MA 106
Tutorial 5 Solutions

D1 T5

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April 07, 2021

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QUESTION 5.1

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QUESTION 5.8

QUESTION 5.9

QUESTION 6.1

QUESTION 6.2

QUESTION 5.1

QUESTION 5.1

Similar to exercise 4.7 and 4.8

QUESTION 5.2

Theorem

Let $\mathbf{A} \in \mathbb{K}^{n\times n}$, and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $\mathbf{A}$. Let g_j be the geometric multiplicity of λ_j for $j = 1, \ldots, k$. Then $g_1 + \cdots + g_k \leq n$. Further, $\mathbf{A}$ is diagonalizable if and only if $g_1 + \cdots + g_k = n$.

You can easily see eigen values are 2,1,2

Just you need to check for nullspace $(\mathbf{A} - \mu\mathbf{I})\mathbf{x} = 0$ or find nullity for $\mu = 2$

$$\begin{bmatrix} 2-\mu & a & b \\ 0 & 1-\mu & c \\ 0 & 0 & 2-\mu \end{bmatrix} \mapsto \begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix}$$

For nullity equal to 2 we need rank=1 hence R_2 must to be a scalar multiple of R_1

$\frac{a}{-1} = \frac{b}{c} \implies$ b=-ac

QUESTION 5.3

Take $(\mathbf{A} - \mu\mathbf{I})$ and perform $R_{2i} \mapsto R_{2i} + R_{2i-1}/\mu$ $\forall$ i=1 to k
There was a catch that $\mu \neq 0$ (how would you prove that). Hint (find nullity of A)
It's a Upper triangular matrix and whose det is product of diagonal entries
$\mu^k(\mu + 1/\mu)^k = 0 \implies (\mu^2 + 1)^k = 0 \implies \mu = \pm i$
We know that GM $\leq$ AM hence GM is k for both eigen value

OR

Find Nullity of $(\mathbf{A} - i\mathbf{I})$ by performing $R_{2i} \mapsto R_{2i} - iR_{2i-1}$ $\forall$ i=1 to k . we easily get GM=k for both
Characteristic polynomial is $(\mathbf{A}^2 + 1)^k = 0$

QUESTION 5.4

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For forward part,

$$\lambda \|\mathbf{x}\|^2 = \lambda < x, x >=< x, \lambda x >=< x, Ax >$$

Transformation property: $< Ax, y >= (Ax)^* y = x^*(A^* y) < x, A^* y >$

$$\lambda \|\mathbf{x}\|^2 =< A^* x, x >$$

Take conjugate on both sides

$$\overline{\lambda} \|\mathbf{x}\|^2 = \overline{< A^* x, x >}$$

$$\overline{\lambda} \|\mathbf{x}\|^2 =< x, A^* x >$$

Similarly prove the backward part (Try it)

QUESTION 5.4

Other method:

$|\mathbf{A} - \lambda \mathbf{I}| = 0$. Choose $\mathbf{B} = \mathbf{A} - \lambda \mathbf{I}$ and we get det(B) = 0

We can claim that det(B*)=0. So $\mathbf{B}^* = \mathbf{A}^* - \overline{\lambda}\mathbf{I}$.

Now $|\mathbf{A}^* - \overline{\lambda}\mathbf{I}| = 0$ hence $\overline{\lambda}$ is an eigen value of $\mathbf{A}*$

QUESTION 5.5

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$\mathbf{A}x = 0 \implies A^*Ax = 0 \implies x \in N(A^*A)$

$N(A) \subseteq N(A^*A)$

Now consider

$A * Ax = 0 \implies x^*A^*Ax = 0 \implies (Ax)^*Ax = 0 \implies Ax = 0 \implies x \in N(A)$

$N(A^*A) \subseteq N(A)$

Hence $N(A) = N(A^*A)$

All part follows from this because geometric multiplicity of 0 is nullity of the matrix.

QUESTION 5.6

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By the Gerschgorin Theorem we know $|\lambda - a_{jj}| \leq \sum_{j \neq k} |a_{jk}|$
Lets calculate $\sum_{j \neq k} |a_{jk}|$
For j=1 it's 1 + $\sqrt{2}$
For j=2 it's 2
For j=3 it's 1 + $\sqrt{2}$

QUESTION 5.7

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By the Gerschgorin Theorem we know $|\lambda - a_{jj}| \leq \sum_{j \neq k} |a_{jk}|$

We have $\lambda - a_{jj} > -\sum_{j} |a_{jk}| \mapsto \text{I}$

We already have that $|a_{jj}| > \sum_{k \neq j} |a_{jk}| \implies a_{jj} - \sum_{k \neq j} |a_{jk}| > 0 \mapsto \text{II}$

From I and II we get $\lambda > 0$ hence the matrix is invertible

QUESTION 5.8

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consider λ to be max of all eigen value
$\alpha_2 \geq \|\mathbf{A}\mathbf{x}\| = \|\lambda \mathbf{x}\| = |\lambda|$

By the Gerschgorin Theorem we know $|\lambda - a_{jj}| \leq \sum_{j \neq k} |a_{jk}|$
$||\lambda| - |a_{jj}|| \leq |\lambda - a_{jj}| \leq \sum_{j \neq k} |a_{jk}|$
$||\lambda| - |a_{jj}|| \leq \sum_{j \neq k} |a_{jk}| \implies |\lambda| - |a_{jj}| \leq \sum_{j \neq k} |a_{jk}|$
$|\lambda| = |a_{jj}| + \sum_{j \neq k} |a_{jk}| \leq \alpha_\infty$

Eigen values of $\mathbf{A}$ and $\mathbf{A}^T$ are same and performing same operations as we did above we can say $|\lambda| \leq \alpha_1$

QESTION 5.8

You can leave this slide since matrix norm isn't covered or won't be covered

Other method (An Important General result):

Let $(\lambda, \mathbf{x})$ be eigen pair s.t $\rho(\mathbf{A}) = max|\lambda|$

Find $\mathbf{y} \neq 0$ s.t $\mathbf{xy}^*$ is a non zero matrix , $\|.\|$ is a matrix norm

$$\lambda\mathbf{x} = \mathbf{Ax} \implies \lambda\mathbf{xy}^* = \mathbf{Axy}^* \implies |\lambda|\|\mathbf{xy}^*\| = \|\mathbf{Axy}^*\| \leq \|\mathbf{A}\|\|\mathbf{xy}^*\| \implies \rho(\mathbf{A}) \leq \|\mathbf{A}\|$$

QUESTION 5.9

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Part.a) You need to use $\|\mathbf{x}+\mathbf{y}\|^2 =< \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} >=$

<x,x+y>+<y,x+y> = <x,x> + <x,y> + <y,x> + <y,y>*and*

Similarly for the other term $\|\mathbf{x}-\mathbf{y}\|^2 =< \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} >=$

<x,x-y>+<-y,x-y> = <x,x> - <x,y> - <y,x> + <y,y>

Part.b) $(\mathrm{Re} < x, y >) = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$ where $\theta \in [0, \pi]$

QUESTION 6.1

QUESTION 6.1

QUESTION 6.2

QUESTION 6.2

Let W be the subspace of $\mathbb{K}^{4\times 1}$ spanned by the vectors $\mathbf{x}_1 := \begin{bmatrix} 1 & -1 & 2 & 0 \end{bmatrix}^T$, $x_2 := \begin{bmatrix} 1 & 1 & 2 & 0 \end{bmatrix}^T$ and $\mathbf{x}_3 := \begin{bmatrix} 3 & 0 & 0 & 1 \end{bmatrix}^T$ Let us apply the G-S OP.

Let $u_1 := \frac{x_1}{\|x_1\|} = \frac{\begin{bmatrix} 1 & -1 & 2 & 0 \end{bmatrix}^T}{\sqrt{6}}$

, $u_2 := \frac{x_2 - P_{u_1}(x_2)}{\|x_2 - P_{u_1}(x_2)\|} = \frac{\begin{bmatrix} 1 & 5 & 2 & 0 \end{bmatrix}^T}{\sqrt{30}}$

$u_3 := \frac{x3 - P_{u_1}(x3) - P_{u_2}(x3)}{\|x3 - P_{u_1}(x3) - P_{u_2}(x3)\|} = \frac{\begin{bmatrix} 12 & 0 & -6 & 5 \end{bmatrix}^T}{\sqrt{205}}$

QUESTION 6.2

You can check for yourself that $\{u_1, u_2, u_3\}$ is an orthonormal basis

To extend $\{u_1, u_2, u_3\}$ to an orthonormal basis for V := $\mathbb{K}^{4\times 1}$, we look for $u_4 := \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}^T$ which is orthogonal to the set $\{x1, x2, x3\}$ where $\|u_4\| = 1$. Try on your own

Questions?

Contact me via 190100051@iitb.ac.in

Thank You