Modern Physics

week 24 Notes

* Black body radiation

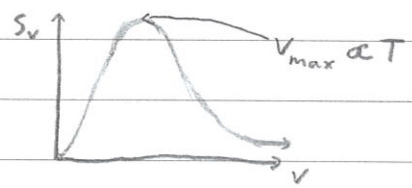
- Classically: energy flux irradiated by black body given by Stefan - Boltzmann Law

$$\boxed{S = \sigma T^4} \qquad \sigma = \text{Stefan-Boltzmann const.}$$

$$\boxed{S = \varepsilon \sigma T^4} \qquad (\text{for non-black body, } \varepsilon < 1. \quad \varepsilon = 1 \Rightarrow \text{perfect absorber})$$

- Black body radiation spectrum

↳ Total power radiated: $\int_0^\infty S_\nu \, d\nu = S = \sigma T^4$



↳ Classical calculation of spectrum using equipartition theorem doesn't fit the curve — "ultraviolet catastrophe"

↳ With QM: $S_\nu = \dfrac{2\pi h}{c^2} \dfrac{\nu^3}{e^{h\nu/kT} - 1}$, $h = 6.62618 \times 10^{-34}$ J·s

↳ works <u>if</u> oscillators have energy $E = nh\nu$, where $n = 0, 1, 2 \ldots$

- Calculation:

↳ From stat mech, $S_\nu = \dfrac{2\pi \nu^2}{c^2} \bar{E}$, $\bar{E}$ = avg. energy $= \dfrac{\sum_{n=1}^{\infty} nh\nu e^{-\frac{nh\nu}{kT}}}{\underbrace{\sum_{m=1}^{\infty} e^{-\frac{mh\nu}{kT}}}_{\text{Normalization}}}$

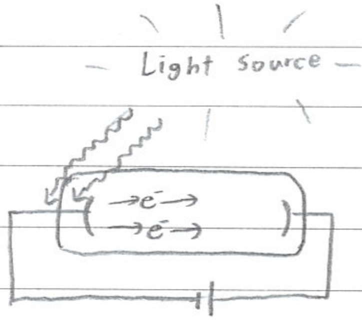
↳ Let $x = e^{-h\nu/kT} \Rightarrow E = \dfrac{\sum_{n=1}^{\infty} n x^n}{\sum_{m=1}^{\infty} x^m}$

$\Rightarrow \boxed{E = \dfrac{h\nu}{e^{h\nu/kT} - 1}}$. Avg. energy at high $h\nu$ is small, avoids ultraviolet catastrophe.

$\Rightarrow \boxed{S_\nu = \dfrac{2\pi \nu^2}{c^2} \dfrac{h\nu}{e^{h\nu/kT} - 1}}$. Integrate to get Stefan - Boltzmann Law.

* The Photoelectric Effect

- Einstein: e.m. radiation quantized ⇒ photons



Electrons ejected by $\underbrace{\text{stream of photons}}_{\text{Light}}$

↳ Not explainable by classical wave theory of light

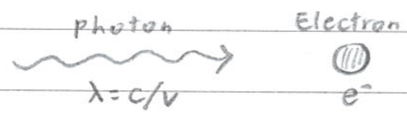
↳ Einstein: suppose that light is quantized & localized in space

Then all of $h\nu$ can be transferred to 1 electron

* Compton Effect

- Another demonstration of the quantized nature of light (cf. "wave" in classical physics)

- Before:

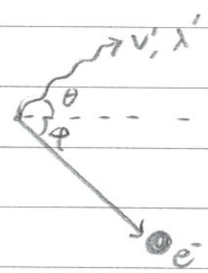


$\lambda = c/\nu$

$E = h\nu,\ p = h\nu/c$

$$\begin{cases} E = h\nu \\ E^2 = (mc^2)^2 + (pc)^2 \end{cases} \Rightarrow p = \frac{h\nu}{c}$$

- After:



- Conserve momentum: $\frac{h\nu}{c} = \frac{h\nu}{c}\cos\theta + P_e \cos\varphi$

$$0 = \frac{h\nu}{c}\sin\theta - P_e \sin\varphi$$

- Conserve energy: $h\nu + m_e c^2 = h\nu' + m_e c^2 + K_e$

$\therefore \frac{1}{h\nu'} = \frac{1}{h\nu} + \frac{1}{m_e c^2}(1-\cos\theta)$, or $\boxed{\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)}$

Modern Physics

Week 26 Notes

* Mathematical Structure of QM

- Previously: Intensity $= I = \frac{1}{2}\varepsilon_0 |E|^2$, $\lim_{N \to \infty} \frac{n(x)}{N}$

- Now: probability $= p(x) = |\psi(x)|^2$, $\lim_{N \to \infty} \frac{n(x)}{N}$

  ↳ $\psi(x)$: quantum state or wavefunction for 1 photon

  ↳ Mathematically like an EM wave, but represents something different

- From Optics: $\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = c^2 \nabla^2 \vec{E}(\vec{r},t)$

  ↳ EM wave solution for double slit experiment must satisfy this equation

  ↳ Since $\vec{E}(\vec{r},t) \to \psi(\vec{r},t)$,

$$\frac{\partial^2 \psi(\vec{r},t)}{\partial t^2} = c^2 \nabla^2 \psi(\vec{r},t)$$

  ↳ Wave equation for a photon

- Some Postulates

1. Quantum State: quantum state of system is represented mathematically by normalized wavefunction ψ. Contains all information that can be known about a system

   ↳ Wavefunction yields probabilities. No longer a well-defined classical state

   ↳ Wavefunction is normalized: $\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$

   ↳ Wavefunction represented in Hilbert space (Crudely: space of all wavefunctions)

2. Observables (more on this later)
3. Measurement

   ↳ Outcome of measurement of a quantum system yields 1 of possible outcomes in accordance to probability measured from state

$$p(x) = |\psi(x)|^2 \quad \leftarrow \text{Born Probability Rule}$$

※ Modulus square: $|\psi|^2 = \psi^* \psi$, $*$ is complex conjugate

- Born Probability Rule for state vectors:

$$\boxed{P_\psi(\omega) = |\langle \omega | \psi \rangle|^2}$$ ω = outcome, ψ = state object is in

↳ $\psi(x) = \langle x | \psi \rangle$

* Two-State Observables: Spin 1/2

- Stern-Gerlach Experiment
- Every particle has magnetic moment $\vec{\mu}$
  ↳ within $\vec{B}$ field it has potential energy $\vec{U} = -\vec{\mu} \cdot \vec{B}$
  ↳ Force on particle is $\vec{F} = \nabla(\vec{\mu} \cdot \vec{B})$

- In S-G, $\vec{B} \sim B_0 z \hat{k}$ (Magnitude grows as z grows)
- Thus $\vec{F} \to F_z = \frac{\partial}{\partial z}(\mu_z B_z) \approx \mu_z \frac{\partial B_z}{\partial z}$

- Classically: $\vec{\mu}$ distributed isotropically, distribution along z should be continuous
  ↳ But that doesn't happen

- Two possible outcomes: $+\frac{\hbar}{2}, -\frac{\hbar}{2}$
  ↳ Require 2D Hilbert space spanned by basis kets $|\uparrow\rangle_z, |\downarrow\rangle_z$

* Two-state observables: polarization

- Malus's Law: $I(\theta) = I_0 \cos^2\theta$
  - ↳ Probability of transmission: $p(\theta) = \cos^2\theta$
  - ↳ Basis kets: $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- Represent linearly polarized photon: $|\theta\rangle = \cos\theta\,|H\rangle + \sin\theta\,|V\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

- Since state vectors may be complex, general form is

$$|\theta, \phi\rangle = \cos\theta\,|H\rangle + e^{i\varphi}\sin\theta\,|V\rangle = \begin{pmatrix} \cos\theta \\ e^{i\varphi}\sin\theta \end{pmatrix}$$

- Another set of bases $\frac{\pi}{4}$ from HV:
  - ↳ $|P\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$, $|M\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$
  - ↳ HV and PM are complementary.
  - ↳ Can only have perfect knowledge of outcome in 1 basis at a time.
  - ↳ If we know outcome in HV, we don't know what it is in PM

- RL basis (right & left circularly polarized light)
  - ↳ $|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)$, $|L\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle)$

4. Time Evolution: time evolution of closed quantum system is governed by appropriate wave equation

↳ Allowed transformations called unitary transformations

↳ Criteria for unitary transformations: preserve normalization of probability

* Discrete Observables

- Ex. flipping a coin (heads or tails)
- $\psi_{coin} = \psi_H + \psi_T$

↳ $P = |\psi_H + \psi_T|^2 = |\psi_H|^2 + |\psi_T|^2 + \psi_H^* \psi_T + \psi_H \psi_T^*$

↳ Cross terms must vanish, so the wavefunctions need to be orthogonal vectors.

- Introduce complex vector space $\mathcal{H}$, allows components to take on complex values.

↳ Ket vector: $|\psi\rangle = |\psi_H\rangle + |\psi_T\rangle$

$|\psi\rangle = \psi_H |H\rangle + \psi_T |T\rangle$

$\boxed{|H\rangle, |T\rangle \text{ are real vectors}}$

↳ Bra vector: $\langle\psi| = \psi_H^* \langle H| + \psi_T^* \langle T|$

$\boxed{\text{And not that kind of bra}}$

- Generalize dot product to inner product within $\mathcal{H}$

↳ $\langle\psi|\psi\rangle = |\psi_H|^2 \langle H|H\rangle + |\psi_T|^2 \langle T|T\rangle + \psi_H^* \psi_T \langle H|T\rangle + \psi_H \psi_T^* \langle T|H\rangle$

$= |\psi_H|^2 + |\psi_T|^2$

※ If the basis vectors $|H\rangle$ and $|T\rangle$ are orthonormal

- Ex. $\psi_H = \frac{i}{2}$, $\psi_T = \frac{\sqrt{3}}{2}$, $|\psi\rangle = \frac{i}{2}|H\rangle + \frac{\sqrt{3}}{2}|T\rangle$

↳ Probability to come up heads:

$\langle H|\psi\rangle = \frac{i}{2}\langle H|H\rangle + \frac{\sqrt{3}}{2}\langle T|H\rangle = \frac{i}{2}$

So we see that $\psi_H = \frac{i}{2}$

↳ $P_\psi(H) = |\langle H|\psi\rangle|^2 = |\frac{i}{2}|^2 = \frac{1}{4}$

Modern Physics

week 27 Notes

*Mathematics of the State vectors & Hilbert spaces

- Inner product: generalization of dot product
  - ↳ Properties of $\langle u|v\rangle \in C$ for $|u\rangle, |v\rangle, |w\rangle \in V$ and $\lambda, \mu \in C$
    1. Distributive: $\langle u|(|v\rangle + |w\rangle)\rangle = \langle u|v\rangle + \langle u|w\rangle$
    2. Scalar distributivity: $\langle u|(\lambda|v\rangle)\rangle = \lambda\langle u|v\rangle$
    3. Complex conjugate: $\langle u|v\rangle^* = \langle v|u\rangle$
    4. Vector norm: $\langle u|u\rangle \geq 0$ & $\in R$
       - ↳ $\langle u|u\rangle = 0 \iff |u\rangle = 0$
       - ↳ $\||u\rangle\| = \sqrt{\langle u|u\rangle}$

    - ↳ Normalized vectors: $\langle u|u\rangle = 1$
    - ↳ Orthogonal vectors: $\langle u|v\rangle = 0$

- Hilbert space, $\mathcal{H}$
  - ↳ Linear vector space (10 axioms) for which an inner product is defined as above

- Dual vector space (adjoint space), $V^\dagger$
  - ↳ For each ket vector $|u\rangle \in V$, there is a unique bra vector $\langle u| \in V^\dagger$

- Matrix form of bra & ket vectors
  - ↳ with orthonormal basis, $|u\rangle = u_{11}|e_1\rangle + u_{12}|e_2\rangle + \ldots + u_{N1}|e_N\rangle$
  - ↳

$$|u\rangle = \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ \vdots \\ u_{N1} \end{pmatrix}, \quad \langle v| = (v_{11}, v_{12}, v_{13} \ldots v_{1N})$$

  - ↳ Hermitian Conjugate: if matrix A has elements a_{nm}, $A^\dagger$ has elements a^*_{mn}
    (Flip elements about diagonal & complex conjugate all elements)
  - ↳ Hermitian operator: $X^\dagger = X$

※ e.g. $|u\rangle = \begin{pmatrix} 1 \\ i \\ 3+2i \\ e^{-i\pi/6} \end{pmatrix}$, so $\langle u| = \begin{pmatrix} 1 & -i & 3-2i & e^{i\pi/6} \end{pmatrix}$

※ General form of inner product:

↳ $|u\rangle = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}$, $|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$

↳ $\langle u|v\rangle = \begin{pmatrix} u_1^* & u_2^* & \dots & u_N^* \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = u_1^* v_1 + u_2^* v_2 + \dots + u_N^* v_N$

- Postulate 1: quantum state
  ↳ Quantum state of system is represented mathematically by normalized vector $|\varphi\rangle$ in a Hilbert space $\mathcal{H}$.
  ↳ Quantum state contains all info that can be known about system

* Superpositions

- Quantum state is a vector in Hilbert space
  ↳ Possible that $|\varphi\rangle$ is not parallel to outcome in some basis
  ↳ Superposition, e.g. $|+\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$

- Global phase factors: $e^{i\delta}$ multiplying entire state
  ↳ Does not affect probabilities
  ↳ e.g. For $|\psi\rangle = e^{i\delta}(a|\alpha\rangle + b|\beta\rangle)$,

$$p(\omega \text{ in } \psi) = |\langle\omega|\psi\rangle|^2 = \langle\omega|\psi\rangle^* \langle\omega|\psi\rangle$$
$$= e^{-i\delta}(a\langle\omega|\alpha\rangle + b\langle\omega|\beta\rangle)^* \, e^{i\delta}(a\langle\omega|\alpha\rangle + b\langle\omega|\beta\rangle)$$
$$= |\langle\omega|(a|\alpha\rangle + b|\beta\rangle)\rangle|^2$$

※ Can be used to simplify some states

※ But relative phase factors are important

* Multiparticle states

- Alice & Bob have a photon
- How to represent polarization state of both?

- Alice's state: $|H\rangle_A \in \mathcal{H}_A$
- Bob's state: $|V\rangle_B \in \mathcal{H}_B$

- To consider as single system, need joint quantum states of both photons
- Given by tensor product state

$$\boxed{|H\rangle_A \otimes |V\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B}$$

↳ Simplified notation: $|H\rangle_A \otimes |V\rangle_B \equiv |H\rangle_A |V\rangle_B \equiv |H\rangle |V\rangle \equiv |HV\rangle$

- Properties of tensor product
  1. Scalar multiplication: $\lambda(|u\rangle|v\rangle) = (\lambda|u\rangle)|v\rangle = |u\rangle(\lambda|v\rangle)$
  2. Distributivity: $(|u\rangle + |v\rangle)|w\rangle = |u\rangle|w\rangle + |v\rangle|w\rangle$
  3. Inner product: $\langle uv|u'v'\rangle = \langle u|u'\rangle\langle v|v'\rangle$
     ↳ For $|u\rangle|v\rangle$, $|u'\rangle|v'\rangle$

※ Dimension of product space is the product of dimensions of subspaces

※ If joint probability cannot be factorized → entangled states
  ↳ If 1 outcome is measured, state of the other is determined
  ↳ But prior to measurement, we can't think of them independently

Modern Physics

week 27 Notes

* Transformation of states

- Consider state $|+\rangle$ impinging on polaroid oriented along H
  - ↳ Probability to transmit is $\frac{1}{2}$
  - ↳ State after polarizer: $\frac{1}{\sqrt{2}}|H\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}$

↳ $$|\psi\rangle_{after} = \underbrace{\begin{pmatrix}1&0\\0&0\end{pmatrix}}_{\text{outer product}}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}_{|+\rangle} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\boxed{|\psi\rangle_{after} = |H\rangle\langle H|+\rangle = \frac{1}{\sqrt{2}}|H\rangle}$$

↳ outer product $|H\rangle\langle H|$ is an operator (matrix) projecting out the H-component

- $|H\rangle\langle H|$ is operator projecting out H-component
- $|V\rangle\langle V|$ is operator projecting out V-component

- $|H\rangle\langle H| + |V\rangle\langle V| = \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$, identity
  - ↳ Probability adds to 1
  - ↳ Resolution of identity: consider orthonormal, complete set of basis vectors in $\mathcal{H}$, $\{|\omega_i\rangle\}$, $i=1,\dots n$.

    Sum of projectors is the identity operator

$$\boxed{\sum_{i=1}^{n}|\omega_i\rangle\langle\omega_i| = \hat{1}}$$

- Example: state $|H\rangle$ sent through polarizers $+, R, H, -$

↳ $|\psi\rangle_{after} = \underbrace{|-\rangle\langle -|}_{\text{Outer prod.}}H\rangle\langle H|R\rangle\langle R|+\rangle\langle +|H\rangle$

↳ ... $|\psi\rangle_{after} = |-\rangle\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1-i}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1-i}{4\sqrt{2}}|-\rangle$

Remember to complex conjugate when going from bra → ket or ket → bra

↳ $P(-\text{ in }\psi_{after}) = |\langle -|\psi_{after}\rangle|^2 = \left|\frac{1-i}{4\sqrt{2}}\right|^2 = \frac{2}{(4\sqrt{2})^2} = \frac{1}{16}$

- Some quantum gates (operators)

1. Identity operator: $\hat{1}$. Trivially does nothing to state
   ↳ Sum of projectors in any basis, e.g. $\hat{1} = |H\rangle\langle H| + |V\rangle\langle V|$

2. Swap operator: $\hat{X}$. Swaps H and V
   ↳ $\hat{X} = \underset{\text{output}}{|H\rangle}\underset{\text{input}}{\langle V|} + |V\rangle\langle H| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
   ↳ "Takes V, gives H. Takes H, gives V."

3. Hadamard gate: $\hat{H}$. Converts HV to PM and back
   ↳ $\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

	$\hat{H}\|H\rangle = \|+\rangle$	$\hat{H}\|+\rangle = \|H\rangle$
	$\hat{H}\|V\rangle = \|-\rangle$	$\hat{H}\|-\rangle = \|V\rangle$

- Eigenvectors & eigenvalues
  ↳ Want to identify operator that doesn't change state (up to an overall const.)
  ↳ Outcomes:
  - HV basis: $H = +1$, $V = -1$
  - PM basis: $P = +1$, $M = -1$
  - RL basis: $R = +1$, $L = -1$

  ↳ Construct operator that, in 1 basis, returns that state multiplied by the outcome.
  ↳ Such operators are called observables

  ↳ E.g. Observable for HV basis:
  $$\hat{O}_{HV} \equiv \hat{Z} = \underset{\text{output}}{|H\rangle}\underset{\text{input}}{\langle H|} \underbrace{- |V\rangle}_{\text{output}}\underset{\text{input}}{\langle V|} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  ↳ For PM: $\hat{O}_{PM} \equiv \hat{X} = |+\rangle\langle +| - |-\rangle\langle -| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  ↳ For RL: $\hat{O}_{RL} \equiv \hat{Y} = |R\rangle\langle R| - |L\rangle\langle L| = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

↳ $\hat{O}|\omega_i\rangle = \omega_i|\omega_i\rangle$

eigenvalue eigenvector

"ω_i is eigenvalue corresponding to eigenvector $|\omega_i\rangle$ of observable $\hat{O}$."

※ physically observable outcomes are real numbers, so the eigenvalues of any observable must be real.

※ i.e. Hermitian operators have real eigenvalues

$$\hat{\Omega} \equiv \sum_{i=1}^{n} \omega_i |\omega_i\rangle\langle\omega_i| , \quad \omega_i \text{ is outcome}$$

$$\omega_i \in \mathbb{R}$$

Modern Physics

Week 28 Notes

* Compatible vs. incompatible observables

- Example
  - ↳ State vector $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is eigenvector of $\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ w/ eigenvalue +1
  - ↳ State vector $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not eigenvector of $\hat{Z}$

↳ eigenvalue is a definite state in that basis. If measured in that basis, corresponding outcome will occur with prob. 1

↳ Measuring along HV affects PM values: e.g. projecting along H then M disturbs HV value, no longer definite value

↳ Measurement of 1 affects other: $\hat{X}\hat{Z}|H\rangle = \hat{X}|H\rangle = |V\rangle$

$$\hat{Z}\hat{X}|H\rangle = \hat{Z}|V\rangle = -|V\rangle$$

※ Operating on state vector ≠ measurement

- If a state has definite values of 2 distinct observables, the operators obey

$$\boxed{\hat{A}\hat{B} - \hat{B}\hat{A} = 0}$$

↳ e.g. state $|a,b\rangle$ that's an eigenstate of $\hat{A}, \hat{B}$

$$\hat{A}|a,b\rangle = a|a,b\rangle$$

$$\hat{B}|a,b\rangle = b|a,b\rangle$$

$$(\hat{A}\hat{B} - \hat{B}\hat{A})|a,b\rangle = b\hat{A}|a,b\rangle - a\hat{B}|a,b\rangle = 0$$

- The commutator: $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$
  - ↳ If commutator vanishes, observables are compatible
  - ↳ The 2 properties can be simultaneously known

※ Incompatible observables: spin along x, y, z

position & momentum

* Expectation Values

- In Classical physics: $E(A) = \langle A \rangle = \sum_{i=1}^{n} a_i P(A = a_i)$
- In Quantum physics: need to find E.V. of quantum observable $\hat{A}$ in state $|\psi\rangle$

$$\langle \hat{A} \rangle_\psi = \sum_{i=1}^{n} a_i |\langle a_i | \psi \rangle|^2 = \sum_{i=1}^{n} a_i \langle \psi | a_i \rangle \langle a_i | \psi \rangle$$

$$\langle \psi | \hat{A} | \psi \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \psi | a_i \rangle \langle a_i | \hat{A} | a_j \rangle \langle a_j | \psi \rangle$$

a_i are outcomes

$$\vdots$$

$$\boxed{\langle \hat{A} \rangle_\psi = \langle \psi | \hat{A} | \psi \rangle}$$ Expectation value of $\hat{A}$

- Other statistics:

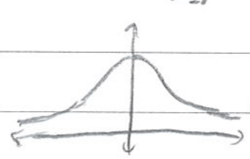
↳ $\langle \hat{A} \rangle_\psi = \langle \psi | \hat{A} | \psi \rangle$, $\langle \hat{A}^2 \rangle_\psi = \langle \psi | \hat{A}^2 | \psi \rangle$

1. $\mathrm{var}(\hat{A})_\psi = \langle \hat{A}^2 \rangle_\psi - (\langle \hat{A} \rangle_\psi)^2$
2. $\mathrm{cov}(\hat{A}\hat{B})_\psi = \langle \hat{A}\hat{B} \rangle_\psi - \langle \hat{A} \rangle \langle \hat{B} \rangle_\psi$
3. $\mathrm{corr}(\hat{A}\hat{B})_\psi = \dfrac{\mathrm{cov}(\hat{A}\hat{B})}{\sqrt{\mathrm{var}(\hat{A})_\psi \mathrm{var}(\hat{B})_\psi}}$

* The Uncertainty Principle

- For 2 incompatible observables, we can't construct a state that has sharp values of those observables simultaneously
- Consider eigenstate $|a_2\rangle$ of $\hat{A}$:

$$\langle \hat{A} \rangle_{|a_2\rangle} = a_2 \quad , \quad \mathrm{var}(\hat{A})_{|a_2\rangle} = a_2^2 - a_2^2 = 0$$



Variance measures "splay" about mean

If var = 0, no splay.

- Uncertainty of observables = Standard dev. = $\Delta\hat{A} = \sqrt{var(A)}$

- Consider 2 incompatible observables $\hat{A}, \hat{B}$:

↳ If $|a\rangle$ is an eigenstate of $\hat{A}$ but not $\hat{B}$,

$\hat{A}|a\rangle = a|a\rangle$, $\Delta_a \hat{A} = \sqrt{var A} = 0$

$[\hat{A},\hat{B}] \neq 0$, $\hat{B}|a\rangle = b|\varphi\rangle$ and $|\varphi\rangle$ is not along $|a\rangle$ direction

↳ Possibility 1: $|\varphi\rangle$ is along $|a_\perp\rangle$ direction, $|\varphi\rangle \sim |a\rangle$

$var(B) = \langle a|\hat{B}^2|a\rangle - \langle a|\hat{B}|a\rangle\langle a|\hat{B}|a\rangle$

$= b\langle a|\hat{B}|a_\perp\rangle$

$\equiv bc\langle a|\psi\rangle$, where $\hat{B}|a_\perp\rangle = c|\psi\rangle$

$|\psi\rangle \neq |a_\perp\rangle$ by our assumption

thus $bc\langle a|\psi\rangle \neq 0$, $\boxed{var(B) \neq 0}$

↳ Possibility 2: $|\varphi\rangle$ is not along $|a_\perp\rangle$ direction, $|\varphi\rangle \nsim |a_\perp\rangle$

$var(B) = d\langle a|\hat{B}|\varphi\rangle - d^2\langle a|\varphi\rangle\langle a|\varphi\rangle$

$\hat{B}|\varphi\rangle \nsim |\varphi\rangle$ from our assumption

Let $\hat{B}|\varphi\rangle = e|\psi\rangle$

⋮

$$\boxed{var(B) = de\langle a|\psi\rangle - d^2\langle a|\varphi\rangle\langle a|\varphi\rangle \neq 0}$$

- Generalized Uncertainty Principle: $\boxed{\Delta_\psi \hat{A}\ \Delta_\psi \hat{B} \geq \frac{1}{2}\left|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle\right|}$

※ Looking ahead: Heisenberg Uncertainty Principle

$\hat{X} = x$, $\hat{P} = -i\hbar\frac{\partial}{\partial x}$

$[\hat{x},\hat{p}] = -i\hbar\left(x\psi_x(x) - \psi(x) + x\psi_x(x)\right) = i\hbar\psi(x)$

$\rightarrow [\hat{x},\hat{p}] = i\hbar\hat{1}$, $\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}}$

Modern physics

week 29 Notes

* plane wave states

- Maxwell's wave equation: $\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2} = 0$

$$\frac{\partial^2 B(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 B(x,t)}{\partial t^2} = 0$$

↳ Cosine solution: $E(x,t) = E_0 \cos(kx - \omega t)$, $\omega = kc$

↳ Exponential solution: $E(x,t) = E_0 e^{i(kx-\omega t)}$

- Wavefunction for single photon: $\Psi(x,t) = Ae^{i(kx-\omega t)}$

↳ $\lambda = \frac{h}{p} \longrightarrow p = \hbar k$

↳ $E = hf \longrightarrow E = \hbar\omega$

$$\boxed{\Psi(x,t) = Ae^{\frac{i}{\hbar}(px - Et)}}$$

plane wave state in terms of energy & momentum

Energy operator

※ Let $\hat{E} = i\hbar\frac{\partial}{\partial t}$, an operator (linear, Hermitian)

- Take derivative of plane wave state:

$$\frac{\partial\Psi}{\partial t} = -\frac{iE}{\hbar}\Psi$$

$$\left(i\hbar\frac{\partial}{\partial t}\right)\Psi = E\Psi \longrightarrow \boxed{\hat{E}\Psi = E\Psi}$$

↳ plane wave is an eigenstate of energy operator $\hat{E}$ w/ eigenvalue E.

Momentum operator

※ Let $\hat{P} = -i\hbar\frac{\partial}{\partial x}$, an operator

- Take derivative of plane wave state:

$$\frac{\partial\Psi}{\partial x} = \frac{ip}{\hbar}\Psi \longrightarrow \boxed{\hat{P}\Psi = p\Psi}$$

↳ Plane wave is eigenstate of momentum operator $\hat{P}$ w/ eigenvalue p.

- Maxwell's wave equation with $\hat{E}$ and $\hat{P}$

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi(x,t)}{\partial t^2} = 0$$

$$-i\hbar\frac{\partial}{\partial x}\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x,t) - \frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)\left(i\hbar\frac{\partial}{\partial t}\right)\psi(x,t) = 0$$

$$\hat{P}^2\psi(x,t) - \frac{1}{c^2}\hat{E}^2\psi(x,t) = 0$$

$$\boxed{\hat{E}^2\psi(x,t) = c^2\hat{P}^2\psi(x,t)}$$

※ Using $\hat{E}\psi = E\psi$, $\hat{P}\psi = p\psi$: $E^2\cancel{\psi(x,t)} = c^2p^2\cancel{\psi(x,t)}$

$\boxed{E = pc}$ ← Relativistic energy of photon

* QM Postulates

1. Quantum state is represented by normalized ket
2. Observables are represented by operators acting on kets
3. Measurement results in the system jumping to 1 of the eigenkets of the operator
   ↳ Probability obtained via Born: $|\langle a|\psi\rangle|^2$

* Class notes

- Matrix elements of A (eigenkets a', a''): $\langle a''|A|a'\rangle = a'\,\delta_{a'a''}$
  ↳ If Kronecker delta $\delta_{a'a''} = 0$ when not on diagonal, A is a diagonal matrix
  $a' \neq a''$
  ↳ If A, B commute, eigenvectors are same
  B has eigenvalues $\langle a'|B|a'\rangle$

Modern Physics

week 32 Notes

* The Schrödinger Equation

- Loose derivation

↳ Classical energy of electron: $E = \frac{p^2}{2m} + V(x,t)$

↳ Limit to time-independent potential: $E = \frac{p^2}{2m} + V(x)$

↳ $E\psi = \frac{p^2}{2m}\psi + V(x)\psi$, and $\hat{p}\psi = p\psi$, $\hat{E}\psi = E\psi$, so

$$\boxed{\hat{E}\psi = \frac{\hat{p}^2}{2m}\psi + V(x)\psi}$$

↳ Substitute operator expressions: $\hat{p} \equiv -i\hbar\frac{\partial}{\partial x}$, $\hat{E} \equiv i\hbar\frac{\partial}{\partial t}$

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi}$$

The Schrödinger Equation for a non-relativistic object in 1-dimensional, time-independent potential

- Cases

1. Free particle in 1D: $i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi$

2. Harmonic oscillator: $i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi + \frac{1}{2}m\omega^2x^2\psi$

(Time dependent form for now)

3. Particle in 3D potential: $i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar}{2m}\nabla^2\psi + V(x,y,z)\psi$

↳ $p^2 = \vec{p}\cdot\vec{p} = p_x^2 + p_y^2 + p_z^2$

↳ $\hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 = -\hbar^2\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)}_{\text{Laplacian, } \nabla^2}$

4. Hydrogen atom: $i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar}{2\mu}\nabla^2\psi - k\frac{e^2}{r}\psi$

↳ Simplified.

↳ μ is reduced mass.

• Rewrite SE as $\boxed{i\hbar \frac{\partial}{\partial t}\Psi = \hat{H}\Psi}$

Remember,
$i\hbar \frac{\partial}{\partial t}\Psi = E\Psi$

↳ $\hat{H}$ is the Hamiltonian: $\hat{H} = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + V(x)$

↳ Hermitian (real eigenvalues, b.c. energies are real numbers)

※ How to think of SE

↳ Determines which states Ψ are permissible depending on system (form of $\hat{H}$)

↳ Allows one to find energy spectrum of system (given Ψ, $\hat{H}$ returns energy of that state)

↳ Gives time evolution of Ψ: given $\Psi(t)$, gives $\Psi'(t)$

• Time evolution: let $\hat{U}$ be a time development operator

↳ $\hat{U}(t', t)\Psi(t) = \Psi(t')$

↳ $\hat{U}(t+dt, t)\Psi(t) = \Psi(t+dt) = \Psi(t) + dt \left.\frac{\partial \Psi}{\partial t}\right|_t$

↳ From SE, $\frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar}\hat{H}\Psi$, so $\Psi(t+dt) = \Psi(t) - dt\frac{i}{\hbar}\hat{H}\Psi(t)$

$$\Psi(t+dt) = \left(\hat{1} - dt\frac{i}{\hbar}\hat{H}\right)\Psi(t)$$

$$\boxed{\hat{U}(t+dt, t) = \hat{1} - dt\frac{i}{\hbar}\hat{H}}$$

※ In general for $\hat{H} \neq \hat{H}(t)$,

$$\boxed{\hat{U}(t,0) = \hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}}$$

※ Unitary evolution: $\hat{U}^{\dagger}\hat{U} = e^{\frac{i}{\hbar}\hat{H}t} e^{-\frac{i}{\hbar}\hat{H}t} = \hat{1}$

Unitary to preserve normalization of probability: $\int \Psi^*(x,t)\Psi(x,t)dx = 1$

$$\int \Psi^*(x,t+dt)\Psi(x,t+dt)\,dx = 1 = \int \Psi^*(x,t)\underbrace{\hat{U}^{\dagger}\hat{U}}\Psi(x,t)dx$$

⚠ When measurement is performed, smooth evolution is broken. Not a unitary evolution.

✱ Free particle solution of Schrödinger Equation

- $i\hbar \frac{\partial}{\partial t}\Psi = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\Psi$

↳ Assume definite energy E: $i\hbar\frac{\partial}{\partial t}\Psi = E\Psi$

↳ Simple solution $\Psi = e^{-iEt/\hbar}$, but can't be full solution since $\Psi'' = 0$

↳ Assume general solution $\Psi = Ae^{i(bx - \frac{E}{\hbar}t)} + Be^{-i(bx + \frac{E}{\hbar}t)}$

w/ some factor b

↳ Now substitute into $\hat{E}\Psi = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\Psi$

$$\hat{E}\Psi = E\Psi = -\frac{\hbar}{2m}\left[-b^2Ae^{i(bx-\frac{E}{\hbar}t)} - b^2Be^{i(bx+\frac{E}{\hbar}t)}\right]$$

$$\boxed{E\Psi = b^2\frac{\hbar}{2m}\Psi, \quad b = \pm\sqrt{\frac{2mE}{\hbar^2}}}$$

∴ The general solution is

$$\boxed{\begin{aligned}\Psi &= Ae^{i(kx-\frac{E}{\hbar}t)} + Be^{-i(kx+\frac{E}{\hbar}t)} \\ k &= \sqrt{\frac{2mE}{\hbar^2}}\end{aligned}}$$

k = wavenumber, $\frac{2\pi}{\lambda}$

※ Note: $E = hf = \hbar\omega$, so $\Psi = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$

- Interpretation: 2 waves, 1 traveling right and 1 traveling left?

↳ Substitute $E = \frac{\hbar^2k^2}{2m}$, $\Psi = Ae^{ik(x-\frac{\hbar k}{2m}t)} + Be^{-ik(x+\frac{\hbar k}{2m}t)}$

↳ Velocity of wave: $V = \frac{\hbar k}{2m} = \sqrt{\frac{E}{2m}}$

↳ But classically, $E = \frac{1}{2}mv^2$, $\boxed{V_{classical} = \sqrt{\frac{2E}{m}} = 2V_{quantum}}$

• Another Problem

↳ For probability density $p(x,t) = \psi^* \psi$, we want $1 = \int_{-\infty}^{\infty} p(x,t)\, dx$

↳ But $\psi^* \psi = \left(Ae^{-i(kx-\omega t)} + Be^{i(kx+\omega t)}\right)\left(Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}\right)$

$$= A^2 + B^2 + AB\left(e^{i(2kx)} + e^{-i(2kx)}\right)$$

$$= A^2 + B^2 + 2AB\cos(kx)$$

$$\int_{-\infty}^{\infty} \psi^* \psi\, dx = \infty$$

∴ Free particle state with definite energy is not normalizable (unphysical, not possible by QM)

↳ But not useless: realistic solutions can be created as sums of free particle states of different energies. (diff. k)

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi(k)\, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}\, dk}$$

Can be normalized if $\Phi(k)$ has correct form

*Time-Independent Schrödinger Equation

- $E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi$

↳ Use $E\psi = i\hbar\frac{\partial}{\partial t}\psi$, and let $\psi(x,t) = g(t)\psi(x)$ ← separable

↳ So $E\cancel{\psi(x)}g(t) = i\hbar\cancel{\psi(x)}\frac{\partial g(t)}{\partial t}$

$\frac{\partial g(t)}{\partial t} = -i\frac{E}{\hbar}g$, solution $\boxed{g(t) = e^{-iEt/\hbar}}$

∴ Full solution involves a phase shift $g(t)$: $\psi(x,t) = e^{-iEt/\hbar}\psi(x)$

※ If external potential doesn't change → Energy Conserved (constant)

If external potential changes → work done on particle, E changes.

- Note: particles in definite energy states have time-independent probability density.

↳ For $\psi(x,t) = e^{-iEt/\hbar}\psi(x)$, $p(x) = \psi^*\psi$ is time-independent

↳ But QM particles can be superpositions of states.

↳ If $\psi = \psi_1 + \psi_2$, $p(x,t) = \psi_1^*\psi_1 + \psi_2^*\psi_2 + \underbrace{\psi_1^*\psi_2 + \psi_2^*\psi_1}_{\text{Different energies}}$

↳ So probability density fluctuates in time

*Properties of the wavefunction

- Constraints in order to solve SE

1. Probability is normalized: $\int_{-\infty}^{+\infty} p(x,t)\,dx = \int_{-\infty}^{+\infty}\psi^*\psi\,dx = 1$
2. As $x \to \pm\infty$, $p(x,t)$ and $\psi \to 0$ (to be normalizable)
3. Wavefunction must be continuous
4. The derivative of the wavefunction is continuous

↳ Otherwise energy is undefined b.c. of the $\frac{\partial^2}{\partial x^2}\psi$ term

*Solving the Schrödinger Equation: Infinite Well

- Process:
  1. Graph of potential v. position
  2. Write SE for regions w/ same V
  3. Write general form of sol'n for those regions
  4. Apply conditions from before:
     - ↳ Drop any terms that blow up as $x \to \pm\infty$
     - ↳ Match wavefunctions when 2 regions meet (and their derivatives)
     - ↳ Normalize overall solution
  5. Include time-dependent portion, $e^{-iEt/\hbar}$

- Cases for V
  1. $V=0$
     - ↳ SE becomes $E\psi = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi \longrightarrow \frac{d^2\psi}{dx^2} = -k^2\psi$, $k=\sqrt{\frac{2mE}{\hbar^2}}$
     - ↳ Solution: $\psi(x) = Ae^{ikx} + Be^{-ikx}$
       $= C\cos(kx) + D\sin(kx)$

  2. $E > V$
     - ↳ SE becomes $E\psi = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi + V\psi$
       $\frac{d^2\psi}{dx^2} = -k^2\psi$, $k = \sqrt{\frac{2m(E-V)}{\hbar^2}}$
     - ↳ Solution: $\psi(x) = Ae^{ikx} + Be^{-ikx}$
       $= C\cos(kx) + D\sin(kx)$

  3. $E < V$
     - ↳ Solution: $\psi(x) = Ae^{+\kappa x} + Be^{-\kappa x}$, $\kappa = \sqrt{\frac{2m(V-E)}{\hbar^2}}$