MA 111 Tutorial 4
Solutions

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Tutorial Solutions

3.

Theorem 1 (Green's Theorem)

1. Let D be a bounded region in R^2 with a positively oriented boundary C consisting of a finite number of non-intersecting simple closed piecewise continuously differentiable curves.
2. Let Ω be an open set in R^2 such that $D \cup C \subset \Omega$ and let $F1 : \Omega \to R$ and $F2 : \Omega \to$ be C^1 functions.

Then the following holds:
$\int_C F_1\, dx + F_2\, dy = \int\int_D (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\, dx\, dy$

Observe that the integral is defined on R^2 which is simply connected.
Thus we may apply Green's Theorem to get:
$F_1 = 2xe^{-y} \implies \frac{\partial F_1}{\partial y} = -2xe^{-y}$ $F_2 = 2y - x^2 e^{-y} \implies \frac{\partial F_2}{\partial x} = -2xe^{-y}$
Thus the integrand: $(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})$ is identically zero
$\implies$ line integral is path-independent.

Tutorial Solutions Contd.

5.

C_1 be any line segment on the line: $x = 2y$

Take C_2 to be the path :$(x, 0)|x \in [0, \frac{\pi}{2}])$

6. (a)

Evaluating the integrand of Green's Theorem:

$(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})$

$= (ye^{xy} + ye^{xy} + xy^2e^{xy}) - (2ye^{xy} + xy^2e^{xy}) = 0.$

Thus, **F** is conservative. Evaluating its scalar function:

$\frac{\partial f}{\partial y} = F_2 \implies f(x, y) = \int F_2 \, dy = ye^{xy} + g(x)$

Similarly: $\frac{\partial f}{\partial x} = F_1 \implies f(x, y) = \int F_1 \, dx = ye^{xy} + h(y)$

Equating both equations, we find that $g(x) = h(y) \implies g(x) = h(y) = c$ where c is a constant.

Tutorial Solutions Contd.

7. (a) Using the method above, find that a possible scalar function f is given by: $f(x, y, z) = x^2yz - cos(x) + c$
Now, applying Fundamental Theorem of Calculus: $\int_a^b \nabla f = f(b) - f(a)$
Define $a = (1, 0, 0)$ and $b = (-1, 0, \pi^4)$
$\implies \int_a^b F = \int_a^b \nabla f = f(b) - f(a) = -cos(1) + cos(1) = 0$

9.(a)

$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$

9.(b)

Construct a path C given by $c(t) = (cos(t), sin(t))$. Recall that the line integral of a field **F** along C is given by:
$\int_C F(c(t)) \cdot c'(t)\, dt = \int_{t=0}^{2\pi} (-sint(t), cost(t)) \cdot (-sin(t), cos(t))\, dt$
$\implies \int_C F(c(t)) \cdot c'(t)\, dt = \int_{t=0}^{2\pi} sin^2(t) + cos^2(t)\, dt = 2\pi$

9.(c)

Therefore, **F** is clearly not conservative despite satisfying the derivative condition due to domain not being simply connected(owing to (0,0))

10. To show that $\mathbf{F} = f(r)\mathbf{r}$ is a conservative field, we will show that $F = \nabla g$ where g is a scalar function.

Construct $g(r, \theta, \phi) = \int_0^r sf(s) \cdot ds$

Thus, by Fundamental Theorem of Calculus: $\frac{\partial g}{\partial x} = rf(r)\frac{\partial r}{\partial x} = xf(r)$

By symmetry, we can claim that: $\frac{\partial g}{\partial y} = yf(r)$ and $\frac{\partial g}{\partial z} = zf(r)$.

$\nabla g = (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} = (x\hat{i} + y\hat{j} + z\hat{k})f(r) = \mathbf{r}f(r) = \mathbf{F}$

$$r = (x^2 + y^2 + z^2)^{1/2},$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

If $\mathbf{F}$ is to be $\nabla\phi$ for some ϕ, then we must have $\phi_x = f(r)x, \phi_y = f(r)y, \phi_z = f(r)z$; that is,

$$\begin{aligned}
\phi_x = xf(r) &= \frac{x}{r}rf(r) = \frac{\partial r}{\partial x}rf(r), \\
\phi_y = yf(r) &= \frac{y}{r}rf(r) = \frac{\partial r}{\partial y}rf(r), \\
\phi_z = zf(r) &= \frac{z}{r}rf(r) = \frac{\partial r}{\partial z}rf(r).
\end{aligned}$$

Now it can be seen that $\phi(x, y, z) = \int_{t_0}^{r} tf(t)dt$, with some t_0 fixed, satisfies all the desired equations.

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