Modern Physics

Reading week Notes

*Intro to Modern Physics

- physics
  1. Quantitative description of universe as precise as necessary
  2. Quantitative description of how objects evolve in time and interact amongst themselves

Assumptions:

↳ Objects have independent existence. Not property of human measurements

↳ Causality: Effects are preceded by their cause

↳ Determinism: if system evolves one way, then it will evolve the exact same way when returned to exact initial state.

↳ Space is 3D & Euclidean

↳ Time can be thought of as 1D space, identical at all points in space.

- Kinematics: math-y description of object's motions
- Dynamics: causations of kinematics (momentum, force, etc.)
- Newton's Laws: relate dynamics to kinematics

*Galilean Transformations

- Consider 2 frames of reference S and S'

Standard config: at time $t=t'=0$, $x'=x=y'=y=z'=z=0$

Event P observed & recorded by observer in S': x', y', z', t'

↳ what values does observer in S measure for P?

↳ Time universal: $t=t'$

↳ No relative motion in y or z

$$\boxed{\begin{aligned} x&=x'+vt' \\ y&=y' \\ z&=z' \\ t&=t' \end{aligned}}$$

Galilean Transformation

$$\boxed{\begin{aligned} x'&=x-vt \\ y&=y' \\ z&=z' \\ t&=t' \end{aligned}}$$ ← "what is (x,y,z,t) in S'?"

Inverse GT

- Inertial Reference Frames (IRF): where Newton's Laws are valid
  - ↳ e.g. $\vec{F} = \frac{d\vec{p}}{dt}$
  - ↳ If S and S' are IRFs, then $\vec{F} = \frac{d\vec{p}}{dt}$ and $\vec{F}' = \frac{d\vec{p}'}{dt'}$

- Galilean Principle of relativity: laws of physics are same for all observers provided that the observers are in IRFs
  - ↳ Galilean Transformations ensures that momentum is conserved between IRFs.

- Event: "A point in space at an instant in time"
- CCC: Consistent Causality Criterion
  - ↳ what is a cause and effect must be same in all reference frames
- PIE: principle of invariant event
  - ↳ what occurs at evt. must be invariant to all observers
- PSR: principle of special relativity
  - ↳ Laws of physics are same in all IRFs
  - ↳ Proposed by Einstein in 1905

*Galilean Relativity & Maxwell's Equations

- Thought experiment:

  ↳ 2 beams of counter propagating electrons (e^-) and positrons (e^+)

  Charge density $\lambda_{e^-} + \lambda_{e^+} = \lambda - \lambda = 0$

  But current $2\lambda v$ to right (positive charges move)

$-q \longrightarrow +v\hat{i}$

Frame S

Put particle w/ charge $-q$, speed $v\hat{i}$. At P at $t=t'$

$$\vec{F} = -q(\vec{v} \times \vec{B}) = +qvB\hat{j} \quad \text{(away from beam)}$$

Frame S'

In Frame S' (moving at $\vec{v} = +v\hat{i}$ w.r.t. frame S)

↳ Same charge density & current, mag. field

$$\vec{F} = -q(\vec{0} \times \vec{B}) = 0 \quad ??? \quad \text{(obj. remains at } P\text{)}$$

Logical contradiction

*Michelson-Morley Experiment

- Maxwell's Wave Eqn: $\dfrac{\partial^2 \vec{E}}{\partial t^2} = c^2 \dfrac{\partial^2 \vec{E}}{\partial x^2}$

  ↳ ~~Maxwell's view: EM wave is undulation of~~ ether

  ↳ Ether: everywhere?

  Idea: light travels at 3×10^8 m/s w.r.t. ether

  ether is at rest in frame of univ. as a whole

*Principle of Special Relativity: Time Dilation

- Q: what do you see if you travel along light beam at same speed?
  - ↳ At this frame, $\vec{E}$ and $\vec{B}$ are static
  - ↳ Not a sol'n to Maxwell's Equations

  - ↳ Einstein: extend principle of relativity

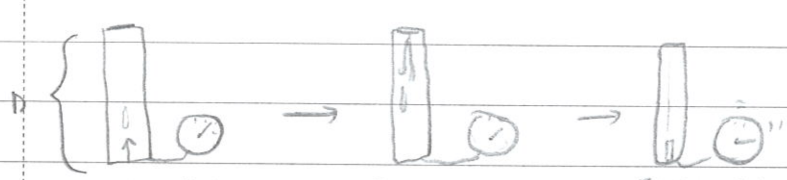
- Principle of special Relativity 
  - ↳ <u>All</u> laws of physics are the same for all observers provided that observers are in inertial reference frames (IRF)

  - ↳ + Postulate

    Since Maxwell's theory stipulates that EM waves travel at $c = 3\times10^8$ m/s...

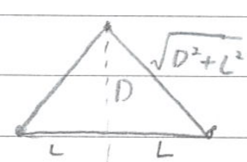
    $\underbrace{\text{All inertial observers see light to travel at } c = 3\times10^8 \text{ m/s}}_{\text{Constancy of speed of light}}$

- Consequences
  - ↳ Alice & Bob have light clocks (emit light → come back → tick)
  - ↳ Synchronized clocks



Alice is at rest (in her IRF)

Bob moves ←

↳ Path of light: 

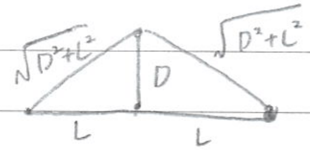
(Alice sees the light following longer path)

↳ Constancy of speed of light → time between ticks is longer

- Math

↳ $\Delta t'_B$ = Bob's time as measured from Bob's frame

Δt_B = Bob's time as measured from Alice's frame



$$\left.\begin{aligned}\Delta t'_B &= \frac{2D}{c}\\ \Delta t_B &= \frac{2\sqrt{D^2+L^2}}{c}\end{aligned}\right\} \Delta t_B > \Delta t'_B$$

$$L = v\frac{\Delta t_B}{2} \quad (v = \text{Bob's speed})$$

Using this value for L, $\Delta t_B\sqrt{1-\frac{v^2}{c^2}} = \frac{2D}{c} = \Delta t'_B$

$$\boxed{\Delta t_B = \gamma \Delta t'_B, \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \geq 1}$$

$$\boxed{t_{motion} = \frac{1}{\gamma} t_{rest}}$$

Time Dilation

(Rate at which time passes is not absolute)

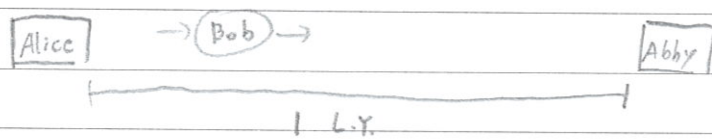
Principle of reciprocal velocity

※ Reverse the argument: Alice moves → at $\vec{v}$

↳ Now Bob sees Alice's clock to tick slower than his...

- Another consequence: length contraction

↳ Bob travels at $v = \frac{c}{2}$ between Alice and Abby



Frame A: Bob takes 2 years

For $v = \frac{c}{2}$, $\gamma = \frac{1}{\sqrt{1-\frac{1}{4}}} \approx 1.15$, $\Delta t = 2y = \gamma \Delta t' = 1.15\,\Delta t'$

$\Delta t' = 1.7\,y$ (1.7 years from Bob's view)

Use principle of reciprocal velocity + principle of invariant event (physical occurrences at single event are invariant in all frames)

- Frame B
  - ↳ Trip takes 1.7 years
  - ↳ Principle of reciprocal velocity; Bob sees Alice & Abby to travel at $\frac{c}{2}$ in ←
  - ↳ $L' = v\Delta t' = 0.87$ L.Y. ← from Bob's perspective

$$\boxed{L' = \sqrt{1-\frac{v^2}{c^2}}\, L = \frac{L}{\gamma}} \qquad \boxed{L_{motion} = \frac{L_{rest}}{\gamma}}$$

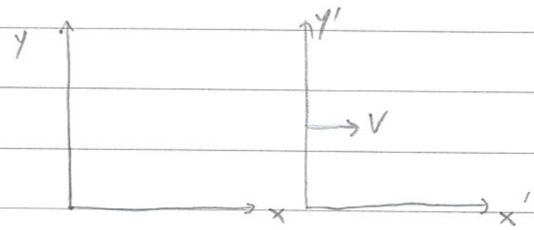
Length Contraction

⚠ Contraction only occurs along direction of motion

Symmetrical: B sees A's length contracted & vice versa

- Another consequence: lack of simultaneity
  - ↳ If some events look simultaneous in one frame, they won't be in another moving frame.

Lorentz transformations



$$\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}}$$

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

Explains why no simultaneity

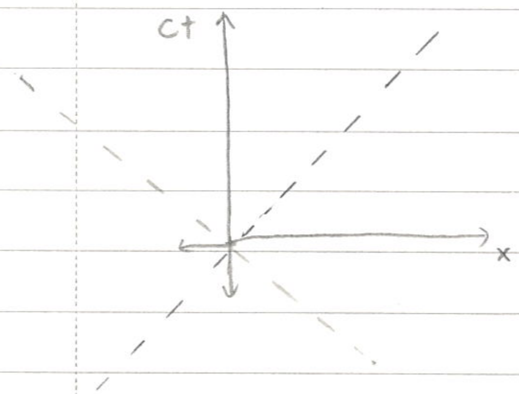
Modern Physics

Week 18 Notes

* Spacetime Diagrams

- Movement through space determines movement through time
- Combine 3D space & 1D time → 4D manifold "spacetime"

  ↳ Points in this space are events.



y-axis: ct [m]
x-axis: x [m]
} same units

Slope: $\frac{ct}{x} = \frac{c}{v}$

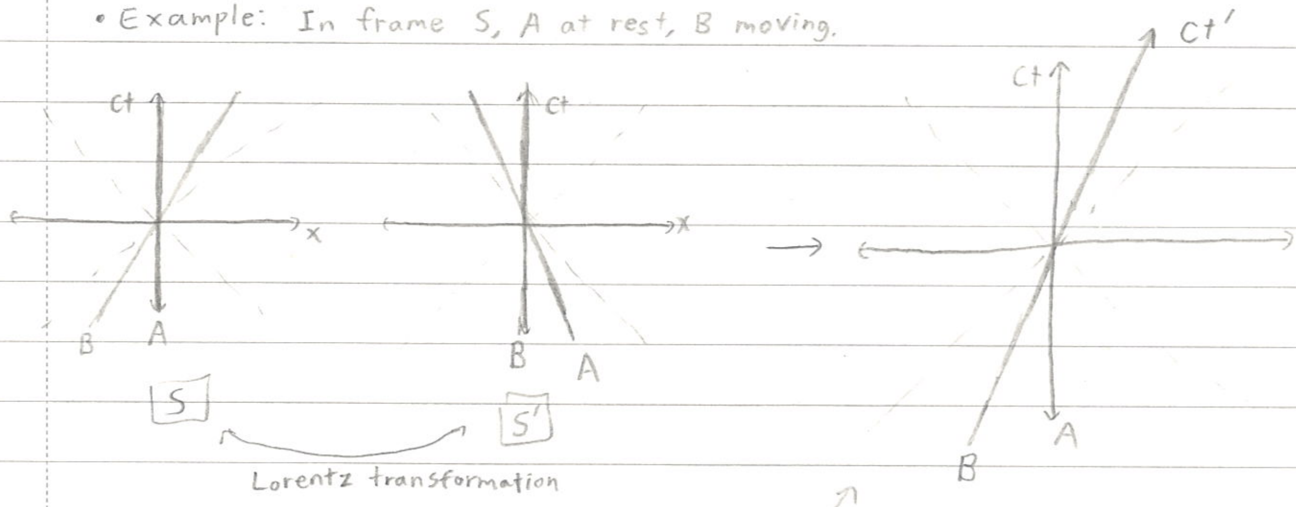
- Light is a 45° line: $\Delta x = c\Delta t$, $v = \frac{\Delta x}{\Delta t} = c$

  ↳ From PSR, 45° in all frames

- Diagram represents metric space (worldlines can be measured in meters)

- Obj. at rest: vertical line, no Δx

- Example: In frame S, A at rest, B moving.



Todo: adding an x' axis (not just rotated)

Modern Physics

Week 18 Notes

* No Causal relations faster than c

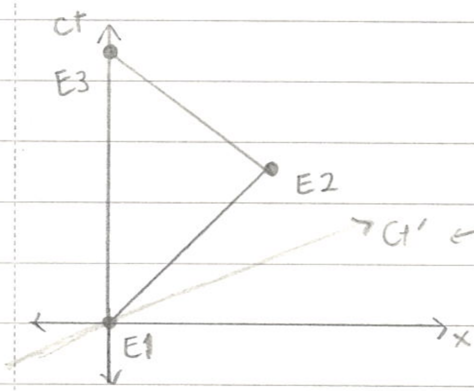
start off with
- Constancy of speed of light: all frames measure c in vacuum to be $c = 3\times10^8$ m/s
- Causality: in any frame, causes precede effects in time.
- Reciprocal Velocity Postulate: If A measures B to travel with $\vec{\beta}$, B measures A to move with $-\vec{\beta}$.

- Scenario: laser pulse reflects off mirror, pops balloon.

  Init. Cause (E1): laser emitts laser pulse

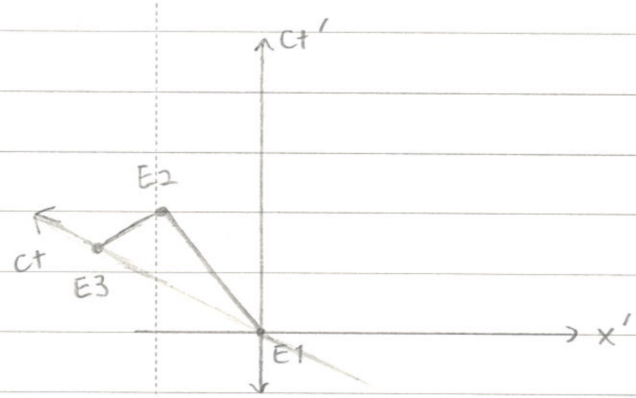
  Cause (E2): laser pulse reflects off mirror

  Effect (E3): laser pulse pops balloon.



← Spacetime diagram

← Fictitious frame faster than c



← In this fictitious frame, E2 happens after E3.

↳ An observer could've removed mirror, changing E3: $E2 \Rightarrow E3$

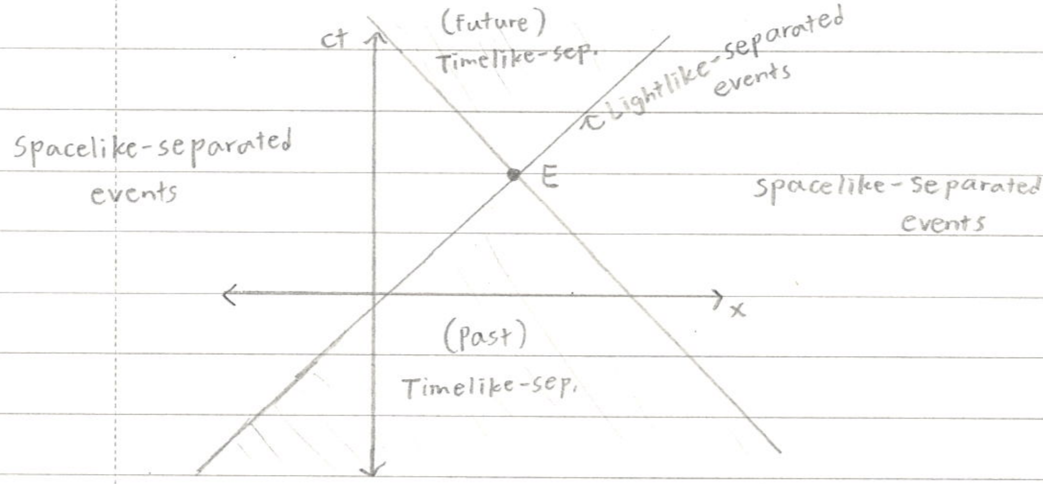
But in S', E3 already happened.

\+ No retrocausality.

∴ No causal relations can be transmitted faster than c.

* Back to spacetime diagrams:

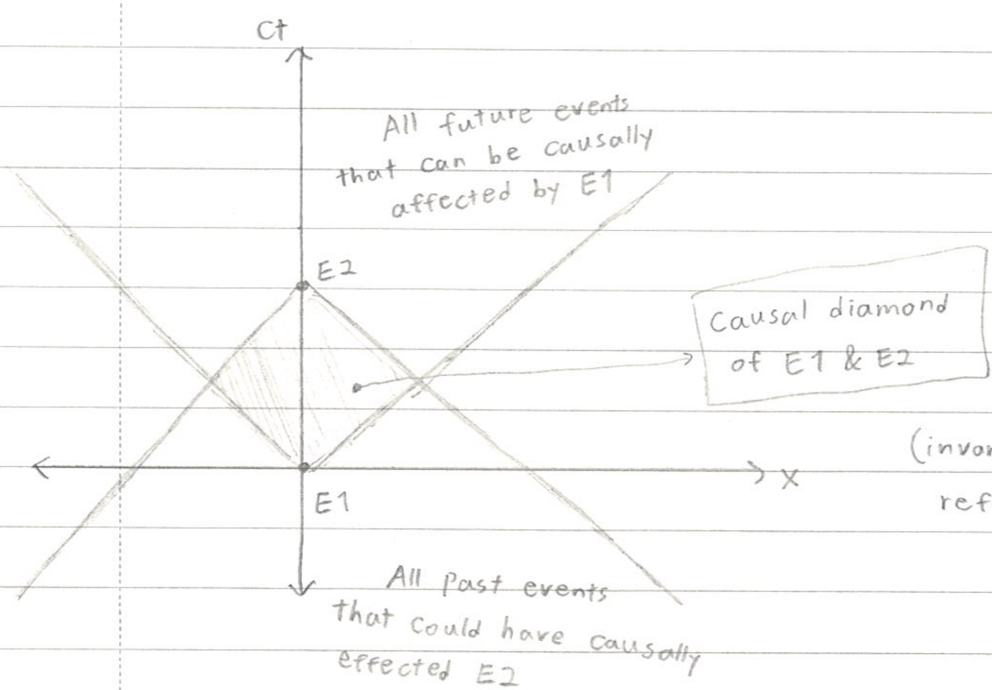
- "No Causal influence faster than c"



↳ If E and another evt. can be connected by a line..

w.r.t. Horizontal
- $> 45°$: timelike separated, can be causally related.
- $= 45°$: lightlike separated, can be causally related.
- $< 45°$: spacelike separated, cannot be causally related.

- Consistent Causality Criterion [CCC]: causal relation between 2 events must remain invariant in all reference frames.

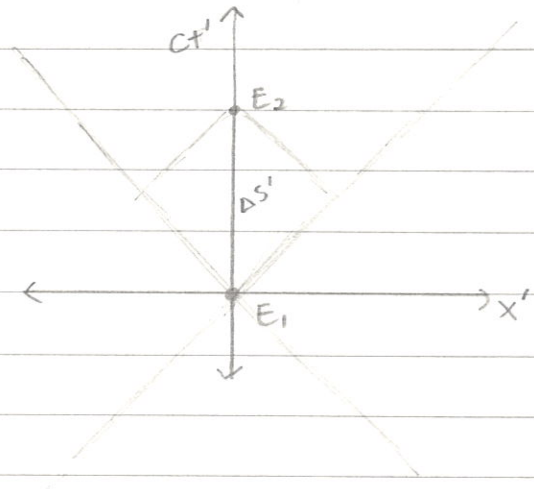


$$Vol(CD) = \sum_{e_i \in CD} Vol(e_i)$$

$Vol(CD)$ const.

(invariant as you change reference frames)

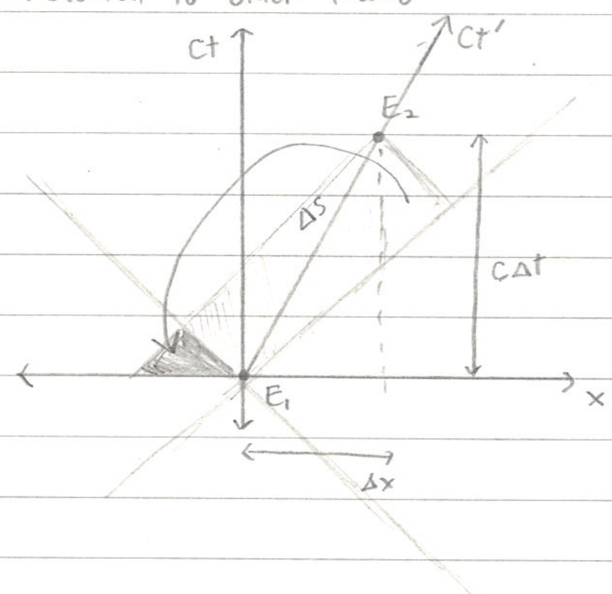
* Geometry of spacetime



$\Delta s' = c\Delta t'$

$Area_{CD} = \frac{1}{2}(\Delta s)^2$

- Switch to other frame



$Area_{CD} = \frac{1}{2}(c\Delta t)^2 - \frac{1}{2}(\Delta x)^2 = \frac{1}{2}(\Delta s)^2$

$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$

- Metric formula in spacetime:

$$\boxed{\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2}$$

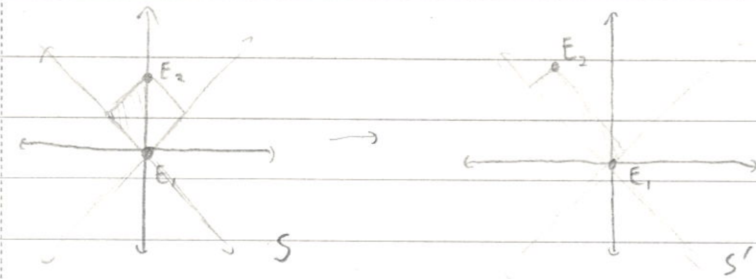
Cf. Euclidean; $\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$

Modern Physics

Week 19 Notes

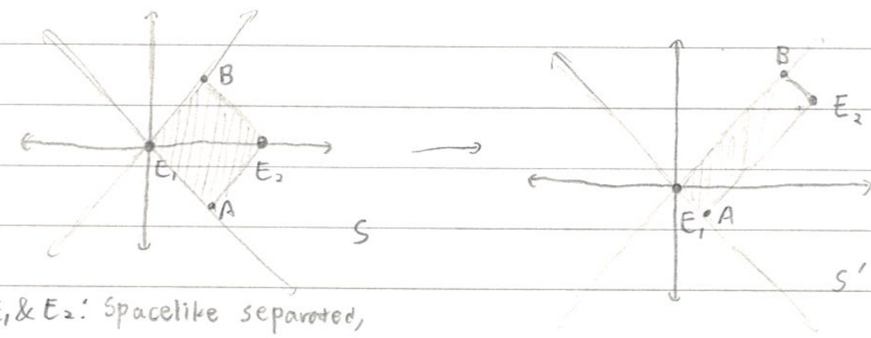
* More with Spacetime Diagrams

- Time Dilation



Time between E_1 & E_2 are different in different frames.

- Length contraction



E_1 & E_2: Spacelike separated, same time

A & B: events.

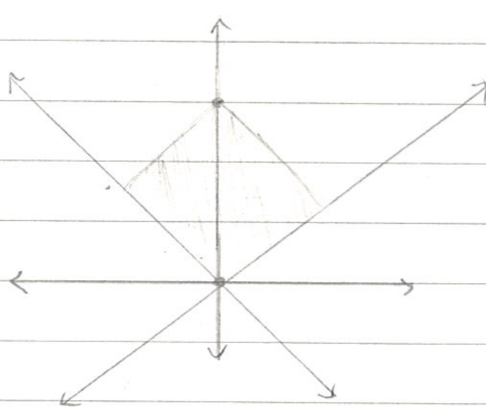
Not simultaneous in different frames

+ Length contraction

※ Above also shows us x' axis with $E_1 - E_2$ line.

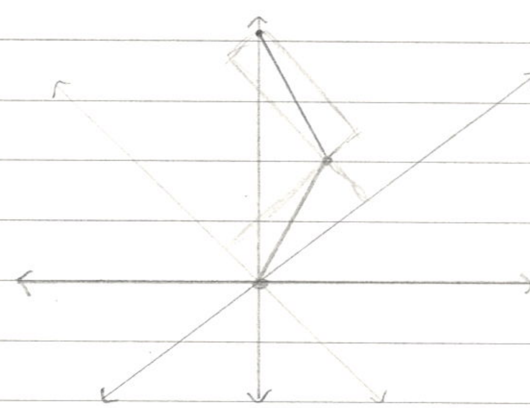
Find ct' axis such that $\frac{\Delta x'}{c\Delta t'} = 1$.

- Twin Paradox (Area $= \frac{1}{2}c\tau^2$, τ = proper time)



Alice stationary

$Area_{CD} \propto \Delta s^2 \propto (c\Delta t)^2$



Bob goes to planet & back

$\sum Area_{CD}$ Smaller

Less Δt for Bob

* Causal Diamonds: addition of velocity

- Area $= \frac{1}{2}\Delta s^2 = \frac{1}{2} c^2 \Delta\tau^2$

  ↳ τ = proper time, time in frame where 2 evt.s are at same loc.

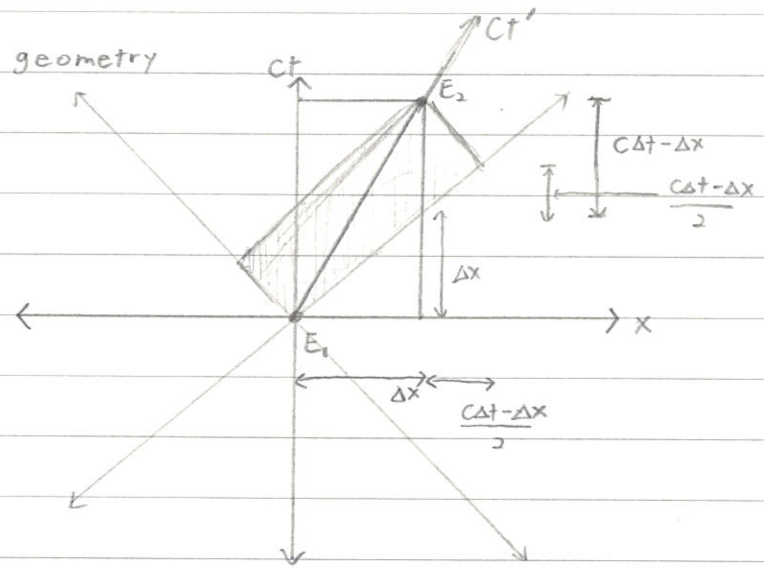
- In another frame S, $\Delta s^2 = c^2\Delta t^2 - \Delta x^2$

$$= c^2\Delta t^2 (1-\beta^2), \quad \beta = \frac{\Delta x}{c\Delta t}$$

$$\boxed{\begin{gathered} c^2\Delta t^2 (1-\beta^2) = c^2\Delta\tau^2 \\ c\Delta t = \gamma\, c\Delta\tau \end{gathered}} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Time dilation relation

- Some geometry



One side: $\frac{1}{\sqrt{2}}(c\Delta t - \Delta x)$, other side: $\frac{1}{\sqrt{2}}(c\Delta t + \Delta x)$

※ Purely geometric — not an actual interpretation of these intervals.

$$\boxed{\begin{gathered} \frac{1}{\sqrt{2}}\Delta s\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{and} \quad \frac{1}{\sqrt{2}}\Delta s\sqrt{\frac{1+\beta}{1-\beta}} \\ \Delta s^2 = c^2\Delta t^2 - \Delta x^2 \end{gathered}} \rightarrow \begin{gathered} \Delta s^2 = \sqrt{\frac{1+\beta}{1-\beta}}(c\Delta t + \Delta x) \cdot \\ \sqrt{\frac{1-\beta}{1+\beta}}(c\Delta t - \Delta x) \end{gathered}$$

Start at S → change to another travelling at β w.r.t. S.

New CD has these sides.

- Addition of velocities relation

$$\boxed{[\underbrace{\beta_1 + \beta_2}_{\beta}] = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}}$$

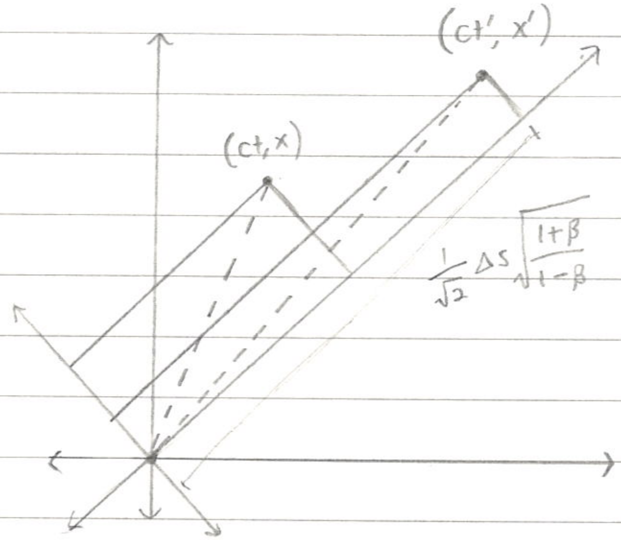
(derive from CD)

$$\frac{1}{2}\Delta s\sqrt{\frac{1+\beta}{1-\beta}} = \frac{1}{2}\Delta s\sqrt{\frac{1+\beta'}{1-\beta'}}\sqrt{\frac{1+\beta_r}{1-\beta_r}}$$

↳ Useful in transforming from 1 frame to another

* Lorentz Transformation

- Isometry of ct-x plane
- Analog of rotations in Euclidean plane



$$\Delta s^2 = \sqrt{\frac{1+\beta}{1-\beta}}\,(c\Delta t + \Delta x)\cdot\sqrt{\frac{1-\beta}{1+\beta}}\,(c\Delta t - \Delta x)$$

$$= \gamma(1+\beta)(c\Delta t + \Delta x)\cdot\gamma(1-\beta)(c\Delta t - \Delta x)$$

Also, $\Delta s^2 = (c\Delta t' + \Delta x')\cdot(c\Delta t' - \Delta x')$

⋮

$$\cancel{2}\, c\Delta t' = \cancel{2}\,\gamma(c\Delta t + \beta\Delta x)$$

$$\left.\begin{array}{l}\boxed{\Delta x' = \gamma(\Delta x + \beta c\,\Delta t)}\\ \boxed{c\Delta t' = \gamma(c\Delta t + \beta\,\Delta x)}\end{array}\right\} \text{ or } \begin{pmatrix} c\Delta t' \\ \Delta x' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} c\Delta t \\ \Delta x \end{pmatrix}$$

* More on isometry

↳ Recall metric eq.: $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$

↳ Each coord. enters as Δq^2, has linear transformation

3 spatial + 1 temporal translations

↳ Also, Euclidean metric is embedded $(\Delta x^2 + \Delta y^2 + \Delta z^2)$,

Total 10 isometries (Poincare group)

3 spatial isometries

↳ Lorentz transformations are those mixing time coord. w/ spatial coordinate: $\Lambda_{ct,x}(\beta_x)$, $\Lambda_{ct,y}(\beta_y)$, $\Lambda_{ct,z}(\beta_z)$

* Isometry = transformation preserving metric

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

* Assumptions so far

1. Isotropy of space: no preferred direction in space
2. PIE (principle of invariant event): what occurs at evt. must be fact, agreed upon by all observers regardless of motion
3. RV (reciprocal velocity postulate)
4. Lack of simultaneity

* Lorentz transforms (primed = moving frame, unprimed = lab frame)

$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Modern Physics

week 14 Notes

* Geometry of spacetime

- Manifold $\mathcal{M}$: restricted form of a set of points (topological space)
  - ↳ Connected Hausdorff space
    1. Connected: no disjoint sections, reach any point from another by smooth transition through manifold
    2. Hausdorff: given topological space X and open sets O_1 and O_2

$$\forall x \neq y \in X, \exists O_1 \,\&\, O_2 \text{ s.t. } x \in O_1 \,\&\, y \in O_2 \;\&\; O_1 \cap O_2 = \phi$$

"For 2 distinct points in X, open sets can be defined around each such that they don't intersect"

We can always burrow down to super close points & still find points between them.

↳ Endowed w/ 1+ coordinate charts/systems, O, where each chart is like $\mathbb{R}^n$.

↳ To cover manifold, charts overlap. When diff. charts overlap, an overlap function relates 1 set of coordinates to another.

↳ If we have complete set of coordinate charts + overlap functions that are infinitely differentiable, we have a differentiable manifold.

- Metric spaces
  - ↳ If distance function d can be defined on coord. chart w/ properties below, it's a metric space.

$$d(x,y) = d(y,x) \geq 0 \quad \forall x, y \in \mathcal{M}$$
$$d(x,y) = 0 \iff x = y$$
$$d(x,z) \leq d(x,y) + d(y,z)$$

- Riemannian metric space: has metric form that allows finding distances & angles
  - ↳ In general, a Riemannian metric function in n-dim. space takes form

$$dl^2 = \sum_{i,j=1}^{n} g_{ij}\, dx^i dy^j$$

  - ↳ Array $\{g_{ij}\}$ forms metric tensor
  - ↳ If all g_{ij} equals same const → Euclidean space
  - ↳ Going forth, only consider spaces w/ only diagonal terms

$$dl^2 = \sum_{i=1}^{n} g_{ii}\, dx^i dx^i$$

  - ↳ Metric func. informs us how to take product of 2 vectors (generalization of dot product)

- Inner product properties

$$\langle \vec{w}, \vec{v} \rangle = \langle \vec{v}, \vec{w} \rangle$$

$$\langle a\vec{w} + b\vec{v}, \vec{u} \rangle = a\langle \vec{w}, \vec{u} \rangle + b\langle \vec{v}, \vec{u} \rangle$$

$$\langle \vec{w}, \vec{w} \rangle > 0 \text{ if } \|\vec{w}\| \neq 0$$

- Onto spacetime
  - ↳ Relax constraints: pseudo-Riemannian metric space
    - $d(x,y) = d(y,x)$ can be real / imaginary
    - $d(x,y) = 0 \iff x = y$ no longer necessarily true
    - $d(x,z) \leq d(x,y) + d(y,z)$

*Geometry of spacetime II

- Find metric:

$x^0 = ct$

$x^1 = x$

$x^2 = y$

$x^3 = z$

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}\, dx^\mu dx^\nu$$

$$= g_{00}\, dx^0 dx^0 + g_{11}\, dx^1 dx^1 + g_{22}\, dx^2 dx^2 + g_{33}\, dx^3 dx^3$$

$$= g_{00}\, c^2 (dt)^2 + g_{11} (dx)^2 + g_{22} (dy)^2 + g_{33} (dz)^2$$

* Greek indices run over all spacetime coordinates

Latin indices run over only spatial coordinates

- Transformations that hold in both Newtonian physics & spacetime
  1. Time translations: $\hat{T}(\epsilon)$ i.e. $\hat{T}(\epsilon)t \to t+\epsilon$
  2. Spatial translations: $\hat{X}(\epsilon)$ i.e. $\hat{X}(\epsilon)x \to x+\epsilon$ (+ for y & z)
  3. Spatial rotations: $\hat{R}_z(\theta)$
     - ↳ $\hat{R}_z(\theta)(x,y,z) \to (\cos\theta\, x + \sin\theta\, y,\ -\sin\theta\, x + \cos\theta\, y,\ z)$
     - ↳ Plus x & y

↳ These leave Euclidean metric ($dL^2 = dx^2 + dy^2 + dz^2$) invariant.

+ invariant under rotation, so $g_{11} = g_{22} = g_{33}$

$$\boxed{\therefore ds^2 = g_{00}\, c^2 dt^2 + g_{ii}\left[dx^2 + dy^2 + dz^2\right]}$$

	Event 1	→ Event 2	
Frame S	$t_1 = 0$	$t_2 = t$, $dt = t$	$ds^2 = g_{00}(ct)^2 + g_{ii}(ct)^2$
	$x_1 = 0$	$x_2 = ct$, $dx = ct$	
Frame S'	$t_1' = 0$	$t_2' = t'$, $dt' = t'$	$ds^2 = g_{00}(ct')^2 + g_{ii}(ct')^2$
	$x_1' = 0$	$x_2' = ct'$, $dx' = ct'$	

Invariant, also $g_{00}, g_{ii} \neq 0$ and $t \neq t'$

Only way is $g_{00} = -g_{ii} \equiv g$

Embodies geometry of spacetime

$$\boxed{ds^2 = g\left[c^2 dt^2 - (dx^2 + dy^2 + dz^2)\right]}$$

$|g| = 1$, $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

* Poincaré group

Time translations: $\hat{T}(\epsilon)$: $\hat{T}(\epsilon)t \rightarrow t+\epsilon$

Spatial translations: $\hat{X}(\epsilon)$: $\hat{X}(\epsilon)x \rightarrow x+\epsilon$

$\hat{Y}(\epsilon)$: $\hat{Y}(\epsilon)y \rightarrow y+\epsilon$

$\hat{Z}(\epsilon)$: $\hat{Z}(\epsilon)z \rightarrow z+\epsilon$

Spatial rotations: $\hat{R}_z(\theta)$: $\hat{R}_z(\theta)(x,y,z) = (\cos\theta\, x + \sin\theta\, y, -\sin\theta\, x + \cos\theta\, y, z)$

$\hat{R}_x(\theta)$: $\hat{R}_x(\theta)(x,y,z) = (x, \cos\theta\, y + \sin\theta\, z, -\sin\theta\, y + \cos\theta\, z)$

$\hat{R}_y(\theta)$: $\hat{R}_y(\theta)(x,y,z) = (\cos\theta\, x - \sin\theta\, z, y, \sin\theta\, x + \cos\theta\, z)$

Lorentz transforms: $\hat{\Lambda}(\beta_x)$: $\hat{\Lambda}(\beta_x)(ct,x,y,z) \rightarrow (ct', x', y, z)$

$\hat{\Lambda}(\beta_y)$: $\hat{\Lambda}(\beta_y)(ct,x,y,z) \rightarrow (ct', x, y', z)$

$\hat{\Lambda}(\beta_z)$: $\hat{\Lambda}(\beta_z)(ct,x,y,z) \rightarrow (ct', x, y, z')$

- Isometries: transformations that leave metric invariant

- Any combo of the Poincaré group will leave metric invariant.

- Metrics (in 1D, $\Delta s^2 = c^2\Delta t^2 - \Delta x^2$)
  - ↳ For lightlike-separated: $c\Delta t = \Delta x$, $\Delta s = 0$
  - ↳ For timelike-separated: $c\Delta t > \Delta x$, $\Delta s^2 > 0$
  - ↳ For spacelike-separated: $c\Delta t < \Delta x$, $\Delta s^2 < 0$
    - ↳ Δs is imaginary

In all frames b.c. invariant

Modern Physics

week 20 Notes

* Kinematical effects from invariance of interval

- Constancy of speed of light: $\Delta s = 0$ is invariant, all frames see light to travel at c

- Time dilation: Alice at rest, Bob moving

	E_1 Light emitted	E_2 light received	
A	$ct_1 = 0$	$ct_2 = c\Delta t_A$	$\Delta s^2 = (c\Delta t_A)^2 - (v\Delta t_A)^2$
	$x_1 = 0$	$x_2 = v\Delta t_A$	
B	$ct_1' = 0$	$ct_2' = c\Delta t_B$	$\Delta s^2 = (c\Delta t_B)^2$
	$x_1' = 0$	$x_2' = 0$	

$$\boxed{\Delta t_A = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Delta t_B}$$

- Length contraction

	E_1 Bob passes left edge	E_2 Bob passes right edge	
A	$ct_1 = 0$	$ct_2 = c\frac{L}{v}$	$\Delta s^2 = \left(c\frac{L}{v}\right)^2 - L^2$
	$x_1 = 0$	$x_2 = L$	
B	$ct_1' = 0$	$ct_2' = c\frac{L'}{v}$	$\Delta s^2 = \left(c\frac{L'}{v}\right)^2 - 0$
	$x_1' = 0$	$x_2' = 0$	

$$\boxed{L' = L\sqrt{1-\frac{v^2}{c^2}}}$$

- Simultaneity

↳ Frame A: A_1 & A_2 at rest, L apart. Bob moves right at $\vec{v}$

E_1: A_1 flashes light at $t=0$

E_2: A_2 flashes light at $t=0$

	E_1	E_2	
A	$ct_1 = 0$	$ct_2 = 0$	$\Delta s^2 = 0 - L^2$
	$x_1 = 0$	$x_2 = L$	
B	$ct_1' = 0$	$ct_2' = c\Delta t'$	$\Delta s^2 = (c\Delta t')^2 - (L' - v\Delta t')^2$
	$x_1' = 0$	$x_2 = L' - v\Delta t'$	

$L \neq L'$, so $\Delta t' \neq 0$

* Addition of velocity

- Frame S: obj. travelling at $\vec{v} = v\hat{i}$

  Frame S': moving at $\vec{v}_{rel} = v_r\hat{i}$ within Frame S

  Relation to observed velocity in frame S', $\vec{v}'$?

- Lorentz transformations: $\vec{v} = \frac{\Delta x}{\Delta t}\hat{i} = \frac{\gamma(\Delta x' + v_r \Delta t')}{\gamma(\Delta t' + \frac{v_r}{c^2}\Delta x')}\hat{i}$ Invert Lorentz transform using $-v$ for v

$$\boxed{v = \frac{v' + v_r}{1 + \frac{v_r v'}{c^2}}}$$ Addition of velocities*

*When velocity of obj. & frame are in same direction

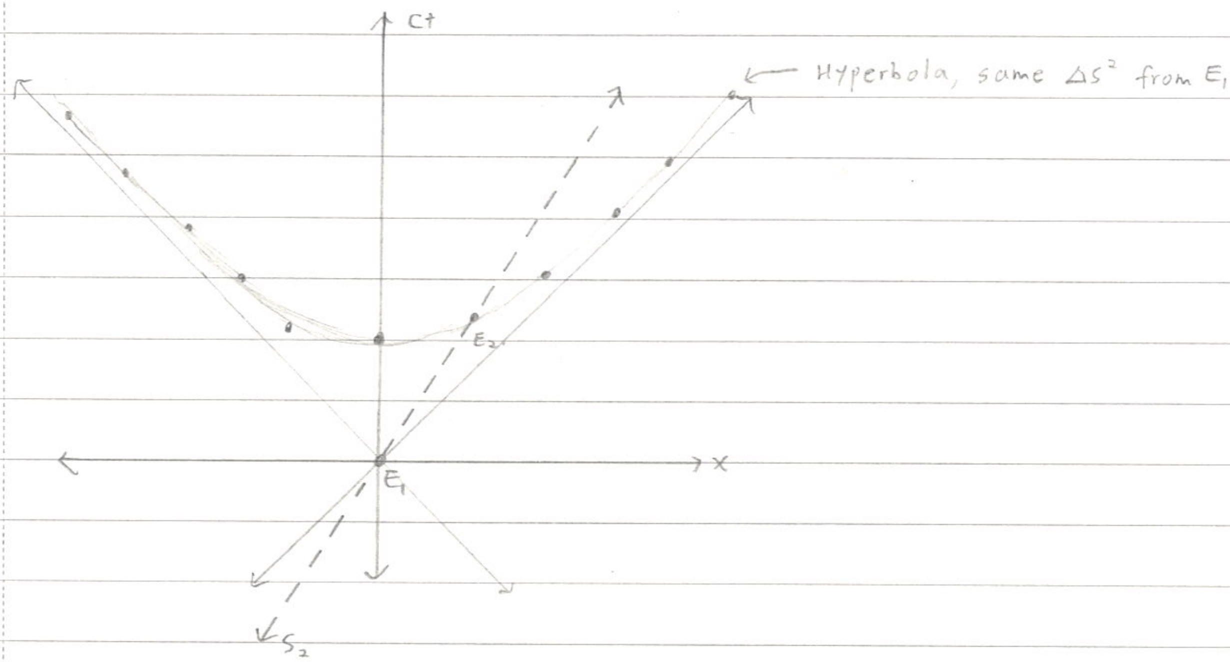
- with $\beta = \frac{v}{c}$: $\boxed{\beta = \frac{\beta' + \beta_r}{1 + \beta'\beta_r}}$

* Observations v. Measurements

- observation: general process of looking w/ eyes & noting what was seen
  - ↳ Referenced to single observer at single evt.
  - ↳ Can pertain to far-away events
  - ↳ "Observation of light arriving from other past light like separated evt.s

- Measurement: recording of facts at a single evt. in reference frames
  - ↳ Necessarily occurs at 1 evt.
  - ↳ Single measurement cannot relate 2 distinct evt.s
  - ↳ Single evt. can be measured in ∞ # of diff. frames
  - ↳ Stricter than an observation

* Invariant hyperbola & proper time



- S_2: frame where $x_2 = x_1$, evt.s occur at same location.
- Time that passes in frame where evt.s are at same location: proper time (τ)
  - ↳ Thus $\Delta s = c\Delta\tau$ for 2 events
  - ↳ All frames agree on what the proper clock reads at 2 events

- Principle of maximal aging
  - ↳ Straight worldline between 2 events: greatest τ elapsed (greatest interval between events)

$$\Delta s_{1-2} = \int_{E_1}^{E_2} \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$$

  - ↳ "An obj. is free (no net force on it) if elapsed time between any 2 events along its worldline, measured within its proper frame, is greater than the elapsed time measured between the events within any other frame."
  - ↳ b.c. obj. at rest has straight world line.

Modern Physics

week 21 Notes

*4-Vectors

- For spacetime: 4 components (0, 1, 2, 3)

↳ Notation: V^μ

*Latin indices i, j, etc. = 1, 2, 3

Greek indices μ, ν, etc. = 0, 1, 2, 3

- Vector space: $V^n(F) = \{\vec{A}, \vec{B}, \dots\}$ form vector space of dim. n over field F (real, complex, quaternion, etc.) iff. they satisfy:

1. Commutativity: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
2. Associativity: $\vec{v} + (\vec{w} + \vec{u}) = (\vec{v} + \vec{w}) + \vec{u}$
3. Additive identity: $\exists$ a vector $\vec{0}$ s.t. $\vec{v} + \vec{0} = \vec{v}$
4. Additive inverse: for every $\vec{v}$, unique inverse $-\vec{v}$ s.t. $\vec{v} + (-\vec{v}) = \vec{0}$

5. Distributive: $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$

$(a+b)\vec{v} = a\vec{v} + b\vec{v}$

$a(b\vec{v}) = (ab)\vec{v} = (ba)\vec{v}$

$0\vec{v} = \vec{0}$

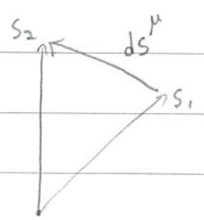
6. Also, $a\vec{0} = \vec{0}$ and $(-1)\vec{v} = -\vec{v}$

*Addition/subtraction results in element within $V^n(F)$

*Multiplication by scalars results in element within $V^n(F)$

*Multiplication of vectors (to be defined - inner product produces scalars)

- The interval



$$\|ds^\mu\|^2 = ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

- 4-velocity: $\frac{\Delta \text{interval}}{\Delta \text{"time"}} = \frac{\Delta \text{interval}}{\Delta \text{parameter}}$ $\boxed{\vec{U} = \gamma(c, \vec{v})}$

↳ Parametrized by proper time 4-velocity: $\frac{ds}{d\tau}$

↳ Magnitude $= \|u^\mu\|^2 = \|\frac{ds^\mu}{d\tau}\|^2 = \frac{c^2dt^2 - dx^2 - dy^2 - dx^2}{d\tau^2}$

↳ $d\tau = \frac{ds}{c}$, $\boxed{\|u^\mu\| \text{ is always } c}$

- Lorentz Transformation in matrix form

↳
$$ds^\mu = \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} = \hat{\Lambda}(\beta_x) ds'^\mu = \begin{pmatrix} \gamma(cdt' + \frac{vdx'}{c}) \\ \gamma(dx' + vdt') \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} \gamma(cdt' + \beta_x dx') \\ \gamma(dx' + \beta_x c dt') \\ dy' \\ dz' \end{pmatrix}$$

↳
$$ds^\mu = \hat{\Lambda}(\beta_x) ds'^\mu = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}$$

※ Proper notation for Lorentz transform:

$$ds^\mu = \sum_{\mu=0}^{3} \Lambda^\nu_\mu(\beta) ds'^\mu \equiv \Lambda^\nu_\mu(\beta) ds'^\mu$$ ← Einstein notation Avoid the Σ

- Relativistic Momentum (4-momentum)

$\boxed{p^\mu = mv^\mu}$ ← $\boxed{m \text{ will be examined later.}}$

↳
$$p^\mu = \left(mc\frac{dt}{d\tau}, m\frac{dx}{d\tau}, m\frac{dy}{d\tau}, m\frac{dz}{d\tau}\right), \quad d\tau = \frac{dt}{\gamma}$$
$$= (mc\gamma, \gamma p_x, \gamma p_y, \gamma p_z)$$

↳ Lorentz Transforms:
$$\Lambda(\beta_1) = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma mc \\ \gamma mv \\ 0 \\ 0 \end{pmatrix}$$

- 1st Component of 4-momentum: $p^0 = \gamma mc$
  - ↳ If $\beta \ll 1$, $\gamma = (1-\beta^2)^{1/2} \approx 1 + \left(-\frac{1}{2}\right)(-\beta^2)$

$$= 1 + \frac{1}{2}\beta^2 = 1 + \frac{1}{2}\frac{v^2}{c^2}$$

$$p^0 = \gamma mc = mc\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \dots\right) \approx mc + \frac{1}{2}mv^2\frac{1}{c}$$

$$\boxed{cp^0 \approx mc^2 + \frac{1}{2}mv^2}$$

↳ Relativistic energy: $\boxed{E = cp^0 = \gamma mc^2}$

↳ If $\beta = 0$, $E_0 = mc^2$, rest energy of object w/ mass m

- Rewrite:

↳ 4-momentum: $p^\mu = (\gamma mc, \gamma p_x, \gamma p_y, \gamma p_z) = \left(\frac{E}{c}, p^x, p^y, p^z\right)$

↳ Energy-momentum 4-vector: $q^\mu = cp^\mu$; (E, cp^x, cp^y, cp^z)

※ Since $p^\mu = mv^\mu$, $\|p^\mu\|^2 = (mc)^2$

Since $q^\mu = cp^\mu$, $\|q^\mu\|^2 = (mc^2)^2$

Solve $\|q^\mu\|^2$ for E: $E = \sqrt{(mc^2)^2 + (cp^x)^2 + (cp^y)^2 + (cp^z)^2}$

- Relativistic conservation of momentum

$$p_1^\mu = p_2^\mu \implies \begin{pmatrix} E/c \\ p^x \\ p^y \\ p^z \end{pmatrix} = \begin{pmatrix} E'/c \\ p'^x \\ p'^y \\ p'^z \end{pmatrix}$$

Mass is not conserved in relativistic dynamics.
Energy is conserved.

↳ Massive particles $\begin{cases} \text{At rest: } E_0 = mc^2 \\ \text{At } \beta_x\text{: } E = \gamma mc^2 = \sqrt{(mc^2)^2 + (p^x c)^2} \end{cases}$

Massless particles $\{$ Energy: $E = \sqrt{(p^x c)^2 + (p^y c)^2 + (p^z c)^2} = pc$

Identities $\begin{cases} p^x = mu^x = m\dfrac{dx}{d\tau} = \gamma m\dfrac{dx}{dt} = \gamma mc \cdot \dfrac{1}{c}\dfrac{dx}{dt} = \gamma\beta mc \\ \gamma = \dfrac{E}{mc^2},\ \gamma\beta = \dfrac{p^x c}{mc^2} \longrightarrow \beta = \dfrac{\gamma\beta}{\gamma} = \dfrac{p^x c}{E} \end{cases}$ | $p^x c = \gamma\beta mc^2$

Modern Physics

Week 22 Notes

* Notes on Specifics

- Warning: $E = mc^2$ might be misleading
  - ↳ Alternative: $E = \sqrt{(mc^2)^2 + (pc)^2}$ (applies in all scenarios)
  - or $E_0 = mc^2$ (obj. at rest)
  - $E = \gamma mc^2$ (for massive bodies)

- Misconception: objects can't pass c b.c. mass gets large and prevents them from being 'pushed' faster
  - ↳ Mass is a Lorentz scalar, doesn't change w/ speed: $m \neq \gamma m_0$

- Mass is not conserved in relativistic physics, but energy is.
- Mass ≠ energy: one conserved, other isn't
- Space ≠ time: due to causal structure, you can always identify time dimension. Metric distinguishes them.

* Gravity & Special Relativity

- Newton's Universal Law of Gravitation: $\vec{F} = -G \frac{m_1 m_2}{r^2} \hat{r}$
  - ↳ Instantaneous influence???

- Equivalence Principle
  - ↳ Observer can't distinguish 1) Free fall in uniform gravitational field
    2) Floating in deep space

  ② is an IRF, and since they're equivalent, ① is an IRF.
  - ↳ Physics in a free-falling frame with uniform gravitational field is equivalent to physics in an IRF (without gravity)
  - ↳ or: physics in a frame w/ uniform gravity (fixed reference point) is equivalent to physics in accelerating reference frame

Modern Physics

Week 23 Notes

* Geodesics

- In the presence of a gravitational field, worldlines between 2 timelike events that have max. proper time (geodesics) are curved.
- Observers following timelike geodesics:
  1. Maximal proper time
  2. No net non-gravitational force
  3. Satisfy Newton's laws, when conducting experiments
  4. Frame fixed to observer is a local IRF

* In ordinary space (not pseudo-Riemannian), geodesics are lines of min. distance

- Non-Euclidean geometry
  ↳ Gaussian Curvature: $K = \frac{1}{R_1 R_2}$
  ↳ Classify:
  1) $K > 0$: $\sum_{i=1}^{3} \theta_i > \pi$
  2) $K = 0$: $\sum_{i=1}^{3} \theta_i = \pi$
  3) $K < 0$: $\sum_{i=1}^{3} \theta_i < \pi$

  Triangles on the surface

* Some metrics (flat)

- 2D Euclidean Plane in polar coordinates: $dl^2 = dr^2 + r^2 d\theta^2$
- 2D surface of sphere: $dl^2 = R^2 d\theta^2 + R^2 \sin^2 d\varphi^2$
- 3D Euclidean space in spherical coords: $dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 d\varphi^2$
- 4D flat spacetime (cartesian): $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$
- 4D flat spacetime (spherical): $ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 d\varphi^2$

- General curved metric spaces: $dl^2 = g_{11} dx_1^2 + g_{22} dx_2^2$ (2D)

  $dl^2 = g_{11} dx_1^2 + g_{22} dx_2^2 + g_{33} dx_3^2$ (3D)

* Schwarzschild Solution

- Solution for spacetime outside a spherically symmetric mass, M

(Exterior solution)

1. Use spherical coordinates
2. Solution is outside mass, so $G^{\mu\nu} = 0$ (Einstein's equations take this form)
3. As $r \to \infty$, metric should approach flat spacetime metric
4. As $M \to 0$, metric should approach flat spacetime metric
5. Spherical symmetry
6. Must satisfy Einstein's equation (not checking this here)

$$ds^2 = g_{tt} c^2 dt^2 - g_{rr} dr^2 - h_\theta r^2 d\theta^2 - h_\varphi r^2 \sin^2\theta \, d\varphi^2$$

Spherical symmetry $\longrightarrow$ $ds^2 = g_{tt} c^2 dt^2 - g_{rr} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$

$\rightarrow$ $g_{tt} = g_{tt}(r,t)$, $g_{rr} = g_{rr}(r,t)$

$$\therefore \quad ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$