MA 108
Tutorial 1 Solution

D1 T5

GYANDEV GUPTA

May 12, 2021

IIT BOMBAY

QUESTION 1

QUESTION 2

QUESTION 4

QUESTION 7

QUESTION 8

QUESTION 10

QUESTION 1

QUESTION 1e

Classify the following equations (order, linear or non-linear):

(v)$(1+y^2)\frac{d^2y}{dt^2}+t\frac{d^6y}{dt^6}+y=e^t$.

Order is 6 ($\frac{d^6y}{dt^6}$) term

Non Linear (y^2) term

QUESTION 2

QUESTION 2e

Formulate the differential equations represented by the following functions by eliminating the arbitrary constants a, b and c. Also state the order of the equations obtained. :

(v) $y = asinx + bcosx + a \mapsto (1)$

Differentiating both sides wrt x

$$\frac{dy}{dx} = acosx - bsinx \mapsto (2)$$

Multiply (1) by sin x and (2) by cos x and add,

$$ysinx + \frac{dy}{dx}cosx = asin^2x + acos^2x + asinx$$

QUESTION 2e

$$y sin x + \frac{dy}{dx} cos x = a(1 + sin x)$$

$$a = \frac{y sin x + \frac{dy}{dx} cos x}{1 + sin x} \mapsto (3)$$

Differentiating both sides of (2) wrt x

$$\frac{d^2y}{dx^2} = (a sin x + b cos x) \mapsto (4)$$

Adding (1) and (4)

$$y + \frac{d^2y}{dx^2} = a$$

From equation (4),

$$y + \frac{d^2y}{dx^2} = \frac{ysinx + \frac{dy}{dx}cosx}{1 + sinx}$$

QUESTION 2f

(vi) $y = a(1-x^2) + bx + cx^3$

$$y = a(1-x^2) + bx + cx^3$$

Differentiating both sides wrt x

$$\frac{dy}{dx} = -2ax + b + 3cx^2 \mapsto (1)$$

Differentiating both sides wrt x

$$\frac{d^2y}{dx^2} = -2a + 6cx \mapsto (2)$$

Differentiating both sides wrt x

QUESTION 2f

$$\frac{d^3y}{dx^3} = 6c \mapsto (3)$$

From (2) and (3),

$$\frac{d^2y}{dx^2} = -2a + x\frac{d^3y}{dx^3} \mapsto (4)$$

From (3), (4) and (1),

$$\frac{dy}{dx} = x(\frac{d^2y}{dx^2} - x\frac{d^3y}{dx^3}) + b + \frac{x^2}{2}\frac{d^3y}{dx^3}$$

Using all of the above equations,

$$y = \frac{-1}{2}\left(\frac{d^2y}{dx^2} - x\frac{d^3y}{dx^3}\right)(1 - x^2) + \left(\frac{dy}{dx} - x\left(\frac{d^2y}{dx^2} - x\frac{d^3y}{dx^3}\right) - \frac{x^2}{2}\frac{d^3y}{dx^3}\right)x + \left(\frac{1}{6}\frac{d^3y}{dx^3}\right)x^3$$

Order is 3

QUESTION 4

QUESTION 4

Prove that a curve with the property that all its normals pass through a point is a circle.

Let (x_0, y_0) be a point on the curve Equation of normal at (x_0, y_0) is ::

$$\frac{x_0 - x}{y_0 - y} = -\frac{dy}{dx}$$

Let the point (a, b) always satisfy this equation, i.e. (x_0, y_0) $\boxed{?}$ (x, y)

$$-\frac{x - a}{y - b} = \frac{dy}{dx}$$

$$-(x - a)dx = (y - b)dy$$

$$\int -(x - a)dx = \int (y - b)dy$$

QUESTION 4

$$-(x-a)^2 + c = (y-b)^2$$

$$(x-a)^2 + (y-b)^2 = c$$

This is a real curve iff c = r^2 , where r is a real number

Therefore, it is the equation of a circle with center at (a, b).

QUESTION 7

QUESTION 7

Let ϕ_i be a solution of $y' + ay = b_i(x)$ for i=1,2. Show that $\phi_1 + \phi_2$ satisfies $y' + ay = b_1(x) + b_2(x)$. Use this result to find the solutions of $y' + y = sinx + 3cos2x$ passing through the origin.

The proof follows immediately from the fact that the equation is linear and hence,

$$(\phi_1 + \phi_2)' + a(\phi_1 + \phi_2)$$

$$= (\phi_1)' + (\phi_2)' + a\phi_1 + a\phi_2$$

$$((\phi_1)' + a\phi_1) + ((\phi_2)' + a\phi_2) = b_1(x) + b_2(x)$$

Let $b_1 = sinx$ and $b_2 = 3cos2x$ For ϕ_1, we have,

$$((\phi_1)' + a\phi_1) = sinx$$

Multiplying both sides by e^{ax} i.e the Integrating factor,

$$(\phi_1 e^{ax})' = e^{ax} sinx$$

$$\phi_1 e^{ax} = \int e^{ax} sinxdx = e^{ax} \frac{(asinx - cosx)}{a^2 + 1} + c$$

$$\phi_1 = \frac{(asinx - cosx)}{a^2 + 1} + c'$$

Similarly,

$$\phi_2 = \frac{3(acos2x + 2sin2x)}{a^2 + 4}$$

The required solutions are ϕ_1 + ϕ_2 with a=1 and (0,0) is the I.C.'s

QUESTION 8

QUESTION 8

Obtain the solution of the following differential equations:
(iii) $y' = y(y^2 - 1)$, with $y(0) = 2$ or $y(0) = 1$, or $y(0) = 0$

$$\frac{dy}{dx} = y(y^2 - 1)$$

$$\int \frac{dy}{y(y-1)(y+1)} = \int dx$$

Using partial fractions,

$$\left(\frac{A}{y} + \frac{B}{y-1} + \frac{C}{y+1}\right) = \frac{1}{y(y-1)(y+1)}$$

QUESTION 8

A = -1, B = C = $\frac{1}{2}$

$$-lny + \frac{1}{2}ln|y-1| + \frac{1}{2}ln|y+1| = x + c$$

On putting values for (x, y) as (0, 2),(0, 1) and (0, 0), we get the respective particular solutions

QUESTION 10

QUESTION 10a

Show that the differential equation $\dfrac{dy}{dx} = \dfrac{ax+by+m}{cx+dy+n}$ where a, b, c, d, m and n are constants can be reduced to $\dfrac{dy}{dx} = \dfrac{ax+by}{cx+dy}$ if $ad - bc \neq 0$. Then Replace x = X+h, y= Y+k

$$\frac{dY}{dX} = \frac{aX + bY + m + ah + bk}{cX + dY + n + ch + dk}$$

Now if we compare with the form we want we need m +ah+bk=0 and n + ch+dk =0 To get (h,k) s.t both equations are zero the condition which we discussed in MA106 is $ad - bc \neq 0$. To solve them find (h,k) and then put y=vx since it's a homogenous equation and then solve

find the general solution of [(i)] $(1+x-2y)+y'(4x-3y-6)=0$
(h,k)=(3,2)

$$\frac{dY}{dX}=-\frac{X-2Y}{4X-3Y}$$

put Y=VX we get

$$V+X\frac{dV}{dX}=-\frac{1-2V}{4-3V}$$

$$X\frac{dV}{dX}=\frac{3V^2-2V-1}{4-3V}$$

$$-2\frac{dX}{X}=\frac{(6V-8)dV}{3V^2-2V-1}$$

$$-2\frac{dX}{X} = \frac{(6V-2)dV}{3V^2-2V-1} + \frac{-6dV}{3V^2-2V-1}$$

Separating variables and integrate it and replace V=Y/X

$$ln(3Y^2-2XY-X^2) - \frac{3}{2}ln(\frac{V-1}{V+1/3}) = c$$

$$ln(3Y^2-2XY-X^2) - \frac{3}{2}ln(\frac{Y-X}{Y+X/3}) = c$$

Replace X=x-3 and Y=y-2 to get the solution

QUESTION 10b

[(ii)] $y' = \frac{y-x+1}{y-x+5}$
This cant be reduced since ad-bc = 0. Now put y-x = Y and x=X and then solve

$$\frac{dY}{dX} = \frac{Y+1}{Y+5}$$

$$(1 + \frac{4}{Y+1})dY = dX$$

Now integrate both sides and replace Y = y-x, X=x

$$y - x + 4ln(y - x + 1) = x + c$$

QUESTIONS?

Contact me via 190100051@iitb.ac.in

THANK YOU