

# Mathematical Methods of Classical Mechanics - V. I. Arnold

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## Part I

# Newtonian Mechanics

## 1 Experimental Facts (p. 3 - 11)

Certain experimental facts form the basis of classical mechanics. We can't verify them for certain, but they hold approximately, according to accurate tests.

1. **Space and time:** Space is 3D and Euclidean. Time is 1D.
2. **Galileo's Principle of Relativity:** There exist inertial coordinates such that
  - Laws of physics are same in all inertial coordinate systems.
  - Coordinate systems in uniform rectilinear motion w.r.t an inertial one is also inertial.
3. **Newton's Principle of Determinacy:** Initial state of a mechanical system uniquely determines its motion.

### 1.1 The Galilean Group

Let  $\mathbb{R}^n$  denote an  $n$ -dimensional real vector space.

An **affine  $n$ -dimensional space**, denoted  $A^n$ , is similar to  $\mathbb{R}^n$  but has no “fixed origin.” The group  $\mathbb{R}^n$  acts on  $A^n$  as the **group of parallel displacements**.

$$a \rightarrow a + \mathbf{b} , \quad a \in A^n, \quad \mathbf{b} \in \mathbb{R}^n, \quad a + \mathbf{b} \in A^n$$

The sum of two points on  $A^n$  is not defined, but their difference is a vector in  $\mathbb{R}^n$ . The distance between points of an affine space  $A^n$  can be defined using the scalar product:

$$\|x - y\| = \sqrt{(x - y, x - y)}$$

An affine space with this distance function is called a **Euclidean space**, denoted  $E^n$ .

The Galilean spacetime structure has 3 elements:

1. **The universe:** A 4D affine space  $A^4$ . Points of  $A^4$  are called *events*. Parallel displacements of  $A^4$  form the vector space  $\mathbb{R}^4$ .
2. **Time:** A linear mapping  $t : \mathbb{R}^4 \rightarrow \mathbb{R}$  from a parallel displacement to the “time axis.” The time interval between  $a, b \in A^4$  is  $t(b - a)$ .
3. **Distance Between Simultaneous Events:** Given by  $\|a - b\| = \sqrt{(a - b, a - b)}$ . This is a scalar product on  $\mathbb{R}^3$ . The space of simultaneous events is thus a 3D Euclidean space  $E^3$ .

A Galilean space is a space  $A^4$  that has a Galilean spacetime structure.

The Galilean group is the group of all transformations of a Galilean space which preserves its structure.

Elements of a Galilean group are called **Galilean transformations**: affine transformations on  $A^4$  which preserve the time interval & distance between simultaneous events.

The **Galilean coordinate space** is the direct product  $\mathbb{R} \times \mathbb{R}^3$  of the  $t$  axis with the 3D vector space  $\mathbb{R}^3$ . There are 3 examples of Galilean transformations of this space:

1. Uniform motion with velocity  $\vec{v}$ :  $g_1(t, \vec{x}) = (t, \vec{x} + \vec{v}t)$
2. Translation of origin:  $g_2(t, \vec{x}) = (t + s, \vec{x} + \vec{s})$
3. Rotation of coordinate axes:  $g_3(t, \vec{x}) = (t, G\vec{x})$ ,  $G : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is orthogonal transformation.

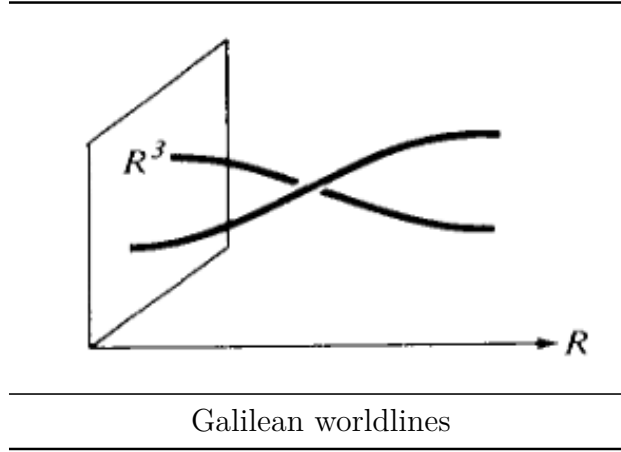
Every Galilean transformation of the space  $\mathbb{R} \times \mathbb{R}^3$  can be written as the composition of the above, thus the dimension of the Galilean group is  $3 + 4 + 3 = 10$ .

Let  $M$  be a set. A one-to-one correspondence  $\phi_1 : M \rightarrow \mathbb{R} \times \mathbb{R}^3$  is called a **Galilean coordinate system** on the set  $M$ .

## 1.2 Motion, Velocity, and Acceleration

A motion in  $\mathbb{R}^n$  is a differentiable mapping  $\mathbf{x} : I \rightarrow \mathbb{R}^n$ , where  $I$  is an interval on the real axis. The image of a mapping  $\mathbf{x} : I \rightarrow \mathbb{R}^n$  is called a **trajectory** or **curve** in  $\mathbb{R}^n$ . The velocity vector is the derivative  $\dot{\mathbf{x}}(t_0)$ , while the acceleration vector is the second derivative  $\ddot{\mathbf{x}}(t_0)$ .

Let  $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^3$  be a motion in  $\mathbb{R}^3$ . The graph of this mapping ( $\mathbb{R}^3$  against  $\mathbb{R}$ ) is a curve in  $\mathbb{R} \times \mathbb{R}^3$ . A curve in Galilean space that appears in some Galilean coordinate system (as the graph of a motion) is called a **world line**.



Consider a system with  $n$  points. In Galilean space, this gives  $n$  world lines, described by  $n$  mappings  $\mathbf{x}_i : \mathbb{R} \rightarrow \mathbb{R}^3$  in a Galilean coordinate system. In total, we have

$$\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^N, \quad N = 3n$$

as the total motion of our system with  $n$  points.

Newton's principle of determinacy states that all motions of a system are uniquely determined by their initial positions and velocities. Acceleration is not needed: it's determined by the initial position and velocities. I.e. there's a function  $\mathbf{F} : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N$  such that

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$$

This is called **Newton's equation**. By the theorem of existence and uniqueness (recall from diff eq), the initial conditions  $\mathbf{F}$ ,  $\mathbf{x}(t_0)$ , and  $\dot{\mathbf{x}}(t_0)$  uniquely determine a motion. The form of  $\mathbf{F}$  can be determined experimentally.

Galileo's principle of relativity requires that Galilean spacetime structure must be invariant w.r.t the group of Galilean transformations. This is a condition on Newton's equation, and leads to three properties of spacetime:

1. **Invariance under time translations:** Laws of nature remain constant regardless of time.

2. **Invariance under spatial translations:** Space is homogeneous; has the same properties at all of its points.
3. **Invariance under spatial rotations:** Space is isotropic; no preferred directions.

We may also introduce the “potential energy”  $U$  to write Newton’s equation. Let  $E^{3n} = E^3 \times \cdots \times E^3$  be the configuration space of a system of  $n$  points in the Euclidean space  $E^3$ . Let  $U : E^{3n} \rightarrow \mathbb{R}$  be a differentiable function. The motion of the  $n$  points (of masses  $m_1, \dots, m_n$ ) is given by the system of differentiable equations

$$m_i \ddot{\mathbf{x}}_i = -\frac{\partial U}{\partial \mathbf{x}_i}, \quad i = 1, \dots, n$$

where  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$  and  $\partial U / \partial \mathbf{x} = (\partial U / \partial \mathbf{x}_1, \dots, \partial U / \partial \mathbf{x}_n)$ . The equations of motion for many other mechanical systems can be written in this form.