SC 617

Quiz-Week2

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1. Consider a scalar valued function, f (t), that is f $(\cdot)$: $\mathbb{R} \rightarrow \mathbb{R}$. Suppose f(t) is bounded from below and non-increasing. Then f(t) has a finite limit as t $\rightarrow \infty$. What is the finite limit ? [1]

Solution: Since f is bounded from below, its infimum f_m exists, i.e.,

$$f_m = inf_{0 \leq t \leq \infty} f(t)$$

which implies that there exists a sequence $\{t_n\} \in \mathbb{R}^+$ such that $lim_{n \rightarrow \infty}$ f$(t_n) = f_m$. This, in turn, implies that given any $\epsilon_0 > 0$ there exists an integer N $>$ 0 such that

$$|f(t_n) - f_m| < \epsilon_0, \forall n \geq N$$

Because f is nonincreasing, there exists an $n_0 \geq N$ such that for any $t \geq t_{n_0}$ and some $n_0 \geq N$ we have

$$f(t) \leq f(t_{n_0})$$

and

$$|f(t) - f_m| \leq |f(t_{n_0}) - f_m| < \epsilon_0$$

for any $t \geq t_{n_0}$. Because $\epsilon_0 > 0$ is any given number, it follows that $lim_{t \rightarrow \infty} f(t) = f_m$.

2. What is the definition of uniformly continuous ?

Solution: A function f : [0,∞) $\mapsto \mathbb{R}$ is uniformly continuous on [0,∞) if for any given $\epsilon_0 > 0$ there exists a $\delta(\epsilon_0)$ such that $\forall$t$_0$, t [0,∞) for which

$$|t - t_0| < \delta(\epsilon_0)$$

we have

$$|f(t) - f(t_0)| < \epsilon_0$$

3. If you have $\dot{e}_1 = e_2$
$\dot{e}_2 = -k_1(t)e_1 - k_2(t)e_2$
Can you prove convergence of $e_1, e_2 \rightarrow 0$ as the case discussed in class. Any additional assumption state it.

Solution: Assume that $\dot{k}_1$ is ≤ 0
$k_1, k_2 > 0$ & bounded
$\dot{k}_1$ and $\dot{k}_2$ are bounded
Define an Energy Functional V(t) $= \frac{k_1(t)e_1^2(t)}{2} + \frac{e_2^2(t)}{2} \geq 0$

[1] Ref. P. Ioannou and J. Sun, Robust Adaptive Control, Upper Saddle River, NJ: Prentice Hall 1996

$$\dot{\mathrm{V}} = k_1 e_1 \dot{\mathrm{e}}_1 + \frac{\dot{k_1} e_1^2}{2} + e_2 \mathrm{e}_2$$
$$\dot{\mathrm{V}} = k_1 e_1 \dot{\mathrm{e}}_1 + \frac{\dot{k_1} e_1^2}{2} + e_2(-k_1 e_1 - k_2 e_2)$$
$$\dot{\mathrm{V}} = \frac{\dot{\mathrm{k}}_1 e_1^2}{2} - k_2 e_2^2 \leq 0$$

Lets use Barbalat's Lemma,to prove that as $t \to \infty, e_1 \to 0, and, e_2 \to 0$.
Proof:
1. Since V (t) is lower bounded ($V0$) and non-increasing ($V(t) \leq 0$), implies that $V_\infty := lim_{t \to \infty} V(t) < \infty$ (i.e. the limit exists).
2. $V(t) \leq V(0) \implies V$ is bounded and k_1, k_2 are also bounded $\implies e_1, e_2$ are bounded $\implies e_1, e_2 \in L_\infty$.
3. $\int_0^\infty \frac{dV}{dt} dt = \int_0^\infty \frac{\dot{\mathrm{k}}_1 e_1^2}{2} - k_2 e_2^2 dt \implies V_\infty - V(0) = \int_0^\infty \frac{\dot{\mathrm{k}}_1 e_1^2}{2} - k_2 e_2^2 dt \leq -\int_0^\infty k_2 e_2^2 dt \implies \int_0^\infty k_2 e_2^2 dt \leq V(0) - V_\infty$ since k_2 is lower bounded, take infimum to be k_2^{inf}, $\int_0^\infty e_2^2 dt \leq (V(0) - V_\infty)/k_2^{inf}$ so $||e_2||_2$ bounded above by a finite quantity $\implies e_2 \in L_2$
4. Since $\dot{\mathrm{e}}_2 = k_2 e_2 - k_1 e_1$ and e_1, e_2, k_1, k_2 are bounded $\implies \dot{\mathrm{e}}_2$ is bounded $\implies \dot{\mathrm{e}}_2 \in L_\infty$
Now, $e_2 \in L_\infty \cap L_2$ and $\dot{\mathrm{e}}_2 \in L_\infty \implies e_2$ is Uniformly Continuous, then, using the Corollary of Barbalat's Lemma, we can say that $e_2 \to 0$.
5. $\int_0^\infty \frac{de_2}{dt} dt = e_2(\infty) - e_2(0) = -e_2(0) \implies e_2$ is integrable.
Taking derivative of Equation $\dot{\mathrm{e}}_2 = -k_1(t) e_1 - k_2(t) e_2 \implies \ddot{e}_2 = -k_1 \dot{e}_1 - \dot{k}_1 e_1 - \dot{k}_2 e_2 - k_2 \dot{e}_2$, all the term are bounded $\implies \ddot{e}_2 \in L_\infty \implies e_2$ is Uniformly Continuous.
Thus, using the Barbalat's Lemma we can say that $\dot{e}_2 \to 0$.
6. Since $\dot{e}_2 = -k_2 e_2 - k_1 e_1$, & as t $\to \infty, \dot{e}_2 \to 0, e_2 \to 0$ and k_1 and k_2 are bounded $\implies e_1 \to 0$