SC 617

Quiz-Week4

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1. Comment on definiteness, radial unboundedness and decrescent properties

Solution: i. $V(x) = \frac{x^2}{1+x^2}$
$V(t,0) = 0, \forall t \in \mathbb{R}^+$
Choose $\phi(x) = \frac{x^2}{1+x^2}$ and $\exists \phi \in \kappa$ such that $V(t,x) \geq \phi(|x|)\ \forall t \in \mathbb{R}^+\ \forall x \in \mathbb{B}_r$
So its a Positive Definiteness
Because $V(x) \leq 1$ one cannot find a function $\phi(|x|) \in$ KR to satisfy $V(x) \geq \phi(|x|) \forall x \in \mathbb{R}$. Hence, V is not radially unbounded
Choose $\phi(x) = x^2$ and $\exists \phi \in \kappa$ such that $V(t,x) \leq \phi(|x|)\ \forall t \in \mathbb{R}^+\ \forall x \in \mathbb{B}_r$
Hence it is decrescent

ii. $V(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_2^4}{2}$ $\geq \frac{x_1^4}{4}$
$V(t,0) = 0, \forall t \in \mathbb{R}^+$
Choose $\phi(x) = \frac{x^2}{4}$ and $\exists \phi \in \kappa$ such that $V(t,x) \geq \phi(|x|)\ \forall t \in \mathbb{R}^+\ \forall x \in \mathbb{B}_r$
So its a Positive Definiteness
Choose $\phi(x) = \frac{x^2}{4}$ and $\exists \phi(|x|) \in$ KR to satisfy $V(x) \geq \phi(|x|) \forall x \in \mathbb{R}^n$. Hence, V is radially unbounded
Choose $\phi(x) = x^4$ and $\exists \phi \in \kappa$ such that $V(t,x) \leq \phi(|x|)\ \forall t \in \mathbb{R}^+\ \forall x \in \mathbb{B}_r$
Hence it is decrescent

iii. $V(x_1, x_2) = \frac{x_1^2}{1+t} + x_2^2$
$V(t,0) = 0, \forall t \in \mathbb{R}^+$
Choose $\phi(x) = \frac{x^2}{4}$ and $\exists \phi \in \kappa$ such not that $V(t,x) \geq \phi(|x|)\ \forall t \in \mathbb{R}^+\ \forall x \in \mathbb{B}_r$
So its a Positive Definiteness
Choose $\phi(x) = \frac{x^2}{4}$ and $\exists \phi(|x|) \in$ KR to satisfy $V(x) \geq \phi(|x|) \forall x \in \mathbb{R}^n$. Hence, V is not radially unbounded
Choose $\phi(x) = x^4$ and $\exists \phi \in \kappa$ such that $V(t,x) \leq \phi(|x|)\ \forall t \in \mathbb{R}^+\ \forall x \in \mathbb{B}_r$
Hence it is decrescent

iv. $V(x_1, x_2) = k(1 - cos(x_1)) + \frac{x_2^4}{2}$
$V(t,0) = 0, \forall t \in \mathbb{R}^+$ and assuming K>0
Since $V(x_1, x_2) > 0\ \forall x\ \in \mathbb{R}^n - \{0\}$ $((-\pi, \pi) \times \mathbb{R})$
So its a Positive Definiteness
V is not radially unbounded because as $x \to \infty$ V is not tending to ∞ due to $1 - cosx_1$ term oscillates between 0 and 2 when $x_2 = 0, and, x_1 \to \infty$
Choose $\phi(x) = x^4 + 2K$ and $\exists \phi \in \kappa$ such that $V(t,x) \leq \phi(|x|)\ \forall t \in \mathbb{R}^+\ \forall x \in \mathbb{B}_r$
Hence it is decrescent

v.$V(x_1, x_2) = x_1^2 + tanh^2(x_2)$ where $tanh^2(x_2) = (\frac{e^{x_2}-1}{e^{x_2}+1})^2$
$V(t,0) = 0, \forall t \in \mathbb{R}^+$

Since $V(x_1, x_2) > 0 \ \forall x \ \in \mathbb{R}^n - \{0\}$
So its a Positive Definiteness
V is not radially unbounded because as $x \to \infty$ V is not tending to ∞ due to $\frac{e^{x_2}-1}{e^{x_2}+1}$ term has a finite limit when $x_1 = 0, and, x_2 \to \infty$
Choose $\phi(x) = x^2$ and $\exists \phi \in \kappa$ such that $V(t, x) \leq \phi(|x|) \ \forall t \in \mathbb{R}^+ \ \forall x \in \mathbb{B}_r$
Hence it is decrescent

2. State a Lyapunov Instability Theorem ? [1]

Solution: We have $x_e = 0$ and let $V : D \to \mathbb{R}$ having following properties
i. V(0) = 0
ii. $\exists x_0 \in \mathbb{R}^n$, arbitarily close to x_e, such that $V(x_0) > 0$
iii. $\dot{V} > 0, \forall x \in U$, where the set $U = \{x \in D : ||x|| \leq \epsilon \text{ and } V(x) > 0\}$
Then we can say that its unstable

3. Use above theorem to prove that the given system is unstable for c>0.

$$\dot{x}_1 = x_2 + cx_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = -x_1 + cx_2(x_1^2 + x_2^2)$$

Solution:

$$\dot{x}_1 = x_2 + cx_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = -x_1 + cx_2(x_1^2 + x_2^2)$$

Choose $V(x_1, x_2) = x_1^2 + x_2^2$
i. So we have V(0) = 0
ii. Now take a point $||x - x_e|| = ||x|| < \epsilon$ where $\epsilon > 0$ and $x \in \mathbb{R}^2$ we also have $V(x) = x_1^2 + x_2^2 > 0$ which is always positive other than at 0
iii. $\dot{V} = c(x_1 + x_2)^2 > 0 \ \forall x$ in set U, here set U describes the set of $\mathbb{R}^2 \backslash 0$ points
Hence the system is unstable

[1] https://nptel.ac.in/content/storage2/courses/101108047/module13/Lecture%2031.pdf