Next week's quiz

- Online again (save the trees!)
- But it will be *different!*
- JUST ONE QUESTION
- A *problem* where you have to write (the idea of) an algorithm

Lecture 8

COMP 3760

Solving problems with graph algorithms

Topological Sorting (Text chapter 4.2)

Greedy Algorithms (Text chapter 9)

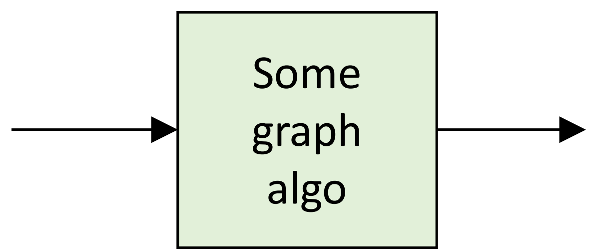
Solving problems with graph algorithms

How can we use graph algorithms to solve problems

Two strategies

1. Modify a known graph algorithm
    - Technically we have already done this
    - DFS and BFS in class notes did not perform output
    - But we tracked it in the examples

2. Use a known graph algorithm as a black box
    - Black box needs input
    - Black box gives output



- Bonus strategy: 

Example: Connected components

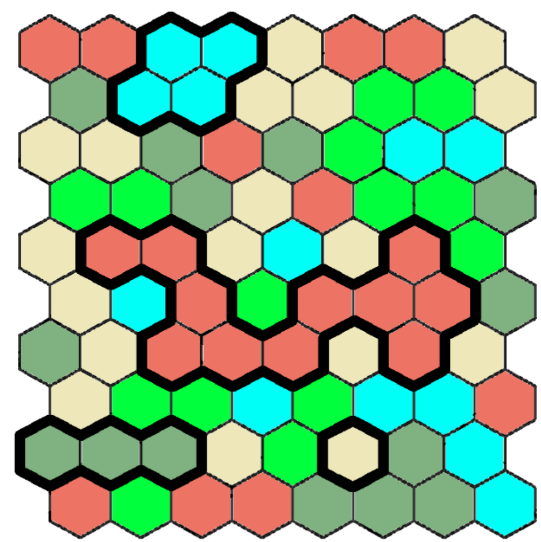
- Problem: Given a graph G, how many *connected components* does G have?

- Strategy 1: *Modify a known graph algorithm*

- Solution idea: Use either DFS or BFS
  - Add a counter to the "main loop"
  - Count how many times (from main) the helper function is called
  - Return the counter

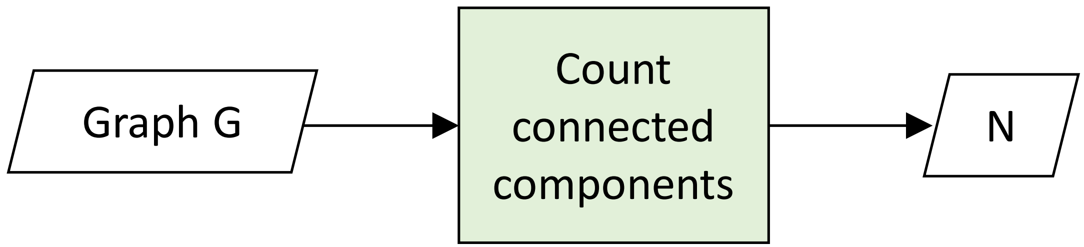
Example 2: Count map regions

- A board game is played on hex tiles.
- Tiles are drawn at random from a large supply.
- Adjacent tiles of the same colour make up a *region*.
- Input data is a list of the colours for all tiles:
    - colour(0,0) = red
    - colour(4,4) = aqua
- Problem: determine how many regions are on the map.
    - This map has 38 regions →



Solution idea

- Strategy 2: Use the "connected components" algorithm as a black box

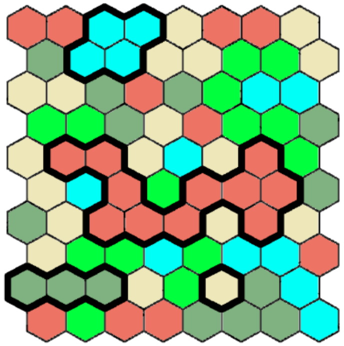


- We need to construct a clever graph – *just the right graph*
  - It will encapsulate or model or represent our input problem in some way

Finding the right graph

We are given a list of colour/point data:

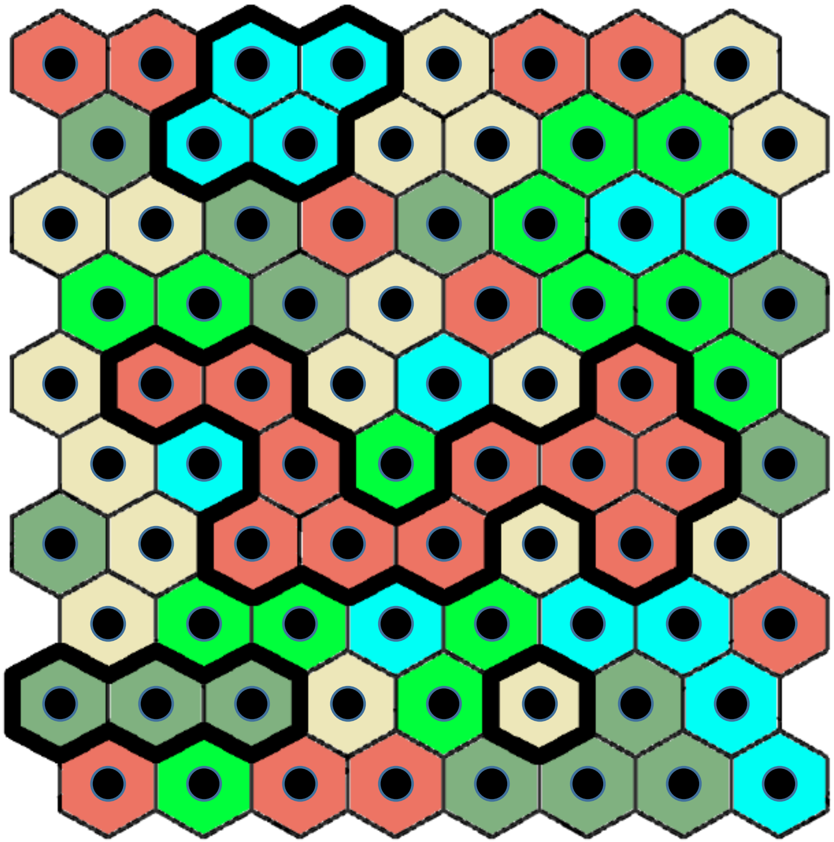
colour(0,0) = red
colour(0,1) = red
colour(0,2) = aqua
colour(0,3) = aqua
colour(0,4) = beige
...
colour(9,7) = aqua



- For a graph we need vertices and edges
- Vertices represent *things* and edges represent *relationships between things*

- The things we have are tiles
  - Idea: represent each tile as a vertex
  - Every vertex "label" will be a point (the grid location)

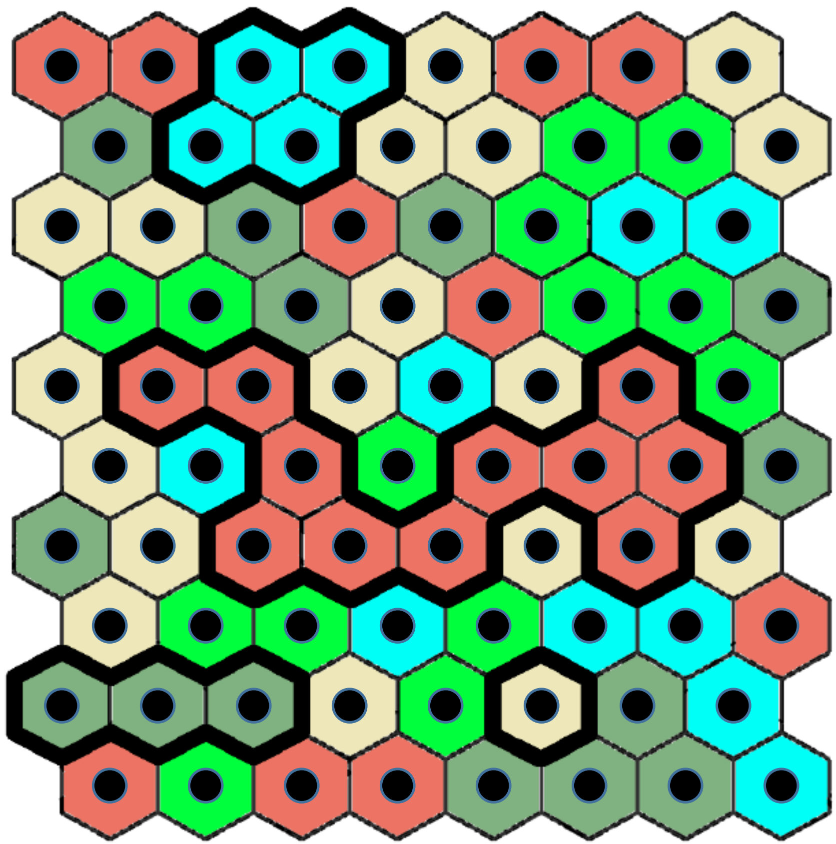
Every tile is a vertex



V = {(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (1,0), (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (2,0), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (3,0), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (3,7), (4,0), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (4,7), (5,0), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (5,7), (6,0), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6), (6,7), (7,0), (7,1), (7,2), (7,3), (7,4), (7,5), (7,6), (7,7), (8,0), (8,1), (8,2), (8,3), (8,4), (8,5), (8,6), (8,7), (9,0), (9,1), (9,2), (9,3), (9,4), (9,5), (9,6), (9,7) }

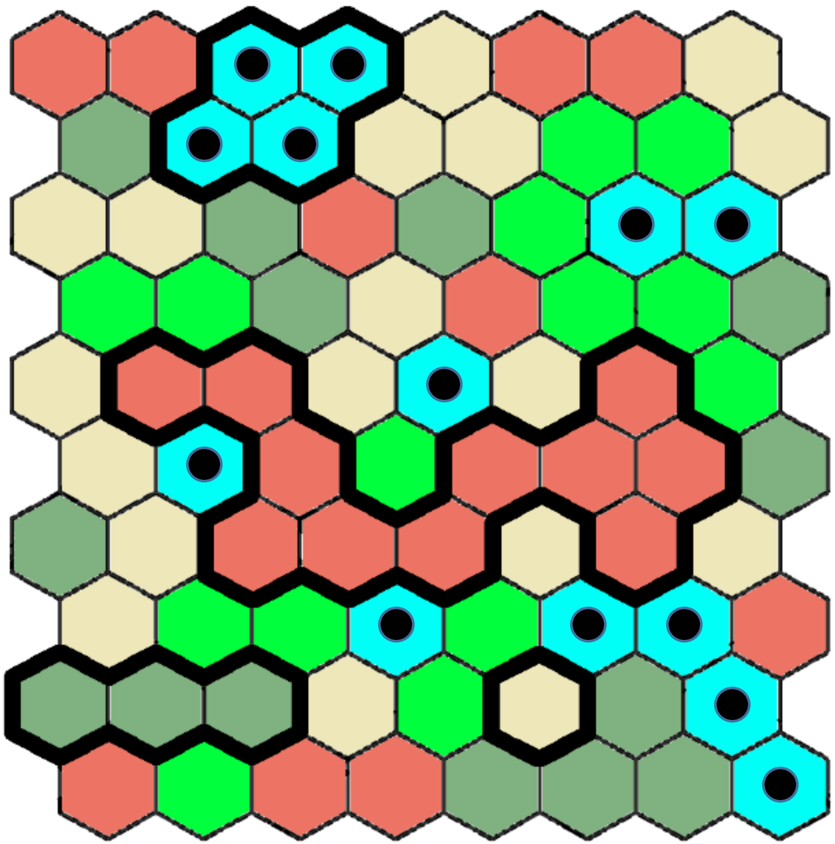
Now how about edges?

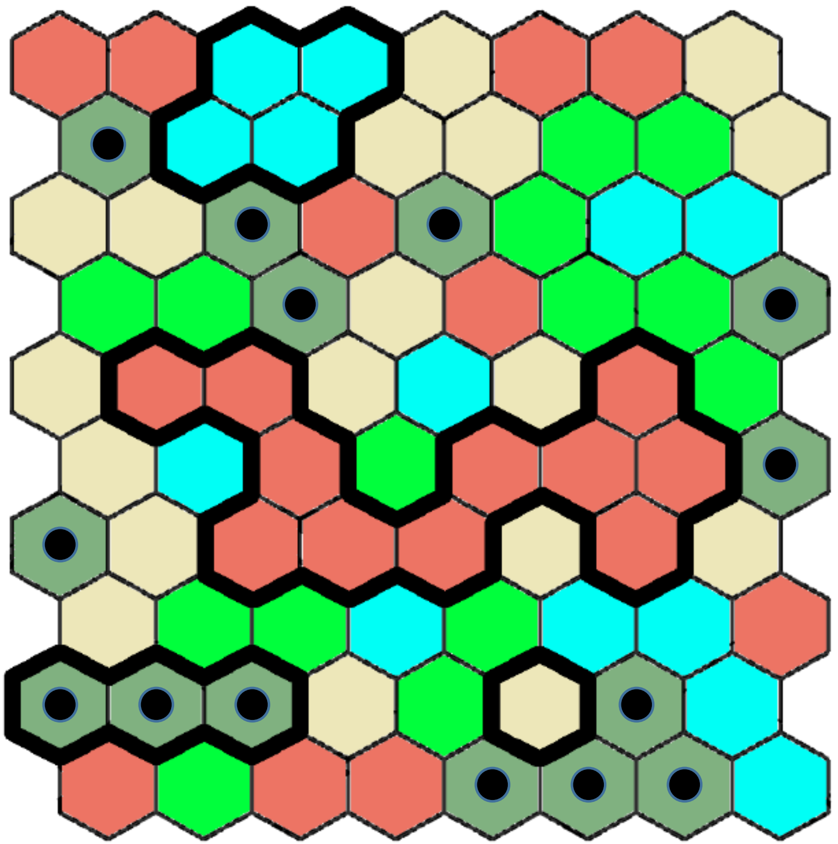
- Idea: Connect vertices of the same colour

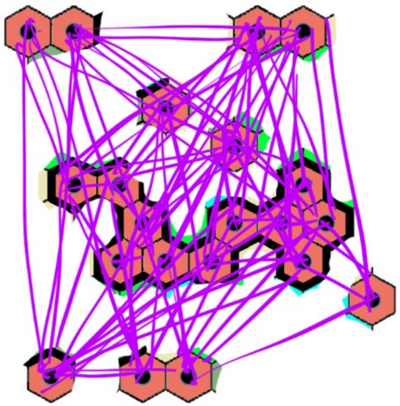


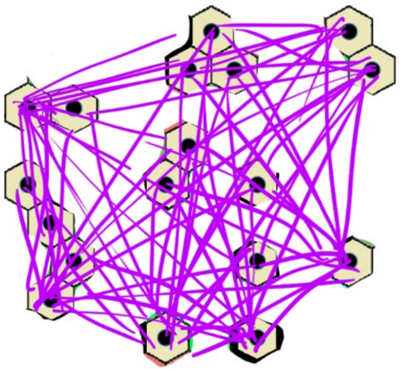
Now how about edges?

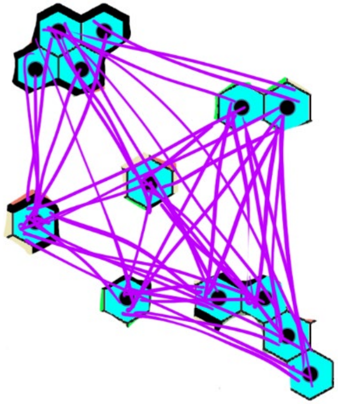
- Idea: Connect vertices of the same colour

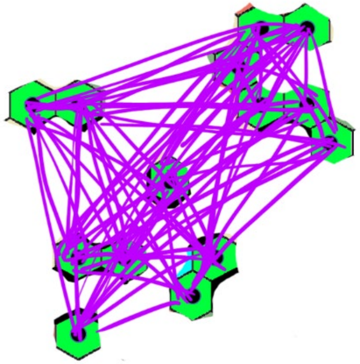


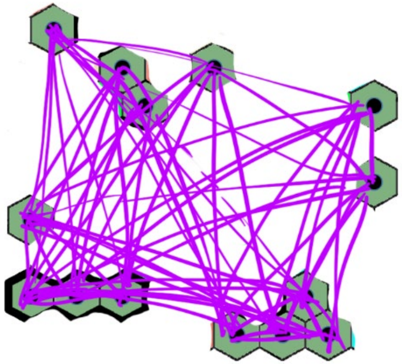




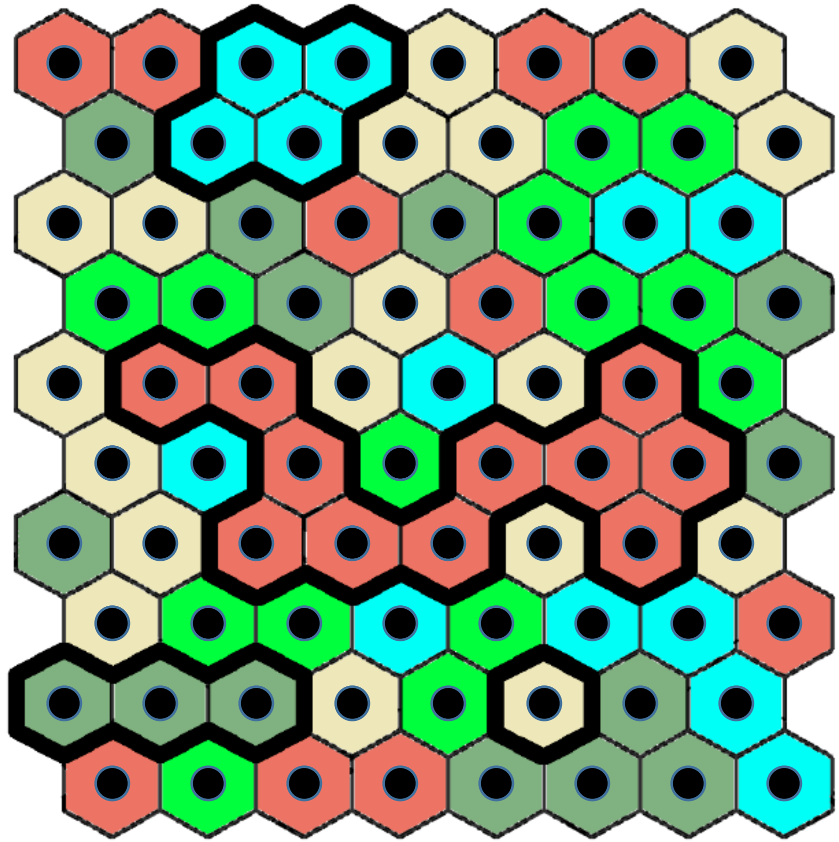




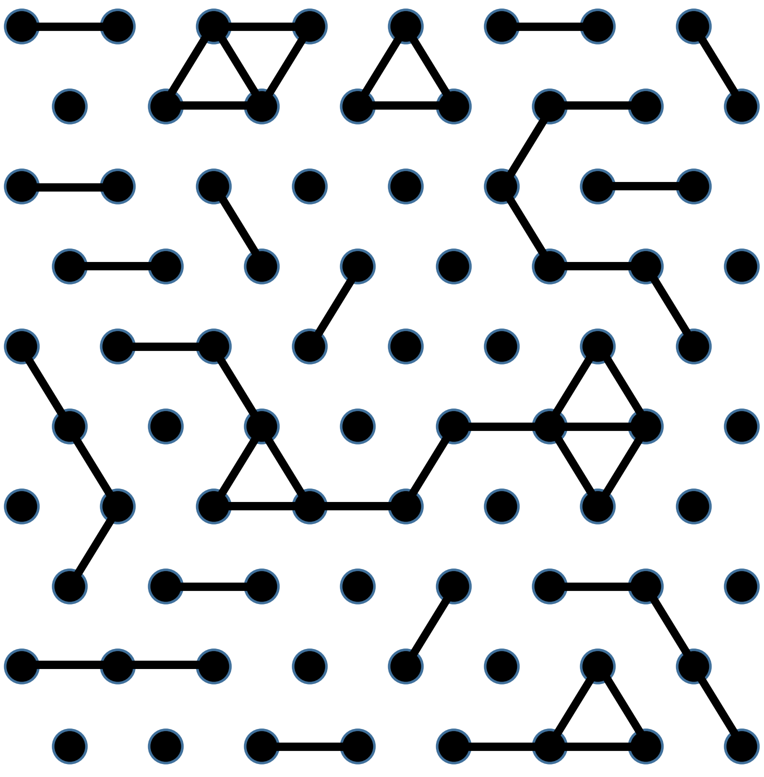




Idea 2: Same colour AND adjacent



Idea 2: Same colour AND adjacent



The Solution

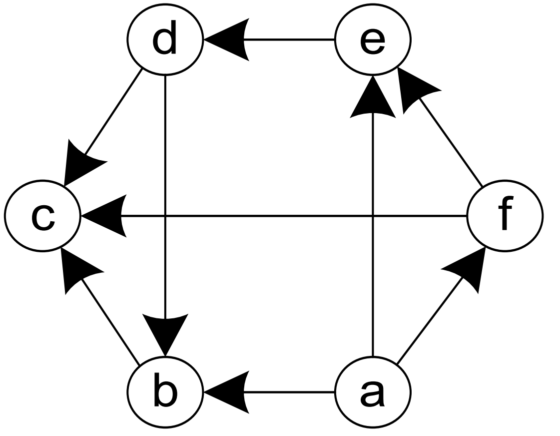
- Step 1: Define a Graph G=(V, E) as follows:
  - The vertices of G are the tiles of the map; each is represented by a grid location aka point (0,0) to (m,n)
  - There is an edge between two vertices u and v iff
    - u and v have the same colour AND
    - u and v are in adjacent locations
- Step 2: Run “Count Connected Components” on G
- Step 3: The output of step 2 is the final answer

Graph Algorithms: Topological sorting

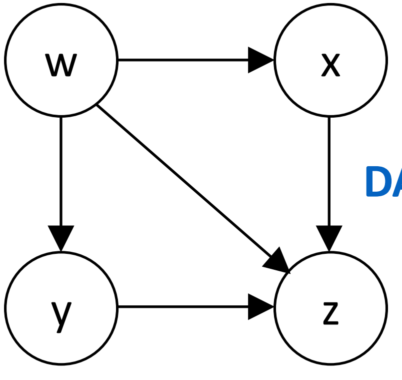
Textbook: Chapter 4.2

Directed acyclic graphs (DAGs)

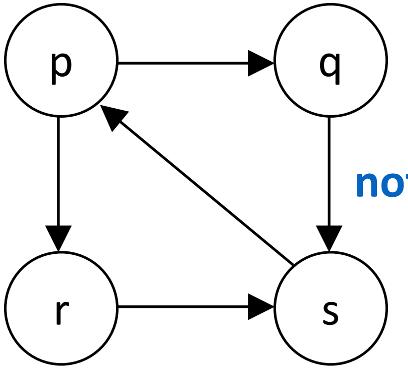
- A **directed graph** is a graph whose edges are directional or one-way



- A **directed acyclic graph** is a directed graph that contains no cycles



DAG



not a DAG

Topological sort problem

- Given a set of tasks with dependencies (precedence constraints), *e.g.*, “task A must be completed before task B”, …
- … find a linear ordering of the tasks that satisfies all dependencies

Example: Getting dressed

- Suppose you need to wear all these items:

  - Belt
  - Jacket
  - Pants
  - Shirt
  - Shoes
  - Socks
  - Suspenders
  - Tie
  - Underwear

- Some of these items must come before others

Example: Getting dressed

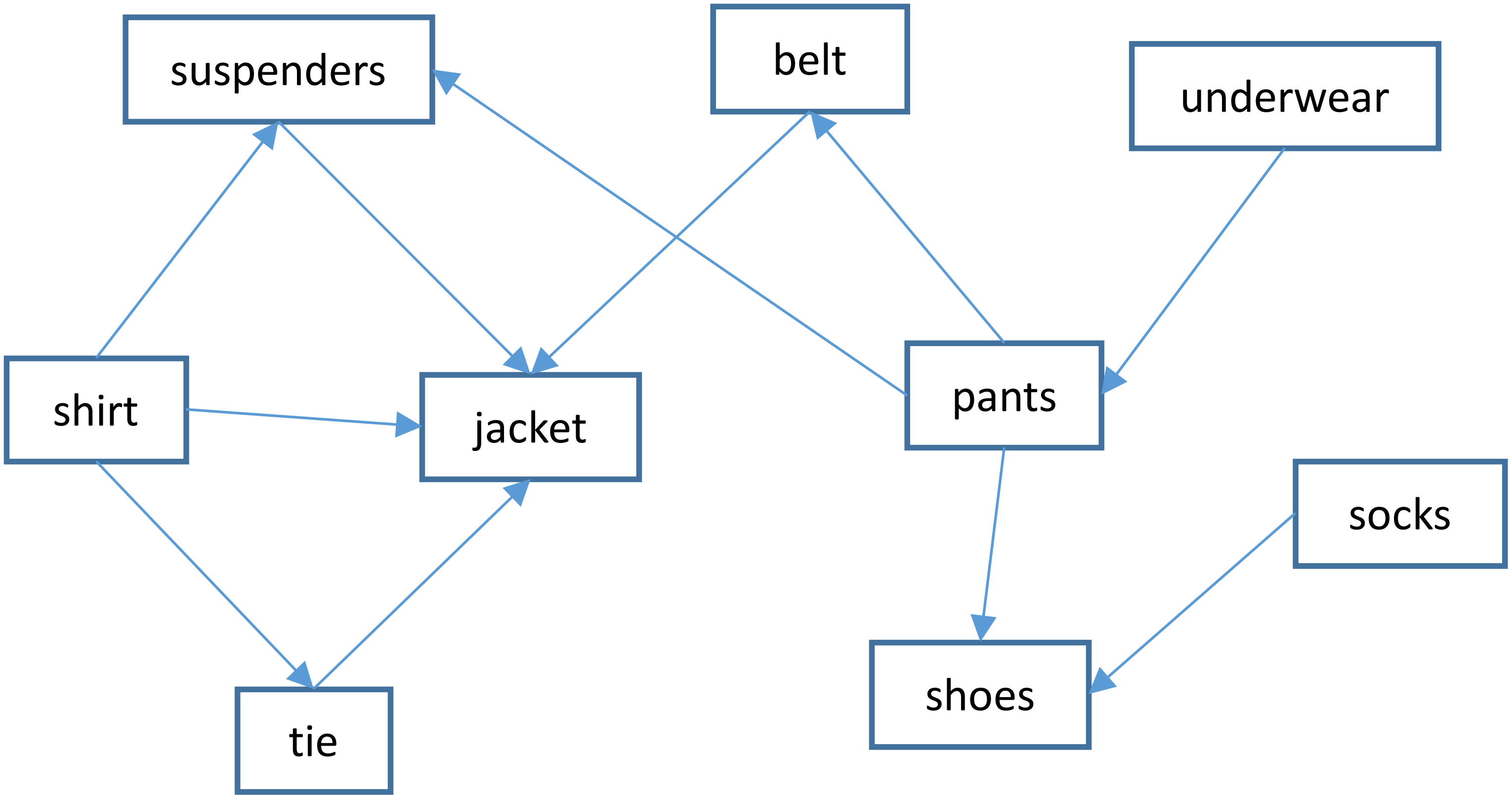
- Socks before shoes
- Shirt before suspenders
- Pants before suspenders
- Pants before shoes
- Pants before belt
- Shirt before tie
- Shirt before jacket
- Suspenders before jacket
- Belt before jacket
- Tie before jacket
- Underwear before pants

Represent the problem as a graph

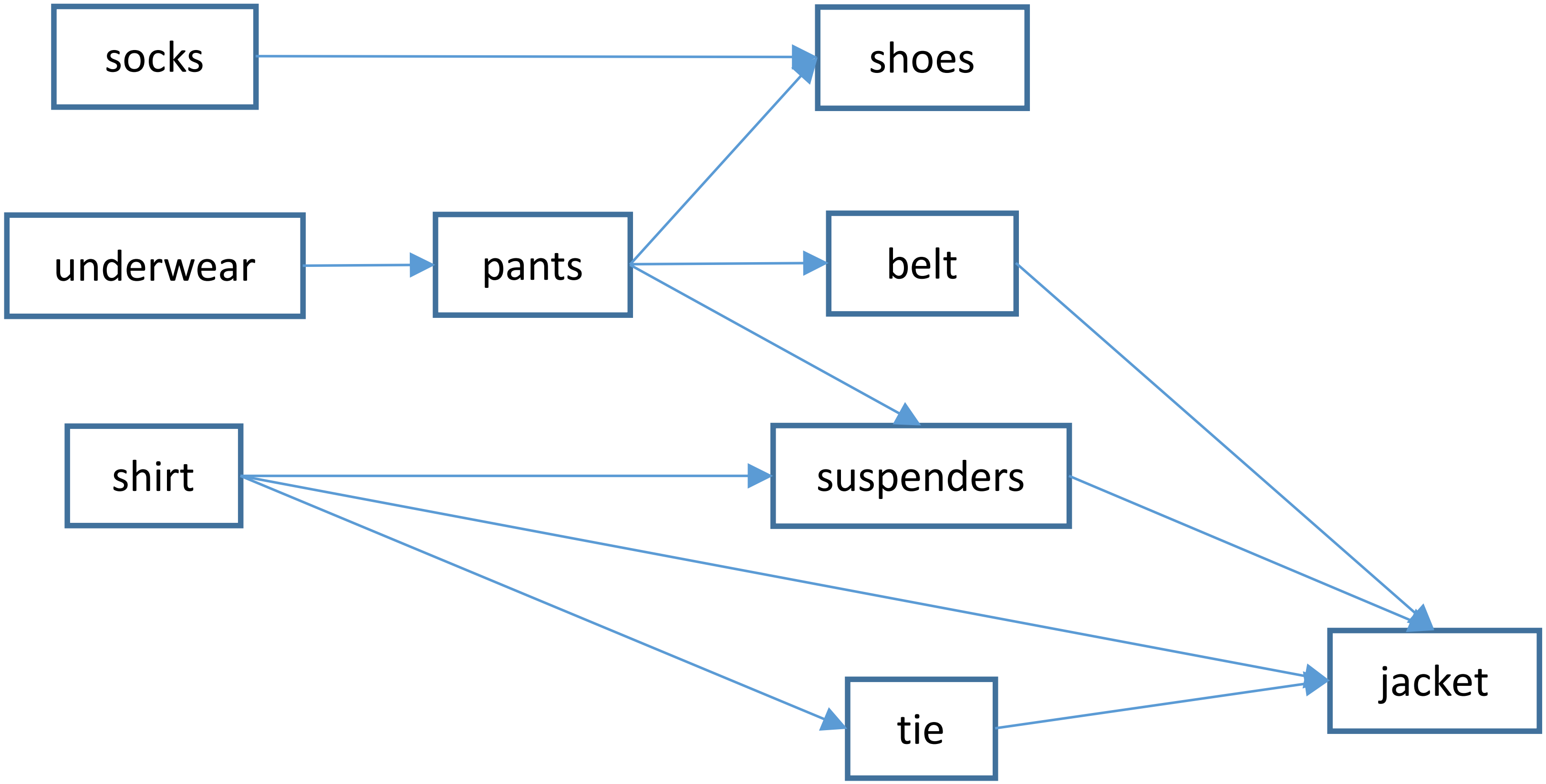
1. V = vertices are the items (tasks)
2. E = edges are the dependencies (constraints) between tasks

- an edge ($v \rightarrow w$) means:
  - w is dependent on v, OR (in other words)
  - Task v comes before task w

Clothing graph



Eyeballing a solution



socks … underwear … shirt … pants … shoes … belt … suspenders … tie … jacket

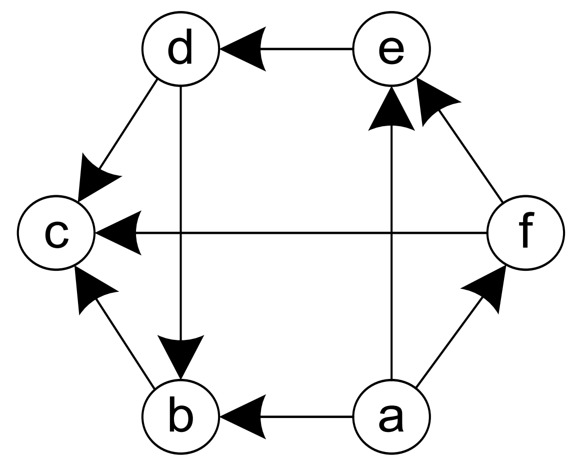
Topological Sort Algo 1: Use Depth First Search

1. Apply DFS to G
   - Starting at any vertex
   - No, really: ANY vertex
2. The order in which vertices become dead ends is the *reverse* of a topological sort order
   - Why?

Example 1

- Assume you have a set of 6 tasks (a, b, c, d, e, f) with the following dependencies:
  - a must be done before b, e, f
  - b must be done before c
  - d must be done before b and c
  - e must be done before d
  - f must be done before c and e

- <u>Step 1</u>: Construct a directed graph to represent the problem (verify it is a DAG)

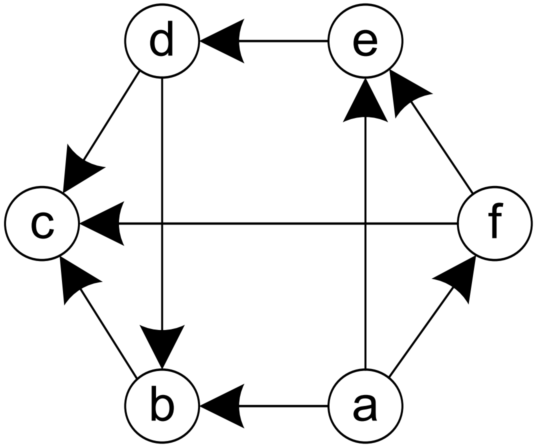


Example 1 (cont)

- <u>Step 2</u>: Apply DFS

  Order vertices become dead ends:

  c b d e f a



- <u>Step 3</u>: Reverse this to get topological sort order:

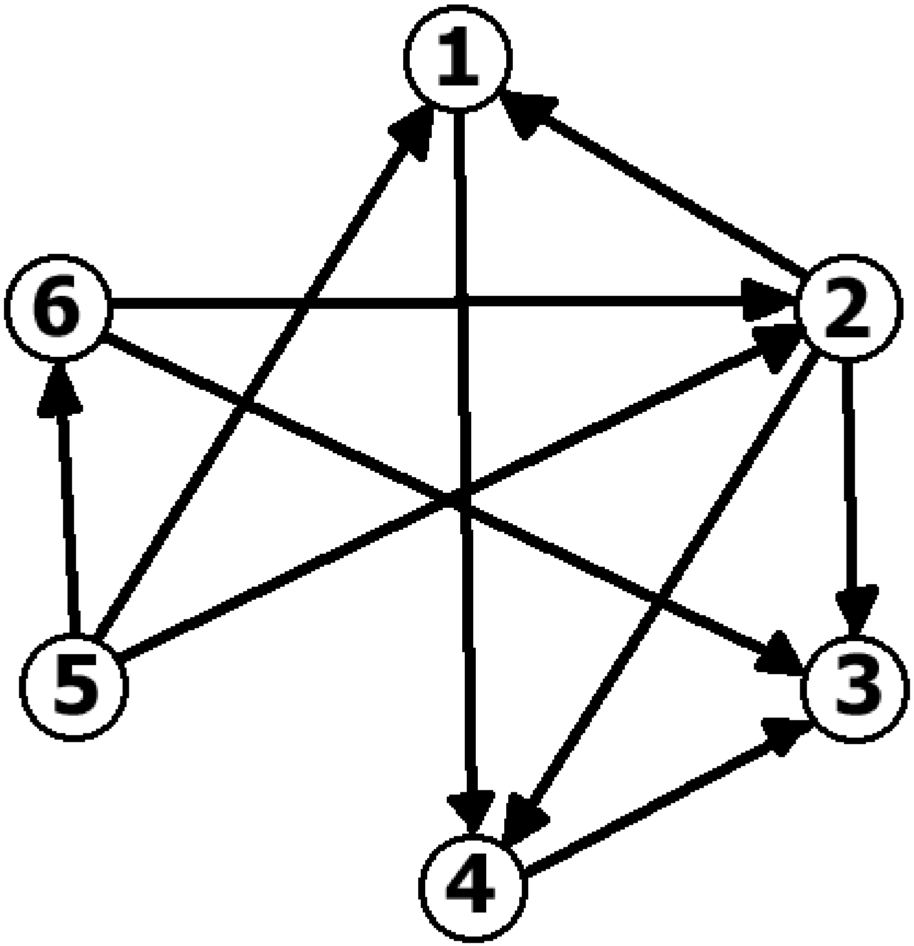
  a f e d b c

Example 2

2 1 (2 before 1)	4 3 (4 before 3)
1 4 (1 before 4)	5 2 (5 before 2)
2 3 (2 before 3)	5 1 (5 before 1)
5 6 (5 before 6)	6 3 (6 before 3)
2 4 (2 before 4)	6 2 (6 before 2)

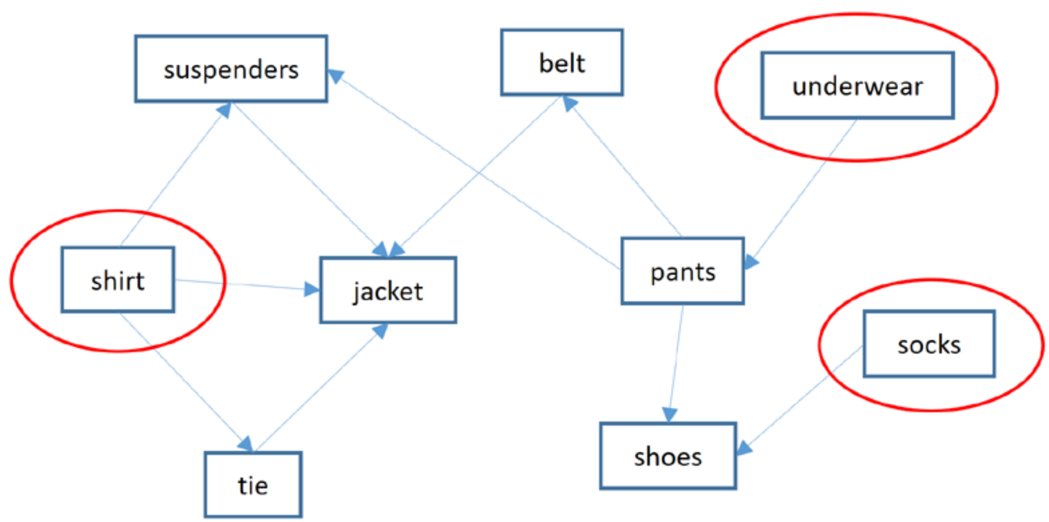
- Step 1: draw the graph (and verify it is a DAG)
- Step 2: apply DFS, get “dead-end” order
- Step 3: reverse this to get topological sort order

Example 2 (cont)



TopoSort Algorithm 2: Decrease (by 1) and conquer

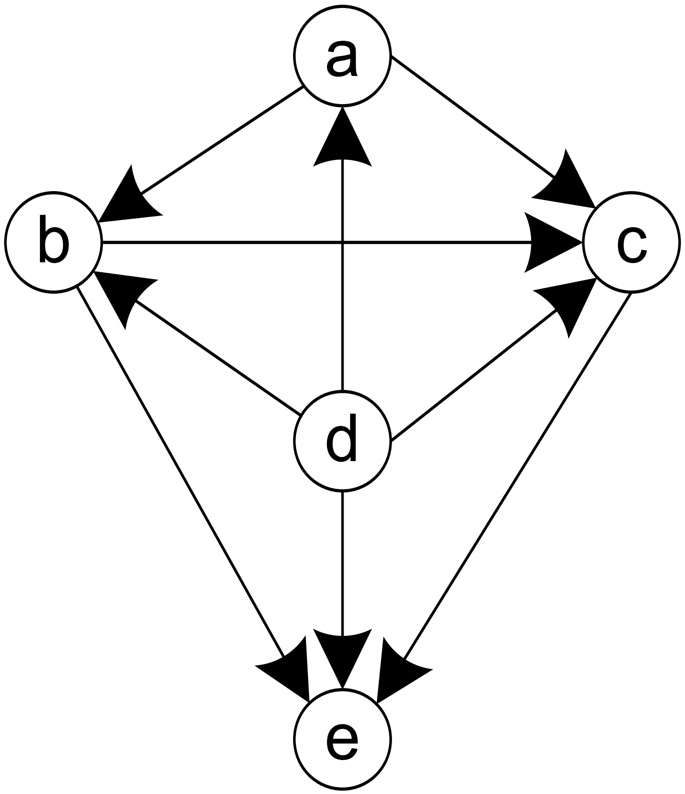
- Key observation:
  - If a vertex v in the dependency graph G has no incoming arrows (*i.e.* in-degree(v) == 0), then v does not have any dependencies
  - It follows that any v that does not have dependencies is a candidate to be visited next in topological order
  - *i.e.* any of these can go first →

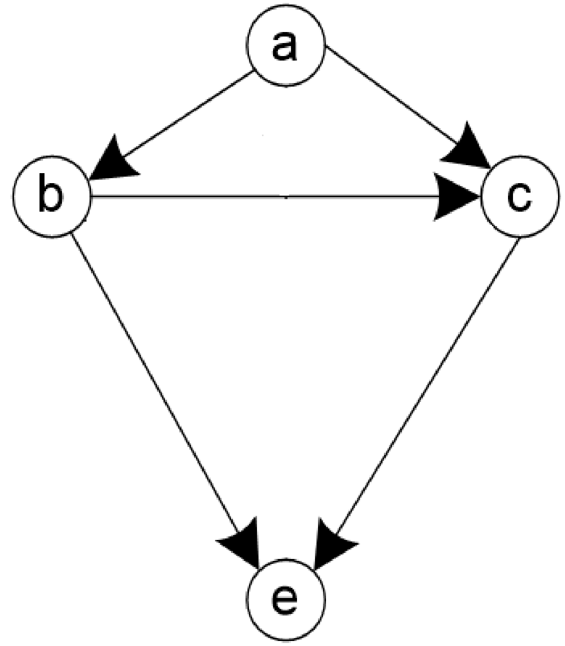


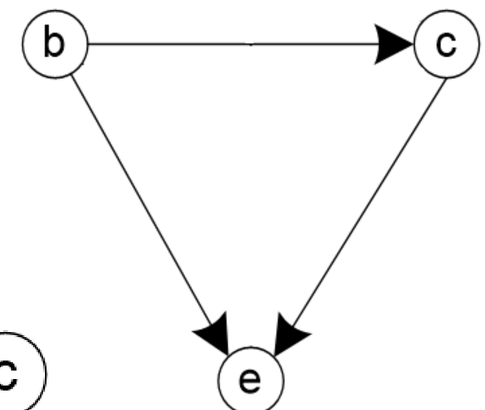
Idea of the algorithm

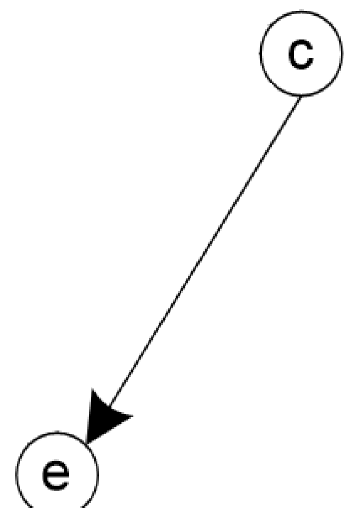
- Identify a $v \in V$ that has in-degree=0
- Delete v and all edges coming out of it
- Repeat until done
- The topological order is the order the vertices are deleted
- If there are $v \in V$, but no v has in-degree=0, the graph G is not a DAG (no feasible solution exists)

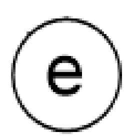
Example 3











Algorithm details

- Use a set to store the candidate vertices
  - *I.e.* the vertices with in-degree = 0
  - Any ordered set will do, e.g. TreeSet.
- Use an ordered list to store the delete order
  - Any list type will do, e.g. ArrayList, always adding to the end

TopoSort “Decrease by one” pseudocode

```
Algorithm TopoSort(G)
   create an empty ArrayList A
   create an empty TreeSet Candidates
   add all v with inDegree=0 to Candidates
   while Candidates is not empty
      v = Candidates.first()
      add v to A
      for each vertex w adjacent to v
         remove edge (v,w) from G
         if w has inDegree=0
            add w to Candidates
      remove vertex v from G
   if there are no vertices remaining in G
      solution is in A
   else
      no solution exists
```

Practice problems

- Chapter 4.2, page 142, question 1
- Chapter 5.3, page 185, questions 5 & 6

Greedy Algorithms

Textbook: Chapter 9

Making change

- Imagine you're a shop clerk giving change and you want to use the *smallest number of coins*

- Strategy:
  - Always select the biggest feasible coin
- Example: 37 cents
  - 1 quarter (need 12 more cents)
  - 1 dime (need 2 more cents)
  - 2 pennies (2 *what*?)

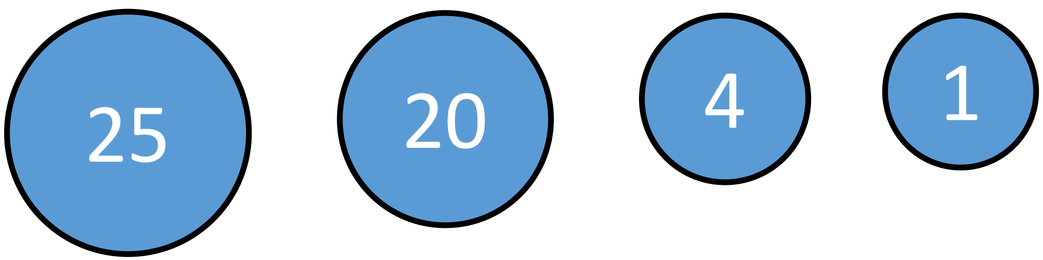
This is a “greedy algorithm”

Making change—algorithm

```
Algorithm MakeChange(N)
    sum = 0
    coins = {}    // set of coins to be returned
    while sum < N do
        choose the largest coin X with value <= (N-sum)
        sum += X.value
        coins += {X}
    endwhile
    return coins
END
```

Does this algorithm always give the best result?

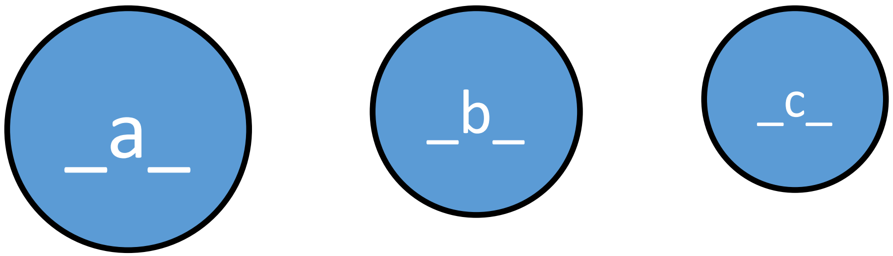
- For US/Canadian coins, yes
  - With or without pennies
- But what if your coins were:



- And you had to give 28 cents?
  - Greedy algorithm result: 25, 1, 1, 1 → 4 coins
  - But there is a 3-coin answer

Puzzle

- Make a “smaller” counterexample
  - What if your coins were:



  - And you had to make __x__ cents?

- I.e., find a,b,c,x so that the greedy algorithm gives a 3-coin answer, even though a 2-coin answer exists

Moral of the story

- Greedy algorithms do not always give optimal general solutions to problems
- But sometimes they do

Optimization problems and decision problems

- An optimization problem is one in which you want to find not just *any* solution, but the *best* solution
  - As opposed to decision problem – “does a solution exist?”
  - Decision problem has a yes/no answer
  - Optimization problem is about minimizing or maximizing

- Greedy algorithms attempt to solve *optimization problems*

Remember the Knapsack problem

- Optimization version:
  - Given N items with weights + values, and a knapsack with carrying capacity W, what is the greatest overall *value* of stuff the thief can steal?
- Decision version:
  - Given N items with weights + values, and a knapsack with carrying capacity W, can the thief steal $V worth of stuff?

Greedy algorithms

- For solving *optimization problems*
- Construct a solution through a sequence of choices
- Always choose the best option available “right now”
    - The “best” choice is the one that gets us closest to an optimal solution (e.g. take the biggest feasible coin)
- You hope that by choosing a *local* optimum at each step, you will end up at a *global* optimum

Greedy algorithms

- Greedy choice properties:
  - *Feasible: Must satisfy the problem's constraints*
    - If you are making change for 17 cents, you don't pick a quarter
  - *Locally optimal: Best local choice among all feasible choices available on that step*
    - If you are making change for 14 cents, you pick a dime, not a nickel
    - Assumption: it is "reasonably efficient" to determine this (think about the Knapsack Problem – how to find the "best" choice)
  - *Irrevocable: Once made, it cannot be changed during subsequent steps of the algorithm*

Greedy algorithms

- We will examine greedy algorithms for the following problems:

  - Finding a minimum spanning tree (MST) of a graph
    - Prim’s algorithm
    - Kruskal’s algorithm

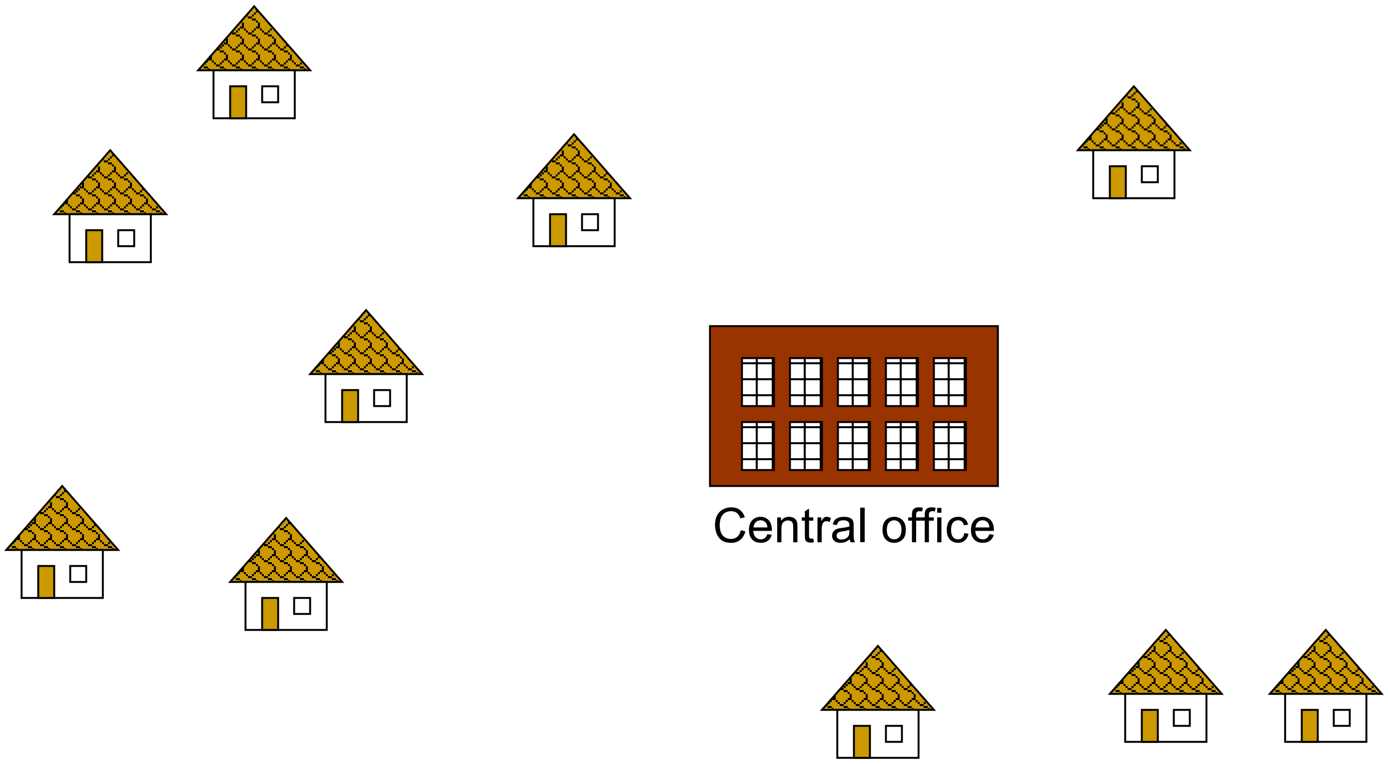
  - Finding Shortest Paths from a Single Source in a graph
    - Dijkstra’s algorithm

  - Coloring a graph

Greedy algorithm TL/DR

1. Iteratively construct a solution
2. At each step select the “best” item to add
   - Idea for how to select the best should be “simple”

A real-world problem: Build a (physical) network



SIDEBAR (helpful background)

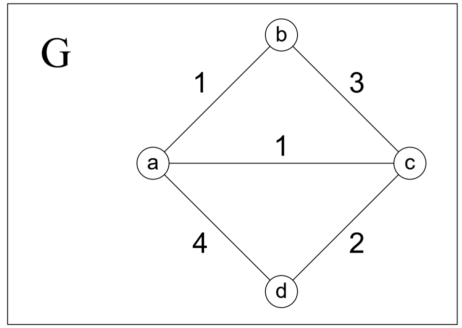
Minimum Spanning Trees

- A **minimum spanning tree** (MST) is a subgraph of a connected, undirected, weighted graph ***G***, such that
  - it includes all the vertices ("spanning")
  - it is acyclic ("tree")
  - the total cost associated with the edges is the minimum among all possible spanning trees

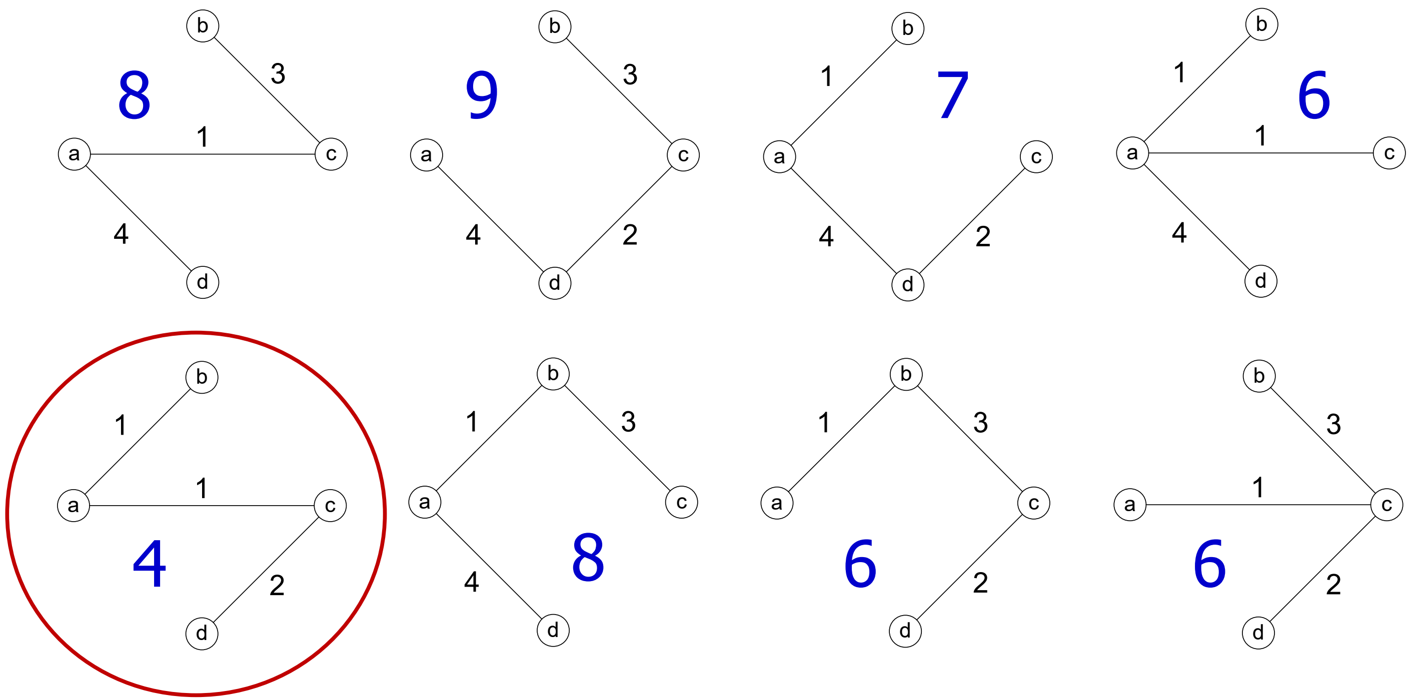
- MST may not be unique

MSTs (cont'd)

Consider all the spanning trees of G:



The weight of each spanning tree is given by the sum of its edges ...



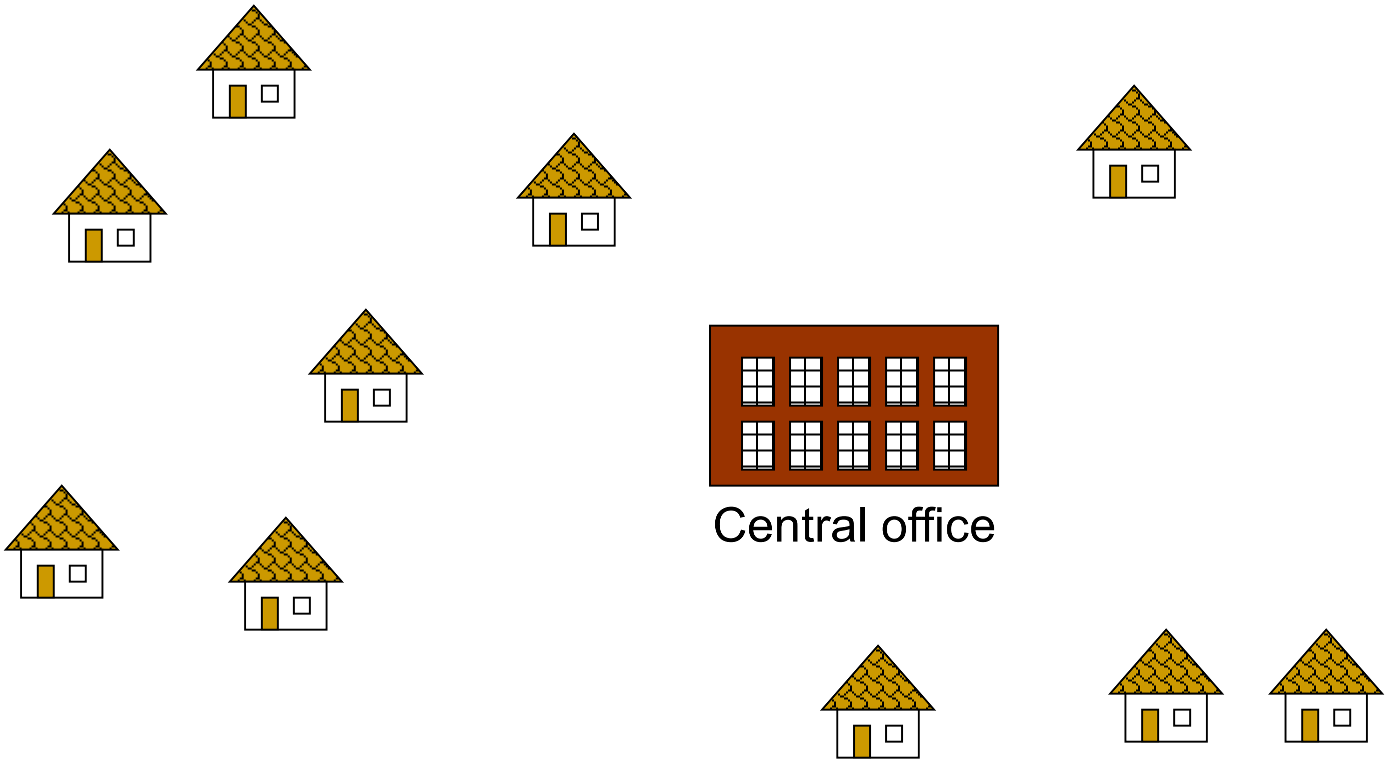
Minimum Spanning Tree of G is this graph, and it has a weight of 4.

SIDEBAR (helpful background)

If you do MST on a complete graph:

- The result:
  - Is a *tree* (obvs)
    - therefore *connected*
  - connects *all the nodes*
  - using *the minimum cost*

Back to our little village

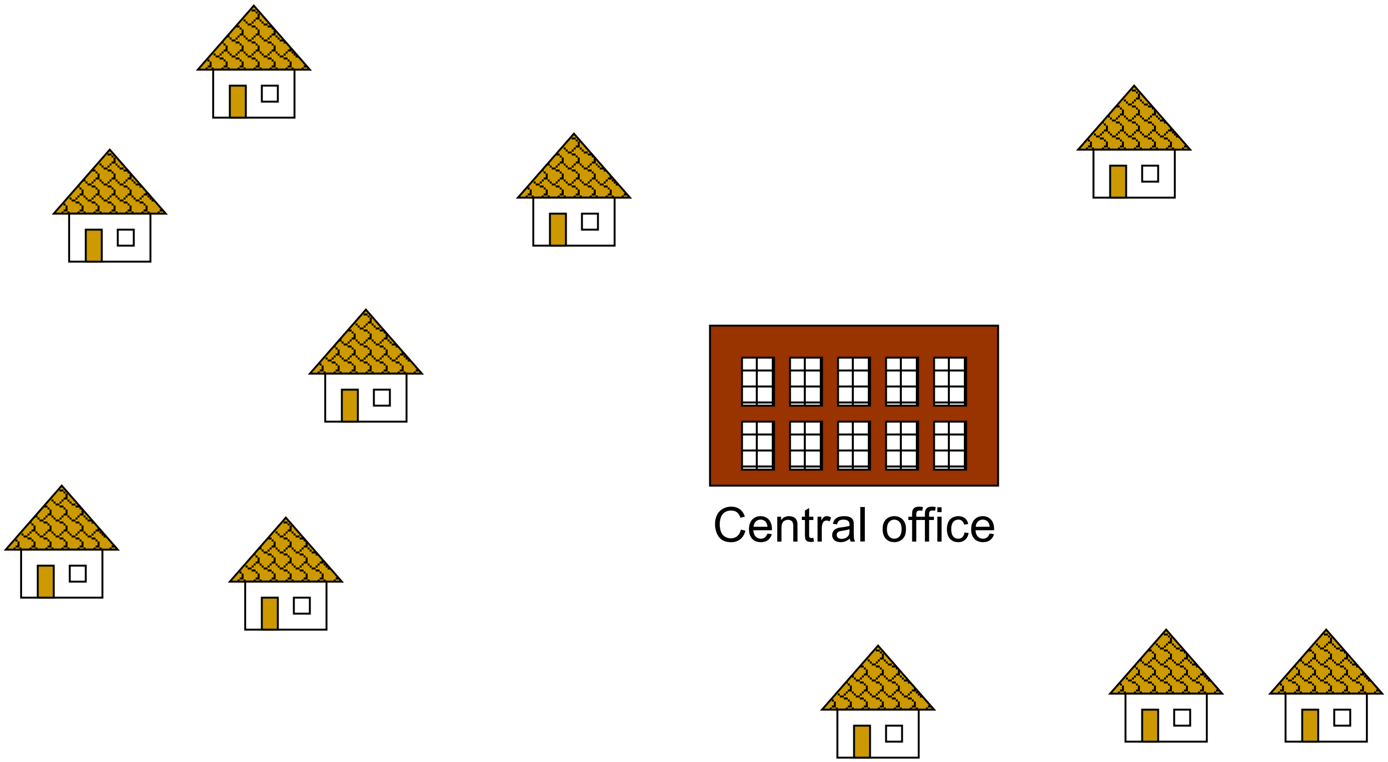


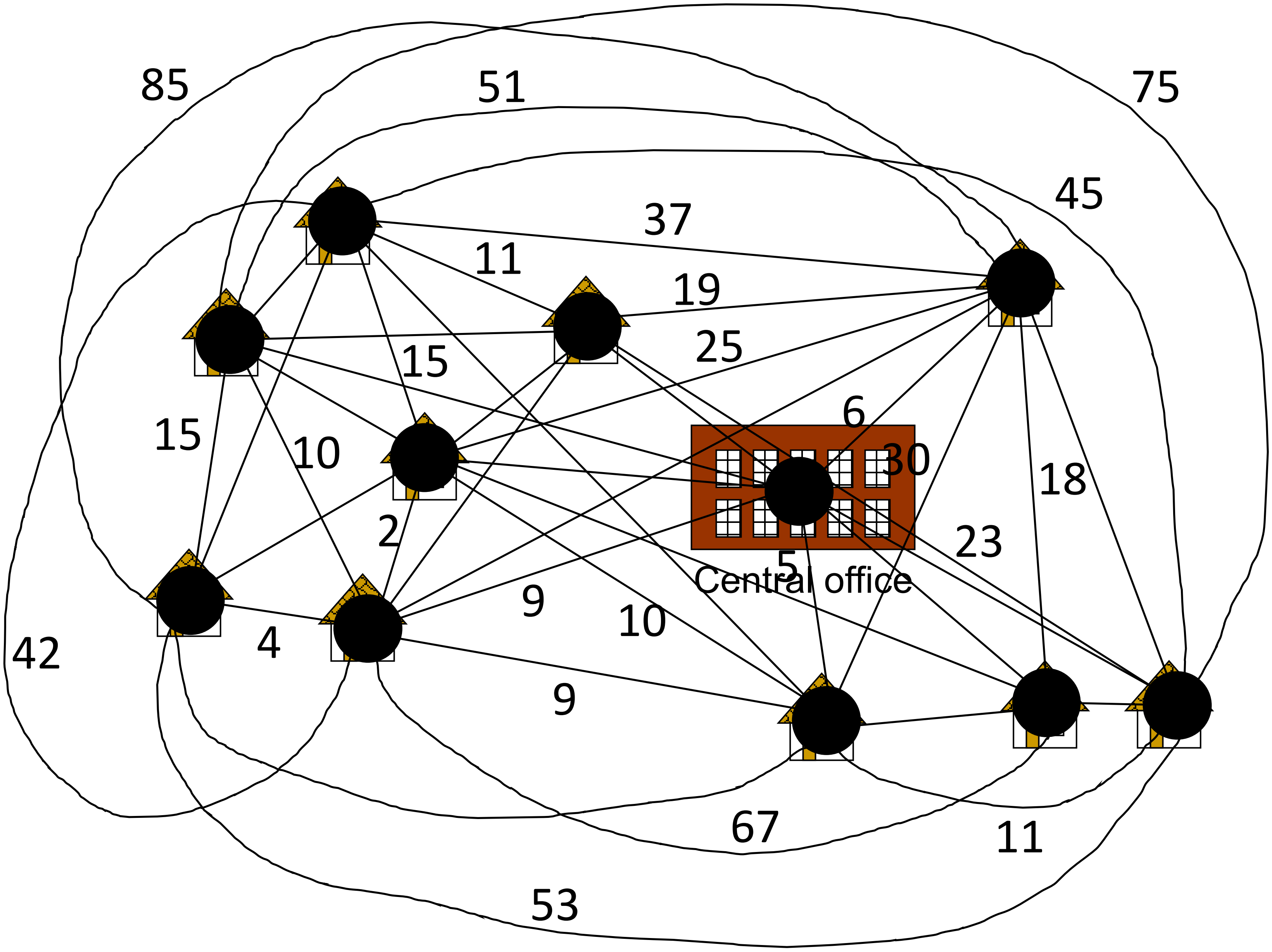
Let’s solve this problem using MST

Represent it as a graph

- Vertices are all the nodes to be connected
- One edge for *every possible* connection
  - I.e. the complete graph of N vertices
- Each edge has a “weight” associated with it
  - Cost of running a wire from node A to node B

- Now find the MST
  - How does this solve the problem?
    - Spanning tree → all nodes are connected
    - Lowest cost tree → cheapest possible network


Central office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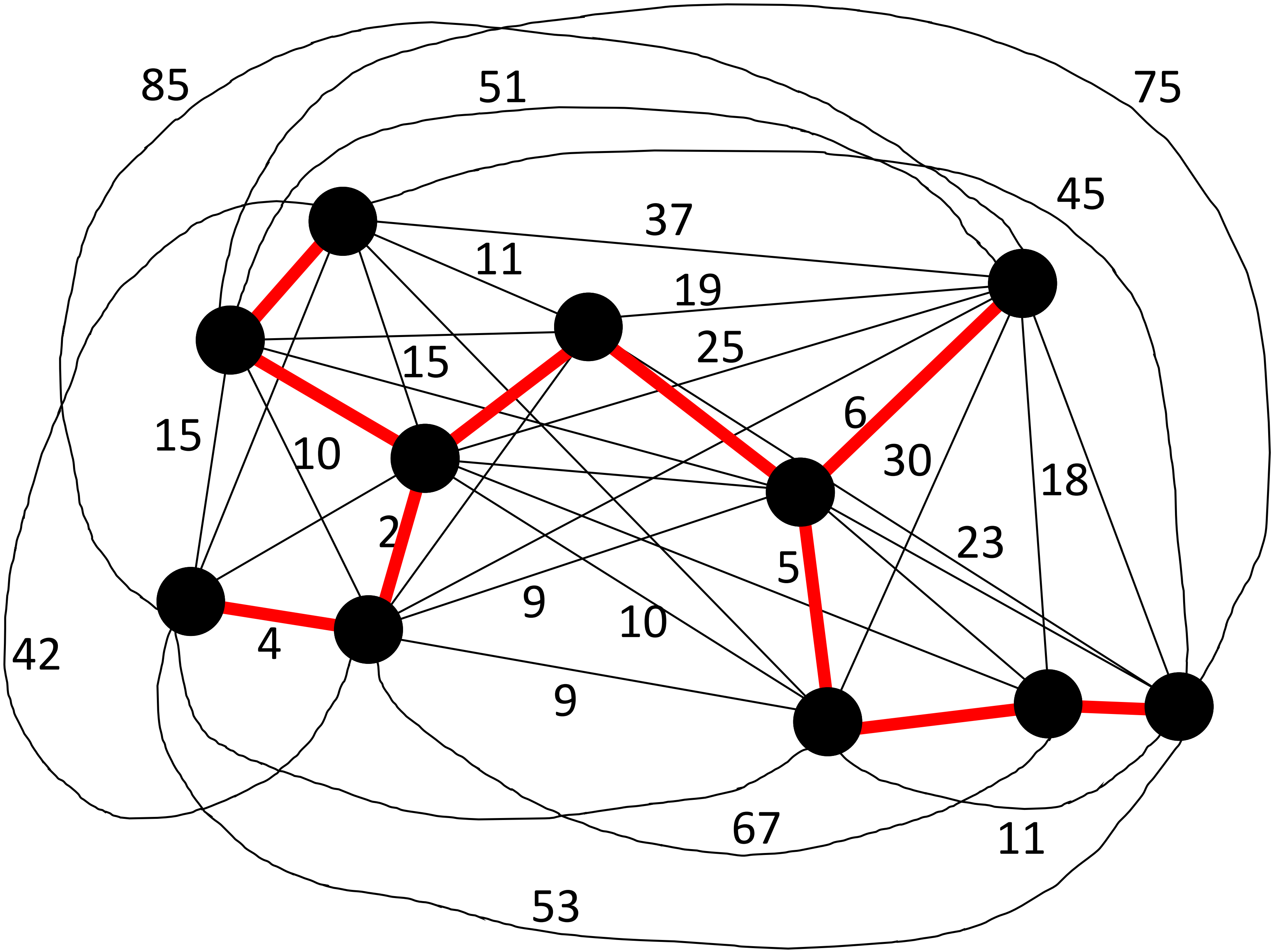
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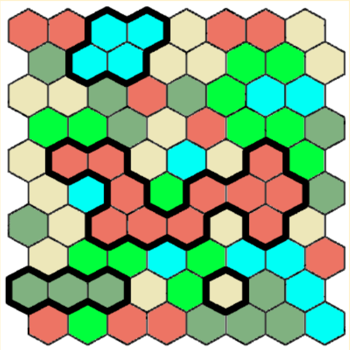
META

Reminder: Solving problems with graphs, strategy 2

1. Represent the problem “cleverly” as a graph
2. Feed the graph to a Graph Algorithm
3. Use the output to determine the answer to your problem

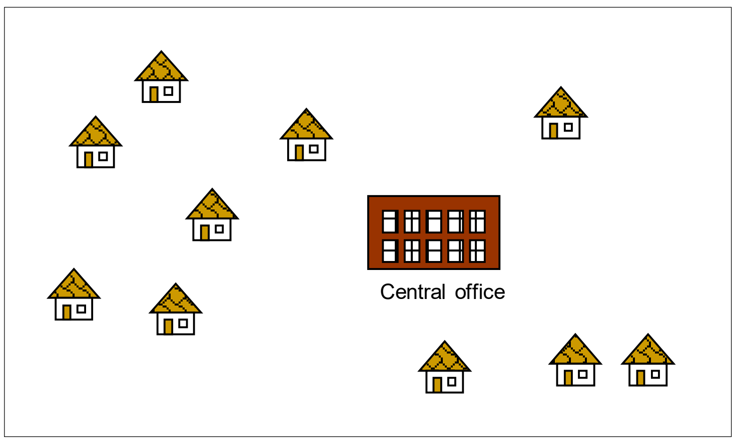
We also used Strategy 2 with the “counting map regions” problem (different Graph Algorithm)



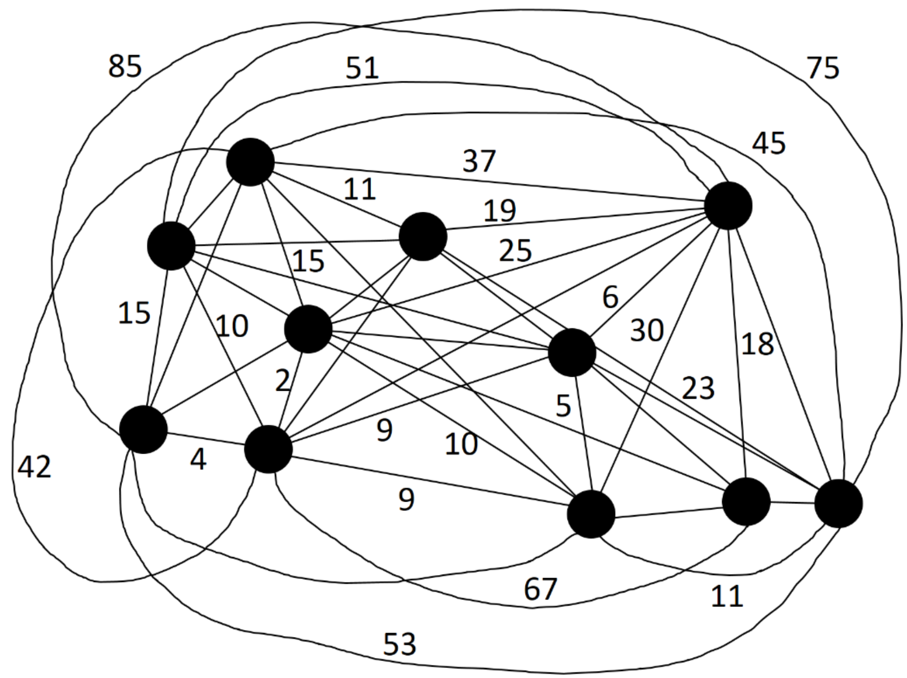
META

Example: our little village

1. Represent the problem “cleverly” as a graph



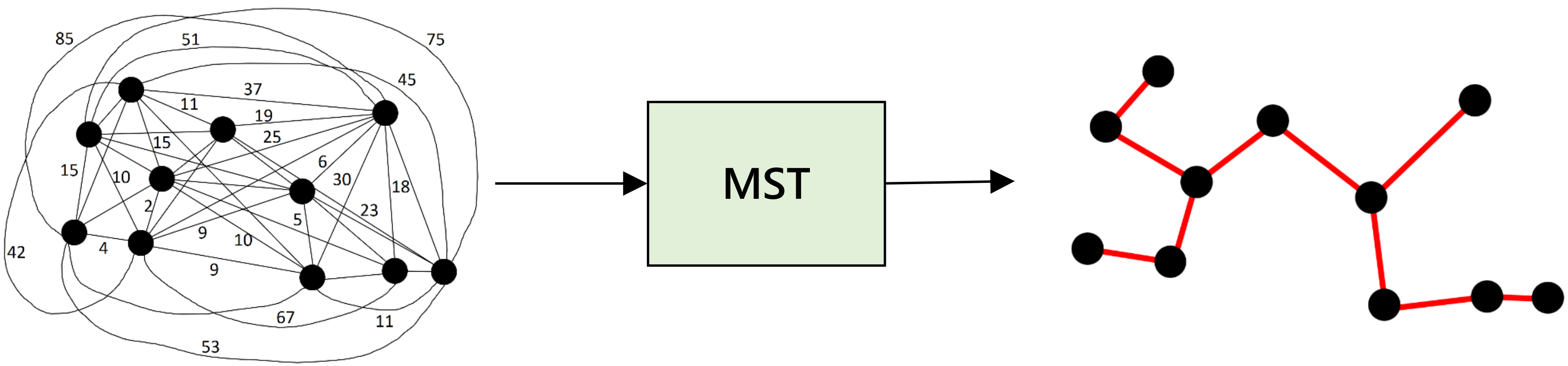
(complete, weighted graph)



META

Example: our little village

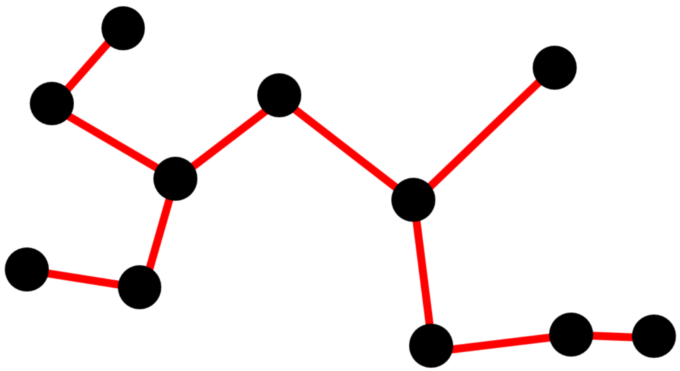
2. Feed the graph to a Graph Algorithm

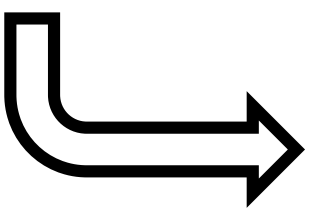


META

Example: our little village

3. Use the output to determine the answer to your problem





Solution:
Connect house A to B
Connect house B to C
Connect house C to D
Connect house D to Central
…
etc.

Whew.

- Now we still need one of these:

MST

- In fact, we're going to look at two of them:

Prim

Kruskal

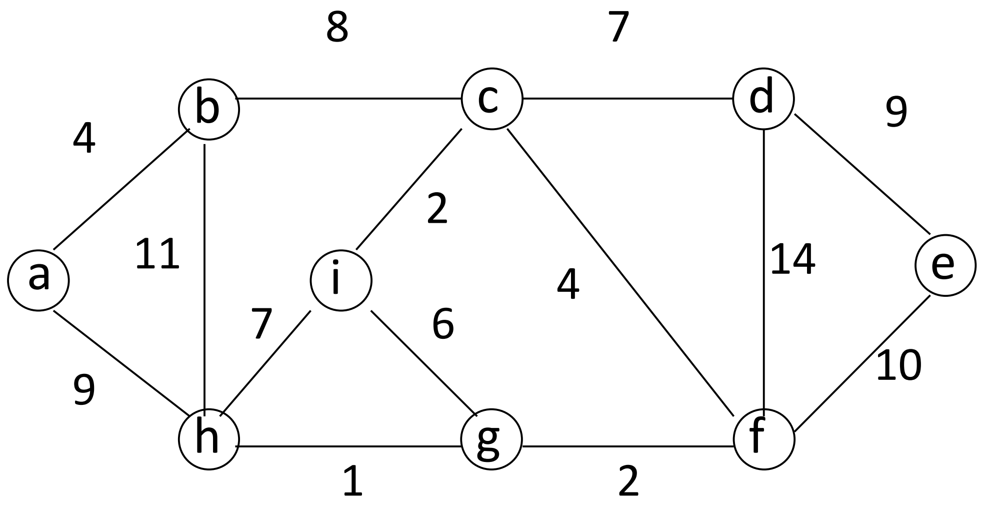
Greedy Algorithms: Prim's Algorithm

Textbook: Chapter 9.1

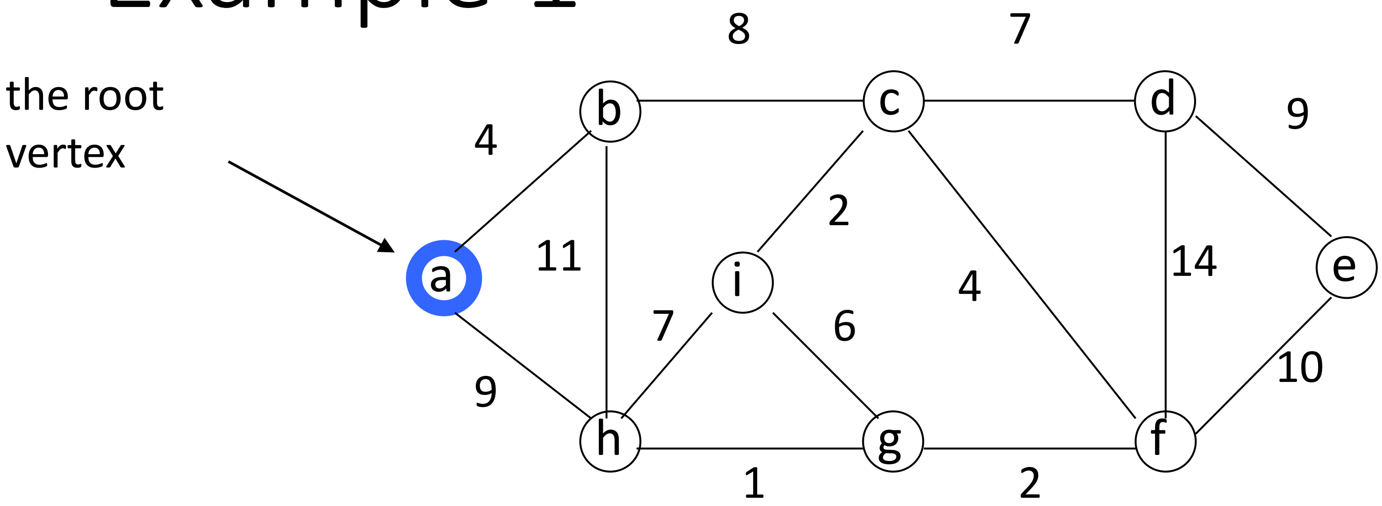
Prim's algorithm

```
Algorithm Prim(G)

    V_T ← {v_0}                          // init tree with one (arbitrary) vertex

    E_T ← ∅                              // init tree with no edges

    for i ← 0 to |V|-1 do                // loop until all vertices added to tree
        find a min-weight edge e=(u,v) from E
           where u is in V_T (in the tree)
           and v is in V-V_T (not yet in the tree)

        V_T ← V_T ∪ {v}                   // add the vertex v to the tree

        E_T ← E_T ∪ {e}                   // add the edge (u,v) to the tree

    return T = (V_T, E_T)
```

Example 1



Example 1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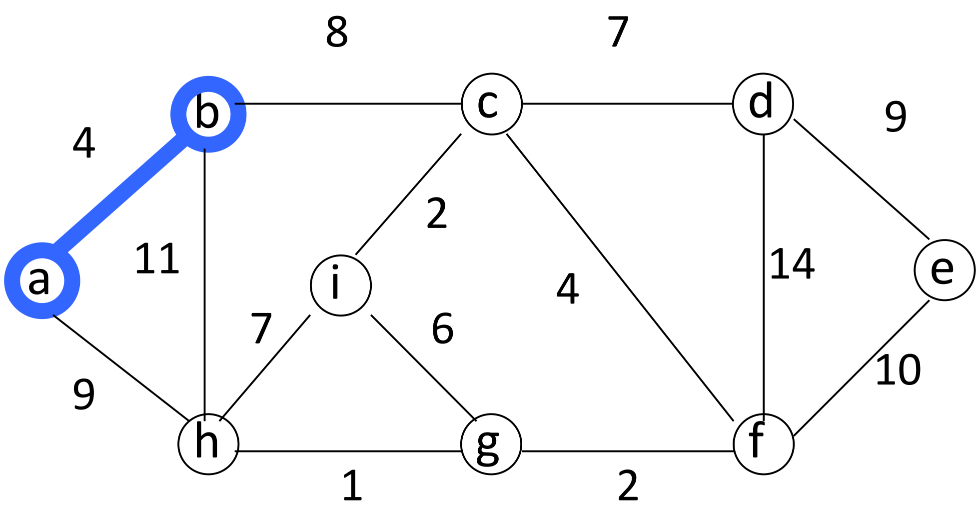
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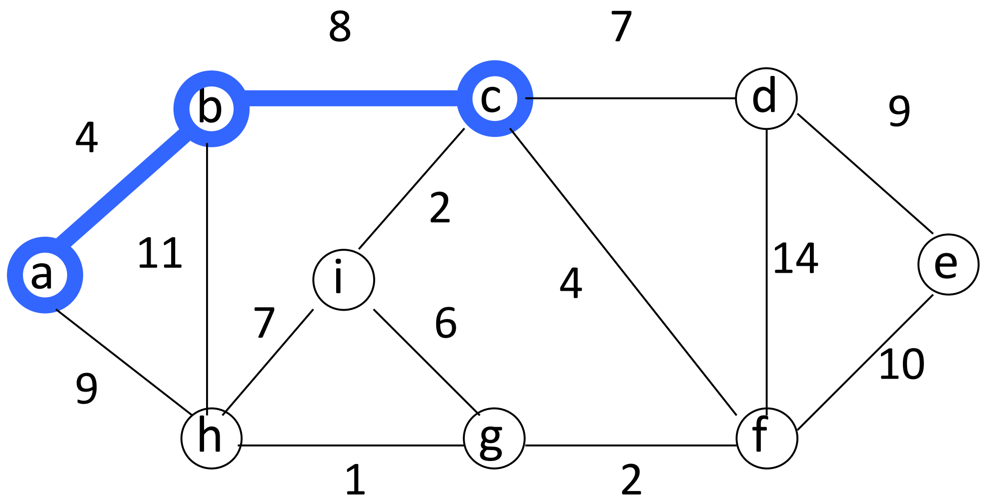

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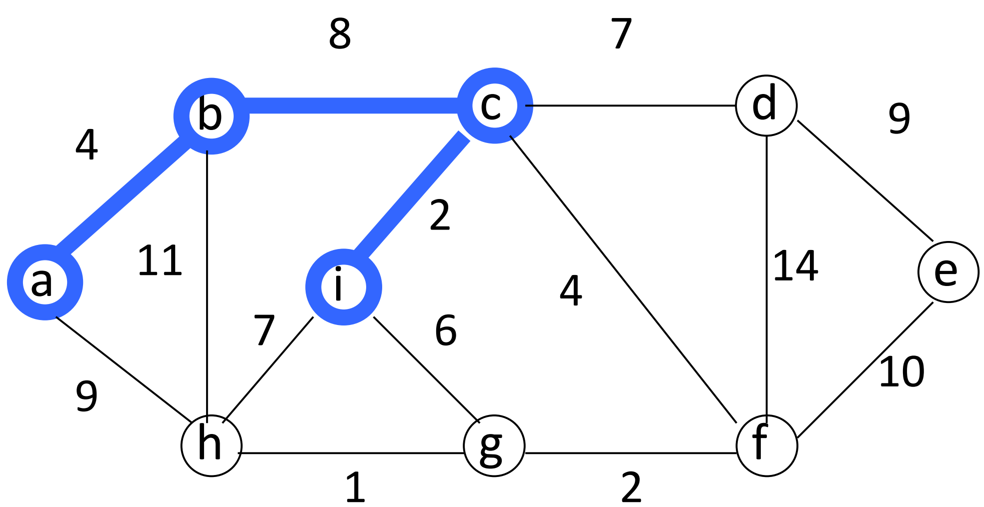

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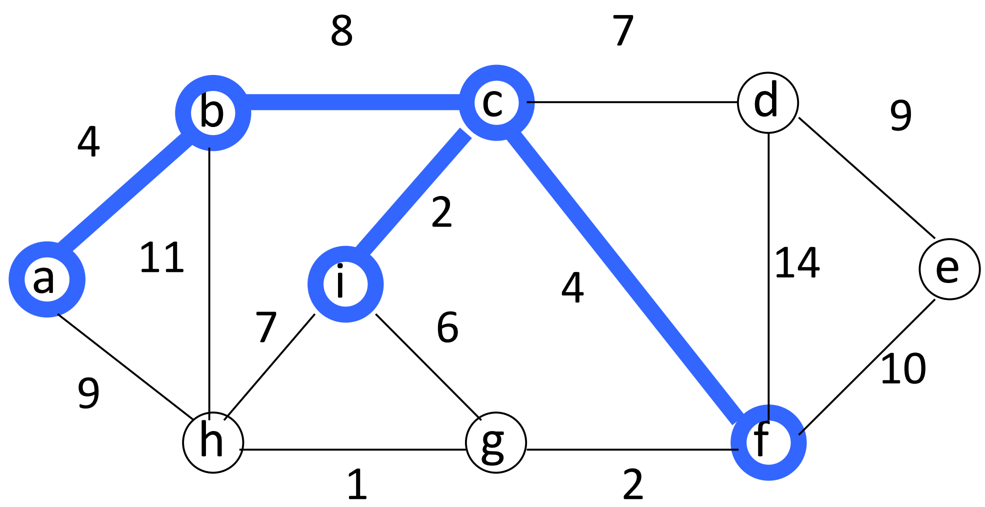
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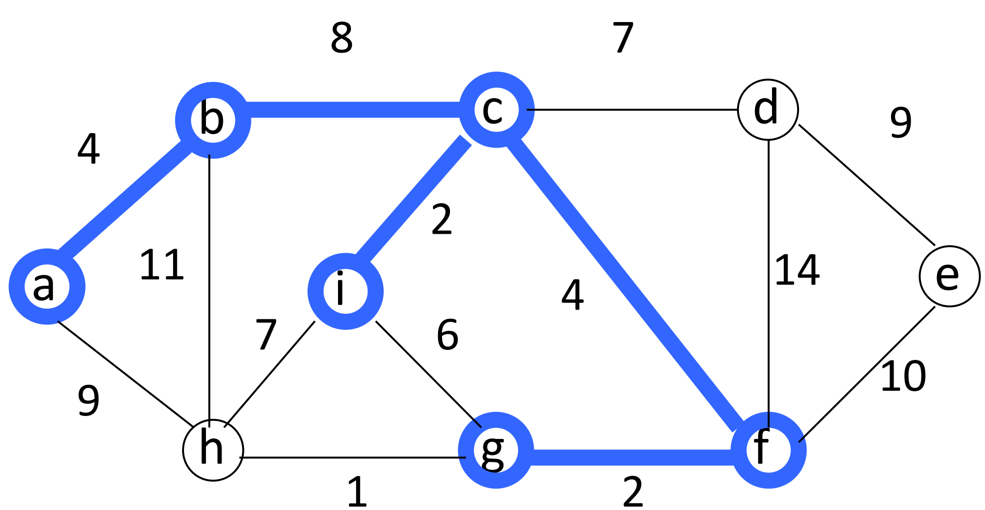
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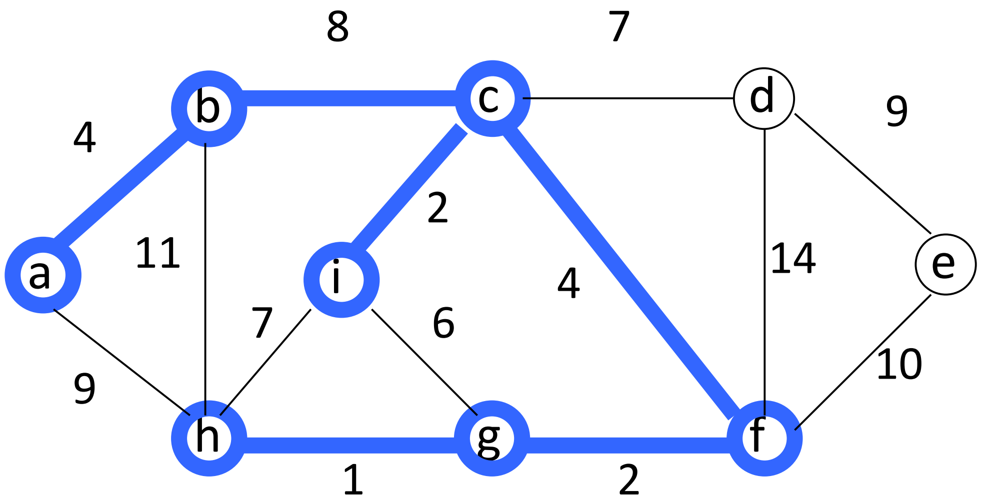
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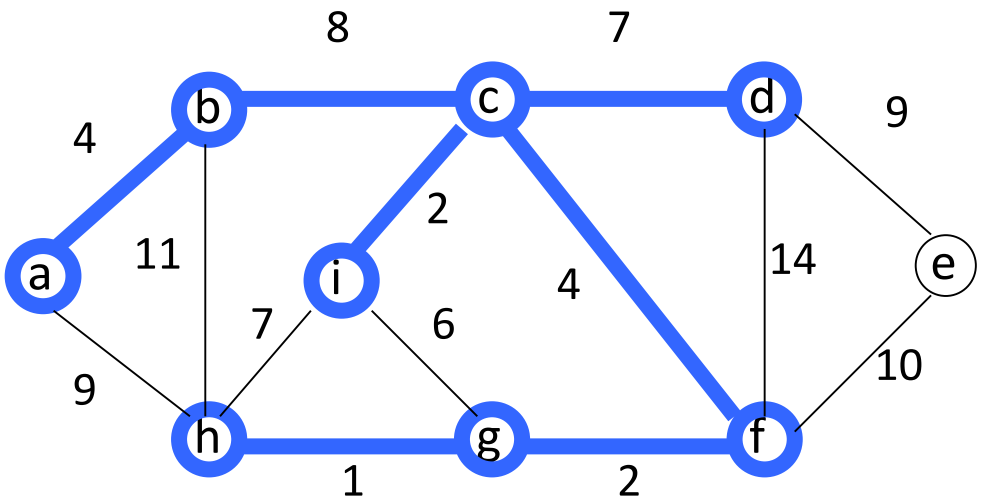
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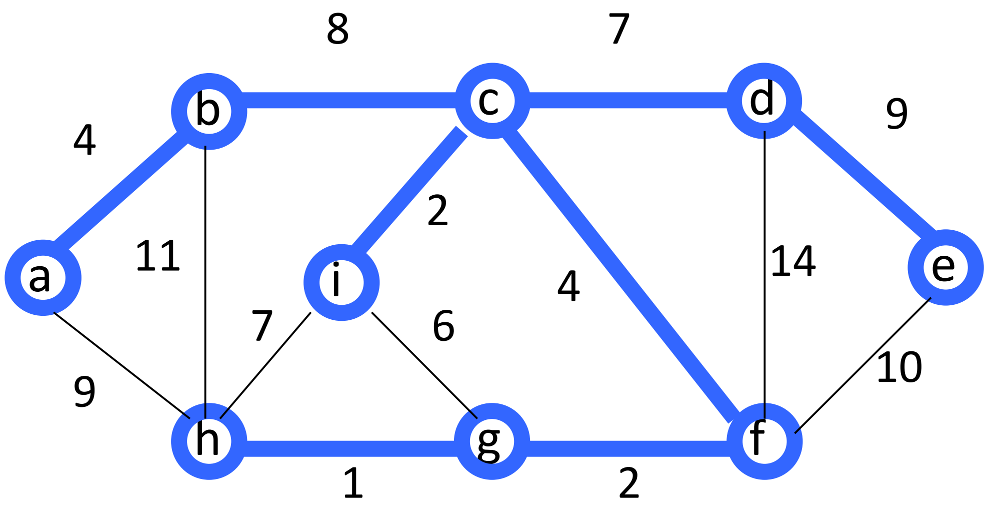

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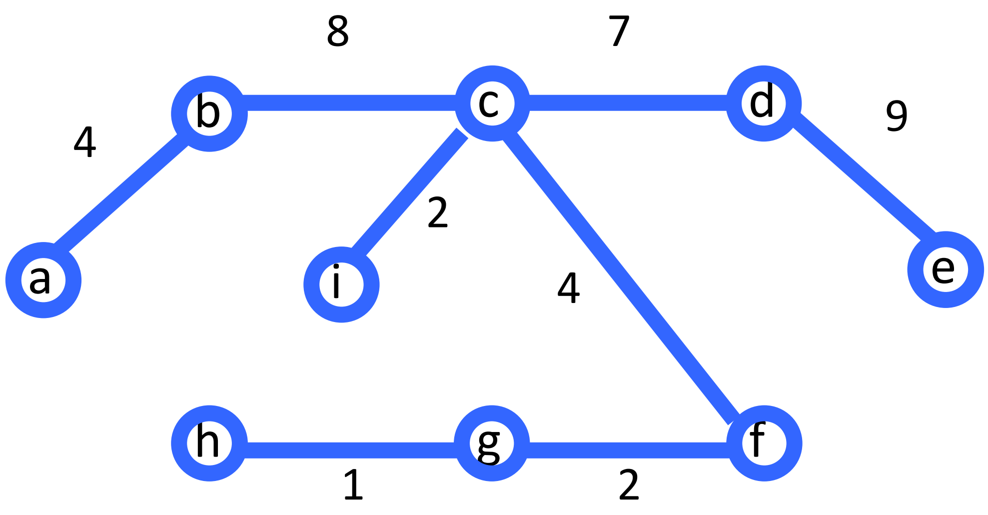

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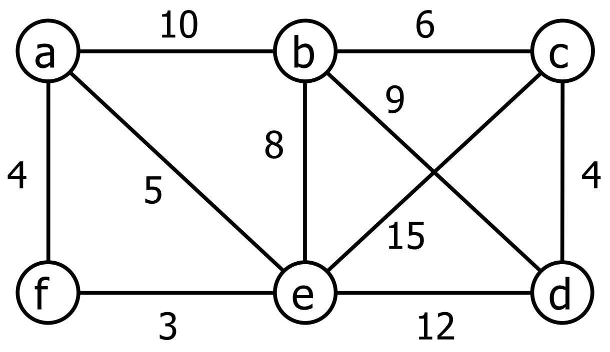
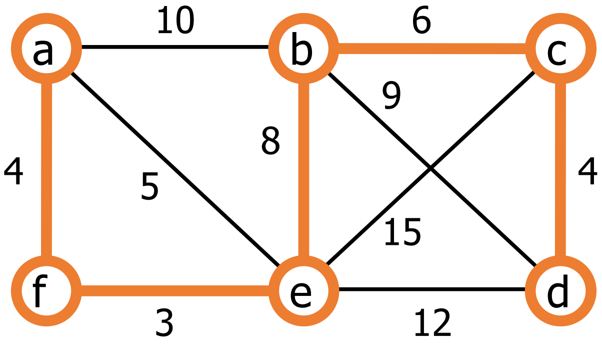
Prim's algorithm

```
Algorithm Prim(G)

    V_T ← {v_0}                              // init tree with one (arbitrary) vertex

    E_T ← ∅                                  // init tree with no edges

    for i ← 0 to |V|-1 do                    // loop until all vertices added to tree
        find a min-weight edge e=(u,v) from E
           where u is in V_T (in the tree)
           and v is in V-V_T (not yet in the tree)

        V_T ← V_T ∪ {v}                      // add the vertex v to the tree

        E_T ← E_T ∪ {e}                      // add the edge (u,v) to the tree

    return T = (V_T, E_T)
```

Example 2



Example 2



Greedy Algorithms: Kruskal’s Algorithm

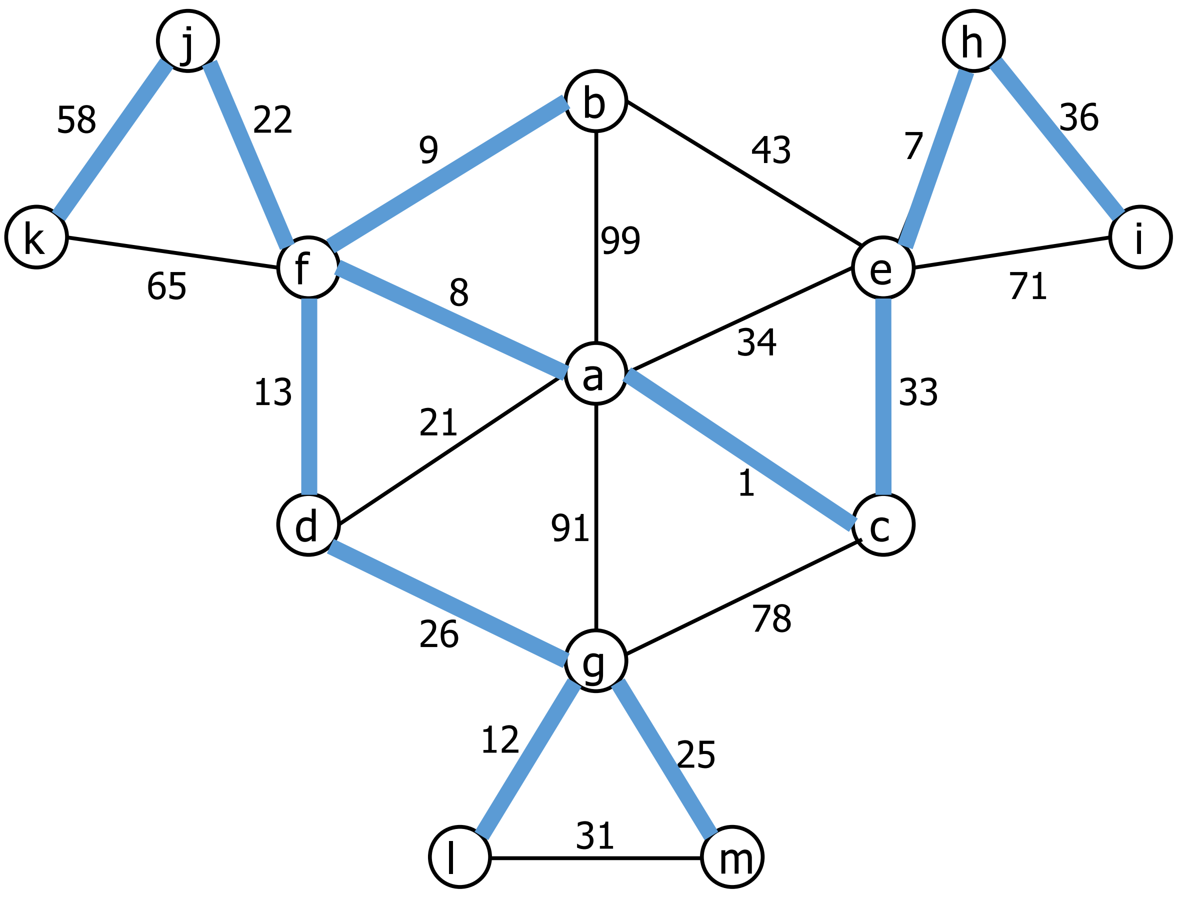
Textbook: Chapter 9.2

Context

- Another one of several “greedy algorithms” we are examining:
  - Minimum Spanning Tree of a graph
    - Prim’s algorithm
    - Kruskal’s algorithm
  - Shortest Paths from a Single Source in a graph
    - Dijkstra’s algorithm
  - Graph coloring

Kruskal's (overview)

- Repeatedly add a minimum-weight edge that does not introduce a cycle
- Example:



Kruskal’s algorithm (basic idea)

```
Kruskal(G)
    sort edges of E in ascending order by weight
    V_T ← V                            // T has all the vertices of G
    E_T ← ∅                            // start with no edges in T
    count ← 0
    k ← 0                              // index over edges of G
    while count < |V|-1 do             // done when T has this many edges
        k ← k + 1
        if E_T ∪ {e_k} is acyclic      // safe to add this edge to T?
            E_T ← E_T ∪ {e_k}          // ...then add it
            count ← count + 1
    return T = (V_T, E_T)
```

These two bits are “efficiency challenges”

Kruskal's algorithm: Implementation challenges

1. Sort the edges
   - We know several O(NlogN) methods
   - Which will serve us well?
2. Determine if adding an edge would create a cycle
   - Maybe use a DFS or BFS to test for a cycle?
     - These are $O(N^2)$ and we have to do it O(N) times
     - Can we improve on $O(N^3)$?

     - The answer is Yes, with a clever data structure

SIDEBAR (helpful background)

Disjoint Subsets (aka “Union-Find”)

- A collection of disjoint subsets – any element can only be in one subset at any time
- Operations on a DS:
  - Makeset(x) – creates a new subset with the element x
  - Find(x) – returns the subset that contains x
  - Union(x,y) – merges the subsets containing x and y together

SIDEBAR
(helpful
background)

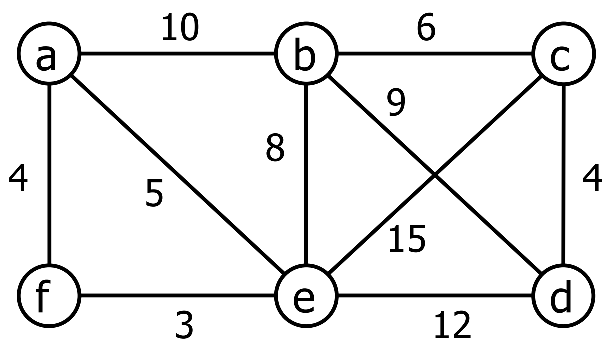
DS/Union-Find Example

```
for each element x in {1,2,3,4,5,6,7,8}
    makeset(x)
        → DS is now {1} {2} {3} {4} {5} {6} {7} {8}
union(2,7)
        → DS is now {1} {2,7} {3} {4} {5} {6} {8}
union(1,4)
        → DS is now {1,4} {2,7} {3} {5} {6} {8}
y ← find(4)
        → y is now {1,4}
union(y,3)
        → DS is now {1,4,3} {2,7} {5} {6} {8}
x ← find(1)
        → x is now {1,4,3}
y ← find(7)
        → y is now {2,7}
union(x,y)
        → DS is now {1,4,3,2,7} {5} {6} {8}
```

Kruskal’s with disjoint subsets

- Maintain DS of vertices in the spanning tree T
- Initially each vertex is a separate subset
- When an edge (u,v) is added to T:
  - DS.union(u,v)
- Each subset is a connected component
  - It’s also a tree – a subset of the eventual MST
- If u,v are in the same subset *do not add edge*
  - It would create a cycle
- At the end there will be only one subset in DS
  - T is a single connected component

Another Kruskal example (using disjoint subsets)



- After the initialization
- PQ contains sorted list of edges
- DS has one subset for each vertex

PQ

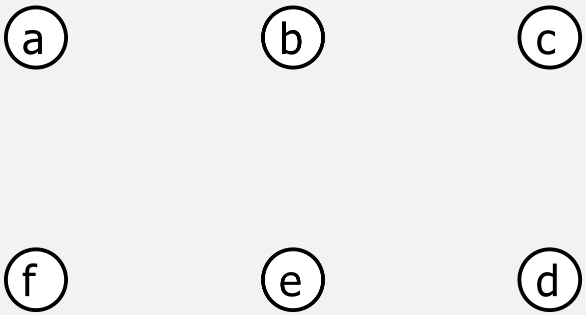
key: value

3:ef
4:af
4:cd
5:ae
6:bc
8:be
9:bd
10:ab
12:de
15:ce

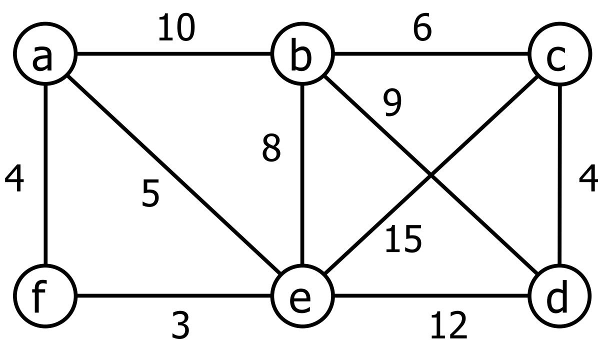
Subsets

{a} {b} {c} {d} {e} {f}

Solution



Another Kruskal example (using disjoint subsets)



- After iteration 1
- edge ef has been added
- e, f subsets merged

PQ

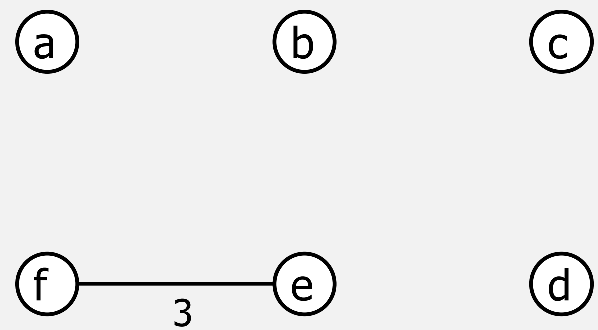
<u>key: value</u>

3:ef
4:af
4:cd
5:ae
6:bc
8:be
9:bd
10:ab
12:de
15:ce

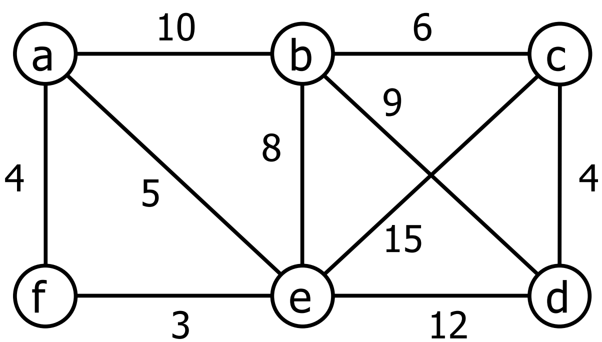
Subsets

{a} {b} {c} {d} {e} {f}
{a} {b} {c} {d} {e,f}

Solution



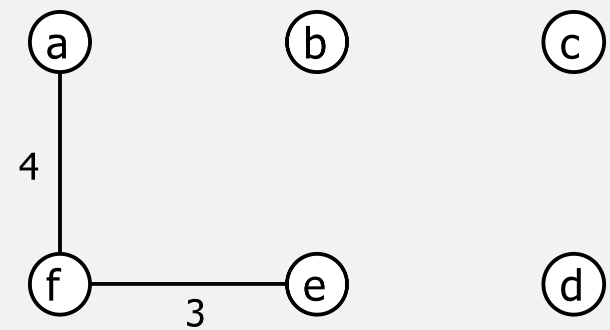
Another Kruskal example (using disjoint subsets)



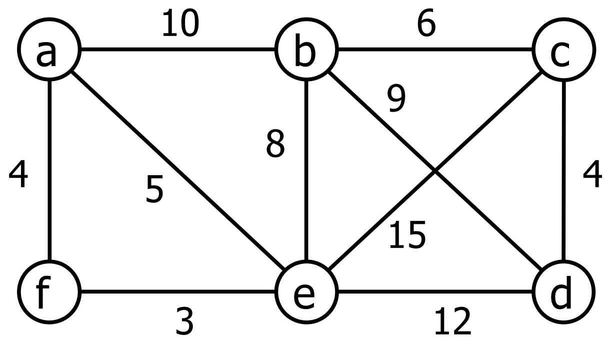
- After iteration 2
- edge af has been added
- a, f subsets merged

PQ	Subsets
<u>key: value</u>	`{a} {b} {c} {d} {e} {f}`
`3:ef`	`{a} {b} {c} {d} {e,f}`
`4:af`	**`{a,e,f} {b} {c} {d}`**
`4:cd`	
`5:ae`	
`6:bc`	
`8:be`	
`9:bd`	
`10:ab`	
`12:de`	
`15:ce`	

Solution



Another Kruskal example (using disjoint subsets)



- After iteration 3
- edge cd has been added
- c, d subsets merged

PQ	Subsets
key: value	{a} {b} {c} {d} {e} {f}
3:ef	{a} {b} {c} {d} {e,f}
4:af	{a,e,f} {b} {c} {d}
4:cd	**{a,e,f} {b} {c,d}**
5:ae	
6:bc	
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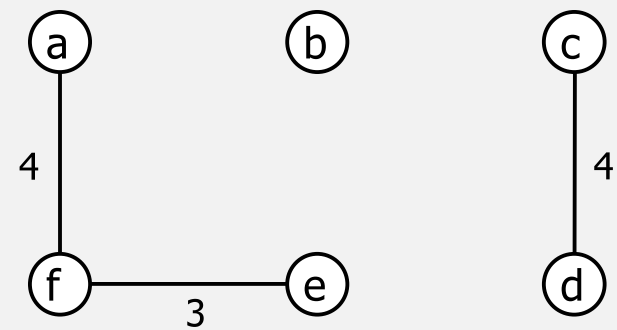
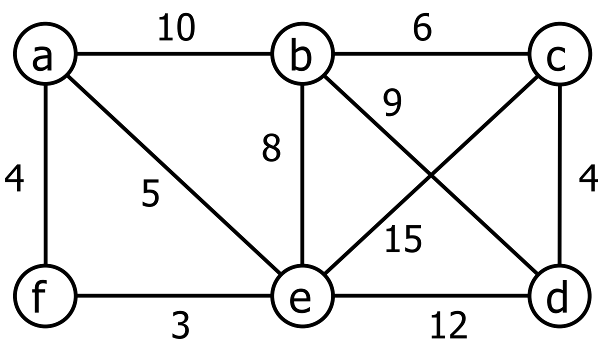
9:bd	
10:ab	
12:de	
15:ce	

Solution



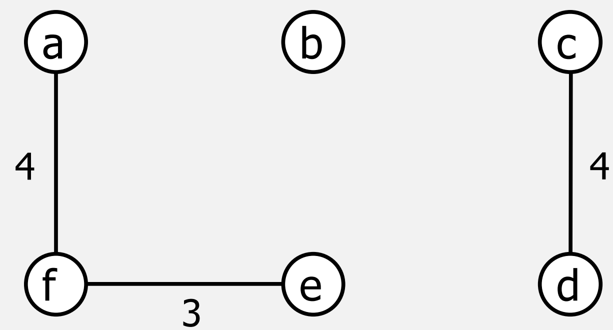
Another Kruskal example (using disjoint subsets)



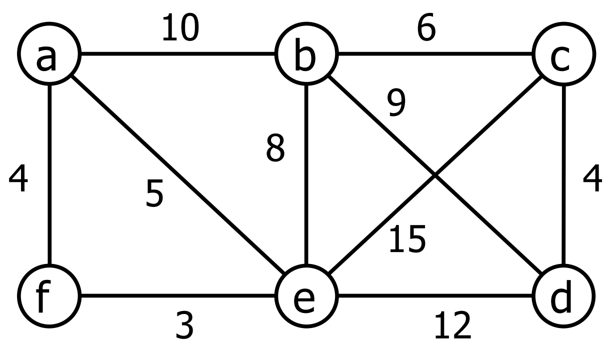
- *No change in iteration 4*
- *a and e are in the same subset*
- *edge ae is not added because it would cause a cycle*

PQ	Subsets
<u>key: value</u>	`{a} {b} {c} {d} {e} {f}`
`3:ef`	`{a} {b} {c} {d} {e,f}`
`4:af`	`{a,e,f} {b} {c} {d}`
`4:cd`	`{a,e,f} {b} {c,d}`
`5:ae`	**`{a,e,f} {b} {c,d}`**
`6:bc`	
`8:be`	
`9:bd`	
`10:ab`	
`12:de`	
`15:ce`	

Solution



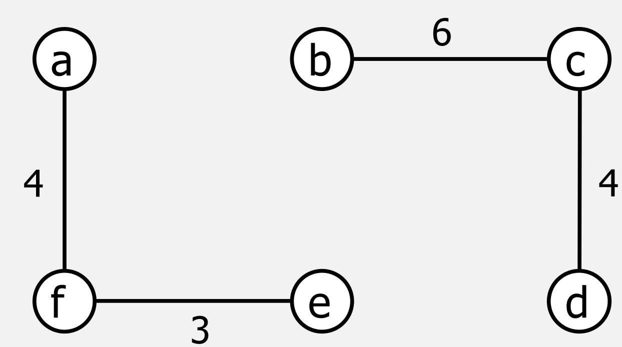
Another Kruskal example (using disjoint subsets)



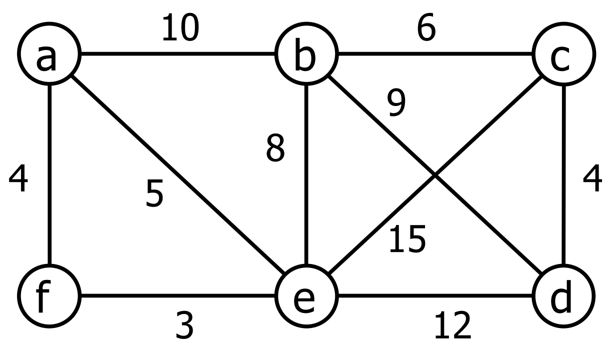
- After iteration 5
- edge bc has been added
- b, c subsets merged

PQ	Subsets
key: value	{a} {b} {c} {d} {e} {f}
3:ef	{a} {b} {c} {d} {e,f}
4:af	{a,e,f} {b} {c} {d}
4:cd	{a,e,f} {b} {c,d}
5:ae	{a,e,f} {b} {c,d}
6:bc	**{a,e,f} {b,c,d}**
8:be	
9:bd	
10:ab	
12:de	
15:ce	

Solution



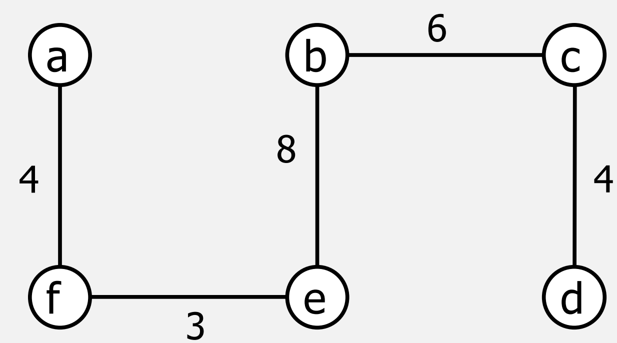
Another Kruskal example (using disjoint subsets)



- After iteration 6
- edge be has been added
- N-1 edges added, main loop ends
- algorithm returns solution

PQ	Subsets
key: value	{a} {b} {c} {d} {e} {f}
3:ef	{a} {b} {c} {d} {e,f}
4:af	{a,e,f} {b} {c} {d}
4:cd	{a,e,f} {b} {c,d}
5:ae	{a,e,f} {b} {c,d}
6:bc	{a,e,f} {b,c,d}
8:be	**{a,e,f,b,c,d}**
9:bd	
10:ab	
12:de	
15:ce	

Solution



Kruskal’s algorithm with PQ + disjoint subsets

```
Algorithm Kruskal(G)
   Add all vertices in G to T    // add v's but don’t add e's
   Create a priority queue PQ    // will hold candidate edges
   Create a collection DS        // disjoint subsets
   for each vertex v in G do
      DS.makeset(v)
   for each edge e in G do
      PQ.add(e.weight, e )       // PQ of edges by min weight
   while T has fewer than n-1 edges do
       (u,v) ← PQ.removeMin()   // get next smallest edge
       cu ← DS.find(u)
       cv ← DS.find(v)
       if cu ≠ cv then           // be sure u,v are not in
          T.addEdge(u,v)              // the same subset
          DS.union(cu, cv)
   return T
```

Efficiency of Kruskal's

- With an efficient union-find algorithm, the slowest thing is the initial sort on edge weights
  - O(|E| log(|E|))
  - Remember that |E| is (in the worst case) $|V|^2$
  - So this is also O($|V|^2$ log(|V|))
  - Since we usually use N as the number of vertices in a graph, this is O(N^2 logN)

Prim’s and Kruskal’s TL/DR

- Same problem: Minimum Spanning Tree (MST)
- Both are greedy algorithms
- Both add edges one at a time
  - Prim’s greedy choice: smallest edge that extends the tree
  - Kruskal’s: smallest edge that doesn’t make a cycle

Greedy Algorithms: Dijkstra’s Algorithm

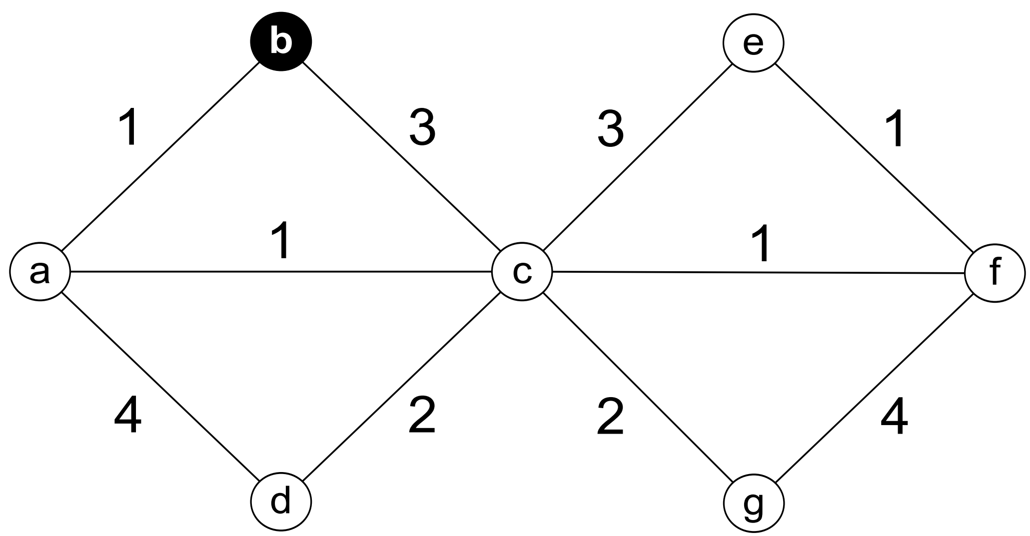
Textbook: Chapter 9.3

Context

- Another one of several "greedy algorithms" we will examine:
  - Minimum Spanning Tree of a graph
    - Prim's algorithm
    - Kruskal's algorithm
  - Shortest Paths from a Single Source in a graph
    - Dijkstra's algorithm
  - Graph coloring

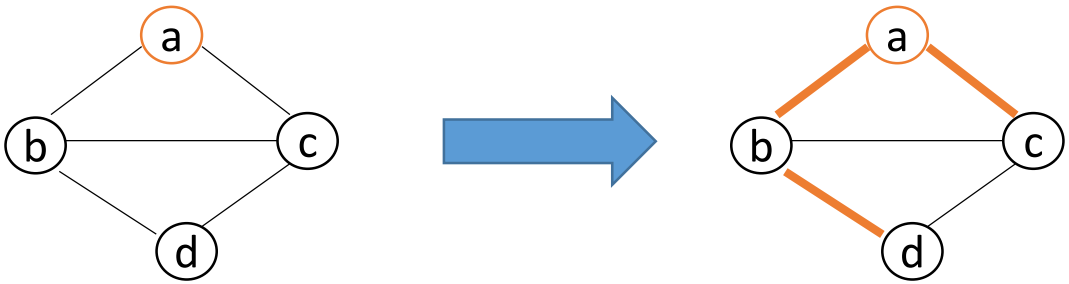
Problem:
Single-source Shortest Paths

- Find the shortest path from a chosen vertex (the *source*) to every other vertex

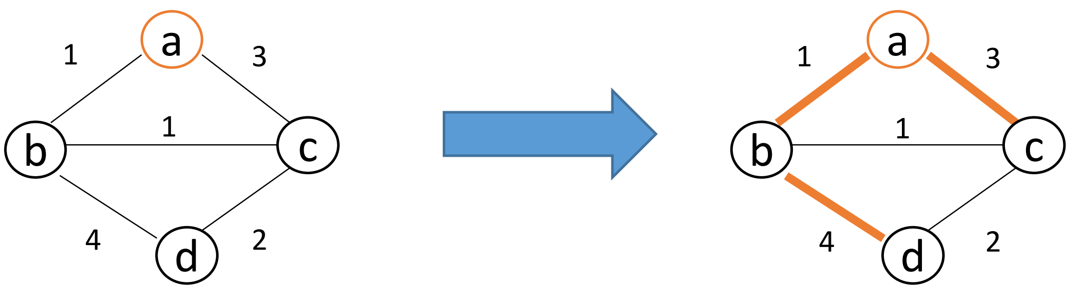


What about BFS?

- Simple/basic BFS already does this for an unweighted graph:



- … but not for weighted graphs. Consider the distance between a and d:



a-b-d has length 5 in BFS tree, but shortest path is 4 (a-b-c-d)

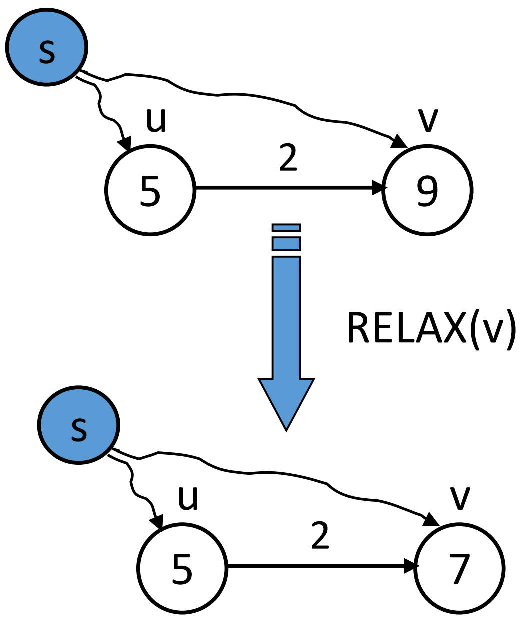
- *Algorithm to find shortest paths in weighted graphs needs to consider the weight on the edge before including it in the solution*

Idea of Dijkstra’s algorithm

- Remember the best-known shortest distances for all vertices
  - Initially “infinity” for all
- Choose the nearest unprocessed vertex
  - Definition of “nearest” tbd
- Look at all of its neighbors
- Update their known shortest distances (“Relax”)
- Repeat

Relaxation

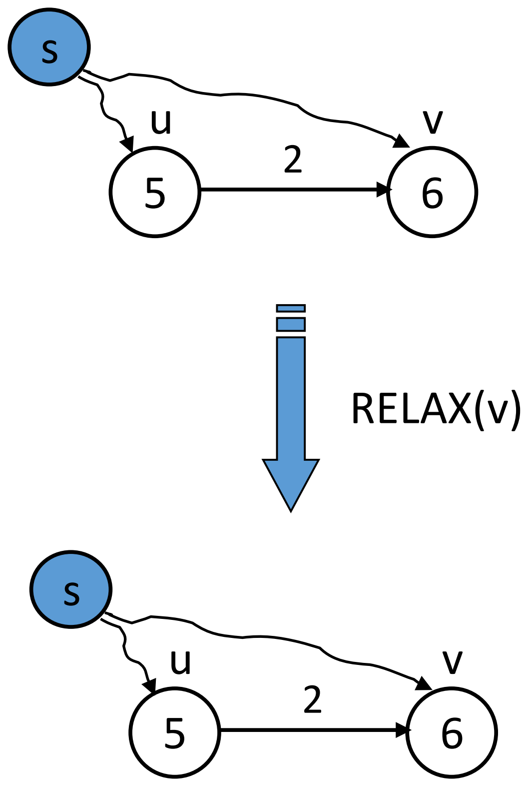
- Dijkstra refers to “relaxing” a vertex
- Meaning: update the best known shortest path to v



We are at an intermediate stage:
So far we “know” that we
can get from s to u with cost 5
and from s to v with cost 9

Using the new information about
edge (u,v) we now know there is a
cheaper path to v

Relaxation – another example

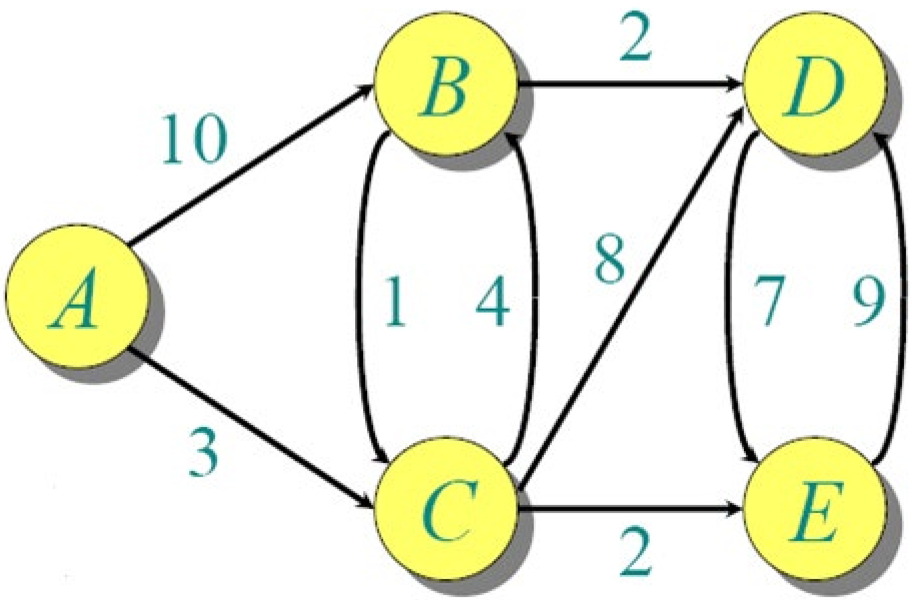


5+2 is no better than 6

No improvement,
so no change this time

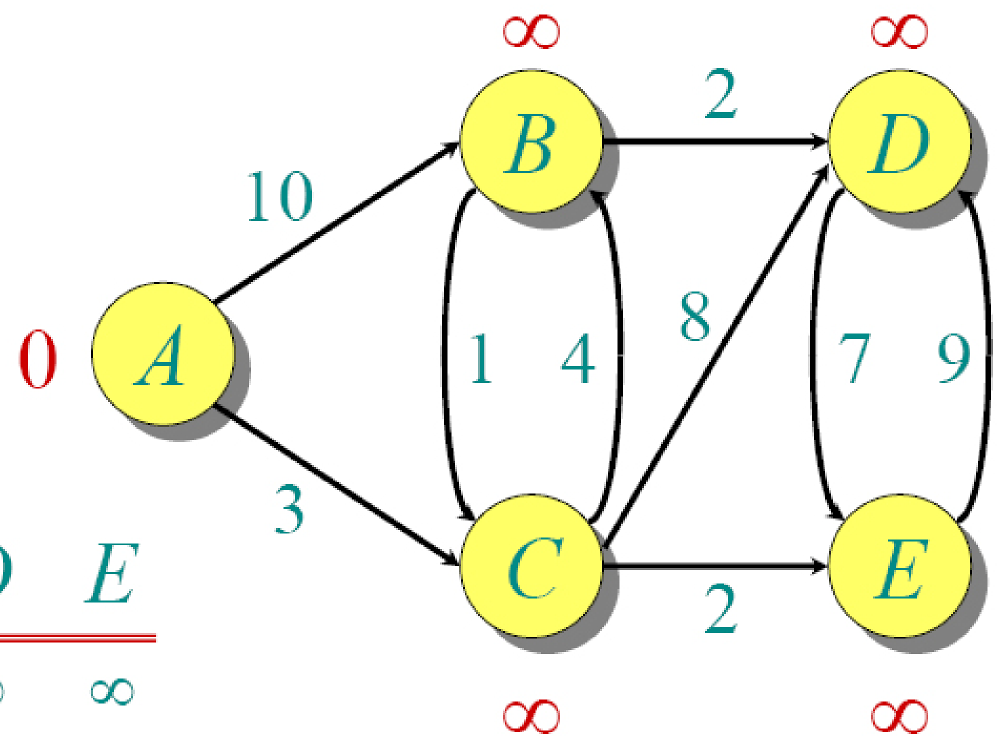
Dijkstra Example

Find the shortest paths from A to all other vertices



Dijkstra Example

Initialize:

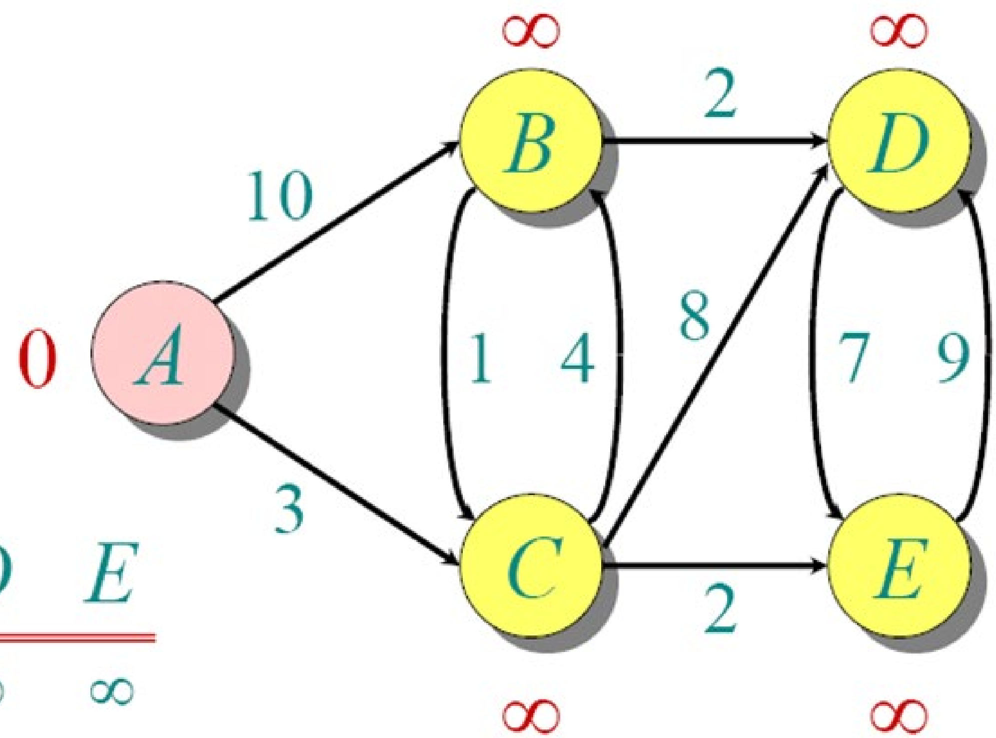


(dist)

Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: {}

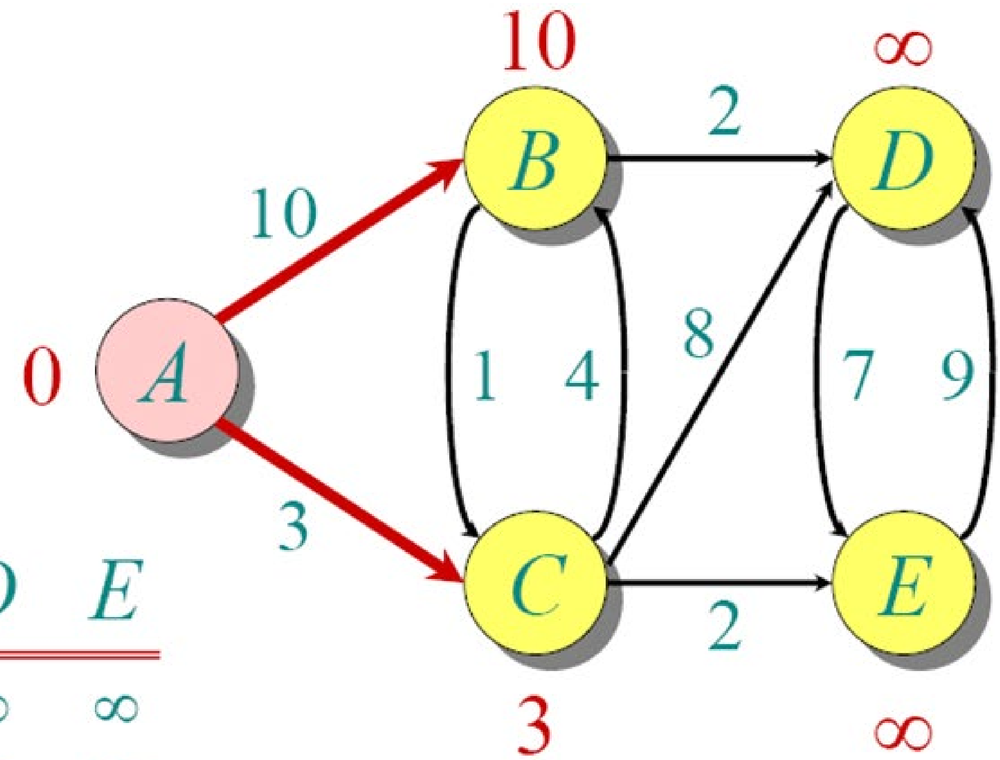
Dijkstra Example



(dist)

Q:	A	B	C	D	E
	0	∞	∞	∞	∞

Dijkstra Example

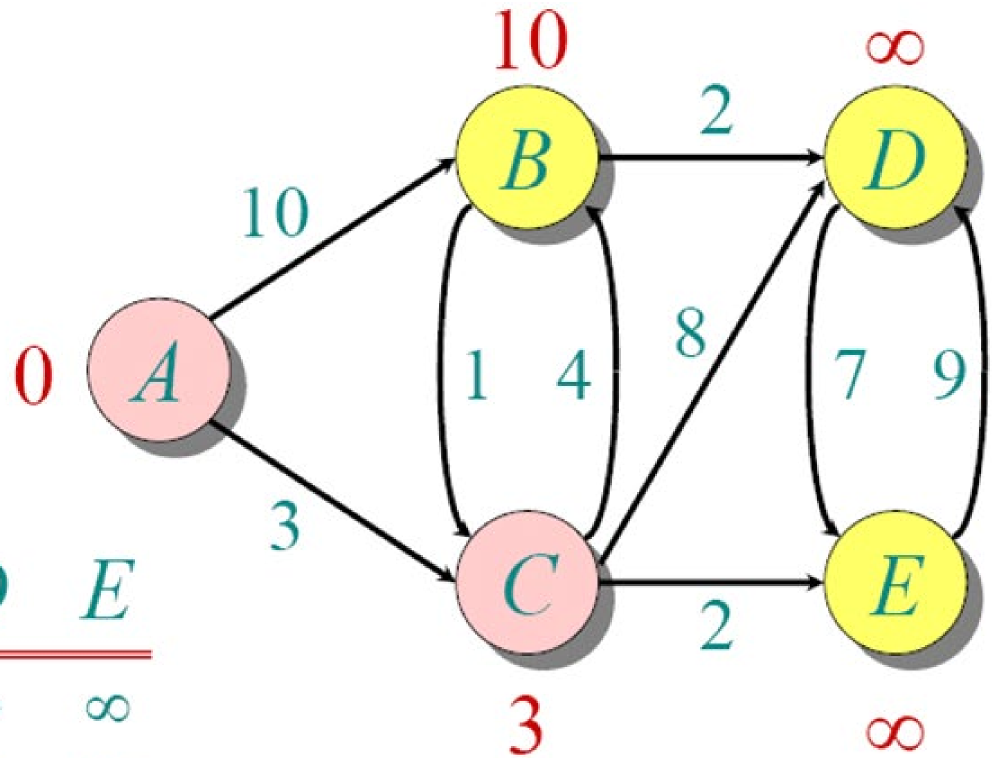


(dist)

Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A }

Dijkstra Example

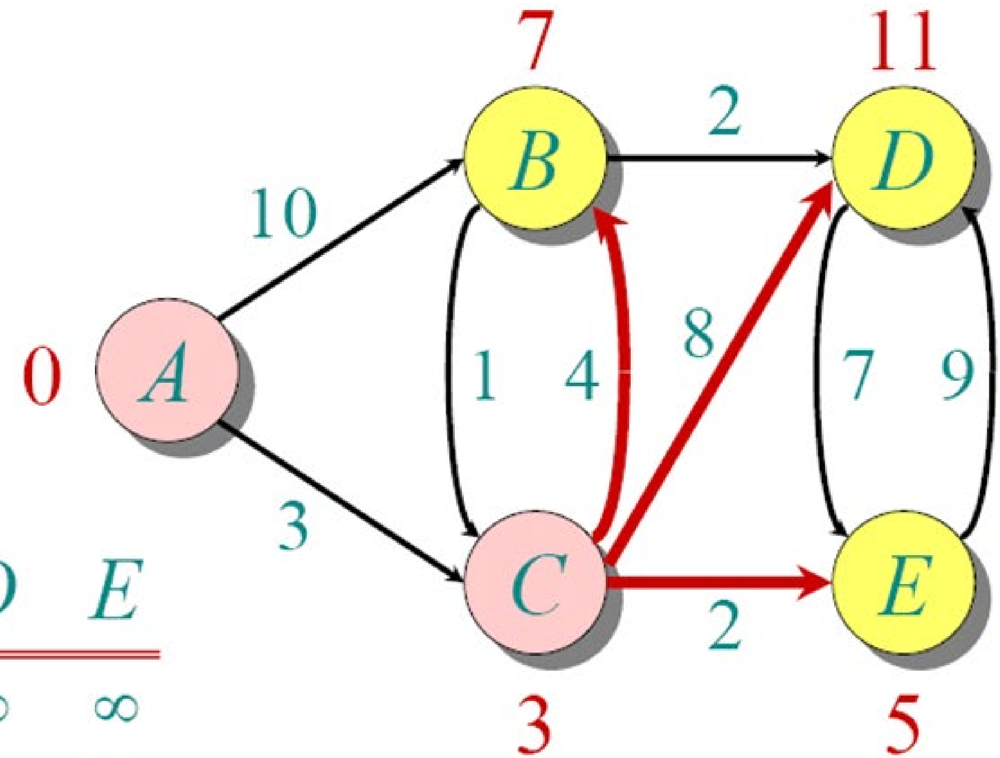


(dist)

Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A, C }

Dijkstra Example

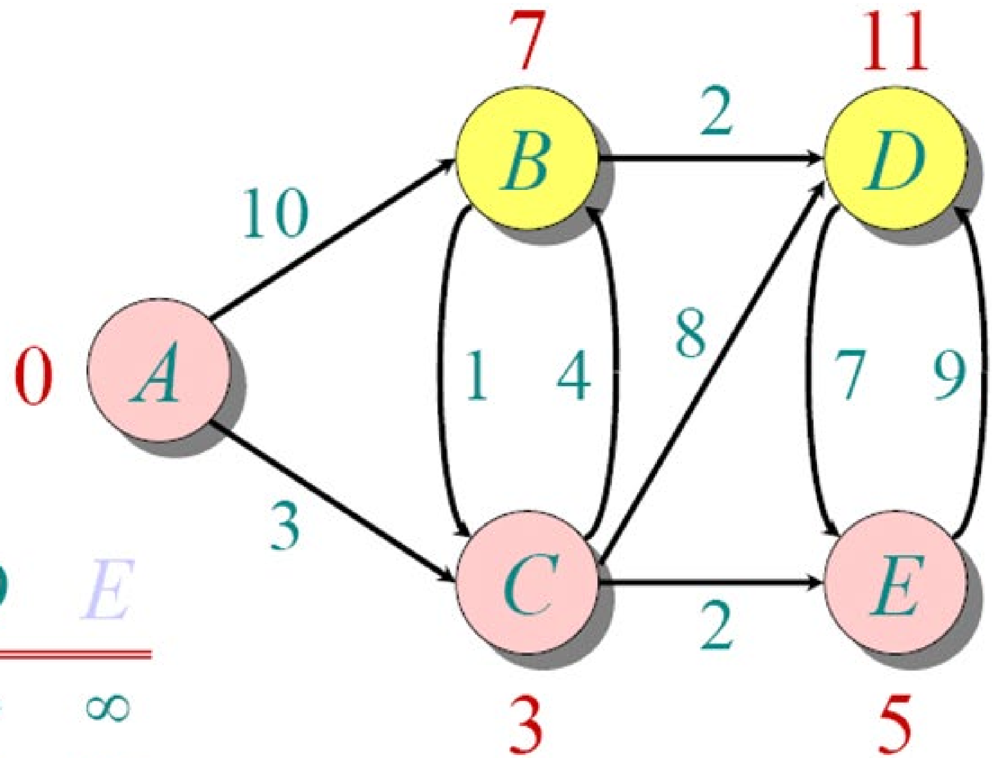


(dist)

Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { A, C }

Dijkstra Example

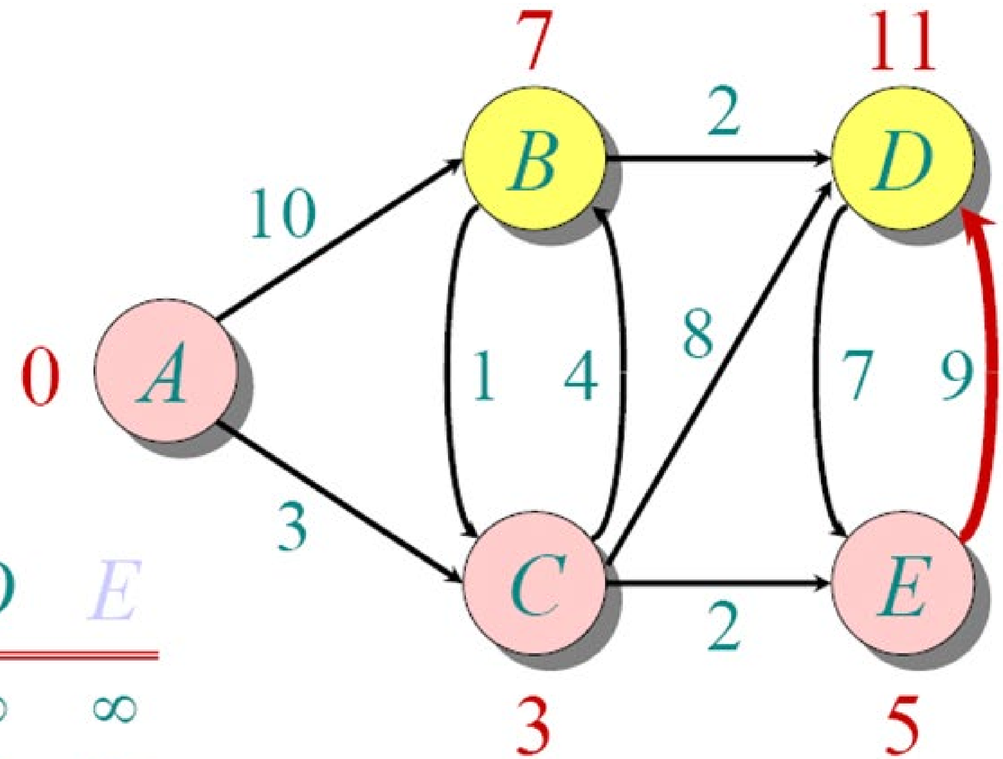


(dist)

Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { A, C, E }

Dijkstra Example

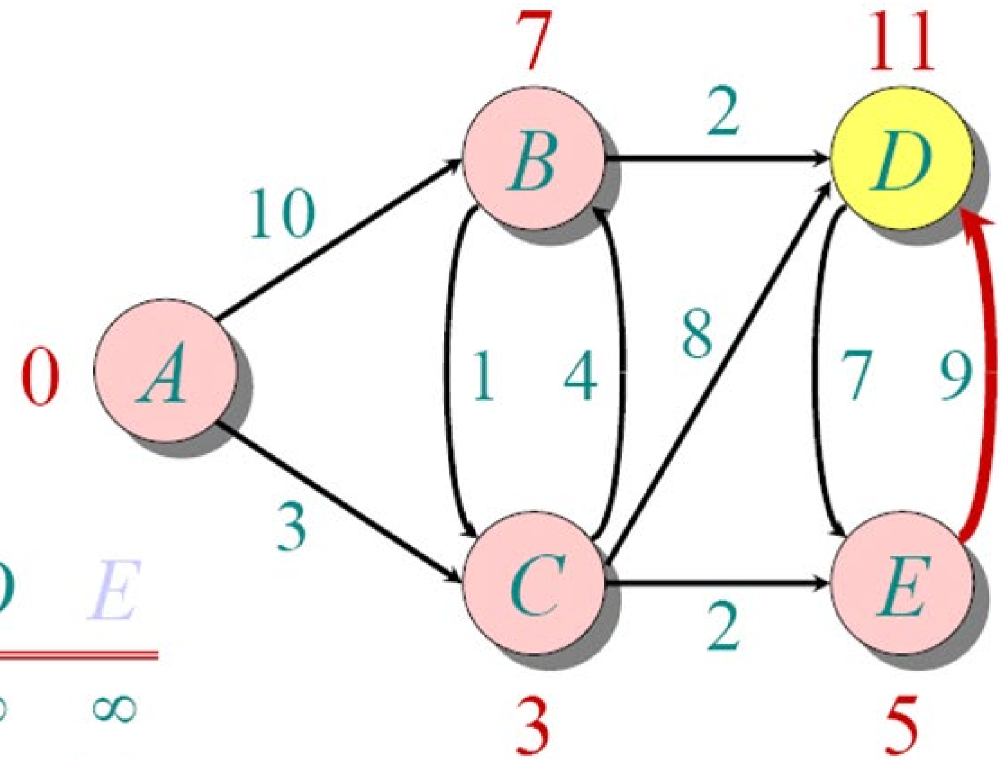


(dist)

Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E }

Dijkstra Example



(dist)

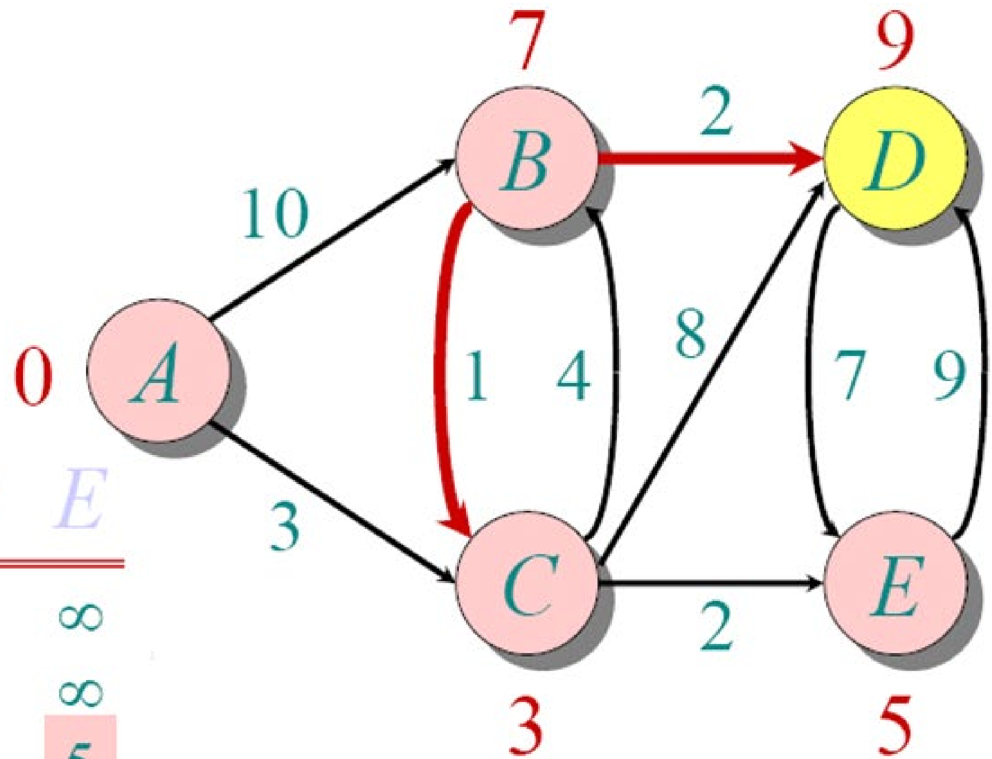
Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E, B }

Dijkstra Example

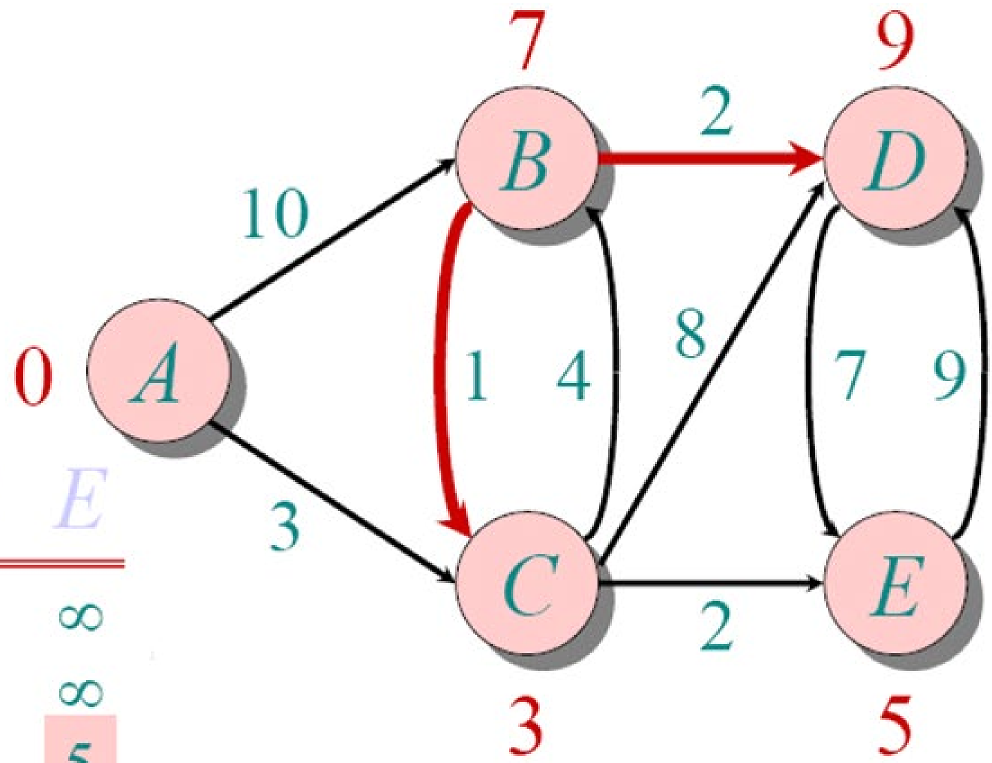
(dist)

Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	



S: { A, C, E, B }

Dijkstra Example



(dist)

Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { A, C, E, B, D }

Dijkstra’s Algorithm

- Builds a tree of shortest paths rooted at the starting vertex
- This is a greedy algorithm: it adds the closest vertex, then the next closest, and so on (until all vertices have been added)

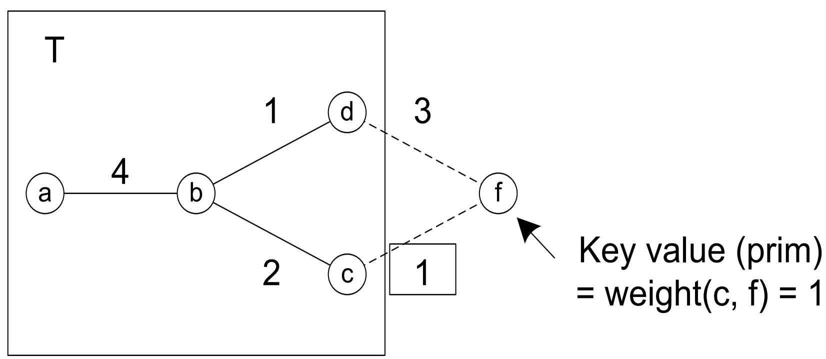
<u>High-level pseudocode:</u>

```
Initialise d and prev
Add all vertices to a PQ with distance from source as the key
While there are still vertices in PQ
    Get next vertex u from the PQ
    For each vertex v adjacent to u
        If v is still in PQ, relax v

Relax(v):
    if d[u] + w(u,v) < d[v]
        d[v] ← d[u] + w(u,v)
        prev[v] ← u
        PQ.updateKey(d[v], v)
```

Similarity of Dijkstra to Prim

- Both accumulate a tree T of edges from G
- Each iteration: select the minimum priority edge adjacent to the tree that has been built so far
- In Prim's the priority of an edge is simply the weight of the edge



- In Dijkstra's the "priority" is the weight of the edge *(u, v)* <u>plus</u> the distance from the start to the parent of *v*

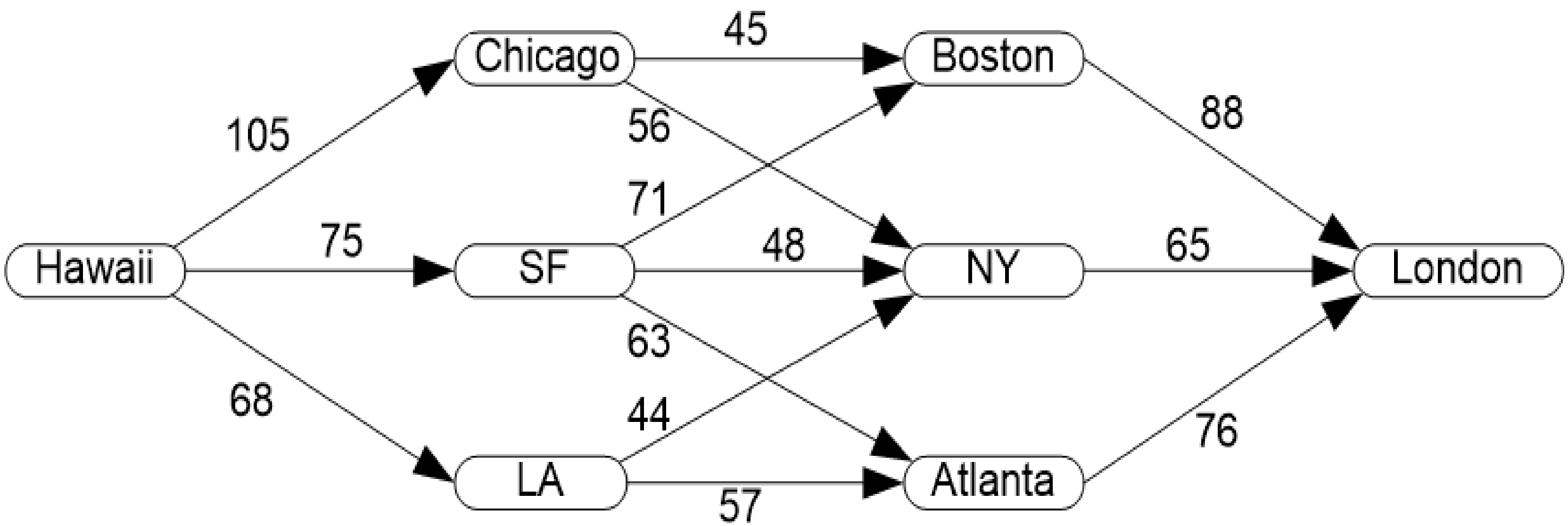
Sample application of Dijkstra’s

- Suppose London wants fresh pineapples from Hawaii.
- There are no direct flights, but many possible connections.
- What is the best possible route to minimize overall shipping cost?

Input: Shipping costs, city to city

- Honolulu to Chicago 105
- Honolulu to San Francisco 75
- Honolulu to Los Angeles 68
- Chicago to Boston 45
- Chicago to New York 56
- San Francisco to Boston 71
- San Francisco to New York 48
- San Francisco to Atlanta 63
- Los Angeles to New York 44
- Los Angeles to Atlanta 57
- Boston to London 88
- New York to London 65
- Atlanta to London 76

Graph model of the problem

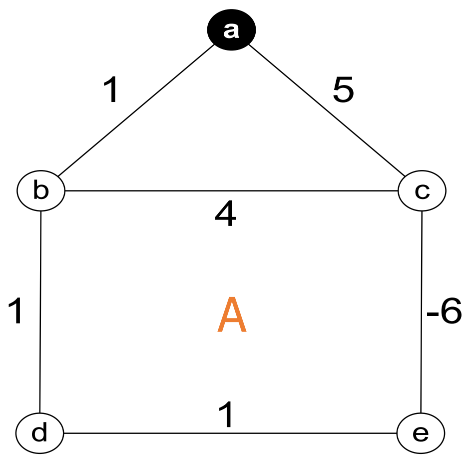


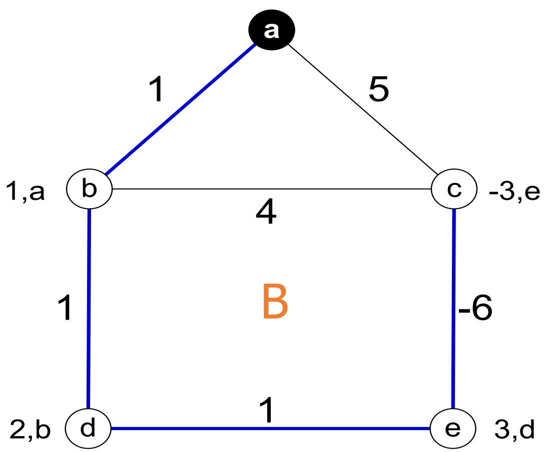
Apply Dijkstra's algorithm to find the cheapest cost from Hawaii to London
(bonus: cheapest cost to all the other cities, too)

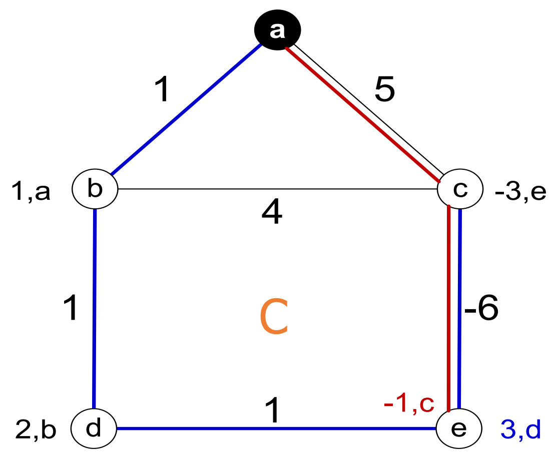
Dijkstra limitation: negative weight edges

- Dijkstra’s algorithm doesn’t work with negative weight edges
- If we added a new edge to T, and it had a negative weight, then there could exist a shorter path (through this new vertex) to vertices already in T

- For example, consider graph A below.
  - Graph B is the result of running Dijkstra's algorithm on A.
  - But clearly there exists a path such as a-c-e in graph C that is shorter than the path found in B. Therefore Dijkstra's algorithm did not work on this graph that has a negative edge weight.







Greedy Algorithms: Graph Coloring

Textbook: Mentioned several times, but not covered in-depth. Look in the index under “graph coloring”.

How many colors do you really NEED?

Map coloring

- Problem: Color the regions on a map
  - Regions that share a border must be different colors
  - Meeting at a single point is not a border
- As a decision problem:
  - Can this map be colored with N colors?
- As an optimization problem:
  - What is the minimum number of colors needed to color this map?

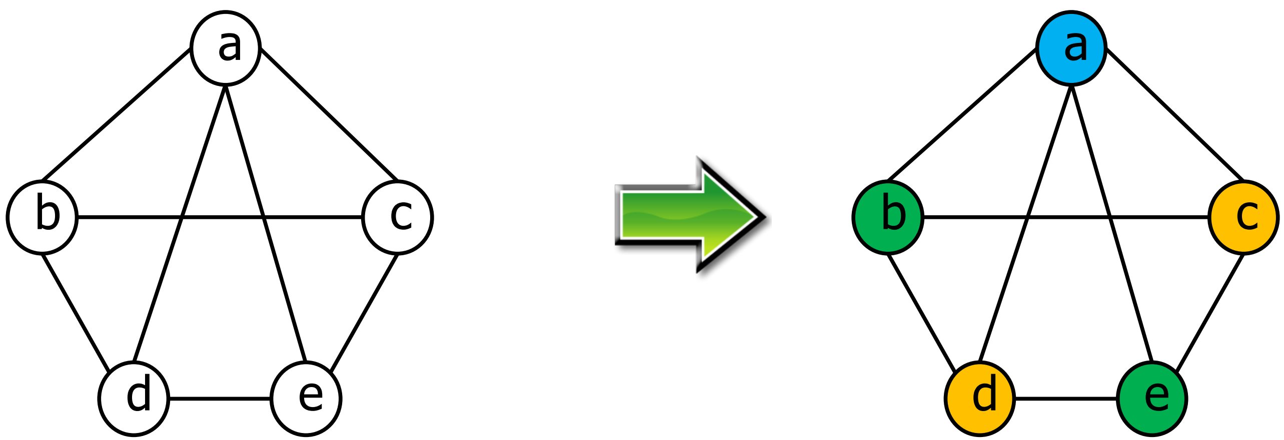
Graph representation

- One vertex for each region
- Edge between regions if they share a border

- Problem re-stated as a graph problem:
  - Assign colors to the vertices of a graph so that no adjacent vertices are the same color

Graph coloring problem

- Color a graph with as few colors as possible such that no two adjacent vertices are the same color
- Example:

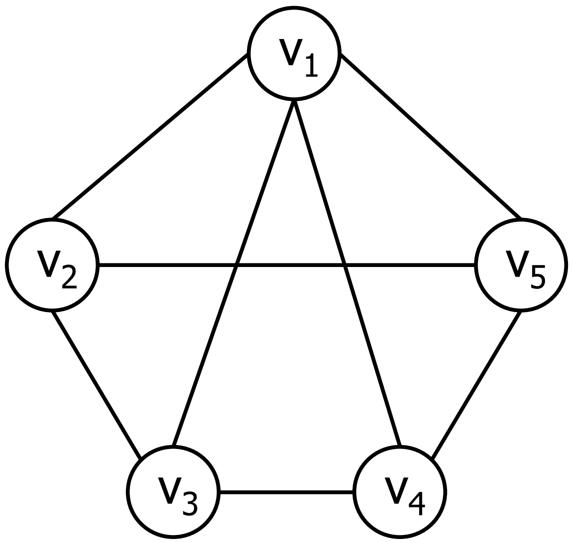


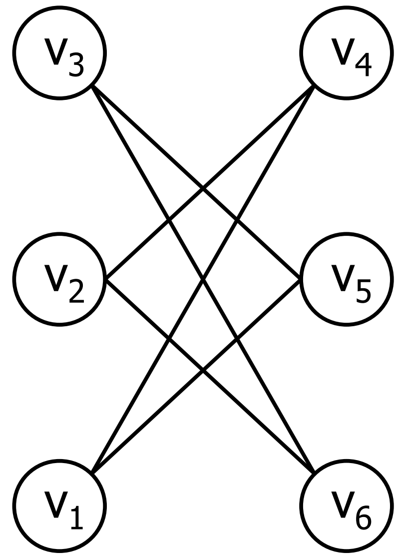
We say that this graph is *3-colorable*

Graph coloring – greedy algorithm

- Start with just one color
- Consider the vertices in a specific order $v_1, \ldots, v_n$
- For each v_i , assign the first available color not used by any of v_i's neighbours
- If all colors are in use by neighbours, add a new color

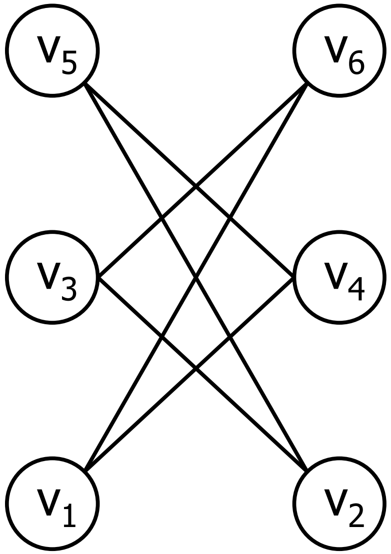
Examples





Is this algorithm optimal?

- Consider the previous graph but with vertices numbered differently



- Needed only two colors before
- The order of considering the vertices matters
- Greedy algorithms do not always yield optimal solutions
- But like brute-force, they are often worth considering because they may be easy to implement

Puzzle – just for fun!

- Make a graph that represents a planar map and that *requires* 4 colors

Practice problems

1. Chapter 9.1, page 324, question 9
2. Chapter 9.2, page 331, questions 1,2
3. Chapter 9.3, page 337, questions 1,2,4