MA111 (IIT Bombay) Weekly Quiz 5 : Green's theorem , February 20, 2021

1. Let $\mathbf{F}(x,y) = \dfrac{(y+e^x)}{\pi}\mathbf{i} + \dfrac{(2x+\sin(y^2))}{\pi}\mathbf{j}$ be defined on $\mathbb{R}^2$ and let C be the circle $x^2+y^2=r^2$. Find $\int_C \mathbf{F}.\mathbf{ds}$. The parameter r is given.

   Answer. Use Green's theorem. Let $\mathbf{F}(x,y) = F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}$ for all $(x,y) \in \mathbb{R}^2$, where

   $$F_1(x,y) = \frac{(y+e^x)}{\pi}, \quad F_2(x,y) = \frac{(2x+\sin(y^2))}{\pi}.$$

   Then for all $(x,y) \in \mathbb{R}^2$,

   $$\frac{\partial F_2}{\partial x}(x,y) = \frac{2}{\pi}, \quad \frac{\partial F_1}{\partial y}(x,y) = \frac{1}{\pi}.$$

   Then, using Green's theorem for the region $D = \{(x,y) \mid x^2+y^2 \le r^2\}$ enclosed by C:

   $$\int_C \mathbf{F}.\mathbf{ds} = \int\int_{x^2+y^2\le r^2} \Big[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\Big]\, dxdy$$
   $$= \frac{1}{\pi}\int\int_{x^2+y^2\le r^2} 1\, dxdy = \frac{1}{\pi}\text{Area}(D) = r^2.$$

   Thus $\int_C \mathbf{F}.\mathbf{ds} = r^2$.

2. Let $\mathbf{F}(x,y) = (e^y\cos x, e^y \sin x)$ defined in $\mathbb{R}^2$. (Mark all correct statements below).

   [1]

   (a) Then $\mathbf{F}$ is a conservative field.
   (b) $\operatorname{div}\mathbf{F} \neq 0$ in $\mathbb{R}^3$.
   (c) Let C be the unit circle parametrized by $(\cos t, \sin t)$ for all $t \in [0,2\pi]$. Let $\mathbf{n}$ be the outward unit normal to C. Then $\int_C \mathbf{F}.\mathbf{n}\, ds = 0$.

   Ans. a) and c).

   a) correct. $\phi(x,y) = e^y \sin x$ and then $\mathbf{F} = \operatorname{grad}\phi$. So $\mathbf{F}$ is conservative.

   b) wrong. because $\operatorname{div}\mathbf{F}(x,y) = -e^y\sin x + e^y \sin x = 0$.

   c) correct. Using divergence from of Green's theorem: $\oint_C \mathbf{F}.\mathbf{n}\, ds = \int\int_D \operatorname{div}\mathbf{F}\, dxdy = 0$.

3. Let $\mathbf{F}(x,y) = (e^y\cos x, e^y \sin x)$ defined in $\mathbb{R}^2$. (Mark all correct statements below).

   (a) Then $\mathbf{F}$ is a conservative field.
   (b) $divF = 0$ in $\mathbb{R}^3$.
   (c) Let C be the unit circle parametrized by $(\cos t, \sin t)$ for all $t \in [0,2\pi]$. Let $\mathbf{n}$ be the outward unit normal to C. Then $\int_C \mathbf{F}.\mathbf{n}\, ds \neq 0$.

   Ans. a) and b).

   Same logic as Qn 2.

4. Mark the correct answer: The value of $\int_C -ydx + xdy$, where C is the triangle with vertices $P=(0,0)$, $Q=(0,a)$ and $R=(a,0)$: the path is traversed from P to Q, then from Q to R and then from R to P. For given parameter $a>0$.

(a) a^2.

(b) $-a^2$.

Answer. Use Green's theorem. Note that the orientation of C is clock-wise, i.e., negative oriented. So,

$$\int_C -ydx + xdy = -\int_{-C} -ydx + xdy,$$

and now using Green's theorem for D, the region bounded by the triangle C

$$\int_{-C} -ydx + xdy = \int_D \left[\frac{\partial}{\partial x}[x] - \frac{\partial -y}{\partial y}\right] dxdy = 2\text{Area}(D) = 2\frac{a^2}{2} = a^2.$$

Thus $\int_C -ydx + xdy = -a^2$.