Acoustophoresis and Standing Waves

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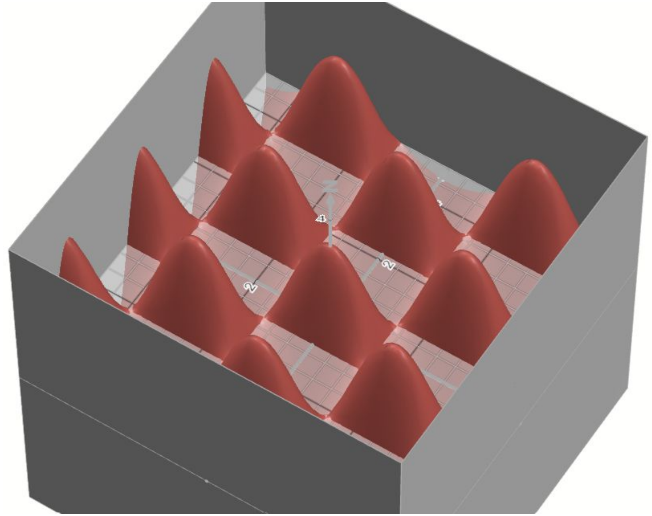
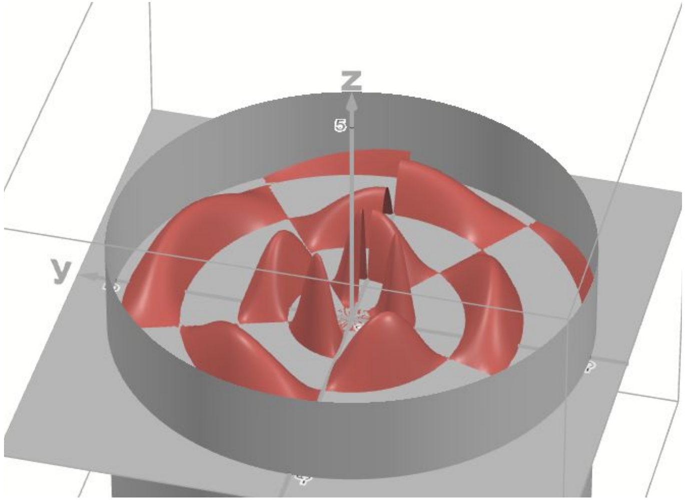


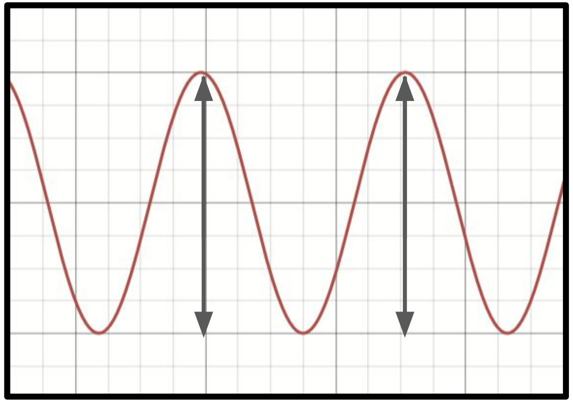


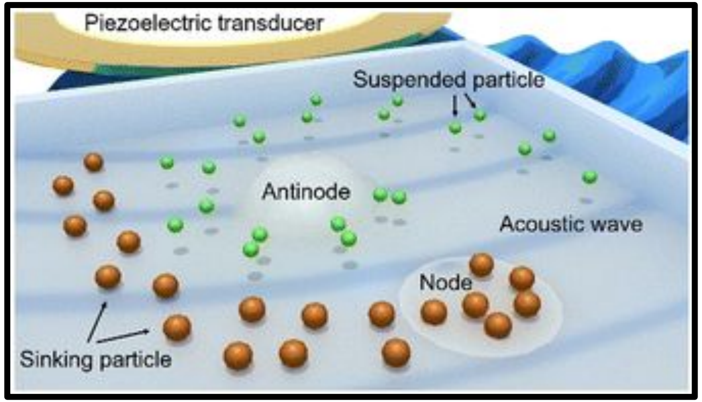
Background

- Wave equation models propagation of a wave (c = propagation speed)

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f = c^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right)$$

- Standing waves are used in acoustophoresis to separate particles





What standing wave boundaries are most helpful for acoustophoresis?

Square Case

- The wave equation is the simplest form

$$\frac{\partial^2 f}{\partial t^2} = c^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right)$$

- Assume a solution of the form $\boxed{f(x, y, t) = X(x)Y(y)T(t)}$
- Let $\boxed{\omega^2 = -\frac{1}{T}\frac{d^2T}{dt^2}}$ and $\boxed{m^2 = -\frac{1}{Y}\frac{d^2Y}{dy^2}}$
- This reduces the wave equation to a set of 3 independent 2nd-order differential equations.
- Multiplying them together, $\boxed{f(x, y, t) = \cos\left(\frac{\sqrt{\omega^2 - c^2m^2}}{c}x\right)\cos(my)\cos(\omega t)}$

Circular Case

- The wave equation in polar coordinates is

$$\boxed{\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f = c^2 \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \right)}$$

- Assuming $\boxed{f(r,\theta,t) = R(r)\Theta(\theta)T(t)}$ and making substitutions,

$$\boxed{0 = \frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\frac{\omega^2}{c^2} - \frac{m^2}{r^2} \right) R}$$

$$\boxed{\omega^2 = -\frac{1}{T}\frac{d^2 T}{dt^2}} \quad \boxed{m^2 = -\frac{1}{\Theta}\frac{d^2 \Theta}{d\theta^2}}$$

$$\downarrow$$

$$\boxed{0 = u^2 \frac{d^2 R}{du^2} + u\frac{dR}{du} + \left(u^2 - m^2 \right) R}$$

$$\boxed{u = \omega r / c}$$

Circular Case (cont'd)

$$0 = u^2 \frac{d^2R}{du^2} + u\frac{dR}{du} + \left(u^2 - m^2\right) R$$

$$u = \omega r / c$$

- This is the Bessel equation, and its solution has the form

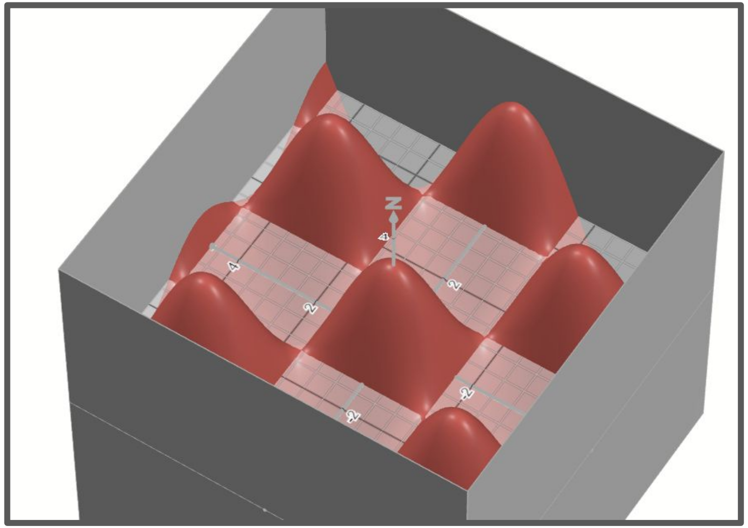
$$J_m(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{u}{2}\right)^{2k+m}$$

- Now we can get our solution:

$$f(r,\theta,t) = CR(r)\Theta(\theta)T(t) = CJ_m(u)\cos(m\theta)\cos(\omega t)$$

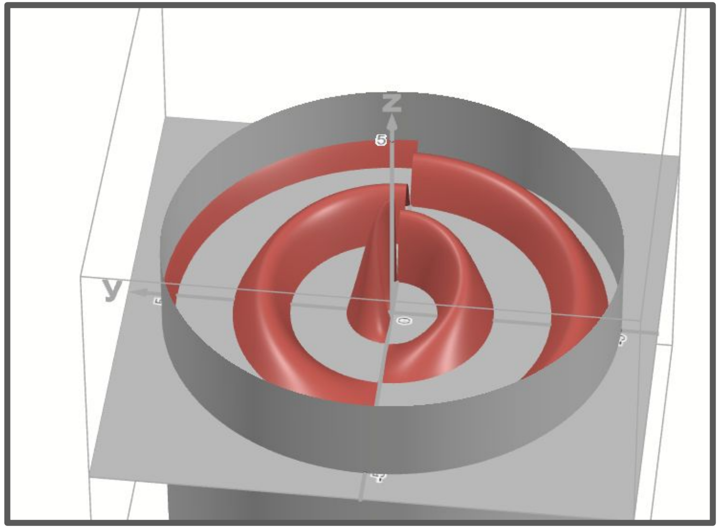
$$f(r,\theta,t) = C\cos(m\theta)\cos(\omega t)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{\omega r}{2c}\right)^{2k+m}$$

Conclusions



Square boundary

- Uniform peak heights
- Always a peak at the origin
- Better for few number of particles



Circular boundary

- Variable heights on each “ring”
- z=0 around origin if m >> ω or m >> c
- Better for spectrum of particle weights

Sources

Images

- https://pubs.rsc.org/en/content/articlelanding/2024/lc/d4lc00277f

Materials

- Circular membrane case
  https://www-eng.lbl.gov/~shuman/NEXT/MATERIALS&COMPONENTS/MISC/Standing-Waves-on-a-Circular-Membrane.pdf

- Bessel functions
  https://math.libretexts.org/Bookshelves/Differential_Equations/A_First_Course_in_Differential_Equations_for_Scientists_and_Engineers_(Herman)/04%3A_Series_Solutions/4.06%3A_Bessel_Functions

- Nice video
  https://www.youtube.com/watch?v=6yV17h26IIc