MA 108 ODE | Spring 2021 | Quiz I | 21-05-2021

Name: Division:
Roll No: Tutorial Batch:

Circle the correct answer in the multiple choice questions.

(1) The DE

$$\left(\frac{d^2y}{dx^2}\right)^{2021} + e^x y \frac{dy}{dx} = \frac{d^4y}{dx^4}$$

has order

(a) 2 (♣) 4 (c) 2021 (d) 4042.

(2) The DE

$$\left(\frac{d^2y}{dx^2}\right)^{2021} + e^x \frac{dy}{dx} = \frac{d^4y}{dx^4}$$

is linear.

(a) True (♣) False.

(3) For the IVP

$$\frac{dy}{dx} = 2xy^2,\ y(0) = 1,$$

starting with $\phi_0 \equiv 1$, the third Picard iterate is $\phi_3(x) = 1 + x^2 + x^4 + x^6 + \alpha x^8 + \beta x^{10} + \gamma x^{12} + \delta x^{14}$ where α is

(a) 1 (♣) $\frac{2}{3}$ (c) 6 (d) $\frac{16}{3}$.

$$\begin{aligned}
\phi_1(x) &= 1 + \int_0^x f(s, \phi_0(s))ds = 1 + \int_0^x 2s\ ds \\
&= 1 + x^2 \\
\phi_2(x) &= 1 + \int_0^x f(s, \phi_1(s))ds = 1 + \int_0^x 2s(1+s^2)^2 = 1 + \int_0^x (2s + 4s^3 + 2s^5)ds \\
&= 1 + x^2 + x^4 + \frac{x^6}{3} \\
\phi_3(x) &= 1 + \int_0^x 2s\left(1 + s^2 + s^4 + \frac{s^6}{3}\right)^2 ds = 1 + \int_0^x \left(2s + 4s^3 + 6s^5 + \frac{16}{3}s^7 + \dots\right) ds \\
&= 1 + x^2 + x^4 + x^6 + \frac{2}{3}x^8 + \dots
\end{aligned}$$

(4) Consider a small disk in which the DE

$$\left(\frac{1}{y} + \tan y\right) + \left(\frac{1}{x} - \cot x\right)\frac{dy}{dx} = 0$$

is well-defined. An integrating factor for this DE in that disk is given by

(a) $xy\cos x \sin y$ (♣) $xy \sin x \cos y$ (c) $\frac{\sin x \cos y}{xy}$ (d) $\frac{\cos x \sin y}{xy}$.

Since the options are given one can check directly which one is an integrating factor. Another way is to observe that the DE is separable. So solve it by

$$\int \frac{dy}{\frac{1}{y} + \tan y} = -\int \frac{dx}{\frac{1}{x} - \cot x}$$

which gives

$$\log(y \sin y + \cos y) = -\log(x \cos x - \sin x) + \text{constant};$$

i.e.,

$$(x \cos x - \sin x)(y \sin y + \cos y) = c.$$

Differentiating, we get

$$y \cos y (x \cos x - \sin x)\frac{dy}{dx} - x \sin x (y \sin y + \cos y) = 0.$$

This is same as

$$xy \sin x \cos y \left[\left(\frac{1}{y} + \tan y\right) + \left(\frac{1}{x} - \cot x\right)\frac{dy}{dx}\right] = 0.$$

Remark: Some students might just guess the answer based on symmetry without actually doing it! That's okay!

(5) Solve the IVP

$$\frac{dy}{dx} = \frac{1}{6e^y - 2x}, \; y(2) = 0,$$

uniquely for $x \in (2 - \varepsilon, 2 + \varepsilon)$ for some $\varepsilon > 0$. Note that this solution, say $y(x)$, in fact solves the DE for any $x > 0$. Now among the options below, the answer closest to $y(40)$ is

(a) 1 (♣) 3 (c) 5 (d) 7.

The equation

$$-1 + (6e^y - 2x)\frac{dy}{dx} = 0$$

is exact after multiplying with the IF

$$\mu(x, y) = e^{2y}.$$

Thus the solution is

$$-xe^{2y} + 2e^{3y} = c.$$

Since $y(2) = 0$, we get

$$-xe^{2y} + 2e^{3y} = 0.$$

Thus,

$$y = \log(x/2).$$

As $2 < e < 3$, the answer is 3 (as $2^3 < \frac{40}{2} < 3^3$).

(6) For a solution $y(x)$ of the IVP

$$\frac{dy}{dx} + \frac{y - x}{y + x} = 0, \; y(1) = 1,$$

$y(0)$ equals

(a) 1 (♣) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2.

This is Tut 1 Q 8 (v). Solution is
$$(y+x)^2 - 2x^2 = 2.$$
So
$$y(0) = \sqrt{2} \text{ or } -\sqrt{2}.$$
Remark: Even if $-\sqrt{2}$ is an option, still $\sqrt{2}$ is the right answer. We've
$$y = -x \pm \sqrt{2+2x^2}$$
and since $y(1) = 1$ is given we in fact have
$$y = -x + \sqrt{2+2x^2}$$
and so $y(0) = \sqrt{2}$.[a]

[a] Thanks to Aryaman for pointing this out.

(7) The dimension of the solution space of
$$x^2y'' - 4xy' + 6y = 0$$
defined on the interval $(-1,1)$ is
(a) 1 (b) 2 (♣) 3 (d) 4.

For instance, $\{x^2, x^3, x^2|x|\}$ is a basis. Note that the dimension theorem does not apply.

(8) There exist solutions f, g of
$$y'' + p(x)y = 0$$
such that $W(f,g;x) \equiv 1$.
(♣) True (b) False.

In this case, Wronskian is a constant, and moreover,
$$W(af+bg, cf+dg; x) = (ad-bc)W(f,g;x).$$