MA111 (IIT Bombay) Weekly Quiz 6 : Surface integrals, Stokes theorem and Gauss divergence theorem, February 21, 2021

1. Let $\mathbf{F}(x,y,z) = a(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ be defined on $\mathbb{R}^3$. Let W be a simple solid region in $\mathbb{R}^3$ with boundary $\partial W = S$, a positively oriented closed surface. Find the value of $\alpha \in \mathbb{R}$, such that

   $$\int\int_S \mathbf{F}.\mathbf{dS} = \alpha \text{Volume}(W).$$

   Here the parameter a is given.

   Answer: Use Gauss' divergence theorem:

   $$\operatorname{div} \mathbf{F}(x,y,z) = 3a, \quad \forall (x,y,z) \in \mathbb{R}^3.$$

   Then,

   $$\int\int_S \mathbf{F}.\mathbf{dS} = 3a \int\int\int_W dxdydz = 3a\text{Volume}(W),$$

   so $\alpha = 3a$.

2. Let $\mathbf{F}(x,y,z) = \frac{-\alpha y}{\pi}\mathbf{i} + \frac{z^2}{\pi}\mathbf{k}$ be defined in $\mathbb{R}^3$. Evaluate $\int_C \mathbf{F}.\mathbf{ds}$, where C is curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$. The curve C is oriented counter clock-wise when viewed from above. Here the parameter α is given.

   Answer. The surface enclosed by C is

   $$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 1 \quad z = 2-y\}.$$

   The parametrization for S

   $$\phi(x,y) = (x,y,2-y), \quad \forall\,(x,y) \in \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}.$$

   Now $\phi_x(x,y) \times \phi_y(x,y) = (0,-1,1)$ and $\operatorname{curl} \mathbf{F}(x,y,z) = (0,0,\frac{\alpha}{\pi})$.

   The orientation of C is matching with the induced orientation due to the above choice of normal direction to S.

   Thus,

   $$\int_C \mathbf{F}.\mathbf{ds} = \int\int_S \operatorname{curl} \mathbf{F}.\mathbf{dS} = \int\int_{x^2+y^2\leq 1} \frac{\alpha}{\pi}\, dxdy = \alpha.$$

3. Let the surface S be the part of a cylinder defined by

   $$x^2+z^2=a^2, \quad z \geq 0, \quad 0 \leq y \leq h.$$

   Mark all correct statements below.

   (a) A parametrization of S can be defined by $\phi(u,v) = (a\cos u, v, a\sin u)$ for $u \in [0,2\pi]$, $v \in [0,h]$.

   (b) A parametrization of S can be defined by $\phi(u,v) = (a\cos u, v, a\sin u)$ for $u \in [0,\pi]$, $v \in [0,h]$.

   (c) A parametrization of S can be defined by $\phi(u,v) = (a\cos u, v, a\sin u)$ for $u \in [0,\frac{\pi}{2}]$, $v \in [0,h]$.

(d) Let $\mathbf{F}(x, y, z) = (\frac{-x}{a}, 0, \frac{-z}{a})$ be defined for all $(x, y, z) \in S$. The vector field $\mathbf{F}$ is an orientation to S.

(e) Let $\mathbf{F}(x, y, z) = (\frac{x}{a}, \frac{y}{a}, 0)$ be defined for all $(x, y, z) \in S$. The vector field $\mathbf{F}$ is an orientation to S.

(f) The plane $z = a$ is a tangent plane to S at $(0, 0, a)$.

Here parameters a and h are given.

Answer. b), d), f) are correct.