Mathematical Methods of Classical Mechanics – V. I. Arnold

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Contents

Part I

Newtonian Mechanics

1 Experimental Facts (p. 3 - 11)

Certain experimental facts form the basis of classical mechanics. We can't verify them for certain, but they hold approximately, according to accurate tests.

1. **Space and time**: Space is 3D and Euclidean. Time is 1D.
2. **Galileo's Principle of Relativity**: There exist inertial coordinates such that
   - Laws of physics are same in all inertial coordinate systems.
   - Coordinate systems in uniform rectilinear motion w.r.t an inertial one is also inertial.
3. **Newton's Principle of Determinacy**: Initial state of a mechanical system uniquely determines its motion.

1.1 The Galilean Group

Let $\mathbb{R}^n$ denote an n-dimensional real vector space.

An **affine n-dimensional space**, denoted A^n, is similar to $\mathbb{R}^n$ but has no "fixed origin." The group $\mathbb{R}^n$ acts on A^n as the **group of parallel displacements**.

$$a \to a + \mathbf{b}\ , \quad a \in A^n,\ \mathbf{b} \in \mathbb{R}^n,\ a + \mathbf{b} \in A^n$$

The sum of two points on A^n is not defined, but their difference is a vector in $\mathbb{R}^n$. The distance between points of an affine space A^n can be defined using the scalar product:

$$\|x - y\| = \sqrt{(x - y, x - y)}$$

An affine space with this distance function is called a **Euclidean space**, denoted E^n.

The Galilean spacetime structure has 3 elements:

1. **The universe**: A 4D affine space A^4. Points of A^4 are called *events*. Parallel displacements of A^4 form the vector space $\mathbb{R}^4$.
2. **Time**: A linear mapping $t : \mathbb{R}^4 \to \mathbb{R}$ from a parallel displacement to the "time axis." The time interval between $a, b \in A^4$ is $t(b - a)$.
3. **Distance Between Simultaneous Events**: Given by $\|a - b\| = \sqrt{(a - b, a - b)}$. This is a scalar product on $\mathbb{R}^3$. The space of simultaneous events is thus a 3D Euclidean space E^3.

A Galilean space is a space A^4 that has a Galilean spacetime structure.

The Galilean group is the group of all transformations of a Galilean space which preserves its structure.

Elements of a Galilean group are called **Galilean transformations**: affine transformations on A^4 which preserve the time interval & distance between simultaneous events.

The **Galilean coordinate space** is the direct product $\mathbb{R} \times \mathbb{R}^3$ of the t axis with the 3D vector space $\mathbb{R}^3$. There are 3 examples of Galilean transformations of this space:

1. Uniform motion with velocity $\vec{v}$: $g_1(t, \vec{x}) = (t, \vec{x} + \vec{v}t)$
2. Translation of origin: $g_2(t, \vec{x}) = (t + s, \vec{x} + \vec{s})$
3. Rotation of coordinate axes: $g_3(t, \vec{x}) = (t, G\vec{x})$, $G : \mathbb{R}^3 \to \mathbb{R}^3$ is orthogonal transformation.

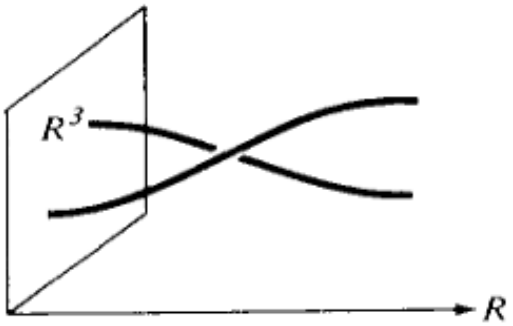
Every Galilean transformation of the space $\mathbb{R} \times \mathbb{R}^3$ can be written as the composition of the above, thus the dimension of the Galilean group is $3 + 4 + 3 = 10$.

Let M be a set. A one-to-one correspondence $\phi_1 : M \to \mathbb{R} \times \mathbb{R}^3$ is called a **Galilean coordinate system** on the set M.

1.2 Motion, Velocity, and Acceleration

A motion in $\mathbb{R}^n$ is a differentiable mapping $\mathbf{x} : I \to \mathbb{R}^N$, where I is an interval on the real axis. The image of a mapping $\mathbf{x} : I \to \mathbb{R}^N$ is called a **trajectory** or **curve** in $\mathbb{R}^N$. The velocity vector is the derivative $\dot{\mathbf{x}}(t_0)$, while the acceleration vector is the second derivative $\ddot{\mathbf{x}}(t_0)$.

Let $\mathbf{x} : \mathbb{R} \to \mathbb{R}^3$ be a motion in $\mathbb{R}^3$. The graph of this mapping ($\mathbb{R}^3$ against $\mathbb{R}$) is a curve in $\mathbb{R} \times \mathbb{R}^3$. A curve in Galilean space that appears in some Galilean coordinate system (as the graph of a motion) is called a **world line**.



Galilean worldlines

Consider a system with n points. In Galilean space, this gives n world lines, described by n mappings $\mathbf{x}_i : \mathbb{R} \to \mathbb{R}^3$ in a Galilean coordinate system. In total, we have

$$\mathbf{x} : \mathbb{R} \to \mathbb{R}^N \ , \quad N = 3n$$

as the total motion of our system with n points.

Newton's principle of determinacy states that all motions of a system are uniquely determined by their initial positions and velocities. Acceleration is not needed: it's determined by the initial position and velocities. I.e. there's a function $\mathbf{F} : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ such that

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$$

This is called **Newton's equation**. By the theorem of existence and uniqueness (recall from diff eq), the initial conditions $\mathbf{F}$, $\mathbf{x}(t_0)$, and $\dot{\mathbf{x}}(t_0)$ uniquely determine a motion. The form of $\mathbf{F}$ can be determined experimentally.

Galileo's principle of relativity requires that Galilean spacetime structure must be invariant w.r.t the group of Galilean transformations. This is a condition on Newton's equation, and leads to three properties of spacetime:

1. **Invariance under time translations**: Laws of nature remain constant regardless of time.
2. **Invariance under spatial translations**: Space is homogeneous; has the same properties at all of its points.
3. **Invariance under spatial rotations**: Space is isotropic; no preferred directions.

We may also introduce the "potential energy" U to write Newton's equation. Let $E^{3n} = E^3 \times \cdots \times E^3$ be the configuration space of a system of n points in the Euclidean space E^3. Let $U : E^{3n} \to \mathbb{R}$ be a differentiable function. The motion of the n points (of masses $m_1, \ldots, m_n$) is given by the system of differentiable equations

$$m_i \ddot{\mathbf{x}}_i = -\frac{\partial U}{\partial \mathbf{x}_i} \ , \quad i = 1, \ldots, n$$

where $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_k})$ and $\partial U / \partial \mathbf{x} = (\partial U / \partial \mathbf{x_1}, \ldots, \partial U / \partial \mathbf{x_k})$. The equations of motion for many other mechanical systems can be written in this form.

2 Investigation of the Equations of Motion (p. 15 - 50)

This chapter deals with configurations for which Newton's equations can be solved.

2.1 Systems with 1 Degree of Freedom

Systems with 1 degree of freedom are described by the differential equation

$$\ddot{x} = f(x) \ , \quad x \in \mathbb{R}$$

The kinetic and potential energies are given by

$$T = \frac{1}{2}\dot{x}^2 \ , \quad U(x) = -\int_{x_0}^{x} f(\xi)\, d\xi$$

The total energy is $E = T + U$. In general, this is a function of x and $\dot{x}$: $E(x, \dot{x})$.

Theorem (law of conservation of energy): the total energy of points moving according to $\ddot{x} = f(x)$ is conserved. That is, $E(x(t), \dot{x}(t))$ is independent of t.

Proof:

$$\frac{d}{dt}(T + U) = \dot{x}\ddot{x} + \frac{dU}{dx}\dot{x} = \dot{x}(\ddot{x} - f(x)) = 0$$

2.1.1 Phase Plane

Note that the equation $\ddot{x} = f(x)$ is equivalent to the system of 2 equations: $\dot{x} = y$ and $\dot{y} = f(x)$. The **phase plane** is a plane with coordinates x and y. A **phase point** is a point on the phase plane. Our system of differential equations determines a vector field on the phase plane, called the phase velocity **vector field**.

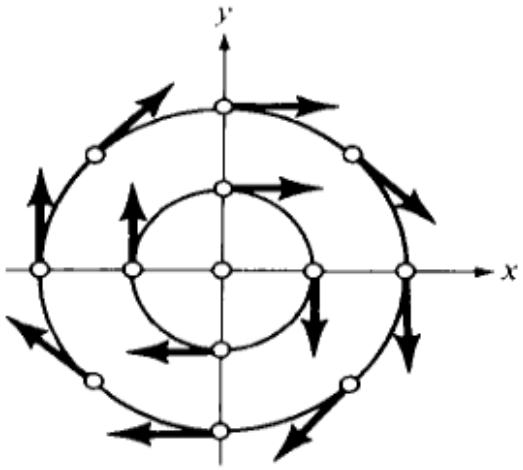
A particular solution $\phi : \mathbb{R} \to \mathbb{R}^2$ is the motion of a phase point in the phase plane, mapping time to the velocity vector. The image of ϕ is called the **phase curve**, given by

$$x = \phi(t) \ , \quad y = \dot{\phi}(t)$$

By existence & uniqueness, there's only one phase curve through every phase point. Note that a phase curve could consist of just one point. Such a point is called the **equilibrium position**, and the phase velocity vector there is zero.

From the law of conservation of energy, total energy must be conserved on a phase curve. Thus each phase curve lies on the energy level set $E(x, y) = h$.

As an example, consider the basic equation of oscillations, $\ddot{x} = -x$. The phase plane is



Phase plane of basic oscillation

2.1.2 Phase Flow

Let M be a point in the phase plane. Assume that any solution to the system can be extended along the time axis. The resulting phase point at time t can be written as

$$M(t) = g^t M(0)$$

This is a mapping of the phase plane to itself, $g^t : \mathbb{R}^2 \to \mathbb{R}^2$. We say g^t is a **diffeomorphism**: it's a one-to-one differentiable mapping with a differentiable inverse.

The diffeomorphisms g^t form a group: $g^{t+s} = g^t \circ g^s$, identity g^0, inverse g^{-t}, etc. The mapping $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by $g(t, M(0)) = g^t M(0)$, is also differentiable. Combined, we say that the transformations g^t form a **one-parameter group of diffeomorphisms** on the phase plane. This group is called the **phase flow**.

2.2 Systems with 2 Degrees of Freedom

These become more difficult. A system with 2 degrees of freedom is defined by the system of differential equations

$$\ddot{x} = f(x) \,, \quad x \in E^2$$

Here $f(x)$ is a vector field on the plane. A system is **conservative** if there exists a function $U : E^2 \to \mathbb{R}$ such that $f = -\partial U / \partial x$. Then the equation of motion becomes

$$\ddot{x} = -\frac{\partial U}{\partial x}$$

Theorem (law of conservation of energy): The total energy of a system is conserved, and given by

$$\frac{dE}{dt} = 0$$

$$E = \frac{1}{2}\dot{x}^2 + U(x) \,, \quad \dot{x}^2 = (\dot{x}, \dot{x})$$

Proof: Use the equation of motion and form of E to get

$$\frac{dE}{dt} = (\dot{x}, \ddot{x}) + (\partial U/\partial x, \dot{x}) = 0$$

Corollary: If at the initial moment the total energy is E, all trajectories will lie in the region where $U(x) \leq E$. In a system with 1 degree of freedom, it's always possible to find $U(x) = -\int f(\xi)d\xi$. But if we have 2 degrees of freedom, this isn't always possible.

I.e. there might not be a $U(x)$ such that $f(x) = -\partial U/\partial x$, which means the system isn't conservative.

An interesting example is **Lissajous figures**. Consider a system of small oscillations with 2 degrees of freedom:

$$\ddot{x}_1 = -x_1\ ,\ \ \ddot{x}_2 = -\omega^2 x_2$$

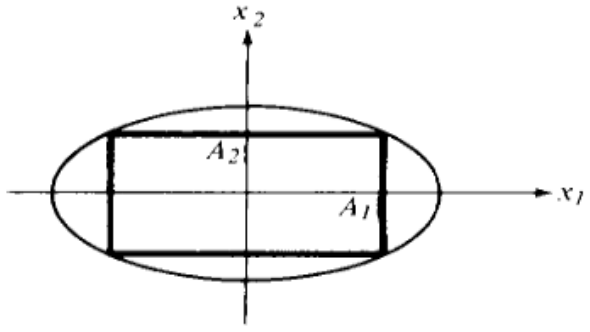
The potential energy (U) and total energy (E) are given by

$$U = \frac{1}{2}x_1^2 + \frac{1}{2}\omega^2 x_2^2$$
$$E = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + U(x_1, x_2)$$

From the law of conservation of energy, all motions will take place inside the ellipse $U(x_1, x_2) \leq E$. If we split into 1-dimensional systems in x_1 and x_2, the following quantities are conserved:

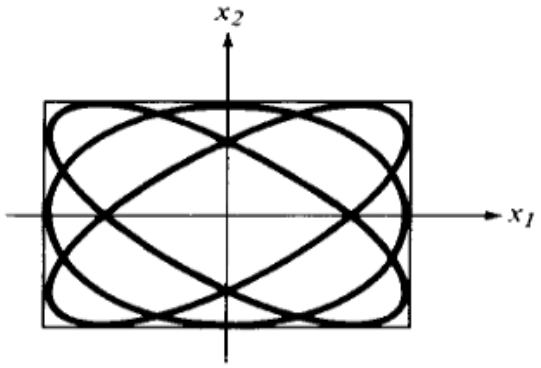
$$E_1 = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}x_1^2\ ,\ \ E_2 = \frac{1}{2}\dot{x}_2^2 + \frac{1}{2}x_2^2$$

Thus we have the bounds $|x_1| \leq \sqrt{2E_1(0)}$ and $|x_2| \leq \sqrt{2E_2(0)}$. We have the elliptical boundary from the 2-dimensional case and a rectangular boundary from the 1-dimensional cases.



Lissajous curve boundaries

Depending on the value of ω, we get some interesting shapes.



Lissajous curve example with $\omega = 1$

2.3 Conservative Force Fields

Here we want to connect work with potential energy. The work of a constant force F is defined as the scalar product of the path S:

$$A = (F, S) = |F||S| \cos\theta$$

Furthermore, we said a system is **conservative** if there exists a function $U : E^2 \to \mathbb{R}$ such that $f = -\partial U/\partial x$, for some force f.

Theorem: A vector field F is conservative if and only if the work along any path M depends only on the endpoints of the path.

Proof: First, suppose the work of a field F doesn't depend on the path. Then

$$U(M) = -\int_{M_0}^{M} (F, dS)$$
$$F = -\frac{\partial U}{\partial x}$$

So we see that the field is conservative. Next, suppose the field F is conservative and U is its potential energy. Then

$$\int_{M_0}^{M} (F, dS) = -U(M) + U(M_0)$$

So the work doesn't depend on the path.

A vector field in the plane E^2 is called **central** with center 0 if it's invariant w.r.t the group of motions of the plane that fix 0. All vectors of a central field lie on rays through 0. The magnitude of the vector field at a point depends only its distance from the center. For example, $F = -k\hat{r}/r^3$ is central.

Every central field is conservative, and its potential energy depends only on its distance to the center: $U = U(r)$.

2.4 Angular Momentum

The invariance of an equation of a mechanical problem w.r.t some group of transformations **always implies a conservation law**. For example, we saw previously that energy is conserved because the system is invariant under a time translation. The central field is invariant w.r.t the group of rotations. This implies a conserved quantity that we call **angular momentum**.

The angular momentum of a point of mass relative to the point 0 (central point) is the cross product

$$M = r \times \dot{r}$$

where r is the radius vector beginning at the center of the field.

Theorem (law of conservation of angular momentum): Under motions in a central field, the angular momentum M relative to the center of the field doesn't change with time.

Proof: We know that $M = r \times \dot{r}$. Taking the derivative, $\dot{M} = \dot{r} \times \dot{r} + r \times \ddot{r}$. Since the field is central, r and $\ddot{r}$ are colinear. Thus $\dot{M} = 0$.

We can use polar coordinates to express angular momentum. Some care is required to differentiate constants and vectors. Parametrize with r and φ. Let $\mathbf{e}_r$ be the unit radius vector and $\mathbf{e}_\varphi$ be the unit angle vector. Then we have the radius vector $\mathbf{r} = r\mathbf{e_r}$. Differentiate to get

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e_r} + r\dot{\mathbf{e}}_\mathbf{r} = \dot{r}\mathbf{e_r} + r\dot{\varphi}\mathbf{e}_\varphi$$

Thus the angular momentum is

$$\begin{gathered}\mathbf{M} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times \dot{r}\mathbf{e_r} + \mathbf{r} \times r\dot{\varphi}\mathbf{e}_\varphi \\ \mathbf{M} = \mathbf{r} \times r\dot{\varphi}\mathbf{e}_\varphi = r^2\dot{\varphi}(\mathbf{e_r} \times \mathbf{e}_\varphi) \\ |\mathbf{M}| = r^2\dot{\varphi}\end{gathered}$$

This is a familiar result from first-year physics: $L = I\omega$, etc.

2.5 Motion in a Central Field

Conservation of angular momentum allows us to reduce motion in a central field to a problem with 1 degree of freedom. Thus, the motion in a central field can be completely determined.

Consider the motion of a point mass in a central field on the plane:

$$\ddot{r} = -\frac{\partial U}{\partial r}\ , \ \ U = U(r)$$

Theorem: For the motion of a point mass in a central field, the equation of motion is the same as the 1-dimensional problem with potential energy

$$V(r) = U(r) + \frac{M^2}{2r^2}$$

Proof: Using $\dot{\mathbf{r}} = \dot{r}\mathbf{e_r} + r\dot{\varphi}\mathbf{e}_\varphi$ with $\dot{\mathbf{e}_r} = \dot{\varphi}\mathbf{e}_\varphi$ and $\dot{\mathbf{e}_\varphi} = -\dot{\varphi}\mathbf{e_r}$, we have

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\varphi}^2)\mathbf{e_r} + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\mathbf{e}_\varphi$$

Since the field is central, we know that

$$\frac{\partial U}{\partial \mathbf{r}} = \frac{\partial U}{\partial r}\mathbf{e_r}$$
$$\ddot{r} - r\dot{\varphi}^2 = -\frac{\partial U}{\partial r}\ ,\ \ 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0$$

From conservation of angular momentum, we can say $\dot{\varphi} = M/r^2$. Therefore,

$$\ddot{r} = -\frac{\partial U}{\partial r} + r\frac{M^2}{r^4}$$
$$\ddot{r} = -\frac{\partial V}{\partial r}\ ,\ \ V = U + \frac{M^2}{2r^2}$$

The total energy is given by $E = \dot{r}^2/2 + V(r)$. This is conserved, thus $\dot{r} = dr/dt = \sqrt{2(E - V(r))}$. We know that $\dot{\varphi} = d\varphi/dt = M/r^2$. Substituting dt, we get

$$\frac{d\varphi}{dr} = \frac{M}{r^2\sqrt{2(E - V(r))}}$$

This is the equation of the orbit in polar coordinates.

As an example, we'll try to solve Kepler's problem for celestial motion. Consider the central field with potential $U = -k/r$ (it turns out that this and $U = ar^2$ are the only cases for which all bounded orbits are closed). Anyways, then $V(r) = -k/r + M^2/(2r^2)$. The general formula for φ above gives

$$\begin{aligned}\varphi &= \int \frac{M}{r^2\sqrt{2(E - V(r))}}\,dr \\ &= \arccos\left(\frac{M/r - k/M}{\sqrt{2E + k^2/M^2}}\right)\end{aligned}$$

Let $p = M^2/k$ and $e = \sqrt{1 + 2EM^2/k^2}$. Then we get

$$\varphi = \frac{\arccos\left(p/r - 1\right)}{e}$$
$$r = \frac{p}{1 + e\cos\varphi}$$

This is the focal point of a conic section, specifically an ellipse for $e < 1$. Thus the motion of planets describe ellipses, with the sun at one focus.

2.6 Motion in Three-Space

Here we want to define angular momentum relative to an axis. In an axially symmetric field, angular momentum is still conserved. First, consider motion in a conservative field

$$\ddot{\mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}} \ , \ \ U = U(\mathbf{r}) \ , \ \ \mathbf{r} \in E^3$$

The law of conservation of energy holds: $dE/dt = 0$, where $E = \frac{1}{2}\dot{\mathbf{r}}^2 + U(\mathbf{r})$.

Recall that every central field is conservative, and the angular momentum doesn't change: for $M = \mathbf{r} \times \dot{\mathbf{r}}$, we have $d\mathbf{M}/dt = 0$. For motion in a central field, every orbit is planar. To see this, note that

$$\mathbf{M} \cdot \mathbf{r} = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{r} = 0$$

Thus all orbits lie in the plane perpendicular to $\mathbf{M}$.

We say a vector field in E^3 is **axially symmetric** if it's invariant w.r.t the group of spatial rotations that fix every point of some axis. The moment M_z is the projection of the moment of the vector $\mathbf{F}$ applied at $\mathbf{r}$ onto the z axis (or any axis of symmetry).

$$M_z = \mathbf{e_z} \cdot (\mathbf{r} \times \mathbf{F})$$

Theorem: For motion in a conservative field with axial symmetry around the z axis, the moment of velocity relative to the z axis is conserved.

Proof: Since $\ddot{\mathbf{r}} = \mathbf{F}$, both $\mathbf{r}$ and $\ddot{\mathbf{r}}$ must lie in a plane passing through the z axis. Thus their cross product is perpendicular to $\mathbf{e_z}$, and

$$\dot{M}_z = \mathbf{e_z} \cdot (\dot{\mathbf{r}} \times \dot{\mathbf{r}}) + \mathbf{e_z} \cdot (\mathbf{r} \times \ddot{\mathbf{r}}) = 0$$

2.7 Motion of a System of n points

First, write Newton's equation for the force acting on each point (mass m_i and radius vector $\mathbf{r}_i \in E^3$).

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i$$

For some systems, the force acting on two points are equal in magnitude and opposite in direction. These are called **forces of interaction** (e.g. universal gravitation). If all forces acting on a point in the system are forces of interaction, then the system is **closed**. The force acting on the i-th point of a close system is

$$\mathbf{F}_i = \sum_{j=1,\ j \neq i}^{n} \mathbf{F}_{ij}$$

Here, $\mathbf{F}_{ij}$ is the force with which the j-th point acts on the i-th.

If the system is not closed, the force acting on the i-th point can be represented with the **external force** $\mathbf{F}'_i$:

$$\mathbf{F}_i = \sum \mathbf{F}_{ij} + \mathbf{F}'_i$$

2.7.1 Conservation of Momentum

The momentum of a system is the vector

$$\mathbf{P} = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i$$

Theorem: The rate of change of a system's momentum is equal to the sum of all external forces acting on points of the system.

Proof: Note that

$$\frac{d\mathbf{P}}{dt} = \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{n} \mathbf{F}_i = \sum_{i,j} \mathbf{F}_{ij} + \sum_{i} \mathbf{F}'_i$$

Further, $\sum_{i,j} \mathbf{F}_{ij} = 0$ since for forces interaction, $F_{ij} = -F_{ji}$. Thus we find that the change in momentum is equal to the sum of external forces.

$$\frac{d\mathbf{P}}{dt} = \sum_{i} \mathbf{F}'_i$$

This also implies that the **momentum of a closed system is conserved**. To simplify calculations, we can treat the total momentum of the system as the momentum of a particle lying at the center of mass $\mathbf{r} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}$ with mass $\sum m_i$.

2.7.2 Conservation of Angular Momentum

The angular momentum of a system (relative to 0) is defined as

$$\mathbf{M} = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i$$

Theorem: The rate of change of a system's angular momentum is equal to the sum of the moments of external forces acting on the system's points.

Proof: Find the derivative as

$$\frac{d\mathbf{M}}{dt} = \sum_{i=1}^{n} \dot{\mathbf{r}}_i \times m_i \dot{\mathbf{r}}_i + \sum_{i=1}^{n} \mathbf{r}_i \times m_i \ddot{\mathbf{r}}_i$$

$$\frac{d\mathbf{M}}{dt} = \sum_{i=1}^{n} \left[\mathbf{r}_i \times \left(\sum_{i \neq j} \mathbf{F}_{ij} + \mathbf{F}'_i \right) \right]$$

Since $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, we have that $(\mathbf{r}_i \times \mathbf{F}_{ij}) + (\mathbf{r}_j \times \mathbf{F}_{ji}) = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = 0$. So the $\mathbf{F}_{ij}$ sum vanishes and we're left with

$$\frac{d\mathbf{M}}{dt} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}'_i$$

Thus, **angular momentum of a closed system is conserved**.

2.7.3 Conservation of Energy

The kinetic energy of a system of mass points is defined as

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$$

Theorem: The increase in a system's kinetic energy is equal to the sum of the work of all forces acting on the system's points.

Proof: First, find dT/dt as

$$\frac{dT}{dt} = \sum_{i=1}^{n} m_i(\dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i) = \sum_{i=1}^{n} \dot{\mathbf{r}}_i \cdot m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{n} \dot{\mathbf{r}}_i \cdot \mathbf{F}_i$$

Then we can find the change in kinetic energy as

$$T(t) - T(t_0) = \int_{t_0}^{t} \frac{dT}{dt}\, dt = \sum_{i=1}^{n} \int_{t_0}^{t} \dot{\mathbf{r}}_i \cdot \mathbf{F}_i\, dt$$

$$T(t) - T(t_0) = \int_{r(t_0)}^{r(t)} \mathbf{F} \cdot d\mathbf{r}$$

It can be seen that the increase in kinetic energy is equal to the work of $\mathbf{F}$ along the path $\mathbf{r}(t)$.

Now consider potential energy. From previous definitions, a system is **conservative** if the forces only depend on the location of a point in the system ($\mathbf{F} = \mathbf{F}(r)$) and the work of $\mathbf{F}$ along any path depends only on the endpoints of the path. A system is conservative if and only if there exists a potential energy $U(\mathbf{r})$ such that

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{r}}$$

Consider the forces acting on points in a system:

$$\mathbf{F}_i = \sum_{i \neq j} \mathbf{F}_{ij} + \mathbf{F}_i'$$

Theorem: If the forces of interaction only depend on distance, such that $f_{ij} = f_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$ for $\mathbf{F}_{ij} = -\mathbf{F}_{ji} = f_{ij}\mathbf{e}_{ij}$, then the forces are conservative.

Proof: Now, the potential energy of interaction is

$$U_{ij}(\mathbf{r}) = \int_{r_0}^{r} f_{ij}(\rho)\, d\rho$$

The potential energy of the interaction of all the points is

$$U(\mathbf{r}) = \sum_{i>j} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$$

If the external forces are also conservative, then $\mathbf{F}'_i = -\partial U'_i/\partial \mathbf{r}_i$. The system here is conservative, and the total potential energy is $U(\mathbf{r}) = \sum_{i>j} U_{ij} + \sum_i U'_i$. Furthermore, note that the kinetic energy is the same at two different times if the potential energy is conservative:

$$T(t_1) - T(t_0) = \int_{\mathbf{r}(t_0)}^{\mathbf{r}(t_1)} \mathbf{F} \cdot d\mathbf{r} = U(\mathbf{r}(t_0)) - U(\mathbf{r}(t_1))$$

Thus, for a system with a conservative potential energy, the total mechanical energy

$$E = T + U = \sum_i \frac{1}{2}\dot{\mathbf{r}}_i^2 + \sum_{i>j} U_{ij} + \sum_i U'_i$$

is conserved.

Part II

Lagrangian Mechanics

3 Variational Principles (p. 55 - 74)

Lagrangian mechanics uses a manifold ("configuration space") and a function on its tangent bundle ("Lagrangian function") to define its system. A conservation law is defined by every 1-parameter group of diffeomorphisms of the configuration space which fixes the Lagrangian function.

The Newtonian potential system is a special case of a Lagrangian system: the configuration space is Euclidean, and the Lagrangian function is $L = T - V$.

3.1 Calculus of Variations

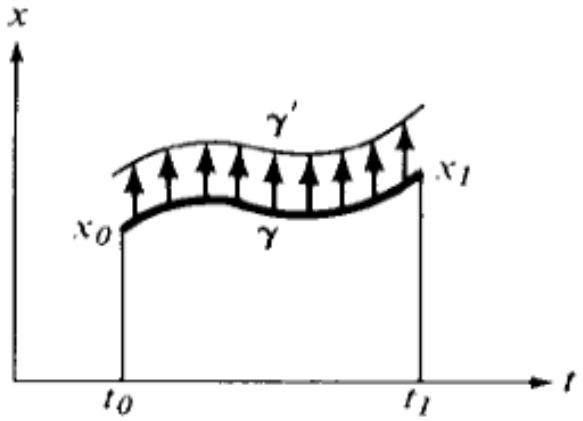
3.1.1 Functionals

Consider functions whose domains are in an infinite-dimensional space: the space of curves. Such functions are called **functionals**. The calculus of variations examines the extremals of functionals.

In general, a functional is **any mapping from the space of curves to** $\mathbb{R}$. For example, the length of a curve in the Euclidean plane is a functional:

$$\gamma = \{(t, x) \mid x(t) = x_0, t_0 \le t \le t_1\}$$
$$\Phi(\gamma) = \int_{t_0}^{t_1} \sqrt{1 + \dot{x}^2}\, dt$$

Now we consider an "approximation" $\gamma' = \{(t, x) \mid x = x(t) + h(t)\}$. Call this $\gamma' = \gamma + h$. Also consider the resulting increment in the functional: $\Phi(\gamma + h) - \Phi(\gamma)$.



Variation of a curve (Arnold 56)

A functional Φ is called **differentiable** if $\Phi(\gamma + h) - \Phi(\gamma) = F + R$, where F depends linearly on h and $R(h, \gamma) = O(h^2)$. We call $F(h)$ the **differential**.

By "depending linearly on h," we mean: $F(h_1 + h_2) = F(h_1) + F(h_2)$ and $F(ch) = cF(h)$ for fixed γ.

If Φ is differentiable, its differential is uniquely defined. The differential $F(h)$ of a functional Φ can also be called its **variation**, and h is a **variation of the curve**.

Theorem: The functional $\Phi(\gamma) = \int_{t_0}^{t_1} L(x, \dot{x}, t)\, dt$ is differentiable, and its differential is given by

$$F(h) = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] h\, dt + \left(\frac{\partial L}{\partial \dot{x}} h\right)\Big|_{t_0}^{t_1}$$

Proof: Find the difference in the functional as

$$\begin{aligned}\Phi(\gamma + h) - \Phi(\gamma) &= \int_{t_0}^{t_1} \Big[L(x + h, \dot{x} + \dot{h}, t) - L(x, \dot{x}, t)\Big]\, dt \\ &= \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x} h + \frac{\partial L}{\partial \dot{x}} \dot{h}\right] dt + O(h^2)\end{aligned}$$

The right hand side can be represented as $F(h) + R$, where $R = O(h^2)$ and $F(h)$ is the integral. Using integration by parts, we can also find

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{x}} \dot{h}\, dt = -\int_{t_0}^{t_1} h \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) dt + \left(h\frac{\partial L}{\partial \dot{x}}\right)\Big|_{t_0}^{t_1}$$

$$F(h) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} h + \frac{\partial L}{\partial \dot{x}} \dot{h}\right) dt = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] h\, dt + \left(\frac{\partial L}{\partial \dot{x}} h\right)\Big|_{t_0}^{t_1}$$

3.1.2 Euler-Lagrange Equations

An **extremal** of a differentiable functional $\Phi(\gamma)$ is a curve γ such that $F(h) = 0$ for all h. It's kind of like x being a stationary point of a function f if $df/dx = 0$. But here, we're finding the "stationary point" across *functionals*, not points.

Theorem: The curve $\gamma : x = x(t)$ is an extremal of the functional $\Phi(\gamma) = \int_{t_0}^{t_1} L(x, \dot{x}, t)\, dt$ on the space of curves passing through $x(t_0) = x_0$ and $x(t_1) = x_1$ if and only if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{along the curve } x(t)$$

Proof: First, we prove the "if" direction. Assume that γ is an extremal. Recall from the previous theorem that

$$F(h) = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] h\, dt + \left(\frac{\partial L}{\partial \dot{x}} h\right)\Big|_{t_0}^{t_1}$$

Since we're finding the extremal of the functional with set endpoints t_0 and t_1, the variation h must be 0 at those points: $h(t_0) = h(t_1)$. Otherwise we're not considering the same functional. Thus the second term vanishes, and $F(h) = 0$ for an extremal, so

$$\int_{t_0}^{t_1} f(t)h(t)\, dt = 0\ , \quad f(t) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}$$

This must hold for all h, thus $f(t) = 0$.

Proving the converse is easy: if $f(t) = 0$, then obviously $F(h) = 0$.

The equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

is called the **Euler-Lagrange equation** for the functional $\Phi = \int_{t_0}^{t_1} L(x, \dot{x}, t)\, dt$. The Euler-Lagrange equation must be satisfied along the curve γ for γ to be an extremal of Φ.

3.2 Lagrange's Equations

Recall Newton's equations of dynamics:

$$m_i \ddot{r}_i = -\frac{\partial U}{\partial r_i}$$

Theorem (Hamilton's principle of least action): Motions of the system described above coincide with the extremals of the functional

$$\Phi(\gamma) = \int_{t_0}^{t_1} L\, dt$$

where $L = T - U$ is the difference between kinetic and potential energy.

Proof: From $U = U(\mathbf{r})$ and $T = \sum \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$, we have

$$\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = m_i \dot{\mathbf{r}}_i\ , \quad \frac{\partial L}{\partial \mathbf{r}_i} = -\frac{\partial U}{\partial \mathbf{r}_i}$$

We see that the Euler-Lagrange equation holds:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m_i \ddot{\mathbf{r}}_i + \frac{\partial U}{\partial \mathbf{r}_i} = 0$$

Thus a motion is an extremal of the functional $\int L\, dt$. In any system of coordinates, the Euler-Lagrange equation written in that coordinate system is satisfied.

The terminology is as follows.

1. Lagrange function/Lagrangian: $L(\mathbf{q}, \dot{\mathbf{q}}, t) = T - U$
2. Generalized coordinates: q_i
3. Generalized momenta: $\partial L/\partial \dot{q}_i = p_i$
4. Generalized forces: $\partial L/\partial q_i = F_i$
5. Action: $\int_{t_0}^{t_1} L(\mathbf{q}, \dot{\mathbf{q}}, t)\, dt$

So the previous theorem is finding the extremal of the action functional $\int L\, dt$. In many cases, the action is not just an extremal, but a **minimum value**. Thus the evolution of the system follows the path that minimizes the action, which is why the theorem is sometimes called "Hamilton's principle of least action."

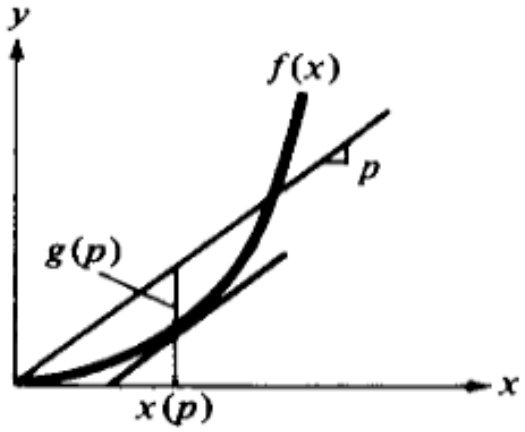
There's a neat connection between conservation laws and the Lagrangian. Call a coordinate q_i **cyclic** if it doesn't enter into the Lagrangian: $\partial L/\partial q_i = 0$. The generalized momentum p_i corresponding to a cyclic coordinate is conserved: $p_i = \text{const}$.

3.3 Legendre Transformations

Legendre transformations transforms functions on a vector space to functions on a dual space. In physics, they're used to convert functions of one quantity to functions of its conjugate quantity: E.g. position to momentum, temperature to entropy.

Let $y = f(x)$ be a convex function: $f''(x) > 0$. Draw f in the xy-plane. Now consider lines of the form $y = px$. Take the point $x = x(p)$ for which the curve is farthest from the straight line vertically: that is, the choice of x maximizes $px - f(x)$. Now we can find one point of g, **the Legendre transformation of** f, as $g(p) = px - f(x)$.

The point $x(p)$ is defined by the condition $\partial F/\partial x = 0$ where $F(p, x) = px - f(x)$, i.e. $f'(x) = p$.



Legendre transformation (Arnold 62)

For example, consider $f(x) = x^2$. Then $F(p, x) = px - x^2$, and by finding the stationary point, we have $x(p) = \frac{1}{2}p$. Now we find the Legendre transformation of f as

$$g(p) = px - f(x) = \frac{1}{2}p^2 - x^2 = \frac{1}{4}p^2$$

For a system with many variables, the procedure is the same. Let $f(\mathbf{x})$ be a convex function of $\mathbf{x} = (x_1, \ldots, x_n)$. The Legendre transformation $g(\mathbf{p})$ of $f(\mathbf{x})$ is defined by the n equalities $F(\mathbf{p}, \mathbf{x}) = \max_{\mathbf{x}} F(\mathbf{p}, \mathbf{x})$, over each component $\mathbf{x}$. Here, $F(\mathbf{p}, \mathbf{x}) = \mathbf{p} \cdot \mathbf{x} - f(\mathbf{x})$ and $\partial f / \partial \mathbf{x} = \mathbf{p}$.

The Legendre transformation is involutive: i.e. If the Legendre transformation of f is g, then the Legendre transformation of g will be f.

Two functions f and g that are Legendre transformations of each other are called **dual in the sense of Young**. Note that in the Legendre transform, $F(x, p) = px - f(x)$ is always less than or equal to $g(p)$. Thus we have Young's inequality:

$$px \leq f(x) + g(p)$$

3.4 Hamilton's Equations

A Legendre transformation can be used to turn a Lagrangian system of differential equations into a Hamiltonian system of equations ("canonical equations").

Hamilton's equations are useful because 1) They allow us to describe systems in terms of position and momentum (quantum mechanics, anyone?), and 2) They simplify the n second-order differential equations of Lagrange to $2n$ first-order differential equations.

Consider the system of Lagrange's equations $\dot{\mathbf{p}} = \partial L / \partial \mathbf{q}$, where $\mathbf{p} = \partial L / \partial \dot{\mathbf{q}}$ with a given Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. Assume this is convex w.r.t the second argument $\dot{\mathbf{q}}$.

Theorem: The system of Lagrange's equations is equivalent to the system of $2n$ first-order equations (Hamilton's equations)

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$$
$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$$

where $H(\mathbf{p}, \mathbf{q}, t) = \mathbf{p}\dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is the Legendre transform of the Lagrangian w.r.t $\dot{\mathbf{q}}$. The function H is called the **Hamiltonian**. Thus, if $\mathbf{q}(t)$ satisfies Lagrange's equations, then $(\mathbf{p}(t), \mathbf{q}(t))$ satisfies Hamilton's equations.

If the equations are mechanical, so that the Lagrangian has the form $L = T - U$, then the Hamiltonian is the total energy: $H = T + U$. The proof of this uses Euler's theorem on homogeneous functions.

Anyway, as an example, consider 1-dimensional motion:

$$\ddot{q} = -\frac{\partial U}{\partial q}$$

Then $T = \frac{1}{2}\dot{q}^2$ and $U = U(q)$, so the Hamiltonian is $H = \frac{1}{2}p^2 + U(q)$. As before, consider the generalized momentum $p = \partial L/\partial q$. Also note that $\ddot{q} = \dot{p}$, so $\dot{q} = p$. Hamilton's equations take the form

$$\dot{p} = -\frac{\partial U}{\partial q}$$
$$\dot{q} = p$$

For a system whose Hamiltonian doesn't depend on time ($\partial H/\partial t = 0$), the law of conservation of energy in a Hamiltonian system takes the form

$$\frac{dH}{dt} = 0$$

which makes sense, as $H = T + U = E_{total}$. But to prove this, differentiate:

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial \mathbf{p}}\left(\frac{d\mathbf{p}}{dt}\right) + \frac{\partial H}{\partial \mathbf{q}}\left(\frac{d\mathbf{q}}{dt}\right) + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial \mathbf{p}}\left(-\frac{\partial H}{\partial \mathbf{q}}\right) + \frac{\partial H}{\partial \mathbf{q}}\left(\frac{\partial H}{\partial \mathbf{p}}\right) + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t}\end{aligned}$$

3.5 Cyclic Coordinates

We can exploit the symmetry of a given problem to reduce the coordinates the Hamiltonian depends upon. If a coordinate q_1 doesn't enter into the Hamiltonian $H(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$, it's a **cyclic coordinate**.

Let q_1 be a cyclic coordinate. Then the variation of the remaining coordinates w.r.t time is the same as the Hamiltonian system with $n - 1$ independent coordinates:

$$H(p_2, \ldots, p_n, q_2, \ldots, q_n, t)$$

This allows us to simplify mechanical systems a lot.

3.6 Liouville's Theorem

Here we show that the phase flow of Hamilton's equations preserves the phase volume.

Consider the $2n$-dimensional space with coordinates $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$. This is called the **phase space**. Hamilton's equations give a vector field $(-\partial H/\partial \mathbf{q}, \partial H/\partial \mathbf{p})$ associated with each point $(\mathbf{p}, \mathbf{q})$ of phase space.

The **phase flow** is the 1-parameter group of phase space transformations:

$$g^t : (\mathbf{p}(0), \mathbf{q}(0)) \mapsto (\mathbf{p}(t), \mathbf{q}(t))$$

Theorem (Liouville's theorem): The phase flow preserves volume. For any region D, we have

$$\text{volume}(g^t D) = \text{volume}(D)$$

Theorem (Liouville's theorem, general): Consider a system of ordinary differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_n)$. Assume that the solution can be extended to the whole time axis. Let $\{g^t\}$ be the group of transformations

$$g^t(\mathbf{x}) = \mathbf{x} + \mathbf{f}(\mathbf{x}) + O(t^2) \ , \quad t \to 0$$

Let $D(t)$ be a region in $\mathbf{x}$-space. If $\nabla \cdot \mathbf{f} = 0$, then g^t preserves volume:

$$\text{volume}(D(t)) = \text{volume}(D(0)) \ , \quad D(t) = g^t D(0)$$

Note that for Hamilton's equations, we have

$$\nabla \cdot \mathbf{f} = \frac{\partial}{\partial \mathbf{p}} \left(-\frac{\partial H}{\partial \mathbf{q}} \right) + \frac{\partial}{\partial \mathbf{q}} \left(\frac{\partial H}{\partial \mathbf{p}} \right) = 0$$

Thus Liouville's theorem (version 1) is proven.

By Liouville's theorem, a Hamiltonian system cannot have an asymptotically stable equilibrium position. That would mean that the divergence of $\mathbf{f} = (\dot{\mathbf{p}}, \dot{\mathbf{q}})$ is nonzero at the equilibrium position.

3.7 Poincaré's Recurrence Theorem

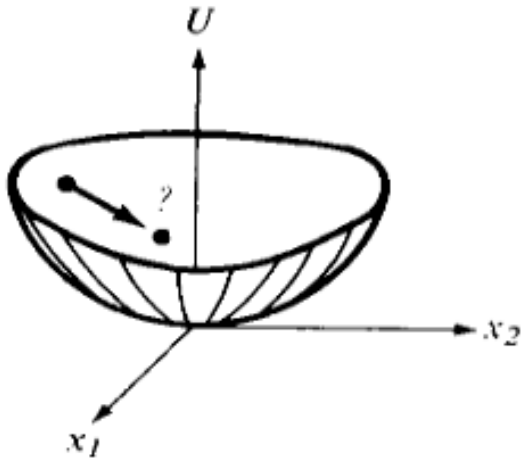
Theorem: Let g be a volume-preserving continuous one-to-one mapping that maps a bounded region $D \subset E^3$ to itself: $gD = D$. Then in any neighborhood U of any point of D, there exists a point $x \in U$ which returns to U, i.e. $g^t x \in U$ for some $t > 0$.

Proof: Consider images in the neighborhood of U: U, gU, g^2U, etc. All of these have the same volume by Liouville's theorem. If they never intersect, D would have infinite volume. Thus, for some $k, l \geq 0$ with $k > l$,

$$g^k U \cap g^l U \neq \emptyset \quad \longrightarrow \quad g^{k-l} U \cap U \neq \emptyset$$

Consider a point $y = g^{k-l}x$ in this intersection, where $x \in U$. Then $g^{k-l}x \in U$, and we've proved Poincaré's theorem.

That is, almost every moving point returns repeatedly to the vicinity of its original position. The details of the motion may not be known, but this conclusion can still be drawn.



Example: the exact motion of a ball in an asymmetrical cup is unknown. But Poincaré's theorem predicts that it'll return to a neighborhood of the original position (Arnold 71)

There seems to be a conflict with the 2nd Law of Thermodynamics: if you open a partition separating two chambers (the left one full of gas, the right one a vacuum), then eventually all the gas will return to the left side? But this isn't a problem since the Poincaré recurrence time is extremely long, maybe longer than the duration of the solar system's existence.

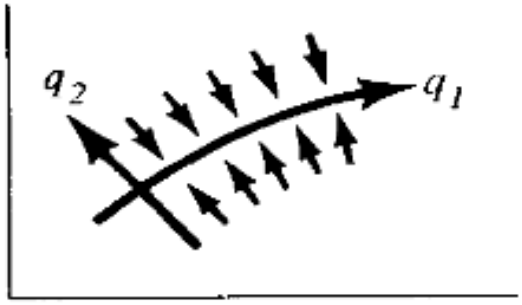
4 Lagrangian Mechanics on Manifolds (p. 75 - 97)

Here, the differentiable manifold and its tangent bundle are introduced. We construct Lagrangian functions on the tangent bundle, which defines a Lagrangian "holonomic system" on a manifold.

4.1 Holonomic Constraints

Imagine a smooth curve γ in a plane. If there's a strong force field in the neighborhood of γ directed towards the curve, a moving point will be close to γ. If the field is infinite, the point will be on γ. In that case, we say a constraint is put on the system.

Introduce the curvilinear coordinates q_1 and q_2 in a neighborhood of γ. q_1 is in the direction of γ and q_2 is the distance from the curve. Consider a system with potential energy $U_N = Nq_2^2 + U_0(q_1, q_2)$.



Constraint as an infinitely strong field (Arnold 76)

Theorem: Let $q_1 = \varphi(t, N)$ be the evolution of q_1 under the field U_N. As $N \to \infty$, the limit

$$\lim_{N\to\infty} q_1 = \lim_{N\to\infty} \varphi(t, N) = \psi(t)$$

exists, and $q_1 = \psi(t)$ satisfies Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L_*}{\partial \dot{q}_1}\right) = \frac{\partial L_*}{\partial q_1}$$

where $L_*(q_1, \dot{q}_1) = T|_{q_2=\dot{q}_2=0} - U|_{q_2=0}$, and T is the kinetic energy of the motion along γ.

This even works in 3 dimensions. If the potential energy has the form

$$U = U_0(\mathbf{q_1}, \mathbf{q_2}) + N\mathbf{q_2}^2$$

then as $N \to \infty$, a motion on γ follows Lagrange's equations with the Lagrangian

$$L_* = T|_{\mathbf{q_2}=\dot{\mathbf{q}}_\mathbf{2}=0} - U_0|_{\mathbf{q_2}=0}$$

Let γ be an m-dimensional surface in the $3n$-dimensional configuration space of the points $\mathbf{r}_1, \ldots, \mathbf{r}_n$ with masses $m_1, \ldots, m_n$. Let $\mathbf{q} = (q_1, \ldots, q_m)$ be some coordinates on γ, such that $\mathbf{r}_i = \mathbf{r}_i(\mathbf{q})$. The system described by the equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}}\,, \quad L = \frac{1}{2}\sum m_i \dot{\mathbf{r}}_i^2 + U(\mathbf{q})$$

is called a system of n points with $3n - m$ ideal **holonomic constraints**. The surface γ is called the **configuration space of the system with constraints**.

4.2 Differentiable Manifolds

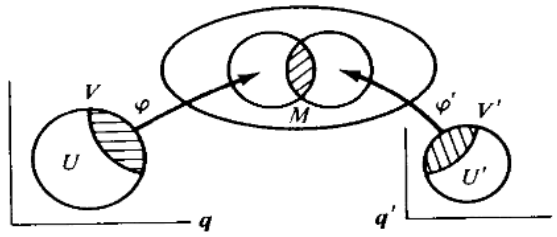
4.2.1 Definitions

The configuration space of a system with constraints is a differentiable manifold. Manifolds are spaces that locally resemble Euclidean space.

A set M is given the structure of a differentiable manifold if it's provided with a countable collection of **charts**, so that every point is represented in at least one chart.

Define a **chart** to be an open set U in the Euclidean coordinate space $\mathbf{q} = (q_1, \ldots, q_n)$, together with the one-to-one mapping φ of U onto some subset of M. That is, $\varphi : U \to \varphi U \subset M$. A chart acts as a local coordinate system and "map" of a manifold to $\mathbb{R}^n$.

Let p and p' be two points in the charts U and U', respectively. Assume that p and p' have neighborhoods $V \subset U$ and $V' \subset U'$ with the same image in M. Then we can get the mapping $\varphi'^{-1}\varphi : V \to V'$. This is a mapping of one Euclidean space onto another. The charts U and U' are **compatible** if the function $\varphi'^{-1}\varphi$ is infinitely differentiable.



Compatible charts (Arnold 78)

An **atlas** is a union of compatible charts. This union of all charts $\{(U_i, \varphi_i)\}$ must cover the entire manifold: $\bigcup U_i = M$. Intuitively, think of covering the surface of earth with a lot of paper sheets ("charts").

Two atlases are **equivalent** if their union is also an atlas. A differentiable manifold is a **class of equivalent atlases**.

For example, Euclidean space $\mathbb{R}^n$ is a manifold, and its atlas consists of one chart.

The sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ is a manifold, and its atlas consists of two charts supplied by stereographic projection.

The circle S^1 is a manifold, and its atlas can be supplied by two charts in the angular coordinates $\varphi : \mathbb{R}^1 \to S^1$. For example, $U_1 = (-\pi, \pi)$, $U_2 = (0, 2\pi)$. Note that a single chart can't cover the entire circle, because φ must be one-to-one. Thus, we need two charts to fully cover the manifold.

The configuration space of a planar double-pendulum is the direct product of two circles. This turns out to be the torus: $T^2 = S^1 \times S^1$.

The dimension of the configuration space is called the **number of degrees of freedom**. For example, a system of k rods in the plane has a configuration space of dimension k, so k degrees of freedom.

We say that a manifold M is an embedded k-dimensional sub-manifold of Euclidean space $\mathbb{R}^n$ if in every neighborhood U of every point $\mathbf{x} \in M$, there are $n-k$ functions $f_i = U - \mathbb{R} \mid i \in [0 \dots n-k]$ such that the intersection of U with M is given by $f_i = 0 \mid i \in [0 \dots n-k]$, and the vectors $\nabla f_i \mid i \in [0 \dots n-k]$ at $\mathbf{x}$ are linearly independent. That is, when U is on the manifold, the $n-k$ "extraneous components" are zero, but the gradient vectors must still form a basis.

Every manifold can be embedded in some Euclidean space.

4.2.2 Tangent Space & Tangent Bundle

If M is a k-dimensional manifold embedded in E^n, then at every point $\mathbf{x}$, there's a k-dimensional tangent space $TM_{\mathbf{x}}$. Here, $TM_{\mathbf{x}}$ is the orthogonal complement to $\{\nabla f_i \mid i = 0 \dots n-k\}$. Intuitively, vectors on the tangent space are tangent to the "curve" of the manifold at $\mathbf{x}$. These vectors can also be defined as "velocity vectors":

$$\dot{\mathbf{x}} = \lim_{t \to 0} \frac{\varphi(t) - \varphi(0)}{t} \ , \ \ \varphi(0) = \mathbf{x}, \varphi(t) \in M$$

We can also define the tangent vetors without saying anything about embedded manifolds. Let U be a chart of an atlas for a manifold M with coordinates $q_1, \dots, q_n$. The components of the tangent vector to the curve $\mathbf{q} = \varphi(t)$ are the numbers $\xi_1, \dots, \xi_n$, where

$$\xi_i = \left. \frac{d\varphi_i}{dt} \right|_{t=0}$$

The union of the tangent spaces to M at the various points $\mathbf{x}$ has a differentiable manifold structure. The dimension is twice that of M. We call this the manifold the **tangent bundle** of M, denoted

$$TM = \bigcup_{\mathbf{x} \in M} TM_{\mathbf{x}}$$

Let $q_1, \dots, q_n$ be local coordinates on M, and $\xi_1, \dots, \xi_n$ be the components of the tangent vector: $\xi_i = dq_i$. The local coordinate system on TM can be defined with the $2n$ components $(q_1, \dots, q_n, \xi_1, \dots, \xi_n)$.

We call the mapping $p : TM \to M$, taking a tangent vector ξ to the point $\mathbf{x} \in M$ where the vector is tangent to M, the **natural projection**.

The inverse image of a point $\mathbf{x} \in M$, $p^{-1}(\mathbf{x})$, is the tangent space $TM_{\mathbf{x}}$. This space is called the **fiber of the tangent bundle over the point** $\mathbf{x}$.

4.2.3 Riemannian Manifold

If M is a manifold embedded in Euclidean space, the Euclidean metric lets us measure lengths, angles, etc. Here we extend that notion using the tangent space $TM_{\mathbf{x}}$.

A differentiable manifold with a fixed positive-definite quadratic form $\langle \xi, \xi \rangle$ on every tangent space $TM_{\mathbf{x}}$ is called a **Riemannian manifold**. The quadratic form is called the **Riemannian metric**. Let U be a chart of an atlas for M with coordinates $q_1, \dots, q_n$. Then a Riemannian metric is given by

$$ds^2 = \sum_{i,j=1}^{n} a_{ij}(q) dq_1 dq_j \ , \ \ a_{ij} = a_{ji}$$

4.2.4 The Derivative Map

Let $f : M \to N$ be a mapping from manifold M to manifold N. We say f is differentiable if in local coordinates on M and N it's given by differentiable functions.

The derivative of a differentiable mapping $f : M \to N$ at a point $\mathbf{x} \in M$ is the linear map of the tangent spaces

$$f_{*\mathbf{x}} : TM_{\mathbf{x}} \to TN_{\mathbf{f}(\mathbf{x})}$$

For example, let $\mathbf{v} \in TM_{\mathbf{x}}$, a velocity vector. Consider the curve $\varphi : \mathbb{R} \to M$ with $\varphi(0) = \mathbf{x}$ and $(d\varphi/dt)_{t=0} = \mathbf{v}$. Then $f_{*\mathbf{x}}\mathbf{v}$ is the velocity vector of the curve $f \circ \varphi : \mathbb{R} \to N$.

$$f_{*\mathbf{x}}\mathbf{v} = \left.\frac{d}{dt}\right|_{t=0} f(\varphi(t))$$

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be coordinates in a neighborhood of $\mathbf{x} \in M$, and $\mathbf{y} = (y_1, \ldots, y_n)$ be coordinates in a neighborhood of $\mathbf{y} \in N$. Let ξ be the set of components of the vector $\mathbf{v}$, and η be the set of components of the vector $\mathbf{f}_{*\mathbf{x}}\mathbf{v}$. Then

$$\eta = \frac{\partial \mathbf{y}}{\partial \mathbf{x}}\xi \ , \quad \eta_i = \sum_i \frac{\partial y_i}{\partial x_i}\xi_i$$

Each component simply transforms according to $\partial y/\partial x$.

4.3 Lagrangian Dynamical Systems

Let M be a differentiable manifold, TM its tangent bundle, and $L : TM \to \mathbb{R}$ a differentiable function. The map $\gamma : \mathbb{R} \to M$ is called a motion in the Lagrangian system with configuration manifold M and Lagrangian function L if γ is an extremal of the functional

$$\Phi(\gamma) = \int_{t_0}^{t_1} L(\dot{\gamma})\, dt$$

where $\dot{\gamma} \in TM_{\gamma(t)}$ is the velocity vector in the tangent bundle.

Let $L(\mathbf{q}, \dot{\mathbf{q}})$ be the expression for the function $L : TM \to \mathbb{R}$ in the coordinates $\mathbf{q}, \dot{\mathbf{q}}$ on the tangent bundle. The evolution of local coordinates $\mathbf{q} = (q_1, \ldots, q_n)$ on a manifold (under motion in a Lagrangian system) satisfies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}}$$

4.3.1 Natural Systems

A Lagrangian system on a Riemannian manifold is called **natural** if the Lagrangian is equal to the difference in kinetic and potential energies.

$$L = T - U$$

$$T = \frac{1}{2}\langle \mathbf{v}, \mathbf{v} \rangle \ \mid \ \mathbf{v} \in TM_{\mathbf{x}} \ , \ \ U : M \to \mathbb{R}$$

Now consider the case of holonomic constraints (given earlier). Consider the $3n$-dimensional configuration space of n free points. The embedded Riemannian manifold M with potential energy U can be used to write the potential energy of the system as $U_{sys} = U + N\mathbf{q_2}^2$, which grows rapidly outside M. So a particle on M would tend to stay on the manifold.

To solve:

1. Determine configuration manifold; introduce coordinates $q_1, \ldots, q_k$ in a neighborhood of each point on the manifold.
2. Find the kinetic energy as a quadratic form of the generalized velocities:

$$T = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j$$

3. Construct Lagrangian $L = T - U(\mathbf{q})$ and solve Lagrange's equations.

This can be used to find, for example, geodesics on surfaces of rotation. Assume constant potential energy U on the manifold, and $U_{sys} = U + N\mathbf{q_2}^2$ outside.

4.3.2 Non-Autonomous Systems

Until now, we've been looking at autonomous Lagrangian systems. A non-autonomous Lagrangian system depends on time. Both the kinetic and potential energies might also depend on time.

$$L : TM \times \mathbb{R} \to \mathbb{R} \;\Big|\; L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$$
$$T : TM \times \mathbb{R} \to \mathbb{R} \;\;,\;\; U : M \times \mathbb{R} \to \mathbb{R} \;\Big|\; T = T(\mathbf{q}, \dot{\mathbf{q}}, t) \;\;,\;\; U = U(\mathbf{q}, t)$$

A system of n mass points (constrained by holonomic constraints that depend on t) are defined using a time-dependent submanifold of the free system's configuration space. This is given by the mapping

$$i : M \times \mathbb{R} \to E^{3n} \;,\;\; i(\mathbf{q}, t) = \mathbf{x}$$

For a fixed $t \in \mathbb{R}$, this defines an embedding $M \to E^{3n}$. Then the Lagrangian $L = T - U(\mathbf{q})$ for autonomous systems can be used here.

Essentially, we've shown that the Lagrangian can also be applied to time-dependent systems.

4.4 Noether's Theorem

Let M be a smooth manifold and $L : TM \to \mathbb{R}$ be a smooth function on its tangent bundle. Let $h : M \to M$ be a smooth map. We say that a Lagrangian system (M, L) **admits the mapping** h if for any tangent vector $\mathbf{v} \in TM$,

$$L(h_* \mathbf{v}) = L(\mathbf{v})$$

For example, if $M = \{(x_1, x_2)\}$ and $L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - U(x_2)$, the system admits the translation $h : (x_1, x_2) \to (x_1 + s, x_2)$ along the x_1-axis, but generally doesn't admit a translation in the x_2-axis.

Noether's Theorem: If the system (M, L) admits the one-parameter group of diffeomorphisms $h^s : M \to M$ with $s \in \mathbb{R}$, then the Lagrangian system of equations corresponding to L has a first integral $I : TM \to \mathbb{R}$. That is, $\frac{dI}{dt} = 0$ along the curve.

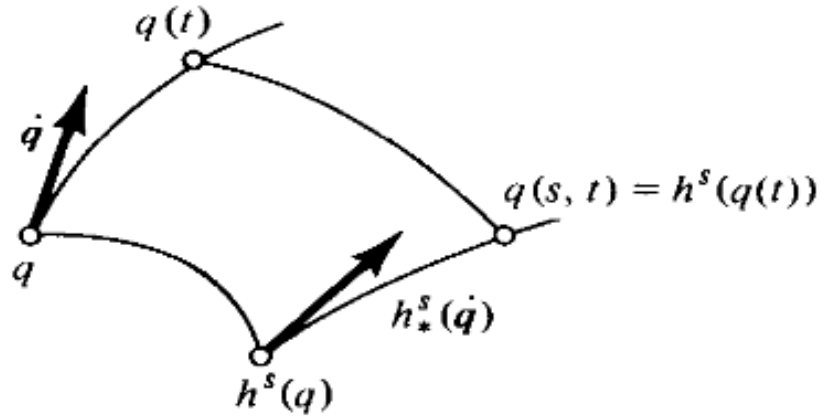
In local coordinates q on M, the integral is

$$I(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{dh^s(\mathbf{q})}{ds}\bigg|_{s=0}$$

Proof: Let $M = \mathbb{R}^n$ be the coordinate space and $\varphi : \mathbb{R} \to M$, $\mathbf{q} = \varphi(t)$ be a solution to Lagrange's equations. Since h^s preserves L (given), $h^s \circ \varphi : \mathbb{R} \to M$ also satisfies Lagrange's equations for any s.

Consider the mapping $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$, given by $\mathbf{q} = \Phi(s, t) = h^s(\varphi(t))$. Denote t-derivatives with a dot and s-derivatives with a primed mark. Note that since h^s preserves the Lagrangian,

$$0 = \frac{\partial L(\Phi, \dot{\Phi})}{\partial s} = \frac{\partial L}{\partial \mathbf{q}} \Phi' + \frac{\partial L}{\partial \dot{\mathbf{q}}} \dot{\Phi}'$$



Noether's theorem (Arnold 89)

Since $\Phi|_{s\,\text{const}}$ preserves the Lagrangian for any fixed s, it must satisfy Lagrange's equation

$$\frac{d}{dt}\left[\frac{\partial L\left(\Phi(s,t), \dot{\Phi}(s,t)\right)}{\partial \dot{\mathbf{q}}}\right] = \frac{\partial L\left(\Phi(s,t), \dot{\Phi}(s,t)\right)}{\partial \mathbf{q}}$$

So we can let $\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}}$. Return to the Lagrangian a few steps back and substitute. Note that s is held constant, so $\mathbf{q} = \Phi(s, t)$ and $\dot{\mathbf{q}} = \dot{\Phi}(s, t)$. The combination is $\dot{\mathbf{q}}' = d\mathbf{q}'/dt$.

$$0 = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) + \frac{\partial L}{\partial \dot{\mathbf{q}}}\left(\frac{d}{dt}\mathbf{q}'\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\mathbf{q}'\right) = \frac{dI}{dt}$$

Note: the first integral $I = (\partial L/\partial \dot{\mathbf{q}})\mathbf{q}'$ is defined with local coordinates $\mathbf{q}$. I is the rate of change of $L(\mathbf{v})$ when the vector $\mathbf{v} \in TM_{\mathbf{x}}$ varies with velocity $(d/ds)|_{s=0}\, h^s\mathbf{x}$.

In plain English, Noether's theorem is: for every continuous symmetry in a physical system, there's an associated conserved quantity.

Noether's theorem is the general theorem from which the laws of conservation (momentum, angular momentum, etc.) can be derived. For example, to derive the conservation of momentum, consider the system of point masses:

$$L = \sum \frac{1}{2} m_i \dot{\mathbf{x}}_i^2 - U(\mathbf{x}) \ , \ \ \mathbf{x}_i = x_{i1}\mathbf{e_1} + x_{i2}\mathbf{e_2} + x_{i3}\mathbf{e_3}$$

Assume that the system admits translations along the $\mathbf{e_1}$ axis, $h^s : \mathbf{x}_i \to \mathbf{x}_i + s\mathbf{e_1}$. Since $(d/ds)|_{s=0}\, h^s \mathbf{x}_i = \mathbf{e_1}$, the first integral is

$$I = \sum \frac{\partial L}{\partial \dot{\mathbf{x}}} \frac{dh^s \mathbf{x}_i}{ds} = \sum \frac{\partial L}{\partial \dot{\mathbf{x}}} \mathbf{e_1} = \sum m_i \dot{x}_{i1}$$

This is conserved since $dI/dt = 0$ by Noether's theorem. Above is the first component of the momentum vector. So there's the law of conservation of momentum.

What about angular momentum? Let h^s be a rotation around the $\mathbf{e_1}$ axis by an angle s. Then $(d/ds)|_{s=0}\, h^s \mathbf{x}_i = \mathbf{e_1} \times \mathbf{x}_i$, thus the first integral is

$$I = \sum_i \frac{\partial L}{\partial \dot{\mathbf{x}}_i} (\mathbf{e_1} \times \mathbf{x}_i) = \sum_i (m_i \dot{\mathbf{x}}_i) \cdot (\mathbf{e_1} \times \mathbf{x}_i) = \sum_i (\mathbf{x}_i \times m_i \dot{\mathbf{x}}_i) \cdot \mathbf{e_1}$$

Since $dI/dt = 0$ by Noether's theorem, the quantity $(\mathbf{x}_i \times m_i \dot{\mathbf{x}}_i)$ is conserved. That's the law of conservation of angular momentum.

The law of conservation of energy is related to a time symmetry: If L doesn't depend on time, then E is conserved.

4.5 D'Alembert's Principle

This is a new definition of a system of point masses with holonomic constraints. But it's equivalent to the definition given previously.

Consider the holonomic system (M, L), where M is a surface in 3-dimensional space $\{\mathbf{x}\}$ and

$$L = \frac{1}{2} m \dot{\mathbf{x}}^2 - U(\mathbf{x})$$

The mass point $\mathbf{x}$ of the mass must remain on M. But consider a motion $\mathbf{x}(t)$ of the point and Newton's equation $m\ddot{\mathbf{x}} + (\partial U/\partial \mathbf{x}) = 0$. If there are no external forces, $U = 0$ and the motion is a straight line, thus it can't always lie on M.

So we introduce a new force that "forces" the point to stay on M. This is called the **constraint force** and is given by

$$\mathbf{R} = m\ddot{\mathbf{x}} + \frac{\partial U}{\partial \mathbf{x}}$$

Then Newton's equations are satisfied again:

$$m\ddot{\mathbf{x}} = -\frac{\partial U}{\partial \mathbf{x}} + \mathbf{R}$$

Tangent vectors to the configuration manifold are called **virtual variations**. The D'Alembert-Lagrange principle states that for any virtual variation ξ, the work of the constraint force on ξ is zero.

$$\mathbf{R} \cdot \xi = \left(m\ddot{\mathbf{x}} + \frac{\partial U}{\partial \mathbf{x}} \right) \cdot \xi = 0$$

For a system of points $\mathbf{x}_i$ with masses m_i, we have $\mathbf{R}_i = m_i \ddot{\mathbf{x}}_i + (\partial U / \partial \mathbf{x}_i)$. According to D'Alembert's principle, $\sum (\mathbf{R}_i \cdot \xi_i) = 0$. The sum of the works of the constraint forces on **any virtual variation** $\{\xi_i\} \in TM_{\mathbf{x}}$ is zero.

Constraints with the property above (like $\mathbf{R}$) are called **ideal**.

We can also derive conservation laws from D'Alembert's principle. For example, assume translation along the $\mathbf{e_1}$-axis is among the virtual variations. The sum of the work of the constraint forces on this variation is zero:

$$\sum (\mathbf{R}_i \cdot \mathbf{e_1}) = \left(\sum \mathbf{R}_i \right) \cdot \mathbf{e_1} = 0$$

This implies that the sum of the 1st component of the constraint forces are zero: $\sum \mathbf{R}_{i1} = 0$. Considering the constraint forces as external forces, this must mean that the first component of momentum, $\mathbf{P}_1 = \sum d\mathbf{R}_{i1}/dt$, is conserved.

4.5.1 Equivalence with the Variational Principle

D'Alembert's principle is equivalent with the variational principle introduced earlier. Let $M \in \mathbb{R}^N$ be a submanifold of Euclidean space. Let $\mathbf{x} : \mathbb{R} \to M$ be a curve with $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$. The curve $\mathbf{x}$ is called a conditional extremal of the action functional

$$\Phi = \int_{t_0}^{t_1} \left(\frac{\dot{\mathbf{x}}^2}{2} - U(\mathbf{x}) \right) dt$$

if the differential $\delta\Phi = 0$ for **variations consisting of nearby curves** joining $\mathbf{x}_0$ to $\mathbf{x}_1$ in M.

Theorem: A curve $\mathbf{x} : \mathbb{R} \to M \subset \mathbb{R}^N$ is a conditional extremal of the action iff. it satisfies D'Alembert's equation

$$\left(\ddot{\mathbf{x}} + \frac{\partial U}{\partial \mathbf{x}} \right) \cdot \xi = 0 \,, \quad \forall \xi \in TM_{\mathbf{x}}$$

So the variational principle (principle of least action) is equivalent to D'Alembert's principle. The principle of least action is the "integral form," while D'Alembert's principle only considers constraints at a point.

4.5.2 Equilibrium Positions

An alternative account of D'Alembert's principle can be stated in terms of equilibrium positions.

Suppose that a point mass moves along a smooth surface M under the force $\mathbf{f} = -\partial U/\partial \mathbf{x}$. We call a point $\mathbf{x}_0 \in M$ an **equilibrium position** iff. the force is orthogonal to the surface at $\mathbf{x}_0$. That is, $\mathbf{f}(\mathbf{x}_0) \cdot \xi = 0$ for all $\xi \in TM_{\mathbf{x}_0}$. This is the case in D'Alembert's equation where $\ddot{\mathbf{x}} = 0$.

Denote $-m\ddot{\mathbf{x}}$ as the **force of inertia**.

Theorem (alternative account of D'Alembert's principle): If the forces of inertia $-m\ddot{\mathbf{x}}$ are added to the acting forces, $\mathbf{x}$ becomes an equilibrium position.

This can be seen from D'Alembert's equation below, since

$$\left(\ddot{\mathbf{x}} + \frac{\partial U}{\partial \mathbf{x}}\right) \cdot \xi = 0 \,, \quad \forall \xi \in TM_{\mathbf{x}}$$
$$(-m\ddot{\mathbf{x}} + \mathbf{f}) \cdot \xi = 0$$

So we can reduce a problem about motion into a problem about equilibrium.