MA 111
Tutorial 5 Solutions

D1 T5

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QUESTION 1

QUESTION 1(i)

Green's Theorem:

1. Let D be a bounded region in R^2 with a positively oriented boundary C consisting of a finite number of non-intersecting simple closed piecewise continuously differentiable curves.
2. Let Ω be an open set in R^2 such that $D \cup C \subset \Omega$ and let $F1 : \Omega \to R$ and $F2 : \Omega \to$ be C^1 functions.

Then the following holds: $\int_C F_1\, dx + F_2\, dy = \int\int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy$

QUESTION 1(i)

Verfiy Green's Theorem:-

To evaluate: $\int_{\partial D} F_1(x,y)\,dx + F_2(x,y)\,dy$

Here, $F_1(x,y) = -xy$, $F_2(x,y) = xy$ on $D =: \{(x,y) \mid 0 \leq y \leq 1 - x^2 and\ 0 \leq x\}$

let ∂D (**positively oriented**) $= C_1 + C_2 + C_3$ where,

$C_1(t) : x(t) = t,\ y(t) = (1-t^2)\ [t = 1 \to 0]$,

$C_2(t) : x(t) = 0,\ y(t) = t\ [t = 1 \to 0]$ and $C_3(t) : x(t) = t,\ y(t) = 0\ [t = 0 \to 1]$

$\therefore \int_{\partial D}(F_1(x,y)\,dx + F_2(x,y)\,dy) = \int_{C_1}(F_1(x,y)\,dx + F_2(x,y)\,dy) + \int_{C_2}(F_1(x,y)\,dx + F_2(x,y)\,dy) + \int_{C_3}(F_1(x,y)\,dx + F_2(x,y)\,dy)$

$\therefore \int_{\partial D}(F_1(x,y)\,dx + F_2(x,y)\,dy) = \int_1^0(-t(1-t^2)(1) + t(1-t^2)(-2t))\,dt + \int_1^0(0+0)\,dt + \int_0^1(0+0)\,dt = \frac{31}{60} \Rightarrow \boxed{1}$

QUESTION 1(i)

To evaluate: $\int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy$

We know, $\frac{\partial F_2}{\partial x} = \frac{\partial (xy)}{\partial x} = y$ and $\frac{\partial F_1}{\partial y} = \frac{\partial (-xy)}{\partial y} = -x$

$$\therefore \int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy = \int_D (y - (-x))\, dx\, dy = \int_0^1 \left[\int_0^{1-x^2} (x+y)\, dy\right]\, dx$$
$$\therefore \int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy = \int_0^1 \left[(xy)\Big|_{y=0}^{y=1-x^2} + (\frac{y^2}{2})\Big|_{y=0}^{y=1-x^2}\right]\, dx$$
$$\therefore \int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy = \int_0^1 \left[0.5 - x^2 + \frac{x^4}{2} + x - x^3\right]\, dx = \frac{31}{60} \Rightarrow \boxed{2}$$

from $\boxed{1}$ and $\boxed{2}$,

$$\int_{\partial D} F_1(x,y)\, dx + F_2(x,y)\, dy = \int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy$$

Hence, Green's theorem is verified.

QUESTION 2

QUESTION 2a

QUESTION 2b

QUESTION 2c

We must ensure that the curve is positively oriented to apply Green's theorem. Thus, paramaterise the boundary of the disk as :

$\partial D = (2cost(t), 2sin(t))|t \in [0, 2\pi]$

Thus ∂D is positively oriented. Further, we may claim that:

$$\int_{\partial D} y^2 dx + xdy = -\int_{\partial R} y^2 dx + xdy$$

Now applying Green's theorem yields: $(\int_{\partial R} y^2 dx + xdy) = \int \int_{x^2+y^2 \leq 4} (1 - 2y)\, dx\, dy$

$$\int_0^{2\pi} \int_{r=0}^{2} (1 - 2rsin(\theta))r = -4\pi$$

QUESTION 3

QUESTION 3a

We define area as: $\int\int dA = \int_D \int 1\, dx\, dy$

Observe that applying Green's Theorem on $F = (-y, x)$ yields:

$$\int_{\partial D} (x\, dy - y\, dx) = \int\int 2\, dx\, dy = 2\int\int dA$$

$$\implies \int\int dA = \frac{1}{2}\int_{\partial D} (x\, dy - y\, dx)$$

Now, to evaluate line integral, we paramaterise ∂D as follows $c(t) = (x(t), y(t))$ where:

$x(t) = r(t)cos(\theta(t))$

$y(t) = r(t)sin(\theta(t))$

QUESTION 3a Contd.

Now $\int\int dA = \frac{1}{2}\int_{\partial D}(x\,dy - y\,dx) = \frac{1}{2}\int_{\partial D}(x(t)y'(t) - y(t)x'(t))\,dt$

$$\implies \int\int dA = \frac{1}{2}\int_{\partial D} r^2\,d\theta$$

QUESTION 3b

Area $=$

$$\frac{1}{2}\int_{\partial D} r^2\,d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (a)^2 cos(2\theta)\,d\theta = \frac{a^2}{2}$$

QUESTION 4

QUESTION 4a

QUESTION 4b

QUESTION 4c

QUESTION 5

QUESTION 5

Consider $F_1 = xe^{-y^2}$ and $F_2 = -x^2ye^{-y^2} + \frac{1}{x^2+y^2}$

Now applying Green's Theorem to get:

$$I = \int\int_D -2xy^2e^{-y^2} - \frac{2x}{(x^2+y^2)^2} + 2xye^{-y^2}.dA \quad (1)$$

$$I = \int_a^b \int_{-\pi}^{\pi} r.(r^2 \sin 2\theta e^{-r^2 cos^2\theta} - 2r^3 \sin\theta cos^2\theta e^{-r^2 cos^2\theta} - 2\sin\theta r^{-3}).d\theta.dr \quad (2)$$

Since integrand is an odd function,

$$I = \int_a^b 0.dr = 0 \quad (3)$$

QUESTION 6

QUESTION 6

QUESTION 7

QUESTION 7

QUESTION 8

QUESTION 8

$$\mathbf{F} = \nabla(x^2 - y^2) = (2x\hat{i} - 2y\hat{j})$$

Thus

$$div(\mathbf{F}) = \frac{\partial 2x}{\partial x} - \frac{\partial 2y}{\partial y} = 2 - 2 = 0$$

Applying Flux-Divergence Theorem:

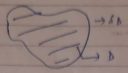
$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS = \int \int_D div(\mathbf{F})\, d(x, y) = 0$$

QUESTION 9

QUESTION 9a

Question 9

(i)



$\phi : \mathbb{R}^2 \to \mathbb{R}$ is C^2

$\phi = \phi(x, y)$

div(grad ϕ)

$$= \nabla \cdot \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} \right)$$

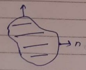
$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$\therefore$ div(grad ϕ) $= \nabla^2 \phi$

(ii) To Prove:

$$\iint \nabla^2 \phi \, d(x,y) = \oint_{\partial D} \frac{\partial \phi}{\partial n} \, ds$$



ϕ is a scalar

$\nabla^2 \phi = \nabla \cdot (\nabla \phi)$

Let $\vec{F} = \nabla \phi$

By Divergence theorem for $\vec{F}$ in D

$$\boxed{\iint_D \nabla \cdot \vec{F} \, d(x,y) = \int_{\partial D} \vec{F} \cdot \vec{n} \, ds}$$

$$\iint_D \nabla \cdot (\nabla \phi) \, d(x,y) \qquad \int_{\partial D} \nabla \phi \cdot \vec{n} \, ds$$

$$\Rightarrow \iint_D \nabla^2 \phi \, d(x,y) = \Rightarrow \int_{\partial D} \frac{\partial \phi}{\partial n} \, ds$$

$$\left(\because \nabla \phi \cdot \vec{n} = \frac{\partial \phi}{\partial n} \right)$$

↳ must have been covered in MA 109.

QUESTION 9c

(iii) $$\oint_{\partial D} \frac{\partial \phi}{\partial n}\, ds = \iint_D \nabla \cdot (\nabla \phi)\, d(x,y)$$

$$\phi = e^x \sin y$$

$$\nabla \phi = e^x \sin y\, \hat{i} + e^x \cos y\, \hat{j}$$

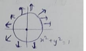
$$\nabla \cdot (\nabla \phi) = e^x \sin y - e^x \sin y = 0$$

$$\therefore \oint_{\partial D} \frac{\partial \phi}{\partial n}\, ds = \iint_D 0\, d(x,y) = 0$$

QUESTION 10

QUESTION 10a

10) (i) $\oint_C \frac{x\,dy - y\,dx}{x^2+y^2}$



$$\Omega = \{(x,y) \mid x^2+y^2 > 1\}$$

Let C be a closed curve in Ω enclosing the origin.

Now, let $P(x,y) = \frac{-y}{x^2+y^2}$

& $Q(x,y) = \frac{x}{x^2+y^2}$

(To be able to apply Green's Theorem)

Let D denote the closed and bounded subset of Ω enclosed by C.

Note that, C does not pass through the origin & encloses it as well.

So, there exists $\varepsilon > 0$ such that the disc of radius "ε" centred at $(0,0)$ lies completely inside D.

QUESTION 10a Contd.

Let this disc be D_ε and curve C_ε.

$\therefore$ Field (P, Q) is defined everywhere outside D_ε and within D.

$$\therefore \oint Pdx + Qdy = \left[\oint_{C} Pdx + Qdy\right] + \left[\oint_{C_\varepsilon} Pdx + Qdy\right]$$

Now, using Green's Theorem,

$$\Rightarrow \oint_{\partial D} Pdx + Qdy = \iint_{D} (Q_x - P_y)\, d(x,y)$$

$$\left(\because \boxed{\partial D = C \cup (-C_\varepsilon)}\right) \quad = \underline{0} \qquad [Q_x = P_y].$$

$$\Rightarrow \oint_{C} Pdx + Qdy = -\oint_{C_\varepsilon} Pdx + Qdy$$

* For simplicity, take C_ε as a circle of radius ε and centre (0, 0)

QUESTION 10a Contd.

$\Rightarrow$ Using the parametrization,

$x = \varepsilon\cos\theta \; ; \; y = \varepsilon\sin\theta \qquad \theta \in [2\pi, 0]$

$\Rightarrow \oint_{C_\varepsilon} P\,dx + Q\,dy = 2\pi$

$\therefore \oint_{C} P\,dx + Q\,dy = \boxed{-2\pi}$ Ans

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\implies \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = 0$$

Since, any closed curve Ω not enclosing the origin constitues a simply-connected closed curve, we may apply Green's Theorem:

$$\int_C F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$\implies \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = 0$$

QUESTION 10c

Consider $F = (F_1, F_2)$ and $F_1 = \frac{\partial ln(r)}{\partial y}$ and $F_2 = -\frac{\partial ln(r)}{\partial x}$.

Here $r = \sqrt{x^2 + y^2}$

$\implies F_1 = \frac{\partial ln(r)}{\partial y} = \frac{\partial ln(\sqrt{x^2+y^2})}{\partial y} \implies F_1 = \frac{y}{x^2+y^2}$

Similarly $F_2 = \frac{-x}{x^2+y^2}$ Thus, now you may use results derived from parts (a) and (b) to arrive at the answer.

QUESTION 11

QUESTION 11a

QUESTION 11b

QUESTION 12

QUESTION 12

$$curl(\mathbf{F}) = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\hat{i} + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})\hat{j} + \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\hat{k}$$

Here $F = (F_1, F_2, F_3)$; $F_1 = f(x), F_2 = g(y), F_3 = h(z)$

$$\because \frac{\partial f(x)}{\partial y} = \frac{\partial f(x)}{\partial z} = 0$$

Thus clearly the $\hat{k}$ component of $curl(\mathbf{F}) = 0$. By symmetry, other components are also 0 $\implies$ $curl(\mathbf{F}) = \mathbf{0}$

QUESTION 13

QUESTION 13

$$div(\mathbf{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Here $F = (F_1, F_2, F_3)$; $F_1 = f(y,z), F_2 = g(x,z), F_3 = h(x,y)$

$$div(\mathbf{F}) = \frac{\partial f(y,z)}{\partial x} + \frac{\partial g(x,z)}{\partial y} + \frac{\partial h(x,y)}{\partial z} = 0$$