MA 108
Tutorial 5 Solutions

D1 T5

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QUESTION 5.1

QUESTION 5.2

QUESTION 5.3

QUESTION 5.4

QUESTION 5.5

QUESTION 5.10

QUESTION 5.1

QUESTION 5.1.vi

Generally for complex a and b the following holds:

$$\mathcal{L}(t^a e^{-bt}) = \frac{\Gamma(a+1)}{(s+b)^{a+1}}$$

$$\mathcal{L}(1+te^{-t})^3 = \mathcal{L}(1) + \mathcal{L}(t^3 e^{-3t}) + 3\mathcal{L}(te^{-t}) + 3\mathcal{L}(t^2 e^{-2t})$$

$$\implies \frac{1}{s} + \frac{6}{(s+3)^4} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3}$$

QUESTION 5.2

QUESTION 5.2.iv

We know the roots of $z^4 = a^4$ are z = $a\frac{(1 \pm i)}{\sqrt{2}}, a\frac{(-1 \mp i)}{\sqrt{2}}$. Using this
For $s^4 = 4a^4$ are s = $a(1 \pm i), a(-1 \mp i)$
Using Partial Fraction Method

$$\frac{s^3}{s^4 + 4a^4} = \frac{A}{s + a(1+i)} + \frac{B}{s + a(1-i)} + \frac{C}{s - a(1+i)} + \frac{D}{s - a(1-i)}$$

Now find A,B,C,D by any of the method one of the way would be to put s=0,+-a or else you can use root method

QUESTION 5.2.iv

We get A=B=C=D=1/4

$$\frac{s^3}{s^4+4a^4} = \frac{1}{4}(\frac{1}{s+a(1+i)} + \frac{1}{s+a(1-i)} + \frac{1}{s-a(1+i)} + \frac{1}{s-a(1-i)})$$

$$\mathcal{L}^{-}1(\frac{s^3}{s^4+4a^4}) = \frac{1}{4}(e^{-ax-iax} + e^{-ax+iax} + e^{ax+iax} + e^{ax-iax}) = cos(ax)cosh(ax)$$

QUESTION 5.3

$$y'' - 2y' - 3y = 10sinh(2t) = 5e^{2t} - 5e^{-2t}$$

Given the I.C's and applying Laplace Transform

$$(s^2 - 2s - 3)\mathcal{L}(y) - 4 = \frac{5}{s-2} - \frac{5}{s+2} = \frac{20}{(s+2)(s-2)}$$

$$\mathcal{L}(y) = \frac{4}{(s-3)(s+1)} + \frac{20}{(s+2)(S+1)(s-2)(s-3)}$$

Use Partial Fraction Method

QUESTION 5.3.vi

$$\mathcal{L}(y) = \frac{1}{(s-3)} - \frac{1}{(s+1)} - \frac{1}{(s+2)} + \frac{5}{3(S+1)} - \frac{5}{3(s-2)} + \frac{1}{(s-3)}$$

Take inverse Laplace Transform

$$y(t) = e^{3t} - e^{-t} - e^{-2t} + \frac{5e^{-t}}{3} - \frac{5e^{2t}}{3} + e^{3t}$$

$$y(t) = 2e^{3t} - e^{-2t} + \frac{2e^{-t}}{3} - \frac{5e^{2t}}{3}$$

QUESTION 5.4

Apply Laplace Transform on both the equality

$$s^2Y_1 + Y_2 - sy_1(0) - y_1'(0) = \frac{-5s}{s^2+4}$$

$$s^2Y_1 + Y_2 = \frac{s^3 + s^2 - s + 4}{s^2+4}$$

$$s^2Y_2 + Y_1 - sy_2(0) - y_2'(0) = \frac{5s}{s^2+4}$$

$$s^2Y_2 + Y_1 = \frac{-s^3 + s^2 + s + 4}{s^2+4}$$

From this two relation we get

$$Y_1 + Y_2 = \frac{2}{s^2+1} \implies y_1(t) + y_2(t) = 2sin(t)$$

$$Y_1 - Y_2 = \frac{2s}{s^2+4} \implies y_1(t) - y_2(t) = 2cos(2t)$$

$y_1(t) = sint + cos2t$ and $y_2(t) = sint - cos2t$

QUESTION 5.5

We use the binomial theorem to obtain the infinite series of $\sqrt{a^2+x^2}$

$$\mathcal{L}^{-}1(\frac{1}{\sqrt{s^2+a^2}}) = \mathcal{L}^{-}1\frac{1}{s}\sum_{j=0}^{\infty}\frac{(-1)^j(1.3.5.7.9...(2j-1))a^{2j}}{2^j s^{2j} j!}$$

$$\implies \mathcal{L}^{-}1\frac{1}{s}\sum_{j=0}^{\infty}\frac{(-1)^j(2j)!a^{2j}}{4^j s^{2j}(j!)^2}$$

$$\implies \sum_{j=0}^{\infty}\frac{(-1)^j(at)^{2j}}{4^j(j!)^2} = J_o(at)$$

QUESTION 5.10

Lets consider ,

$$ln(\frac{s+a}{s+b}) = ln(1+\frac{a-b}{s+b}) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}(\frac{a-b}{s+b})^{j+1}$$

$$\implies \sum_{j=0}^{\infty}\frac{(-1)^j(a-b)^{j+1}}{(j+1)!}\mathcal{L}(e^{-bx}x^j)$$

$$\mathcal{L}^{-1}ln(\frac{s+a}{s+b}) = \frac{-e^{-bx}}{x}\sum_{j=0}^{\infty}\frac{(x(b-a))^{j+1}}{(j+1)}! = \frac{e^{-bx}-e^{-ax}}{x}$$

QUESTION 5.10

Similarly for ,

$$\mathcal{L}^{-1} ln(\frac{s+a}{s+b})(\frac{s+c}{s+d}) = \frac{e^{-bx} - e^{-ax} + e^{-dx} - e^{-cx}}{x}$$

For our case a=-2+i, c=-2-i , b=-1+2i, d=-1-2i

***[Point to note: Series only converges for s>|a-b|+|b|] ***

Questions?

Contact me via 190100051@iitb.ac.in

Thank You