Notes for Complex Analysis

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Contents

0 Reading Week

0.1 Algebra of Complex Numbers

Complex numbers are an extension of $\mathbb{R}$. We adjoin $\sqrt{-1}$ and allow addition & multiplication. The set of complex numbers is denoted $\mathbb{C}$.

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

For every complex number $z \in \mathbb{C}$, the real part is denoted $\mathrm{Re}(z)$ and the imaginary part is denoted $\mathrm{Im}(z)$.

$$z = a + bi \longrightarrow \begin{cases} \mathrm{Re}(z) = a \\ \mathrm{Im}(z) = b \end{cases}$$

We denote the absolute value or "modulus" of z as $|z|$. It's pretty simple, $|z|^2 = a^2 + b^2$ for $z = a + bi$.

$$|z|^2 = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2$$

The Inverse: For every $z \in \mathbb{C}$ that is nonzero, there exists a $z^{-1} \in \mathbb{C}$ such that $zz^{-1} = 1$. Namely,

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

The Complex Conjugate: For every $z = a + bi \in \mathbb{C}$, the complex conjugate is $\bar{z} = a - bi$. Addition, subtraction, multiplication, and division work as expected, which is nice.

$$\begin{aligned} |z|^2 &= z\overline{z} \\ \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} \\ \overline{z_1 - z_2} &= \overline{z_1} - \overline{z_2} \\ \overline{z_1 z_2} &= \overline{z_1} \cdot \overline{z_2} \\ \overline{z_1 / z_2} &= \overline{z_1} / \overline{z_2} \end{aligned}$$

The Triangle Inequalities: Cf. triangle inequalities for $\mathbb{R}$

$$\begin{aligned} |z_1 + z_2| &\leq |z_1| + |z_2| \\ |z_1 - z_2| &\geq \Big| |z_1| - |z_2| \Big| \end{aligned}$$

0.2 Polar Coordinates

Let $z = r(\cos\theta + i\sin\theta)$. Then there are multiple values of θ that can create a given z (e.g. θ, $\theta + 2\pi$, ...)

A possible θ for a given z is called an argument of z, denoted $\arg(z)$.

$$\arg(z) = \{\theta \mid z = r(\cos\theta + i\sin\theta)\}$$

A unique argument θ of z satisfying $-\pi < \theta \leq \pi$ is called the principal argument of z, denoted $\mathrm{Arg}(z)$. It can be thought of as a function from $\mathbb{C}$ (excluding 0) to a number between $-\pi$ and π.

$$\mathrm{Arg} : \mathbb{C} \setminus \{0\} \longrightarrow (-\pi, \pi]$$

We can represent a complex number $z \in \mathbb{C}$ in polar coordinates, using $r = |z|$ and $\theta \in \arg(z)$. It's often easier to multiply and divide complex numbers in this form.

$$z = re^{i\theta}$$

1 Week 1 (Jan 12 - 16)

1.1 Roots of Complex Numbers

- All nonzero real numbers have 2 square roots in $\mathbb{C}$.
- All nonzero elements of $\mathbb{C}$ have a square root in $\mathbb{C}$ (cf: not the case in $\mathbb{R}$).
- Note that 0 has only 1 square root: 0.

For any positive, nonzero integer n and $z \in \mathbb{C}$, all n-th roots of z lie in $\mathbb{C}$. The set of n-th roots of $z = re^{i\theta}$ are given by

$$\left\{ \sqrt[n]{r} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} \;\middle|\; k = 0, 1, 2, ..., n-1 \right\}$$

Proof: Let $a_k = \sqrt[n]{r} e^{i(\theta + 2\pi k)/n}$. Then a calculation shows that $a_k^n = re^{i\theta} = z$ as expected.

We also need to show that all $a_0, a_1, ..., a_{n-1}$ are distinct. Proof by contradiction: assume $a_r = a_s$ and $0 \leq r < s \leq n-1$. Then $(\theta + 2\pi r)/n = (\theta + 2\pi s)/n - 2\pi j$ for some integer j. Solving, we should get

$$\frac{2\pi(r-s)}{n} = 2\pi j$$

But this is impossible, since $s - r < n$. This is a contradiction, so all $a_0, a_1, ..., a_{n-1}$ must be distinct.

We know that a polynomial of degree n has at most n zeros. Apply this to $x^n - z$ to see that z has at most n n-th roots. So $a_0, a_1, ..., a_k$ are all the n-th roots of z.

Geometrically, if $n > 0$ and $z \neq 0$, then the elements of the set $z^{1/n}$ lie at the vertices of a regular n-gon centered at the origin. Each vertex is equidistant from 0, and the angle between successive vertices is $2\pi/n$.

A special case is the n-th roots of 1, or "unity." These are given by

$$\left\{e^{i0}, e^{2\pi i/n}, e^{4\pi i/n} ..., e^{2(n-1)\pi i/n}\right\}$$

If $\omega^n = 1$ but $\omega^k \neq 1$ if $k < n$, then ω is a *primitive n-th root of unity.* For example, let $\omega = e^{2\pi i/n}$. $\omega^n = 1$, but not for any $0 < k < n$.

We can define the set of n-th roots using primitive n-th roots. Let $z \neq 0$, $\alpha^n = z$, and ω be a primitive n-th root of unity. Then

$$z^{1/n} = \left\{\alpha, \alpha\omega, \alpha\omega^2, ..., \alpha\omega^{n-1}\right\}$$

Proof: For any k, $(\alpha\omega^k)^n = z(1^k) = z$, so $\alpha\omega^k$ is indeed an n-th root.

The n-th roots are also unique. To prove this, assume $\alpha\omega^s = \alpha\omega^r$ for $0 \leq r < s \leq n-1$. Then $\omega^{s-r} = 1$, but $0 < s - r < n$, so ω can't be a primitive n-th root of unity.

1.2 Open vs. Closed Sets

Given $z_0 \in \mathbb{C}$ and $\epsilon \in \mathbb{R}$, we define the ε-neighborhood of z_0 as the subset

$$\left\{z \;\middle|\; |z - z_0| < \varepsilon\right\}$$

We also define the deleted ε-neighborhood as the subset

$$\left\{z \;\middle|\; 0 < |z - z_0| < \varepsilon\right\}$$

The ε-neighborhood is like a circular region around z_0, except the outline. The deleted ε-neighborhood is that minus z_0.

We call $z_0 \in \mathbb{C}$ as an interior point of some set S if there exists an ε-neighborhood of z_0 contained in S.

Similarly, $z_0 \in \mathbb{C}$ is an exterior point of some set S if there exists an ε-neighborhood of z_0 contained in the complement of S, namely $\mathbb{C} \setminus S = \{z \in \mathbb{C} \mid z \notin S\}$.

There also exists points that are neither interior nor exterior points of S. This is the boundary of S, denoted ∂S. A point z_0 is in ∂S iff. every ε-neighborhood of z_0 intersects both S and $\mathbb{C} \setminus S$.

A set S is open if no boundary points lie in S (think of intervals from Calc BC).

$$S \ \cap \ \partial S = \emptyset$$

Theorem: S is open iff. every point in S is an interior point.

Proof: If $S = \emptyset$, there's nothing to prove, so assume $S \neq \emptyset$.

- First direction: Let $z \in S$. If S is open, then $z \notin \partial S$, but since $z \in S$, z must be an interior point.
- Second direction: If every $z \in S$ is an interior point, then for every z, we have $z \notin \partial S$. Use the notation $S = \{z\}$, the set of all z. Then $S \ \cap \ \partial S = \emptyset$, so S must be open.

A set S is closed if its boundary lies in S (again, think of closed intervals from Calc BC).

$$\partial S \subset S$$

For example, $\{z \bigm| |z-1| \leq 2\}$ has the boundary given by $|z-1| = 2$.

For any given S, the set $S \cup \partial S$ is closed. We call $S \cup \partial S$ the *closure* of S. The closure of S is the intersection of all closed sets containing S.

Note 1: A set can be neither open nor closed. Back to the Calc BC example, think of the interval $[0, 1)$. For sets, it's similar. A set could contain just a few of its boundary points.

Note 2: If S is both open and closed, $S = \mathbb{C}$ or $S = \emptyset$.

We call S connected if for $z_1, z_2 \in S$, there exists a polygonal path in S, composed of straight line segments, that connects z_1 and z_2.

S is bounded if for some $r \in \mathbb{R}$, S can be constrained in a finite disk:

$$S \subset \left\{z \Bigm| |z| < r\right\}$$

We call z_0 an accumulation point of S if every deleted ε-neighborhood of z_0 contains points of S (except z_0, per definition). It's kind of like the "center" of a set, I guess.

Theorem: If S is closed, then S contains all its accumulation points.

Proof: Suppose z is an accumulation point of a closed set S, and $z \notin S$. Then each ε-neighborhood of z contains a point not in S. But every ε-neighborhood of z must contain some point in S, since z is an accumulation point of S. Every ε-neighborhood of z contains both points in S and points not in S, thus $z \in \partial S$.

Since S is closed, $\partial S \subset S$, so $z \in S$. This contradicts our assumption that $z \notin S$, which completes the proof.

A domain is defined to be a set that's both open and connected (think of intervals on the real line in Calc). This can conflict with our usual definition of "domain" as the inputs to a function, but they're different. Be careful.

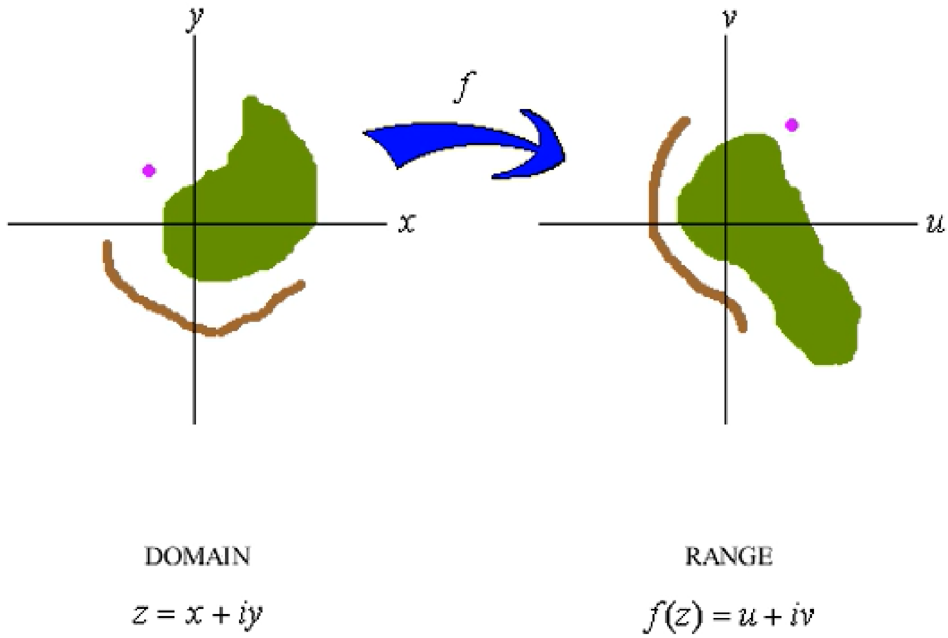
A region is a domain plus some (maybe none, maybe all) of its boundary points.

1.3 Complex Functions

A function f defined on set A is a rule that assigns to each $a \in A$ an element $f(b) \in B$.

In Complex Analysis, we'll consider functions of a complex value: $A \subset C$.

Graphing is a bit tricky because we have to map from a complex plane $\longrightarrow$ complex plane, total 4 dimensions. Use colors to get around this.



Graphing Complex-Valued Functions. Consider points, lines, or curves in the domain, parametrize them, and map with the function.

2 Week 2 (Jan 19 - 23)

2.1 Limits of Complex Functions

Suppose f (function of a complex variable) is defined in some deleted ϵ-neighborhood of z_0, and w_0 is some complex number. Then $\lim_{z \to z_0} f(z) = w_0$ means

> For every $\epsilon > 0$, there exists $\delta > 0$ such that for every z in the domain of f, $0 < |z - z_0| < \delta$ implies $|f(z) - w_0| < \epsilon$

An equivalent formulation can be made in terms of neighborhoods:

> For any ϵ-neighborhood of w_0, there exists a deleted δ-neighborhood of z_0 such that f takes every point in this deleted δ-neighborhood into the ϵ-neighborhood of w_0.

Baically, as the input z gets arbitrarily close to z_0, the function's output gets arbitrarily close to w_0. If that happens, w_0 is the limit.

Limit theorems are pretty much the same from Real Analysis. Suppose that $\lim_{z\to z_0} f_1(z) = w_1$ and $\lim_{z\to z_0} f_2(z) = w_2$. Then

i) Limits are unique. If for some $w \in \mathbb{C}$, we have $\lim_{z\to z_0} f_1(z) = w$, then $w = w_1$.

ii) $\lim_{z\to z_0}(f_1 + f_2)(z) = w_1 + w_2$

iii) $\lim_{z\to z_0}(cf_1)(z) = cw_1$

iv) $\lim_{z\to z_0}(f_1 \cdot f_2)(z) = w_1 w_2$

v) $\lim_{z\to z_0}\left(\frac{f_1}{f_2}\right)(z) = \frac{w_1}{w_2}$, given $w_2 \neq 0$

We can also reduce complex-valued limits to real-valued ones.

Theorem: Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$. Then

$$\lim_{z\to z_0} f(z) = w_0 \text{ if and only if}$$

$$\lim_{(x,y)\to(x_0,y_0)} u = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v = v_0$$

Limits of polynomials are similar to before. Given $P(z) = c_0 + c_1 z + \cdots + c_n z^n$,

$$\lim_{z\to z_0} P(z) = P(z_0)$$

If $\lim_{z\to z_0} f(z)$ exists, then it can be computed by considering f restricted to *any* path of inputs that get arbitrarily close to z_0. The specific path doesn't matter.

For example, when we try to find the limit $\lim_{z\to 0} f(z)$ for $f(z) = \mathrm{Re}(z)/z$, we can consider two paths:

1. Path such that $\mathrm{Re}(z) = z$ (pure real), and z gets arbitrarily close to 0.
2. Path such that $\mathrm{Re}(z) = 0$ (pure imaginary), and z gets arbitrarily close to 0.

In the first case, $\lim_{z\to 0} f(z) = 1$, but in the second case, $\lim_{z\to 0} f(z) = 0$. Thus the limit of f doesn't exist at $z = 0$.

2.2 Limits Involving ∞

For complex numbers, there's no distinction between $-\infty$ and $+\infty$. When we say $z \to \infty$, we just mean that z gets arbitrarily far away from 0, so $|z| \to +\infty$.

There are three basic cases for limits:

i) $\lim_{z\to z_0} f(z) = \infty$

For every $\epsilon > 0$, there exists $\delta > 0$ such that for every z in the domain of f, $0 < \|z - z_0\| < \delta$ implies $\|f(z)\| > \frac{1}{\epsilon}$

ii) $\lim_{z\to\infty} f(z) = w$

For every $\epsilon > 0$, there exists $\delta > 0$ such that for every z in the domain of f, $\|z\| > \frac{1}{\delta}$ implies $\|f(z) - w\| < \epsilon$

iii) $\lim_{z\to\infty} f(z) = \infty$

For every $\epsilon > 0$, there exists $\delta > 0$ such that for every z in the domain of f, $\|z\| > \frac{1}{\delta}$ implies $\|f(z)\| > \frac{1}{\epsilon}$

Note that $\infty \notin \mathbb{C}$. But it's a useful concept to use. We can rewrite limits involving ∞ with real numbers by taking the reciprocal. So for example, limit (i) and (ii) could be rewritten as

$$\lim_{z\to z_0} f(z) = \infty \longrightarrow \lim_{z\to z_0} \frac{1}{f(z)} = 0$$
$$\lim_{z\to\infty} f(z) = w \longrightarrow \lim_{z\to 0} f\left(\frac{1}{z}\right) = w$$

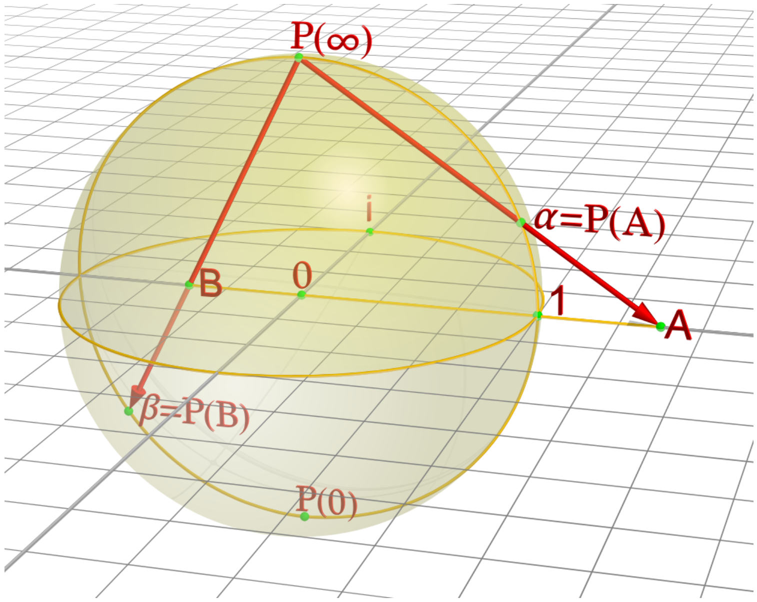
Example: Show that $\lim_{z\to\infty} f(z) = \infty$, where $f(z) = \frac{1+z+iz^4}{z^2-2}$.

Rewriting, we want to prove that

$$\lim_{z\to 0} \frac{1}{f(1/z)} = 0$$

which is easily seen by substituting and using the appropriate limit theorems.

An interpretation of ∞ can be done with a Riemann sphere.



https://upload.wikimedia.org/wikipedia/commons/3/32/Riemann_sphere1.svg

The plane in which the sphere lies is the complex plane. The North Pole of the sphere is ∞. We see that all points in the plane correspond to points on the sphere in some small disk around the North Pole.

$$\left\{ z \;\middle|\; |z| > \frac{1}{\epsilon} \right\}$$

2.3 Continuity

We say that f is continuous at z_0 if $f(z_0)$ is defined, the limit $\lim_{z \to z_0} f(z)$ exists, and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

For every $\epsilon > 0$, there exists $\delta > 0$ such that for every z in the domain of f, $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \epsilon$

If f is continuous at all points in some set D, we say that f is continuous on/throughout D. For example, polynomials are continuous on/throughout $\mathbb{C}$.

Continuity theorems are the same from Real Analysis. If f and g are continuous at z_0, then

i) $f + g$ is continuous at z_0

ii) $f \cdot g$ is continuous at z_0

iii) f/g is continuous at z_0, assuming $g(z_0) \neq 0$

Theorem: Suppose g is continuous at z_0 and f is continuous at $g(z_0)$. Then $f \circ g$ is continuous at z_0.

Proof: We have to show that $\lim_{z \to z_0} f(g(z)) = f(g(z_0))$. Let $\epsilon > 0$. Since f is continuous at $g(z_0)$, there exists $\delta_1 > 0$ such that for all w in the domain of f,

$$|w - g(z_0)| < \delta_1 \implies |f(w) - f(g(z_0))| < \epsilon$$

Since g is continuous at z_0, there exists $\delta > 0$ such that for all z in the domain of g,

$$|z - z_0| < \delta \implies |g(z) - g(z_0)| < \delta_1$$

Thus for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(g(z)) - f(g(z_0))| < \epsilon$$

Theorem: Suppose $z_0, w_0 \in \mathbb{C}$ and f is continuous at z_0. If $f(z_0) \neq w_0$, then there exists a neighborhood N around z_0 such that $f(z) \neq w_0$ for all $z \in N$.

Essentially, if $f(z_0) \neq w_0$, we can find a neighborhood around z_0 such that f is not w_0 for any points in that neighborhood.

Proof: Since $f(z_0) \neq w_0$, there exists $\epsilon > 0$ such that

$$|w_0 - f(z_0)| > \epsilon$$

Since f is continuous at z_0, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$$

Let N be the δ-neighborhood around z_0. The above two statements imply that no $z \in N$ satisfies $f(z) = w_0$.

2.3.1 Uniform Continuity

If f is a complex function, continuous on a closed and bounded D, then $|f|$ attains a maximum on D.

If f is continuous on a closed and bounded set D, then $|f|$ is uniformly continuous on D. In epsilon-delta form,

$$\boxed{\begin{array}{c} f \text{ is uniformly continuous on } D \text{ if } \textit{for all } \epsilon > 0, \text{ there exists } \delta > 0 \\ \text{such that for every } z_1, z_2 \in D, |z_1 - z_2| < \delta \text{ implies } |f(z_1) - f(z_2)| < \epsilon \end{array}}$$

The difference with normal continuity is that uniform continuity has a "universal delta" that applies to *all* ϵ, not just *each* ϵ.

Note: A closed and bounded set is called "compact."

3 Week 3 (Jan 26 - 30)

3.1 Derivatives

Let f be defined in some neighborhood of z_0. We define the derivative of f at z_0 similarly to the real-valued counterpart:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{option 1}$$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{option 2}$$

We can also denote the derivative w.r.t z as $\frac{d}{dz} f$.

The same properties hold as for real-valued derivatives:

i) $(f + g)'(z_0) = f'(z_0) + g'(z_0)$

ii) $(f \cdot g)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$

iii) $(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}, \quad g(z_0) \neq 0$

iv) $(cf)'(z_0) = cf'(z_0)$

v) $\frac{d}{dz} c = 0$

vi) $\frac{d}{dz} z^n = nz^{n-1}, \quad n \in \mathbb{Z}^+$

Theorem (Chain Rule): Suppose g is differentiable at z_0 and f is differentiable at $g(z_0)$. Then $f \circ g$ is differentiable at z_0, and

$$(f \circ g)'(z_0) = f'(g(z_0)) \cdot g'(z_0)$$

Proof: Using the definition of the derivative,

$$\begin{aligned}
\lim_{h\to 0}\frac{f\left(g(z_0+h)\right)-f\left(g(z_0)\right)}{h} &= \lim_{h\to 0}\frac{f\left(g(z_0+h)\right)-f\left(g(z_0)\right)}{g(z_0+h)-g(z_0)}\frac{g(z_0+h)-g(z_0)}{h}\\
&= \lim_{h\to 0}\frac{f\left(g(z_0+h)\right)-f\left(g(z_0)\right)}{g(z_0+h)-g(z_0)}\cdot g'(z_0)
\end{aligned}$$

It suffices to show that

$$\lim_{h\to 0}\frac{f\left(g(z_0+h)\right)-f\left(g(z_0)\right)}{g(z_0+h)-g(z_0)} = f'\left(g(z_0)\right)$$

Let $\epsilon > 0$. Since f is differentiable $g(z_0)$, there exists $\delta_1 > 0$ such that for all w in the domain of f,

$$|w-g(z_0)| < \delta_1 \implies \left|\frac{f(w)-f(g(z_0))}{w-g(z_0)} - f'(g(z_0))\right| < \epsilon$$

We also know that g is continuous at z_0 (since it's differentiable there), i.e. $\lim_{h\to 0} g(z_0+h) = g(z_0)$. So there exists $\delta > 0$ such that

$$|h| < \delta \implies |g(z_0+h)-g(z_0)| < \delta_1$$

Let $w = g(z_0+h)$, a point in the domain of f. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|h| < \delta \implies |g(z_0+h)-g(z_0)| \implies \left|\frac{f(w)-f(g(z_0))}{w-g(z_0)} - f'(g(z_0))\right| < \epsilon$$

So we've shown that

$$\begin{gathered}
\lim_{h\to 0}\frac{f\left(g(z_0+h)\right)-f\left(g(z_0)\right)}{g(z_0+h)-g(z_0)} = f'\left(g(z_0)\right)\\
\lim_{h\to 0}\frac{f\left(g(z_0+h)\right)-f\left(g(z_0)\right)}{h} = f'(g(z_0))\cdot g'(z_0)
\end{gathered}$$

Complex functions are often not differentiable. E.g. Let f be differentiable and nonzero at z_0. Then $g(z) = \overline{z}f(z)$ is not differentiable at z_0.

If we try to take the derivative, we get

$$\begin{aligned} g'(z_0) &= \lim_{h\to 0} \frac{(\overline{z_0+h})f(z_0+h) - \overline{z_0}f(z_0)}{h} \\ &= z_0\left(\frac{f(z_0+h)-f(z_0)}{h}\right) + \frac{\overline{h}}{h}f(z_0+h) \end{aligned}$$

The specific "path" we take to approach $h \to 0$ shouldn't matter if the derivative is well-defined. However,

$$\lim_{h\to 0} \frac{\overline{h}}{h} f(z_0+h) = f(z_0) \qquad (h \text{ is real})$$

$$\lim_{h\to 0} \frac{\overline{h}}{h} f(z_0+h) = -f(z_0) \qquad (h \text{ is imaginary})$$

Thus $g(z) = \overline{z}f(z)$ isn't differentiable at z_0.

3.2 Cauchy-Riemann Equations

The Cauchy-Riemann equations must be satisfied by any differentiable function. It's a necessary condition for the differentiability of f. Later, we'll make it a sufficient condition.

Theorem: Suppose $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, and f is differentiable at z_0. Then the function must satisfy

$$\begin{cases} u_x(x_0,y_0) = v_y(x_0,y_0) \\ u_y(x_0,y_0) = -v_x(x_0,y_0) \end{cases}$$

Proof: Since f is differentiable at z_0, the following limit must exist.

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

Let $h = s + it$. Then by substitution,

$$\begin{aligned} f(z_0+h) &= f((x_0+s)+i(y_0+t)) \\ &= u(x_0+s, y_0+t) + iv(x_0+s, y_0+t) \end{aligned}$$

Now we try to find the limit:

$$
\begin{aligned}
f'(z_0) &= \lim_{s+it\to 0} \frac{f(z_0+s+it)-f(z_0)}{s+it} \\
&= \lim_{s+it\to 0} \left(\frac{u(x_0+s, y_0+t) - u(x_0, y_0)}{s+it} + i\frac{v(x_0+s, y_0+t) - v(x_0, y_0)}{s+it} \right)
\end{aligned}
$$

We can restrict $s+it$ to the real axis, so $t=0$. Then the derivative is

$$
\begin{aligned}
f'(z_0) &= \lim_{s\to 0} \left(\frac{u(x_0+s, y_0) - u(x_0, y_0)}{s} \right) + i \lim_{s\to 0} \left(\frac{v(x_0+s, y_0) - v(x_0, y_0)}{s} \right) \\
&= u_x(x_0, y_0) + iv_x(x_0, y_0)
\end{aligned}
$$

We can also restrict $s+it$ to the imaginary axis, so $s=0$. Then the derivative is

$$
\begin{aligned}
f'(z_0) &= \lim_{t\to 0} \left(\frac{u(x_0, y_0+t) - u(x_0, y_0)}{it} \right) + i \lim_{s\to 0} \left(\frac{v(x_0, y_0+t) - v(x_0, y_0)}{it} \right) \\
&= -iu_y(x_0, y_0) + v_y(x_0, y_0)
\end{aligned}
$$

For the derivative to be well-defined, both results must be equal, thus

$$
\begin{cases}
u_x(x_0, y_0) = v_y(x_0, y_0) \\
u_y(x_0, y_0) = -v_x(x_0, y_0)
\end{cases}
$$

We can also make the Cauchy-Riemann equations a sufficient condition for differentiability by adding an extra condition.

Let $f(z) = u(x,y) + iv(x,y)$ be defined in a neighborhood of $z_0 = x_0 + iy_0$. Suppose u_x, u_y, v_x, and v_y are exist throughout the neighborhood and are continuous at (x_0, y_0).

Then f is differentiable at z_0 if and only if u and v satisfy the Cauchy-Riemann equations at (x_0, y_0) .

There's also a polar version of the Cauchy-Riemann equations, which can be derived with $x = r\cos\theta$ and $y = r\sin\theta$.

Theorem: If $f(z) = u(r,\theta) + iv(r,\theta)$ and f is differentiable at $z_0 = r_0 e^{i\theta_0}$, then

$$\begin{cases} u_r(r_0,\theta_0) = \frac{1}{r_0} v_\theta(r_0,\theta_0) \\ \frac{1}{r_0} u_\theta(r_0,\theta_0) = -v_r(r_0,\theta_0) \end{cases}$$

Moreover,

$$f'(z_0) = e^{-i\theta_0}\Big(u_r(r_0,\theta_0) + iv_r(r_0,\theta_0)\Big)$$

Proof: Rewrite the partial derivatives as

$$\begin{aligned} &1) \quad \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x \cos\theta + u_y \sin\theta \\ &2) \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -u_x r\sin\theta + u_y r\cos\theta \end{aligned}$$

Similarly for v,

$$\begin{cases} v_r = v_x \cos\theta + v_y \sin\theta \\ v_\theta = -v_x r\sin\theta + v_y r\cos\theta \end{cases}$$

If we use the Cauchy-Riemann conditions $u_x = v_y$ and $u_y = -v_x$, it becomes clear that $ru_r = v_\theta$ and $u_\theta = -rv_r$.

3.3 Analytic Functions

We say f is analytic on some set S if there exists an open set A such that $S \subseteq A$ and f is differentiable at each point of A.

In other words, f is analytic on S if every point in S has a neighborhood on which f is differentiable.

i) If S is open, then "f is analytic on S" just means "f is differentiable at each point of S."

ii) "f is analytic at z_0" just means "f is differentiable in a neighborhood of z_0."

Note: "Analytic at z_0" is a stronger condition than "differentiable at z_0." Analyticity tells us something about the function's behavior.

Properties of analytic functions:

i) If f and g are analytic at D, then so are $f+g$, $f \cdot g$, and f/g (as long as $g(z) \neq 0$ in D).

ii) If g is analytic on D and f is analytic on the image of g, then $f \circ g$ is analytic on D.

If f is analytic throughout $\mathbb{C}$, then it's called "entire." E.g. polynomials.

We call z_0 a singular point (singularity) of f is f is not analytic at z_0, but in every neighborhood of z_0 there exists a point at which f is analytic.

- For example, 0 is a singularity for $f(z) = 1/z$.
- However, 0 isn't a singularity of $g(z) = |z|^2$, because $g(z)$ isn't analytic anywhere.

Theorem: Suppose f is analytic on D and $f'(z) = 0$ for all of $z \in D$. Then f is constant on D.

Proof: From the Cauchy-Riemann equations,

$$f'(z) = \begin{cases} u_x(x,y) + iv_x(x,y) \\ v_y(x,y) - iu_y(x,y) \end{cases}$$

Since $f'(z) = 0$ on D, $u_x = u_y = v_x = v_y = 0$ on D. It follows that the gradient $\nabla u = \nabla v = \vec{0}$ on D.

Let $r(t) = (x(t), y(t))$ for $0 \leq t \leq 1$ be a parametrized line in D, from $z_0 = x_0 + iy_0$ to $z_1 = x_1 + iy_1$. Then $u(r(t))$ is a real-valued function with $u(r(0)) = u(x_0, y_0)$ and $u(r(1)) = u(x_1, y_1)$

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = \nabla u \cdot r'(t) = 0$$

Thus $u(r(t))$ must be constant. Since the parametrized line was a general one on D, we conclude that u must be constant on D. Similar reasoning for v shows that $f = u + iv$ is constant on D.

3.4 Harmonic Functions

The Cauchy-Riemann equations are very restrictive conditions for a function to be analytic on a region. Which functions can be real or imaginary components of an analytic function?

Theorem: If $f = u + iv$ is analytic on D, and the 1st and 2nd partials of u and v are continuous throughout D, then

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ v_{xx} + v_{yy} = 0 \end{cases}$$

Proof: From Cauchy-Riemann, $u_x = v_y$ and $u_y = -v_x$ throughout D. Thus

$$(u_x)_y = (v_y)_x = (v_x)_y = (-u_y)_y$$

Similar reasoning for v can show that $v_{xx} = -v_{yy}$.

Functions that satisfy the above condition are called harmonic. More specifically, suppose ϕ is a real-valued function of 2 real variables x and y. Further, let ϕ have continuous 1st and 2nd partials onn D. Then ϕ is harmonic on D if

$$\phi_{xx} + \phi_{yy} = 0 \qquad \text{(Laplace's equation)}$$

Cauchy-Riemann equations imply that only harmonic functions can be real or imaginary parts of analytic functions.

$$\boxed{\text{If } f = u + iv \text{ is analytic on } D\text{, then } u \text{ and } v \text{ are harmonic on } D.}$$

If u, v are harmonic on D, and u, v satisfy the Cauchy-Riemann equations, then v is a *harmonic conjugate* of u.

Properties:

i) The harmonic conjugate isn't symmetric. "v is the harmonic conjugate of u" doesn't imply that "u is the harmonic conjugate of v."

ii) If v is the harmonic conjugate of u on D, then $-u$ is the harmonic conjugate of v on D.

iii) If v_1, v_2 are both harmonic conjugates of u on D, then $v_1 - v_2$ is constant throughout D.

iv) If u, v are both harmonic conjugates of each other on D, then u, v are both constant throughout D.

Theorem: $f = u + iv$ is analytic on D iff. v is a harmonic conjugate of u on D.

Theorem: If u is harmonic on a simply connected domain D, then there exists a harmonic conjugate v on D.

4 Week 4 (Feb 2 - 6)

4.1 Complex Expontential

We define the complex exponential function as

$$f(z) = e^x (\cos y + i \sin y), \quad z = x + iy$$

This is the counterpart of the real-valued exponential function $f(x) = e^x$. In fact, when $z = x \in \mathbb{R}$, the complex exponential reduces to e^x.

Properties of the complex exponential:

i) f is entire: it's analytic throughout $\mathbb{C}$. Thus $\text{Re}(z)$ and $\text{Im}(z)$ have continuous partials satisfying the Cauchy-Riemann equations.

ii) $f' = f$, to see this:

$$f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y = f(z)$$

iii) $|e^z| = e^x$, assuming $e^z \neq 0$.

iv) $\arg(z) = \{y + e\pi n \mid n = 0, \pm 1, \pm 2, \dots\}$

v) $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for any $z_1, z_2 \in \mathbb{C}$

vi) $(e^x)^n = e^{nz}$ for any integer n.

Note that there are differences with the real-valued exponential. For example, $e^x > 0$ for $x \in \mathbb{R}$, but e^z can be negative for $z \in \mathbb{C}$. The classic example is $e^{\pi i} = -1$.

Additionally, e^z is *not* one-to-one. As seen in property (iv), it has period $2\pi i$. So no single-valued inverse exists, but we'll see a multi-valued log later.

There are potential ambiguities in notation. For example, given e^{iy}, should that be interpreted as a complex exponential? In this case both Euler's formula and the complex exponential return the same answer:

$$e^{iy} = \cos y + i \sin y \qquad \text{(Euler)}$$
$$e^{iy} = e^0 (\cos y + i \sin y) \qquad \text{(Exponential)}$$

However, consider $e^{1/n}$. This could either mean the complex exponential, or the roots of e.

$$e^{1/n} = \{z \in \mathbb{C} \mid z^n = e\} \qquad \text{(Roots)}$$
$$e^{1/n} = e^{1/n}\left(\cos 0 + i \sin 0\right) = \sqrt[n]{e} \qquad \text{(Exponential)}$$

This semester, the convention will be that

If $z \neq e$, then $z^{1/n}$ denotes the n n-th roots of z.

If $z = e$, then $z^{1/n}$ denotes $\sqrt[n]{e}$.

4.2 Complex Trigonometric Functions

We also define the complex trig functions as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

These reduce to the real-valued sin and cos if $z = x \in \mathbb{R}$.

Properties (for all $z \in \mathbb{C}$):

i) $\sin(-z) = \sin(z)$
$\cos(-z) = \cos(z)$

ii) $\frac{d}{dz} \sin z = \cos z$
$\frac{d}{dz} \cos z = -\sin z$

iii) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$
$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

iv) $\sin^2 z + \cos^2 z = 1$

v) sin and cos are both periodic with period 2π.

There are some differences from the real-valued sin and cos. For example, if $x \in \mathbb{R}$, then $|\sin x| \leq 1$ and $|\cos x| \leq 1$. However, in $\mathbb{C}$, the trig functions aren't bounded.

We can also write the complex trig functions in terms of the hyperbolic sine and cosine.

Recall that

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad , \quad \cosh y = \frac{e^y + e^{-y}}{2}$$

Thus for $z = x + iy$,

$$\begin{aligned} \sin(z) &= \sin(x)\cosh(y) + i\cos(x)\sinh(y) \\ \cos(z) &= \cos(x)\cosh(y) - i\sin(x)\sinh(y) \end{aligned}$$

Note: the hyperbolic sine and cosine aren't equal to the complex sine and cosine. They look pretty similar, though.

4.3 Complex Logarithm

Given $z \neq 0$, we define the complex logarithm as

$$\log z = \left\{ \ln|z| + i\theta \;\middle|\; \theta \in \arg(z) \right\}$$

The complex logarithm is the inverse of the complex exponential. $w \in \log z$ iff. $e^w = z$.

The complex logarithm is a *set* of values, not a single value. This makes sense, as the complex exponential is multi-valued. So for some $z = re^{i\theta}$, we have

$$\begin{aligned} \log z &= \ln r + i\theta \\ &= \ln r + i(\theta + 2\pi) \\ &= \ln r + i(\theta + 4\pi) \\ &= \cdots \end{aligned}$$

So there are some subtleties in notation. For example, $z = e^{\log z}$ doesn't make sense because the log is multi-valued. Instead, we say $z \in e^{\log z}$. Additionally, the equation $\log e^z = \ln|e^z| + i\arg(e^z)$ is describing an equality of *sets* generated by log and arg.

We can also define a single-valued logarithm. Recall that the principal argument $\operatorname{Arg}(z)$ is the unique $\theta \in \arg(z)$ such that $-\pi < \theta < \pi$. Then the single-valued principal logarithm is

$$\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$$

Alternatively, we can define a branch of the multi-valued complex log to make it single-valued.

Denote the branch as

$$L_\alpha(z) = \ln r + i\theta \quad , \quad r > 0, \alpha < \theta < \alpha + 2\pi$$

The branch is defined, single-valued, and analytic at all points in its domain excluding $\theta = \alpha$. The "normal" complex logarithm corresponds to $L_{-\pi}$.

$$L_{-\pi} = \ln r + i\theta \quad , \quad r > 0, -\pi < \theta \leq \pi$$

Theorem: The branch of the complex log

$$L_\alpha(z) = \ln r + i\theta \quad , \quad r > 0, \alpha < \theta < \alpha + 2\pi$$

is analytic at every point in its domain of definition, and its derivative at any such z is $1/z$.

Proof: Let $z = u(r, \theta) + iv(r, \theta)$. Then

$$\begin{aligned} \operatorname{Re}\left(L_\alpha(z)\right) &= u(r, \theta) = \ln r \\ \operatorname{Im}\left(L_\alpha(z)\right) &= v(r, \theta) = \theta \end{aligned}$$

u and v satisfy the polar version of Cauchy-Riemann, since

$$u_r = \frac{1}{r} v_\theta \quad , \quad \frac{1}{r} u_\theta = -v_r$$

Thus the branch L_α is analytic. Further, for every function $f(z) = u(r, \theta) + iv(r, \theta)$, we have

$$f'(z) = e^{-i\theta}\left(u_r(r, \theta) + iv_r(r, \theta)\right)$$

Substituting $f(z) = L_\alpha(z)$,

$$\frac{d}{dz} L_\alpha(z) = e^{-i\theta}\left(\frac{1}{r} + i \cdot 0\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

We've shown that L_α is analytic, and its derivative at any point z is $1/z$.

Properties of the complex log:

1. $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ for all $z_1, z_2 \in \mathbb{C}$ and $z_1, z_2 \neq 0$

- This is really a statement about sets.
- This doesn't hold for log branches $L_\alpha(z)$.

2. $\log(z_1/z_2) = \log(z_1) - \log(z_2)$ for $z_1, z_2 \neq 0$
3. $\log(z^n) \neq n \log(z)$ in general for $z \neq 0$ and $n \in \mathbb{Z}$
   - A difference with the real logarithm.
   - However, $e^{n \log z} = e^{\log(z^n)} = z^n$. Equal when exponentiated.
4. $z^{1/n} = e^{\frac{1}{n}\log z}$ for any nonzero z and integer n
   - Recall that $z^{1/n} = \{w \mid w^n = z\}$, so this is another statement about sets.

5 Week 5 (Feb 9 - 13)

5.1 Generalized Exponentiation

Suppose $z, c \in \mathbb{C}$ and $z \neq 0$. Then we define the general complex exponent as

$$z^c = e^{c \log z}$$

Note that since the complex log is multi-valued, the exponent is also multi-valued.

For example, 1^i can be found as

$$\begin{aligned} 1^i &= e^{i \log 1} \\ &= e^{i(\ln 1 + i(2\pi n))}\ ,\ \ n = 0, \pm 1, \pm 2, \dots \\ &= e^{-2\pi n}\ \ n = 0, \pm 1, \pm 2, \dots \end{aligned}$$

Setting $n = 0$ gives us the real-valued exponent.

To obtain a single-valued exponent, we use a branch of the complex log.

$$L_\alpha(z) = \ln(r) + i\theta \quad | \quad r > 0, \alpha < \theta < \alpha + 2\pi$$

$$z^c = e^{cL_\alpha(z)} \tag{branch of z^c}$$

Theorem: The branch of $f(z) = z^c$ is analytic on its domain, and $f'(z) = cz^{c-1}$, where z^{c-1} is defined using the same branch L_α.

Proof: The domain of $f(z) = z^c = e^{cL_\alpha(z)}$ is the same as the domain of L_α. Since L_α is analytic on its domain and the exponential function is analytic everywhere, $f(z) = z^c = e^{cL_\alpha(z)}$ is analytic on its domain.

From the chain rule,

$$\begin{aligned}
\frac{d}{dz}z^c &= \frac{d}{dz}e^{cL_\alpha(z)} \\
&= e^{cL_\alpha(z)} \cdot \frac{c}{z} \\
&= c \cdot \frac{e^{cL_\alpha(z)}}{e^{L_\alpha(z)}} \\
&= ce^{(c-1)L_\alpha(z)} \\
&= cz^{c-1}
\end{aligned}$$

The branch here is defined using the same L_α branch as for $f(z) = z^c$.

5.2 Calculus of Complex Functions (of Real Variables)

Eventually, we want to compute integrals of the form

$$\int_C f(z)\ dz$$

where f is a complex-valued function and C is a curve in $\mathbb{C}$. Note that we can parametrize complex-valued functions in terms of real variables: e.g. $w(t) = u(t) + iv(t)$

5.2.1 Differentiation

If u, v are defined and differentiable on some $a \leq t \leq b$, then $w(t) = u(t) + iv(T)$ is differentiable on that interval, and the derivative is

$$w'(t) = u'(t) + iv'(t)$$

Suppose $z_0 \in \mathbb{C}$, and $w_1(t) = u_1(t) + iv_1(t)$ and $w_2(t) = u_2(t) + iv_2(t)$ are differentiable, complex-valued functions on $a \leq t \leq b$. Then for all $a \leq t \leq b$,

i) $\frac{d}{dt}(w_1 + w_2)(t) = w_1'(t) + w_2'(t)$

ii) $\frac{d}{dt}(z_0 w_1(t)) = z_0 w_1'(t)$

iii) $\frac{d}{dt}(w_1 \cdot w_2)(t) = w_1'(t)w_2(t) + w_1(t)w_2'(t)$

iv) $\frac{d}{dt}e^{z_0 t} = z_0 e^{z_0 t}$

Most of the same properties in the real-valued counterpart hold in the complex-valued realm as well. However, not all of them. For example, the mean-valued theorem doesn't hold for complex functions. You can see this with $w(t) = e^{it} = \cos t + i \sin t$, for $0 \leq t \leq 2\pi$. The derivative is $w'(t) = ie^{it}$.

$$|w'(t)| = 1 \quad , \quad \frac{w(2\pi) - w(0)}{2\pi - 0} = 0$$
$$w'(t) \neq \frac{w(2\pi) - w(0)}{2\pi - 0}$$

5.2.2 Integration

If $w(t) = u(t) + iv(t)$, then

$$\int_a^b w(t)\ dt = \int_a^b u(t)\ dt + i \int_a^b v(t)\ dt$$

$w(t)$ is integrable on $a \leq t \leq b$ if both u and v are integrable on $a \leq t \leq b$, i.e. if u and v are piecewise continuous on $a \leq t \leq b$.

Theorem (Fundamental Theorem of Calculus): Let $w_1(t) = u_1(t) + iv_1(t)$ and $w_2(t) = u_2(t) + iv_2(t)$. Suppose w_1, w_2 are continuous and $w_2'(t) = w_1(t)$ for all $a \leq t \leq b$. Then

$$\int_a^b w_1(t)\ dt = w_2(b) - w_2(a)$$

Proof: $w_2' = w_1$ implies $u_2' = u_1$ and $v_2' = v_1$. Therefore,

$$\begin{aligned}\int_a^b w_1(t)\ dt &= \int_a^b u_1(t)\ dt + i \int_a^b v_1(t)\ dt \\ &= u_2(b) - u_2(a) + i(v_2(b) - v_2(a)) \\ &= w_2(b) - w_2(a)\end{aligned}$$

This is the Fundamental Theorem of Calculus for complex-valued functions of a real variable.

5.2.3 Arcs

We call a curve $w(t) = u(t) + iv(t)$ with $a \leq t \leq b$ an *arc* if both u, v are continuous on $a \leq t \leq b$. An arc is notated $z(t) = x(t) + iy(t)$, for $a \leq t \leq b$.

- An arc is *simple* if it doesn't cross itself: $z(t_1) \neq z(t_2)$ unless $t_1 = t_2$.
- An arc is a *simple closed curve* (Jordan curve) if only its endpoints meet: $z(t_1) = z(t_2)$ iff. $\{t_1, t_2\} = \{a, b\}$.

Arcs have a natural orientation, given by the direction traversed by its output as the input increases.

In general, $\int_C f(z)\ dz$ is independent of the arc C's parametrization. However, the signs flip if the orientations are different:

$$\int_C f(z)\ dz = -\int_{-C} f(z)\ dz$$

An arc is differentiable if $x(t), y(t)$ have continuous derivatives for all $a \leq t \leq b$. If $z(t) = x(t) + iy(t)$ is a differentiable arc, its length is defined as

$$L = \int_a^b |z'(t)|\ dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\ dt$$

A differentiable arc is *smooth* if $z'(t)$ is continuous on $a \leq t \leq b$ and nonzero for $a < t < b$.

A *contour* is an arc composed of a finite number of smooth arcs, joined end-to-end. Thus a contour is like a "piecewise smooth" arc.

A *simple closed contour* $z(t)$ for $a \leq t \leq b$ is a contour that's also a simple closed curve. That is, $z(a) = z(b)$, and there's no other intersections.

Theorem (Jordan Curve Theorem): A set of points on a simple closed contour C is the boundary of 2 disjoint domains—one bounded, one not.

It helps to think of the bounded points as the "interior" of the shape defined by the contour. The unbounded points are the "exterior."

Proof: Too hard.