# MA 111
# Tutorial 3 Solutions

D1-T5

GYANDEV GUPTA

February 3, 2021

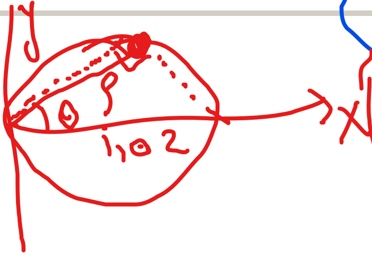
IIT BOMBAY

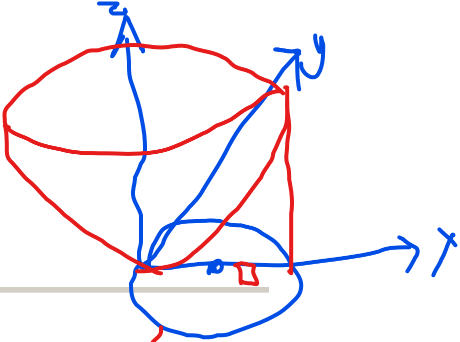
# Outline

$$\iint z \, dx \, dy$$

# QUESTION I

$$\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} \rho^3 \, d\rho \, d\theta$$




$x = 1 + \rho\cos\theta$

$y = \rho\sin\theta$

$0 \le \rho \le 1$

$0 \le \theta \le 2\pi$

# QUESTION I

## QUESTION I1

# QUESTION I1

$x^2 + y^2 = 2x \Rightarrow (x-1)^2 + y^2 = 1$ The region inside the cylinder $\{(x, y) \mid x^2 + y^2 = 2x$ can easily be parametrized as

# QUESTION I1

$x^2 + y^2 = 2x \Rightarrow (x-1)^2 + y^2 = 1$ The region inside the cylinder $\{(x, y) \mid x^2 + y^2 = 2x$ can easily be parametrized as

$$\begin{cases} x = 1 + \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \tag{1}$$

# QUESTION I1

$x^2 + y^2 = 2x \Rightarrow (x-1)^2 + y^2 = 1$ The region inside the cylinder $\{(x, y) \mid x^2 + y^2 = 2x$ can easily be parametrized as

$$\begin{cases} x = 1 + \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \tag{1}$$

This gives

$$z = x^2 + y^2 \Rightarrow z = (1 + \rho\cos\theta)^2 + (\rho\sin\theta)^2 = 1 + 2\rho\cos\theta + \rho^2 \tag{2}$$

# QUESTION I1

$x^2 + y^2 = 2x \Rightarrow (x-1)^2 + y^2 = 1$ The region inside the cylinder $\{(x, y) \mid x^2 + y^2 = 2x$ can easily be parametrized as

$$\begin{cases} x = 1 + \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \tag{1}$$

This gives

$$z = x^2 + y^2 \Rightarrow z = (1 + \rho\cos\theta)^2 + (\rho\sin\theta)^2 = 1 + 2\rho\cos\theta + \rho^2 \tag{2}$$

Thus, our region is given by $\mathcal{D} = \{z(\rho, \theta) \mid 0 \leq z \leq 1 + 2\rho\cos\theta + \rho^2\}$.

# QUESTION I1

$x^2 + y^2 = 2x \Rightarrow (x-1)^2 + y^2 = 1$ The region inside the cylinder $\{(x, y) \mid x^2 + y^2 = 2x$ can easily be parametrized as

$$\begin{cases} x = 1 + \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \tag{1}$$

This gives

$$z = x^2 + y^2 \Rightarrow z = (1 + \rho\cos\theta)^2 + (\rho\sin\theta)^2 = 1 + 2\rho\cos\theta + \rho^2 \tag{2}$$

Thus, our region is given by $\mathcal{D} = \{z(\rho, \theta) \mid 0 \leq z \leq 1 + 2\rho\cos\theta + \rho^2\}$.

The volume is thus

$$V = \iint\limits_{\mathcal{D}} z(x, y)\, dx\, dy = \iint\limits_{\mathcal{D}} z(\rho, \theta) \left|J_{x,y}(\rho, \theta)\right|\, d\rho\, d\theta \tag{3}$$

# QUESTION I1 (Contd.)

Invoking the Jacobian

$$J_{xy}(\rho, \theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

# QUESTION I1 (Contd.)

Invoking the Jacobian

$$J_{xy}(\rho, \theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{vmatrix} = \rho \tag{4}$$

# QUESTION I1 (Contd.)

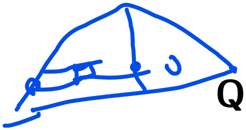
Invoking the Jacobian

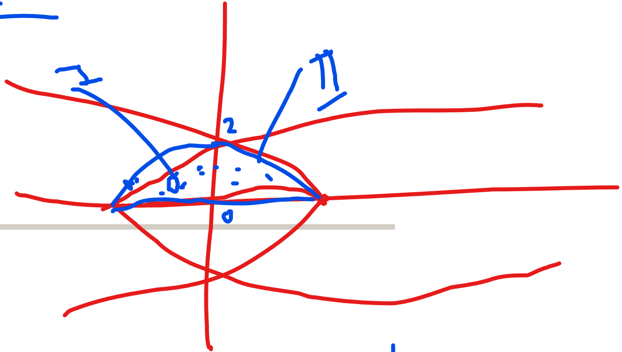
$$J_{xy}(\rho,\theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{vmatrix} = \rho \tag{4}$$

Equation 3 becomes

$$V = \iint\limits_{\mathcal{D}} z(\rho,\theta)\,|J_{x,y}(\rho,\theta)|\ d\rho\, d\theta = \int_0^{2\pi}\int_0^1 (1+2\rho\cos\theta+\rho^2)(\rho)\,d\rho\,d\theta = \boxed{\frac{3\pi}{2}}$$

$x = u^2 - v^2 \quad y = 2uv$






# QUESTION I

## QUESTION I2

$\frac{y^2}{4} - 1$

$y = \sqrt{4(-x+1)}$

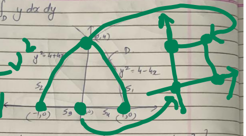
$$-\int_0^2 \int_{\frac{y^2}{4}-1}^{0} y \, dx \, dy + \int_0^1 -4(x-1) \, dx$$

I. 2 $\quad I = \iint_D y \, dx \, dy$

$D =$

$x = u^2 - v^2$

$y = 2uv$




$$D = \left\{ (x,y) \mid 0 \le y \le 2, \ \frac{y^2-4}{4} \le x \le \frac{4-y^2}{4} \right\}$$

Changing variables :- $\quad x = u^2 - v^2$

$\qquad\qquad\qquad\qquad y = 2uv$

Why? Let's look at the boundaries

$S_1$: $\quad 0 \le y \le 2 \qquad x = \frac{4-y^2}{4} = 1 - \frac{y^2}{4}$

$\quad \hookrightarrow x = 1 - v^2 \qquad y = 2v \qquad v \in [0,1]$

$S_2$: $\quad 0 \le y \le 2 \qquad x = -\left(\frac{4-y^2}{4}\right)$

$\quad x = u^2 - 1 \qquad y = 2u \qquad u \in [0,1]$

Guessing $\quad x = u^2 - v^2 \qquad y = 2uv$ from above

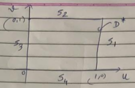
Motto? To obtain a square / rectangle in u-v plane.

$x^2 =$

# QUESTION I2 Contd

$$\int_0^{2\pi}\int^{c} \alpha\cos\theta + r^3\sin\theta$$

$$= 0$$




~~$D^* = [0,1]$~~

$D^* = [0,1] \times [0,1]$

$$J = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$$

$|J| = 4u^2 + 4v^2$

$\therefore dx\,dy = 4(u^2+v^2)\,du\,dv$

$I = \iint_D y\,dx\,dy = \iint_{D^*} 2uv \times 4(u^2+v^2)\,du\,dv$

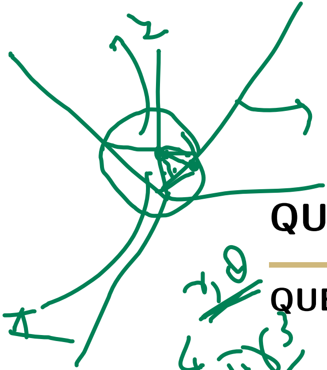
$I = 8\int_0^1\int_0^1 (u^3v + uv^3)\,du\,dv$

$I = 8\int_0^1 \left(\frac{v}{4} + \frac{v^3}{2}\right) dv$

$= 8\left(\frac{1}{8} + \frac{1}{8}\right)$

$= 2$

$\boxed{I = 2}$



$z = \sqrt{x^2+y^2}$

# QUESTION I

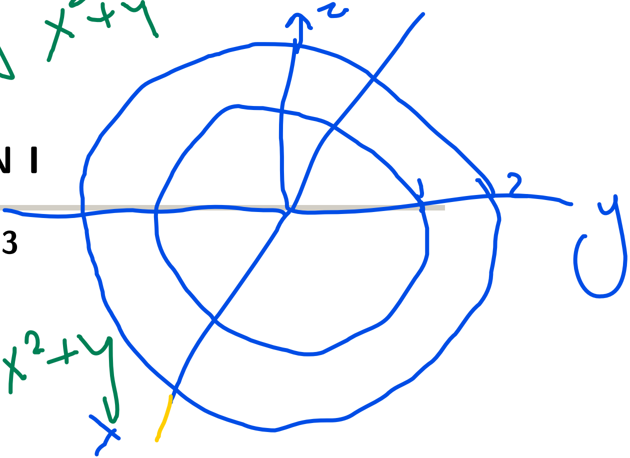
$\cancel{r, \theta}$

## QUESTION I3

$\frac{4}{3}\pi r^3$

2

$x^2+y$




$\int\int x^2+y^2 \, dx \, dy$

# QUESTION I3

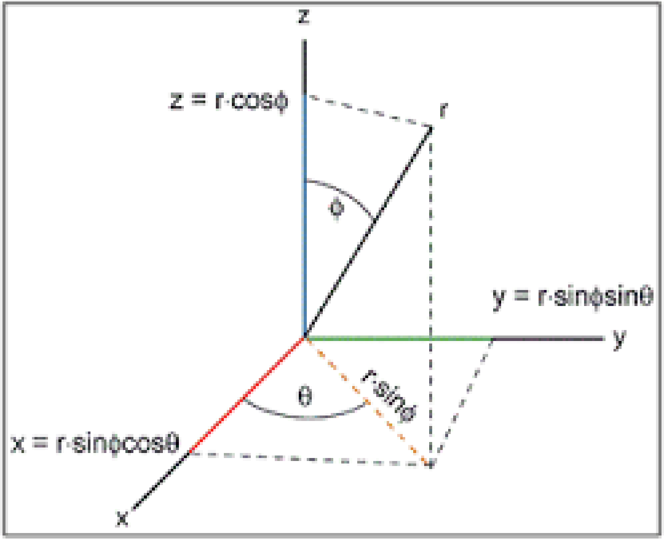
# QUESTION I

## QUESTION I4

# QUESTION I4

# QUESTION I

## QUESTION I5




Standard Spherical Coordinates

# QUESTION I5

Let's first look at how limits are setup in standard spherical coordinates:

$$\theta \in [0, 2\pi],\ \phi \in [0, \pi]$$

Comparing these to the values in the question, we see: $\theta \in [0, \pi/2]$, $\phi \in [0, \pi/2]$ $\implies$ that on the $xy$ plane, position vector only moves in the first quadrant (determined by limits of $\theta$) while the revolution about $z$ only occurs above the $xy$ plane (determined by limits of $\phi$). Thus, our solid lies in the first octant (i.e $x, y, z \geq 0$).
Observe that $\rho \in [1, 2] \implies 1 \leq x^2 + y^2 + z^2 \leq 4$

# QUESTION I

## QUESTION I6

Applying spherical coordinates ($\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$, $\rho \in [r, R]$) and incorporating the Jacobian, we get:

$$\iiint_F \frac{1}{(x^2+y^2+z^2)^{\frac{n}{2}}}\, dV = \int_0^{\pi}\int_0^{2\pi}\int_r^{R} \frac{\rho^2 sin(\phi)}{\rho^{\frac{2*n}{2}}}\, d\rho\, d\theta\, d\phi = 4\pi[\frac{R^{3-n}-r^{3-n}}{3-n}]$$

$n \neq 3$

What will be the solution when $n = 3$?

$$\int\int\int_r^R \frac{dr}{r}\, d\theta\, d\phi \quad \ln(r)$$

# QUESTION II

# QUESTION II

## QUESTION II1

# QUESTION II1

# QUESTION II

## QUESTION II2

# QUESTION II2

# QUESTION II

## QUESTION II3

# QUESTION II3

# QUESTION II

## QUESTION II4

# QUESTION II4

# QUESTION II

## QUESTION II5

# QUESTION II5

# QUESTION II

## QUESTION II6

# QUESTION II6

$$\int y\,dx + z\,dy + x\,dz \qquad \theta \in [0, 2\pi)$$

$$x = \cos\theta \quad y = \sin\theta$$

$$z = \cos\theta\sin\theta$$

# QUESTION II

## QUESTION II7

# QUESTION II7

# QUESTION II

## QUESTION II8

$$\int \vec{F} \cdot dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\vec{F} = \Delta G$$
$$= \frac{\alpha}{2}(x^2 + y^2 + z^2)$$

$$= G(f) - G(i)$$
$$= 0$$

Date ______ Page ______

8. Let the force field be $\alpha\hat{i} + \beta\hat{j} + r\hat{k}$, where $\alpha, \beta, r$ are constants.

Work done $= \int_C F \cdot ds$

$C = x^2 + y^2 = 1$ (assume $z = 0$)

Parameterize $C$ with $\theta$.

$x = \cos\theta,\ y = \sin\theta \quad ; \theta \in [0, 2\pi]$

$$\text{Work done} = \int_0^{2\pi} F(c(\theta)) \cdot c'(\theta) \cdot d\theta$$
$$= \int_0^{2\pi} (\alpha\hat{i} + \beta\hat{j} + r\hat{k}) \cdot (\cos\theta\hat{i} + \sin\theta\hat{j}) \cdot d\theta$$
$$= \int_0^{2\pi} (\alpha\cos\theta + \beta\sin\theta) \cdot d\theta$$
$$= 0$$

---

Is it true for $F(x, y, z) = \alpha(x\hat{i} + y\hat{j} + z\hat{k})$

↳ $F$ is a conservative force field, as

$$F = \Delta G$$
$$\text{where } G = \frac{\alpha}{2}(x^2 + y^2 + z^2)$$

So, work done will be zero.

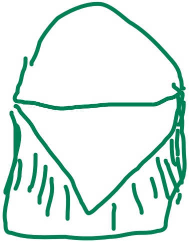
# QUESTION II

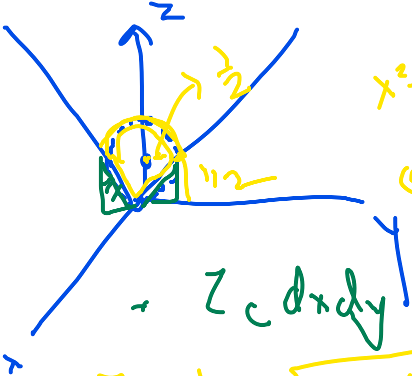
## QUESTION II9

$z_s \qquad z_c$

$x^2+y^2 = \frac{1}{4}$

$\theta = \pi/4$






$\int\int z_s \, \cancel{dx\,dy}$

$z_c \, dx\,dy$

$z = \frac{1}{2} + \sqrt{x^2+y^2}$

$\int\int$

# QUESTION II

## QUESTION II10

$\left(2x - 2y\right) \cdot dx\hat{i} + dy\hat{j}$

$(t, t^3) \quad \int 2x\,dx - 2y\,dy$

$$\sum \frac{\partial fg}{\partial x_i} \qquad \sum_{i=1}^{3} \frac{\partial fg}{\partial x_i} \qquad x_i \quad x_1 = x \quad x_2 = y \quad x_3 = z$$

$$f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i}$$

$$f\nabla g + g\nabla f$$

$$|x| + |y| = 1$$

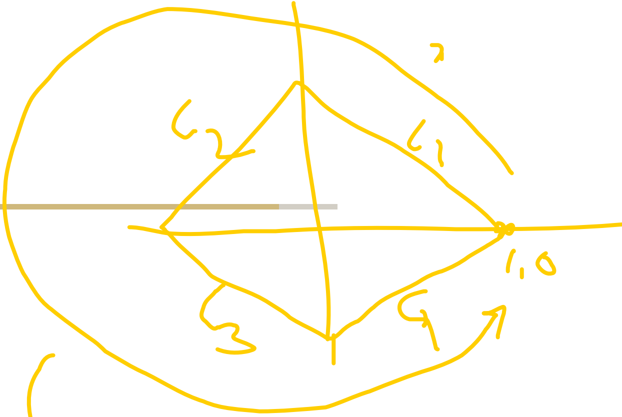
# QUESTION II

$$x + y = 1$$

$$-x + y = 1$$

## QUESTION II11

$$\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$



# QUESTION II11

The required integral can be broken down into individual integral by defining it into separate lines :

The required integral can be broken down into individual integral by defining it into separate lines : $\int_{C_1} \frac{dx+dy}{|x|+|y|} + \int_{C_2} \frac{dx+dy}{|x|+|y|} + \int_{C_3} \frac{dx+dy}{|x|+|y|} + \int_{C_4} \frac{dx+dy}{|x|+|y|}$

# QUESTION II11

The required integral can be broken down into individual integral by defining it into separate lines : $\int_{C_1} \frac{dx+dy}{|x|+|y|} + \int_{C_2} \frac{dx+dy}{|x|+|y|} + \int_{C_3} \frac{dx+dy}{|x|+|y|} + \int_{C_4} \frac{dx+dy}{|x|+|y|}$

For $C_1$ : x+y=1 and $|x| + |y| = x + y = 1$

$\int_{C_1} \frac{dx+dy}{|x|+|y|} = \int_1^0 \frac{dx}{1}$ - $\int_1^0 \frac{dx}{1} = 0$

# QUESTION II11

The required integral can be broken down into individual integral by defining it into separate lines : $\int_{C_1} \frac{dx+dy}{|x|+|y|} + \int_{C_2} \frac{dx+dy}{|x|+|y|} + \int_{C_3} \frac{dx+dy}{|x|+|y|} + \int_{C_4} \frac{dx+dy}{|x|+|y|}$

For $C_1$ : x+y=1 and $|x| + |y| = x + y = 1$

$\int_{C_1} \frac{dx+dy}{|x|+|y|} = \int_1^0 \frac{dx}{1}$ - $\int_1^0 \frac{dx}{1} = 0$

For $C_2$ : -x+y=1 and $|x| + |y| = -x + y = 1$

$\int_{C_2} \frac{dx+dy}{|x|+|y|} = \int_0^{-1} \frac{dx}{1} + \int_0^{-1} \frac{dx}{1} =$ -2

The required integral can be broken down into individual integral by defining it into separate lines : $\int_{C_1} \frac{dx+dy}{|x|+|y|} + \int_{C_2} \frac{dx+dy}{|x|+|y|} + \int_{C_3} \frac{dx+dy}{|x|+|y|} + \int_{C_4} \frac{dx+dy}{|x|+|y|}$

For $C_1$ : x+y=1 and $|x| + |y| = x + y = 1$

$\int_{C_1} \frac{dx+dy}{|x|+|y|} = \int_1^0 \frac{dx}{1}$ - $\int_1^0 \frac{dx}{1} = 0$

For $C_2$ : -x+y=1 and $|x| + |y| = -x + y = 1$

$\int_{C_2} \frac{dx+dy}{|x|+|y|} = \int_0^{-1} \frac{dx}{1} + \int_0^{-1} \frac{dx}{1} =$ -2

For $C_3$ : x+y=-1 and $|x| + |y| = -x - y = 1$

$\int_{C_1} \frac{dx+dy}{|x|+|y|} = \int_{-1}^0 \frac{dx}{1}$ - $\int_{-1}^0 \frac{dx}{1} = 0$

# QUESTION II11

The required integral can be broken down into individual integral by defining it into separate lines : $\int_{C_1} \frac{dx+dy}{|x|+|y|} + \int_{C_2} \frac{dx+dy}{|x|+|y|} + \int_{C_3} \frac{dx+dy}{|x|+|y|} + \int_{C_4} \frac{dx+dy}{|x|+|y|}$

For $C_1$ : x+y=1 and $|x| + |y| = x + y = 1$

$\int_{C_1} \frac{dx+dy}{|x|+|y|} = \int_1^0 \frac{dx}{1}$ - $\int_1^0 \frac{dx}{1} = 0$

For $C_2$ : -x+y=1 and $|x| + |y| = -x + y = 1$

$\int_{C_2} \frac{dx+dy}{|x|+|y|} = \int_0^{-1} \frac{dx}{1} + \int_0^{-1} \frac{dx}{1} =$ -2

For $C_3$ : x+y=-1 and $|x| + |y| = -x - y = 1$

$\int_{C_1} \frac{dx+dy}{|x|+|y|} = \int_{-1}^0 \frac{dx}{1}$ - $\int_{-1}^0 \frac{dx}{1} = 0$

For $C_4$ : x-y=1 and $|x| + |y| = x - y = 1$

$\int_{C_1} \frac{dx+dy}{|x|+|y|} = \int_0^1 \frac{dx}{1} + \int_0^1 \frac{dx}{1} = 2$

## QUESTION II11 (Contd.)

Hence $\boxed{\int_C \frac{dx + dy}{|x| + |y|} = 0 - 2 + 0 + 2 = 0}$

# QUESTION II

## QUESTION II12

$\frac{dW}{dh} = 0$ $\quad a =$

$(xy\,\hat{i} + x^2 y^3\,\hat{j})$

$x = t \quad y = t^b$

$\int xy\,dx + x^2 y^2\,dy$

$0 \le t \le 1$

$W =$

# QUESTION II12

Recall that the line line integral of a Field **F** along a path $c(t)$ is given by : $\int_C F(c(t))c'(t)dt$. Here $c(t) = (t, at^b)(y = ax^b)$. Thus the work $W$ is given by:

$$W = \int_C F(c(t))c'(t)dt = \int_0^1 (tat^b, t^6(at^b)^2) \cdot (1, abt^{b-1})\, dt = \frac{a}{b+2}(1 + \frac{a^2 b}{3})$$

Now, for $W$ to be independent of $b$, $\frac{\partial W}{\partial b} = 0$. On solving this, we get $\frac{(b+2)a^2 - (3+a^2 b)}{(b+2)^2} = 0 \implies a = \sqrt{\frac{3}{2}}$

# That's All Folks!