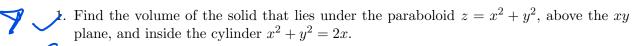
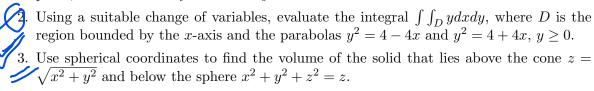
MA111 (IIT Bombay) Tutorial Sheet 3: Change of variables, Line integrals, January 29, 2021

I Multiple integrals and change of variables





- 4. Use cylindrical coordinates to evaluate $\int \int \int_W (x^2+y^2) dz dy dx$, where

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid -2 \le x \le 2, \quad -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}, \quad \sqrt{x^2 + y^2} \le z \le 2\}.$$

✓5. Describe the solid whose volume is given by the integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin \phi d\rho d\phi d\theta, \qquad \qquad$$

and evaluate the integral.

6 Find $\iiint_F \frac{1}{(x^2+y^2+z^2)^{n/2}}dV$, where F is the region bounded by the spheres with center the origin and radii r and R, 0 < r < R.

II Vector analysis and line integrals

1. Let f, g be differentiable functions on \mathbb{R}^2 . Show that

i.
$$\nabla (fg) = f\nabla g + g\nabla f$$
;

ii.
$$\nabla f^n = n f^{n-1} \nabla f$$
;

iii.
$$\nabla (f/g) = (g\nabla f - f\nabla g)/g^2$$
 whenever $g \neq 0$.

- 2. Let \mathbf{a}, \mathbf{b} be two fixed vectors, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:
 - (i) $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for any integer n.

(ii)
$$\mathbf{a} \cdot \nabla \left(\frac{1}{r} \right) = -\left(\frac{\mathbf{a} \cdot \mathbf{r}}{r^3} \right)$$
.

(iii)
$$\mathbf{b} \cdot \nabla \left(\mathbf{a} \cdot \nabla \left(\frac{1}{r} \right) \right) = \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}.$$

3. Calculate the line integral of the vector field

$$\mathbf{F}(x,y) = (x^2 - 2xy)\mathbf{i} + (y^2 - 2xy)\mathbf{j}$$

from (-1,1) to (1,1) along $y=x^2$.

4. Calculate the line integral of

$$\mathbf{F}(x,y) = (x^2 + y^2)\mathbf{i} + (x - y)\mathbf{j}$$

once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.

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Remark Often line integral of a vector field \mathbf{F} along a 'geometric curve' C is represented by $\int_C \mathbf{F}.\mathbf{ds}$. A geometric curve C is a set of points in the plane or in the space that can be traversed by a parametrized path in the given direction.

To evaluate $\int_C \mathbf{F.ds}$, choose a convenient parametrization \mathbf{c} of C traversing C in the given direction and then

$$\int_C \mathbf{F}.\mathbf{ds} := \int_C \mathbf{F}.\mathbf{ds}.$$

' \oint_C ' means the line integral over a closed curve C.

5. Calculate the value of the line integral

$$\oint_C \frac{(x+y)dx - (x-y)dy}{x^2 + y^2}$$

where C is the curve $x^2 + y^2 = a^2$ traversed once in the counter clockwise direction.

6. Calculate

$$\oint_C ydx + zdy + xdz$$

where C is the intersection of two surfaces z = xy and $x^2 + y^2 = 1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.

- 7. Let the curve C be given by $x^2 + y^2 = 1, z = 0$. Let \mathbf{c}_1 be a parametrization defined by $\mathbf{c}_1(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. Find the line integral of $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$ along this curve. Also find the line integral along the curve parametrized by $\mathbf{c}_2(t) = (\cos t, -\sin t)$, for $t \in [0, \pi]$.
- 8. Show that a constant force field does zero work on a particle that moves once uniformly around the circle: $x^2+y^2=1$. Is this also true for a force field $\mathbf{F}(x,y,z)=\alpha(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$, for some constant α .
 - 9. Let $C: x^2 + y^2 = 1$. Find

$$\oint_C \operatorname{grad}(x^2 - y^2) \cdot \mathbf{ds}.$$

10. Evaluate

$$\int_C \operatorname{grad}(x^2 - y^2) \cdot \mathbf{ds},$$

where C is $y = x^3$.

1. Compute the line integral

$$\oint_C \frac{dx + dy}{|x| + |y|}$$

where C is the square with vertices (1,0),(0,1),(-1,0) and (0,-1) traversed once in the counter clockwise direction.

12. A force $F = xy\mathbf{i} + x^6y^2\mathbf{j}$ moves a particle from (0,0) onto the line x = 1 along $y = ax^b$ where a, b > 0. If the work done is independent of b find the value of a.

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