MA 111 Tut 1
Integrals of Dirichlet and Thomae Functions

January 2021

1 DIRICHLET FUNCTION

The Dirichlet function is defined as

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Is it integrable in $x \in [a, b]$?

Solution:

Consider any partition

$$\mathcal{P}_n = \{x_0, x_1, \ldots, x_n\}$$

where $a = x_0 < x_1 < \cdots < x_n = b$. Define

$$m_j = \inf_{x \in [x_{j-1}, x_j]} f(x) \quad \text{and} \quad M_j = \sup_{x \in [x_{j-1}, x_j]} f(x)$$

Note that $[x_{j-1}, x_j]$ contains infinitely many rationals and infinitely many irrationals. Thus,

$$m_j = 0 \quad \text{and} \quad M_j = 1 \quad \forall\, j \in \{1, 2, \ldots, n\}$$

Now,

$$L(f, \mathcal{P}_n) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}) = \sum_{j=1}^{n} 0 \times (x_j - x_{j-1}) = 0$$

$$U(f, \mathcal{P}_n) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}) = \sum_{j=1}^{n} 1 \times (x_j - x_{j-1}) = \sum_{j=1}^{n} (x_j - x_{j-1}) = x_n - x_0 = b - a$$

This is true for all partitions $\mathcal{P}_n$. That is,

$$U(f, \mathcal{P}_n) = b - a \quad \text{and} \quad L(f, \mathcal{P}_n) = 0 \quad \forall\, \mathcal{P}_n$$

This gives

$$U(f, \mathcal{P}) = \sup_{\mathcal{P}_n} \{U(f, \mathcal{P}_n)\} = \sup_{\mathcal{P}_n} \{b - a\} = b - a$$
$$L(f, \mathcal{P}) = \inf_{\mathcal{P}_n} \{U(f, \mathcal{P}_n)\} = \inf_{\mathcal{P}_n} \{0\} = 0$$
$$U(f, \mathcal{P}) \neq L(f, \mathcal{P})$$

Hence, **the Dirichlet Function is <u>NOT INTEGRABLE</u>**.

2 THOMAE FUNCTION

The Thomae function is defined as

$$f(x) = \begin{cases} \dfrac{1}{q} & x = \dfrac{p}{q} \in \mathbb{Q},\ p, q \in \mathbb{N},\ gcd(p, q) = 1 \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Is it integrable in $x \in [a, b]$?

<u>**Solution:**</u>
Consider some $\epsilon > 0$. Let

$$\mathcal{S} = \{x \mid f(x) > \epsilon, a \leq x \leq b\}$$

Note that <u>$\mathcal{S}$ has finitely many elements</u>. This is because, $f(x) > \epsilon \Rightarrow x \in \mathbb{Q}$ and $\frac{1}{q} > \epsilon \Rightarrow q < \frac{1}{\epsilon}, q \in \mathbb{N}$. Thus, q can have only finitely many values. For a given q, p can take only finitely many values, since $a \leq x \leq b \Rightarrow a \leq \frac{p}{q} \leq b \Rightarrow aq \leq p \leq bq, p \in \mathbb{N}$. Let the number of elements in $\mathcal{S}$ be N. Consider the partition

$$\mathcal{P}_\epsilon = \{x_0, x_1, \ldots, x_n\}$$

where $a = x_0 < x_1 < \cdots < x_n = b$, so that

$$x_j - x_{j-1} < \frac{\epsilon}{N} \ \forall\ j \in \{1, 2, \ldots, n\} \quad \text{that is } ||\mathcal{P}_\epsilon|| < \frac{\epsilon}{N}$$

Since $\mathcal{S}$ has only N elements, they belong to at-most $2N$ of the intervals $[x_{j-1}, x_j],\ j \in \{1, 2, \ldots, n\}$.
Define

$$m_j = \inf_{x \in [x_{j-1}, x_j]} f(x) \quad \text{and} \quad M_j = \sup_{x \in [x_{j-1}, x_j]} f(x)$$

Note that $[x_{j-1}, x_j]$ contains infinitely many irrationals and so $m_j = 0\ \forall\ j \in \{1, 2, \ldots, n\}$. Now,

$$U(f, \mathcal{P}_\epsilon) - L(f, \mathcal{P}_\epsilon) = \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} M_j \times (x_j - x_{j-1})$$

Now, we divide the this $U(f, \mathcal{P}_\epsilon) - L(f, \mathcal{P}_\epsilon)$ into two parts, one that contains j such that $[x_{j-1}, x_j]$ contains some $x \in \mathcal{S}$, i.e., $[x_{j-1}, x_j] \cap \mathcal{S} \neq \emptyset$ and the other that satisfies $[x_{j-1}, x_j] \cap \mathcal{S} = \emptyset$. This gives

$$U(f, \mathcal{P}_\epsilon) - L(f, \mathcal{P}_\epsilon) = \sum_{\substack{j=1 \\ [x_{j-1}, x_j] \cap \mathcal{S} \neq \emptyset}}^{n} (M_j)(x_j - x_{j-1}) + \sum_{\substack{j=1 \\ [x_{j-1}, x_j] \cap \mathcal{S} = \emptyset}}^{n} (M_j)(x_j - x_{j-1})$$

If $x \in [x_{j-1}, x_j]$, where$[x_{j-1}, x_j] \cap \mathcal{S} = \emptyset$, then $f(x) < \epsilon \ \forall \ x \in [x_{j-1}, x_j] \Rightarrow M_j < \epsilon$. Also, in general, $M_j < 1$, as $M_j = \dfrac{1}{q}, \ q \in \mathbb{N}$.

$$\text{Consider the first part of the sum} \quad \sum_{\substack{j=1 \\ [x_{j-1}, x_j] \cap \mathcal{S} \neq \emptyset}}^{n} (M_j)(x_j - x_{j-1})$$

This set contains at-most $2N$ values of j. Also, $M_j < 1$ and $x_j - x_{j-1} < \frac{\epsilon}{N}$. Thus

$$\sum_{\substack{j=1 \\ [x_{j-1}, x_j] \cap \mathcal{S} \neq \emptyset}}^{n} (M_j)(x_j - x_{j-1}) < \sum_{\substack{j=1 \\ [x_{j-1}, x_j] \cap \mathcal{S} \neq \emptyset}}^{n} (x_j - x_{j-1}) < (2N)\left(\frac{\epsilon}{N}\right) < 2\epsilon \quad (1)$$

$$\text{Consider the second part of the sum} \quad \sum_{\substack{j=1 \\ [x_{j-1}, x_j] \cap \mathcal{S} = \emptyset}}^{n} (M_j)(x_j - x_{j-1})$$

For this set, $M_j < \epsilon$. Thus

$$\sum_{\substack{j=1 \\ [x_{j-1}, x_j] \cap \mathcal{S} = \emptyset}}^{n} (M_j)(x_j - x_{j-1}) < \sum_{\substack{j=1 \\ [x_{j-1}, x_j] \cap \mathcal{S} = \emptyset}}^{n} \epsilon \times (x_j - x_{j-1}) < \epsilon(x_n - x_0) = \epsilon(b-a) \quad (2)$$

??+**??** gives

$$U(f, \mathcal{P}_\epsilon) - L(f, \mathcal{P}_\epsilon) = \sum_{\substack{j=1 \\ [x_{j-1}, x_j] \cap \mathcal{S} \neq \emptyset}}^{n} (M_j)(x_j - x_{j-1}) + \sum_{\substack{j=1 \\ [x_{j-1}, x_j] \cap \mathcal{S} = \emptyset}}^{n} (M_j)(x_j - x_{j-1}) < 2\epsilon + (b-a)\epsilon < (b-a+2)\epsilon$$

This can be made arbitrarily small for smaller and smaller values of ϵ.
Hence, **the Thomae Function is <u>INTEGRABLE</u>**.