e}_2 + 1\mathbf{e}_3 + 1\mathbf{e}_4$$

$$\mathbf{M}_F^E(T) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

QUESTION 4.3

Part(ii). Try whether the set E' and set F' forms a basis set? Indeed yes they form

$T(\mathbf{e}_1 + \mathbf{e}_2) = \begin{bmatrix} 2 & 1 & 1 & 2 \end{bmatrix}^T =$

$0(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) + 0(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) + 1(\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_1) + 1(\mathbf{e}_4 + \mathbf{e}_1 + \mathbf{e}_2)$

$T(\mathbf{e}_2 + \mathbf{e}_3) = \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}^T =$

$0(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) + 1(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) + 0(\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_1) + 1(\mathbf{e}_4 + \mathbf{e}_1 + \mathbf{e}_2)$

$T(\mathbf{e}_3 + \mathbf{e}_1) = \begin{bmatrix} 1 & 1 & 2 & 2 \end{bmatrix}^T =$

$0(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) + 1(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) + 1(\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_1) + 0(\mathbf{e}_4 + \mathbf{e}_1 + \mathbf{e}_2)$

$$\mathbf{M}_{F'}^{E'}(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

QUESTION 4.4

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Proposition

Let $\mathbf{A}, \mathbf{B} \in \mathbb{K}^{n \times n}$. Then $\mathbf{A} \sim \mathbf{B}$ if and only if there is an ordered basis E for $\mathbb{K}^{n \times 1}$ such that $\mathbf{B}$ is the matrix of the linear transformation $T_{\mathbf{A}} : \mathbb{K}^{n \times 1} \to \mathbb{K}^{n \times 1}$ with respect to E.

In fact, $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ if and only if the columns of $\mathbf{P}$ form an ordered basis, say E, for $\mathbb{K}^{n \times 1}$ and $\mathbf{B} = \mathbf{M}_E^E(T_{\mathbf{A}})$.

Using the theorem we get $\mathbf{E} = \{P_1, P_2, P_3, P_4\}$

QUESTION 4.5

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For $|\mathbf{A} - \mu\mathbf{I}| = 0 = (\mu - \lambda)^3$ its true for all vector $\mathbf{x} = (x_1, x_2, x_3)$ and hence eigen space is $\mathbb{R}^3$

For $|\mathbf{B} - \mu\mathbf{I}| = 0 = (\mu - \lambda)^3$ and for corresponding eigen vector $\mathbf{x} = (x_1, x_2, x_3)$

Solve $(\mathbf{B} - \lambda\mathbf{I})\mathbf{x} = 0 \implies x_2 = 0$ and hence eigen space is $\mathbb{R}^2$

For $|\mathbf{C} - \mu\mathbf{I}| = 0 = (\mu - \lambda)^3$ and for corresponding eigen vector $\mathbf{x} = (x_1, x_2, x_3)$

Solve $(\mathbf{B} - \lambda\mathbf{I})\mathbf{x} = 0 \implies x_2 = 0, x_3 = 0$ and hence eigen space is $\mathbb{R}$

QUESTION 4.6

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Check $|\mathbf{A}-3\mathbf{I}|=0$, we get $\det\begin{bmatrix}0 & 0 & 0\\ -2 & 1 & 2\\ -2 & 1 & 2\end{bmatrix}=0$

$$\mathbf{A}\mathbf{x}=3\mathbf{x}$$

$$\begin{bmatrix}3 & 0 & 0\\ -2 & 4 & 2\\ -2 & 1 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=3\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

We get $-2x_1+x_2+2x_3=0$. So all eigen vectors of form $\mathbf{x}=x_1(1,2,0)+x_3(0,-2,1)$ where $x_3,x_1\in\mathbb{R}$

QUESTION 4.6

To prove $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$

$$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We get the eigen value to be 6.

QUESTION 4.7

For $|\mathbf{A} - \mu \mathbf{I}| = 0$,

$$\det\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} - \mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \cos\theta - \mu & -\sin\theta \\ \sin\theta & \cos\theta - \mu \end{bmatrix}\right) = 0$$

$$\mu^2 - 2\mu\cos\theta + 1 = 0 \implies \mu = \cos\theta \pm i\sin\theta$$

Let $\mathbf{x} = (x_1, x_2)$ where $x_1, x_2 \in \mathbb{C}$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mu \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

QUESTION 4.7

We get $\cos\theta x_1 - \sin\theta x_2 = (\cos\theta - i\sin\theta)x_1 \implies x_2 = ix_1$

We get $\mathbf{x} = x_1(1, i)$ where $x_1 \in \mathbb{C}$

For other eigen value $\cos\theta x_1 + \sin\theta x_2 = (\cos\theta + i\sin\theta)x_1 \implies x_2 = -ix_1$

We get $\mathbf{x} = x_1(1, -i)$ where $x_1 \in \mathbb{C}$

$\mathbf{P} := \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ and Check it $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \cos\theta - i\sin\theta & 0 \\ 0 & \cos + i\sin\theta \end{bmatrix}$

QUESTION 4.8

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$Rank\mathbf{A} = 1$ and $Nullity\mathbf{A} = n - 1$

Eigen vector = $\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$ for eigen value=n

To find $|\mathbf{A} - \mu\mathbf{I}| = 0$, Swap all rows inititially ($R_i \iff R_{n+1-i}$ $\forall$ i=1 to n) and perform $R_1 \mapsto \sum_{i=1}^{n} R_i$ and take (n-μ) common and then $R_k \mapsto R_k - R_1 \forall$ k=2 to n and then expand via last column

we get $\mu^{n-1}(\mu - n) = 0 \implies \mu$ = 0 GM is n-1 ,μ = n GM is 1

Now find eigen vectors corresponding to all eigen values $(\mathbf{A} - \mu\mathbf{I})\mathbf{x} = 0$ we get

For $\mu = 0$,v = { $\mathbf{x}$: $\sum_{i=1}^{n} x_i = 0$}

For $\mu = n$ we get $\mathbf{v} = x_1(1, 1, 1, 1....)^T \forall x_1 \in \mathbb{R}$

QUESTION 4.8

$$P := \begin{bmatrix} -1 & -1 & -1 & . & . & -1 & 1 \\ 1 & 0 & 0 & . & . & 0 & 1 \\ 0 & 1 & 0 & . & . & 0 & 1 \\ 0 & 0 & 1 & . & . & 0 & 1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 & 1 \end{bmatrix}$$

Perform $\mathbf{P}^{-1}\mathbf{AP}$ to get to a diagonal matrix $(0, 0, 0, ..., n)^T$

Questions?

Contact me via 190100051@iitb.ac.in

Thank You