Lecture 6

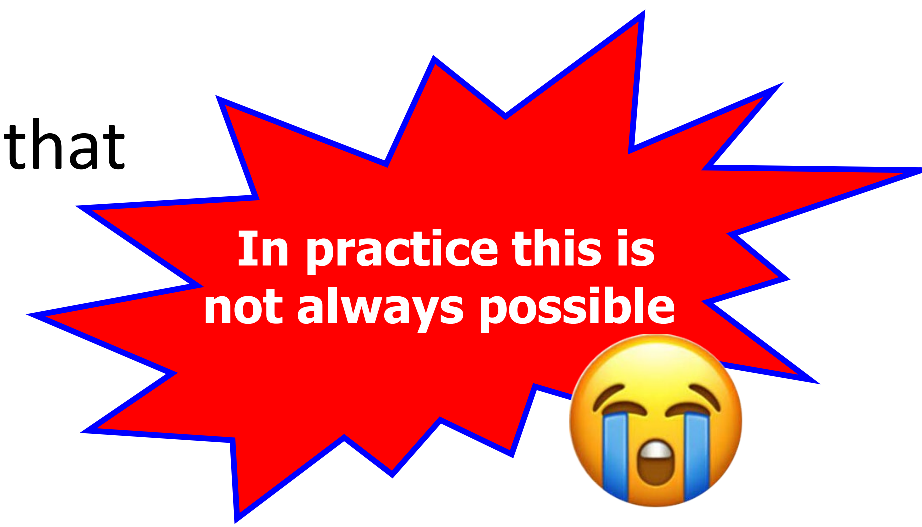
COMP 3760

Space/time trade-offs

Text chapter 7

Space/time tradeoffs

- **Space** refers to the memory consumed by an algorithm to complete its execution
- **Time** refers to the required time for an algorithm to complete the execution
- The best algorithm is one that
  - Requires less memory
  - AND takes less time



Space/time tradeoffs

- We have to sacrifice one at the cost of the other.

- If space is our constraint, then we have to choose an algorithm that requires less space at the cost of more execution time. (example: Bubble Sort, Selection Sort)

- If time is our constraint then we have to choose an algorithm that takes less time to complete its execution at the cost of more space. (example: MergeSort)

Types of space/time tradeoffs

1. **Input enhancement:** preprocess the input to store some info to be used later in solving the problem
    - Comparison Counting Sort
    - Distribution Counting Sort
    - String Matching (improved algorithm)

2. **Pre-structuring:** use extra space to facilitate faster access to the data
    - Hashing
    - Hash Function
    - Collision Handling
    - Efficiency of Hashing

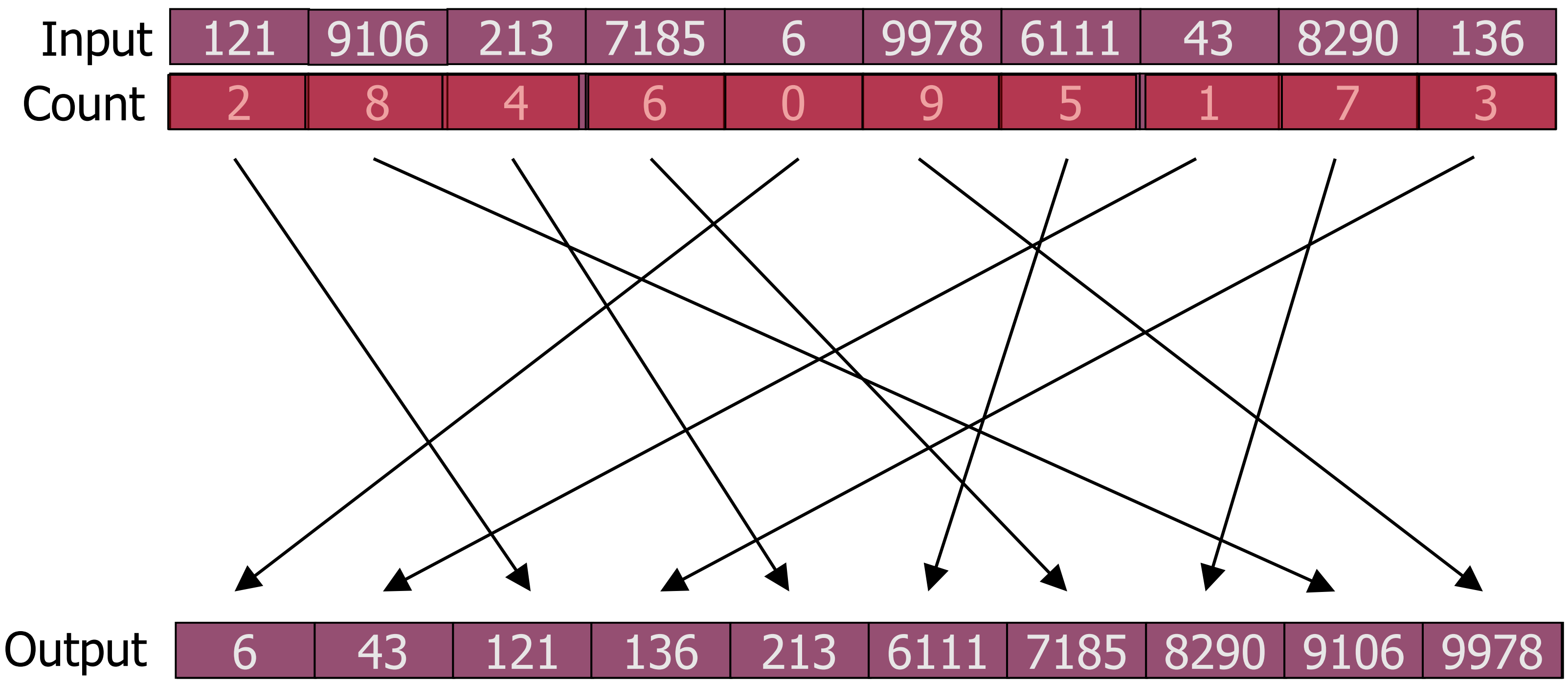
Comparison Counting Sort

- Idea: for each element of a list to be sorted, count the total number of elements smaller than this element and record the results in a table.

Input	121	9106	213	7185	6	9978	6111	43	8290	136
Count	2	8	4	6	0	9	5	1	7	3

Comparison Counting Sort

- Now move each input element to its corresponding position



Comparison Counting Sort

```
Algorithm ComparisonCountingSort(A[0..n-1])
 for i ← 0 to n-2
    for j ← i+1 to n-1
       if input[i] < input[j]
          Count[j]++
       else
          Count[i]++
 for i ← 0 to n-1
    output[Count[i]] ← input[i]
```

Efficiency of CCS

- It's $O(n^2)$
  - Of course we know a couple of algorithms that are $O(n\log n)$: MergeSort and HeapSort

Types of space/time tradeoffs

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   - Distribution Counting Sort
   - String Matching

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Distribution Counting Sort

- Suppose we need to sort an array with a “small” set of known values

8	5	7	6	7	8	5	8	6	8	8	5

Distribution Counting Sort

- Idea: count how many of each number...

8	5	7	6	7	8	5	8	6	8	8	5

- ...and determine the distribution from that
  - three 5's → positions 0 to 2
  - two 6's → positions 3 to 4
  - two 7's → positions 5 to 6
  - five 8's → positions 7 to 11 (11 is n-1)

5	5	5	6	6	7	7	8	8	8	8	8

Distribution Counting Sort

Algo DistributionCountingSort (A[0.. n-1])

for $j \leftarrow 0$ **to** $u\text{-}l$ **do**

$\quad C[j] \leftarrow 0$

for $i \leftarrow 0$ **to** $n\text{-}1$ **do**

$\quad C[A[i]\text{-}l] \leftarrow C[A[i]\text{-}l] + 1$

for $j \leftarrow 1$ **to** $u\text{-}l$ **do**

$\quad C[j] \leftarrow C[j\text{-}1] + C[j]$

for $i \leftarrow n\text{-}1$ **downto** 0 **do**

$\quad j \leftarrow \mathrm{A}[i]\text{-}\ l$

$\quad S[C[j]\ \text{-}1] \leftarrow \mathrm{A}[i]$

$\quad C[j] \leftarrow C[j] - 1$

return S

Distribution Counting Sort-example

u:14
l:11

A:

14	11	13	14	13

Size: $u - l + 1 = k$

C:

#of11	#of12	#of13	#of14

One "bucket" for each different value we might encounter

S:

This will be the sorted array

Loop 1: initialization

A:	14	11	13	14	13

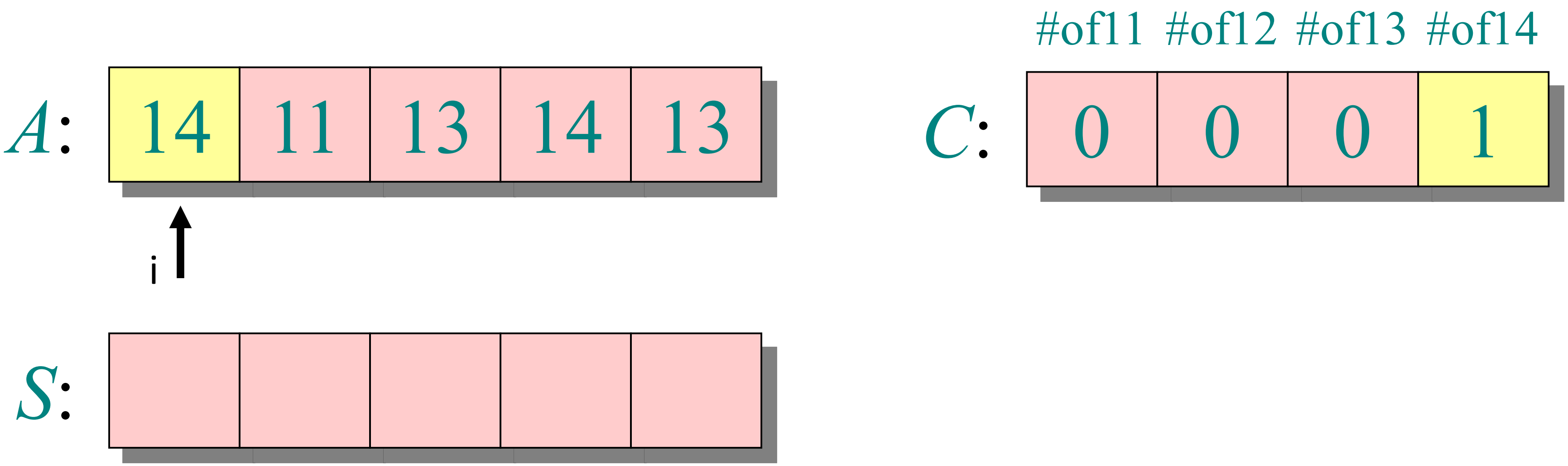
	#of11	#of12	#of13	#of14
C:	0	0	0	0

S:					

1. **for** $j \leftarrow 0$ **to** $u\text{-}l$ **do**

$C[j] \leftarrow 0$

Loop 2: count



2. for $i \leftarrow 0$ **to** $n\text{-}l$ **do**

$C[A[\,i]\text{-}l] \leftarrow C[A[\,i]\text{-}l] + 1$

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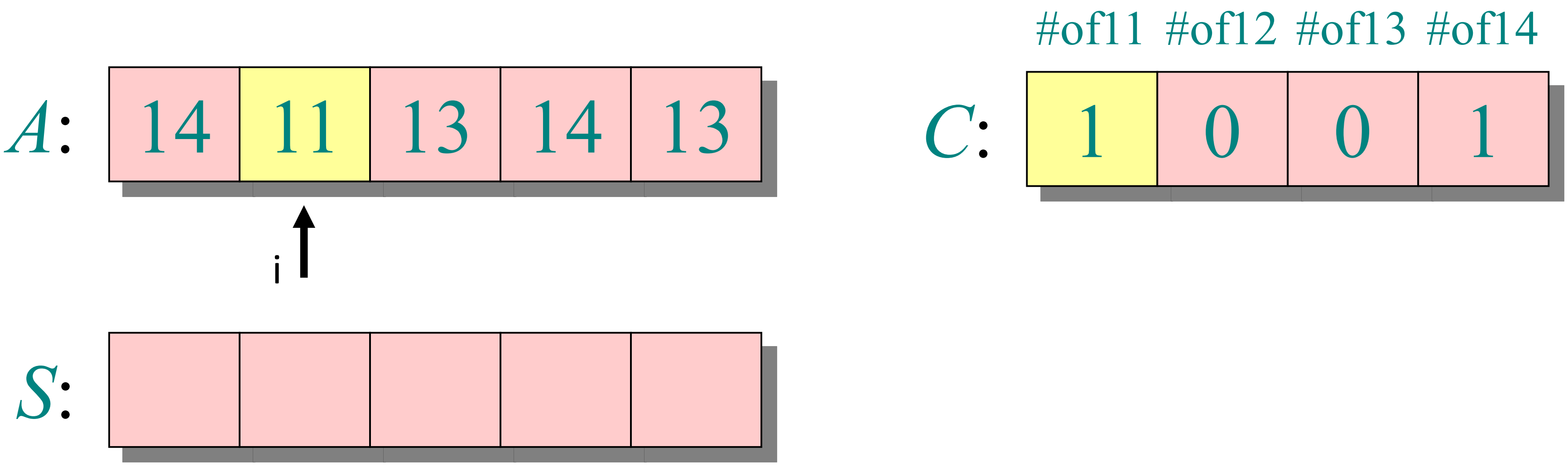

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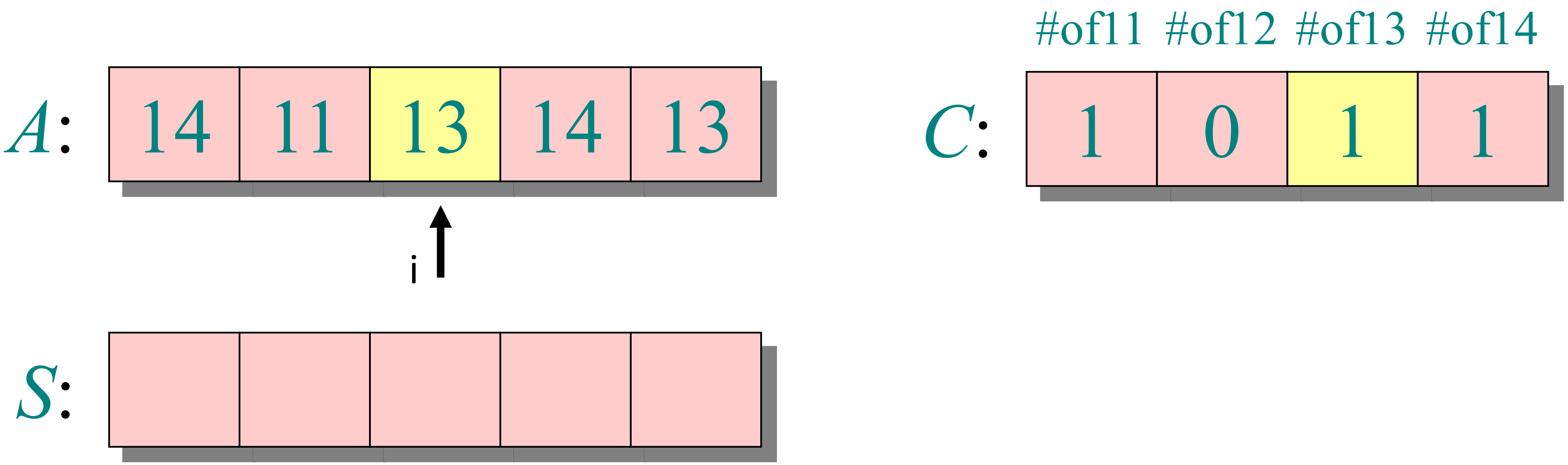

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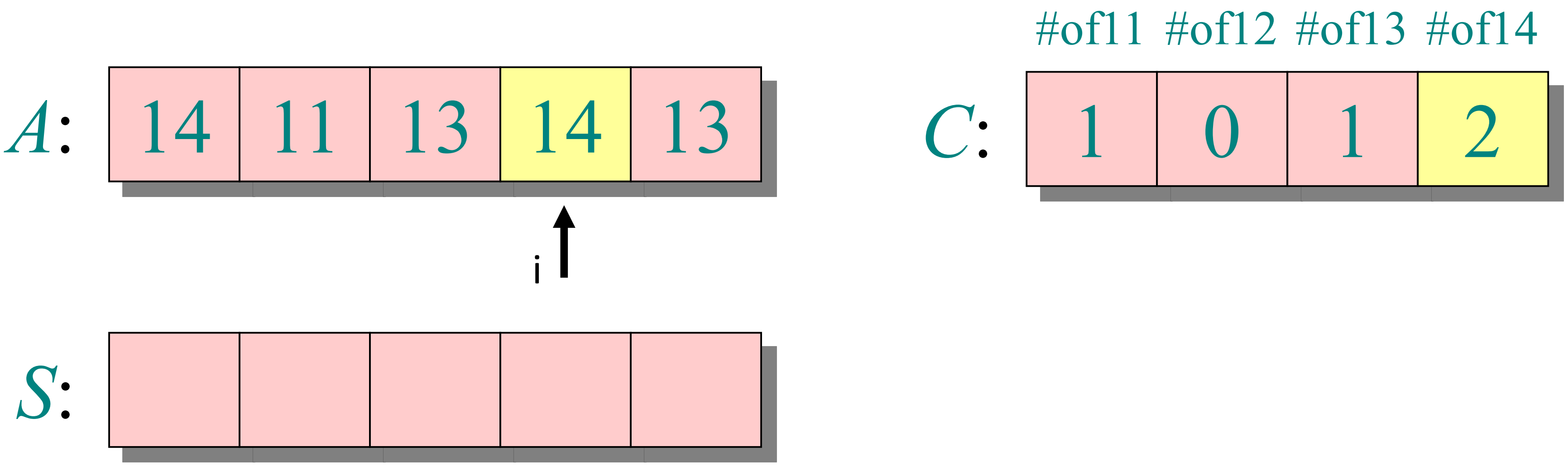

2. **for** $i \leftarrow 0$ **to** $n\text{-}l$ **do**

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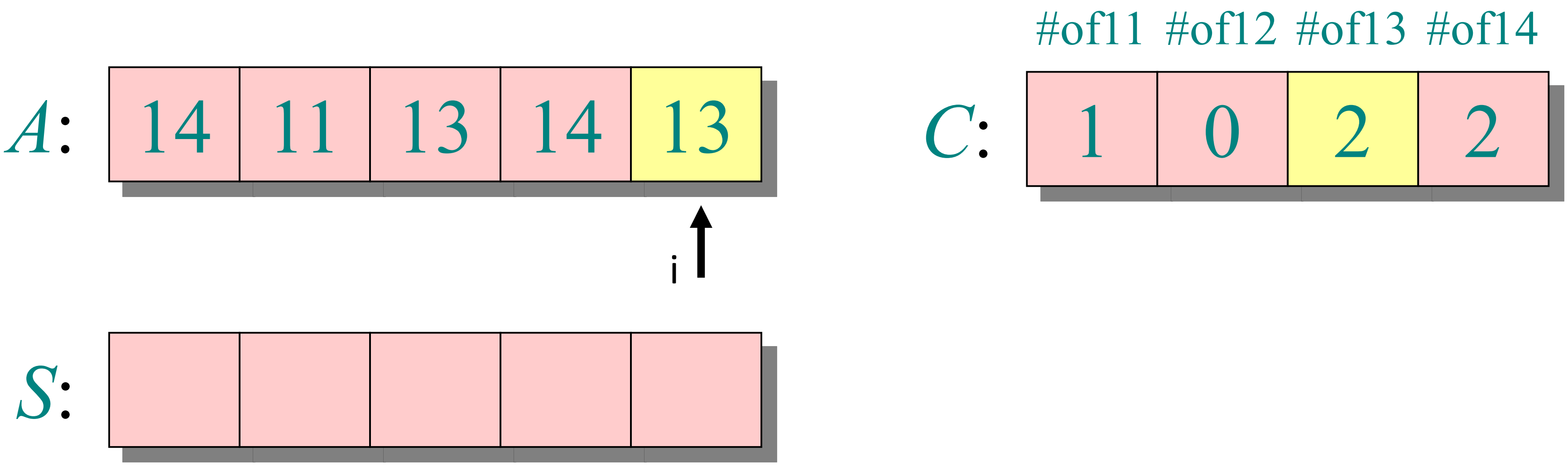
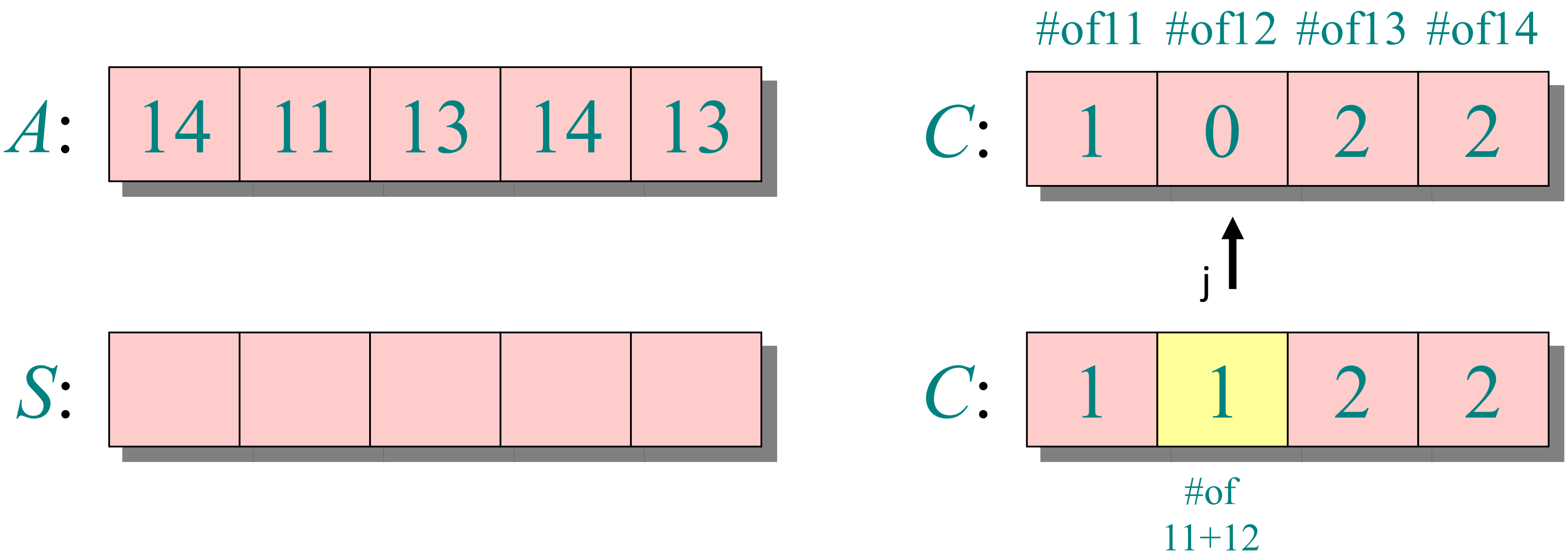
Loop 2: count



2. **for** $i \leftarrow 0$ **to** $n\text{-}1$ **do**

$$C[A[\,i]\text{-}l] \leftarrow C[A[\,i]\text{-}l] + 1$$

Loop 3: compute running sum



3. **for** $j \leftarrow 1$ **to** $u\text{-}l$ **do**

$$C[j] \leftarrow C[j\text{-}1] + C[j]$$

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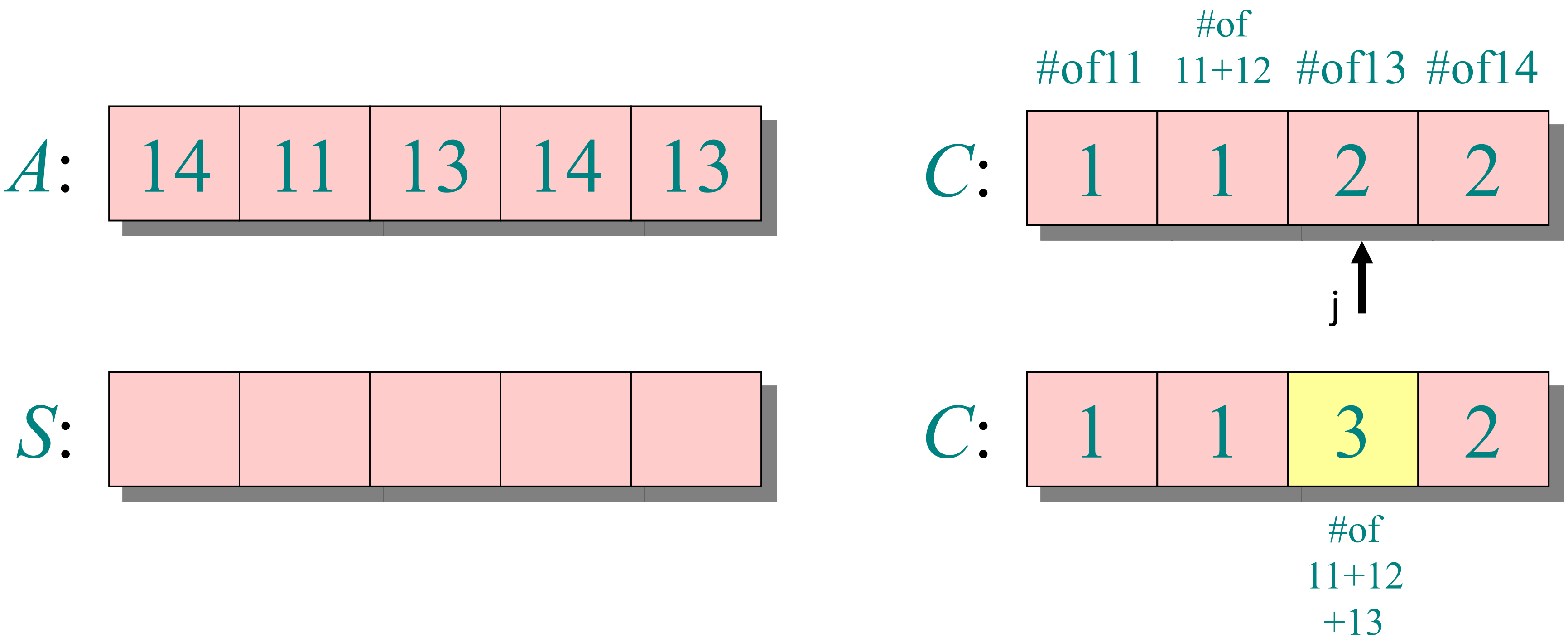

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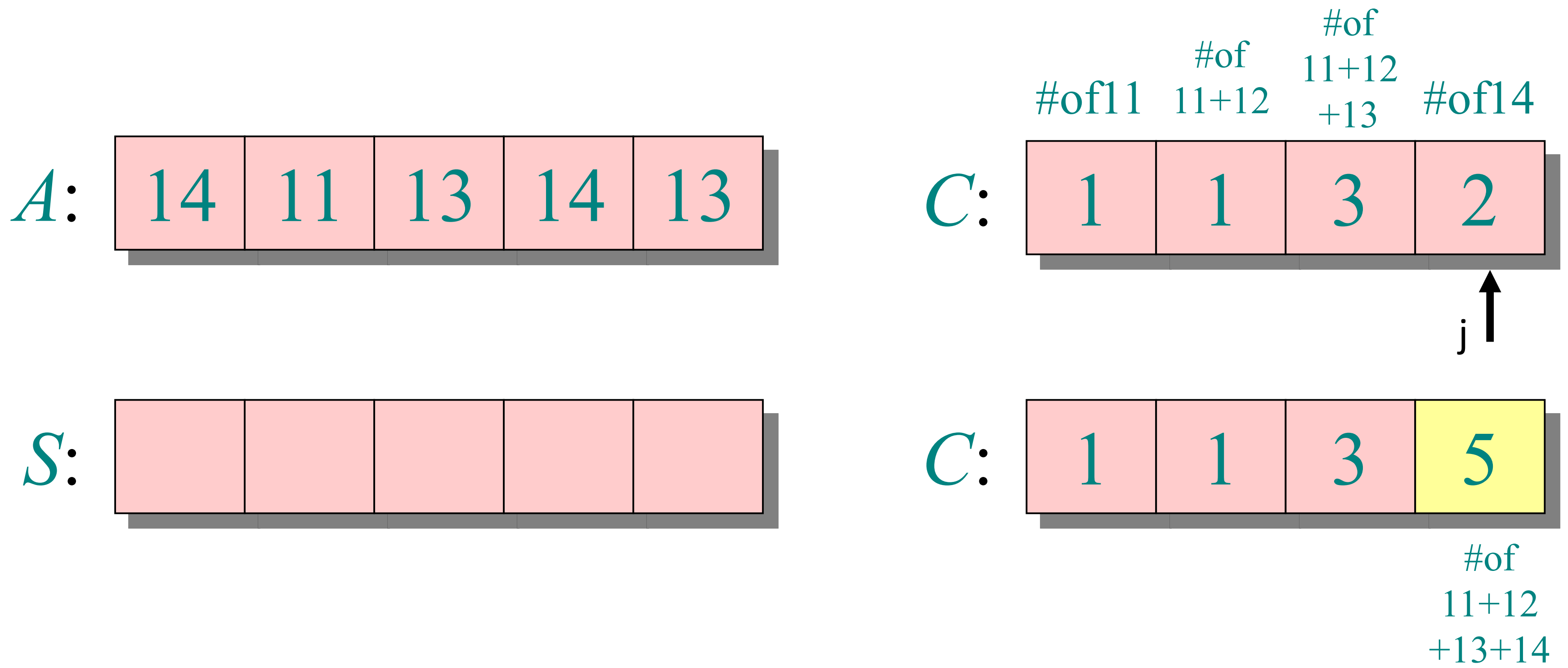

3. for $j \leftarrow 1$ **to** $u\text{-}l$ **do**

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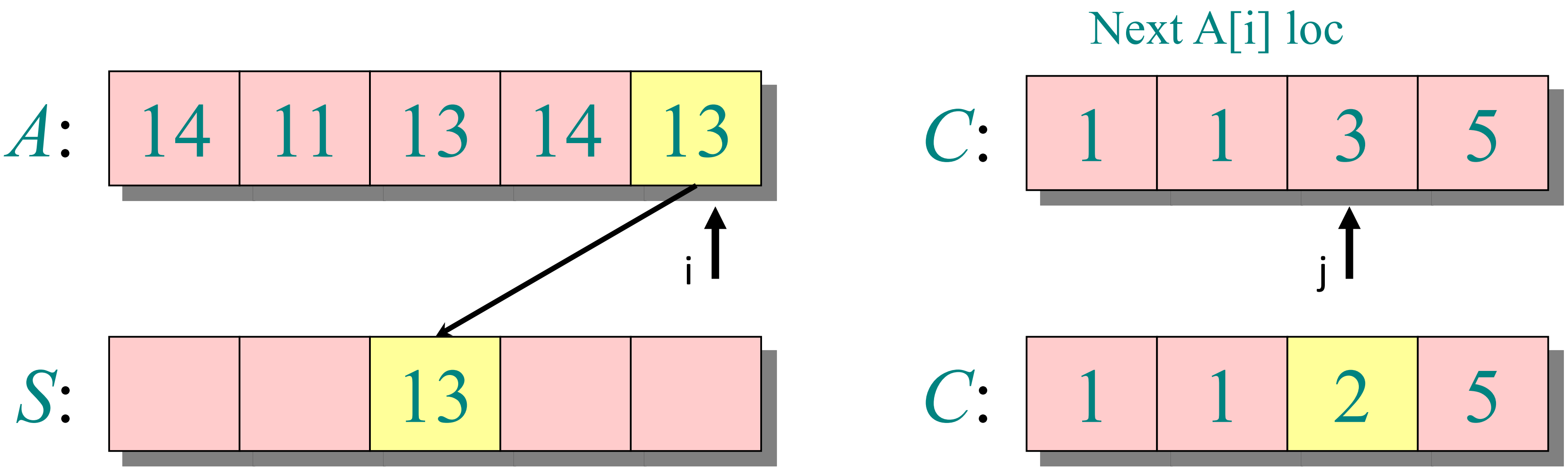
Loop 4: re-arrange



4. for $i \leftarrow n\text{-}1$ **downto** 0 **do**

$j \leftarrow \mathrm{A}[i] - l$

$S[C[j] - 1] \leftarrow \mathrm{A}[i]$

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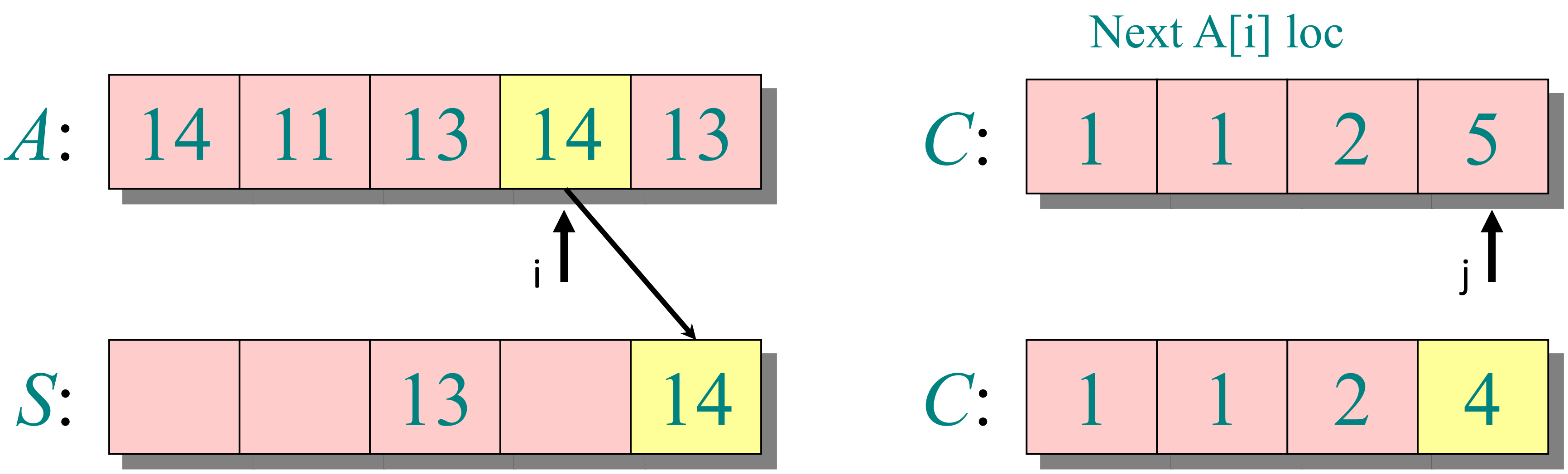
Loop 4: re-arrange



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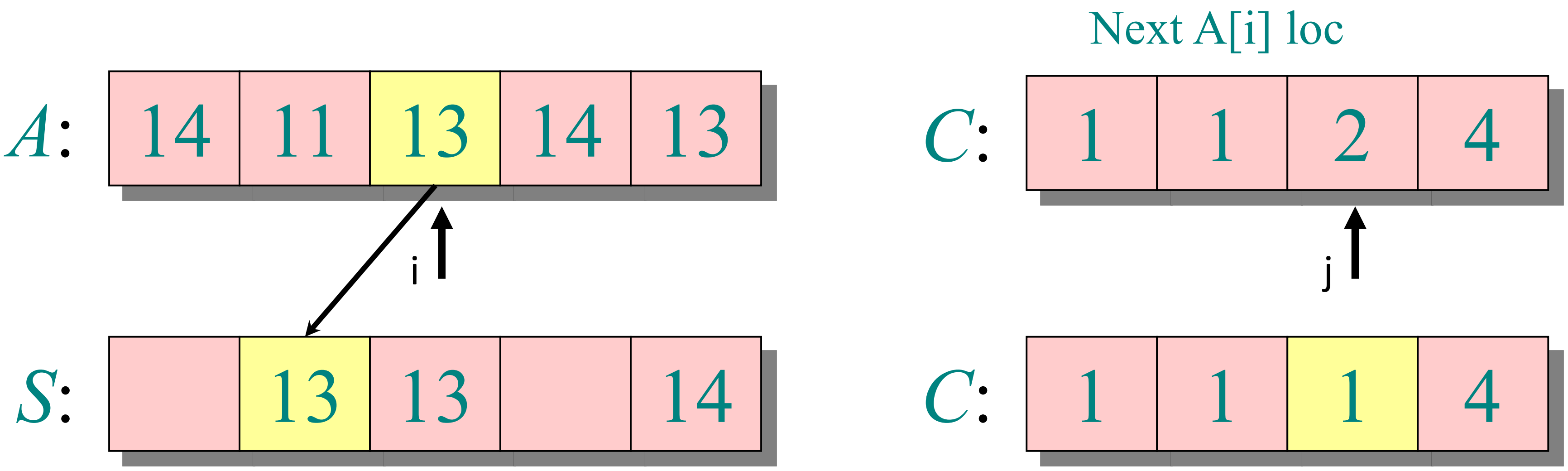

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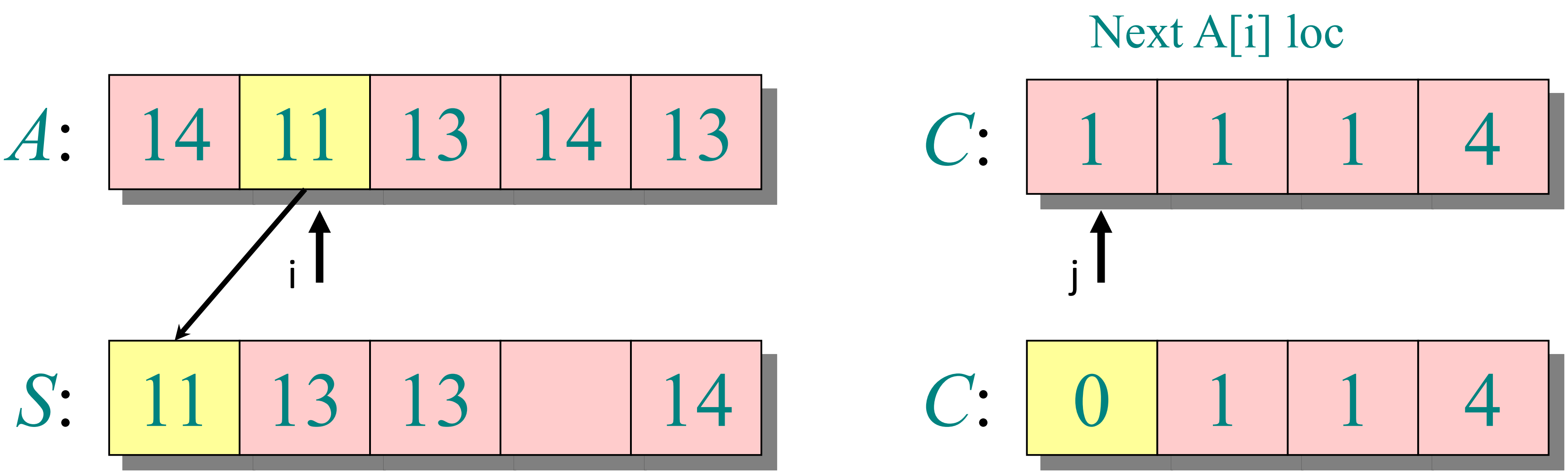

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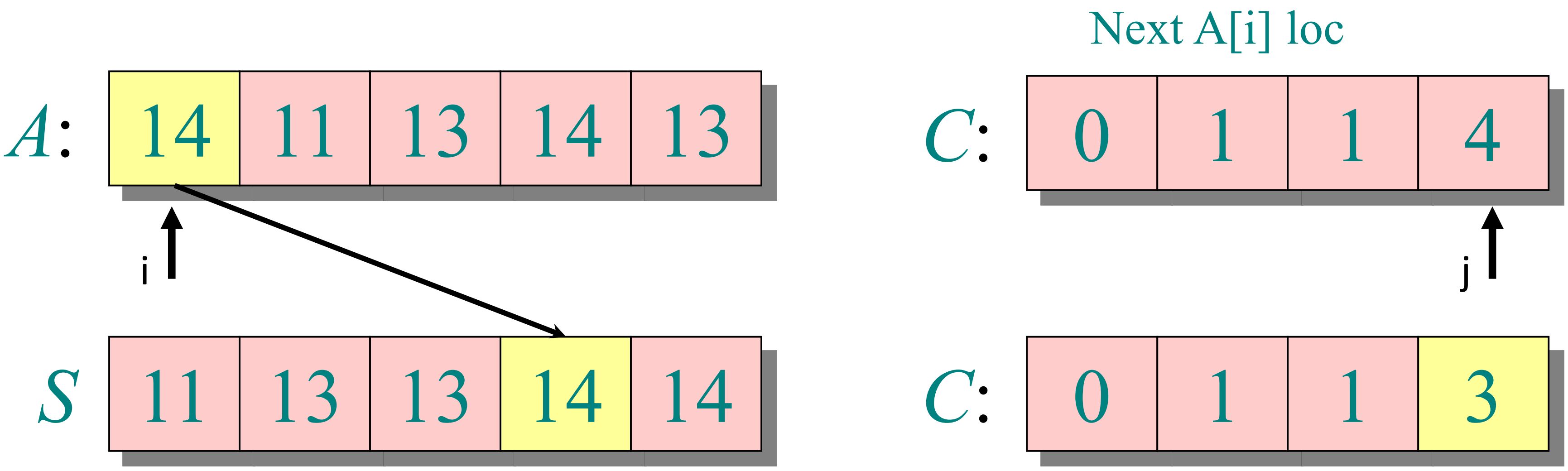

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$j \leftarrow \mathrm{A}[i] - l$

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$C[j] \leftarrow C[j] - 1$

Algo DistributionCountingSort (A[0.. n-1])

$O(k)$
: **for** $j \leftarrow 0$ **to** $u\text{-}l$ **do**
: $\quad C[j] \leftarrow 0$

$O(n)$
: **for** $i \leftarrow 0$ **to** $n\text{-}1$ **do**
: $\quad C[A[i]\text{-}l] \leftarrow C[A[i]\text{-}l] + 1$

$O(k)$
: **for** $j \leftarrow 1$ **to** $u\text{-}l$ **do**
: $\quad C[j] \leftarrow C[j\text{-}1] + C[j]$

$O(n)$
: **for** $i \leftarrow n\text{-}1$ **downto** 0 **do**
: $\quad j \leftarrow \mathrm{A}[i]\text{-}\, l$
: $\quad S[C[j]\ \text{-}1] \leftarrow \mathrm{A}[i]$
: $\quad C[j] \leftarrow C[j] - 1$

$O(n + k)$
: **return S**

Efficiency of Distribution Counting Sort

- If the range of input values is roughly <= the number of input values
  - … then this algorithm is O(n)

- This is really, really good!
  - But it is a *special-purpose* algorithm
  - Significant constraint on the *range* of input values

Types of space/time tradeoffs

1. **Input enhancement:** preprocess the input to store some info to be used later in solving the problem
    - Comparison Counting Sort
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2. **Pre-structuring:** uses extra space to facilitate faster access to the data.
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String Matching: reminder

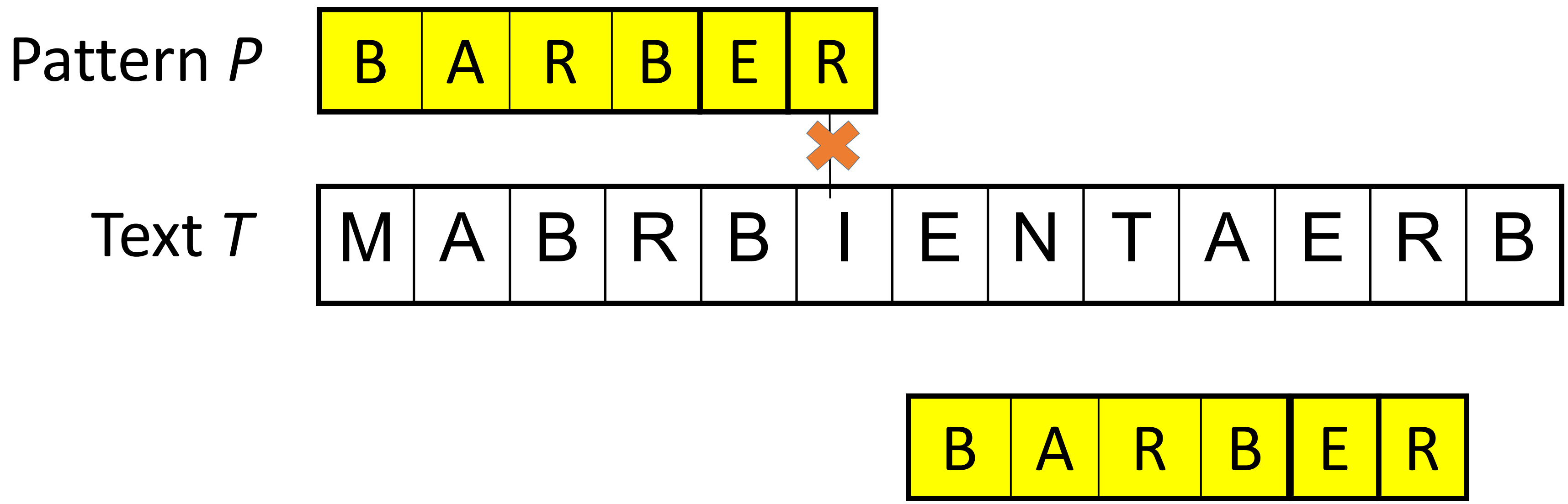
- Pattern: a string of m characters to search for
- Text: a (long) string of n characters to search in
- Brute force algorithm:
  - Align pattern at beginning of text
  - Moving L-R within pattern, compare pattern to text until
    - All characters are found to match (successful search); or
    - A mismatch is detected
  - While pattern is not found and the text is not yet exhausted, shift pattern one position to the right and repeat step 2.
- Time Complexity: $O((n-m+1) \times m)$

Input Enhancement in String Matching

- How can we improve string matching by using the concept of *input enhancement*?

- Useful observation: Whenever we have a mismatch, maybe we can shift the pattern over by *more than one character* before comparing again

Input Enhancement in String Matching



- *Compare the chars* ***right to left***
- *There is no "I" in BARBER, so we should shift the pattern all the way past the "I"*
- Determine the number of shifts by looking at the character of the text that is aligned against the last character of the pattern

String Matching: Key Observation

- Consider, as an example, searching for the pattern BARBER in some text. Here is *a moment in time*:

$$s_0 \quad \cdots \qquad\qquad\qquad\qquad c \quad \cdots \quad s_{n-1}$$

B A R B E R

- *When a mismatch occurs, look at the Text character that is aligned with the rightmost character of P*

String Matching: Four cases

- Text char *c* never appears in the Pattern
- Text char *c* appears in the Pattern but *not last*
- Text char *c* appears last in the Pattern but *only that one time*
- Text char *c* appears last in the Pattern *and other times*

String Matching: Four cases

- Case 1: If the Text char *c* never appears in the Pattern...

$s_0 \quad \ldots \quad S \quad \ldots \quad s_{n-1}$

B A R B E R

B A R B E R

...we can shift Pattern by its entire length

String Matching: Four cases

- Case 2: If the Text char *c* appears in the Pattern but *not last*...

$s_0 \quad \ldots \quad\quad\quad\quad\quad\quad\quad\quad \mathrm{B} \quad\quad\quad\quad\quad\quad\quad\quad \ldots \quad s_{n-1}$

$\neq$

B A R B E R

B A R B E R

...we can shift to align the last occurrence of *c* in Pattern with *c* in Text

String Matching: Four cases

- Case 3: If the Text char *c* appears last in the Pattern but *only that one time*...

s_0	$\ldots$				M	E	R							$\ldots$	s_{n-1}
					$\nparallel$	$\parallel$	$\parallel$								
		L	E	A	D	E	R								
								L	E	A	D	E	R		

...we can shift Pattern by its entire length

String Matching: Four cases

- Case 4: If the Text char *c* appears last in the Pattern *and other times*...

```
s_0  ...            A R           ...  s_{n-1}
                    ≠ ||
          R E O R D E R
                R E O R D E R
```

...we can shift to align the second-to-last occurrence of *c* in Pattern with *c* in Text

The Strategy

- How can we use this observation for input enhancement?

- Strategy:
  - Create a “shift table”
    - One entry for each possible value in the *input alphabet*
  - Shift table will indicate the number of positions to shift the pattern

Table	**2**	**5**	**7**	**2**	**7**	**7**	**3**	**…**	**7**	**4**	**7**
	0	**1**	**2**	**3**	**4**	**5**	**6**		**23**	**24**	**25**
	“A”	**“B”**	**“C”**	**“D”**	**“E”**	**“F”**	**“G”**		**“X”**	**“Y”**	**“Z”**

The Shift Table

- How to construct the shift table:
  - it will have a size equal to the number of elements in the input alphabet (so we have to know this in advance!)

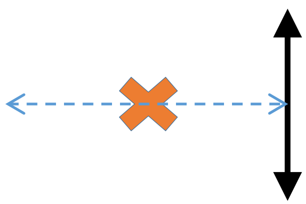
$$t(c) = \begin{cases} \text{distance from } c\text{'s rightmost occurrence in pattern} \\ \text{among its first } m\text{-1 characters to its right end} \\ \\ \text{pattern's length } m\text{, otherwise} \end{cases}$$

The Shift Table

- Example:
  - assume our alphabet is {A B C D E F G H I J}
  - assume our pattern is IDIGDAB (m=7)

  - When a mismatch occurs, what character is aligned with our pattern?

… text … text … text … text … text … text … text … text … text … text … text …



I D I G D A B

Table										
	1	7	7	2	7	7	3	7	4	7
	0	1	2	3	4	5	6	7	8	9
	"A"	"B"	"C"	"D"	"E"	"F"	"G"	"H"	"I"	"J"

Using the shift table …

- Example:

there is a mismatch on the first compare, so we lookup **table[“D”]**, which returns **2**, so we shift by 2 …

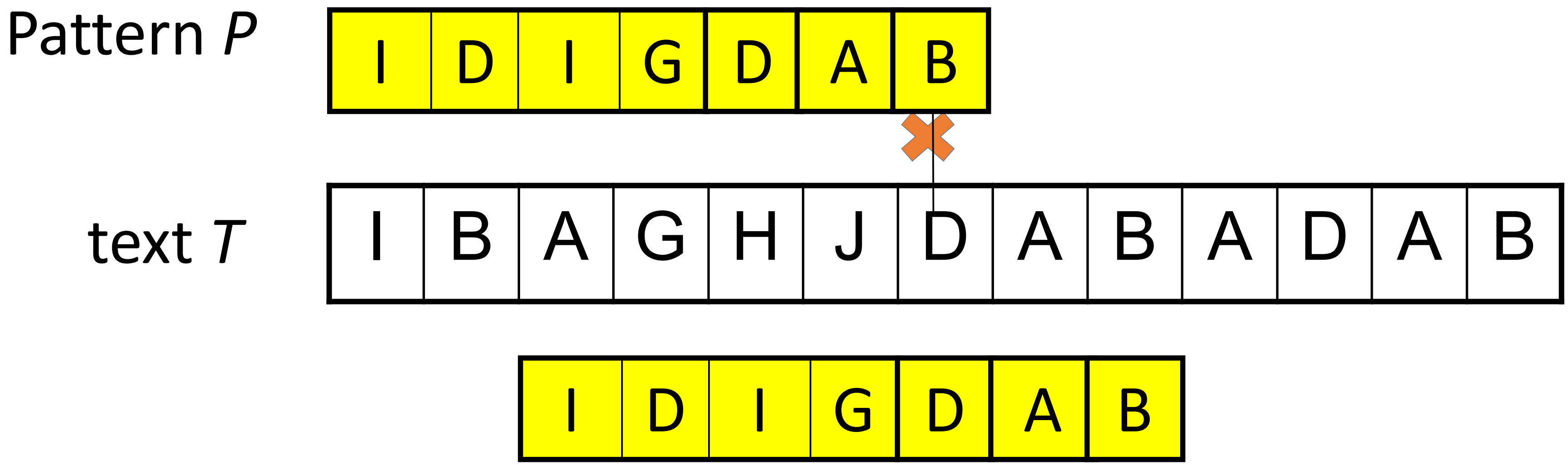


Table	1	7	7	2	7	7	3	7	4	7
	0	1	2	3	4	5	6	7	8	9
	“A”	“B”	“C”	“D”	“E”	“F”	“G”	“H”	“I”	“J”

Using the shift table …

- **Example:**

there is a mismatch, so we lookup **table[“B”]**, which returns **7**, so we shift by 7 .

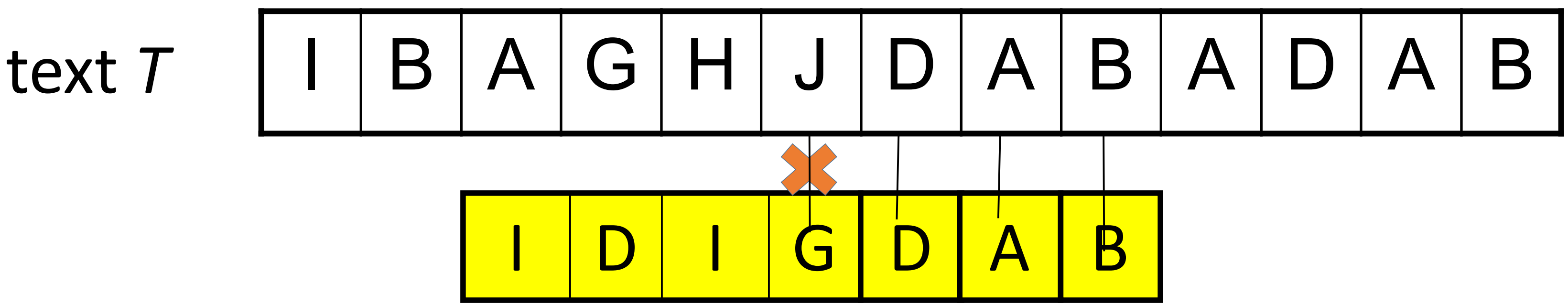


Table										
	1	7	7	2	7	7	3	7	4	7
	0	1	2	3	4	5	6	7	8	9
	“A”	“B”	“C”	“D”	“E”	“F”	“G”	“H”	“I”	“J”

More details about this algorithm in the textbook.

(it is called Horspool's algorithm)

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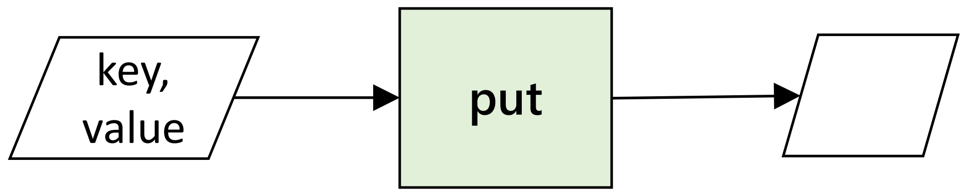
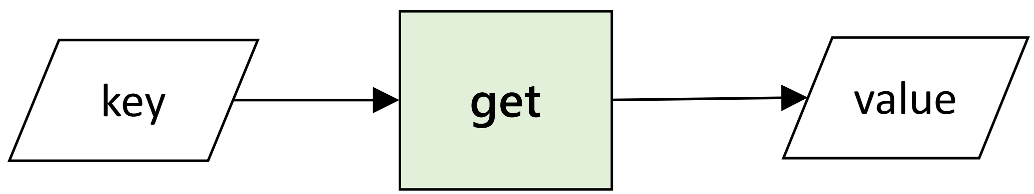
You know about HashMaps

- Map.put(key, value)



- value = Map.get(key)



Today we are looking inside these boxes.

Fast Storage of Keyed Records

Goal: want some way to do fast storage/lookups/retrieval of information, based on an arbitrary key

eg: `key = A00043526`
`value = Jimmy`

Let's consider traditional data structures ...

<u>Array:</u> How would you use an array (or arrays) to store this

- use either 2 1D arrays or 1 2D array or an array of objects
  - store key in a sorted array (for fast retrieve)
  - use the second array (or column) to store the record or a pointer to the record ... or ...
- alternatively, create an object 'Employee', and store in an array of objects

Using Sorted Array

Two 1D Arrays ...

1	A00043522
2	A00666666
3	
4	
	⋮
n-1	
n	

1	Jimmy
2	beelzebub
3	
4	
	⋮
n-1	
n	

One 2D Array ...

1	A00043522	Jimmy
2	A00666666	beelzebub
3		
4		
	⋮	⋮
n-1		
n		

Using Sorted Array (2)

Inserting a new element ... eg: ***insert(A00099999, "foo")***

1	A00043522	Jimmy
2	A00066666	beelzebub
3	A00100000	186A0
4	A00111111	Bob
5	A00123456	n(n+1)/2
6	A00444444	bertcubed
7	A00666666	Beelzebub
8		
9		
10		

Using Sorted Array (3)

Inserting a new element … eg: ***insert(A00099999, "foo")***

1	A00043522	Jimmy
2	A00066666	beelzebub
3	A00100000	186A0
4	A00111111	Bob
5	A00123456	n(n+1)/2
6	A00444444	bertcubed
7	A00666666	Beelzebub
8		
9		
10		

```
find location
     - (use binary search)
     - O(logn) operation
```

Using Sorted Array (4)

Inserting a new element … eg: ***`insert(A00099999, "foo")`***

1	A00043522	Jimmy
2	A00066666	beelzebub
3		
4	A00100000	186A0
5	A00111111	Bob
6	A00123456	n(n+1)/2
7	A00444444	bertcubed
8	A00666666	Beelzebub
9		
10		

`find location`

- `(use binary search)`
- `O(logn) operation`

`create space`

- `(move existing elements)`
- `O(n) operation`

Using Sorted Array (5)

Inserting a new element … eg: ***insert(A00099999, "foo")***

1	A00043522	Jimmy
2	A00066666	beelzebub
3	A00099999	foo
4	A00100000	186A0
5	A00111111	Bob
6	A00123456	n(n+1)/2
7	A00444444	bertcubed
8	A00666666	Beelzebub
9		
10		

find location
- **(use binary search)**
- **O(logn) operation**

create space
- **(move existing elements)**
- **O(n) operation**

put the new element
- direct access to array
- O(1) operation

Overall efficiency is:

$$O(\log n)+O(n)+O(1) = O(n)$$

Using Sorted Array (6)

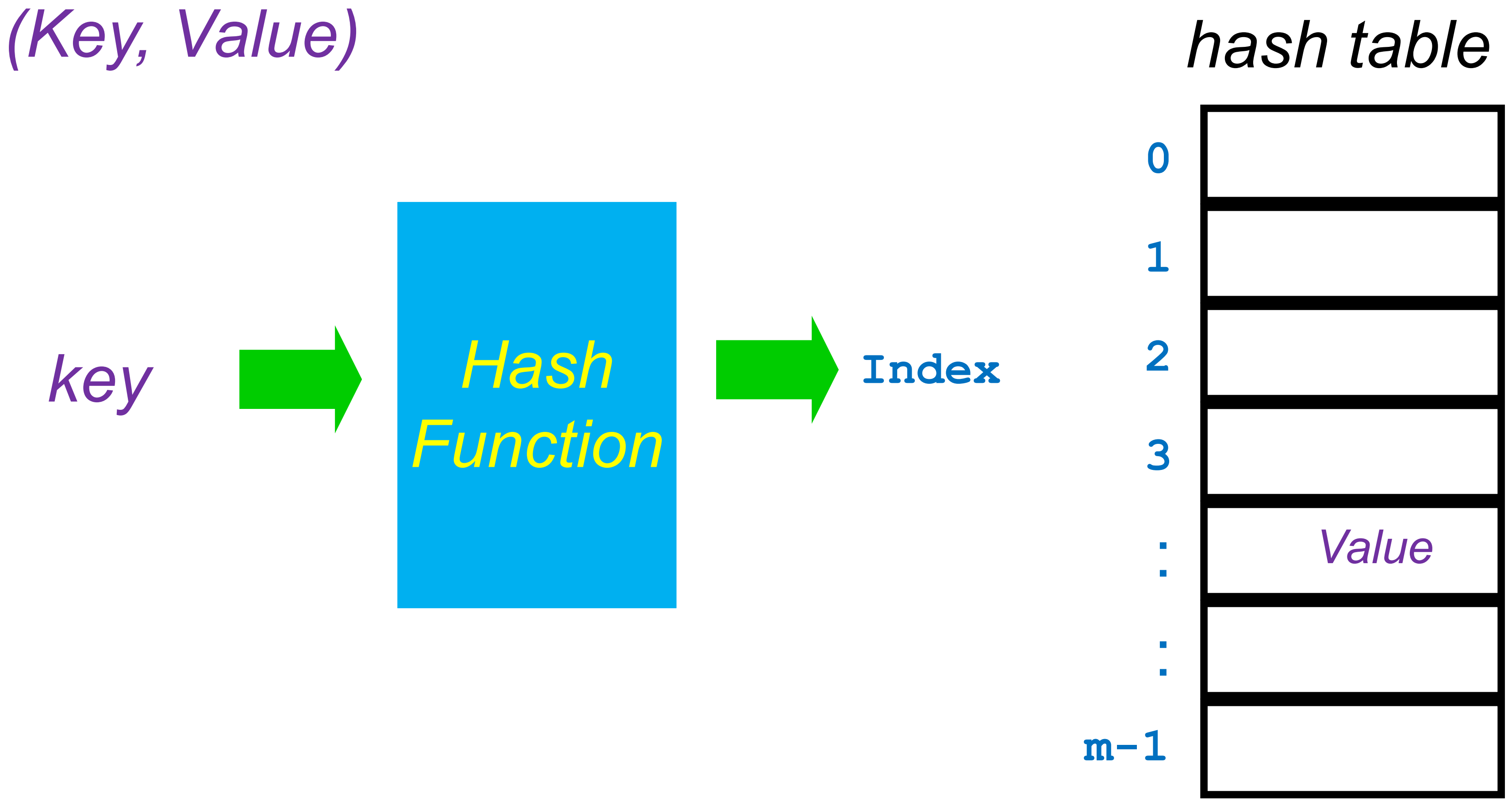
- *Search/retrieval* is $\boldsymbol{O(logn)}$
- *Insertion/deletion* is $\boldsymbol{O(n)}$

What if we use an unsorted Array:

- *Insertion* will be much faster – O(1)
- *Search, retrieve* will be slower – O(n)
- *Deletion* will be the same O(n)

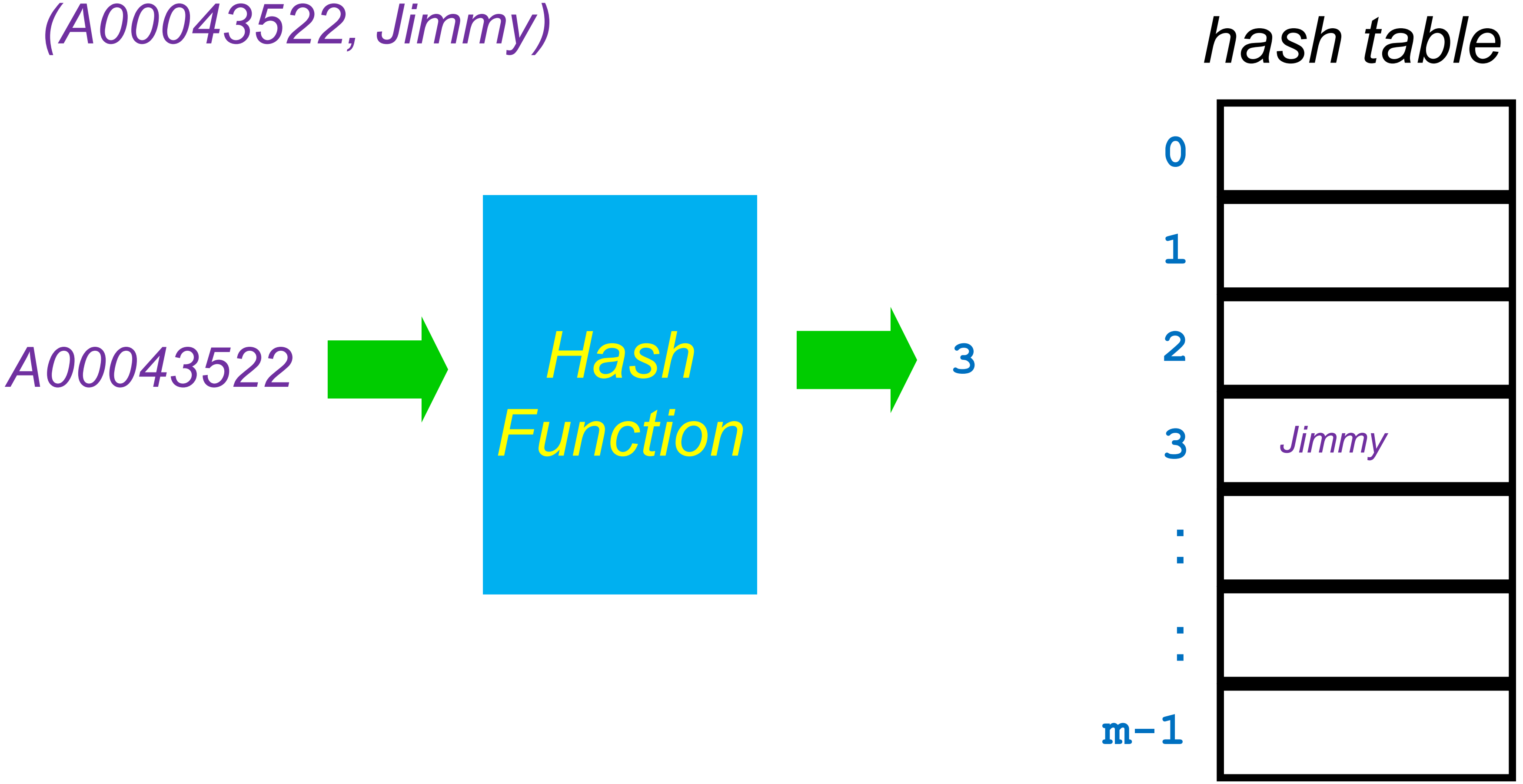
- *So how to get better performance ... ?*
    - *Hashing*

Hashing/ Hash Table



Example

(A00043522, Jimmy)



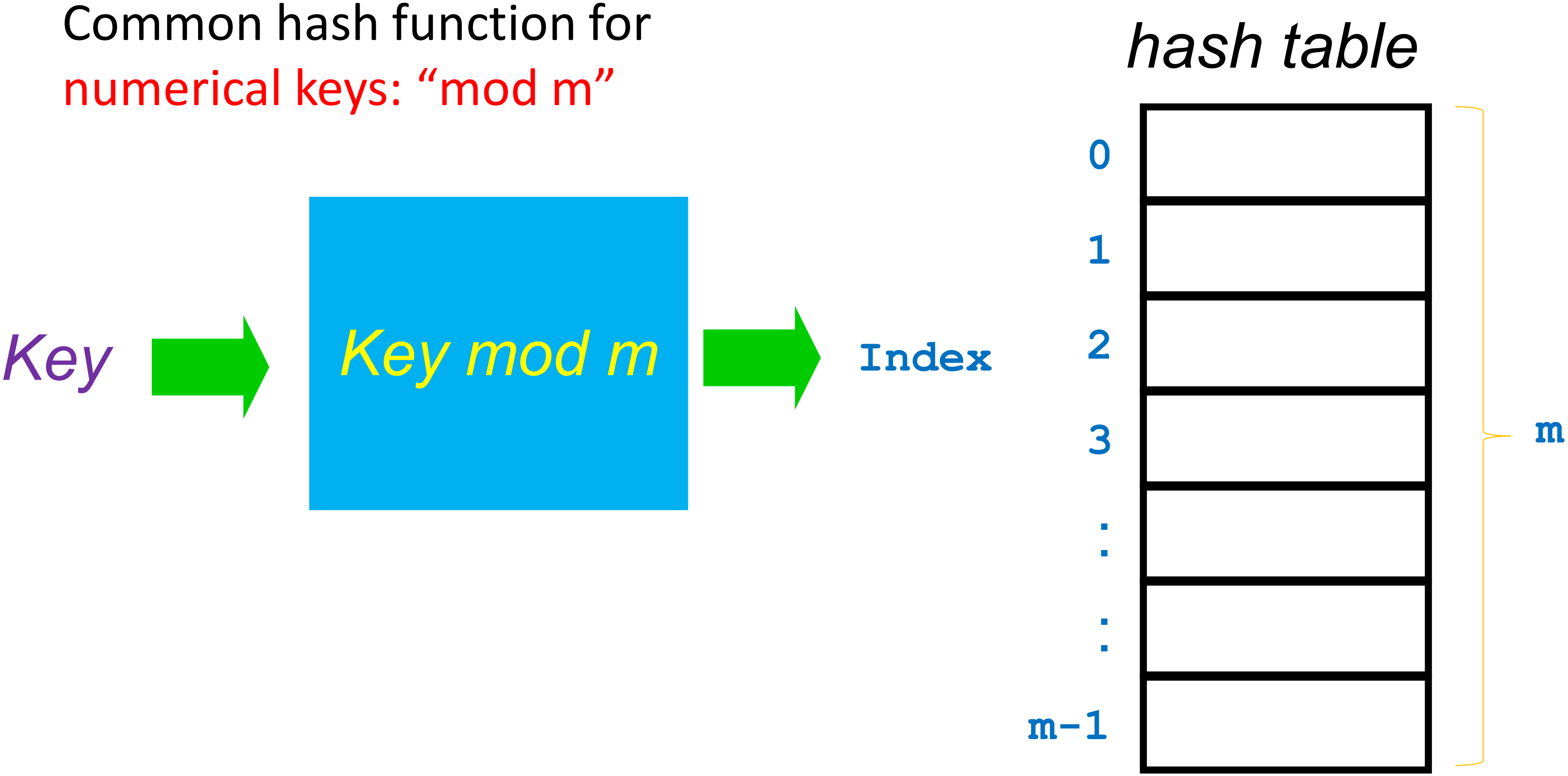
Hashing

- Each item has a unique key.
- Use a large array called a Hash Table.
- Use a Hash Function that maps keys to an index in the Hash Table.

`f(key) = index`

Hash Functions

Common hash function for
numerical keys: "mod m"

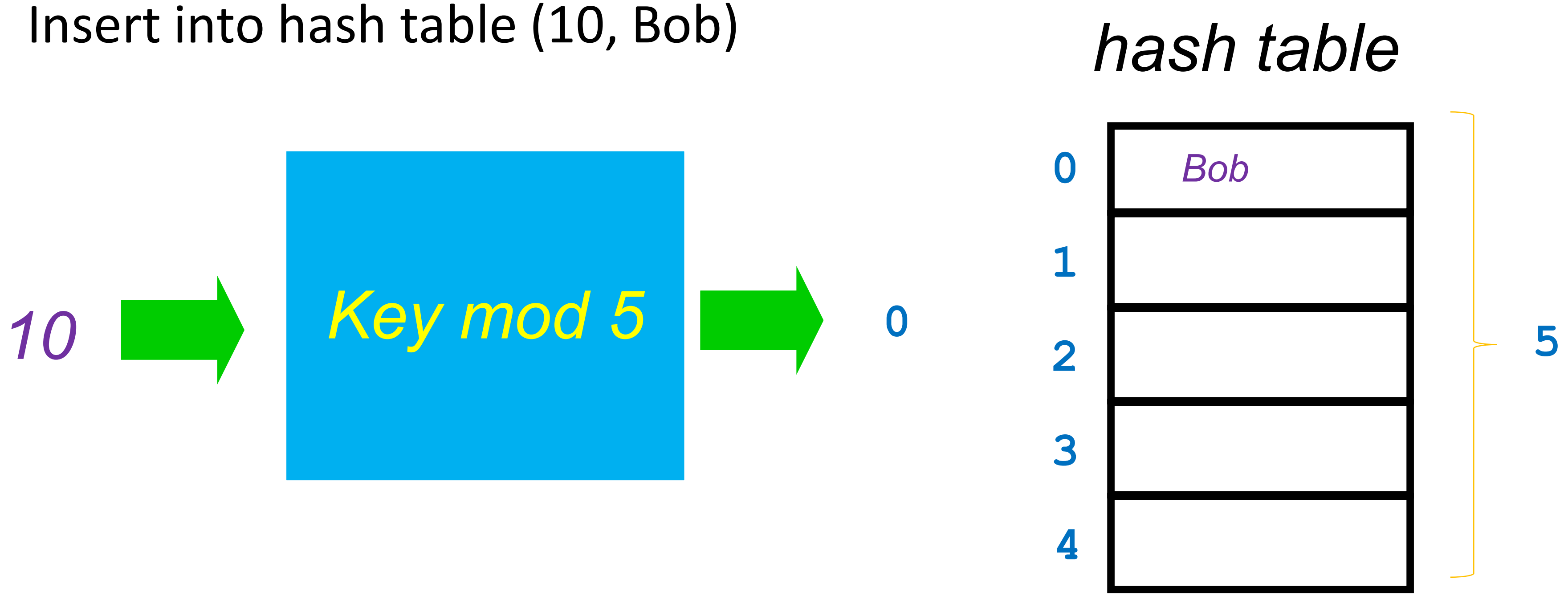


Hash Functions

Example

assume m=5

Insert into hash table (10, Bob)



Hash Functions

- What do we do if our key is not a number?
  - *answer: map it to a number!*

- Example

  assume m=5

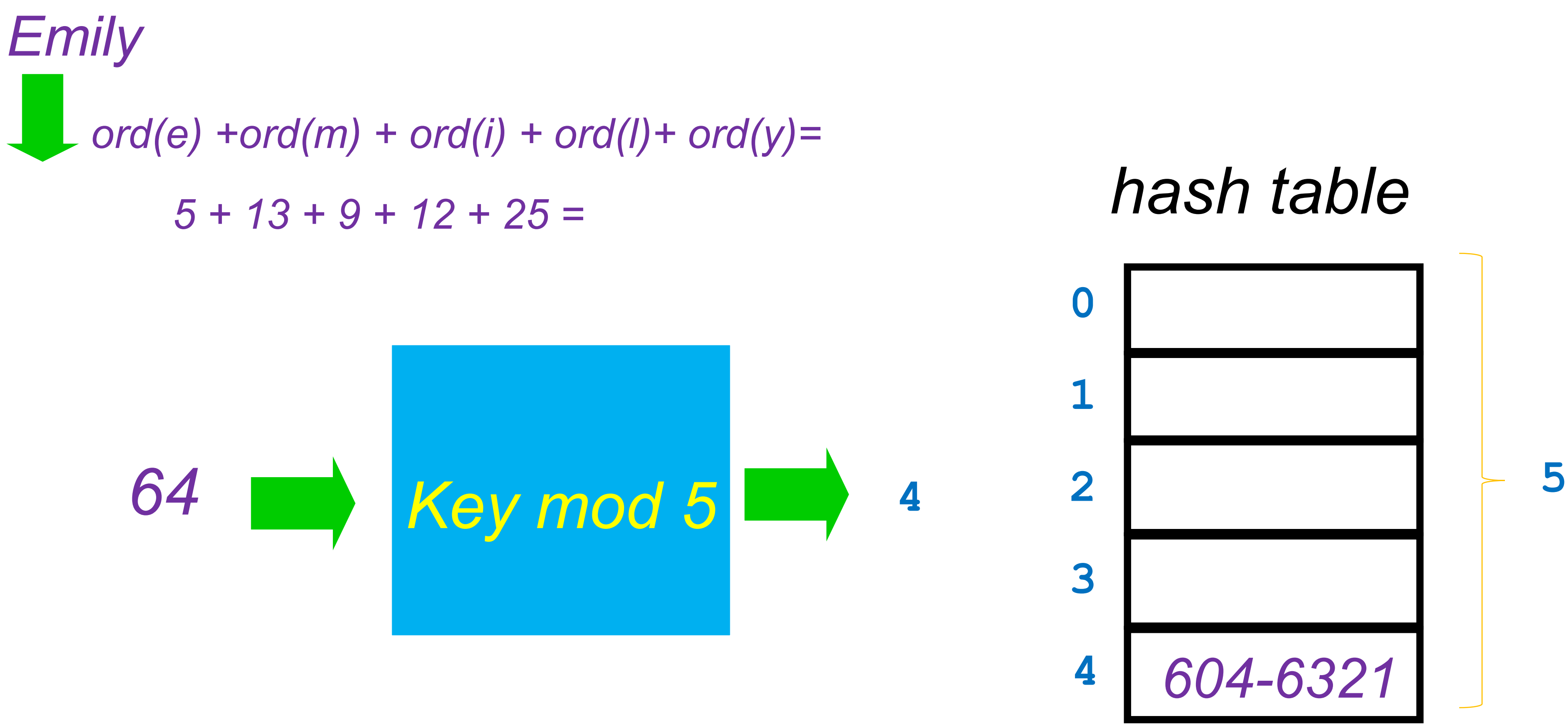
  Insert into hash table (Emily, 604-6321)

Hash Functions

Example

assume m=5

Insert into hash table (Emily, 604-6321)



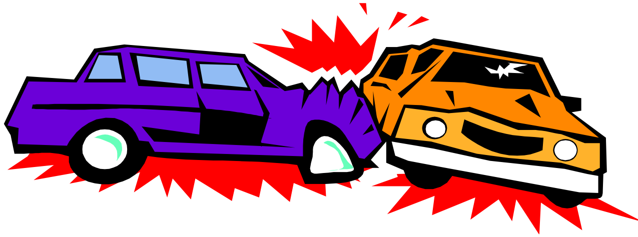
Hash Functions

- Sample hash function for string keys:

```
h ← 0                             // input is a string S of length s
for i ← 0 to s-1 do               // c_i is the char in i^th posn of S
    h ← h + ord(c_i)              // ord(c_i) is the relative posn
                                  //          of c_i in the alphabet
hashcode ← h mod numBuckets       // map sum of posns into range
```

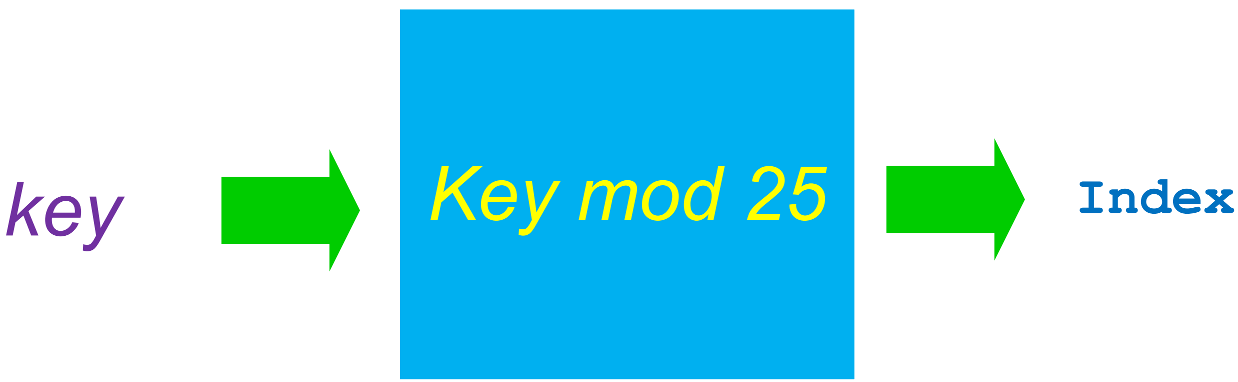
the actual hashcode depends on the number of buckets

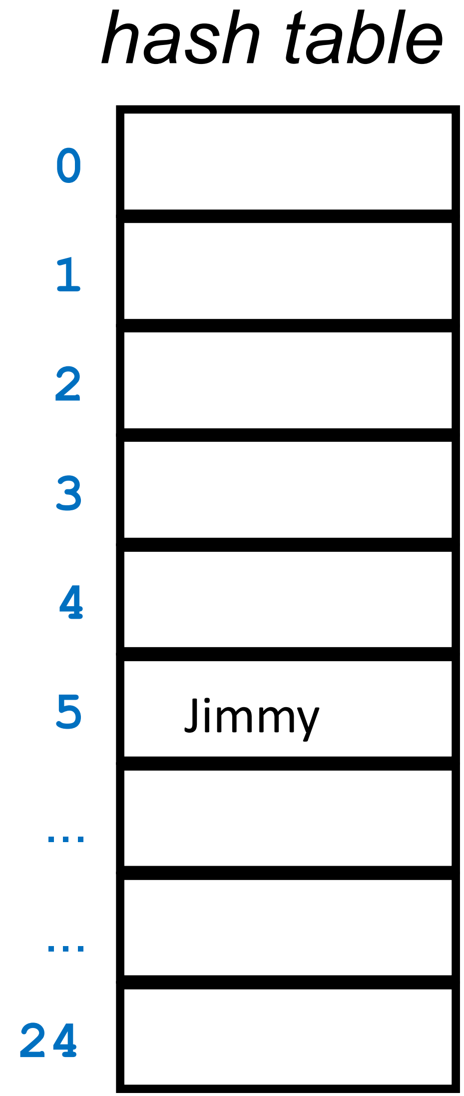
- This is similar to “H1” on the Lab



Collisions

Collisions occur when different keys are mapped to the same bucket





1.Insert into hash table (30, Jimmy)
index = 30 mod 25 = 5

2. Insert into hash table (105, Anthony)
index = 105 mod 25 = 5

Collision Handling

Two strategies to handle collisions:

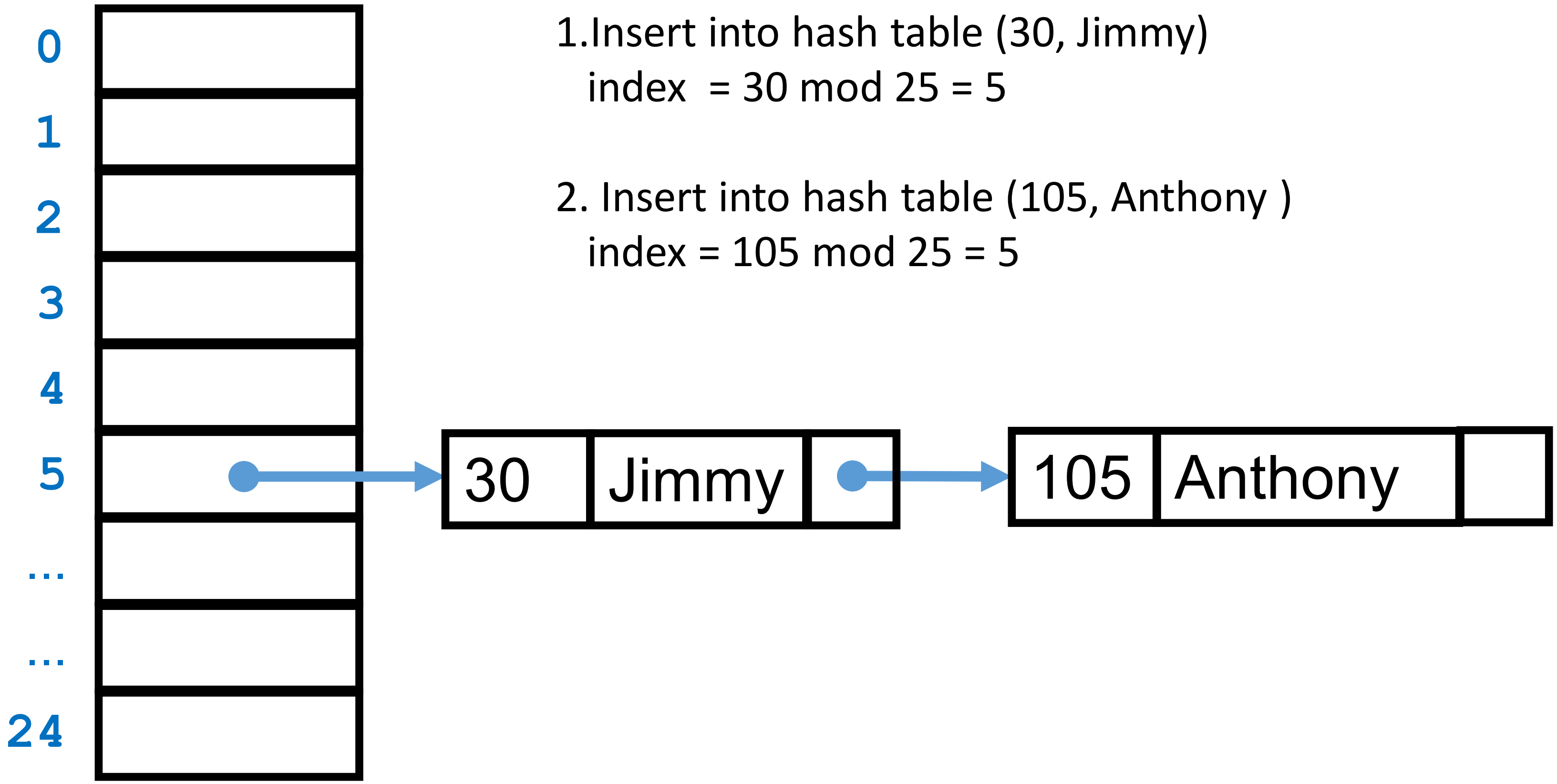
1. Separate Chaining
2. Closed Hashing

Collision Handling: Separate Chaining

- Each bucket in the table points to a *list* of entries that map there

1.Insert into hash table (30, Jimmy)
index = 30 mod 25 = 5

2. Insert into hash table (105, Anthony)
index = 105 mod 25 = 5



Separate chaining Example 1

- Use the hash function h(i) = i mod 7
- Draw the Separate chaining hash table resulting from inserting following keys and values:

  (44, red)

  (12, orange)

  (23, yellow)

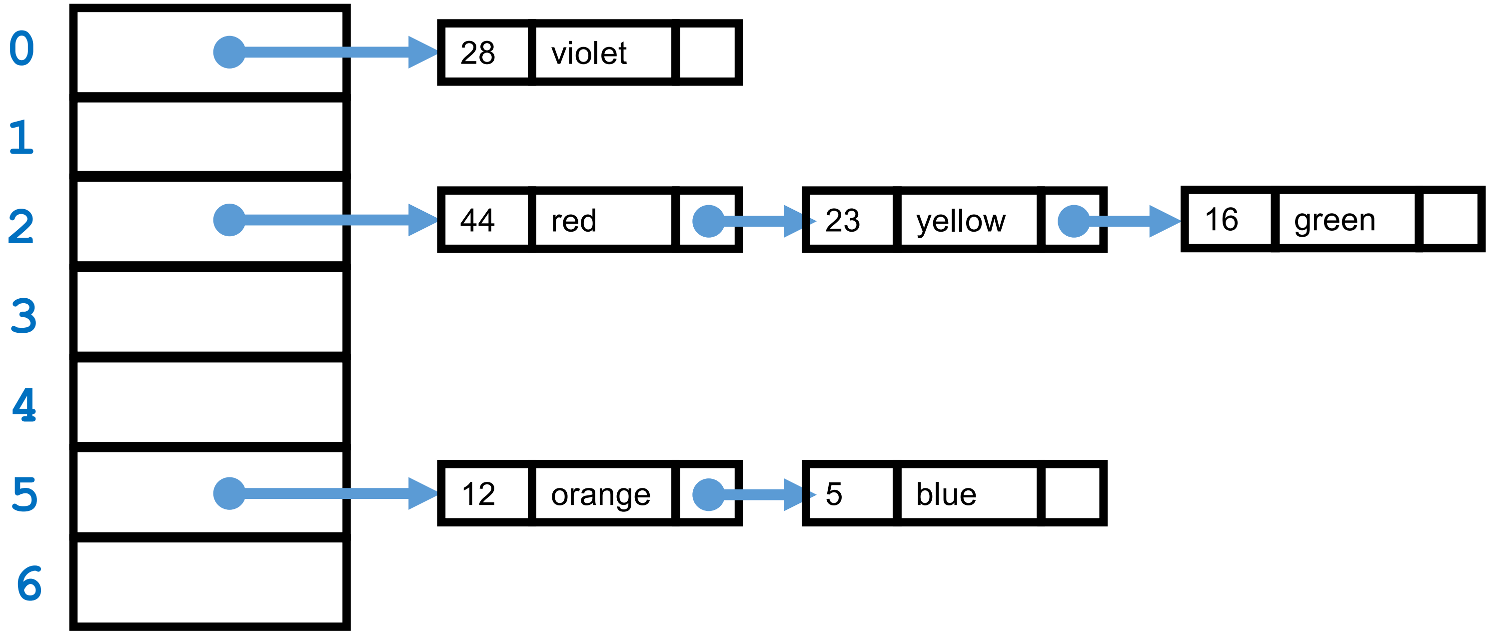
  (16, green)

  (5, blue)

  (28, violet)

(44, red)
(12, orange)
(23, yellow)
(16, green)
(5, blue)
(28, violet)

hash function h(i) = i mod 7

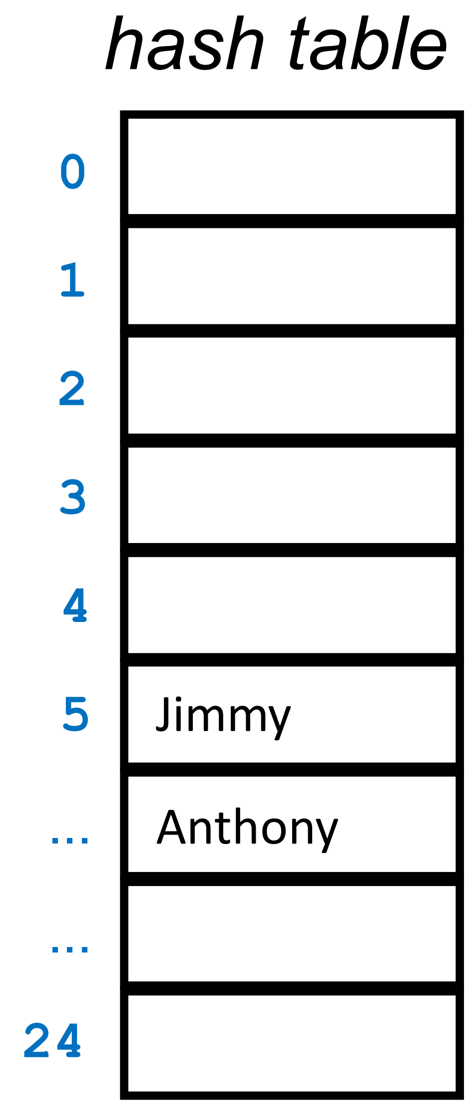


Strategy 2: Closed Hashing with Linear Probing

- It works like this:
  - compute the hash
  - if the bucket is empty, store the value in it
  - if there is a collision, linearly scan for next free bucket and put the key there
    - note: treat the table as a circular array

- Note: important - with this technique the size of the table must be at least n (or there would not be enough room!)

Linear Probing



1.Insert into hash table (30, Jimmy)
index = 30 mod 25 = 5

2. Insert into hash table (105, Anthony)
index = 105 mod 25 = 5

Closed Hashing Exercise

- Use the hash function h(i) = i mod 10
- Draw the hash table resulting from inserting following key and values:

(44, mojo)

(12, buzz)

(13, iggy)

(88, flem)

(23, sue)

(16, vern)

(22, sami)

(44, mojo)
(12, buzz)
(13, iggy)
(88, flem)
(23, sue)
(16, vern)
(22, sami)

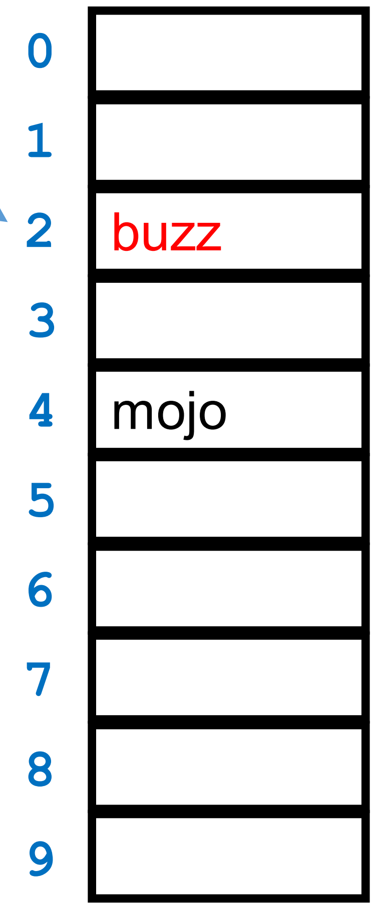
hash function h(i) = i mod 10

44 mod 10 = 4

(44, mojo)
(12, buzz)
(13, iggy)
(88, flem)
(23, sue)
(16, vern)
(22, sami)

hash function h(i) = i mod 10

12 mod 10 = 2



(44, mojo)
(12, buzz)
(13, iggy)
(88, flem)
(23, sue)
(16, vern)
(22, sami)

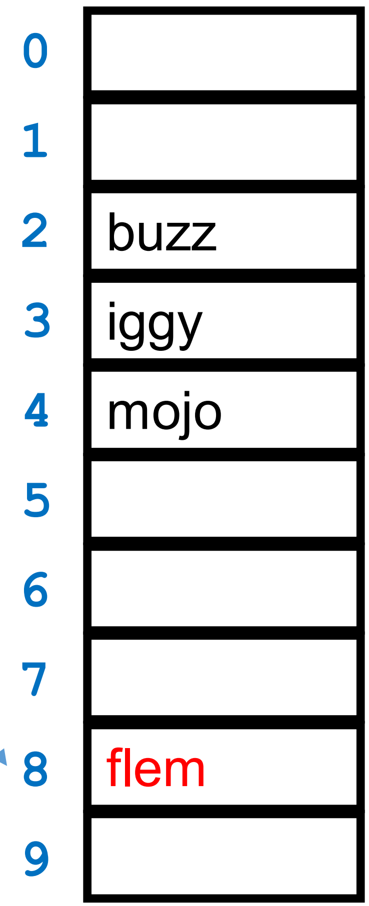
hash function h(i) = i mod 10

13 mod 10 = 3

(44, mojo)
(12, buzz)
(13, iggy)
(88, flem)
(23, sue)
(16, vern)
(22, sami)

hash function h(i) = i mod 10

88 mod 10 = 8

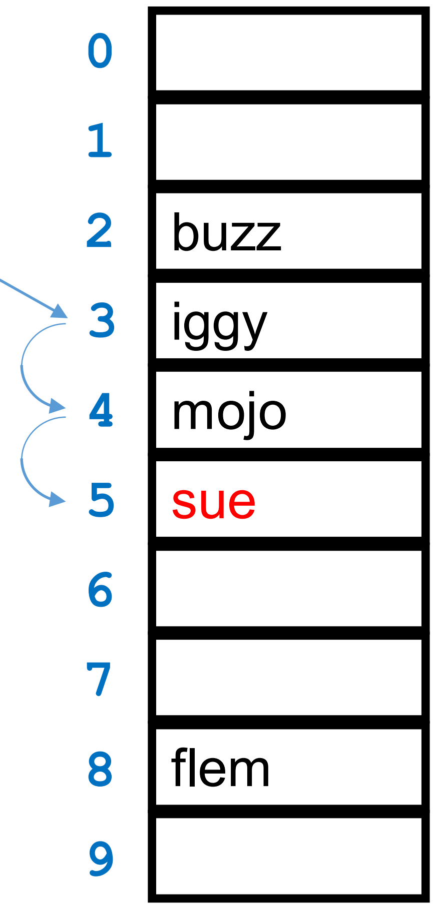


(44, mojo)
(12, buzz)
(13, iggy)
(88, flem)
(23, sue)
(16, vern)
(22, sami)

hash function h(i) = i mod 10

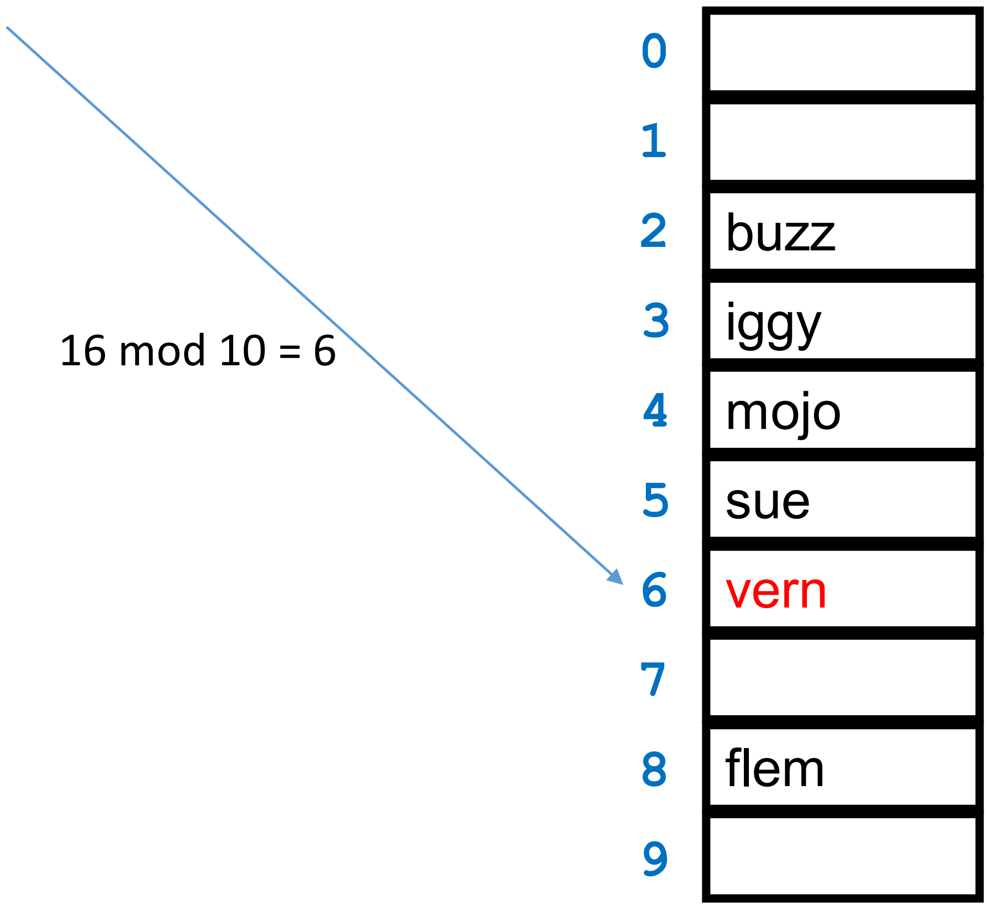
23 mod 10 = 3

Collision!
Followed by 2 probes



(44, mojo)
(12, buzz)
(13, iggy)
(88, flem)
(23, sue)
(16, vern)
(22, sami)

hash function h(i) = i mod 10

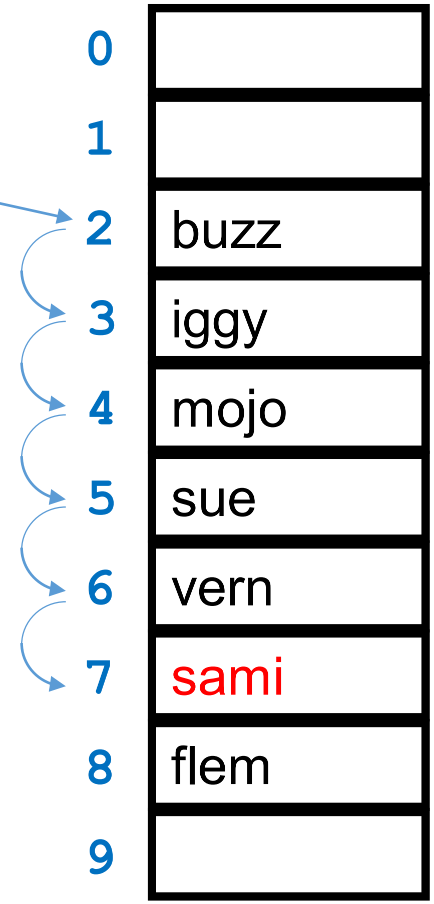


(44, mojo)
(12, buzz)
(13, iggy)
(88, flem)
(23, sue)
(16, vern)
(22, sami)

hash function h(i) = i mod 10

22 mod 10 = 2

Collision!
Followed by 5 probes



Efficiency of Hashing

What is the efficiency of the hashtable structure?

- **insert**(key, value) ... is $\boldsymbol{O(1)}$
- value $\leftarrow$ **get**(key) ... is $\boldsymbol{O(1)}$
- **delete**(key) ... is $\boldsymbol{O(1)}$

- of course there could always be a degenerate case, where (almost) every insert causes a collision to be handled. We could end up with O(n) or even worse.

$\rightarrow$***conclusion : implementation of the hashing function is important***

$\rightarrow$***it must distribute the keys evenly over the buckets***

Hash Functions

- the efficiency of hashing depends on the quality of the **hash function**

  A "good" hash function will

  1. distribute the keys uniformly over the buckets
  2. produce very different hashcodes for similar data

- hashing of numbers is relatively easy, as we just distribute them over the buckets with

  key mod *numBuckets*

Hashing Strings

- most keys are Strings, and Strings are a bit trickier
  - consider the simple algorithm:

```
h ← 0
for i ← 0 to s-1 do
    h ← h + ord(c_i)    // ord(c_i) is the posn of char i
code ← h mod numBuckets
```

- Is that a good hash function?
  - sample: assume numbuckets = 99
    - hash("dog") = 26
    - hash("god") = 26
    - hash("add") = 9
    - hash("dad") = 9

Better String Hash Function

- a better hashcode algorithm for strings

```
alpha  ← |alphabet|       // size of the alphabet used
h ← 0
for i ← 0 to s-1 do
    h ← h + (ascii(c_i) * alpha^(i))   ascii num * char position in the string
code ← h mod numBuckets
```

  - Assuming alpha = 128 (number of ascii codes)
  - Assuming numbuckets = 99
    - dog = 64
    - god = 46
    - add = 26
    - dad = 65

- This is similar to our "H2" on the Lab

Practice problems

1. Chapter 7.1, page 257, questions 3, 7
2. Chapter 7.2, page 267, question 1,2
3. Chapter 7.3, page 275, question 1,2,7