Deriving Maxwell's Equations

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Contents

1 Gauss's Law for Electricity **2**

2 Gauss's Law for Magnetism **2**

3 Faraday's Law **3**

4 Ampère's Law **5**

1 Gauss's Law for Electricity

Maxwell's first equation is Gauss's Law for Electricity, which says that the flux of an electric field over a closed surface is equal to the electric charge enclosed by that surface [1]. Using the electric flux density field $\mathbf{D}$,

$$\iint_S \mathbf{D} \cdot d\mathbf{S} = \textit{total charge enclosed}$$

If ρ is the charge density, then Gauss's Law can be rewritten as

$$\iint_S \mathbf{D} \cdot d\mathbf{S} = \iiint_V \rho \; dV$$

Using the divergence theorem,

$$\iiint_V \nabla \cdot \mathbf{D} \; dV = \iiint_V \rho \; dV$$

Since the volume V is arbitrary, the equation can be satisfied for any volume V.

$$\nabla \cdot \mathbf{D} = \rho$$

2 Gauss's Law for Magnetism

Maxwell's second equation is Gauss's Law for Magnetism, which says that the flux of a magnetic field through a closed surface is always zero [1]. This can be derived using properties of magnetic fields: Magnetic field lines are continuous, so any magnetic field line entering a surface must also leave it. As a result, the net magnetic flux through the surface is zero [1].

Using the magnetic flux density field $\mathbf{B}$, this can be stated as

$$\iint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

From the divergence theorem,

$$\iint_S \mathbf{B} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{B} \; dV$$

$$\iiint_V \nabla \cdot \mathbf{B} \; dV = 0$$

Since the volume V is arbitrary, the equation can be satisfied for any volume V.

$$\nabla \cdot \mathbf{B} = 0$$

3 Faraday's Law

Maxwell's third equation is Faraday's Law of Induction. To derive it, we can use Lenz's Law. Lenz'z Law says that an induced electromagnetic force drives a current along a closed loop, in the direction opposite the change in magnetic flux that caused the electromagnetic force [2]:

$$Induced\ EMF = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}$$

The induced EMF is also equal to the work done by the electric field $\mathbf{E}$ around a circuit C [3], so we can rewrite the equation:

$$Induced\ EMF = \oint_C \mathbf{E} \cdot dL$$

$$\oint_C \mathbf{E} \cdot dL = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}$$

Using Stokes' Theorem,

$$\iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}$$

To calculate the time derivative of a flux integral, we can use the equation given in [4, pg.622], from which [1]

$$\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S} = \iint_S (\nabla \cdot \mathbf{B})\mathbf{v} \cdot d\mathbf{S} - \int_C (\mathbf{v} \times \mathbf{B}) \cdot dL + \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

For a surface S that moves with velocity $\mathbf{v}$.

From Gauss's Law for Magnetism in section 2, we know that $\nabla \cdot \mathbf{B} = 0$, so

$$\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S} = -\int_C (\mathbf{v} \times \mathbf{B}) \cdot dL + \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

We can write out the line integral using the unit tangent vector $\mathbf{T}$ to get

$$\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S} = -\int_C (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{T}\ dL + \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Since $\mathbf{v}$ along a circuit C has the same direction as C, $(\mathbf{v} \times \mathbf{B})$ is perpendicular to C, which means that $(\mathbf{v} \times \mathbf{B})$ is also perpendicular to $\mathbf{T}$. Therefore, $(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{T} = 0$, which simplifies the equation to

$$\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S} = \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

We can substitute this back into the result from Lenz's Law:

$$\iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}$$

[1] There seems to be a mistake/misprint in the article. It uses $\mathbf{F}$ when it should use $\frac{\partial}{\partial t}\mathbf{F}$, which is how it is in the book it cites: `https://www.google.com/books/edition/The_Classical_Theory_of_Electricity_and/9rTQAAAAMAAJ?hl=en&gbpv=1&printsec=frontcover&pg=PA40`

$$\iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Since the surface S is arbitrary, we can remove the integrals to get

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

4 Ampère's Law

Maxwell's fourth equation can be derived from Ampère's Law, which says that the circulation of $\mathbf{H}$ along a circuit L is equal to the total enclosed current [1]:

$$\oint \mathbf{H} \cdot dL = I_{enc}$$

Using Stokes' Theorem, we can rewrite the equation as

$$\iint_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = I_{enc}$$

Let $\mathbf{J}$ be the current density for the surface, so that

$$I_{enc} = \iint_S \mathbf{J} \cdot d\mathbf{S}$$

$$\iint_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \iint_S \mathbf{J} \cdot d\mathbf{S}$$

Since the surface S is arbitrary, the equality of the integral doesn't depend on S, so the equation can be simplified to

$$\nabla \times \mathbf{H} = \mathbf{J}$$

However, an inconsistency occurs if we take the divergence of both sides:

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J}$$

$$0 = \nabla \cdot \mathbf{J}$$

Divergence of a curl is zero, but $\nabla \cdot \mathbf{J}$ might not be zero for a fluctuating current density. To fix this, Maxwell added a new term to Ampère's Law: the displacement current $\frac{\partial \mathbf{D}}{\partial t}$. The Ampère-Maxwell Equation can be stated as

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Now if we take the divergence of both sides:

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t})$$

$$0 = \nabla \cdot \mathbf{J} + \frac{\partial (\nabla \cdot \mathbf{D})}{\partial t}$$

Since $\nabla \cdot \mathbf{D} = \rho$ from Gauss's Law for Electricity in section 1, the equation becomes

$$0 = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}$$

Which is true, since it's equivalent to the continuity equation [5]. The existence of the displacement current was later shown by Heinrich Hertz in a series of experiments that detected electromagnetic waves [6].

References

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[2] Pressbooks. *Lenz's Law*. URL: https://pressbooks.online.ucf.edu/osuniversityphysics2/chapter/lenzs-law/. (accessed: 12.07.2023).

[3] OpenStax. *Induced Electric Fields*. URL: https://phys.libretexts.org/Bookshelves/University_Physics/Book%3A_University_Physics_(OpenStax)/Book%3A_University_Physics_II_-_Thermodynamics_Electricity_and_Magnetism_(OpenStax)/13%3A_Electromagnetic_Induction/13.05%3A_Induced_Electric_Fields. (accessed: 12.07.2023).

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[5] Konstantin K. Likharev. *Continuity Equation and the Kirchhoff Laws*. URL: https://phys.libretexts.org/Bookshelves/Electricity_and_Magnetism/Essential_Graduate_Physics_-_Classical_Electrodynamics_(Likharev)/04%3A_DC_Currents/4.01%3A_Continuity_Equation_and_the_Kirchhoff_Laws. (accessed: 12.08.2023).

[6] D. F. Bartlett and T. R. Corle. *Measuring Maxwell's Displacement Current Inside a Capacitor*. URL: https://homepage.physics.uiowa.edu/~rmerlino/28S04/MaxDcurr.htm.pdf. (accessed: 12.09.2023).