MA 111 Tut 1

January 2021

1 Solutions

1(a):

$$f(x,y) = \lfloor x \rfloor + \lfloor y \rfloor + 1,\ x, y \in R = [0,1] \times [0,1]$$

Consider the partition $\mathcal{P}_n = \{(x_i, y_j) \mid i, j \in \{0,1,2\}\}$. Taking

$$0 = x_0 < x_1 = 1 - \frac{1}{n} < x_2 = 1,\ \text{and}\ 0 = y_0 < y_1 = 1 - \frac{1}{n} < y_2 = 1$$

$$L(\mathcal{P}_n, f) = \sum_{i=0}^{1} \sum_{j=0}^{1} m_{ij}(f)\Delta_{ij} \quad \text{and} \quad U(\mathcal{P}_n, f) = \sum_{i=0}^{1} \sum_{j=0}^{1} M_{ij}(f)\Delta_{ij}$$

But $f(x,y) = 1\ \forall\ x, y \in [0,1) \times [0,1)$,
So

$$m_{ij}(f) = \min_{(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]} f(x,y) = 1\ \forall\ i,j \in \{0,1,2\}$$

Let

$$M_{ij}(f) = \max_{(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]} f(x,y)$$

Thus, $M_{00}(f) = 1,\ M_{01}(f) = M_{10}(f) = 2$ and $M_{11}(f) = 3$
Therefore,

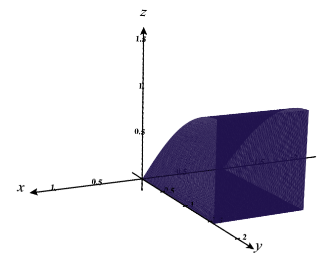
$$L(\mathcal{P}_n, f) = 1\ \forall\ n \in \mathbb{N}$$

$$U\left(\mathcal{P}_n, f\right) = \Delta_{00} + 2\Delta_{01} + 2\Delta_{10} + 3\Delta_{11} = \left(1 - \frac{1}{n}\right)^2 + 2\times 2\times\left(1 - \frac{1}{n}\right).\left(\frac{1}{n}\right) + 3.\left(\frac{1}{n}\right)^2 = 1$$

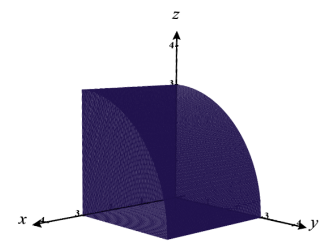
$$\text{Since,}\ \lim_{n\to\infty} U\left(\mathcal{P}_n, f\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^2 = 1 = \lim_{n\to\infty} L\left(\mathcal{P}_n, f\right),$$

$$\int\int_R f\left(x,y\right)\, d\left(x,y\right) = 1$$

2(a):
R = [-1,0] x [0,π/2] with z = f(x,y) = sin y
Using Fubini's Theorem,
$Volume(V) = \iint_R f(x,y)\,dx\,dy$
$V = \int_{y=0}^{\pi/2} (\int_{x=-1}^{0} siny\,dx)\,dy$
$V = \int_{y=0}^{\pi/2} [xsiny]_{x=-1}^{0}\,dy$
$V = \int_{y=0}^{\pi/2} siny\,dy = [cosy]_{y=0}^{\pi/2}$
$V = 1$



2(b):
R = [0,3] x [0,3] with z = f(x,y) = $\sqrt{9-y^2}$
Using Fubini's Theorem,
$Volume(V) = \iint_R f(x,y)\,dx\,dy$
$V = \int_{y=0}^{3} (\int_{x=0}^{3} \sqrt{9-y^2}\,dx)\,dy$
$V = \int_{y=0}^{3} [x\sqrt{9-y^2}]_{x=0}^{3}\,dy$
$V = \int_{y=0}^{3} 3\sqrt{9-y^2}\,dy$
$V = \frac{27\pi}{4}$



3 :

$$f(x,y) = \begin{cases} 1 - \dfrac{1}{q} & x = \dfrac{p}{q}, p, q \in \mathbb{N};\ y \in \mathbb{Q} \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, $f(x,y) \in [0,1]\ \forall\ (x,y) \in R = [0,1] \times [0,1]$. Consider some $\epsilon > 0$. Define for some y

$$\mathcal{S} = \{x \mid 1 - f(x,y) \geq \epsilon, 0 \leq x \leq 1; 0 \leq y \leq 1, y \in \mathbb{Q}\}$$

Thus, $\mathcal{S}$ has all rational numbers $0 \leq \dfrac{p}{q} \leq 1$ that satisfies $q \leq \dfrac{1}{\epsilon}; q \in \mathbb{N}$. Note that $\mathcal{S}$ is a **finite set** and has a finite number of, say l, elements. Now, consider a partition $\mathcal{P}_\epsilon = \{x_0, x_1, \cdots, x_m\} \times \{y_0, y_1, \cdots, y_n\}$ with $x_0 = y_0 = 0$ and $x_m = y_n = 1$ so that $x_j - x_{j-1} < \frac{\epsilon}{l}\ \forall\ j \in \{1, 2, \cdots, m\}$ and $y_k - y_{k-1} < \frac{\epsilon}{l}\ \forall\ k \in \{1, 2, \cdots, n\}$, i.e., $||\mathcal{P}_\epsilon|| < \frac{\epsilon}{l}$. We define some notations for ease

$$\mathcal{P}_{\epsilon jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$$

$$\Delta_{jk} = (x_j - x_{j-1})(y_k - y_{k-1})$$

$$m_{jk} = \inf_{x,y \in \mathcal{P}_{\epsilon jk}} f(x,y) \quad \text{and} \quad M_{jk} = \sup_{x,y \in \mathcal{P}_{\epsilon jk}} f(x,y)$$

Note that $M_{jk} = 1\ \forall\ j,k$. This follows from the fact that between any two numbers, there are infinitely many irrational numbers. $m_{jk} = \min\{1 - \dfrac{1}{q}\}$ for some $q \in \mathbb{N}$. Thus,

$$U(f, \mathcal{P}_\epsilon) - L(f, \mathcal{P}_\epsilon) = \sum_{j=1}^{m}\sum_{k=1}^{n} M_{jk}\Delta_{jk} - \sum_{j=1}^{m}\sum_{k=1}^{n} m_{jk}\Delta_{jk} = \sum_{j=1}^{m}\sum_{k=1}^{n} (M_{jk} - m_{jk})\Delta_{jk}$$

At this point, we shall split this sum into two parts, namely a part that contains at least one point with $x \in \mathcal{S}$ and thus satisfies $M_{jk} - m_{jk} \geq \frac{1}{q} = \epsilon$. However, we do have an upper bound on $M_{jk} - m_{jk}$, namely $M_{jk} - m_{jk} \leq 1$. Note that $\mathcal{S}$ has l elements and hence can be "covered" by at-most $2l$ of the partitions $\{x_0, x_1, \cdots, x_n\}$. Then, using the fact that $M_{jk} - m_{jk} \leq 1$,

$$\sum_{\substack{j=1 \\ \mathcal{S}\cap[x_{j-1},x_j]\neq\emptyset}}^{m} \sum_{k=1}^{n} (M_{jk} - m_{jk})\Delta_{jk} \leq \sum_{\substack{j=1 \\ \mathcal{S}\cap[x_{j-1},x_j]\neq\emptyset}}^{m} \sum_{k=1}^{n} \Delta_{jk} \leq 2l \left(\frac{\epsilon}{l} \times 1\right) \leq 2\epsilon$$

Since there are at the most $2l$ number of $\left(\frac{\epsilon}{l} \times 1\right)$ rectangles $\subseteq \mathcal{P}_{\epsilon jk}$ for some j, k such that $\mathcal{S} \cap [x_{j-1}, x_j] \neq \emptyset$.

In the second part, $m_{jk} > 1 - \frac{1}{q}$ and thus $M_{jk} - m_{jk} < \frac{1}{q} = \epsilon$. The upper limit of summation of Δ_{jk} over all values of j and k is the area of $R = [0,1] \times [0,1]$, which is 1.

$$\sum_{\substack{j=1 \\ \mathcal{S}\cap[x_{j-1},x_j]\neq\emptyset}}^{m} \sum_{k=1}^{n} (M_{jk} - m_{jk})\Delta_{jk} \leq \epsilon \times \sum_{\substack{j=1 \\ \mathcal{S}\cap[x_{j-1},x_j]\neq\emptyset}}^{m} \sum_{k=1}^{n} \Delta_{jk} \leq \epsilon$$

Thus,

$$U(f, \mathcal{P}_\epsilon) - L(f, \mathcal{P}_\epsilon) \leq 2\epsilon + \epsilon \Rightarrow U(f, \mathcal{P}_\epsilon) - L(f, \mathcal{P}_\epsilon) \leq 3\epsilon$$

which can be made arbitrarily small for smaller and smaller values of ϵ. Thus, $f(x, y)$ is **integrable** on $R = [0,1] \times [0,1]$.
What about the iterated integrals? We first consider the iterated integral over x then y and then the other way round.

CASE I: First over x, then over y
Define

$$\Phi^y(x) = f(x, y) \text{ for fixed y}$$

Then, we have

$$\Phi^y(x) = \begin{cases} \begin{cases} 1 - \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q},\ p, q \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases} & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases}$$

Thus, for a given y, $\Phi^y(x)$ is either the constant function 1 or $1-$ the Thomae's function in x, both of which are integrable for $x \in [0,1]$. This yields

$$\int_0^1 \Phi^y(x)\,dx = \begin{cases} \int_0^1 1 - \text{Thomae}(x)\,dx & y \in \mathbb{Q} \\ \int_0^1 1\,dx & y \notin \mathbb{Q} \end{cases}$$

Using the fact that the integral of the Thomae's function over any interval is 0, we get

$$\int_0^1 \Phi^y(x)\,dx = \begin{cases} 1 & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases} \Rightarrow \int_0^1 \Phi^y(x) = 1$$

Now,

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \left(\int_0^1 \Phi^y(x)\,dx \right) dy = \int_0^1 1\,dy = 1$$

Thus, the **iterated integral exists in this case and is equal to 1**.

<u>**CASE II:**</u> **First over y, then over x**

Define

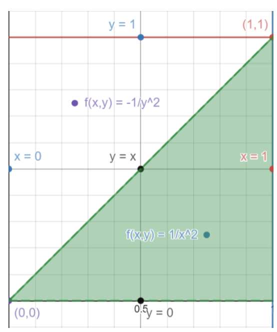
$$\Phi^x(y) = f(x,y) \text{ for fixed x}$$

For a given $x = \frac{p}{q} \in \mathbb{Q}$, let $k_x = \frac{1}{q}$. Then, we have

$$\Phi^x(y) = \begin{cases} \begin{cases} 1 - k_x & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases} & x = \frac{p}{q} \in \mathbb{Q},\ p, q \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

Thus, for a given x, $\Phi^x(y)$ is either the constant function 1 or $1-$ the Dirichlet's function in y. However, the Dirichlet's function is not integrable for $y \in [0,1]$. Thus, the **iterated integral does not exist in this case**.

4



Observe that f is not bounded in $R = [0,1] \times [0,1]$ (why?). Thus, we can immediately claim that f is not integrable in R. However, this *does not* imply that the iterated integrals do not exist.

Evaluating iterated integrals:

For $y = 0$, $f(x,y) = 0$ and thus $\int_{x=0}^1 f(x,y)\,dx = 0$.

For $y \in (0,1]$, $\int_{x=0}^1 f(x,y)\,dx = \int_{x=0}^y f(x,y)\,dx + \int_{x=y}^1 f(x,y)\,dx$.

Then, $\int_{x=0}^1 f(x,y)\,dx = \int_{x=0}^y \frac{-1}{y^2}\,dx + \int_{x=y}^1 \frac{1}{x^2}\,dx$

$\implies \int_{x=0}^1 f(x,y)\,dx = \frac{-1}{y} + \frac{1}{y} - 1 = -1$

Thus, define

$$B(y) = \int_{x=0}^1 f(x,y)\,dx = \begin{cases} 0 & \text{for } y = 0 \\ -1 & \text{for } y \in (0,1) \end{cases}$$

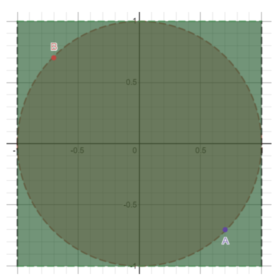
This function is bounded and has a single point of discontinuity, and hence Riemann integrable.

$$\int_{y=0}^{1} \left(\int_{x=0}^{1} f(x,y)\, dx \right)\, dy = \int_{y=0}^{1} B(y)\, dy = -1$$

Similarly, let's evaluate the iterated integral: $\int_{x=0}^{1} (\int_{y=0}^{1} f(x,y)\, dy)\, dx$ by repeating the above procedure.
For $x = 0$, $f(x,y) = 0$ and thus $\int_{y=0}^{1} f(x,y)\, dy = 0$.
For $x \in (0,1]$, $\int_{y=0}^{1} f(x,y)\, dy = \int_{y=0}^{x} f(x,y)\, dy + \int_{y=x}^{1} f(x,y)\, dy$
$\implies \int_{x=0}^{1} f(x,y)\, dx = \int_{y=0}^{x} \frac{1}{x^2}\, dy + \int_{y=x}^{1} \frac{-1}{y^2}\, dy$
$\implies \int_{x=0}^{1} f(x,y)\, dx = \frac{1}{x} + 1 - \frac{1}{x} = 1$
Thus, define

$$D(x) = \int_{x=0}^{1} f(x,y)\, dx = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x \in (0,1) \end{cases}$$

As discussed above, this function is Riemann integrable (why?). Hence, the iterated integral: $\int_{x=0}^{1} \left(\int_{y=0}^{1} f(x,y)\, dy \right)\, dx = \int_{y=0}^{1} D(x)\, dx = 1$
8 :



Let's split the domain into three set E_1, E_2, E_3.
On set $E_1 = \{(x,y) : x^2 + y^2 < 1\}$, We have the function f(x,y) = x+y which is simple continuous function
But on the set E_2 = { (x,y): x^2 + y^2 = 1 } , if we approach any point on E_2 from E_1 we get f(x,y) = x+y
Now if we approach from E_3 = R - (E_1 ∪ E_1) , we get f(x,y) = 0 , which is clearly discontinuous
Note an interesting point when x+y=0 on E_2 i.e. x= $\pm \frac{1}{\sqrt{2}}$ and y= $\mp \frac{1}{\sqrt{2}}$, we have the continuity, since function goes to zero at these two points.
So f(x,y) is discontinuous on E_2 - $\{(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}),(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})\}$
Recall the definition of content zero, we can apply and state that the function is integrable over R

$$\iint_R f(x,y)\,dx\,dy = \iint_{E_1} f(x,y)\,dx\,dy = \iint_{E_1} (x+y)\,dx\,dy = \iint_{E_1} x\,dx\,dy + \iint_{E_1} y\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\,dx\,dy + \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\,dx\,dy = 0$$

2 Answer Key

1. a 1

   b $\frac{7}{6}$

2. a 1

   b $\frac{27\pi}{4}$

4. $\int\limits_{y=0}^{1}\int\limits_{x=0}^{1} f(x,y)dxdy = -1$ and $\int\limits_{x=0}^{1}\int\limits_{y=0}^{1} f(x,y)dydx = 1$

5. a Equal, $4ln(2) - 2$

   b Equal, $\dfrac{sin(1)}{9}$

6. b $\dfrac{(2^{r+1}-1)(2^{s+1}-1)}{(r+1)(s+1)}$

   c 1

7. a 50

   b $\dfrac{45}{4} + \dfrac{15}{2}log\left(\dfrac{3}{2}\right)$