MA111 (IIT Bombay) Tutorial Sheet 3 : Change of variables, Line integrals, January 29, 2021

I Multiple integrals and change of variables

1. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy plane, and inside the cylinder $x^2 + y^2 = 2x$.
2. Using a suitable change of variables, evaluate the integral $\int\int_D y\,dxdy$, where D is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.
3. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.
4. Use cylindrical coordinates to evaluate $\int\int\int_W (x^2 + y^2)dzdydx$, where

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid -2 \leq x \leq 2, \quad -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, \quad \sqrt{x^2 + y^2} \leq z \leq 2\}.$$

5. Describe the solid whose volume is given by the integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin\phi \, d\rho d\phi d\theta,$$

and evaluate the integral.

6. Find $\iiint_F \dfrac{1}{(x^2 + y^2 + z^2)^{n/2}} dV$, where F is the region bounded by the spheres with center the origin and radii r and R, $0 < r < R$.

II Vector analysis and line integrals

1. Let f, g be differentiable functions on $\mathbb{R}^2$. Show that
   i. $\nabla(fg) = f\nabla g + g\nabla f$;
   ii. $\nabla f^n = nf^{n-1}\nabla f$;
   iii. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.
2. Let $\mathbf{a}, \mathbf{b}$ be two fixed vectors, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:
   (i) $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for any integer n.
   (ii) $\mathbf{a} \cdot \nabla \left(\dfrac{1}{r}\right) = -\left(\dfrac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right)$.
   (iii) $\mathbf{b} \cdot \nabla \left(\mathbf{a} \cdot \nabla \left(\dfrac{1}{r}\right)\right) = \dfrac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \dfrac{\mathbf{a} \cdot \mathbf{b}}{r^3}$.
3. Calculate the line integral of the vector field

$$\mathbf{F}(x, y) = (x^2 - 2xy)\mathbf{i} + (y^2 - 2xy)\mathbf{j}$$

from $(-1, 1)$ to $(1, 1)$ along $y = x^2$.

4. Calculate the line integral of

$$\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + (x - y)\mathbf{j}$$

once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.

Remark Often line integral of a vector field $\mathbf{F}$ along a 'geometric curve' C is represented by $\int_C \mathbf{F}.\mathbf{ds}$. A geometric curve C is a set of points in the plane or in the space that can be traversed by a parametrized path in the given direction.

To evaluate $\int_C \mathbf{F}.\mathbf{ds}$, choose a convenient parametrization $\mathbf{c}$ of C traversing C in the given direction and then

$$\int_C \mathbf{F}.\mathbf{ds} := \int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}.$$

'$\oint_C$' means the line integral over a closed curve C.

5. Calculate the value of the line integral

$$\oint_C \frac{(x+y)dx - (x-y)dy}{x^2+y^2}$$

where C is the curve $x^2 + y^2 = a^2$ traversed once in the counter clockwise direction.

6. Calculate

$$\oint_C ydx + zdy + xdz$$

where C is the intersection of two surfaces $z = xy$ and $x^2 + y^2 = 1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.

7. Let the curve C be given by $x^2 + y^2 = 1, z = 0$. Let $\mathbf{c}_1$ be a parametrization defined by $\mathbf{c}_1(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. Find the line integral of $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$ along this curve. Also find the line integral along the curve parametrized by $\mathbf{c}_2(t) = (\cos t, -\sin t)$, for $t \in [0, \pi]$.

8. Show that a constant force field does zero work on a particle that moves once uniformly around the circle: $x^2+y^2 = 1$. Is this also true for a force field $\mathbf{F}(x, y, z) = \alpha(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$, for some constant α.

9. Let $C : x^2 + y^2 = 1$. Find

$$\oint_C \text{grad } (x^2 - y^2) \cdot \mathbf{ds}.$$

10. Evaluate

$$\int_C \text{grad } (x^2 - y^2) \cdot \mathbf{ds},$$

where C is $y = x^3$.

11. Compute the line integral

$$\oint_C \frac{dx + dy}{|x| + |y|}$$

where C is the square with vertices $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$ traversed once in the counter clockwise direction.

12. A force $F = xy\mathbf{i} + x^6y^2\mathbf{j}$ moves a particle from $(0, 0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b find the value of a.