MA 111

Tutorial 5 Solutions

D1 T5

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$$\iint \left(F_{2x} - F_{1y}\right) dx\,dy \quad \int_{C_1} + \int_{1}^{0}{}_{C_2} + \int_{C_3}$$

QUESTION 1

$y = 1 - x^2$

$t,\ 1 - t^2$

$t \in (1, 0)$

$dx = 0$ $\quad$ $dy = 0$ $\quad C_1$

QUESTION 1(i)

Green's Theorem:

1. Let D be a bounded region in R^2 with a positively oriented boundary C consisting of a finite number of non-intersecting simple closed piecewise continuously differentiable curves.
2. Let Ω be an open set in R^2 such that $D \cup C \subset \Omega$ and let $F1 : \Omega \to R$ and $F2 : \Omega \to$ be C^1 functions.

Then the following holds: $\int_C F_1\, dx + F_2\, dy = \iint_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy$

QUESTION 1(i)

Verfiy Green's Theorem:-

To evaluate: $\int_{\partial D} F_1(x,y)\,dx + F_2(x,y)\,dy$

Here, $F_1(x,y) = -xy$, $F_2(x,y) = xy$ on $D =: \{(x,y) \mid 0 \le y \le 1 - x^2 and\ 0 \le x\}$
let ∂D (**positively oriented**) $= C_1 + C_2 + C_3$ where,
$C_1(t) : x(t) = t,\ y(t) = (1-t^2)\ [t = 1 \to 0]$,
$C_2(t) : x(t) = 0,\ y(t) = t\ [t = 1 \to 0]$ and $C_3(t) : x(t) = t,\ y(t) = 0\ [t = 0 \to 1]$

$\therefore \int_{\partial D}(F_1(x,y)\,dx + F_2(x,y)\,dy) = \int_{C_1}(F_1(x,y)\,dx + F_2(x,y)\,dy) +$
$\int_{C_2}(F_1(x,y)\,dx + F_2(x,y)\,dy) + \int_{C_3}(F_1(x,y)\,dx + F_2(x,y)\,dy)$

$\therefore \int_{\partial D}(F_1(x,y)\,dx + F_2(x,y)\,dy) = \int_1^0(-t(1-t^2)(1) + t(1-t^2)(-2t))\,dt +$
$\int_1^0(0+0)\,dt + \int_0^1(0+0)\,dt = \frac{31}{60} \Rightarrow \boxed{1}$

QUESTION 1(i)

To evaluate: $\int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy$

We know, $\frac{\partial F_2}{\partial x} = \frac{\partial (xy)}{\partial x} = y$ and $\frac{\partial F_1}{\partial y} = \frac{\partial (-xy)}{\partial y} = -x$

$\therefore \int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy = \int_D (y - (-x))\, dx\, dy = \int_0^1 \left[\int_0^{1-x^2} (x+y)\, dy\right] dx$

$\therefore \int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy = \int_0^1 \left[(xy)\Big|_{y=0}^{y=1-x^2} + (\frac{y^2}{2})\Big|_{y=0}^{y=1-x^2}\right] dx$

$\therefore \int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy = \int_0^1 \left[0.5 - x^2 + \frac{x^4}{2} + x - x^3\right] dx = \frac{31}{60} \Rightarrow \boxed{2}$

from $\boxed{1}$ and $\boxed{2}$,

$\int_{\partial D} F_1(x,y)\, dx + F_2(x,y)\, dy = \int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy$

Hence, verified Green's theorem.

QUESTION 1b

$$\text{LHS} = \iint_R (e^x + 2x - 2x)\, dx dy = \int_0^1 \left(\int_0^x e^x \, dy \right) dx = 1.$$

and

$$\text{RHS} = \oint_{\partial R} [2xy \, dx + (e^x + x^2)\, dy]$$

$$= \int_0^1 (e+1) dy + \int_1^0 (3t^2 + e^t)\, dt = e + 1 - e = 1.$$

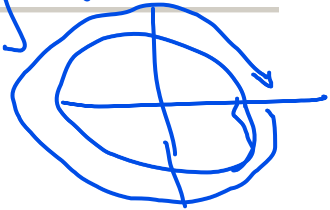
(Observe that f and dy are zero on the horizontal segment of the curve, whereas on the vertical segment $dx = 0$.)

$\int$

QUESTION 2

$Q_x - P_y$

$$\int_{2\pi}^{0}\int (1-2y)\,dx\,dy$$



QUESTION 2a

(i) Here

$$f(x,y) = y^2; g(x,y) = x.$$

Therefore, the given path integral is equal to

$$\iint_R (1-2y)\,dxdy = \int_0^2 \int_0^2 (1-2y)dy\,dx = 4 - 4\int_0^2 dx = 4 - 8 = -4.$$

QUESTION 2b

(ii) Here

$$\iint_R (1-2y)\,dxdy = \iint_R dxdy + \int_{-1}^{1}\int_{-1}^{1}(-2y)dy\,dx = 4+0 = 4.$$

QUESTION 2c

Trivial, make sure the curve is positively oriented to apply Green's theorem

I$=\int\int_D \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}.dA$

(iii) Here

$$\iint_R (1-2y)\,dxdy = \iint_R dxdy + \int_{-2}^{2}\left[\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (-2y)dy\right] dx = 4\pi + 0.$$

QUESTION 3

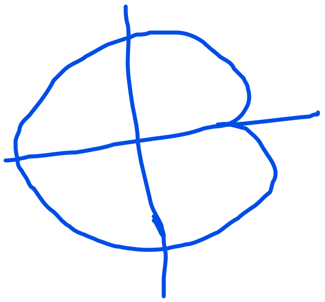
QUESTION 3a

$$\iint_D 1\,dx\,dy \qquad Q_x - P_y = 1 \qquad Q = \frac{x}{2}$$

$$= \int_{\partial D} \frac{x\,dy - y\,dx}{2} \qquad P = -\frac{y}{2}$$

$$=$$

QUESTION 3b



QUESTION 4

QUESTION 4a

(i) The required area is bounded by the curves

$$C_1 : r = a(1 - \cos\theta),\ \ 0 \le \theta \le \pi/2$$

and C_2 which is a portion of the y-axis. In any case, the required area is equal to

$$\frac{1}{2}\oint_C r(\theta)^2\, d\theta.$$

Since θ is a constant along the y-axis, this integral is equal to

$$\frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{a^2}{8}(3\pi - 8).$$

QUESTION 4b

(ii) The required area is

$$\frac{1}{2}\oint_C xdy - ydx.$$

Here the boundary curve consists of the interval $[0, 2\pi]$ and the cycloid above traced in the opposite direction. But the integrand is zero on the x-axis, since both y and dy vanish there. Hence the required area is

$$-\frac{a^2}{2}\int_0^{2\pi}(t - \sin t)d(1 - \cos t) - (1 - \cos t)d(t - \sin t) = 2\pi a^2.$$

QUESTION 4c

(iii) Here we use the polar coordinate form as in the previous exercise:

$$A = \frac{1}{2} \oint_C r^2 d\theta.$$

(This formula follows from Green's Theorem.)

Since θ is a constant on the two axes, this integral is equal to

$$\frac{1}{2} \int_0^{\pi/2} (1 - 2\cos\theta)^2 \, d\theta = \frac{1}{2} \left(\frac{3\pi - 8}{2} \right).$$

$$\int x e^{-y^2} dx + y(-x^2 e^{-y^2})\, dy$$

P Q

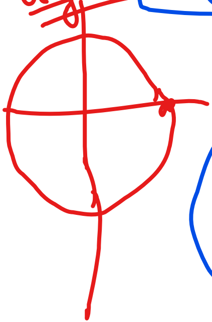
$P = \quad Q =$

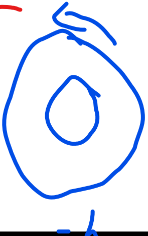
$$\int \frac{dy}{x^2+y^2}$$

$dy = 0$

QUESTION 5

$Q_x - P_y \neq 0$





$x^2 + y^2$

$a < r < b$

$0 < \theta < 2\pi$

$$= 0 \quad \int_{-b}^{b} \frac{dy}{b^2} + \int_{-a}^{a} \frac{dy}{a^2} = 0$$

$$\Rightarrow \boxed{\frac{2}{b} + \frac{2}{a}}$$

QUESTION 5

Observe that

$$xe^{-y^2}dx + (-x^2ye^{-y^2})dy = d(\frac{x^2e^{-y^2}}{2}).$$

Hence the integral of this term along a closed path vanishes. So the given integral is equal to

$$\oint_C \frac{dy}{x^2 + y^2.}$$

We compute this directly. Observe that $dy = 0$ along the two horizontal parts. But then the integral along one vertical segment cancels with that on the other since the integrands are the same and the segments are traced in the opposite direction. So the value of the required integral is equal to 0.

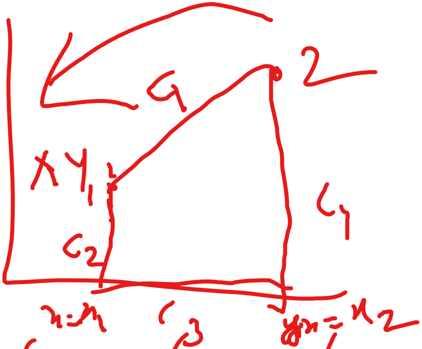
QUESTION 6

QUESTION 6

Take $f = -y^3$ and $g = x^3$ and apply Green's theorem. We get

$$\text{RHS} = \iint_R (3x^2 + 3y^2)\,dxdy = 3I_0.$$

QUESTION 7



$$\int dx\,dy$$

$$\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = \text{Area}$$

$\partial\phi$

$\frac{\partial\phi}{\partial n} = \nabla\phi \cdot \hat{n}$

Gauss Divergence

$$\oint (2x_1 - 2y) \cdot \hat{n} \, ds$$

QUESTION 8

$$\iint \nabla \cdot (2x_1 - 2y) \, dx\,dy$$

$$\iint 2 - 2 = 0$$

$$= 0$$

QUESTION 8

Since $\nabla^2(x^2 - y^2) = 0$, using one of Green's identities (refer to (9),(i)) one has

$$\oint_C \nabla(x^2 - y^2).\mathbf{n}ds = \oint_C \frac{\partial(x^2 - y^2)}{\partial\mathbf{n}}ds = \iint_R \nabla^2(x^2 - y^2)\,dxdy = 0.$$

QUESTION 9

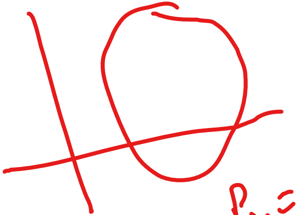
QUESTION 9a

QUESTION 9b

QUESTION 9c

$\ln(r) = \ln\sqrt{x^2+y^2}$

$\frac{1}{2}\frac{\partial \ln(x^2+y^2)}{\partial y}$



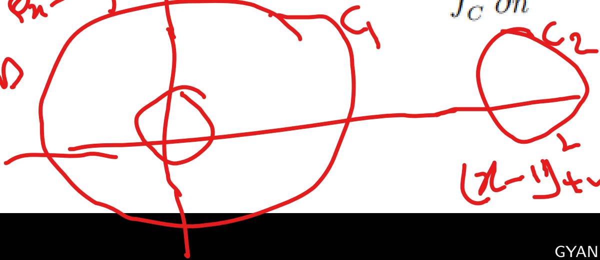
$$\nabla^2 w = 0 \quad \text{hence,} \quad \oint_C \frac{\partial w}{\partial n} dS = 0$$

$\phi_x - \phi_y = 0$

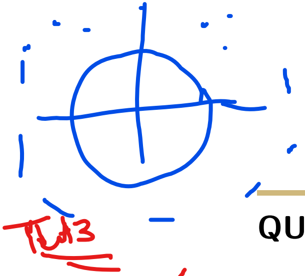
$\therefore = 2\pi$

$\frac{y\,dx}{x^2+y^2}$

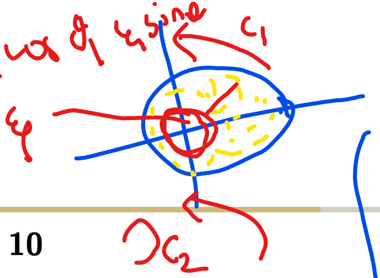
$-\frac{x\,dy}{x^2+y^2}$



$(x-1)^2+y^2=1$



C_2 = Eqn of ellipse



$$\oint \frac{y\,dx - x\,dy}{x^2+y^2}$$

QUESTION 10

$$\int_{C_1} = \int_{C_2}$$

$$\int I - \int_{C_2} I = 0$$

$$-2\pi$$

$C_1 \cup C_2$ $x - y = 0$

QUESTION 10a

Case (a): Suppose the curve does not enclose the origin. Take

$$f(x,y) = \frac{y}{x^2+y^2}, g(x,y) = \frac{x}{x^2+y^2}$$

and apply Green's theorem in the region R boundedby C. So the integral is equal to

$$\iint_R (g_x - f_y)\, dxdy.$$

A simple computation show that $g_x = f_y$ and hence the integral vanishes.

QUESTION 10b

Case (b): Suppose the curve encloses the origin, i.e, $(0,0) \in R$. (Now the above argument does not work!) We choose a small disc D around the origin contained in R and apply Green's theorem in the closure of $R' = R \setminus D$. As before, the double integral vanishes. But since the boundary of R' consists of C and $-\partial D$ it follows that

$$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2} = \oint_{\partial D} \frac{y\,dx - x\,dy}{x^2 + y^2}.$$

We can compute this now by using polar coordinates and see that this is equal to -2π.

We have

$$\frac{\partial(\ln r)}{\partial y} = \frac{y}{x^2 + y^2} \quad \text{and} \quad \frac{\partial(\ln r)}{\partial x} = \frac{x}{x^2 + y^2}.$$

By part (b), the required line integral is -2π.

QUESTION 11

QUESTION 11a

QUESTION 11b

QUESTION 12

QUESTION 12

QUESTION 13

QUESTION 13