MA 111
Tutorial 3 Solutions

D1-T5

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IIT BOMBAY

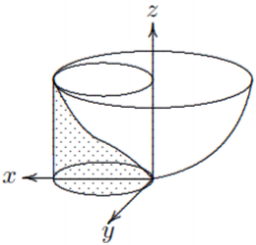
Outline

QUESTION I

QUESTION I

QUESTION I1

The base of cylinder is also the region's projection z on the xy-plane. The boundary of R is the circle $(x-1)^2 + y^2 = 1$. Its polar coordinate equation is r $= 2\cos\theta$



QUESTION I

QUESTION I2

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$$\begin{aligned} y^2 = 4 - 4x \quad &\Leftrightarrow \quad 4u^2v^2 = 4 - 4u^2 + 4v^2 \\ &\Leftrightarrow \quad (u^2-1)(v^2+1) = 0 \\ &\Leftrightarrow \quad u = \pm 1 \end{aligned}$$

Similarly,

$$y^2 = 4 + 4x \Leftrightarrow v = \pm 1$$

so that the preimage of the domain is the square $[0,1] \times [0,1]$ in the uv-plane.

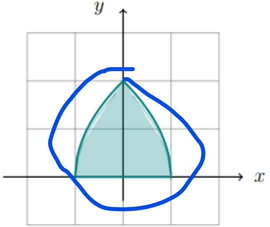
The Jacobian of the transformation is

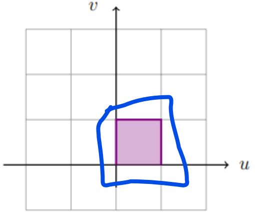
$$\mathrm{Jac}(G) = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 4(u^2+v^2).$$

By the Change of Variables formula, it follows that

$$\begin{aligned} \iint_A y\, dA &= \iint_{[0,1]\times[0,1]} 2uv \cdot 4(u^2+v^2) \\ &= 8 \int_0^1 \int_0^1 (u^3v + uv^3)\, du\, dv = \boxed{2}. \end{aligned}$$

Hint: $x = u^2 - v^2$, $y = 2uv$





QUESTION I

QUESTION I3

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QUESTION I4

QUESTION I4

QUESTION 1

QUESTION 15

QUESTION I5

QUESTION 1

QUESTION 16

QUESTION I6

QUESTION II

QUESTION II

QUESTION II1

QUESTION II1

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QUESTION II

QUESTION II9

QUESTION II9

A parametrization of C is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le 2\pi.$$

Note that the outward unit normal to the circle at $\mathbf{r}(t)$ is the radial vector $\mathbf{n} = \mathbf{r}(t)$. Also,

$$\nabla(x^2 - y^2) = 2x\,\mathbf{i} - 2y\,\mathbf{j}.$$

Thus

$$\begin{aligned}\oint_C \nabla(x^2 - y^2) \cdot d\mathbf{n} &= \int_0^{2\pi} (2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})dt \\ &= \int_0^{2\pi} (-2\sin 2t)dt = 0.\end{aligned}$$

QUESTION II

QUESTION II10

Parameterize C as

$$\mathbf{r}(t) = t\,\mathbf{i} + t^3\,\mathbf{j},\ 0 \leq t \leq 2.$$

Then $\mathbf{r}'(t) = \mathbf{i} + 3t^2\,\mathbf{j}$. Since $\nabla(x^2 - y^2) = 2t\mathbf{i} - 2t^3\mathbf{j}$, we have

$$\int_C \nabla(x^2 - y^2) \cdot d\mathbf{r} = \int_0^2 (2t - 6t^5)dt = 4 - 64 = -60.$$

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For C_1 : x+y=1 and $|x| + |y| = x + y = 1$

$\int_{C_1} \frac{dx+dy}{|x|+|y|} = \int_1^0 \frac{dx}{1}$ - $\int_1^0 \frac{dx}{1} = 0$

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For C_4 : x-y=1 and $|x| + |y| = x - y = 1$

$\int_{C_1} \frac{dx+dy}{|x|+|y|} = \int_0^1 \frac{dx}{1} + \int_0^1 \frac{dx}{1} = 2$

Hence $\boxed{\displaystyle\int_C \frac{dx+dy}{|x|+|y|} = 0-2+0+2=0}$

QUESTION II

QUESTION II12

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$$\begin{aligned}
\text{Work W} &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (xy\mathbf{i} + x^6y^2\mathbf{j})\cdot(\mathbf{i}dx + \mathbf{j}dy)\\
&= \int_0^1 ax^{b+1}dx + \int_0^1 (a^2x^{2b+6})(abx^{b-1})dx\\
&= \frac{a}{b+2} + \frac{a^3b}{3b+6}\\
&= \frac{a}{b+2}\left(1+\frac{a^2b}{3}\right) = a\left(\frac{3+a^2b}{3(b+2)}\right).
\end{aligned}$$

This will be independent of b iff $\frac{dW}{db} = 0$ iff $0 = \frac{(b+2)a^2-(3+a^2b)}{(b+2)^2}$ iff $a = \sqrt{\frac{3}{2}}$ (as $a > 0$).

That's All Folks!