

Mathematical Methods of Classical Mechanics - V. I. Arnold

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Part I

Newtonian Mechanics

1 Experimental Facts (p. 3 - 11)

Certain experimental facts form the basis of classical mechanics. We can't verify them for certain, but they hold approximately, according to accurate tests.

1. **Space and time:** Space is 3D and Euclidean. Time is 1D.
2. **Galileo's Principle of Relativity:** There exist inertial coordinates such that
 - Laws of physics are same in all inertial coordinate systems.
 - Coordinate systems in uniform rectilinear motion w.r.t an inertial one is also inertial.
3. **Newton's Principle of Determinacy:** Initial state of a mechanical system uniquely determines its motion.

1.1 The Galilean Group

Let \mathbb{R}^n denote an n -dimensional real vector space.

An **affine n -dimensional space**, denoted A^n , is similar to \mathbb{R}^n but has no “fixed origin.” The group \mathbb{R}^n acts on A^n as the **group of parallel displacements**.

$$a \rightarrow a + \mathbf{b}, \quad a \in A^n, \quad \mathbf{b} \in \mathbb{R}^n, \quad a + \mathbf{b} \in A^n$$

The sum of two points on A^n is not defined, but their difference is a vector in \mathbb{R}^n . The distance between points of an affine space A^n can be defined using the scalar product:

$$\|x - y\| = \sqrt{(x - y, x - y)}$$

An affine space with this distance function is called a **Euclidean space**, denoted E^n .

The Galilean spacetime structure has 3 elements:

1. **The universe:** A 4D affine space A^4 . Points of A^4 are called *events*. Parallel displacements of A^4 form the vector space \mathbb{R}^4 .
2. **Time:** A linear mapping $t : \mathbb{R}^4 \rightarrow \mathbb{R}$ from a parallel displacement to the “time axis.” The time interval between $a, b \in A^4$ is $t(b - a)$.
3. **Distance Between Simultaneous Events:** Given by $\|a - b\| = \sqrt{(a - b, a - b)}$. This is a scalar product on \mathbb{R}^3 . The space of simultaneous events is thus a 3D Euclidean space E^3 .

A Galilean space is a space A^4 that has a Galilean spacetime structure.

The Galilean group is the group of all transformations of a Galilean space which preserves its structure.

Elements of a Galilean group are called **Galilean transformations**: affine transformations on A^4 which preserve the time interval & distance between simultaneous events.

The **Galilean coordinate space** is the direct product $\mathbb{R} \times \mathbb{R}^3$ of the t axis with the 3D vector space \mathbb{R}^3 . There are 3 examples of Galilean transformations of this space:

1. Uniform motion with velocity \vec{v} : $g_1(t, \vec{x}) = (t, \vec{x} + \vec{v}t)$
2. Translation of origin: $g_2(t, \vec{x}) = (t + s, \vec{x} + \vec{s})$
3. Rotation of coordinate axes: $g_3(t, \vec{x}) = (t, G\vec{x})$, $G : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is orthogonal transformation.

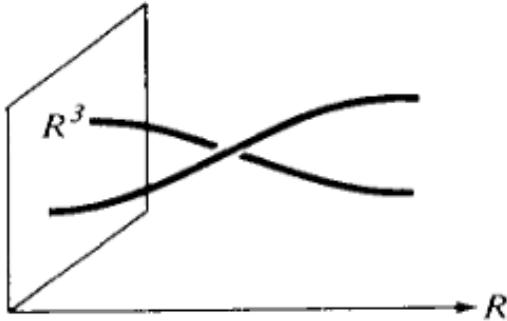
Every Galilean transformation of the space $\mathbb{R} \times \mathbb{R}^3$ can be written as the composition of the above, thus the dimension of the Galilean group is $3 + 4 + 3 = 10$.

Let M be a set. A one-to-one correspondence $\phi_1 : M \rightarrow \mathbb{R} \times \mathbb{R}^3$ is called a **Galilean coordinate system** on the set M .

1.2 Motion, Velocity, and Acceleration

A motion in \mathbb{R}^n is a differentiable mapping $\mathbf{x} : I \rightarrow \mathbb{R}^N$, where I is an interval on the real axis. The image of a mapping $\mathbf{x} : I \rightarrow \mathbb{R}^N$ is called a **trajectory** or **curve** in \mathbb{R}^N . The velocity vector is the derivative $\dot{\mathbf{x}}(t_0)$, while the acceleration vector is the second derivative $\ddot{\mathbf{x}}(t_0)$.

Let $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^3$ be a motion in \mathbb{R}^3 . The graph of this mapping (\mathbb{R}^3 against \mathbb{R}) is a curve in $\mathbb{R} \times \mathbb{R}^3$. A curve in Galilean space that appears in some Galilean coordinate system (as the graph of a motion) is called a **world line**.



Galilean worldlines

Consider a system with n points. In Galilean space, this gives n world lines, described by n mappings $\mathbf{x}_i : \mathbb{R} \rightarrow \mathbb{R}^3$ in a Galilean coordinate system. In total, we have

$$\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^N, \quad N = 3n$$

as the total motion of our system with n points.

Newton's principle of determinacy states that all motions of a system are uniquely determined by their initial positions and velocities. Acceleration is not needed: it's determined by the initial position and velocities. I.e. there's a function $\mathbf{F} : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N$ such that

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$$

This is called **Newton's equation**. By the theorem of existence and uniqueness (recall from diff eq), the initial conditions \mathbf{F} , $\mathbf{x}(t_0)$, and $\dot{\mathbf{x}}(t_0)$ uniquely determine a motion. The form of \mathbf{F} can be determined experimentally.

Galileo's principle of relativity requires that Galilean spacetime structure must be invariant w.r.t the group of Galilean transformations. This is a condition on Newton's equation, and leads to three properties of spacetime:

1. **Invariance under time translations:** Laws of nature remain constant regardless of time.

2. **Invariance under spatial translations:** Space is homogeneous; has the same properties at all of its points.
3. **Invariance under spatial rotations:** Space is isotropic; no preferred directions.

We may also introduce the “potential energy” U to write Newton’s equation. Let $E^{3n} = E^3 \times \cdots \times E^3$ be the configuration space of a system of n points in the Euclidean space E^3 . Let $U : E^{3n} \rightarrow \mathbb{R}$ be a differentiable function. The motion of the n points (of masses m_1, \dots, m_n) is given by the system of differentiable equations

$$m_i \ddot{\mathbf{x}}_i = -\frac{\partial U}{\partial \mathbf{x}_i}, \quad i = 1, \dots, n$$

where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$ and $\partial U / \partial \mathbf{x} = (\partial U / \partial \mathbf{x}_1, \dots, \partial U / \partial \mathbf{x}_k)$. The equations of motion for many other mechanical systems can be written in this form.