MA 111
Tutorial 2 Solutions

TUTORIAL 2
D1-T5

GYANDEV GUPTA

January 27, 2021

IIT BOMBAY

Outline

QUESTION 1

QUESTION 1

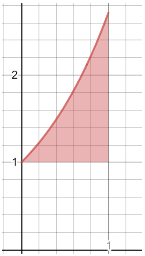
QUESTION 1a

QUESTION 1a

The region is given by $0 \leq x \leq 1$, $1 \leq y \leq e^x$. Can you sketch the region?

QUESTION 1a

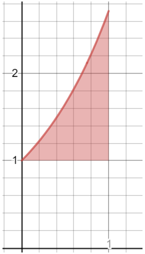
The region is given by $0 \leq x \leq 1$, $1 \leq y \leq e^x$. Can you sketch the region? The region is sketched in figure 1a.1.



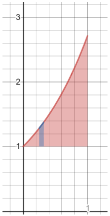
1a.1: The region

QUESTION 1a

The region is given by $0 \leq x \leq 1$, $1 \leq y \leq e^x$. Can you sketch the region? The region is sketched in figure 1a.1. The first iterated integral limits are shown in 1a.2.



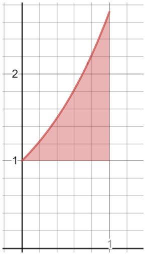
1a.1: The region



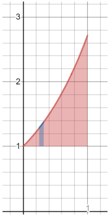
1a.2: The region with the first iterated integral shaded in blue

QUESTION 1a

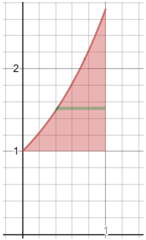
The region is given by $0 \leq x \leq 1$, $1 \leq y \leq e^x$. Can you sketch the region? The region is sketched in figure 1a.1. The first iterated integral limits are shown in figure 1a.2. Figure 1a.3 shows the order of taking the first integral reversed.



1a.1: The region



1a.2: The region with the first iterated integral shaded in blue



1a.3: The region with the first iterated integral shaded in green

QUESTION 1a (Contd.)

From figure 1a.3, it is clear that x lies between the curve and 1. The curve satisfies $y = e^x \Rightarrow x = \ln(y)$. Thus, for a given y, $\ln(y) \leq x \leq 1$.

QUESTION 1a (Contd.)

From figure 1a.3, it is clear that x lies between the curve and 1. The curve satisfies $y = e^x \Rightarrow x = \ln(y)$. Thus, for a given y, $\ln(y) \leq x \leq 1$. Also, $1 \leq y \leq e$.

QUESTION 1a (Contd.)

From figure 1a.3, it is clear that x lies between the curve and 1. The curve satisfies $y = e^x \Rightarrow x = \ln(y)$. Thus, for a given y, $\ln(y) \leq x \leq 1$. Also, $1 \leq y \leq e$.
This yields

$$\int_0^1 \left[\int_1^{e^x} dy \right] dx = \boxed{\int_1^e \left[\int_{\ln(y)}^1 dx \right] dy}$$

QUESTION 1

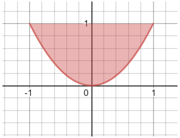
QUESTION 1b

QUESTION 1b

The region is given by $0 \leq y \leq 1$, $-\sqrt{y} \leq x \leq \sqrt{y}$. Can you sketch the region?

QUESTION 1b

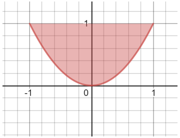
The region is given by $0 \leq y \leq 1$, $-\sqrt{y} \leq x \leq \sqrt{y}$. Can you sketch the region? The region is sketched in figure 1b.1.



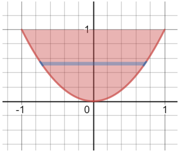
1b.1: The region

QUESTION 1b

The region is given by $0 \leq y \leq 1$, $-\sqrt{y} \leq x \leq \sqrt{y}$. Can you sketch the region? The region is sketched in figure 1b.1. The first iterated integral limits are shown in 1b.2.



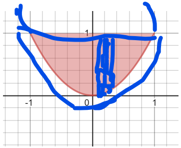
1b.1: The region



1b.2: The region with the first iterated integral shaded in blue

QUESTION 1b

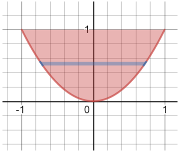
The region is given by $0 \leq y \leq 1$, $-\sqrt{y} \leq x \leq \sqrt{y}$. Can you sketch the region? The region is sketched in figure 1b.1. The first iterated integral limits are shown in figure 1b.2. Figure 1b.3 shows the order of taking the first integral reversed.



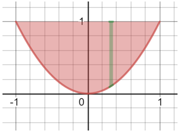
1b.1: The region

$x^2 \leq y \leq 1$

$-1 \leq x \leq 1$



1b.2: The region with the first iterated integral shaded in blue



1b.3: The region with the first iterated integral shaded in green

QUESTION 1b (Contd.)

From figure 1b.3, it is clear that y lies between the curve and 1. The curve satisfies $x = \pm\sqrt{y} \Rightarrow y = x^2$. Thus, for a given x, $x^2 \leq y \leq 1$.

QUESTION 1b (Contd.)

From figure 1b.3, it is clear that y lies between the curve and 1. The curve satisfies $x = \pm\sqrt{y} \Rightarrow y = x^2$. Thus, for a given x, $x^2 \leq y \leq 1$. Also, $-1 \leq x \leq 1$.

QUESTION 1b (Contd.)

From figure 1b.3, it is clear that y lies between the curve and 1. The curve satisfies $x = \pm\sqrt{y} \Rightarrow y = x^2$. Thus, for a given x, $x^2 \leq y \leq 1$. Also, $-1 \leq x \leq 1$.
This yields

$$\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\right]\,dy = \boxed{\int_{-1}^1 \left[\int_{x^2}^1 f(x,y)\,dy\right]\,dx}$$

QUESTION 2

QUESTION 2

QUESTION 2a

QUESTION 2a

$$\text{Let } I = \int_0^\pi \left[\int_x^\pi \frac{\sin y}{y}\, dy \right] dx \tag{1}$$

Note that this is very difficult to integrate.

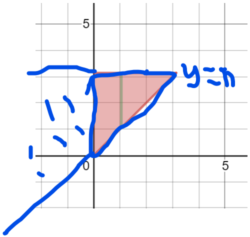
QUESTION 2a

$$\text{Let } I = \int_0^{\pi} \left[\int_x^{\pi} \frac{\sin y}{y} \, dy \right] dx \tag{1}$$

Note that this is very difficult to integrate. How about switching the order of integration?

$$\text{Let } I = \int_0^{\pi} \left[\int_x^{\pi} \frac{\sin y}{y}\, dy \right] dx \qquad (1)$$

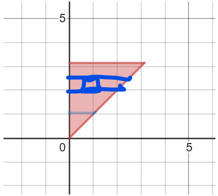
Note that this is very difficult to integrate. How about switching the order of integration?



2a.1: The region with the original order of integration

$x < y < \pi$

$0 < x < \pi$



2a.2: The region with the order of integration swapped

$0 \le x \le y$

$0 \le y \le \pi$

QUESTION 2a (Contd.)

From figure 2a.2, we get the limits as $0 \leq x \leq y$ and $0 \leq y \leq \pi$. Thus,

$$I = \int_0^{\pi} \left[\int_0^{x} \frac{\sin y}{y}\, dx \right] dy = \int_0^{\pi} \left[\frac{\sin y}{y}(x) \Big|_{x=0}^{x=y} \right] dy = \int_0^{\pi} \sin y \, dy = \boxed{2} \tag{2}$$

QUESTION 2

QUESTION 2b

QUESTION 2b

$$\text{Let } I = \int_0^1 \left[\int_y^1 x^2 e^{xy} \, dx \right] dy \tag{1}$$

Note that this is very difficult to integrate.

QUESTION 2b

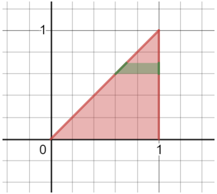
$$\text{Let } I = \int_0^1 \left[\int_y^1 x^2 e^{xy} \, dx \right] dy \tag{1}$$

Note that this is very difficult to integrate. How about switching the order of integration?

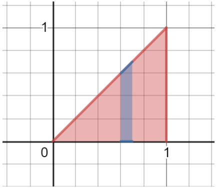
QUESTION 2b

$$\text{Let } I = \int_0^1 \left[\int_y^1 x^2 e^{xy}\, dx \right] dy \tag{1}$$

Note that this is very difficult to integrate. How about switching the order of integration?



2b.1: The region with the original order of integration



2b.2: The region with the order of integration swapped

QUESTION 2b (Contd.)

From figure 2b.2, we get the limits as $0 \leq y \leq x$ and $0 \leq x \leq 1$. Thus,

$$I = \int_0^1 \left[\int_0^x x^2 e^{xy} \, dy \right] dx = \int_0^1 \left[x^2 \frac{e^{xy}}{x} \Bigg|_{y=0}^{y=x} \right] dx = \int_0^1 x e^{x^2} - x \, dx = \boxed{\frac{e-2}{2}} \quad (2)$$

QUESTION 2

QUESTION 2c

QUESTION 2c

$$\text{Let } I = \int_0^2 \tan^{-1}(\pi x) - \tan^{-1}(x)\, dx \tag{1}$$

Note that this is very difficult to integrate.

QUESTION 2c

$$\text{Let } I = \int_0^2 \tan^{-1}(\pi x) - \tan^{-1}(x)\, dx \tag{1}$$

Note that this is very difficult to integrate. How about converting it to a double integral?

QUESTION 2c

$$\text{Let } I = \int_0^2 \tan^{-1}(\pi x) - \tan^{-1}(x)\, dx \tag{1}$$

Note that this is very difficult to integrate. How about converting it to a double integral? Note that

$$\tan^{-1}(\pi x) - \tan^{-1}(x) = \int_0^x \frac{\pi}{1+\pi^2 y^2} - \frac{1}{1+y^2}\, dy \tag{2}$$

QUESTION 2c

$$\text{Let } I = \int_0^2 \tan^{-1}(\pi x) - \tan^{-1}(x)\, dx \tag{1}$$

Note that this is very difficult to integrate. How about converting it to a double integral? Note that

$$\tan^{-1}(\pi x) - \tan^{-1}(x) = \int_0^x \frac{\pi}{1+\pi^2 y^2} - \frac{1}{1+y^2}\, dy \tag{2}$$

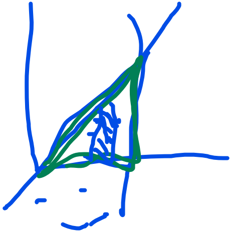
Thus, equation 1 can be written as

$$I = \int_0^2 \left[\int_0^x \frac{\pi}{1+\pi^2 y^2} - \frac{1}{1+y^2}\, dy \right] dx \tag{3}$$

QUESTION 2c (Contd.)

Swapping order of integration (by now, you must get used to it), equation 3 becomes

$$I = \int_0^2 \left[\int_y^2 \frac{\pi}{1+\pi^2 y^2} - \frac{1}{1+y^2} \, dx \right] dy \tag{4}$$



$0 \le y \le 2$

$0 \le x \le 2$

$y \le x \le 2$

$0 \le y \le 2$

QUESTION 2c (Contd.)

Swapping order of integration (by now, you must get used to it), equation 3 becomes

$$I = \int_0^2 \left[\int_y^2 \frac{\pi}{1+\pi^2 y^2} - \frac{1}{1+y^2} \, dx \right] dy \tag{4}$$

This yields

$$I = \int_0^2 \left[\frac{2\pi}{1+\pi^2 y^2} - \frac{2}{1+y^2} - \frac{\pi y}{1+\pi^2 y^2} + \frac{y}{1+y^2} \right] dy \tag{5}$$

Integrating,

$$I = \left(2\arctan(\pi y) - 2\arctan(y) - \frac{\ln(1+\pi^2 y^2)}{2\pi} + \frac{\ln(1+y^2)}{2} \right) \Bigg|_{y=0}^{y=2} \tag{6}$$

$$\boxed{I = -2\arctan(2) + 2\arctan(2\pi) - \frac{\ln(1+4\pi^2)}{2\pi} + \frac{\ln(5)}{2} \approx 0.8274} \tag{7}$$

QUESTION 3

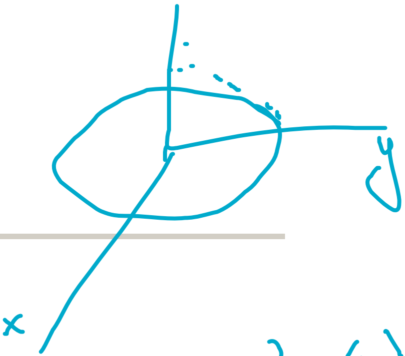
$$2\int_{-b}^{b}\int_{-t}^{t}\int_{-z_0}^{z_0} 1 \, dz\,dx\,dy$$

$$t = a\sqrt{1-\frac{y^2}{b^2}}$$

QUESTION 4

$$z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

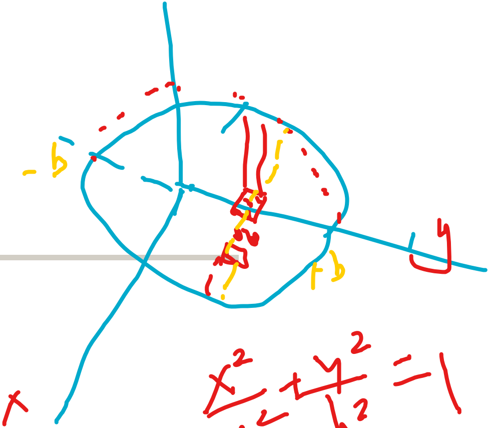
$$= z_0$$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1-\frac{z^2}{c^2} \le 1$$

QUESTION 4

QUESTION 4a



$$\iint_{-a\sqrt{1-\frac{y^2}{b^2}}}^{+} z\,dx\,dy$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

QUESTION 4a SOLUTION I

THERE ARE TWO WAYS TO SOLVE THIS QUESTION.
THIS IS THE FIRST METHOD.

The ellipsoid has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

QUESTION 4a SOLUTION I

THERE ARE TWO WAYS TO SOLVE THIS QUESTION.
THIS IS THE FIRST METHOD.

The ellipsoid has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

Note that we cannot write $z = z(x, y)$, i.e., as a function of x, y since for every value of x and y, there are two values of z. To simplify matters, let us calculate the volume of only the upper half of the ellipsoid, i.e., $z \geq 0$ and then, using symmetry (prove!) arguments, argue that the total volume is twice the volume of the upper ellipsoid.

QUESTION 4a SOLUTION I (Contd.)

From 1, and using $z \geq 0$, we write

$$z = z(x, y) = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \tag{2}$$

QUESTION 4a SOLUTION I (Contd.)

From 1, and using $z \geq 0$, we write

$$z = z(x, y) = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \tag{2}$$

Now, the volume of the upper half ellipsoid, by Cavaliers Principle, can be given by

$$V = \iint\limits_{(x,y)} z(x, y)\, d(xy) = \iint\limits_{(x,y)} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, d(xy) \tag{3}$$

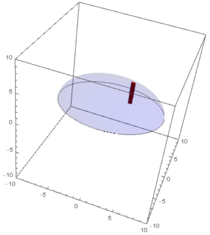
QUESTION 4a SOLUTION I (Contd.)

What are the ranges of integration?

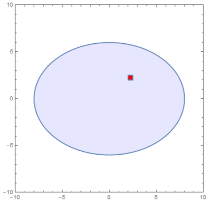
QUESTION 4a SOLUTION I (Contd.)

What are the ranges of integration?

Observe the figures given below. Note that one of the many "small rectangles" over which the integration is carried out is shaded.



4.1: Hemi-Ellipsoid with a small rectangle of integration



4.2: The surface of (x, y)

Let us integrate over y first and then x.

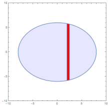
QUESTION 4a SOLUTION I (Contd.)

Let us integrate over y first and then x.

What are the ranges of x and y?

QUESTION 4a SOLUTION I (Contd.)

Let us integrate over y first and then x.
What are the ranges of x and y?

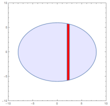


4.3: The limits of y for a given x

Consider the surface of the hemi-ellipsoid as shown in figure 4.1. This immediately tells us that (x, y) lie in the ellipse centred at origin and having semi-major axis length a and semi-minor axis length b. Thus, we have $-a \leq x \leq a$ and $-b \leq y \leq b$.

QUESTION 4a SOLUTION I (Contd.)

Let us integrate over y first and then x.
What are the ranges of x and y?



4.3: The limits of y for a given x

Consider the surface of the hemi-ellipsoid as shown in figure 4.1. This immediately tells us that (x, y) lie in the ellipse centred at origin and having semi-major axis length a and semi-minor axis length b. Thus, we have $-a \leq x \leq a$ and $-b \leq y \leq b$.
Now, for a given x, the range of y is as indicated in figure 4.3. Also, we have an additional constraint, in the form of

$$-b\sqrt{1-\frac{x^2}{a^2}} \leq y^2 \leq b\sqrt{1-\frac{x^2}{a^2}} \tag{4}$$

QUESTION 4a SOLUTION I (Contd.)

Putting the limits from equation 4 in equation 3, we get

$$V = \int_{-a}^{a} \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dy \, dx \tag{5}$$

How do we solve this?

QUESTION 4a SOLUTION I (Contd.)

Putting the limits from equation 4 in equation 3, we get

$$V = \int_{-a}^{a} \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\ dy\, dx \tag{5}$$

How do we solve this?

Set $\mu = b\sqrt{1 - \frac{x^2}{a^2}}$. Note that μ has no dependence on y and hence can be treated as a constant with respect to y. Thus,

$$V = \int_{-a}^{a} \int_{-\mu}^{\mu} \frac{c}{b}\sqrt{\mu^2 - y^2}\ dy\, dx \tag{6}$$

QUESTION 4a SOLUTION I (Contd.)

Recall from your JEE Calculus

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{x^2}{2} \arcsin\left(\frac{x}{a}\right) \tag{7}$$

Recall from your JEE Calculus

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{x^2}{2} \arcsin\left(\frac{x}{a}\right) \tag{7}$$

Applying this result to equation 6, we get

$$V = \int_{-a}^{a} \frac{c}{b} \left(\frac{y\sqrt{\mu^2 - y^2}}{2} + \frac{y^2}{2} \arcsin\left(\frac{y}{\mu}\right) \right) \Bigg|_{y=-\mu}^{y=\mu} dx = \int_{-a}^{a} \frac{c}{b} \times \frac{\pi\mu^2}{2}\, dx \tag{8}$$

QUESTION 4a SOLUTION I (Contd.)

Substituting the value of $\mu = b\sqrt{1 - \frac{x^2}{a^2}}$ in equation 8, we get

$$V = \int_{-a}^{a} \frac{bc\pi}{2}\left(1 - \frac{x^2}{a^2}\right) dx = \frac{bc\pi}{2}\left(x - \frac{x^3}{3a^2}\right)\Bigg|_{x=-a}^{x=a} = \frac{2}{3}\pi abc \tag{9}$$

QUESTION 4a SOLUTION I (Contd.)

Substituting the value of $\mu = b\sqrt{1 - \frac{x^2}{a^2}}$ in equation 8, we get

$$V = \int_{-a}^{a} \frac{bc\pi}{2}\left(1 - \frac{x^2}{a^2}\right)\, dx = \frac{bc\pi}{2}\left(x - \frac{x^3}{3a^2}\right)\Bigg|_{x=-a}^{x=a} = \frac{2}{3}\pi abc \tag{9}$$

This is the volume of the upper hemi-ellipsoid. Hence, the total volume of the ellipsoid is

$$\text{Total Volume} = 2 \times V = 2 \times \frac{2}{3}\pi abc = \boxed{\frac{4}{3}\pi abc}$$

Thus ends the long and tedious method of finding the volume of the ellipsoid.

Is there a cleverer method?

QUESTION 4a SOLUTION II

THERE ARE TWO WAYS TO SOLVE THIS QUESTION.
THIS IS THE SECOND METHOD.

The ellipsoid has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{10}$$

QUESTION 4a SOLUTION II

THERE ARE TWO WAYS TO SOLVE THIS QUESTION.
THIS IS THE SECOND METHOD.

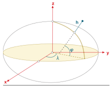
The ellipsoid has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{10}$$

Can you parametrize it?

QUESTION 4a SOLUTION II (Contd.)

Observe the figure given below

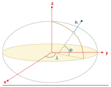


4.4: Parametrization of the ellipsoid

The parametrization can be given as follows. (Check!)

QUESTION 4a SOLUTION II (Contd.)

Observe the figure given below



4.4: Parametrization of the ellipsoid

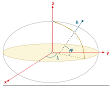
What are the ranges of θ and ϕ?

The parametrization can be given as follows. (Check!)

$$\begin{cases} x = a \sin\theta \cos\phi \\ y = b \sin\theta \sin\phi \\ z = c \cos\theta \end{cases} \tag{11}$$

QUESTION 4a SOLUTION II (Contd.)

Observe the figure given below



4.4: Parametrization of the ellipsoid

The parametrization can be given as follows. (Check!)

$$\begin{cases} x = a\sin\theta\cos\phi \\ y = b\sin\theta\sin\phi \\ z = c\cos\theta \end{cases} \tag{11}$$

What are the ranges of θ and ϕ? It is easy to check that

$$0 \leq \phi \leq 2\pi, \text{ since } (x, y) \text{ lie on the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2} = \sin^2\theta$$

Also, note that $-c \leq z \leq c$ yields $0 \leq \theta \leq \pi$

QUESTION 4a SOLUTION II (Contd.)

The volume can then be given by Cavaliers Principle as

$$V = \iint\limits_{ellipse} z \, d(xy) \tag{12}$$

QUESTION 4a SOLUTION II (Contd.)

The volume can then be given by Cavaliers Principle as

$$V = \iint_{ellipse} z \, d(xy) \tag{12}$$

We now make a change of variables to θ and ϕ. Invoking the Jacobian to equation 11

$$J_{xy}(\theta, \phi) = \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} a \cos\theta \cos\phi & -a \sin\theta \sin\phi \\ b \cos\theta \sin\phi & b \sin\theta \cos\phi \end{vmatrix} = ab \sin\theta \cos\theta \tag{13}$$

QUESTION 4a SOLUTION II (Contd.)

Equation 10 becomes

$$V = \iint\limits_{ellipse} z \, d(xy) = \iint\limits_{(\theta,\phi)} (z(\theta,\phi)) \left|J_{xy}(\theta,\phi)\right| \, d(\theta\phi) \tag{14}$$

QUESTION 4a SOLUTION II (Contd.)

Equation 10 becomes

$$V = \iint\limits_{ellipse} z\, d(xy) = \iint\limits_{(\theta,\phi)} (z(\theta,\phi))\, |J_{xy}(\theta,\phi)|\ d(\theta\phi) \tag{14}$$

Substituting $J_{xy}(\theta,\phi)$ from 13 and z from 11 and putting in the limits, integrating first with respect to θ, then ϕ, we get

$$V = \int_0^{2\pi}\int_0^{\pi} abc \sin\theta \cos\theta\, |\cos\theta|\ d\theta\, d\phi \tag{15}$$

Put $\cos\theta = t$ and integrate to get $V = \boxed{\frac{4}{3}\pi abc}$

QUESTION 5

$(x-1)^2 + y^2 = 1$

$x^2 + y^2 = r^2$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 - 2x + y^2 =$

$r^2 = 2r\cos\theta$

QUESTION 6

$\cos\theta = \frac{x}{2}$ $\theta = \cos^{-1}\left(\frac{x}{2}\right) = \theta_0$

$x = r\cos\theta, \; y = r\sin\theta$

$\iint \frac{\partial x}{\partial u}$

$du\, dv$

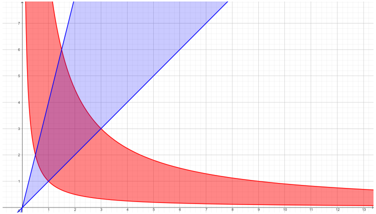
$\iint_A 1 \, dx\, dy$

QUESTION 6

We need

$$I = \iint_{\mathcal{D}} dx\, dy \tag{1}$$

Note that $\mathcal{D} := \{(x, y) \mid 1 \leq xy \leq 9,\ x \leq y \leq 4x\}$. $\mathcal{D}$ is the area common to the two shaded regions in figure 6.1.



6.1: The region $\mathcal{D}$

QUESTION 6 (Contd.)

Note that in $\mathcal{D}$, there is no point that has $x = 0$ or $y = 0$.

Note that in $\mathcal{D}$, there is no point that has $x = 0$ or $y = 0$. Thus, we can parametrize x and y as

$$\begin{cases} x = uv \\ y = \frac{u}{v} \end{cases}$$

Note that in $\mathcal{D}$, there is no point that has $x = 0$ or $y = 0$. Thus, we can parametrize x and y as

$$\begin{cases} x = uv \\ y = \frac{u}{v} \end{cases} \Rightarrow \quad \begin{cases} xy = u^2 \\ \frac{y}{x} = v^2 \end{cases}$$

QUESTION 6 (Contd.)

Note that in $\mathcal{D}$, there is no point that has $x = 0$ or $y = 0$. Thus, we can parametrize x and y as

$$\begin{cases} x = uv \\ y = \frac{u}{v} \end{cases} \Rightarrow \begin{cases} xy = u^2 \\ \frac{y}{x} = v^2 \end{cases} \Rightarrow \begin{cases} 1 \leq u^2 \leq 9 \\ 1 \leq v^2 \leq 4 \end{cases}$$

QUESTION 6 (Contd.)

Note that in $\mathcal{D}$, there is no point that has $x = 0$ or $y = 0$. Thus, we can parametrize x and y as

$$\begin{cases} x = uv \\ y = \frac{u}{v} \end{cases} \Rightarrow \begin{cases} xy = u^2 \\ \frac{y}{x} = v^2 \end{cases} \Rightarrow \begin{cases} 1 \leq u^2 \leq 9 \\ 1 \leq v^2 \leq 4 \end{cases} \Rightarrow \begin{cases} 1 \leq u \leq 3 \\ 1 \leq v \leq 2 \end{cases} \tag{2}$$

Why have we taken only the positive root? Could we have taken the negative root as well?

Note that in $\mathcal{D}$, there is no point that has $x = 0$ or $y = 0$. Thus, we can parametrize x and y as

$$\begin{cases} x = uv \\ y = \frac{u}{v} \end{cases} \Rightarrow \begin{cases} xy = u^2 \\ \frac{y}{x} = v^2 \end{cases} \Rightarrow \begin{cases} 1 \leq u^2 \leq 9 \\ 1 \leq v^2 \leq 4 \end{cases} \Rightarrow \begin{cases} 1 \leq u \leq 3 \\ 1 \leq v \leq 2 \end{cases} \tag{2}$$

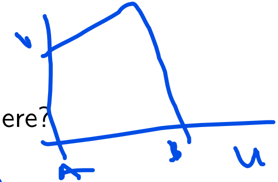
Why have we taken only the positive root? Could we have taken the negative root as well?

We could have taken the negative root as well. We have chosen u and v to be both positive for our convenience. We could have chosen u and v to be both negative as well and would have gotten the same answer. Important point to note is that u and v must have the same sign, else, we will get x and y in III^{rd} Quadrant.

QUESTION 6 (Contd.)

Change of Variables invokes the Jacobian. What is the Jacobian here?

QUESTION 6 (Contd.)



Change of Variables invokes the Jacobian. What is the Jacobian here?

$$J_{xy}(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{2u}{v} \tag{3}$$

Can the Jacobian be negative? What does a negative Jacobian imply?

$$\iint dx\,dy = \iint du\,dv\,\left| J(x,y) \right|$$

QUESTION 6 (Contd.)

Change of Variables invokes the Jacobian. What is the Jacobian here?

$$J_{xy}(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{2u}{v} \tag{3}$$

Can the Jacobian be negative? What does a negative Jacobian imply?

The Jacobian can be negative. A negative Jacobian only means that the "orientation" has been reversed in changing the variables(more about this later!). That is why, we take the absolute value of the Jacobian.

QUESTION 6 (Contd.)

Equation 1 becomes

$$I = \iint_{\mathcal{D}} dx\, dy = \iint_{E} |J_{xy}(u, v)|\; du\, dv = \int_1^3 \int_1^2 \frac{2u}{v}\, du\, dv \tag{4}$$

where $E := \{(u, v) \mid 1 \leq u \leq 3,\ 1 \leq v \leq 2\}$.

Equation 1 becomes

$$I = \iint_{\mathcal{D}} dx\, dy = \iint_{E} |J_{xy}(u,v)|\; du\, dv = \int_1^3 \int_1^2 \frac{2u}{v}\, du\, dv \tag{4}$$

where $E := \{(u,v) \mid 1 \leq u \leq 3,\ 1 \leq v \leq 2\}$.

Solving,

$$I = \int_1^3 \int_1^2 \frac{2u}{v}\, du\, dv = \left(\int_1^3 2u\, du\right)\left(\int_1^2 \frac{1}{v}\, dv\right) = \boxed{8 \ln 2}$$

QUESTION 7

QUESTION 7

QUESTION 7a

QUESTION 7

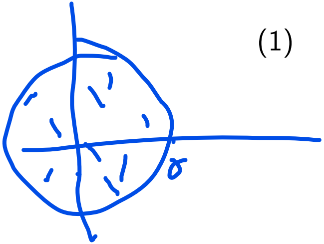
QUESTION 7b

QUESTION 7b

We need

$$I = \lim_{r \to \infty} \iint_{\mathcal{D}_r} e^{-(x^2+y^2)}\, dx\, dy \tag{1}$$

Note that $\mathcal{D}_r := \{(x, y) \mid x^2 + y^2 \leq r^2,\ x \geq 0,\ y \geq 0\}$.



$x = \rho \cos\theta$

$y = \rho \sin\theta$

$\rho^2 \leq r^2$

$\Rightarrow 0 \leq \rho \leq r$

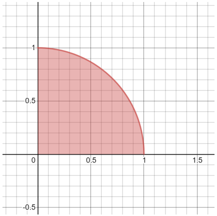
$0 \leq \theta \leq \pi/2$

QUESTION 7b

We need

$$I = \lim_{r \to \infty} \iint_{\mathcal{D}_r} e^{-(x^2+y^2)}\, dx\, dy \tag{1}$$

Note that $\mathcal{D}_r := \{(x, y) \mid x^2 + y^2 \leq r^2,\ x \geq 0,\ y \geq 0\}$. The region is shaded below (Figure 7b.1).



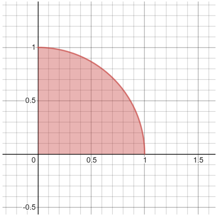
7b.1: The region in Cartesian coordinates

QUESTION 7b

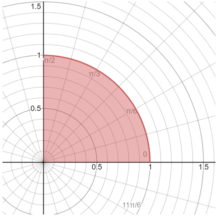
We need

$$I = \lim_{r \to \infty} \iint_{\mathcal{D}_r} e^{-(x^2+y^2)}\, dx\, dy \tag{1}$$

Note that $\mathcal{D}_r := \{(x, y) \mid x^2 + y^2 \leq r^2,\ x \geq 0,\ y \geq 0\}$. The region is shaded below (Figure 7b.1). Figure 7b.2 shows the same region in Polar coordinates.



7b.1: The region in Cartesian coordinates



7b.2: The region in Polar coordinates

QUESTION 7b (Contd.)

Note from Figure 7b.2 that we have the limits $0 \leq \rho r$ and $0 \leq \theta \leq \frac{\pi}{2}$.

QUESTION 7b (Contd.)

Note from Figure 7b.2 that we have the limits $0 \leq \rho r$ and $0 \leq \theta \leq \frac{\pi}{2}$.
Invoking the Jacobian for $x = \rho \cos\theta$ and $y = \rho \sin\theta$

$$J_{xy}(\rho, \theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{vmatrix} = \rho \tag{2}$$

QUESTION 7b (Contd.)

Note from Figure 7b.2 that we have the limits $0 \leq \rho r$ and $0 \leq \theta \leq \frac{\pi}{2}$.
Invoking the Jacobian for $x = \rho \cos\theta$ and $y = \rho \sin\theta$

$$J_{xy}(\rho, \theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{vmatrix} = \rho \tag{2}$$

Equation 1 then becomes

$$I = \lim_{r\to\infty} \iint_{\mathcal{D}_r} e^{-\rho^2} \left|J_{xy}(\rho, \theta)\right| \, d\rho \, d\theta = \lim_{r\to\infty} \int_0^{\pi/2} \int_0^r \rho e^{-\rho^2} \, d\rho \, d\theta = \lim_{r\to\infty} \frac{\pi}{4}(1-e^{-r^2}) = \boxed{\frac{\pi}{4}} \tag{3}$$

QUESTION 7

QUESTION 7c

QUESTION 7

QUESTION 7d

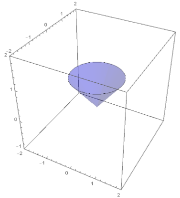
QUESTION 10

$$\underbrace{\sqrt{x^2+y^2} \leq z}_{\sqrt{x^2+y^2} \leq 1} \leq 1$$

$$-1 \leq x \leq 1 \qquad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$$

QUESTION 10

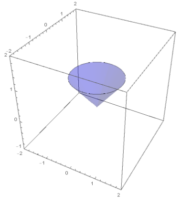
The region
$\mathcal{D} = \{(x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq 1\}$
is traced in Figure 10.



10: The region $\mathcal{D}$

QUESTION 10

The region
$\mathcal{D} = \{(x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq 1\}$
is traced in Figure 10.
Observe that $\sqrt{x^2 + y^2} \leq 1 \Rightarrow x^2 + y^2 \leq 1 \Rightarrow$
$x^2 \leq 1 \Rightarrow \boxed{-1 \leq x \leq 1}$.



10: The region $\mathcal{D}$

QUESTION 10

$z = 0\ 1$

$x^2 + y^2 = 1$

0.01

The region

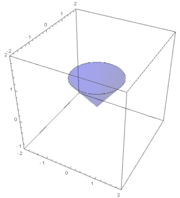
$\mathcal{D} = \{(x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq 1\}$

is traced in Figure 10.

Observe that $\sqrt{x^2 + y^2} \leq 1 \Rightarrow x^2 + y^2 \leq 1 \Rightarrow$

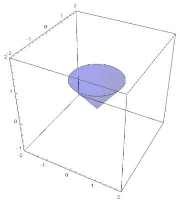
$x^2 \leq 1 \Rightarrow \boxed{-1 \leq x \leq 1}$.

This also yields $\boxed{-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}}$.



10: The region $\mathcal{D}$

The region
$\mathcal{D} = \{(x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq 1\}$
is traced in Figure 10.
Observe that $\sqrt{x^2 + y^2} \leq 1 \Rightarrow x^2 + y^2 \leq 1 \Rightarrow$
$x^2 \leq 1 \Rightarrow \boxed{-1 \leq x \leq 1}$.
This also yields $\boxed{-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}}$.

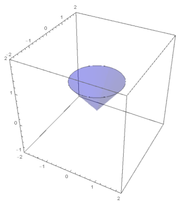


10: The region $\mathcal{D}$

As for $z(x, y)$, we are already provided with $\boxed{\sqrt{x^2 + y^2} \leq z \leq 1}$.

QUESTION 10

The region
$\mathcal{D} = \{(x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq 1\}$
is traced in Figure 10.
Observe that $\sqrt{x^2 + y^2} \leq 1 \Rightarrow x^2 + y^2 \leq 1 \Rightarrow$
$x^2 \leq 1 \Rightarrow \boxed{-1 \leq x \leq 1}$.
This also yields $\boxed{-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}}$.

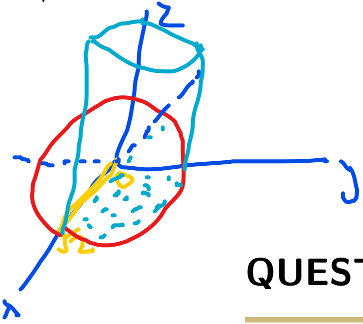


10: The region $\mathcal{D}$

As for $z(x, y)$, we are already provided with $\boxed{\sqrt{x^2 + y^2} \leq z \leq 1}$.
Thus,

$$\boxed{\mathcal{D} = \{(x, y, z) \mid -1 \leq x \leq 1,\ -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2},\ \sqrt{x^2 + y^2} \leq z \leq 1\}}$$

*



$0 \le z \le 2$

$dx\, dy\, dz$

$x^2+y^2 \le z \le 2$

$0 \le y \le \sqrt{2-x^2}$

$0 \le x \le \sqrt{2}$

QUESTION 11

$0 \le z \le 2$

$\cancel{0 \le x \le \sqrt{z-y}}$

$y^2 \le 2-x^2$

$y^2 \le z-x^2$

$\le z$

$\boxed{0 \le y \le \sqrt{z}}$

QUESTION 11

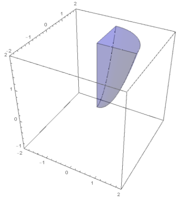
$y^2 \leq z - x^2 \leq z$

$y^2 \leq z$

$-\sqrt{z} \leq y \leq \sqrt{z}$

$0 \leq y \leq \sqrt{z}$

The region is traced in Figure 11. It is the part of the paraboloid enclosed by $x^2 + y^2 \leq z$ in the first octant.



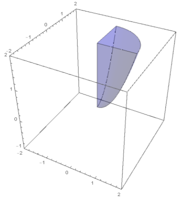
11: The region

QUESTION 11

The region is traced in Figure 11. It is the part of the paraboloid enclosed by $x^2 + y^2 \leq z$ in the first octant.

Observe that for a given z,

$x^2 + y^2 \leq z \Rightarrow 0 \leq y^2 \leq z$. Also, it is given that $y \geq 0 \Rightarrow \boxed{0 \leq y \leq \sqrt{z}}$.



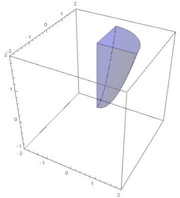
11: The region

QUESTION 11

The region is traced in Figure 11. It is the part of the paraboloid enclosed by $x^2 + y^2 \le z$ in the first octant.

Observe that for a given z,
$x^2 + y^2 \le z \Rightarrow 0 \le y^2 \le z$. Also, it is given that $y \ge 0 \Rightarrow \boxed{0 \le y \le \sqrt{z}}$.

This finally yields $\boxed{0 \le x \le \sqrt{z - y^2}}$.



11: The region

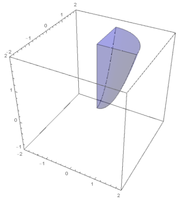
QUESTION 11

The region is traced in Figure 11. It is the part of the paraboloid enclosed by $x^2 + y^2 \leq z$ in the first octant.

Observe that for a given z,
$x^2 + y^2 \leq z \Rightarrow 0 \leq y^2 \leq z$. Also, it is given that $y \geq 0 \Rightarrow \boxed{0 \leq y \leq \sqrt{z}}$.

This finally yields $\boxed{0 \leq x \leq \sqrt{z - y^2}}$.

As for z, it is clear from Figure 11 that $0 \leq z \leq 2$.

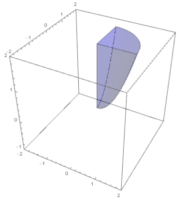


11: The region

The region is traced in Figure 11. It is the part of the paraboloid enclosed by $x^2 + y^2 \leq z$ in the first octant.

Observe that for a given z,
$x^2 + y^2 \leq z \Rightarrow 0 \leq y^2 \leq z$. Also, it is given that $y \geq 0 \Rightarrow \boxed{0 \leq y \leq \sqrt{z}}$.

This finally yields $\boxed{0 \leq x \leq \sqrt{z - y^2}}$.



11: The region

As for z, it is clear from Figure 11 that $0 \leq z \leq 2$.

Thus,

$$I = \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} x \, dz \, dy \, dx = \int_0^{2} \int_0^{\sqrt{z}} \int_0^{\sqrt{z-y^2}} x \, dx \, dy \, dz = \boxed{\frac{8\sqrt{2}}{15} \approx 0.754}$$

That's All Folks!