MA 106
Tutorial 1 Solutions

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QUESTION 1

QUESTION 2

QUESTION 3

QUESTION 4

QUESTION 5

QUESTION 6

QUESTION 1

QUESTION 1

Given B should be symmetric and C should be skew-symmetric such that $\boxed{A = B + C}$.
Take transpose on both sides of this equation. This gives us
$A^T = B^T + C^T \Rightarrow \boxed{A^T = B - C}$. Solve these two boxed equations simultaneously to get $B = \frac{A+A^T}{2}$ and $C = \frac{A-A^T}{2}$.
Thus we have $A = B + C$ and clearly, B is symmetric and C is skew-symmetric.
By our solution, B and C must be unique

Other method

Let $B = \frac{A+A^T}{2}$ and $C = \frac{A-A^T}{2}$. We can see that $A = B + C$. Clearly, B is symmetric and C is skew-symmetric.

B and C are unique solutions.

Proof by contradiction: Let $A = B + C = B_1 + C_1$ where B, B_1 are symmetric and distinct and C, C_1 are skew-symmetric and distinct. Take transpose on all sides. We get, $B - C = B_1 - C_1$. From these two equations, we see that $B = B_1$ and $C = C_1$ which is a contradiction to our previous assumption. Thus, B and C must be unique.

$$C = AB$$
$$c_{ij} = \sum_{k} a_{ik} b_{kj}$$

QUESTION 2

QUESTION 2

(i) AB is a 3×3 matrix. The elements of the second row of AB are given by the expression: $AB_{2,j} = \sum_{k=1}^{2} A_{2,k} B_{k,j}$. Thus, the second row can be written as the linear combination of rows of B as follows:

$A_{21} B_{1j}$

$$3 \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + 4 \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$[3[\quad] + 4[\quad]]$

(ii) Similarly, the second column of AB can be written as as the linear combination of columns of A as follows:

$AB_{i2} = A_{ik} B_{k2}$

$= A_{i1} B_{12}$

$$2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$[2 A_{i1} + 5 A_{i2}]$

$A(A^{-1}) = I$ B

$AB = I$ $j=4$

$c_{ij} = \sum_k a_{ik} b_{kj}$ $j:4$ b_{k4}

$= c_{ij} = [I]$

$[x_1\ x_2\ x_3\ x_4]^T$

QUESTION 3

$\sum_k a_{ik} b_{k4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$a_{i1} b_{14}$

$A^{-1} A = I$

$BA = I$

$\sum_k b_{ik} a_{kj} = c_{ij} = I$

$[b_{4k}] a_{kj} = [0\ 0\ 0\ 1]$

$\begin{bmatrix} \\ \\ \dots \end{bmatrix}$

QUESTION 3

$AA^{-1} = I_4$, Thus we have the following system of equations to get the last column of A^{-1}:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -24 & 8 & -5 \\ 0 & -7 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Solve this to get the last column of A^{-1}

We get: $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} 2.75 & -0.5 & -2.25 & 1 \end{bmatrix}^T$

QUESTION 3(continue)

Since we already know x_4, now we'll have to solve a system of only 3 equations and 3 unknowns.

$$\begin{bmatrix} x_5 & x_6 & x_7 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -24 & 8 & -5 \\ 0 & -7 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve this to get the last row of A^{-1}

We get: $\begin{bmatrix} x_5 & x_6 & x_7 & x_4 \end{bmatrix} = \begin{bmatrix} -1.5 & -0.5 & 0 & 1 \end{bmatrix}$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$d_{ij} = \sum_{k} b_{ik} a_{kj}$$

$$\sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$$

$$d_{ii} = \sum_{k} b_{ik} a_{ki}$$

QUESTION 4

$k = r$

$i = s$

$i = r$

$k = s$

$$\sum_{r} \sum_{s} a_{rs} b_{sr} = \sum_{r} \sum_{s} a_{rs} b_{sr}$$

QUESTION 4

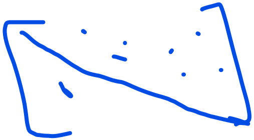
Assume A and B are two upper triangular matrices. For these upper triangular matrices, A_{ij} and $B_{ij} = 0$ for $i > j$. We have to show that $AB_{ij} = 0$ for $i > j$ also holds true.

We have $AB_{ij} = A_i^T B_j$ where A_i^T is the i$^{\text{th}}$ row of A and B_j^T is the j$^{\text{th}}$ column of B.

$$\text{Thus, } AB_{i,j} = A_i^T B_j = \sum_{k=1}^{n} A_{ik} B_{kj}$$

$$= \sum_{k=1}^{j} A_{ik} B_{kj} + \sum_{k=j+1}^{n} A_{ik} B_{kj}$$

QUESTION 4



Now given A, B are upper triangular. So $A_{ij} = 0, B_{ij} = 0$ for $i > j$. Here we are only checking AB_{ij} for $i > j$, so we get $\sum_{k=1}^{j} A_{ik}B_{kj} = 0$ since A_{ik} is zero in the summation. $\sum_{k=j+1}^{n} A_{ik}B_{kj} = 0$ since B_{kj} is zero in the summation.
Similarly we can show that product of two lower triangular matrix is also lower triangular but there we would consider $i < j$ in our analysis.

$C = \underbrace{AB}$ $\quad C^T = B^T A^T$ $\quad l = 3$

$x_1 = c + c_1 x_3 + c_2 x_4$

$x_2 = d + c_1' x_3 + c_2' x_4$

$n = 4$

QUESTION 5

$m = 2,$ $\quad 5 d_1$

3×1 $\quad c_1 x_3 + c_2 x_4$

$x_1 = c + c_1 x_3 + c_2 x_4 + c_3 x_5 + c_4 x_6$

$x_2 =$

x_j $\quad [x_1, x_2, x_3, x_4, \dots x_n] \dots x_n$

Part (a) is trivial.

$$trace(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}$$

$$trace(BA) = \sum_{i=1}^{n} (BA)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} B_{ik} A_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} A_{ki} B_{ik}$$

We have just switched the order of summation as the two summations are over independent axes. Thus we see that $trace(AB) = trace(BA)$ as the two expressions are equivalent

QUESTION 6

QUESTION 6a

We perform the row operations to the augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]$$

$R_4 := R_4 - 2R_1$

REF

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right]$$

$R_2 := R_2 - 2R_1$

QUESTION 6a

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{-1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right]$$

$R_3 := R_3 + 5R_2$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right]$$

QUESTION 6a

Swap R_3 and R_4

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

x

$R_3 = R_3 + 4R_2$

$$\left[\begin{array}{cccccc|c} \boxed{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \boxed{-1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{6} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$0 \neq 1$

The last row of the augmented matrix is inconsistent. So the system has no solution.

QUESTION 6b

(ii) Performing row operations on the augmented matrix,

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

$R_2 := R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

QUESTION 6b

$R_3 := R_3 + R_1$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & \boxed{-8} & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right]$$

$\boxed{R_3 := R_3 + R_2}$

$$\left[\begin{array}{ccc|c} \boxed{2} & 1 & 1 & 5 \\ 0 & \boxed{-8} & -2 & -12 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

So we get $x_3 = 2$. Back-substituting in $8x_2 + 2x_3 = 12$ we get $x_2 = 1$ and back-substituting in $2x_1 + x_2 + x_3 = 5$, we get $x_1 = 1$.

The solution is; $x := \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}^T$

QUESTION 6c

(iii) Here the augmented matrix is [1]

$$\left[\begin{array}{cccc|c} 0 & 2 & -2 & 1 & 2 \\ 2 & -8 & 14 & -5 & 2 \\ 1 & 3 & 0 & 1 & 8 \end{array}\right]$$

Performing the following operations, we get; Swap R_1 and R_3

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 2 & -8 & 14 & -5 & 2 \\ 0 & 2 & -2 & 1 & 2 \end{array}\right]$$

$R_2 := R_2 - 2R_1$

QUESTION 6c

$AX = b$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 0 & -14 & 14 & -7 & -14 \\ 0 & 2 & -2 & 1 & 2 \end{array}\right]$$

x_3, x_4

Then $R_3 := 7R_3 + R_2$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 0 & -14 & 14 & -7 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$b_1 = 0$, $b_2 = 0$, $b_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

There are infinitely many solutions. (Why ?) Think