MA111 TUTORIAL 4

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QUESTION 1

1. Determine whether or not the given set is *a*) open, *b*) path-connected, and *c*) simply-connected.

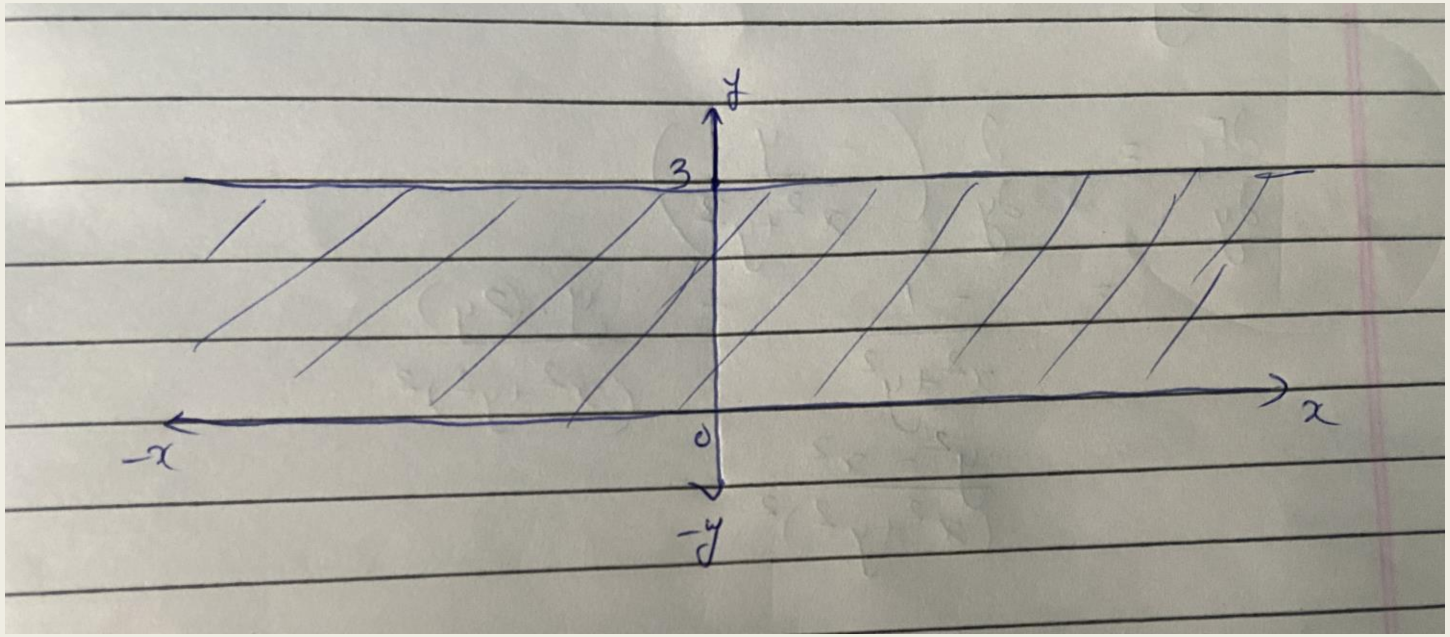
   (a) $D = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 3\}$,

   (c) $D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4, \quad y \geq 0\}$,

   (d) $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (1, 4)\}$.

Question 1.a

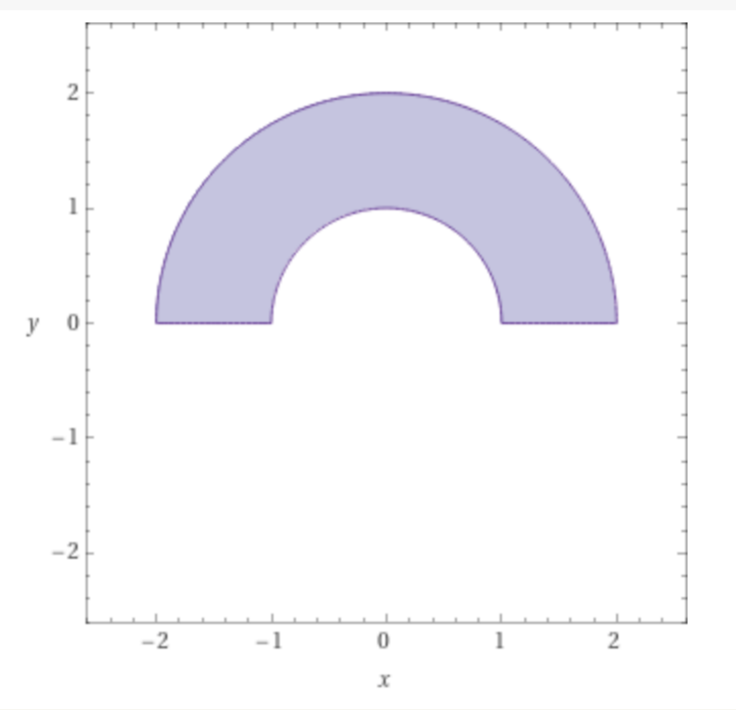
$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 3\}$$



- The set is an open set. (Show by taking a circle in the domain with suitable radius)
- Path connected since any 2 points can be joined by a path.
- Simply connected. Draw any closed curve C in D.

Question 1.c

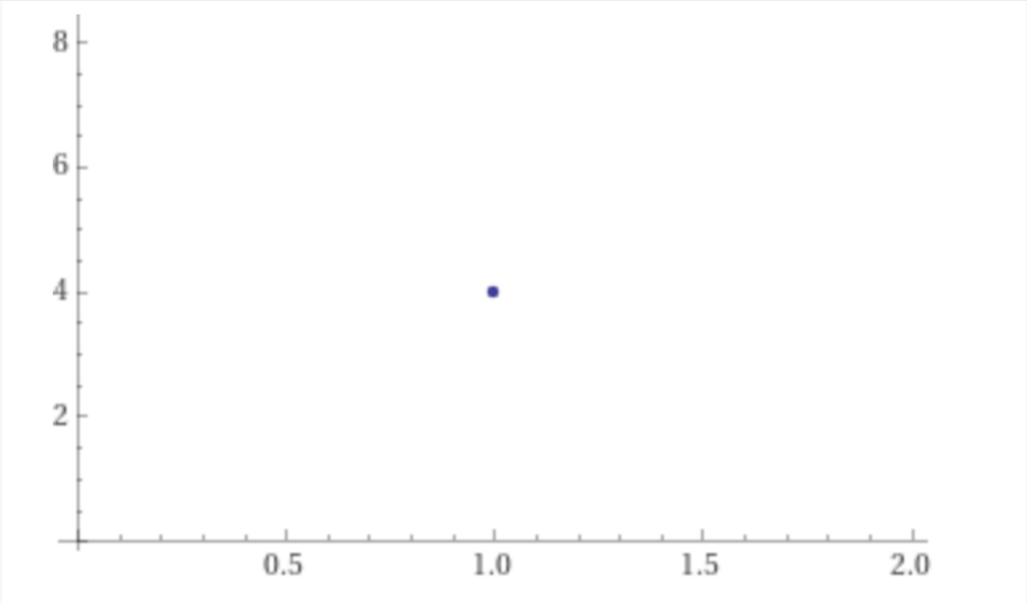
$$D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4, \quad y \geq 0\}$$



- The set is not an open set.
- Path connected since any 2 points can be joined by a path.
- Simply connected since any closed curve in D will have no holes.

Question 1.d

$$D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (1, 4)\}.$$



- The set is an open set.
- Path connected since any 2 points can be joined by a path.
- NOT Simply connected. Draw any closed curve enclosing (1,4).

QUESTION 3

3. Show that the line integral is path-independent and evaluate the integral:

$$\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$$

Question 3

3

$$I = \int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$$

C is any path from $\underset{(x_1, y_1)}{\overset{(a)}{(1,0)}}$ to $\underset{(x_2, y_2)}{\overset{(b)}{(2,1)}}$.

$F_1(x,y) = 2xe^{-y}$ $\qquad$ $F_2(x,y) = 2y - x^2e^{-y}$

$\dfrac{\partial F_1}{\partial y} = -2xe^{-y}$ $\qquad$ $\dfrac{\partial F_2}{\partial x} = -2xe^{-y}$

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

Hence $\vec{F}$ is a conservative vector field.

Question 3

$$\vec{F} = \vec{\nabla} \cdot f$$

$$\therefore F_1 = \frac{\partial f}{\partial x}.$$

$$\frac{\partial f}{\partial x} = +2xe^{-y}.$$

$$\therefore f(x,y) = +x^2 e^{-y} + g(y).$$

$$\frac{\partial f(x,y)}{\partial y} = -x^2 e^{-y} + g'(y). = 2y - x^2 e^{-y}.$$

$$\therefore g'(y) = 2y \qquad \therefore g(y) = y^2 + c.$$

$$\therefore f(x,y) = x^2 e^{-y} + y^2 + c$$

Question 3

$$\therefore f(x,y) = x^2 e^{-y} + y^2 + c$$

$$\therefore I = f(x_2, y_2) - f(x_1, y_1)$$

$$= (4e^{-1} + 1 + c) - (1 + 0 + c)$$

$$\boxed{I = \frac{4}{e}}$$

QUESTION 5

5. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy

$$\int_{C_1} \mathbf{F}.\mathbf{ds} = 0, \quad \int_{C_2} \mathbf{F}.\mathbf{ds} = 1.$$

Question 5

5. $F = \nabla f$, $f = \sin(x - 2y)$

~~If C_1 jo~~

F is obviously conservative.

If C_1 joins points P_1 and P_2,

$$\int_{C_1} F \cdot ds = f(P_2) - f(P_1)$$

Question 5

For $\int_{C_1} F \cdot ds = 0$, take any path joining two point on line $x = 2y$. For example, the path joining $(0,0)$ and $(2,1)$.

For $\int_{C_2} F \cdot ds = 1$, take the path from $(0,0)$ to $(\pi/2, 0)$.

QUESTION 6.A

6. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x, y) = y^2 e^{xy}\mathbf{i} + (1 + xy)e^{xy}\mathbf{j}$, for all $(x, y) \in \mathbb{R}^2$.

Question 6.a

6.a $\vec{F}(x,y) = y^2 e^{xy}\,\hat{i} + (1+xy)\,e^{xy}\,\hat{j} \quad \forall\, (x,y) \in R^2$

The domain R^2 is an open, connected & simply connected domain. Hence we can just ~~chekk~~ check if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \quad \text{on } R^2.$$

$F_1 = y^2 e^{xy}$ $\qquad$ $F_2 = (1+xy)\, e^{xy}$.

Question 6.a

$F_1 = y^2 e^{xy}$ $\qquad$ $F_2 = (1+xy)\, e^{xy}$.

$$\frac{\partial F_1}{\partial y} = 2y e^{xy} + xy^2 e^{xy}$$

$$\frac{\partial F_2}{\partial x} = y(1+xy)e^{xy} + y e^{xy}.$$

$$= 2y e^{xy} + xy^2 e^{xy}.$$

$$\therefore \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

$\therefore \vec{F}(x,y)$ is a conservative vector field.

Question 6.a

$$\vec{F} = \vec{\nabla} f.$$

$$\therefore \frac{\partial f}{\partial x} = F_1 = y^2 e^{xy}.$$

$$\therefore f(x,y) = \int \left(y^2 e^{xy}\right) dx$$

$$= y^2 \frac{e^{xy}}{y} + g(y)$$

$$\therefore f(x,y) = y e^{xy} + g(y).$$

$$\frac{\partial f}{\partial y} = F_2$$

$$\therefore xye^{xy} + e^{xy} + g'(y) = e^{xy} + xye^{xy}.$$

$$\therefore g'(y) = 0 \qquad g(y) = c.$$

$$\therefore \boxed{f(x,y) = ye^{xy} + c.}$$

QUESTION 7.A

7. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$. Find a function f such that $\mathbf{F} = \operatorname{grad} f$ and using it evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{ds}$, where $\mathbf{F}$ and $\mathbf{c}$ are given below:

(a) $\mathbf{F}(x, y, z) = (2xyz + \sin x)\mathbf{i} + x^2 z\mathbf{j} + x^2 y\mathbf{k}$ and $\mathbf{c}(t) = (\cos^5 t, \sin^3 t, t^4)$, $0 \leq t \leq \pi$.

Question 7.a

7.(a) $$F(x, y, z) = (2xyz + \sin x)\,\hat{i} + (x^2 z)\,\hat{j} + (x^2 y)\,\hat{k}$$

Let's try to guess f : $\nabla f = F$

$$\frac{\partial f}{\partial x} = 2xyz + \sin x, \quad \frac{\partial f}{\partial y} = x^2 z, \quad \frac{\partial f}{\partial z} = x^2 y.$$

$f = (-\cos x) + x^2 yz$. satisfies all three.

Question 7.a

$$c(t) = (\cos^5 t, \sin^3 t, t^4)$$

$$c(0) = (1, 0, 0)$$

$$c(1) = (-1, 0, \pi^4)$$

$$\int_c F \cdot ds = (-\cos(1) + 0 + \cos(-1) - 0)$$

$$= 2\cos(1)$$

QUESTION 9

9. Let $S = \mathbb{R}^2 \setminus \{(0,0)\}$. Let

$$\mathbf{F}(x, y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} := F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}.$$

(a) Show that $\frac{\partial}{\partial y}F_1(x, y) = \frac{\partial}{\partial x}F_2(x, y)$ on S.

(b) Compute $\int_C \mathbf{F}.\mathbf{ds}$ where C is the circle: $x^2 + y^2 = 1$.

Question 9

a show: $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial f_2}{\partial x}$ on S.

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{-1}{x^2+y^2} + \frac{y\cdot(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

Hence $\dfrac{\partial F_1(x,y)}{\partial y} = \dfrac{\partial F_2(x,y)}{\partial x}$ on S.

Question 9

b

$\oint_C \vec{F}\cdot d\vec{s}$ C is the circle $x^2+y^2=1$

$x = \cos t$ $y = \sin t$. $t \in [0, 2\pi]$

$$\therefore \oint_C \vec{F}\cdot d\vec{s} = \int_0^{2\pi}\left[F_1(t)\cdot\frac{dx}{dt} + F_2(t)\frac{dy}{dt}\right]dt.$$

$$= \int_0^{2\pi}\left(\underline{-\sin t}\cdot(-\sin t) + \cos t\cdot\cos t\right)dt$$

$$= \int_0^{2\pi} dt$$

$$\oint_C \vec{F}\cdot d\vec{s} = 2\pi \neq 0.$$

Integral on a closed curve is <u>non zero</u>.

Question 9

c From (b), clearly $\vec{F}$ is not path independent & hence NOT conservative.

Also, although $\frac{dF_1}{dy} = \frac{dF_2}{dx}$ in (a), the domain S is not simply connected. Hence we cannot conclude conservative from (a).

QUESTION 10

10. A radial force field is one which can be expressed as $\mathbf{F}(x, y, z) = f(r)\mathbf{r}$ where $\mathbf{r} = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ is the position vector and $r = \|\mathbf{r}\|$. Show that, if f is continuous, $\mathbf{F}$ is conservative in $\mathbb{R}^3$.

    (Hint. Try to guess what the potential function could be, provided f is continuous.)

Question 10

10. We need to show that F is conservative.
It is sufficient to show that there is a function g: ~~$\Delta g = F$~~. $\nabla g = F$

↳ Note that since we need to only show F is conservative, we can just prove existance of g, rather than calculating g in closed form.

↳ $f(s)$ is given to be continuous.

$\Rightarrow$ $s f(s)$ is also continuous.

Hence $s f(s)$ is integrable, i.e. $\exists \cancel{\forall} g(r)$:

$$g(r) = \int_0^r f(s) \cdot s \, ds.$$

↳ Hence F is conservative in $\mathbb{R}^3$.

Question 10

↳ Note that, if you want g in terms of cartesian coordinates, you can first evaluate $g(r) = \int_0^r s f(s)\,ds$, and the substitute $r = \sqrt{x^2+y^2+z^2}$.

Or you can directly write

$$\cancel{g(x,y,z) = \int_0^{\sqrt{x^2+y^2}+z^2} s\ldots}$$

$$g(x,y,3) = \int_0^{\sqrt{x^2+y^2+z^2}} s f(s) . ds .$$