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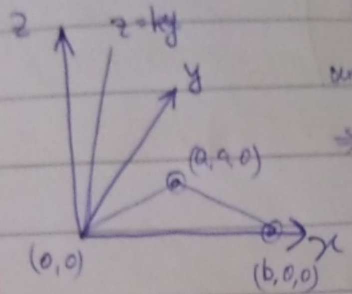
$$\boxed{D : (0,0,0) ; (a,a,0) ; (b,0,0)}$$

1.

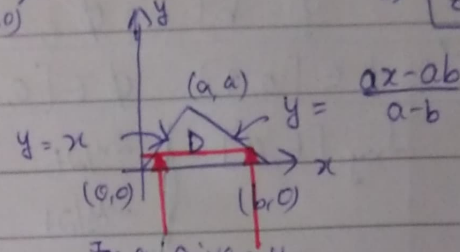


under $\boxed{z = ky}$

$\Rightarrow 0 \le z \le ky \Rightarrow y \geq z/k \Rightarrow \boxed{y \in [z/k, a]}$

$y_{max} = a \Rightarrow z_{max} = ka$

$\Rightarrow \boxed{z \in [0, ka]}$



For a given y,

$x \in \left[y, \dfrac{(a-b)y + ab}{a}\right]$

$$\Rightarrow V = \int_0^{ka}\int_{z/k}^{a}\int_{y}^{[(a-b)y+ab]/a} 1 \, dx\, dy\, dz$$

$$\Rightarrow V = \int_0^{ka}\int_{z/k}^{a}\left(\frac{(a-b)y + ab}{a} - y\right) dy\, dz$$

$$\Rightarrow V = \int_0^{ka}\int_{z/k}^{a} \frac{ab - by}{a}\, dy\, dz = \int_0^{ka}\left(\frac{2aby - by^2}{2a}\right)_{z/k}^{a} dz$$

$$\Rightarrow V = \int_0^{ka} \frac{2a^2b - a^2b - \frac{2abz}{k} + \frac{bz^2}{k^2}}{2a}\, dz$$

$$\Rightarrow V = \int_0^{ka} \frac{a^2bk^2 - 2abzk + bz^2}{2ak^2}\, dz$$

$$\Rightarrow V = \left(\frac{a^2bk^2z - abz^2k + bz^3/3}{2ak^2}\right)_0^{ka}$$

$$\Rightarrow V = \frac{a^3bk^3 - a^3k^3b + a^3k^3b/3}{2ak^2}$$

$$\Rightarrow \boxed{V = \frac{1}{6}ka^2b}$$

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2. $\boxed{D : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$

Parameterize :

$$x = at\cos\phi$$
$$y = bt\sin\phi \, ,$$

$\boxed{D' : t \in [0,1] \quad \phi \in [0, 2\pi]}$

$$J_{xy}[t,\phi] = \begin{vmatrix} \partial x/\partial t & \partial x/\partial\phi \\ \partial y/\partial t & \partial y/\partial\phi \end{vmatrix} = \begin{vmatrix} a\cos\phi & -at\sin\phi \\ b\sin\phi & bt\cos\phi \end{vmatrix}$$

$$\Rightarrow J_{xy}[t,\phi] = abt$$

$$I = \iint_D f(x,y)\,dx\,dy = \iint_{D'} g(t,\phi)\, J_{xy}(t,\phi)\,dt\,d\phi ,$$

where $g(t,\phi) = f(x,y)$

$$f(x,y) = \frac{x^2}{\pi} \Rightarrow g(t,\phi) = \frac{a^2t^2\cos^2\phi}{\pi}$$

$$\Rightarrow I = \iint_{D'} \frac{a^2t^2\cos^2\phi}{\pi} |abt| \, dt\, d\phi$$

$$\Rightarrow J = \int_0^{2\pi}\int_0^1 \frac{a^3b\,t^3\cos^2\phi}{\pi}\, dt\, d\phi$$

$$\Rightarrow J = \int_0^{2\pi} \frac{a^3b\cos^2\phi}{4\pi}\, d\phi = \frac{a^3b\,\cancel{2\pi}}{8\pi} = \boxed{\frac{a^3b}{4}}$$

$y = a + \frac{c-a}{b}x$

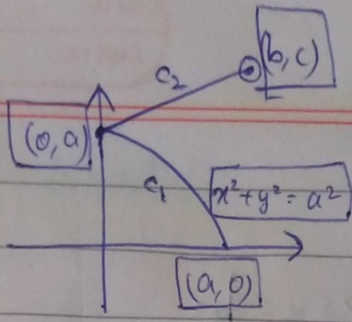
$a + (c-a)t$

$= a + ct - at$

bt

1/3

3.



$C = C_1 \cup C_2$ and

$C_1 \cap C_2$ has content 0

IMPORTANT STEP JUSTIFICATION

IMPORTANT FOR THIS TO HOLD

Therefore,

$$\int_C \bar{F}\cdot d\bar{s} = \int_{C_1} \bar{F}\cdot d\bar{s} + \int_{C_2} \bar{F}\cdot d\bar{s}$$

$$\bar{F} = 3x^2\hat{i} + 3y^2\hat{j}$$

$\Rightarrow$ $\underline{C_1}$

Parameterise : $\bar{s} = (a\cos\phi, a\sin\phi)$, $\phi \in [0, \pi/2]$

$$\Rightarrow \int_{C_1} \bar{F}\cdot d\bar{s} = \int_0^{\pi/2} (3x^2\hat{i} + 3y^2\hat{j})\cdot d\bar{s}$$

$$d\bar{s} = (-a\sin\phi\,\hat{i} + a\cos\phi\,\hat{j})\,d\phi$$

$$\bar{F} = 3x^2\hat{i} + 3y^2\hat{j} = 3a^2\cos^2\phi\,\hat{i} + 3a^2\sin^2\phi\,\hat{j}$$

$$\Rightarrow \bar{F}\cdot d\bar{s} = (-3a^3\cos^2\phi\sin\phi + 3a^3\sin^2\phi\cos\phi)\,d\phi$$

$$\Rightarrow \bar{F}\cdot d\bar{s} = [3a^3(\sin\phi\cos\phi)(\sin\phi - \cos\phi)]\,d\phi$$

$$\Rightarrow \int_{C_1} \bar{F}\cdot d\bar{s} = \int_0^{\pi/2} 3a^3(\sin^2\phi\cos\phi - \cos^2\phi\sin\phi)\,d\phi$$

$$\Rightarrow \boxed{\int_{C_1} \bar{F}\cdot d\bar{s} = 0}$$

$\underline{C_2}$

Parameterise $\bar{s} = (bt, a + ct - at)$, $t \in [0, 1]$

$$\bar{F} = 3b^2t^2\hat{i} + 3(a+ct-at)^2\hat{j}$$

$$d\bar{s} = [b\hat{i} + (c-a)\hat{j}]\,dt$$

$$\Rightarrow \bar{F}\cdot d\bar{s} = (3b^3t^2 + 3(c-a)[a + (c-a)t]^2)\,dt$$

$$\Rightarrow \int_{C_2} \bar{F}\cdot d\bar{s} = \int_0^1 3b^3t^2 + 3(c-a)a^2 + 6a(c-a)^2t + 3(c-a)^3t^2\,dt$$

$$\Rightarrow \int_{C_2} \bar{F}\cdot d\bar{s} = b^3 + 3a^2(c-a) + 3a(c-a)^2 + (c-a)^3$$

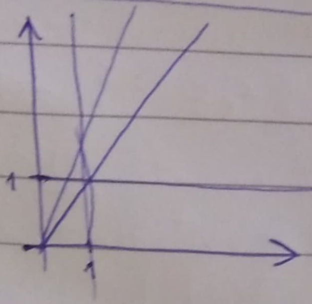
$$= b^3 - a^3 + [a + (c-a)]^3 = b^3 + c^3 - a^3$$

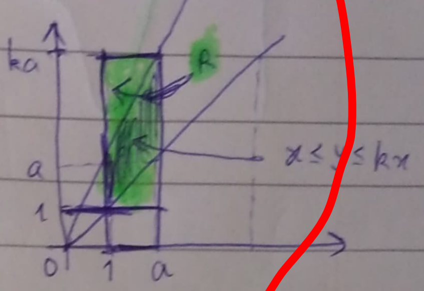
$$\Rightarrow \boxed{\int \bar{F}\cdot d\bar{s} = b^3 + c^3 - a^3}$$

Subjective

1. $f(x,y) = \begin{cases} \dfrac{1}{(x+y)^2} & \boxed{x \le y \le kx} \\ 0 & \text{otherwise} \end{cases}$

$\boxed{R = [1, a] \times [1, ka]}$





① Elementary Region : Type I — IMPORTANT!

② No discontinuities of $f(x,y)$ in R — IMPORTANT!

$\Rightarrow$ Integrable

For a given x, $y \in [x, kx]$, $\forall\, x \in [1, a]$

$$I = \underbrace{\int_1^a \int_x^{kx} f(x,y)\, dx\, dy}_{}$$

EXISTS AND FINITE $\Rightarrow$ Fubini's Theorem is applicable — IMPORTANT!

$$\Rightarrow I = \int_1^a \int_x^{kx} \frac{1}{(x+y)^2}\, dy\, dx = \int_1^a \left(-\frac{1}{x+y}\right)_x^{kx} dx$$

$$\Rightarrow I = \int_1^a \frac{1}{2x} - \frac{1}{(1+k)x}\, dx$$

$$\Rightarrow I = \int_1^a \frac{(k-1)}{2(1+k)x}\, dx = \boxed{\frac{k-1}{2(k+1)} \ln a}$$

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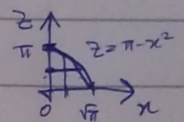
2. This question's answer depends on the values in the question.

(i) $$\int_0^{\sqrt{\pi}}\int_0^{x}\int_0^{\pi-x^2} \frac{\sin 2z}{\pi - z}\, dz\, dy\, dx$$



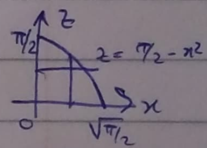
$$= \int_0^{\sqrt{\pi}}\int_0^{\pi-x^2}\int_0^{x} \frac{\sin 2z}{\pi-z}\, dy\, dz\, dx = \int_0^{\sqrt{\pi}}\int_0^{\pi-x^2} \frac{x \sin 2z}{\pi - z}\, dz\, dx$$

$$= \int_0^{\pi}\int_0^{\sqrt{\pi-z}} \frac{x \sin 2z}{\pi - z}\, dx\, dz = \int_0^{\pi} \frac{\sin 2z\,(\pi - z)}{2(\pi - z)}\, dz$$

$$= \int_0^{\pi} \frac{\sin 2z}{2}\, dz = \boxed{0}$$

(ii) $$\int_0^{\sqrt{\pi/2}}\int_0^{x}\int_0^{\frac{\pi}{2}-x^2} \frac{\sin 2z}{\pi/2 - z}\, dz\, dy\, dx = \int_0^{\sqrt{\pi/2}}\int_0^{\pi/2 - x^2}\int_0^{x} \frac{\sin 2z}{\frac{\pi}{2}-z}\, dy\, dz\, dx$$

$$= \int_0^{\sqrt{\pi/2}}\int_0^{\pi/2 - x^2} \frac{x \sin 2z}{\pi/2 - z}\, dz\, dx$$



$$= \int_0^{\pi/2}\int_0^{\sqrt{\pi/2 - z}} \frac{x \sin 2z}{\pi/2 - z}\, dx\, dz = \int_0^{\pi/2} \frac{\sin 2z\,(\pi/2 - z)}{2(\pi/2 - z)}\, dz$$

$$= \int_0^{\pi/2} \frac{\sin 2z}{2}\, dz = \boxed{1/2}$$

(iii) $$\int_0^{1}\int_0^{x}\int_0^{1-x^2} \frac{\sin 2z}{1-z}\, dz\, dy\, dx$$

Not integrable

Fubini not applicable as function is unbounded at $z = 1$. for this quiz, Accept $\frac{1}{4}(1-\cos 2)$ if Fubini is applied.

(iv) Same as above, unbounded at $z = 1/2$

Accept $\frac{1}{4}(1-\cos 1)$ for this quiz