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Module 1

Matrices and Eigenvalue Problems

Module contents

1.1 Systems of Linear Equations: Conceptual Review

The material of this section should be known to you from your previous linear algebra studies. It is repeated here to put the known to you material in the context directly relevant to the new topics that will be introduced in this subject. In particular, concepts of vectors, matrices and determinants provide a convenient framework for dealing with *systems of linear equations*, for example,

$$(1)\ \begin{cases} 2x+3y &= 8 \\ 4x+y &= 6 \end{cases},\quad (2)\ \begin{cases} ax+by &= e \\ cx+dy &= f \end{cases},$$

$$(3)\ \begin{cases} a_1x_1+a_2x_2+a_3x_3 &= y_1 \\ b_1x_1+b_2x_2+b_3x_3 &= y_2 \\ c_1x_1+c_2x_2+c_3x_3 &= y_3 \end{cases}.$$

Such systems can be written in the *matrix-vector form* as

$$(1)\ \underbrace{\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{z}} = \underbrace{\begin{bmatrix} 8 \\ 6 \end{bmatrix}}_{\mathbf{r}},\quad (2)\ \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{z}} = \underbrace{\begin{bmatrix} e \\ f \end{bmatrix}}_{\mathbf{r}},$$

$$(3)\ \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{z}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{r}}.$$

The above conversion is based on a standard *matrix-vector multiplication* that is in turn a series of *row-column multiplications*. For example, the first row $[2\ \ 3]$ in the matrix (1) is multiplied by a column *vector of unknowns* $\begin{bmatrix} x \\ y \end{bmatrix}$ according to the following:

$$\begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2x+3y,$$

which is the left-hand side of the first equation in system (1).

As you can see from the above examples systems of linear algebraic equations are generally written as

$$\mathbf{A}\mathbf{z} = \mathbf{r}\,,$$

where $\mathbf{A}$ is the *coefficient matrix*, $\mathbf{z}$ is the vector of unknowns and $\mathbf{r}$ is the *vector of right-hand sides*.

Let us now recollect how systems of linear algebraic equations are solved. There are several ways we can do that. We follow them in the specific example of the 2×2 system (2) above, identify what various methods have in common and then generalize them to larger systems. We start with a simple elimination.

From the first of equations in system (2) we obtain

$$x = \frac{e - by}{a} \tag{1.1}$$

and then substituting this into the second equation leads to

$$\frac{ce}{a} - \frac{bc}{a}y + dy = f\,. \tag{1.2}$$

After a simple re-arrangement and back-substitution of y into the right-hand side of equation (1.1) we obtain

$$y = \frac{af - ce}{ad - bc}\,, \quad x = \frac{de - bf}{ad - bc}\,. \tag{1.3}$$

The above elimination procedure can be conveniently written in the matrix form, which is known as *Gaussian elimination*. It starts with forming the *augmented matrix* containing equation coefficients and the right-hand sides:

$$\mathbf{A}|\mathbf{r} = \left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right].$$

Dividing all elements in the first row by a and in the second row by c and then subtracting the second row from the first one we obtain

$$\left[\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{e}{a} \\ 1 & \frac{d}{c} & \frac{f}{c} \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{e}{a} \\ 0 & \frac{bc - ad}{ac} & \frac{ce - af}{ac} \end{array}\right].$$

Finally, multiplying the second row by the reciprocal of the second element we obtain the augmented *upper triangular matrix* in the *row echelon form*

$$\left[\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{e}{a} \\ 0 & 1 & \frac{af - ce}{ad - bc} \end{array}\right],$$

which is equivalent to (1.3).

By inspecting solution (1.3) we recognise a number of its features.

- Expressions for x and y have the common denominator that can be recognised as the *determinant* of the 2×2 coefficient matrix $\mathbf{A}$

$$\boxed{\det(\mathbf{A}) = \Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

  Note: we will use square brackets $[\cdots]$ to denote matrices and vertical bars $|\cdots|$ to denote determinants. The determinants can also be conveniently denoted by det or Δ.

- The numerators can also be recognised as determinants of the coefficient matrix, where the column corresponding to a considered variable is replaced with the column vector of the right-hand side:

$$\Delta_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix},$$

- Solution (1.3) then can be written as

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}.$$

  This is known as *Cramer's rule*.

Finally, similar to the solution of a single equation obtained by multiplying both of its sides by the reciprocal of the coefficient

$$kx = l \quad \Longrightarrow \quad k^{-1}kx = x = k^{-1}l,$$

the solution of a system of equations written in a matrix form can be obtained by multiplying it by the *inverse matrix* $\mathbf{A}^{-1}$ defined as $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the *identity matrix* (a matrix with zero off-diagonal elements and unit elements along the *main diagonal*):

$$\mathbf{A}\mathbf{z} = \mathbf{r} \quad \Longrightarrow \quad \mathbf{A}^{-1}\mathbf{A}\mathbf{z} = \mathbf{I}\mathbf{z} \equiv \mathbf{z} = \mathbf{A}^{-1}\mathbf{r}.$$

In particular, from the above equation we conclude that:

> If the inverse matrix exists and the right-hand-side vector $\mathbf{r} = \mathbf{0}$, then the only possible solution of a linear algebraic system of equations is the *trivial solution* $\mathbf{z} = \mathbf{0}$.

For a 2×2 matrix

$$\boxed{\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}.$$

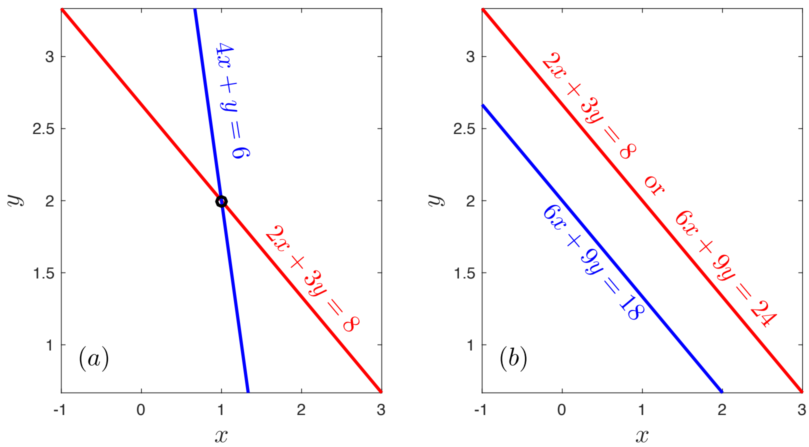


Figure 1.1: Graphical solution of a system of linear equations: (a) unique solution, (b) infinitely many or no solutions.

> The inverse matrix does not exist if the determinant of the original matrix is zero. Such matrices are called *non-invertible*.

Then in our example

$$\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} de - bf \\ af - ce \end{bmatrix},$$

which is again equivalent to solution (1.3).

The inspection of solutions obtained by various methods shows that their expressions necessarily contain division by the determinant of the coefficient matrix. The natural question then is: what happens if this determinant is zero? For example, for a 2×2 matrix this would occur whenever

$$\frac{a}{c} = \frac{b}{d}$$

that is when the corresponding equation coefficients are proportional to each other. A good way to understand this is to refer to a *geometrical meaning of linear algebraic equations*. For example, each of linear equations entering system (1) represents a straight line, see Figure 1.1(a). The two lines intersect at a single point $(x, y) = (1, 2)$ and this is the *unique* solution of system (1). The determinant of the coefficient matrix of system (1) is

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = 2 \cdot 1 - 4 \cdot 3 = -10 \neq 0\,.$$

Now consider a system with proportional coefficients

$$\begin{cases} 2x + 3y &= 8 \\ 6x + 9y &= 24 \end{cases}.$$

The determinant of its coefficient matrix $\det(\mathbf{A}) = 2 \cdot 9 - 6 \cdot 3 = 0$. Clearly, the second equation is simply a multiple of the first one and thus graphically is represented by exactly the same line, see Figure 1.1(b). This can be viewed as "a complete overlap" of the two lines, thus every point that belongs to these lines is a solution of the above system of equations. The total number of solutions is infinite. To find them we cannot use Cramer's rule or matrix inversion because both of them would require dividing by a zero determinant. Thus we are limited to Guassian elimination

$$\begin{bmatrix} 2 & 3 & | & 8 \\ 6 & 9 & | & 24 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & \frac{3}{2} & | & 4 \\ 1 & \frac{3}{2} & | & 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & \frac{3}{2} & | & 4 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

Note that one of the equations disappeared completely. This means that

> whenever the determinant of a system of linear algebraic equations is zero its coefficient matrix consists of *linearly dependent vectors* (columns or rows).

Recollect that

Definition 1.1 *A set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is said to be linearly dependent if the equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$ can be satisfied when constants c_1, c_2, $\cdots$, c_n are not simultaneously zero.*

In a two-dimensional case the linear dependence means that one vector is just a multiple of another.

Since we are left with a single equation for two unknowns, the solution can be written in a parametric form by making one of the uknowns, say x, an arbitrary parameter

$$x = t\,, \quad y = \frac{8}{3} - \frac{2}{3}t \text{ or vectorially } \mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{8}{3} \end{bmatrix}.$$

Note a few important features of the above solution. Firstly, since t is an arbitrary parameter, we can redefine it without loss of generality as $t = 3p$ and then conveniently write the solution as

$$\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix} = p \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{8}{3} \end{bmatrix}.$$

Secondly, the second part of the solution represents an arbitrary point on the line corresponding to the remaining equation. We can choose it freely. For example, since point $(x, y) = (1, 2)$ satisfies the equation, we can write

$$\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix} = p_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

It is easy to check that the above is obtained by simply redefining $p = p_1 + \frac{1}{3}$. This type of manipulations between equivalent results will be commonly used in our further studies. Note also that the vectors multiplied by different parameters (t, p, p_1) that define the multitude of solutions are necessarily linearly dependent and thus equivalent to each other in the context of the material that will follow.

Finally, consider a slightly modified system

$$\begin{cases} 2x + 3y &= 8 \\ 6x + 9y &= 18 \end{cases}.$$

It has the same coefficients, zero determinant but a different right-hand side. Graphically, this corresponds to two parallel straight line with a different intercepts, see Figure 1.1(b). Such lines do not intersect and thus the system of equation has no solutions. Gaussian elimination procedure confirms this:

$$\begin{bmatrix} 2 & 3 & | & 8 \\ 6 & 9 & | & 18 \end{bmatrix} \implies \begin{bmatrix} 1 & \frac{3}{2} & | & 4 \\ 1 & \frac{3}{2} & | & 3 \end{bmatrix} \implies \begin{bmatrix} 1 & \frac{3}{2} & | & 4 \\ 0 & 0 & | & 1 \end{bmatrix}.$$

The inconsistency between the zero last row in the reduced coefficient matrix and non-zero last element in the modified right-hand side demonstrates the contradiction and thus the lack of a solution.

The important conclusions that we should make from considering the above simple examples and that we will need to keep in mind when studying new material in this subject are that for square matrices, or systems of n linear algebraic equations with n unknowns:

- if the determinant of the coefficient matrix of a system of linear algebraic equations is non-zero, such a system has a unique solution that can be found by any of the methods discussed above;

- if the determinant of the coefficient matrix of a system of linear algebraic equations is zero, such a system has either no solutions or infinitely many solutions that can only be found by Gaussian elimination not requiring an explicit division by the determinant;
- if the right-hand side of a system of linear algebraic equations is zero, the system can only have a non-trivial solution if the determinant of the coefficient matrix is zero.

This completes a brief conceptual review relevant to this subject. However, to make progress you will need to be fluent with the generalisations of the techniques mentioned above to systems with larger dimensions (more than two equations) and with several more definitions that will be required when we introduce new material. For your convenience the review including more examples is given in Appendices A.1 and A.3. However, since you have already studied this material previously, you are expected to brush up on it in your own time. It will not be lectured in class in this subject.

1.2 Eigenvalue Problems

1.2.1 Introduction: geometrical meaning of matrix-vector multiplication

Let us consider system (1) written in a matrix-vector form $\mathbf{A}\mathbf{z} = \mathbf{r}$ in a somewhat different context. It can be interpreted as transforming its solution vector $\mathbf{z} = [1, 2]^T$ into the right-hand side vector $\mathbf{r} = [8, 6]^T$, where the superscript T denotes *matrix transposition* (in this particular case a row vector used for convenience of in-line writing becomes a column vector) by multiplying $\mathbf{z}$ by $\mathbf{A}$. As seen from Figure 1.2 multiplication of a vector by a matrix has two effects:

- rotation of a vector and
- stretching of a vector.

We discuss these two effects below in more detail.

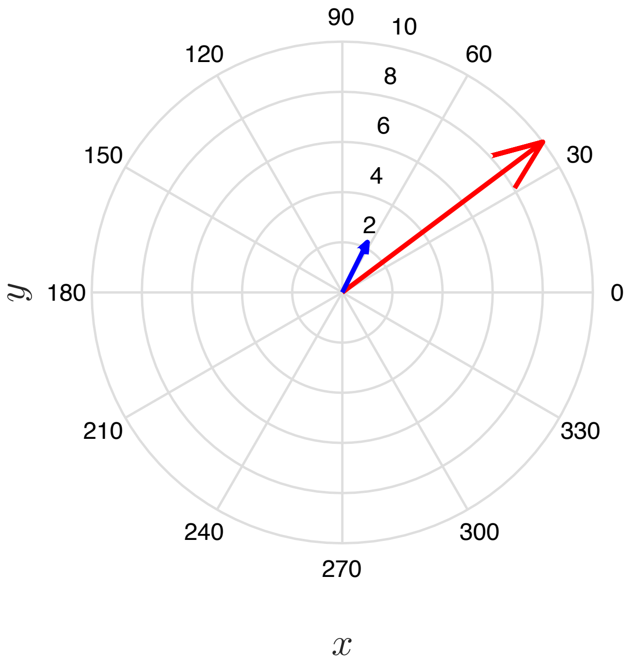


Figure 1.2: Transforming a vector by matrix multiplication.

1.2.2 Rotation matrices

We start with designing a matrix that will rotate a vector by angle $-\theta$ without stretching it. In fact, it is more convenient to solve a slightly different problem: let a vector be fixed and the coordinate system rotate by the angle θ (that is in the opposite direction to the rotation of a vector). For example, the $y_{1,2}$ system in Figure 1.3 can be considered as a rotation of the $x_{1,2}$ system about the origin through an angle θ in a anti-clockwise direction. Let point P have coordinates (x_1, x_2) and (y_1, y_2) in the original and rotated systems, respectively. From the diagram we see that $OS = x_1$, $PS = x_2$, $OR = y_1$ and $PR = y_2$. Then

$$y_1 = OQ + QR = OQ + ST = OS\cos\theta + PS\sin\theta = x_1\cos\theta + x_2\sin\theta$$

i.e. $y_1 = x_1\cos\theta + x_2\sin\theta$. Similarly,

$$y_2 = PT - RT = PT - QS = PS\cos\theta - OS\sin\theta = x_2\cos\theta - x_1\sin\theta\,.$$

This can be expressed as

$$\begin{aligned} y_1 &= \quad x_1\cos\theta + x_2\sin\theta\,, \\ y_2 &= -x_1\sin\theta + x_2\cos\theta\,. \end{aligned}$$

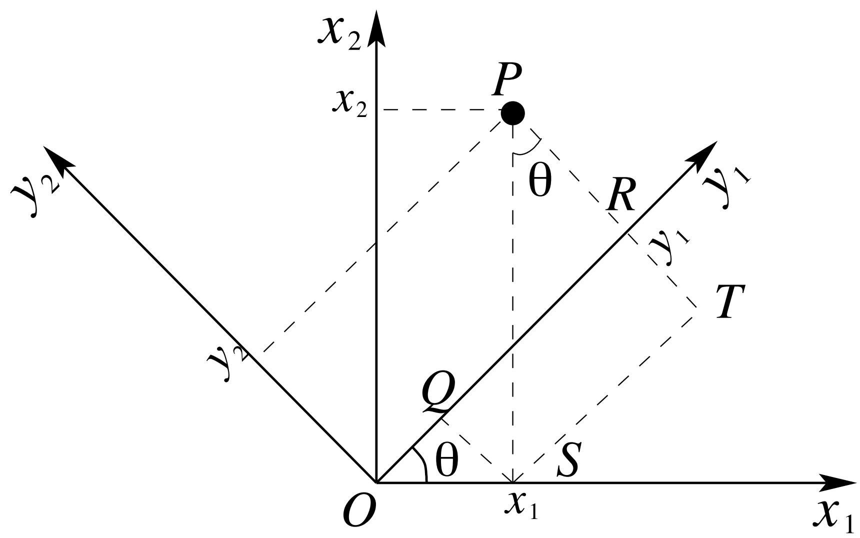


Figure 1.3: Rotation of a coordinate system.

As a matrix equation this is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{R}[\theta] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ,$$

where

$$\boxed{\mathbf{R}[\theta] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}$$

is a *rotation matrix*.

Note: $\det(\mathbf{R}[\theta]) = \cos^2\theta + \sin^2\theta = 1$. Also

$$\mathbf{R}^{-1}[\theta] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \mathbf{R}[-\theta] = \mathbf{R}^T[\theta] .$$

Because of this property, rotation matrices belong to the special class of matrices, the so-called orthogonal matrices.

Definition 1.2 *Matrix* $\mathbf{A}$ *is called orthogonal if* $\mathbf{A}^{-1} = \mathbf{A}^T$ *or, equivalently, if* $\mathbf{A}\mathbf{A}^T = \mathbf{I}$.

The reason why such matrices are called orthogonal becomes clear from the following consideration:

$$(\mathbf{A}\mathbf{A}^T)_{ij} = r_i(\mathbf{A})c_j(\mathbf{A}^T) = r_i(\mathbf{A})r_j(\mathbf{A}) = \mathbf{I}_{ij} ,$$

where $\mathbf{I}_{ij} = 0$ if $i \neq j$ and $\mathbf{I}_{ij} = 1$ if $i = j$. That is $r_i(\mathbf{A})r_j(\mathbf{A}) = 0$ if $i \neq j$ and $r_i(\mathbf{A})r_i(\mathbf{A}) = 1$. Thus distinct rows of an orthogonal matrix are orthogonal unit vectors and so are its column vectors. Since $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$, from $\mathbf{I} = \mathbf{A}\mathbf{A}^T$ we obtain that

$$\det(\mathbf{I}) = \det(\mathbf{A}\mathbf{A}^T) = \det(\mathbf{A})\det(\mathbf{A}^T) = \det(\mathbf{A})^2$$

so that for an orthogonal matrix $\det(\mathbf{A}) = \pm 1$.

Note that rotation matrices are just one example of orthogonal matrices. More general orthogonal matrices can be constructed from n n-dimensional orthogonal vectors.

EXAMPLE **1.1.** *Consider 3 three-dimensional vectors.*

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Check that they are mutually orthogonal first:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= 2 \cdot 3 + 1 \cdot (-2) + 4 \cdot (-1) = 6 - 2 - 4 = 0\,, \\ \mathbf{a} \cdot \mathbf{c} &= 2 \cdot 1 + 1 \cdot 2 + 4 \cdot (-1) = 2 + 2 - 4 = 0\,, \\ \mathbf{b} \cdot \mathbf{c} &= 3 \cdot 1 + (-2) \cdot 2 + (-1) \cdot (-1) = 3 - 4 + 1 = 0\,. \end{aligned}$$

Next construct the corresponding unit vectors:

$$\hat{\mathbf{a}} = \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \quad \hat{\mathbf{b}} = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \quad \hat{\mathbf{c}} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Then

$$\mathbf{A} = \begin{bmatrix} \frac{2}{\sqrt{21}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{21}} & -\frac{2}{\sqrt{14}} & \frac{2}{\sqrt{6}} \\ \frac{4}{\sqrt{21}} & -\frac{1}{\sqrt{14}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

is an orthogonal matrix.

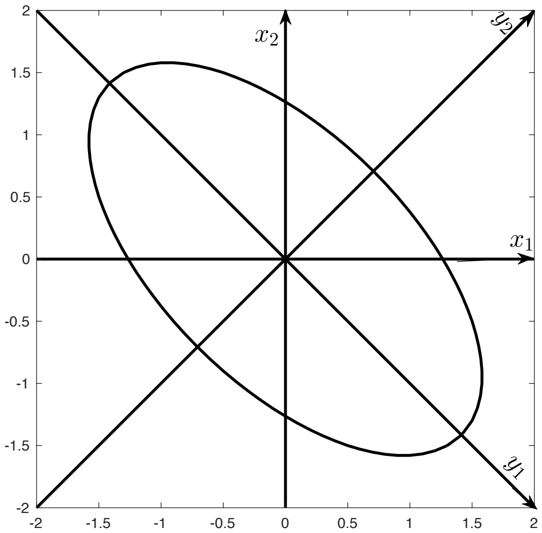


Figure 1.4: Rotated ellipse.

Exercise

EX. 1.1. Verify that vectors $\mathbf{e}_1 = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 10 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{e}_3 = \begin{bmatrix} 5 \\ -6 \\ -8 \end{bmatrix}$ are orthogonal and use them to construct an orthogonal 3×3 matrix.

We will see where general orthogonal matrices are used later in this subject, but here we will illustrate the practical use of their specific representative, the rotation matrix.

EXAMPLE **1.2.** *A curve has equation $5x_1^2 + 6x_1x_2 + 5x_2^2 = 8$ in the x-coordinate system. If the y-coordinate system is obtained by rotating the x-coordinates through $45°$ in a clockwise direction, find the equation to the curve in the y-coordinate system and determine the type of the curve.*

SOLUTION. *First, convert the rotation angle to radians keeping in mind the that a clockwise rotation corresponds to a negative angle: $\theta = -\dfrac{\pi}{4}$. Then writing $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ we obtain $\mathbf{y} = \mathbf{R}\left[-\dfrac{\pi}{4}\right]\mathbf{x}$ so that $\mathbf{x} = \mathbf{R}^{-1}\left[-\dfrac{\pi}{4}\right]\mathbf{y} = \mathbf{R}\left[\dfrac{\pi}{4}\right]\mathbf{y}$. Writing this out in* *full we obtain*

$$\begin{aligned} x_1 &= y_1 \cos\left(\frac{\pi}{4}\right) + y_2 \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(y_1 + y_2)\,, \\ x_2 &= -y_1 \sin\left(\frac{\pi}{4}\right) + y_2 \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-y_1 + y_2)\,. \end{aligned}$$

Substituting these expressions into the equation to the curve gives

$$5 \cdot \frac{1}{2}(y_1 + y_2)^2 + 6 \cdot \frac{1}{2}(y_1 + y_2)(-y_1 + y_2) + 5 \cdot \frac{1}{2}(-y_1 + y_2)^2 = 8\,.$$

Expanding and multiplying both sides by 2 gives

$$5(y_1^2 + 2y_1y_2 + y_2^2) + 6(-y_1^2 + y_2^2) + 5(y_1^2 - 2y_1y_2 + y_2^2) = 16\,,$$

which simplifies to

$$4y_1^2 + 16y_2^2 = 16 \text{ or } \frac{y_1^2}{4} + y_2^2 = 1\,.$$

We can now identify the curve as an ellipse with y_1-intercepts ± 2 and y_2-intercepts ± 1, see Figure 1.4. ■

The main conclusion from the above example is that by rotating the coordinate system we do not change the shape of the curve, but choose a "more convenient viewing direction" for that curve. Algebraically, this leads to a significant simplification of the equation defining the curve. The main remaining question though is: how could we know that the coordinate system had to be rotated by exactly $\theta = -\frac{\pi}{4}$ to achieve this simplification. The answer is not straightforward. We will need to learn quite a bit more about matrices before we can formulate it.

1.2.3 Stretching without rotation: eigenvalues and eigenvectors

Having found a matrix that rotates vectors without stretching them it is logical to ask whether it is possible to find a matrix that would stretch (compress) all vectors without rotating them. Recollect that stretching (compressing) a vector ($\mathbf{x}$) is achieved by multiplying it by a scalar (λ), which can be written as

$$\lambda\mathbf{x} = \lambda\mathbf{I}\mathbf{x}\,,$$

where $\mathbf{I}$ is the identity matrix. Then multiplying a vector by matrix $\mathbf{\Lambda} = \lambda\mathbf{I}$, that is by a diagonal matrix with identical elements λ along the main diagonal, will stretch this vector by the factor of λ.

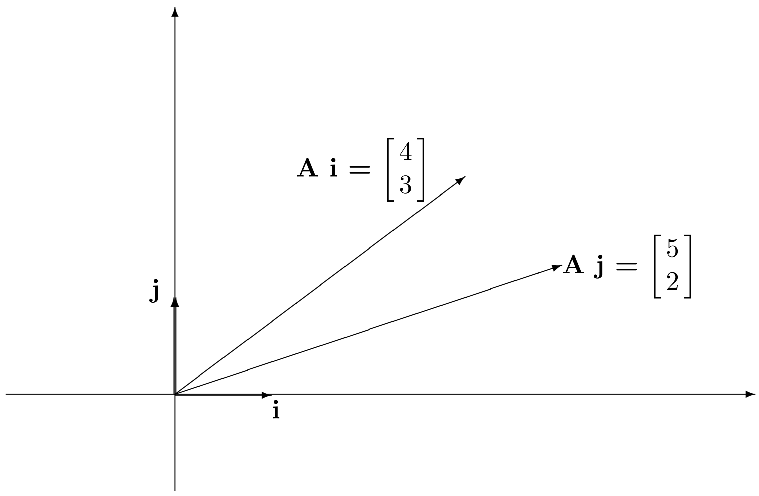


Figure 1.5: Vector transformation via matrix multiplication.

There exist no general matrices with non-zero (real) off-diagonal elements by which a vector can be stretched without rotation. However, let us investigate what happens with *different* vectors multiplied by the *same general square* matrix, for example, $\mathbf{A} = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$. This is not a rotation matrix because its column vectors are not orthogonal: $4 \cdot 5 + 3 \cdot 2 = 26 \neq 0$. If it multiplies unit vectors $\mathbf{i} = [1, 0]^T$ and $\mathbf{j} = [0, 1]^T$, we obtain

$$\mathbf{Ai} = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \text{ and } \mathbf{Aj} = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

In this example the multiplication by matrix $\mathbf{A}$ rotates vector $\mathbf{i}$ in the anti-clockwise direction and vector $\mathbf{j}$ in the clockwise direction, see Figure 1.5. It seems reasonable then to assume that there should be a vector $\mathbf{x}$ with the direction that is unaltered by multiplication by $\mathbf{A}$. Of course, this vector would depend on the matrix itself, its existence is the property of this particular matrix. Such a vector is called an *eigenvector* from German "own": it "belongs" to the matrix.

On the other hand, stretching an arbitrary vector is achieved by multiplying it by a scalar, say, λ, see Figure 1.6. Then we can write $\mathbf{Ax} = \lambda\mathbf{x}$.

Definition 1.3 *A nonzero vector* $\mathbf{x}$ *satisfying* $\mathbf{Ax} = \lambda\mathbf{x}$ *is called an* eigenvector, *and the scalar quantity* λ *is called an* eigenvalue *of matrix* $\mathbf{A}$.

To find the eigenvalues of a matrix we consider the equation $\mathbf{Ax} = \lambda\mathbf{x}$ in the re-arranged form $\mathbf{Ax} - \lambda\mathbf{Ix} = \mathbf{0}$ or $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, where $\mathbf{I}$ is the identity

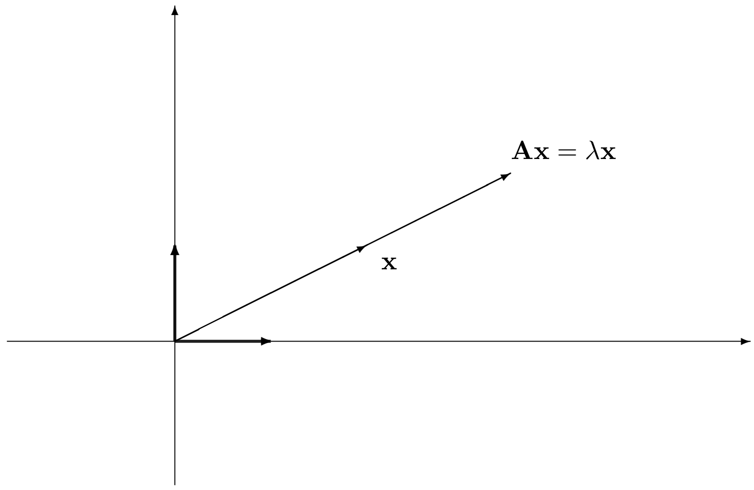


Figure 1.6: Eigenvector.

matrix of the same dimensions as $\mathbf{A}$. This last equation shows that the matrix $\mathbf{A} - \lambda\mathbf{I}$ maps the eigenvector $\mathbf{x}$ to $\mathbf{0}$. This can be restated in terms of linear algebraic equations for the components of vector $\mathbf{x}$. For $\mathbf{x}$ to be nonzero (non-trivial) and yet for its components to satisfy the system of linear equations with the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$ and zero right-hand side this matrix must have zero determinant, see discussion in Section 1.1

$$\boxed{|\mathbf{A} - \lambda\mathbf{I}| = 0}\,.$$

This is called the *characteristic equation of matrix* $\mathbf{A}$.

EXAMPLE **1.3.** *Find all eigenvalues and eigenvectors of matrix* $\mathbf{A} = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$.

SOLUTION. *The characteristic equation is*

$$\left| \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0 \text{ or } \begin{vmatrix} 4-\lambda & 5 \\ 3 & 2-\lambda \end{vmatrix} = 0\,.$$

Thus we obtain $(4-\lambda)(2-\lambda) - 15 = 0$. *Hence* $8 - 6\lambda + \lambda^2 - 15 = 0$ *or* $\lambda^2 - 6\lambda - 7 = 0$ *and* $(\lambda - 7)(\lambda + 1) = 0$. *Therefore,* $\lambda_{1,2} = 7, -1$ *are the two eigenvalues of* $\mathbf{A}$. *Corresponding to each eigenvalue, there is an eigenvector (more accurately, there is an infinite number of eigenvectors because any multiple of an eigenvector is also an eigenvector as we will see below).*

Eigenvectors corresponding to $\lambda = \lambda_1 = 7$. *If* $\begin{bmatrix} x \\ y \end{bmatrix}$ *is an eigenvector corresponding to* $\lambda = 7$, *then* $\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 7 \begin{bmatrix} x \\ y \end{bmatrix}$ *or*

$$\begin{aligned} 4x + 5y &= 7x\,, \\ 3x + 2y &= 7y\,, \end{aligned}$$

which is equivalent to the pair of equations

$$\begin{aligned} -3x + 5y &= 0\,, \\ 3x - 5y &= 0\,. \end{aligned}$$

Note that the two equations are identical up to a multiplicative constant. This is the consequence of the fact that the determinant of the system is set to zero.

Set $y = t_1$, *then* $x = \dfrac{5t_1}{3}$ *and all vectors of the form* $t_1 \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}$ *are eigenvectors corresponding to the eigenvalue* $\lambda_1 = 7$. *For example, setting* $t_1 = 3$ *gives the eigenvector* $\mathbf{e}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. *Note that* $||\mathbf{e}_1|| = \sqrt{34}$ *so the* unit eigenvector *corresponding to* $\lambda = 7$ *is* $\hat{\mathbf{e}}_1 = \begin{bmatrix} \frac{5}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{bmatrix}$.

Eigenvectors corresponding to $\lambda = \lambda_2 = -1$. *If* $\begin{bmatrix} x \\ y \end{bmatrix}$ *is an eigenvector corresponding to* $\lambda = -1$*, then* $\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} x \\ y \end{bmatrix}$ *or*

$$\begin{aligned} 4x + 5y &= -x\,, \\ 3x + 2y &= -y\,, \end{aligned}$$

which is equivalent to the pair of equations

$$\begin{aligned} 5x + 5y &= 0\,, \\ 3x + 3y &= 0\,. \end{aligned}$$

Set $y = t_2$ *to obtain* $x = -t_2$*. Then all eigenvectors corresponding to the eigenvalue* $\lambda = -1$ *can be written in the form* $t_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$*. For example, setting* $t_2 = 1$ *gives the eigenvector* $\mathbf{e}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$*. Note that* $||\mathbf{e}_2|| = \sqrt{2}$ *so that the unit eigenvector corresponding to* $\lambda = -1$ *is* $\hat{\mathbf{e}}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. ■

1.2.4 Properties of eigenvalues and eigenvectors

The eigenvalues λ_i of a square $n \times n$ matrix are the solutions of the characteristic equation $\det(\mathbf{A} - \lambda_i \mathbf{I}) = 0$, $i = 1, 2, \ldots$. This is a polynomial equation of degree n in λ_i. It has exactly n solutions, some or all of which can be complex and some could be given by repeated roots of the equation. Therefore, $n \times n$ matrix has n eigenvalues.

Definition 1.4 *The set of all eigenvalues of a matrix is called the* spectrum of the matrix.

Definition 1.5 *The* spectral radius *of a matrix is the maximum of the magnitudes of the absolute values of its eigenvalues, i.e. the largest of* $|\lambda_1|$, $|\lambda_2|$, $\cdots$, $|\lambda_n|$.

Suppose $\mathbf{A}$ is a square matrix with spectrum λ_1, λ_2, $\cdots$, λ_n. Then the following statements are true (we provide short proofs for some of them, but skip those that require the knowledge that is outside the scope of this subject).

1. The real matrix $\mathbf{A}^T$ has the same spectrum as $\mathbf{A}$.

   Proof: Recollect that the determinants of a matrix and its transpose are equal and take into account that the transpose of a diagonal matrix coincides with that matrix. Then the characteristic equation for a transposed matrix is the the same as that for the original matrix:

   $$\det(\mathbf{A}^T - \lambda\mathbf{I}) = \det(\mathbf{A}^T - \lambda\mathbf{I}^T) = \det((\mathbf{A} - \lambda\mathbf{I})^T) = \det(\mathbf{A} - \lambda\mathbf{I}) = 0\,.$$

   ♠

2. $\mathbf{A}^{-1}$ exists if and only if each eigenvalue in the spectrum of $\mathbf{A}$ is non-zero.

   Proof: If a matrix has a zero eigenvalue, then its characteristic equation becomes $\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{A}) = 0$. Since the determinant of such a matrix is zero, it cannot be inverted. ♠

3. The eigenvalues of $\mathbf{A}^{-1}$ are $\dfrac{1}{\lambda_i}$.

   Proof: Multiply both sides of the eigenvalue problem $\mathbf{Ax} = \lambda_i\mathbf{x}$ by $\mathbf{A}^{-1}$ and divide by λ_i to obtain $\mathbf{A}^{-1}\mathbf{x} = \dfrac{1}{\lambda_i}\mathbf{x}$. Thus $\dfrac{1}{\lambda_i}$, $i = 1, 2, \ldots, n$ are the eigenvalues of $\mathbf{A}^{-1}$. ♠

4. The eigenvalues of $\mathbf{A}^m$ are λ_i^m, where m is a positive integer.

5. **Definition 1.6** *The sum of the diagonal elements of a matrix,* $a_{11} + a_{22} + \cdots + a_{nn}$ *is called the* trace of matrix*, or* diagonal sum of matrix $\mathbf{A}$ *and is denoted by* $\mathrm{Tr}(\mathbf{A})$*.*

   It can be shown that $\mathrm{Tr}(\mathbf{A}) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

6. $\det(\mathbf{A}) = \lambda_1 \cdot \lambda_2 \cdot \;\cdots\; \cdot \lambda_n$.

Linear independence of eigenvectors

Theorem 1.7 *Eigenvectors corresponding to distinct eigenvalues are linearly independent.*

Proof: *We prove this theorem by contradiction. Assume that eigenvalues* λ_1, λ_2, λ_3, $\ldots$, λ_n *of* $n \times n$ *matrix* $\mathbf{A}$ *are distinct, yet the eigenvectors* $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\ldots$, $\mathbf{e}_n$ *are linearly dependent. This means that there exist non-zero constants* c_1, c_2, c_3, $\ldots$, c_m, $m \le n$ *such that*

$$\mathbf{v}_0 = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 \cdots + c_m\mathbf{e}_m = \mathbf{0}\,.$$

Then

$$\mathbf{v}_1 = \mathbf{A}\mathbf{v}_0 = c_1\lambda_1\mathbf{e}_1 + c_2\lambda_2\mathbf{e}_2 + c_3\lambda_3\mathbf{e}_3 \cdots + c_m\lambda_m\mathbf{e}_m = \mathbf{0}\,.$$

Eliminate $c_1\mathbf{e}_1$ *from the above two equations to obtain*

$$\mathbf{v}_2 = c_2(\lambda_2 - \lambda_1)\mathbf{e}_2 + c_3(\lambda_3 - \lambda_1)\mathbf{e}_3 + \cdots + c_m(\lambda_m - \lambda_1)\mathbf{e}_m = \mathbf{0} \text{ and}$$

$$\mathbf{v}_3 = \mathbf{A}\mathbf{v}_2 = c_2\lambda_2(\lambda_2-\lambda_1)\mathbf{e}_2+c_3\lambda_3(\lambda_3-\lambda_1)\mathbf{e}_3+\cdots+c_m\lambda_m(\lambda_m-\lambda_1)\mathbf{e}_m = \mathbf{0}\,.$$

Now eliminate $c_2(\lambda_2 - \lambda_1)\mathbf{e}_2$ *to obtain*

$$\mathbf{v}_4 = c_3(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)\mathbf{e}_3 + \cdots + c_m(\lambda_m - \lambda_2)(\lambda_m - \lambda_1)\mathbf{e}_m = \mathbf{0}\,.$$

By repeating this process $m-1$ *times we obtain*

$$c_m(\lambda_m - \lambda_{m-1})(\lambda_m - \lambda_{m-2})\cdots(\lambda_m - \lambda_1)\mathbf{e}_m = \mathbf{0}\,,$$

which is impossible because $c_m \neq 0$*, all eigenvalues are distinct and* $\mathbf{e}_m \neq \mathbf{0}$ *since it is a non-trivial eigenvector. This contradiction shows that eigenvectors corresponding to distinct eigenvalues cannot be linearly dependent.* ♠

1.2.5 Repeated eigenvalues

While an $n \times n$ matrix always has n eigenvalues, some of them can be repeated. In this case n linearly independent eigenvectors cannot always be found or finding them may require additional steps.

EXAMPLE **1.4.** *Find eigenvalues and eigenvectors of matrix* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

SOLUTION. *The characteristic equation for this matrix is* $(1-\lambda)^2 = 0$ *and its solution is* $\lambda_1 = \lambda_2 = 1$. *Then the eigenvector must satisfy the equation*

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This requires that $x_2 = 0$ *while* x_1 *is arbitrary (but non-zero as the zero vector cannot be considered as an eigenvector). Thus* $\mathbf{e}_1 = t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $t_1 \neq 0$, *but it is impossible to find a second linearly independent eigenvector that is not a multiple of the above.* ■

EXAMPLE **1.5.** *Find eigenvalues and linearly independent eigenvectors of matrix* $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

SOLUTION. *The characteristic equation for this matrix is easiest to obtain by expanding the determinant by the second column that contains two zeros:*

$$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(-\lambda(3-\lambda)+2) = (2-\lambda)^2(1-\lambda) = 0\,.$$

Thus there are one simple eigenvalue $\lambda_1 = 1$ *and one eigenvalue* $\lambda_{2,3} = 2$ *repeated twice.*

For $\lambda_1 = 1$ *we obtain*

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

(after row reduction) so that $\mathbf{e}_1 = t_1 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

For the repeated eigenvalue $\lambda_{2,3} = 2$ *we obtain*

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad x_1 + x_3 = 0$$

(after row reduction) so $x_1 = -x_3$ *and* x_2 *is arbitrary, for example,* $x_2 = 0$. *Then* $\mathbf{e}_2 = t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. *But we could have also chosen a non-zero value for* x_2, *for example,* $x_2 = 1$ *so that* $\mathbf{e}_3 = t_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. *Note that the so chosen vectors* $\mathbf{e}_2$ *and* $\mathbf{e}_3$ *are linearly independent because the zero component of* $\mathbf{e}_2$ *cannot be converted to the corresponding non-zero component of* $\mathbf{e}_3$ *via the multiplication by a constant. Thus for the given* 3×3 *matrix we have been able to obtain three linearly independent eigenvectors even though only two distinct eigenvalues could be found.* ■

1.2.6 Eigenvalues and eigenvectors of the rotation matrix

We have introduced eigenvectors as vectors that "do not rotate" when multiplied by a matrix. On the other hand, we have also introduced a rotation matrix as "the matrix that rotates all vectors". This seems to mean that the rotation matrix cannot have any eigenvectors. However, this is not completely so. Indeed, using a standard procedure we obtain a characteristic equation for the rotation matrix

$$\det(\mathbf{R}[\theta] - \lambda\mathbf{I}) = \begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = \lambda^2 - 2\lambda\cos\theta + 1 = 0\,.$$

Then

$$\lambda_{1,2} = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \mathrm{i}\sin\theta = \exp(\pm\mathrm{i}\theta)\,,$$

where $\mathrm{i} = \sqrt{-1}$. Note that the eigenvalues of the rotation matrix are complex numbers with unit magnitudes except for trivial cases of $\theta = 0$ and $\theta = \pi$. You may remember from your previous studies of complex numbers that multiplication by a complex number with a unit magnitude corresponds not to stretching a vector but to its rotation in a complex plane as indeed the multiplication by the rotation matrix is supposed to do. So the contradiction is resolved by taking into account that the concept of stretching a vector is associated with multiplying it by a real number, while the multiplication by a complex number in general involves both stretching and rotating.

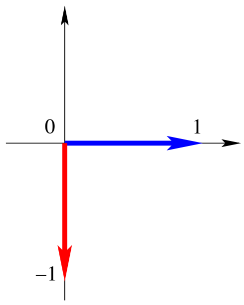


Figure 1.7: Rotation of real parts of eigenvectors of a rotation matrix.

EXAMPLE **1.6.** *Find the eigenvectors of the rotation matrix for $\theta = \dfrac{\pi}{2}$ and discuss their rotation in a complex plane.*

SOLUTION. *In this case the eigenvalues become $\lambda_{1,2} = \pm\mathrm{i}$ and*

$$\begin{bmatrix} \mp\mathrm{i} & 1 \\ -1 & \mp\mathrm{i} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \textit{Then}$$

$$\lambda_1 = \mathrm{i}\,,\ \mathbf{e}_1 = \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}; \quad \lambda_2 = -\mathrm{i}\,,\ \mathbf{e}_2 = t_2\begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix} \quad \textit{and}$$

$$\mathbf{R}\left[\frac{\pi}{2}\right]\mathbf{e}_1 = \begin{bmatrix} \mathrm{i} \\ -1 \end{bmatrix}, \quad \mathbf{R}\left[\frac{\pi}{2}\right]\mathbf{e}_2 = \begin{bmatrix} -\mathrm{i} \\ -1 \end{bmatrix}.$$

That is the eigenvectors of a rotation matrix exist, but they do rotate. To see that consider the real parts of the above eigenvectors and their images after multiplication by the rotation matrix:

$$\Re\{\mathbf{e}_{1,2}\} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Re\left\{\mathbf{R}\left[\frac{\pi}{2}\right]\mathbf{e}_{1,2}\right\} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

They are shown by the blue and red vectors in Figure 1.7, respectively.

EXAMPLE 1.6 demonstrates that while the eigenvector **Definition** 1.3 remains the same, the geometrical meaning of eigenvectors corresponding to complex eigenvalues changes: they do rotate. However, in the majority of practical applications it is the invariant algebraic **Definition** 1.3 of eigenvectors, and not its context-dependent geometrical interpretation, what is important.

Exercises

Ex. 1.2. For the following 2×2 matrices find their characteristic equations, eigenvalues and the corresponding eigenvectors.

(a) $\begin{bmatrix} 3 & 0 \\ -8 & -1 \end{bmatrix}$; (b) $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$; (c) $\begin{bmatrix} 0 & 4 \\ 9 & 0 \end{bmatrix}$; (d) $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$; (e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Ex. 1.3. For the following 3×3 matrices find their characteristic equations, their eigenvalues and the corresponding eigenvectors.

(a) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$; (b) $\begin{bmatrix} 5 & 1 & -2 \\ 1 & 2 & 1 \\ -2 & 1 & 5 \end{bmatrix}$; (c) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$; (d) $\begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}$.

Ex. 1.4. For each of the following matrices

$$\mathbf{A} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix};$$

$$\mathbf{D} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}; \quad \mathbf{E} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

(a) Find the eigenvalues and the corresponding unit eigenvectors.

(b) Find the spectral radius.

(c) Use the trace of the matrix to check that eigenvalues you computed are correct.

Ex. 1.5. Show that if $\mathbf{x}$ is an eigenvector of matrix $\mathbf{A}$, then it is also an eigenvector of $\mathbf{A}^{-1}$.

1.3 Matrix Diagonalisation

Definition 1.8 *Let $\mathbf{A}$ be an $n \times n$ matrix with distinct eigenvalues λ_1, λ_2, …, λ_n and the corresponding eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$. Then matrix $\mathbf{M} = [\mathbf{e}_1 : \mathbf{e}_2 : \cdots : \mathbf{e}_n]$ whose columns are the eigenvectors of matrix $\mathbf{A}$ is called the* modal matrix *of $\mathbf{A}$.*

Note that

$$\mathbf{AM} = [\mathbf{Ae}_1 : \cdots : \mathbf{Ae}_n] = [\lambda_1\mathbf{e}_1 : \cdots : \lambda_n\mathbf{e}_n] .$$

Also, by definition

$$\mathbf{M}^{-1}\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

so that

$$\mathbf{M}^{-1}\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{M}^{-1}\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \mathbf{M}^{-1}\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \mathbf{M}^{-1}\mathbf{A}\mathbf{M} &= \mathbf{M}^{-1}[\lambda_1\mathbf{e}_1 : \cdots : \lambda_n\mathbf{e}_n] = [\lambda_1\mathbf{M}^{-1}\mathbf{e}_1 : \cdots : \lambda_n\mathbf{M}^{-1}\mathbf{e}_n] \\ &= \left[\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} : \cdots : \lambda_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right] = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \mathbf{\Lambda}. \end{aligned}$$

Definition 1.9 *The diagonal matrix* $\mathbf{\Lambda}$ *containing the eigenvalues of matrix* $\mathbf{A}$ *is called the* spectral matrix *of* $\mathbf{A}$ *corresponding to the modal matrix* $\mathbf{M}$.

Note: in forming the modal matrix the eigenvalues and eigenvectors must be given in the same order. If we change the order of the elements of the modal matrix, then the order of the elements of the spectral matrix will change correspondingly.

Definition 1.10 *A transformation of a matrix* $\mathbf{A} \to \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ *is called a* similarity transformation *and the matrices* $\mathbf{A}$ *and* $\mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ *are said to be* similar matrices.

EXAMPLE **1.7.** *For the* 2×2 *matrix* $\mathbf{A} = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$ *the distinct eigenvalues are* $\lambda_1 = 7$, *and* $\lambda_2 = -1$ *with the corresponding eigenvectors* $\mathbf{e}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ *and* $\mathbf{e}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *(see* EXAMPLE 1.3*). Then*

$\mathbf{M} = [\mathbf{e}_1 : \mathbf{e}_2] = \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix}$. *The inverse of* $\mathbf{M}$ *is* $\mathbf{M}^{-1} = \dfrac{1}{8}\begin{bmatrix} 1 & 1 \\ 3 & -5 \end{bmatrix}$.
Therefore, $\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$.

The above example demonstrates that a matrix with distinct eigenvalues is similar to a diagonal matrix. However, it is not always the case that an $n \times n$ has n distinct eigenvalues. Below we consider two examples of finding the eigenvalues of 3×3 matrices. The first has three distinct eigenvalues while the second has only two.

EXAMPLE **1.8.** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}.$$

Construct a modal matrix $\mathbf{M}$ *and the corresponding spectral matrix.*

SOLUTION. *The characteristic equation is* $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. *That is*

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0\,.$$

Expanding the determinant along the top row results in

$$(1-\lambda)((2-\lambda)(3-\lambda)-2) - (2 - 2(2-\lambda)) = 0\,.$$

This leads to $(1-\lambda)((2-\lambda)(3-\lambda)-2) + 2(1-\lambda) = 0$. *Factoring out* $(1-\lambda)$ *leads to* $(1-\lambda)(2-\lambda)(3-\lambda) = 0$ *and we conclude that the eigenvalues are* $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. *If* $\mathbf{e}_i = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *is an eigenvector corresponding to* λ_i, $i = 1, 2, 3$, *then* $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{e}_i = \mathbf{0}$.

Eigenvectors corresponding to $\lambda = \lambda_1 = 1$. *In this case*

$$\mathbf{A} - \lambda_1\mathbf{I} = \mathbf{A} - \mathbf{I} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

and for the corresponding eigenvector $\mathbf{e}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *we must have*

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is equivalent to a system of equations

$$\begin{aligned} -z &= 0\,, \\ x + y + z &= 0\,, \\ 2x + 2y + 2z &= 0\,. \end{aligned}$$

Therefore, $z = 0$ *and* $x = -y$. *Setting* $y = t_1$ *and* $x = -t_1$ *we obtain the eigenvectors of* $\mathbf{A}$ *corresponding to* $\lambda_1 = 1$ *in the form* $\mathbf{e}_1 = t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ *and a unit eigenvector corresponding to* $\lambda = 1$ *is* $\hat{\mathbf{e}}_1 = \dfrac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Eigenvectors corresponding to $\lambda = \lambda_2 = 2$. *In this case*

$$\mathbf{A} - \lambda_2 \mathbf{I} = \mathbf{A} - 2\mathbf{I} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

and for the corresponding eigenvector $\mathbf{e}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *we must have*

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is equivalent to a system of equations

$$\begin{aligned} -x - z &= 0\,, \\ x + z &= 0\,, \\ 2x + 2y + z &= 0\,. \end{aligned}$$

Gaussian elimination leads to the pair of equations

$$\begin{aligned} x + z &= 0\,, \\ 2y + z &= 0\,. \end{aligned}$$

Then setting $z = t_2$ *gives* $y = -\frac{t_2}{2}$ *and* $x = -t_2$. *Therefore, the eigenvectors corresponding to* $\lambda_2 = 2$ *are found in the form* $\mathbf{e}_2 = \frac{t_2}{2}\begin{bmatrix}-2\\1\\2\end{bmatrix}$ *and a unit eigenvector is* $\hat{\mathbf{e}}_2 = \frac{1}{3}\begin{bmatrix}-2\\1\\2\end{bmatrix}$.

Eigenvectors corresponding to $\lambda = \lambda_3 = 3$. *In this case*

$$\mathbf{A} - \lambda_3\mathbf{I} = \mathbf{A} - 3\mathbf{I} = \begin{bmatrix}-2 & 0 & -1\\1 & -1 & 1\\2 & 2 & 0\end{bmatrix}$$

and for the corresponding eigenvector $\mathbf{e}_3 = \begin{bmatrix}x\\y\\z\end{bmatrix}$ *we must have*

$$\begin{bmatrix}-2 & 0 & -1\\1 & -1 & 1\\2 & 2 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

This is equivalent to a system of equations

$$\begin{aligned}-2x - z &= 0\,,\\ x - y + z &= 0\,,\\ 2x + 2y &= 0\,.\end{aligned}$$

Gaussian elimination leads to the pair of equations

$$\begin{aligned}x - y + z &= 0\,,\\ 2y - z &= 0\,.\end{aligned}$$

Then setting $z = t_3$ *gives* $y = \frac{t_3}{2}$ *and* $x = \frac{t_3}{2}$. *Therefore, the eigenvectors corresponding to* $\lambda_3 = 3$ *are found in the form* $\mathbf{e}_3 = \frac{t_3}{2}\begin{bmatrix}-1\\1\\2\end{bmatrix}$ *and a unit eigenvector is* $\hat{\mathbf{e}}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix}$.

Thus for the matrix $\mathbf{A} = \begin{bmatrix}1 & 0 & -1\\1 & 2 & 1\\2 & 2 & 3\end{bmatrix}$ *the eigenvalues are* $\lambda = 1$, $\lambda = 2$, $\lambda = 3$ *and the corresponding eigenvectors can conveniently be taken* *as* $\mathbf{e}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. *Then* $\mathbf{M} = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ *so that it can be verified that* $\mathbf{M}^{-1}\mathbf{AM} = \mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. ■

EXAMPLE **1.9.** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

and discuss whether it can be diagonalised.

SOLUTION. *The characteristic equation is*

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

i.e. $(1-\lambda)((2-\lambda)^2 - 2) - ((2-\lambda)2 - 2) = 0$, *which reduces to* $(1-\lambda)(2-\lambda)^2 = 0$. *Thus, the eigenvalues are* $\lambda_1 = 1$ *and* $\lambda_2 = \lambda_3 = 2$.

Eigenvector corresponding to $\lambda = \lambda_1 = 1$. *The components of an eigenvector* $\mathbf{e}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *satisfy*

$$\begin{aligned} 2y + 2z &= 0\,, \\ y + z &= 0\,, \\ -x + 2y + z &= 0\,. \end{aligned}$$

This gives $z = t_1$, $y = -t_1$ *and* $x = -t_1$ *so that the eigenvectors for* $\lambda_1 = 1$ *are* $\mathbf{e}_1 = t_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Eigenvector corresponding to $\lambda = \lambda_{2,3} = 2$. *The components of an eigenvector* $\mathbf{e}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *satisfy*

$$\begin{aligned} -x + 2y + 2z &= 0\,, \\ z &= 0\,, \\ -x + 2y &= 0\,. \end{aligned}$$

Thus $z = 0$ and $x = 2y$ so the eigenvectors for $\lambda = 2$ are of the form $\mathbf{e}_2 = t_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. *Since only two linearly independent eigenvectors are found, we cannot construct a square modal matrix required for the diagonalisation of the original matrix* $\mathbf{A}$. ■

EXAMPLE 1.9 demonstrates that not all $n \times n$ matrices have exactly n linearly independent eigenvectors. Such matrices cannot be diagonalised.

Exercises

Ex. 1.6. Check that $\lambda = 1$ is a three-times repeated eigenvalue of the matrix $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ and find all linearly independent eigenvectors corresponding to this eigenvalue. Subsequently, conclude whether it is possible to construct a modal matrix $\mathbf{M}$ that can be used to diagonalise the given matrix.

Ex. 1.7. The eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} -1 & 6 & -12 \\ 0 & -13 & 30 \\ 0 & -9 & 20 \end{bmatrix}$ are 5, 2 and -1. Obtain the corresponding eigenvectors. Write down the modal matrix $\mathbf{M}$ and spectral matrix $\mathbf{\Lambda}$. Show that $\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \mathbf{\Lambda}$ and find $\mathbf{A}^4$ using matrix diagonalisation.

1.4 Matrix Powers

Matrix diagonalisation can be used for computing *matrix power*. Indeed, if $\mathbf{A}$ matrix is diagonalisable, then $\mathbf{A} = \mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1}$ and

$$
\begin{aligned}
\mathbf{A}^m &= \underbrace{\mathbf{A} \cdot \mathbf{A} \cdot \ \cdots \ \cdot \mathbf{A}}_{m} = \mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1} \cdot \mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1} \cdot \ \cdots \ \cdot \mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1} \\
&= \mathbf{M}\underbrace{\mathbf{\Lambda} \cdot \ \cdots \ \cdot \mathbf{\Lambda}}_{m}\mathbf{M}^{-1} = \mathbf{M}\mathbf{\Lambda}^m\mathbf{M}^{-1} \,,
\end{aligned}
$$

where the mth power of the diagonal spectral matrix is computed by simply raising its individual elements to the mth power

$$\mathbf{\Lambda}^m = \begin{bmatrix} \lambda_1^m & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^m & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n^m \end{bmatrix}.$$

EXAMPLE **1.10.** *Find $\mathbf{A}^4$, where $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.*

SOLUTION. *The modal and spectral matrices have been found in* EXAMPLE *1.8*: $\mathbf{M} = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ *and* $\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. *To invert matrix* $\mathbf{M}$ *we follow a general procedure for a* 3×3 *matrix that you studied previously.*

1. *It is shorter to calculate the determinant of* $\mathbf{M}$*by expanding along the first column as it contains zero:*

   $$\det(\mathbf{M}) = (-1) \times (1 \times 2 - 1 \times 2) - 1 \times ((-2) \times 2 - (-1) \times 2) = 2\,.$$

   Since the determinant is not zero, the matrix is invertible.

2. *Compute the minors:*
   $M_{11} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0,\ M_{12} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2,\ M_{13} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2,$
   $M_{21} = \begin{vmatrix} -2 & -1 \\ 2 & 2 \end{vmatrix} = -2,\ M_{22} = \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = -2,$
   $M_{23} = \begin{vmatrix} -1 & -2 \\ 0 & 2 \end{vmatrix} = -2,\ M_{31} = \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = -1,$
   $M_{32} = \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} = 0,\ M_{33} = \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} = 1.$

3. *Compute the corresponding cofactors:*
   $C_{11} = 0,\ C_{12} = -2,\ C_{13} = 2,$
   $C_{21} = 2,\ C_{22} = -2,\ C_{23} = 2,$
   $C_{31} = -1,\ C_{32} = 0,\ C_{23} = 1.$

4. *Construct* $\mathbf{C}^T = \begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$ *and write down the inverse*

modal matrix

$$\mathbf{M}^{-1} = \frac{\mathbf{C}^T}{\det(\mathbf{M})} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix}.$$

Finally, compute the answer

$$\mathbf{A}^4 = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}.$$

■

From this example it becomes clear that using matrix diagonalisation to compute matrix power becomes computationally more efficient than a direct matrix multiplication if the exponent $m \gg 3$.

1.5 The Cayley-Hamilton Theorem

The matrix power concept enables us to formulate a very powerful theorem.

Theorem 1.11 (Cayley-Hamilton) *Every matrix satisfies its own characteristic equation.*

We demonstrate the *idea of the proof* (the rigorous proof would require material that is outside the scope of this Module) on the example of a 2×2 matrix $\mathbf{A}$ with a characteristic equation $\lambda^2 + b\lambda + c = 0$ and eigenvalues λ_1 and λ_2. We argue then that $\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{0}$. Let us assume that $\mathbf{A}$ is diagonalisable and $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ is its spectral matrix. It is easy to see that this matrix satisfies the characteristic equation.

$$\begin{aligned} \mathbf{\Lambda}^2 + b\mathbf{\Lambda} + c\mathbf{I} &= \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} + b\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1^2 + b\lambda_1 + c & 0 \\ 0 & \lambda_2^2 + b\lambda_2 + c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

since $\lambda_{1,2}^2 + b\lambda_{1,2} + c = 0$ (the eigenvalues must satisfy the characteristic equation). Now recollect that $\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \mathbf{\Lambda}$ and substitute this into the matrix characteristic equation. After simple factorisation we obtain

$$\begin{aligned}\mathbf{\Lambda}^2 + b\mathbf{\Lambda} + c\mathbf{I} &= \mathbf{M}^{-1}\mathbf{A}\mathbf{M}\mathbf{M}^{-1}\mathbf{A}\mathbf{M} + b\mathbf{M}^{-1}\mathbf{A}\mathbf{M} + c\mathbf{M}^{-1}\mathbf{I}\mathbf{M} \\ &= \mathbf{M}^{-1}(\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I})\mathbf{M} = \mathbf{0}\,.\end{aligned}$$

Finally, by pre- and post-multiplying by $\mathbf{M}$ and $\mathbf{M}^{-1}$ we obtain the desired result.

$$\mathbf{M}\mathbf{M}^{-1}(\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I})\mathbf{M}\mathbf{M}^{-1} = \mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{M}\mathbf{0}\mathbf{M}^{-1} = \mathbf{0}\,.$$

There exist a number of important corollaries to the Cayley-Hamilton theorem. We will derive them next with a view to further applications.

1.5.1 Powers of a 2×2 matrix

Consider the quadratic characteristic equation for a 2×2 matrix again. It has two solutions $\lambda_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$. It is easy to check that they satisfy two general conditions: $\lambda_1 + \lambda_2 = -b$ and $\lambda_1\lambda_2 = c$. On the other hand, according to the eigenvalue properties listed in Section 1.2.4 $\lambda_1 + \lambda_2 = \mathrm{Tr}(\mathbf{A})$ and $\lambda_1\lambda_2 = \det(\mathbf{A})$. Thus

Corollary 1.12 *For a 2×2 matrix* $\boxed{\mathbf{A}^2 - \mathrm{Tr}(\mathbf{A})\mathbf{A} + \det(\mathbf{A})\mathbf{I} = \mathbf{0}}$.

That is $\mathbf{A}^2 = \mathrm{Tr}(\mathbf{A})\mathbf{A} - \det(\mathbf{A})\mathbf{I}$. This expression can be used to find higher powers of $\mathbf{A}$ upon multiplying it by the original matrix. For example,

$$\begin{aligned}\mathbf{A}^3 &= \mathrm{Tr}(\mathbf{A})\mathbf{A}^2 - \det(\mathbf{A})\mathbf{A} = \mathrm{Tr}^2(\mathbf{A})\mathbf{A} - \mathrm{Tr}(\mathbf{A})\det(\mathbf{A})\mathbf{I} - \det(\mathbf{A})\mathbf{A} \\ &= (\mathrm{Tr}^2(\mathbf{A}) - \det(\mathbf{A}))\mathbf{A} - \mathrm{Tr}(\mathbf{A})\det(\mathbf{A})\mathbf{I} = \alpha_1\mathbf{A} + \alpha_0\mathbf{I}\,,\end{aligned}$$

where $\alpha_0 = -\mathrm{Tr}(\mathbf{A})\det(\mathbf{A})$ and $\alpha_1 = \mathrm{Tr}^2(\mathbf{A}) - \det(\mathbf{A})$.

Expressions for higher powers of $\mathbf{A}$ can be obtained in a similar way. The important observation is that in general the final expression for the power of a 2×2 matrix will be in the form

$$\boxed{\mathbf{A}^m = \alpha_1\mathbf{A} + \alpha_0\mathbf{I}}\,, \tag{1.4}$$

where $\alpha_{0,1}$ are some scalar constants.

In particular, after multiplying the expression in Corollary 1.12 by $\mathbf{A}^{-1}$ and rearranging it we obtain

Corollary 1.13 *The inverse of a 2×2 matrix can be computed as*

$$\boxed{\mathbf{A}^{-1} = \frac{\mathrm{Tr}(\mathbf{A})\mathbf{I} - \mathbf{A}}{\det(\mathbf{A})}}\,.$$

Note: the above manipulations remain valid if matrix $\mathbf{A}$ is replaced with its eigenvalue λ. Thus the powers of eigenvalues should satisfy the same modified equation as the matrix itself.

In particular, for distinct eigenvalues λ_1 and λ_2 of a 2×2 matrix we can write

$$\lambda_1^m = \alpha_1\lambda_1 + \alpha_0 \text{ and } \lambda_2^m = \alpha_1\lambda_2 + \alpha_0 \,.$$

Solving these equations for α_0 and α_1 results in

$$\boxed{\alpha_1 = \frac{\lambda_2^m - \lambda_1^m}{\lambda_2 - \lambda_1} \text{ and } \alpha_0 = \frac{\lambda_1^m\lambda_2 - \lambda_1\lambda_2^m}{\lambda_2 - \lambda_1}}\,.$$

If $\lambda_1 = \lambda_2 = \lambda$, we can only write

$$\lambda^m = \alpha_1\lambda + \alpha_0 \,.$$

The second equation can be obtained by differentiating the above functional relationship with respect to λ. This immediately gives

$$\boxed{\alpha_1 = m\lambda^{m-1} \text{ and } \alpha_0 = (1-m)\lambda^m}\,.$$

EXAMPLE **1.11.** *Compute* $\mathbf{A}^{-1}$ *and* $\mathbf{A}^7$ *if* $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$.

SOLUTION. *Check that* $\mathbf{A}$ *has eigenvalues* $\lambda_1 = 2$ *and* $\lambda_2 = 1$, $\mathrm{Tr}(\mathbf{A}) = 3$ *and* $\det(\mathbf{A}) = 2$. *Thus*

$$\mathbf{A}^{-1} = \frac{1}{2}\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

According to the Cayley-Hamilton theorem the matrix power equation

$$\mathbf{A}^7 = \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$$

must hold for both eigenvalues, that is $\lambda_{1,2}^7 = \alpha_1\lambda_{1,2} + \alpha_0$. *Then*

$$1^7 = 1\alpha_1 + \alpha_0 \textit{ and } 2^7 = 2\alpha_1 + \alpha_0$$

with the solution $\alpha_1 = 127$ *and* $\alpha_0 = -126$. *Finally,*

$$\mathbf{A}^7 = 127\mathbf{A} - 126\mathbf{I} = \begin{bmatrix} -253 & 254 \\ -381 & 382 \end{bmatrix}.$$

■

EXAMPLE **1.12.** *Compute* $\mathbf{A}^{15}$ *if* $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$.

SOLUTION. *This matrix has a single repeated eigenvalue* $\lambda_{1,2} = 2$. *Thus even though we know that*

$$\mathbf{A}^{15} = \alpha_1 \mathbf{A} + \alpha_0 \mathbf{I}$$

we can only write one equation for $\lambda_{1,2}$: $2^{15} = 2\alpha_1 + \alpha_0$. *The trick to obtaining the required second equation is to recognise the equation* $\lambda^{15} = \alpha_1 \lambda + \alpha_0$ *as a continuous function of* λ *that we differentiate to obtain* $15\lambda^{14} = \alpha_1$. *Then* $\alpha_1 = 15\times 2^{14}$ and $\alpha_0 = -14\times 2^{15} = -28\times 2^{14}$. *Thus*

$$\mathbf{A}^{15} = 2^{14}\times\left(15\begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix}\right) = 2^{14}\times\begin{bmatrix} 2 & 0 \\ 60 & 2 \end{bmatrix} = 2^{15}\times\begin{bmatrix} 1 & 0 \\ 30 & 1 \end{bmatrix}.$$

■

As you can see the Cayley-Hamilton theorem offers a method of finding powers of a matrix that is alternative to the matrix diagonalisation considered in Section 1.4.

1.5.2 Matrix functions

A straightforward generalisation of a matrix power is a *matrix power series*

$$f(\mathbf{A}) = \sum_{i=0}^{n} c_i \mathbf{A}^i \equiv c_0 \mathbf{I} + c_1 \mathbf{A} + c_2 \mathbf{A}^2 + \cdots + c_n \mathbf{A}^n,$$

where $\mathbf{A}^0 \equiv \mathbf{I}$ and $f(\mathbf{A})$ is referred to as a *matrix function*. Since this series consists entirely of powers of $\mathbf{A}$, in the 2×2 case it could be written simply as $f(\mathbf{A}) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A}$ for some coefficients $\alpha_{0,1}$. This conclusion holds even if $n \to \infty$. Then similarly to the *Taylor series representation* of scalar functions we can introduce the power series representation of matrix functions (the discussion of convergence of such series is outside the scope of this subject). For example,

$$\exp(\lambda t) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n t^n.$$

and we define the *matrix exponential function* as

$$\exp(\mathbf{A}t) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n t^n .$$

Because of (1.4) we expect that for a 2×2 matrix

$$\exp(\mathbf{A}t) = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} .$$

(and this is true for any function of a 2×2 matrix). Then

$$\boxed{\exp(\mathbf{A}0) = \mathbf{I} = \alpha_0(0)\mathbf{I} + \alpha_1(0)\mathbf{A} \text{ thus } \alpha_0(0) = 1 \text{ and } a_1(0) = 0}.$$

EXAMPLE **1.13.** *Find* $\exp(\mathbf{A}t)$ *if* $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

SOLUTION. *The eigenvalues are* $\lambda_1 = 5$ *and* $\lambda_2 = -1$. *They yield* $e^{5t} = \alpha_0 + 5\alpha_1$ *and* $e^{-t} = \alpha_0 - \alpha_1$ *with solutions* $\alpha_0 = \frac{1}{6}\left(e^{5t} + 5e^{-t}\right)$ *and* $\alpha_1 = \frac{1}{6}\left(e^{5t} - e^{-t}\right)$. *Therefore,*

$$\begin{aligned} \exp(\mathbf{A}t) &= \frac{1}{6}\left(e^{5t} + 5e^{-t}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{6}\left(e^{5t} - e^{-t}\right)\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} e^{5t} + 2e^{-t} & e^{5t} - e^{-t} \\ 2\left(e^{5t} - e^{-t}\right) & 2e^{5t} + e^{-t} \end{bmatrix} . \end{aligned}$$

■

EXAMPLE **1.14.** *Find* $\sin(\mathbf{A}t)$ *if* $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

SOLUTION. *There is only one repeated eigenvalue* $\lambda = 1$ *for this matrix. From the power series representation of this function we obtain using Cayley-Hamilton theorem that* $\sin(\mathbf{A}t) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A}$. *The equivalent eigenvalue equation then is* $\sin(\lambda t) = \alpha_0 + \alpha_1\lambda$. *Differentiate this equation with respect to* λ *to obtain the second condition:* $t\cos(\lambda t) = \alpha_1$. *Then substituting* $\lambda = 1$ *we obtain* $\sin t = \alpha_0 + \alpha_1$ *and* $t\cos t = \alpha_1$, *which gives* $\alpha_1 = t\cos t$, *and so it follows that* $\alpha_0 = \sin t - t\cos t$. *Therefore,*

$$\sin(\mathbf{A}t) = (\sin t - t\cos t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\cos t\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sin t & t\cos t \\ 0 & \sin t \end{bmatrix} .$$

■

It is easy to demonstrate the validity of the following general statements.

1. If $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then $f(\mathbf{\Lambda}t) = \begin{bmatrix} f(\lambda_1 t) & 0 \\ 0 & f(\lambda_2 t) \end{bmatrix}$.

2. If two matrices have identical eigenvalues, then the expansion of their powers have identical coefficients.

EXAMPLE **1.15.** *Use the above properties to find* $\cos(\mathbf{A}t)$ *if* $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$.

SOLUTION. *The eigenvalues of matrix* $\mathbf{A}$ *are* $\lambda_1 = -1$ *and* $\lambda_2 = 4$. *The corresponding eigenvectors are* $\mathbf{e}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ *and* $\mathbf{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Then the modal matrix is* $\mathbf{M} = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}$ *with inverse* $\mathbf{M}^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$ *and the spectral matrix is* $\mathbf{\Lambda} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$. *Then*

$$\begin{aligned}
\cos(\mathbf{A}t) &= \alpha_1\mathbf{A} + \alpha_0\mathbf{I} = \alpha_1\mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1} + \alpha_0\mathbf{M}\mathbf{I}\mathbf{M}^{-1} \\
&= \mathbf{M}(\alpha_1\mathbf{\Lambda} + \alpha_0\mathbf{I})\mathbf{M}^{-1} = \mathbf{M}\cos(\mathbf{\Lambda}t)\mathbf{M}^{-1} \\
&= \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \cos(-t) & 0 \\ 0 & \cos(4t) \end{bmatrix}\frac{1}{5}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 3\cos t + 2\cos(4t) & -3\cos t + 3\cos(4t) \\ -2\cos t + 2\cos(4t) & 2\cos t + 3\cos(4t) \end{bmatrix}.
\end{aligned}$$

■

1.5.3 Generalisation for larger matrices

Without detailed discussion note that the applications of Cayley-Hamilton theorem can be extended to larger square matrices. For example, for a 3×3 matrix

$$\boxed{\mathbf{A}^3 - \mathrm{Tr}(\mathbf{A})\mathbf{A}^2 + \frac{1}{2}(\mathrm{Tr}^2(\mathbf{A}) - \mathrm{Tr}(\mathbf{A}^2))\mathbf{A} - \det(\mathbf{A})\mathbf{I} = \mathbf{0}}.$$

It can be shown that for a general $n \times n$ matrix

$$\boxed{\mathbf{A}^m = \alpha_{n-1}\mathbf{A}^{n-1} + \alpha_{n-2}\mathbf{A}^{n-2} + \cdots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}},$$

where α_i, $i = 0, 1, \ldots, n-1$ are scalar constants. The functions of a general $n \times n$ matrix then can be defined in a way similar to that introduced in Section 1.5.1 and Section 1.5.2.

Exercises

Ex. 1.8. Given $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ use the Cayley-Hamilton theorem to evaluate (a) $\mathbf{A}^{-1}$, (b) $\mathbf{A}^2$ and (c) $\mathbf{A}^3$.

Ex. 1.9. Evaluate $\exp(\mathbf{A}t)$ for (a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, (b) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.

Ex. 1.10. Given $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ find $\mathbf{A}^6$, $\exp(\mathbf{A}t)$ and $\sin(\mathbf{A}t)$.

Ex. 1.11. Use the Cayley-Hamilton theorem to prove the following formula for the four square-roots of a 2×2 matrix $\mathbf{A}$ (subject to certain constraints):

$$\sqrt{\mathbf{B}} = \pm \frac{\mathbf{B} + \epsilon\sqrt{\det(\mathbf{B})}\mathbf{I}}{\sqrt{\mathrm{Tr}(\mathbf{B}) + 2\epsilon\sqrt{\det(\mathbf{B})}}},$$

where $\epsilon = \pm 1$. Amazingly, this formula was first published only in 1983.

Hint. Write the Cayley-Hamilton formula for a matrix $\mathbf{A}$, then denote $\mathbf{A}^2 = \mathbf{B}$ so that $\det(\mathbf{A}) = \pm\sqrt{\det(\mathbf{B})}$. Then take the trace of the expression and solve the resulting equation for $\mathrm{Tr}(\mathbf{A})$ in terms of $\mathrm{Tr}(\mathbf{B})$ substituting the result back into the original matrix formula.

1.6 Symmetric Matrices and Their Applications

1.6.1 Definition and properties

Definition 1.14 *A square matrix is symmetric if* $\mathbf{A}^T = \mathbf{A}$.

Example **1.16.** $\mathbf{A} = \begin{bmatrix} 7 & 3 \\ 3 & -4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -5 & 4 \\ -3 & 4 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 8 & 3 & 5 \\ 3 & 0 & -6 \\ 5 & -6 & 1 \end{bmatrix}$.

There are three very important properties of symmetric matrices.

1. The eigenvalues of a real symmetric matrix are always real numbers.

2. The eigenvectors of a symmetric matrix corresponding to distinct eigenvalues are mutually orthogonal and thus the corresponding modal matrix $\mathbf{P}$ formed using the unit eigenvectors is orthogonal.

3. Any $n \times n$ symmetric matrix possesses n mutually orthogonal eigenvectors even if some eigenvalues of this matrix are repeated.

Proof:

1. Recollect that the dot product between complex-valued vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as $(\mathbf{u} \cdot \mathbf{v}) \equiv \mathbf{u}^{*T}\mathbf{v}$, where $*$ denotes complex conjugate. This guarantees that the square of any complex-valued vector $\mathbf{x}$, which defines its norm (length), is a non-negative real number: $(\mathbf{x} \cdot \mathbf{x}) \equiv \mathbf{x}^{*T}\mathbf{x} \equiv ||\mathbf{x}||^2 \geq 0$. Given that the norm of a complex vector is real, we can write $(||\mathbf{x}||^2)^* = ||\mathbf{x}||^2$ and

$$(||\mathbf{x}||^2)^* = (\mathbf{x} \cdot \mathbf{x})^* = (\mathbf{x}^{*T}\mathbf{x})^* = \mathbf{x}^T\mathbf{x}^* = ||\mathbf{x}||^2 = (\mathbf{x} \cdot \mathbf{x}) = \mathbf{x}^{*T}\mathbf{x}\,.$$

   If an eigenvalue λ of a real symmetric matrix is complex, we can write

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \text{ and } \mathbf{A}^*\mathbf{x}^* = \mathbf{A}\mathbf{x}^* = \lambda^*\mathbf{x}^*$$

   because $\mathbf{A}^* = \mathbf{A}$ since it is a real matrix. Then on one hand

$$\mathbf{x}^T\mathbf{A}\mathbf{x}^* = \mathbf{x}^T(\mathbf{A}\mathbf{x}^*) = \mathbf{x}^T(\lambda^*\mathbf{x}^*) = \lambda^*\mathbf{x}^T\mathbf{x}^* = \lambda^*\mathbf{x}^{*T}\mathbf{x} = \lambda^*(\mathbf{x} \cdot \mathbf{x})\,,$$

   on the other

$$\mathbf{x}^T\mathbf{A}\mathbf{x}^* = (A^T\mathbf{x})^T\mathbf{x}^* = (A\mathbf{x})^T\mathbf{x}^* = \lambda\mathbf{x}^T\mathbf{x}^* = \lambda(\mathbf{x} \cdot \mathbf{x})\,.$$

   Thus,

$$\lambda(\mathbf{x} \cdot \mathbf{x}) = \lambda^*(\mathbf{x} \cdot \mathbf{x}) \text{ or } \lambda = \lambda^*$$

   since the eigenvector $\mathbf{x}$ is non-zero and $(\mathbf{x} \cdot \mathbf{x}) > 0$. Therefore, λ is real.

2. Consider two eigenvectors $\mathbf{u}$ and $\mathbf{v}$ of a real symmetric matrix $\mathbf{A}$ corresponding to distinct eigenvalues λ_1 and λ_2, respectively. Then

$$\mathbf{u}^{*T}A\mathbf{v} = \mathbf{u}^{*T}\lambda_2\mathbf{v} = \lambda_2(\mathbf{u} \cdot \mathbf{v})\,.$$

On the other hand,

$$\mathbf{u}^{*T} A\mathbf{v} = (A\mathbf{u})^{*T}\mathbf{v} = \lambda_1 \mathbf{u}^{*T}\mathbf{v} = \lambda_1(\mathbf{u}\cdot\mathbf{v})\,.$$

Therefore,

$$\lambda_1(\mathbf{u}\cdot\mathbf{v}) = \lambda_2(\mathbf{u}\cdot\mathbf{v}) \text{ or } (\lambda_1-\lambda_2)(\mathbf{u}\cdot\mathbf{v}) = 0\,.$$

Since $\lambda_1 \neq \lambda_2$, $(\mathbf{u}\cdot\mathbf{v}) = 0$, that is the eigenvectors corresponding to different eigenvalues are orthogonal.

3. The proof of property 3 is more involved and we leave it outside the scope of this subject.

♠

Exercises

Ex. 1.12. Find the eigenvalues and the corresponding eigenvectors of the symmetric matrices $\mathbf{A} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 7 & 3 & 3 \\ 3 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix}$ and verify that their eigenvectors are mutually orthogonal.

Ex. 1.13. Determine the eigenvalues and corresponding eigenvectors of the symmetric matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Write down the orthogonal modal matrix $\mathbf{P}$ and spectral matrix $\mathbf{\Lambda}$. Show that $\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{\Lambda}$.

1.6.2 Quadratic curves

Symmetric matrices are often used to classify quadratic curves and surfaces.

Definition 1.15 *A* quadratic curve *is a curve in two dimensions with an equation of the form*

$$ax_1^2 + bx_2^2 + 2cx_1x_2 = 1\,.$$

Such an equation can be written in matrix form as

$$[x_1\ x_2]\begin{bmatrix} a & c \\ c & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \quad \text{or} \quad \mathbf{x}^T\mathbf{A}\mathbf{x} = 1\,,$$

where $\mathbf{A} = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (note that the matrix $\mathbf{A}$ is a symmetric matrix).

EXAMPLE **1.17.** *The curve equation* $2x_1^2 + 2x_2^2 - 6x_1x_2 = 1$ *can be written as a matrix equation* $[x_1\ x_2]\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$.

The standard quadratic curves are *circles*, *ellipses* and *hyperbolas*. They have the following *canonical equations*

$$x_1^2 + x_2^2 = r^2\,, \quad \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1\,, \quad \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1\,,$$

respectively. These equations in a matrix form are expressed in terms of a diagonal matrix.

EXAMPLE **1.18.** *The canonical equation for hyperbola* $\dfrac{x_1^2}{4} - \dfrac{x_2^2}{9} = 1$ *can be expressed in a matrix form as* $\mathbf{x}^T\mathbf{A}\mathbf{x} = 1$, *where* $\mathbf{A} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{9} \end{bmatrix}$ *and* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Matrix diagonalisation can be used to convert the algebraic equation for a quadratic curve into its canonical form. We show how and why this works next. First, let us formulate what we wish to achieve in a table format, where general expressions are given in the left column and the equivalent canonical expressions are shown in the right column.

General form	Canonical form
$ax_1^2 + bx_2^2 + 2cx_1x_2 = 1$	$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 1$
$[x_1\ x_2]\begin{bmatrix} a & c \\ c & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$	$[y_1\ y_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 1$
$\mathbf{x}^T\mathbf{A}\mathbf{x} = 1$	$\mathbf{y}^T\mathbf{\Lambda}\mathbf{y} = 1$

From Section 1.3 we know that $\mathbf{\Lambda} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ so that

$$\mathbf{y}^T\mathbf{\Lambda}\mathbf{y} = \mathbf{y}^T\mathbf{M}^{-1}\mathbf{A}\mathbf{M}\mathbf{y} = \mathbf{x}^T\mathbf{A}\mathbf{x}\,.$$

The comparison with the last entry in the left column in the table above suggests that $\mathbf{M}\mathbf{y} = \mathbf{x}$ and then

$$\mathbf{x}^T = (\mathbf{M}\mathbf{y})^T = \mathbf{y}^T\mathbf{M}^T\,.$$

On the other hand to ensure the equivalence of the table entries we must require that $\mathbf{x}^T = \mathbf{y}^T\mathbf{M}^{-1}$ that is

$$\mathbf{M}^T = \mathbf{M}^{-1} \quad \text{or} \quad \mathbf{M}^T\mathbf{M} = \mathbf{I}\,,$$

where $\mathbf{I}$ is the identity matrix. Recollect that columns of matrix

$$\mathbf{M} = [\mathbf{e}_1 : \mathbf{e}_2 : \cdots : \mathbf{e}_n]$$

are the eigenvectors of a symmetric matrix $\mathbf{A}$, thus they are orthogonal and $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$. Then

$$\mathbf{M}^T\mathbf{M} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{e}_2 \\ \mathbf{e}_2 \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} |\mathbf{e}_1|^2 & 0 \\ 0 & |\mathbf{e}_2|^2 \end{bmatrix} = \mathbf{I}\,.$$

This only possible if $|\mathbf{e}_1| = |\mathbf{e}_2| = 1$ that is if unit eigenvectors are used to construct matrix $\mathbf{M}$. To emphasise this in the above procedure we must use matrix $\mathbf{P}$ consisting of unit eigenvectors. In particular, the new coordinate vector $\mathbf{y}$ is obtained from the original coordinate vector $\mathbf{x}$ as

$$\boxed{\mathbf{y} = \mathbf{P}^{-1}\mathbf{x} = \mathbf{P}^T\mathbf{x}}\,.$$

The new coordinates define the so-called principal axes.

Definition 1.16 *The coordinate axes with respect to which the equation of a quadratic curve takes the canonical form are called* principal axes.

The method we have used to express the equation for a quadratic curve with equation $ax_1^2 + bx_2^2 + 2cx_1x_2 = 1$ relative to its principal axes can be summarised as follows:

1. Express the equation of a given curve in matrix form $\mathbf{x}^T\mathbf{A}\mathbf{x} = 1$, where $\mathbf{A}$ is a symmetric matrix.

2. Find the eigenvalues λ_1 and λ_2. The equation to the curve in new coordinates will be $\lambda_1 y_1^2 + \lambda_2 y_2^2 = 1$.

3. Find the *unit* eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ of matrix $\mathbf{A}$.

4. Construct the orthogonal modal matrix $\mathbf{P} = [\mathbf{e}_1 : \mathbf{e}_2]$. Make sure that the order of the eigenvectors is the same as the order of the eigenvalues they correspond to.

5. Transform the coordinates using $\mathbf{y} = \mathbf{P}^T\mathbf{x}$. The direction of the y_1 and y_2 axes will be the same as the directions of the eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively.

EXAMPLE **1.19.** *Identify the curve described by $2x_1^2 + 2x_2^2 - 6x_1x_2 = 1$ using matrix diagonalisation and define the principal axes.*

SOLUTION. *The matrix corresponding to the given equation is* $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$. *Its characteristic equation is* $(2-\lambda)^2 - 9 = 0$. *This simplifies to* $\lambda^2 - 4\lambda - 5 = 0$. *Thus the eigenvalues are* $\lambda_{1,2} = -1, 5$. *This information is sufficient to write down the canonical form of the equation in the transformed coordinates* $-y_1^2 + 5y_2^2 = 1$ *and determine the type of the curve as hyperbola. To determine the required coordinate transformation we need to find the eigenvectors next.*

Eigenvectors corresponding to $\lambda = \lambda_1 = -1$. *Since the matrix equation for the eigenvector in this case is* $\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{0}$, *the eigenvectors are of the form* $t_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and a unit eigenvector is* $\mathbf{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Eigenvectors corresponding to $\lambda = \lambda_2 = 5$. *The eigenvector equation is* $\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{0}$ *and the eigenvectors are of the form* $t_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ *so that a unit eigenvector is* $\mathbf{e}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Since the original matrix is symmetric its eigenvectors are orthogonal and so is the corresponding modal matrix $\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. *Therefore,* $\mathbf{P}^{-1} = \mathbf{P}^T$ *and we obtain* $\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} = \mathbf{\Lambda}$.

Consider the coordinate transformation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{P}^T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *or* $\mathbf{y} = \mathbf{P}^T\mathbf{x}$. *Then we obtain* $\mathbf{x} = \mathbf{P}\mathbf{y}$ *and the equation for the curve* $\mathbf{x}^T\mathbf{A}\mathbf{x} = 1$ *becomes* $(\mathbf{P}\mathbf{y})^T\mathbf{A}\mathbf{P}\mathbf{y} = 1$, *which in turn becomes* $\mathbf{y}^T\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{y} = 1$ *and is equivalent to*

$$\mathbf{y}^T\mathbf{\Lambda}\mathbf{y} = 1 \quad \text{or} \quad -y_1^2 + 5y_2^2 = 1$$

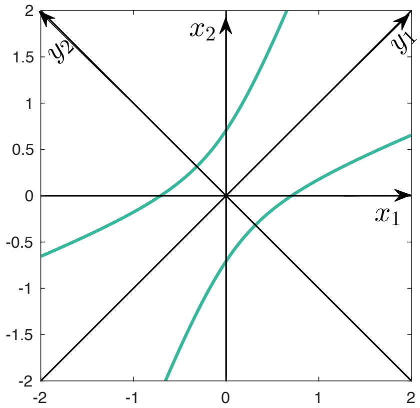


Figure 1.8: Rotated hyperbola in EXAMPLE 1.19.

confirming that this is the equation for a hyperbola in the y coordinates. As discussed above

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{P}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(x_1 + x_2) \\ \frac{-1}{\sqrt{2}}(x_1 - x_2) \end{bmatrix}.$$

Since the transformation matrix $\mathbf{P}$ consists of orthogonal unit vectors, the y-coordinate system is obtained from the x-coordinates via rotation. To see that recollect that the unit coordinate vectors in a rotated coordinate system are $\mathbf{i} = (y_1, y_2) = (1, 0)$ and $\mathbf{j} = (y_1, y_2) = (0, 1)$. Their expressions in the original coordinate system are then given by

$$\mathbf{i} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{P} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\mathbf{j} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{P} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

The principal axes are the lines passing through the origin and containing the above vectors as shown in Figure 1.8. Once the principal axes are established it is much easier to draw the canonical hyperbola with respect to them. ■

EXAMPLE **1.20.** *Let us now revisit* EXAMPLE *1.2 that we looked at in the context of the rotation matrix. The curve considered there was described by* $5x_1^2 + 6x_1x_2 + 5x_2^2 = 8$*. We showed that by rotating the coordinate system by* $\frac{\pi}{4}$ *we were able to convert this equation to a canonical form and determine that this curve was an ellipse. Let us show how the rotation angle was chosen.*

SOLUTION. *The matrix corresponding to the given equation is* $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$. *Its characteristic equation is* $(5-\lambda)^2 - 9 = (2-\lambda)(8-\lambda) = 0$*. Thus the eigenvalues are* $\lambda_{1,2} = 2, 8$*. Therefore, the canonical form of the equation in the transformed coordinates is* $2y_1^2 + 8y_2^2 = 8$ *or* $\frac{y_1^2}{4} + y_2^2 = 1$ *and the described curve is an ellipse. To determine the required coordinate transformation we find the eigenvectors.*

Eigenvectors corresponding to $\lambda = \lambda_1 = 2$. *Since the matrix equation for the eigenvector in this case is* $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{0}$, *the eigenvectors are of the form* $t_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *and a unit eigenvector is* $\mathbf{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$.

Eigenvectors corresponding to $\lambda = \lambda_2 = 8$. *The eigenvector equation is* $\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{0}$ *and the eigenvectors are of the form* $t_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and a unit eigenvector is* $\mathbf{e}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

The corresponding modal matrix is $\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ *and*

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \mathbf{x} = \mathbf{R}[\theta]\mathbf{x}\,,$$

where we recognise that $\cos\left(-\frac{\pi}{4}\right) = -\sin\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ *and thus* $\theta = -\frac{\pi}{4}$*. Therefore, the canonical coordinates* (y_1, y_2) *are obtained* *from the original coordinates* (x_1, x_2) *by rotating them by the angle* $\frac{\pi}{4}$ *clockwise as indeed was shown in* Example *1.2.* ■

1.6.3 Quadratic surfaces

The method of classifying quadratic curves that has been considered in the previous section can be applied when working with *quadratic surfaces*, or *quadratic forms*, in three variables. These are surfaces in three dimensions whose equations are of the form

$$ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x_2 + 2fx_1x_3 + 2ex_2x_3 = 1\,.$$

This can be written in matrix form

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1\,.$$

The eigenvalues λ_1, λ_2 and λ_3 of the matrix $\mathbf{A} = \begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix}$ real and distinct because the matrix is symmetric. The corresponding unit eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are orthogonal so that $\mathbf{P} = [\mathbf{e}_1 : \mathbf{e}_2 : \mathbf{e}_3]$ is an orthogonal matrix. Under the transformation of coordinates $\mathbf{x} = \mathbf{Py}$ the equation for the surface becomes

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1\,.$$

This is called the *canonical form of the equation*. The surface represented by this equation is

- an *ellipsoid* if all λ's are positive, see plot (a) in Figure 1.9;
- a *one sheet hyperboloid* if two of the λ's are positive and one is negative, see plot (b) in Figure 1.9;
- a *two sheet hyperboloid* if one of the λ's is positive and two are negative, see plot (c) in Figure 1.9;
- an *elliptic cylinder* if one of the λ's is zero and two are positive, see plot (d) in Figure 1.9;
- a *hyperbolic cylinder* if one of the λ's is zero and two others have opposite signs, see plot (e) in Figure 1.9.

Definition 1.17 *The expression* $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$ *is called the* canonical form of the quadratic form $ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x_2 + 2fx_1x_3 + 2ex_2x_3$.

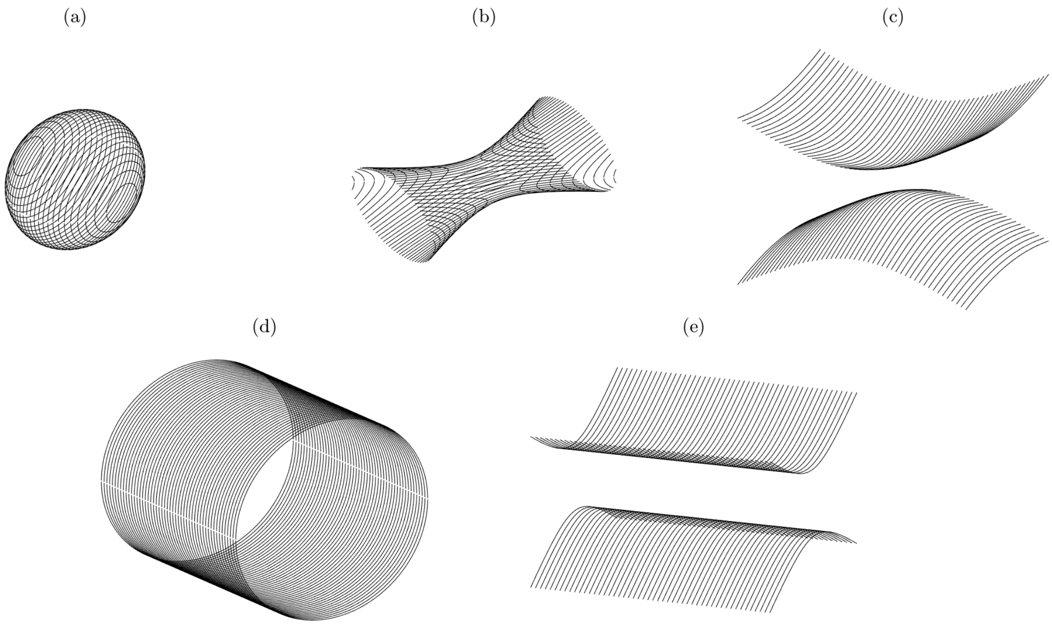


Figure 1.9: Quadratic surfaces.

EXAMPLE **1.21.** *Identify the surface $x_1^2 + 6x_1x_2 - 2x_2^2 - 2x_2x_3 + x_3^2 = 1$ by a suitable coordinate transformation and determine the orthogonal matrix which transforms the equation into canonical form and express the canonical coordinates y_1, y_2 and y_3 in terms of the original coordinates x_1, x_2 and x_3.*

SOLUTION. *The equation can be written in matrix form as*

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1\,.$$

The corresponding characteristic equation is

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0\,.$$

This expands to

$$(1-\lambda) \begin{vmatrix} -2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - 3 \begin{vmatrix} 3 & -1 \\ 0 & 1-\lambda \end{vmatrix} = 0\,,$$

which gives
$(1-\lambda)((2+\lambda)(\lambda-1)-1)-9(1-\lambda) = 0$ *and* $(1-\lambda)(\lambda+4)(\lambda-3) = 0.$

Thus the eigenvalues are 1, 3 and -4*. The equation to the surface relative to its principal axes then is*

$$y_1^2 + 3y_2^2 - 4y_3^2 = 1\,.$$

Thus the surface is a one-sheet hyperboloid.

In order to construct the transformation matrix we find the unit eigenvectors.

Eigenvector corresponding to $\lambda = \lambda_1 = 1$. *We obtain*

$$\begin{bmatrix} 0 & 3 & 0 \\ 3 & -3 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\,.$$

Thus, $x_2 = 0$ *and* $x_3 = 3x_1$*. Therefore, a general eigenvector is* $t_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ *and the unit eigenvector is* $\mathbf{e}_1 = \begin{bmatrix} \dfrac{1}{\sqrt{10}} \\ 0 \\ \dfrac{3}{\sqrt{10}} \end{bmatrix}$.

Eigenvector corresponding to $\lambda = \lambda_2 = 3$. *We have*

$$\begin{bmatrix} 1-3 & 3 & 0 \\ 3 & -2-3 & -1 \\ 0 & -1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

that is

$$\begin{aligned} -2x_1 + 3x_2 &= 0\,, \\ 3x_1 - 5x_2 - x_3 &= 0\,, \\ -x_2 - 2x_3 &= 0\,. \end{aligned}$$

From the last equation we have $x_2 = -2x_3$ *and from the first one* $x_1 = \dfrac{3}{2}x_2$*. Setting* $x_3 = t_2$ *we obtain that a general eigenvector* $t_2 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ *and the unit eigenvector is* $\mathbf{e}_2 = \begin{bmatrix} \dfrac{-3}{\sqrt{14}} \\ \dfrac{-2}{\sqrt{14}} \\ \dfrac{1}{\sqrt{14}} \end{bmatrix}$.

Eigenvector corresponding to $\lambda = \lambda_2 = -4$. *The unit eigenvector is* $\mathbf{e}_3 = \begin{bmatrix} \frac{-3}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \\ \frac{1}{\sqrt{35}} \end{bmatrix}$ *and we conclude that the orthogonal transformation matrix is*

$$\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{14}} & \frac{-3}{\sqrt{35}} \\ 0 & \frac{-2}{\sqrt{14}} & \frac{5}{\sqrt{35}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{35}} \end{bmatrix} \text{ so that } \mathbf{y} = \mathbf{P}^T\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{10}}(x_1 + 3x_3) \\ \frac{1}{\sqrt{14}}(-3x_1 - 2x_2 + x_3) \\ \frac{1}{\sqrt{35}}(-3x_1 + 5x_2 + x_3) \end{bmatrix}.$$

■

Exercises

Ex. 1.14. Reconsider Example 1.2 and show that the rotation matrix used there can be formally obtained by following the procedure of Section 1.6.2. Subsequently, draw the rotated coordinate system (y_1, y_2) obtained here and endeavour to explain the reasons for any differences you notice with Figure 1.4.

Ex. 1.15. Write the following quadratic forms as $V(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and then determine their canonical forms:

(a) $-x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_1x_3 - 8x_2x_3$;
(b) $x_1^2 - 3x_2^2 - 3x_3^2 + 4x_1x_2 + 4x_1x_3 + 12x_2x_3$;
(c) $4x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 6x_2x_3$.

In each case identify the corresponding surface given by $\mathbf{x}^T\mathbf{A}\mathbf{x} = 1$ and determine the corresponding modal matrix.

Ex. 1.16. Reduce the quadratic form

$$2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$$

to canonical form, determine the corresponding modal matrix and identify the surface given by

$$2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 1\,.$$

1.7 Systems of Differential Equations

In the previous sections we considered several examples of geometrical applications of matrices and their eigenvalues and eigenvectors. Another important practical applications of matrices is in solving systems of simultaneous linear differential equations. We will consider it next.

Recall that a *first order linear ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}t} - kx = 0$$

has the solution $x = ae^{\lambda t}$, where a is a constant. A solution of a *second order linear ordinary differential equation with constant coefficients*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + k_1\frac{\mathrm{d}x}{\mathrm{d}t} + k_2x = 0$$

is also obtained as a sum of exponential functions. There is a close link between linear ordinary differential equations with constant coefficients of arbitrary orders. In fact any higher order equation can be rewritten as a system of first order equations. For example, let $x = x_1$ and $\dfrac{dx}{dt} = x_2$. Then the above second order equation can be rewritten as a pair of first order equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2\,,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -k_1x_2 - k_2x_1\,.\end{aligned}$$

More generally, consider systems of simultaneous differential equations of the form

$$\begin{bmatrix}\dfrac{\mathrm{d}x_1}{\mathrm{d}t} &=& ax_1 + bx_2\,,\\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} &=& cx_1 + dx_2\,.\end{bmatrix}$$

This can be written in the matrix form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{Ax} \text{ or simply } \dot{\mathbf{x}} = \mathbf{Ax}\,,$$

where $\mathbf{A} = \begin{bmatrix} a & b\\ c & d\end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1\\ x_2\end{bmatrix}$ and $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \dot{\mathbf{x}} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\dfrac{\mathrm{d}x_1}{\mathrm{d}t}\\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t}\end{bmatrix}$. Assume a solution in the form $\mathbf{x} = \mathbf{c}e^{\lambda t}$. Differentiating we obtain $\dot{\mathbf{x}} = \lambda\mathbf{c}e^{\lambda t}$. Substitution into the equation $\mathbf{Ax} = \dot{\mathbf{x}}$ leads to $\mathbf{Ac}e^{\lambda t} = \lambda\mathbf{c}e^{\lambda t}$. Then dividing through by $e^{\lambda t}$ we have $\mathbf{Ac} = \lambda\mathbf{c}$. Thus the problem of solving the differential equation has been reduced to an eigenvalue problem.

Note: if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the linear equation, then any linear combination $a\mathbf{x}_1 + b\mathbf{x}_2$ is also a solution.

EXAMPLE **1.22.** *Find the* general solution *of the system of equations*

$$\begin{aligned} \dot{x}_1 &= 3x_1 + 4x_2\,, \\ \dot{x}_2 &= 5x_1 + 2x_2 \end{aligned}$$

and then find the specific solution *satisfying the initial conditions* $x_1(0) = 10$, $x_2(0) = 1$.

SOLUTION. *The matrix form of the equation is* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, *where*

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}.$$

First, we find the eigenvalues of $\mathbf{A}$. *The characteristic equation is*

$$\begin{vmatrix} 3-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} = 0\,.$$

This expands to $\lambda^2 - 5\lambda + 6 - 20 = 0$, *that is* $\lambda^2 - 5\lambda - 14 = 0$, *and factorizes to* $(\lambda - 7)(\lambda + 2) = 0$ *so the eigenvalues are* $\lambda_1 = 7$ *and* $\lambda_2 = -2$. *Next, find the corresponding eigenvectors.*

If $\begin{bmatrix} u \\ v \end{bmatrix}$ *is an eigenvector corresponding to* $\lambda_1 = 7$, *then*

$$\begin{bmatrix} 3-7 & 4 \\ 5 & 2-7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

These equations reduce to the single equation $-4u + 4v = 0$ *or* $v = u$. *Thus an eigenvector corresponding to* $\lambda_1 = 7$ *is of the form* $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *for some* c_1.

Similarly, for the eigenvector corresponding to $\lambda_2 = -2$ *we obtain*

$$\begin{bmatrix} 3+2 & 4 \\ 5 & 2+2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

These equations reduce to the single equation $5u + 4v = 0$ *or* $u = -\frac{4}{5}v$. *Thus an eigenvector corresponding to* $\lambda_2 = -2$ *is of the form* $c_2 \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ *for some* c_2.

The general solution of the system of equations then can be written as

$$\mathbf{x} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} e^{7t} + \begin{bmatrix} 4c_2 \\ -5c_2 \end{bmatrix} e^{-2t}$$

or

$$\begin{aligned} x_1 &= c_1 e^{7t} + 4c_2 e^{-2t}\,, \\ x_2 &= c_1 e^{7t} - 5c_2 e^{-2t}\,. \end{aligned}$$

We can determine c_1 *and* c_2 *using the initial conditions* $x_1(0) = 10$ *and* $x_2(0) = 1$*. Substituting these into the general results in*

$$\begin{aligned} 10 &= c_1 e^{7(0)} + 4c_2 e^{-2(0)} = c_1 + 4c_2\,, \\ 1 &= c_1 e^{7(0)} - 5c_2 e^{-2(0)} = c_1 - 5c_2 \end{aligned}$$

that is $c_1 = 6$ *and* $c_2 = 1$*. Thus the specific solution of the initial value problem is*

$$x_1 = 6e^{7t} + 4e^{-2t}\,, \quad x_2 = 6e^{7t} - 5e^{-2t}\,.$$

■

Recollect that the a solution of the first order linear differential equation $\dot{x} = ax$ is an exponential function $x(t) = ce^{at}$, where b is some constant. Similarly, the solution of a system of first order linear differential equations written in a matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ can be given using matrix exponential as $\mathbf{x} = \exp(\mathbf{A}t)\mathbf{c}$, where $\mathbf{c} = [c_1, c_2]^T$ is a constant vector. Given that $\exp(\mathbf{A}0) = \mathbf{I}$ we conclude that constants c_1 and c_2 are the values of x_1 and x_2 at $t = 0$, that is these are the initial conditions to the problem.

Thus the solution of a system of linear t-dependent first order differential equations is obtained by directly multiplying an exponential function of the t-multiplied problem's coefficient matrix by the vector of initial conditions.

EXAMPLE **1.23.** *Solve the sytem of differential equations considered in* EXAMPLE 1.22 *using matrix exponential.*

SOLUTION. *As follows from Cayley-Hamilton theorem*

$$\exp(\mathbf{A}t) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A}\,,$$

where the eigenvalues of matrix $\mathbf{A}$, $\lambda_1 = 7$ *and* $\lambda_2 = -2$ *satisfy*

$$e^{7t} = \alpha_0 + 7\alpha_1 \text{ and } e^{-2t} = \alpha_0 - 2\alpha_1\,.$$

This results in $\alpha_0 = \dfrac{1}{9}(2e^{7t} + 7e^{-2t})$ *and* $\alpha_1 = \dfrac{1}{9}(e^{7t} - e^{-2t})$. *Then*

$$\begin{aligned}\exp(\mathbf{A}t) &= \frac{1}{9}\left(2e^{7t} + 7e^{-2t}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{1}{9}\left(e^{7t} - e^{-2t}\right)\begin{bmatrix}3 & 4\\ 5 & 2\end{bmatrix}\\ &= \frac{1}{9}\begin{bmatrix}5e^{7t} + 4e^{-2t} & 4\left(e^{7t} - e^{-2t}\right)\\ 5(e^{7t} - e^{-2t}) & 4e^{7t} + 5e^{-2t}\end{bmatrix}.\end{aligned}$$

Finally,

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \frac{1}{9}\begin{bmatrix}5e^{7t} + 4e^{-2t} & 4\left(e^{7t} - e^{-2t}\right)\\ 5\left(e^{7t} - e^{-2t}\right) & 4e^{7t} + 5e^{-2t}\end{bmatrix}\begin{bmatrix}10\\ 1\end{bmatrix} = \begin{bmatrix}6e^{7t} + 4e^{-2t}\\ 6e^{7t} - 5e^{-2t}\end{bmatrix}$$

as expected. ■

EXAMPLE **1.24.** *Find the* general solution *of the system of equations*

$$\begin{aligned}\dot{x}_1 &= x_1\,,\\ \dot{x}_2 &= x_1 + x_2\,.\end{aligned}$$

SOLUTION. *The matrix form of the system of equations is* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, *where*

$$\mathbf{A} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}.$$

First, we find the eigenvalues of $\mathbf{A}$. *The characteristic equation is*

$$\begin{vmatrix}1-\lambda & 0\\ 1 & 1-\lambda\end{vmatrix} = 0\,.$$

This expands to $(1-\lambda)^2 = 0$, *that is the matrix has a repeated eigenvalue* $\lambda_1 = \lambda_2 = \lambda = 1$. *Only one eigenvector* $\mathbf{e}_1 = c_1\begin{bmatrix}0\\ 1\end{bmatrix}$, *where*

c_1 is an arbitrary constant, can be found. Thus, the solution procedure described in EXAMPLE 1.22 *is not applicable here, but the use of matrix exponential enables to obtain the answer. As follows from Cayley-Hamilton theorem*

$$\exp(\mathbf{A}t) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A}\,,$$

and, for a repeated eigenvalue (see also EXAMPLE 1.14*)*

$$\exp(\lambda t) = \alpha_0 + \alpha_1 \lambda \text{ and } t\exp(\lambda t) = \alpha_1\,.$$

Then for $\lambda = 1$ we obtain $\alpha_1 = te^t$, $\alpha_0 = e^t - te^t$ and

$$\exp(\mathbf{A}t) = (e^t - te^t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ c_1 te^t + c_2 e^t \end{bmatrix}.$$

Note that in this case the solution contains term te^t in addition to a pure exponential term e^t. To understand why it appears note that the first equation in the system, $\dot{x}_1 = x_1$, is fully decoupled from the second equation and can be solved directly to give $x_1(t) = c_1 e^t$. Subsequently, the second equation can be rearranged to $\dot{x}_2 - x_2 = c_1 e^t$, where the right-hand side can be treated as forcing. It suggests that the solution of the form $x_2 \sim e^t$ should exist. However, substituting this in the left-hand side of the equation makes it zero, which cannot be equal to the function in the right-hand side. Such a situation when the forcing has a functional form similar to that of a natural solution of an equation (that is the solution of the homogeneous version of equation) is called resonance. *At resonance the amplitude of the solution grows linearly with time under the action of forcing and the term te^t appears here.* ■

Exercises

EX. 1.17. Use matrix methods to solve the following systems of differential equations.

(a) $\left\{ \begin{array}{rcl} \dot{x}_1 &=& 3x_1 + 4x_2\,, \\ \dot{x}_2 &=& 4x_1 - 3x_2 \end{array} \right. \quad x_1(0) = 5,\ x_2(0) = 1.$

(b) $\left\{ \begin{array}{rcl} \dot{x}_1 &=& 3x_1 + x_2\,, \\ \dot{x}_2 &=& -2x_1 \end{array} \right. \quad x_1(0) = 9,\ x_2(0) = 3$

(c) $\begin{cases} \dot{x}_1 &= 6x_1 + 2x_2\,, \\ \dot{x}_2 &= 2x_1 + 3x_2 \end{cases}$

(d) $\begin{cases} \dot{x}_1 &= x_1 + x_2\,, \\ \dot{x}_2 &= 3x_1 - x_2 \end{cases}$ $\quad x_1(0) = 1,\ x_2(0) = 2$

(e) $\begin{cases} \dot{x}_1 &= 14x_1 - 10x_2\,, \\ \dot{x}_2 &= 5x_1 - x_2 \end{cases}$ $\quad x_1(0) = 1,\ x_2(0) = 0$

(f) $\begin{cases} \ddot{x}_1 &= -5x_1 + 2x_2\,, \\ \ddot{x}_2 &= 2x_1 - 2x_2 \end{cases}$ $\quad \begin{array}{l} x_1(0) = 5\,,\ x_2(0) = 10\,, \\ \dot{x}_1(0) = \dot{x}_2(0) = -10\sqrt{6} \end{array}$

(g) $\begin{cases} \ddot{x}_1 &= x_1 + 3x_2\,, \\ \ddot{x}_2 &= 2x_1 + 2x_2 \end{cases}$

(h) $\begin{cases} \ddot{x}_1 &= -4x_1 + 2x_2\,, \\ \ddot{x}_2 &= x_2 \end{cases}$

(i) $\begin{cases} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= x_1 + 3x_3\,, \\ \dot{x}_3 &= x_2 \end{cases}$ $\quad x_1(0) = 2,\ x_2(0) = 0,\ x_3(0) = 2.$

1.8 Answers to Selected Exercises

Ex. 1.2. (a) $\lambda_1 = -1$, $\lambda_2 = 3$, $\mathbf{e}_1 = t_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{e}_2 = t_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$;

(b) double eigenvalue $\lambda_{1,2} = 4$, $\mathbf{e}_1 = t_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$;

(c) $\lambda_1 = -6$, $\lambda_2 = 6$, $\mathbf{e}_1 = t_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{e}_2 = t_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$;

(d) $\lambda_1 = -\sqrt{3}\mathrm{i}$, $\lambda_2 = \sqrt{3}\mathrm{i}$, $\mathbf{e}_1 = t_1 \begin{bmatrix} -2 + \mathrm{i}\sqrt{3} \\ 1 \end{bmatrix}$, $\mathbf{e}_2 = t_2 \begin{bmatrix} -2 - \mathrm{i}\sqrt{3} \\ 1 \end{bmatrix}$.

(e) $\lambda_{1,2} = 1$ (double eigenvalue), any vector is the eigenvector of the identity matrix. We can choose two orthogonal unit eigenvectors as $\mathbf{e}_1 = t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = t_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Ex. 1.3. (a) $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\mathbf{e}_1 = t_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = t_2 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$,

$\mathbf{e}_3 = t_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$;

(b) $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 7$, $\mathbf{e}_1 = t_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{e}_2 = t_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

$\mathbf{e}_3 = t_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$;

(c) $\lambda_1 = 5$, $\lambda_{2,3} = 1$ (double eigenvalue), $\mathbf{e}_1 = t_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{e}_2 = t_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$,

$\mathbf{e}_3 = t_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$;

(d) $\lambda_{1,2,3} = 2$ (repeated eigenvalue), $\mathbf{e}_1 = t_1 \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$.

Ex. 1.4. **A**: $\lambda_1 = 1$, $\lambda_2 = 6$, $\mathbf{e}_1 = t_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{e}_2 = t_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$;

B: $\lambda_1 = 2$, $\lambda_2 = 4$, $\mathbf{e}_1 = t_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{e}_2 = t_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

C: $\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 3,\ \mathbf{e}_1 = t_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{e}_2 = t_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},\ \mathbf{e}_3 = t_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix};$

D: $\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 6,\ \mathbf{e}_1 = t_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{e}_2 = t_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{e}_3 = t_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix};$

E: $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3,\ \mathbf{e}_1 = t_1 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix},\ \mathbf{e}_2 = t_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{e}_3 = t_3 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$

Ex. 1.8. $\det(\mathbf{A}) = -1$ and $\mathrm{Tr}(\mathbf{A}) = 2$ so that

(a) $\mathbf{A}^{-1} = \det(\mathbf{A})^{-1}(\mathrm{Tr}(\mathbf{A})\mathbf{I} - \mathbf{A}) = -(2\mathbf{I} - \mathbf{A}) = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix};$

(b) $\mathbf{A}^2 = \mathrm{Tr}(\mathbf{A})\mathbf{A} - \det(\mathbf{A})\mathbf{I} = 2\mathbf{A} + \mathbf{I} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix};$

(c) $\mathbf{A}^3 = \mathrm{Tr}(\mathbf{A})\mathbf{A}^2 - \det(\mathbf{A})\mathbf{A} = 2\mathbf{A}^2 + \mathbf{A} = \begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix}.$

Ex. 1.9. (a) The matrix has a repeated eigenvalue $\lambda = 1$. Write $\exp(\mathbf{A}t) = \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$ and replace $\mathbf{A}$ with an eigenvalue λ to obtain $e^{\lambda t} = \alpha_1\lambda + \alpha_0$. To obtain the second equation differentiate this with respect to λ: $te^{\lambda t} = \alpha_1$. Now substitute $\lambda = 1$ obtaining $\alpha_1 = te^t$ and $\alpha_0 = (1-t)e^t$. Hence

$$\exp(\mathbf{A}t) = te^t\mathbf{A} + (1-t)e^t\mathbf{I} = te^t \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + (1-t)e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}.$$

(b) The matrix has two distinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. Thus $e^{\lambda_{1,2}t} = \alpha_1\lambda_{1,2} + \alpha_0$. Then

$$\begin{aligned} e^t &= \alpha_1 + \alpha_0 \\ e^{2t} &= 2\alpha_1 + \alpha_0 \end{aligned}$$

so that $\alpha_1 = e^{2t} - e^t$ and $\alpha_0 = 2e^t - e^{2t}$. Hence

$$\begin{aligned} \exp(\mathbf{A}t) &= \left(e^{2t} - e^t\right)\mathbf{A} + \left(2e^t - e^{2t}\right)\mathbf{I} \\ &= \left(e^{2t} - e^t\right)\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + \left(2e^t - e^{2t}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ e^{2t} - e^t & e^{2t} \end{bmatrix}. \end{aligned}$$

Ex. 1.12. **A**: $\lambda_1 = 6,\ \lambda_2 = -2,\ \lambda_3 = -3,\ \mathbf{e}_1 = t_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{e}_2 = t_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$

$\mathbf{e}_3 = t_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix};$

$$\mathbf{B}: \lambda_1 = 10,\ \lambda_2 = 2,\ \lambda_3 = 1,\ \mathbf{e}_1 = t_1\begin{bmatrix}2\\1\\1\end{bmatrix},\ \mathbf{e}_2 = t_2\begin{bmatrix}0\\-1\\1\end{bmatrix},\ \mathbf{e}_3 = t_3\begin{bmatrix}-1\\1\\1\end{bmatrix}.$$

Ex. 1.15. (a) $-5y_1^2 + 5y_2^2 + y_3^2$ is one-sheet hyperboloid; modal matrix is

$$\mathbf{M} = \begin{bmatrix}\frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{bmatrix};$$

(b) $-9y_1^2 + 5y_2^2 - y_3^2$ is two-sheet hyperboloid; modal matrix is

$$\mathbf{M} = \begin{bmatrix}0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{bmatrix};$$

(c) $6y_1^2 + 3y_2^2 - y_3^2$ is one-sheet hyperboloid; modal matrix is

$$\mathbf{M} = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\end{bmatrix}.$$

Ex. 1.16. Transformation matrix is $\mathbf{M} = \begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}}\end{bmatrix}$, $3y_1^2 + 3y_2^2 = 1$ is a circular cylinder.

Ex. 1.17. (a) $x_1 = \frac{3}{5}e^{-5t} + \frac{22}{5}e^{5t}$, $x_2 = -\frac{6}{5}e^{-5t} + \frac{11}{5}e^{5t}$;

(b) $x_1 = -12e^t + 21e^{2t}$, $x_2 = 24e^t - 21e^{2t}$;

(c) $x_1 = c_1e^{2t} + 2c_2e^{7t}$, $x_2 = -2c_1e^t + c_2e^{2t}$;

(d) $x_1 = -\frac{1}{4}e^{-2t} + \frac{5}{4}e^{2t}$, $x_2 = \frac{3}{4}e^{-2t} + \frac{5}{4}e^{2t}$;

(e) $x_1 = -e^{4t} + 2e^{9t}$, $x_2 = -e^{4t} + e^{9t}$;

(f) $x_1 = 5\cos t - 6\sqrt{6}\sin t - 4\sin(\sqrt{6}t)$,
$x_2 = 10\cos t - 12\sin t + 2\sin(\sqrt{6}t)$;

(g) $x_1 = e^{-2t}(4c_1 - 2c_2 + 6c_3 - 3c_4) + e^{2t}(4c_1 + 2c_2 + 6c_3 + 3c_4) + 12(c_1 - c_3)\cos t + 12(c_2 - c_4)\sin t$,
$x_2 = e^{-2t}(4c_1 - 2c_2 + 6c_3 - 3c_4) + e^{2t}(4c_1 + 2c_2 + 6c_3 + 3c_4) + 8(c_1 + c_3)\cos t - 8(c_2 - c_4)\sin t$,

(h) $x_1 = 2e^{-t}(c_3 - c_4) + 2e^t(c_3 + c_4) + (10c_1 - 4c_3)\cos(2t) + (5c_2 - 2c_4)\sin(2t)$,
$x_2 = 5e^{-t}(c_3 - c_4) + 5e^t(c_3 + c_4)$;

(i) $x_1 = e^{2t} + e^{-2t}$, $x_2 = 2(e^{2t} - e^{-2t}$, $x_3 = e^{2t} - e^{-2t}$.

Module 2

Vector Calculus: Vector Differentiation

Module contents

2.1 Vector Calculus

This section introduces the subject of *vector calculus*. It deals with two kinds of function, *scalar functions* and *vector functions*. The section assumes that you are fluent with the previously studied material reviewed in Appendix A.3.

2.1.1 Vector and scalar functions and fields

Definition 2.1 *A function*

$$f = f(P)$$

that returns a single (scalar) value is called a scalar function.

Definition 2.2 *A function*

$$\mathbf{v}(P) = v_1(P)\mathbf{i} + v_2(P)\mathbf{j} + v_3(P)\mathbf{k} = (v_1(P), v_2(P), v_3(P))$$

that returns a vector with components v_1, v_2, v_3 in the directions of the unit vectors **i**, **j**, **k** *defined by scalar functions depending on P is called a vector function.*

In applications, the *domain of definition* for such functions is a region, a surface or a curve in space. For example, the atmospheric pressure or temperature are described by scalar functions defined over the surface of the Earth, speed of a projectile is given by a scalar function defined along its trajectory (which is a curve), while the velocity of the same projectile is given by a vector function defined along the trajectory.

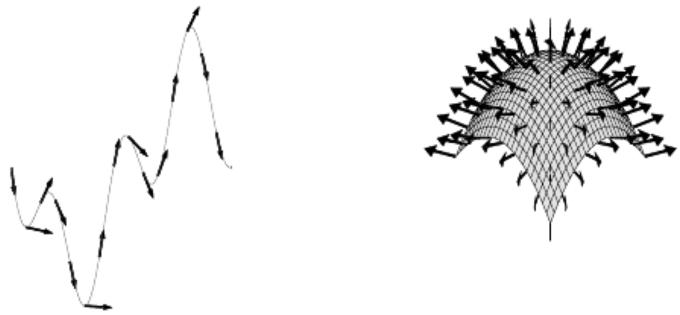


Figure 2.1: Vector fields of tangent vectors to a curve and normal vectors to a surface.

Definition 2.3 *The multitude of all values (vectors) of a scalar (vector) function defined over its domain is called a* scalar field *(*vector field*).*

The atmospheric pressure map is an example of a scalar field and the map of wind directions is an example of a vector field. More geometrical examples of vector fields are shown in Figure 2.1.

Using the description of a point in Cartesian coordinates x, y, z instead of $f(P)$ and $\mathbf{v}(P)$ we can write $f(x,y,z)$ and $\mathbf{v}(x,y,z) = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k} = (v_1(x,y,z), v_2(x,y,z), v_3(x,y,z))$.

Example **2.1.** *Distance in space.*

The distance $f(P)$ of any point from the origin O in space is a scalar function the domain of definition of which is the whole space. The scalar function $f(P)$ defines a scalar field in space, and

$$f(P) = f(x,y,z) = \sqrt{x^2+y^2+z^2}\,,$$

where x, y, z are the coordinates of P.

Example **2.2.** *The vector field of position vectors $\mathbf{r} = (x,y,z)$ is defined in three-dimensional space by a vector function*

$$\begin{aligned}\mathbf{r}(P) = \mathbf{r}(x,y,z) &= r_1(x,y,z)\mathbf{i} + r_2(x,y,z)\mathbf{j} + r_3(x,y,z)\mathbf{k} \\ &= (r_1(x,y,z), r_2(x,y,z), r_3(x,y,z))\,,\end{aligned}$$

where $r_1(x,y,z) = x$, $r_2(x,y,z) = y$ and $r_3(x,y,z) = z$.

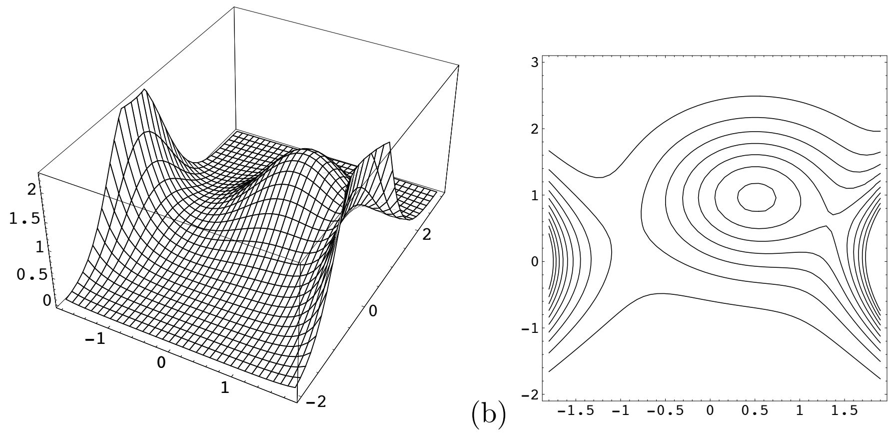


Figure 2.2: Function of two variables as a 3D surface and as a contour plot.

2.1.2 Visualising two-dimensional fields

Scalar fields

A scalar *function of two variables* $z = f(x, y)$ is given by specifying a value of the function for each pair (x, y) in the domain of the function. Therefore, it naturally defines a scalar field. The simplest examples are *linear functions* such as $z = 2x - 3y + 4$, or in general $z = ax + by + c$, where a, b and c are constants. Next in complexity are *quadratic functions* $z = ax^2 + by^2 + cxy + dx + ey + g$. For example, $T = 100 - (x^2 + y^2)$ might represent the temperature distribution in a heated plate. We can graphically represent a function of two variables and, consequently, the corresponding scalar field by a graph of a *surface* in three dimensions, or by a set of contours or level curves. The general idea is that if $z = f(x, y)$ then setting $z = c$ (some constant) gives an equation $c = f(x, y)$ which has a graph in the (x, y)-plane. Letting $z = c_1, c_2, \cdots$ gives a family of curves. In Figure 2.2 (a) and (b) we show the graph of a function as a surface in three dimensions and a contour plot, respectively. The contours are the curves along which the function is constant. They are also referred to as *level curves* of the function—if the the surface is regarded as a hill then the points on a contour or level curve are all at the same height.

EXAMPLE **2.3.** *For the function $z = 20 - x + y$ sketch a few contours and the surface.*

SOLUTION. *We assign different values to z, then the contours are the corresponding curves. For example, if $z = 20$, then $20 = 20 - x + y$. Therefore, the contour is the curve with equation $y = x$. If $z = c$ then*

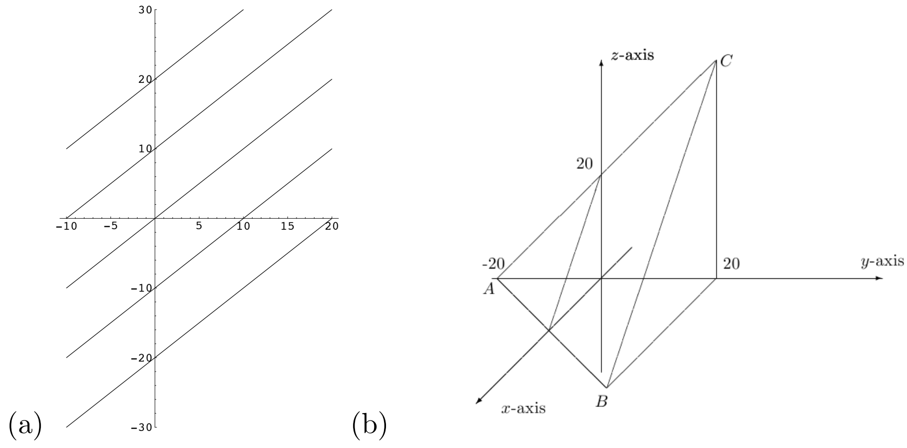


Figure 2.3: Contours of $z = 20 - x + y$.

the contour has equation $y = x + c - 20$ that describes straight lines parallel to the line $y = x$ shown in Figure 2.3 (a) for $z = 0$, 10, 20, 30 and 40. Figure 2.3 (b) shows part of the graph of the function as the triangular plane region ABC. ■

Vector fields

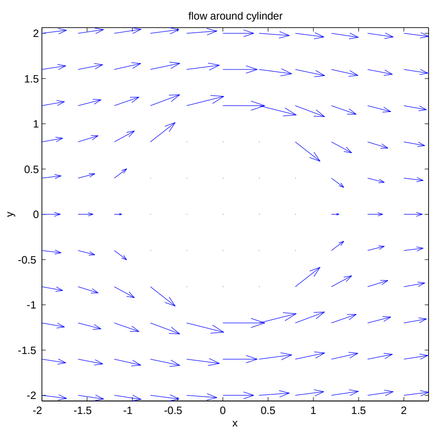


Figure 2.4: Flow velocity vector field around a circular cylinder.

Visualising a two-dimensional vector field requires plotting vectors at each point in a plane in such a way that their lengths are proportional to their computed magnitudes. For example, a vector field of velocities around a solid cylinder placed in a uniform stream of inviscid fluid is shown in Figure 2.4.

A visualisation of three-dimensional fields is usually done by showing several of their two-dimensional cross-sections.

2.1.3 Basic concepts of vector calculus

The basic concepts of calculus such as convergence, continuity and differentiability that you have met in your previous studies in the context of scalar functions can be defined for vector functions in a simple and natural way.

Definition 2.4 (Convergence) *A vector function* $\mathbf{v}(t)$ *is said to have the* limit $\mathbf{a}$ *as* t *approaches* t_0 *if the magnitude of* $\mathbf{v}(t)-\mathbf{a}$ *tends to zero i.e.*

$$\lim_{t\to t_0} ||\mathbf{v}(t)-\mathbf{a}|| = 0\,,$$

In this case we write

$$\lim_{t\to t_0} \mathbf{v}(t) = \mathbf{a}\,.$$

Definition 2.5 (Continuity) *A vector function* $\mathbf{v}(t)$ *is said to be continuous at* $t=t_0$ *if*

$$\lim_{t\to t_0} \mathbf{v}(t) = \mathbf{v}(t_0)\,.$$

Definition 2.6 (Derivative of a vector function of a single variable) *A vector function* $\mathbf{v}(t)$ *is said to be* differentiable *at a point* t *if the following limit exists:*

$$\mathbf{v}'(t) = \lim_{\Delta t\to 0} \frac{\mathbf{v}(t+\Delta t)-\mathbf{v}(t)}{\Delta t}\,.$$

If $\mathbf{v}(t)$ is an instantaneous position vector of a moving particle, then as follows from Figure 2.5 the position difference vector $\mathbf{v}(t+\Delta t)-\mathbf{v}(t)$ connects two points on the particle trajectory and thus is a chord of the trajectory curve. In the limit $\Delta t \to 0$ the two end points of the chord collapse into a single point and the chord approaches the tangent to the trajectory. This defines the *geometrical meaning of the vector derivative*.

In terms of components with respect to a given Cartesian coordinate system, $\mathbf{v}(t)$ is differentiable if and only if its three components are differentiable at t, and

$$\mathbf{v}'(t) = v_1'(t)\mathbf{i} + v_2'(t)\mathbf{j} + v_3'(t)\mathbf{k} = (v_1'(t), v_2'(t), v_3'(t))\,.$$

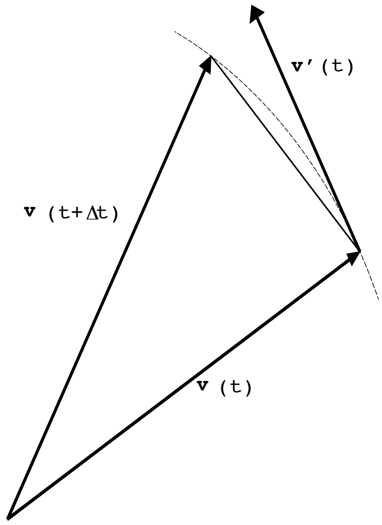


Figure 2.5: Derivative of a vector function.

Here prime denotes differentiation with respect to t. Since the differentiation of a vector function is defined in terms of differentiation of its individual components that are scalar functions, the standard differentiation rules for scalar functions yield corresponding rules for the differentiation of vector functions:

$$\begin{aligned} (c\mathbf{v})' &= c\mathbf{v}'\,, \\ (\mathbf{u}+\mathbf{v})' &= \mathbf{u}'+\mathbf{v}'\,, \\ (\mathbf{u}\cdot\mathbf{v})' &= \mathbf{u}'\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{v}'\,, \\ (\mathbf{u}\times\mathbf{v})' &= \mathbf{u}'\times\mathbf{v}+\mathbf{u}\times\mathbf{v}'\,. \end{aligned}$$

2.2 The Gradient Operator and Directional Derivatives

2.2.1 Rates of change

The rate of change of a function of one variable $y = f(x)$ is defined by its *full derivative*

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\,.$$

For a function of several variables, for example, $z = f(x, y, z)$ we define the rate of change in the x, y and z directions by the respective *partial* *derivatives*

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}, \\
\frac{\partial f}{\partial y} &= \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y}, \\
\frac{\partial f}{\partial z} &= \lim_{\Delta z \to 0} \frac{f(x, y, z+\Delta z) - f(x,y,z)}{\Delta z}.
\end{aligned}$$

2.2.2 Vector operator nabla

Definition 2.7 *The* nabla operator $\boldsymbol{\nabla}$ *is defined as*

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Note the two special features of this operator. Firstly, on its own it does not return any values. It has to be applied to, or operate on, a function to produce a numerical or symbolic output. That is why the term *operator* has been introduced. Secondly, $\boldsymbol{\nabla}$ consists of several components so that this is a vector. Therefore, we will study the three standard vector operations—multiplication by a scalar, scalar (or dot) product and vector (cross) product—applied to the operator vector nabla next[1].

2.2.3 Gradient of a scalar field

We start with the action of a vector operator nabla on a scalar. Namely, we apply operator nabla to a scalar field f. This yields the *gradient of a scalar field*

$$\boxed{\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}.$$

> The gradient operator applies to a scalar field and produces a vector field.

EXAMPLE **2.4.** *The gradient of a function* $f(x,y,z) = x^2 + y^3 + z^4$ *at the point* (x_0, y_0, z_0) *is*

$$\boldsymbol{\nabla} f(x_0, y_0, z_0) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = 2x_0\mathbf{i} + 3y_0^2\mathbf{j} + 4z_0^3\mathbf{k} = \left(2x_0, 3y_0^2, 4z_0^3\right).$$

[1] Operator nabla is sometimes referred to as "del".

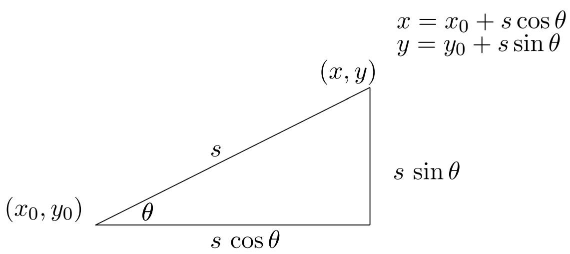


Figure 2.6: Directional derivative.

2.2.4 Gradient of spherically symmetric functions

It is often necessary to find the gradient of functions expressed in the form $f(r)$, where r is the distance of a given point from the origin, $r = \sqrt{x^2 + y^2 + z^2}$. Such functions are called *spherically symmetric*. For example, it is known that the gravity force is inversely proportional to the square of the distance from the centre of the Earth.

Note that $\dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{x}{r}$. Using symmetry arguments we can write straight away that $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$. Therefore, for the gradient of r we obtain

$$\boldsymbol{\nabla} r = \frac{1}{r}(x, y, z) = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}\,.$$

Using the chain rule for an arbitrary spherically symmetric function we obtain then

$$\boldsymbol{\nabla} f(r) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\mathrm{d} f}{\mathrm{d} r}\frac{\partial r}{\partial x}, \frac{\mathrm{d} f}{\mathrm{d} r}\frac{\partial r}{\partial y}, \frac{\mathrm{d} f}{\mathrm{d} r}\frac{\partial r}{\partial z}\right) = \frac{\mathrm{d} f}{\mathrm{d} r}\boldsymbol{\nabla} r\,.$$

Therefore,

$$\boxed{\boldsymbol{\nabla} f(r) = \frac{\mathrm{d} f}{\mathrm{d} r}\boldsymbol{\nabla} r = \frac{\mathrm{d} f}{\mathrm{d} r}\hat{\mathbf{r}}}\,.$$

2.2.5 Directional derivative

We have seen that the gradient is related to the rate of change of f in the directions of the three coordinate axes. These rates are given by the respective partial derivatives. The idea of extending this to arbitrary directions seems natural and leads to the concept of the *directional derivative*.

Consider a point (x_0, y_0) and a line passing through it making an angle θ with the x-axis, see Figure 2.6. It is parallel to the unit vector

$\hat{\mathbf{a}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} = (\cos\theta, \sin\theta)$. The parametric equations for the line can be written as $x(s) = x_0 + s\cos\theta$, $y(s) = y_0 + s\sin\theta$, where parameter s has the meaning of the distance from point (x_0, y_0). Since coordinates (x, y) of an arbitrary point that belongs to the line are functions of s, the rate of change along this line, or directional derivative in the direction of $\hat{\mathbf{a}}$, can be computed using the *chain rule* as

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\,.$$

This expression can be written as a dot product of two vectors

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)\cdot(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot\hat{\mathbf{a}}\,.$$

The first of the vectors is recognised as the gradient vector of $f(x, y)$. Therefore, we obtain the following expression for a directional derivative.

Definition 2.8 *The rate of change of a scalar function in the direction of a unit vector* $\hat{\mathbf{a}}$ *is given by the* directional derivative

$$\boxed{D_{\mathbf{a}}f = \boldsymbol{\nabla}f\cdot\hat{\mathbf{a}}}\,.$$

This expression retains its form in three dimensions with vectors having three components.

The directional derivative applies to a scalar function (field) and produces a scalar value (field).

Example **2.5.** *A function is defined by* $f(x, y) = 3x^2 - xy + 5y^2$. *Calculate* $\boldsymbol{\nabla}f$ *and find the rate of change of* f *in the direction of the vector* $\mathbf{a} = (3, -4)$ *at the point* $(2, 1)$.

Solution. *The partial derivatives are* $\dfrac{\partial f}{\partial x} = 6x - y$ *and* $\dfrac{\partial f}{\partial y} = 10y - x$ *so that* $\boldsymbol{\nabla}f = (6x - y, 10y - x)$. *At the point* $(2, 1)$, $\boldsymbol{\nabla}f(2, 1) = (11, 8)$. *To find the rate of change in the direction of* $\mathbf{a}$ *we need to define the corresponding unit vector. Since* $||\mathbf{a}|| = \sqrt{3^2 + (-4)^2} = 5$,
$\hat{\mathbf{a}} = \left(\frac{3}{5}, -\frac{4}{5}\right)$. *Therefore,* $D_{\mathbf{a}}f = \boldsymbol{\nabla}f\cdot\hat{\mathbf{a}} = (11, 8)\cdot\left(\frac{3}{5}, -\frac{4}{5}\right) = \dfrac{33 - 32}{5} = \dfrac{1}{5}$. ■

EXAMPLE **2.6.** *Find the directional derivative of $f = x^2 + xyz$ at $(1, 2, -1)$ in the direction of $\mathbf{u} = (1, 2, -3)$.*

SOLUTION. *Since $\boldsymbol{\nabla} f(x, y, z) = (2x + yz, xz, xy)$, $\boldsymbol{\nabla} f(1, 2, -1) = (0, -1, 2)$. The magnitude of the given vector is $||\mathbf{u}|| = \sqrt{1+4+9} = \sqrt{14}$. Thus, the unit vector is $\hat{\mathbf{u}} = \frac{1}{\sqrt{14}}(1, 2, -3)$ and we obtain*

$$D_{\mathbf{u}} f(1, 2, -1) = \boldsymbol{\nabla} f(1, 2, -1) \cdot \hat{\mathbf{u}} = -\frac{8}{\sqrt{14}}\,.$$

■

2.2.6 The meaning of the gradient vector

We have shown in Section 2.2.5 that the rate of change of a scalar function f in a particular direction is obtained by taking the scalar product of its gradient vector with the unit vector in the chosen direction. Using the definition of a dot product we can write

$$D_{\mathbf{a}} f = ||\boldsymbol{\nabla} f||\,||\hat{\mathbf{a}}|| \cos\theta = ||\boldsymbol{\nabla} f|| \cos\theta\,.$$

This expression leads to the following conclusions.

1. $D_{\mathbf{a}} f$ attains its maximum (positive) value when $\cos\theta = 1$ that is when $\theta = 0$ and the chosen direction coincides with that of the gradient vector. Therefore, *the gradient vector specifies the direction of the fastest increase of a scalar function and the magnitude of the gradient is the fastest growth rate of the function.*

2. $D_{\mathbf{a}} f$ attains its minimum (negative) value when $\cos\theta = -1$ that is when $\theta = \pi$ and the chosen direction is opposite to that of the gradient vector. Therefore, *a scalar function experiences the fastest decrease in the direction opposite to that of its gradient vector and the maximum decrease rate is given by the negative of magnitude of the gradient.*

3. $D_{\mathbf{a}} f = 0$ when $\cos\theta = 0$ that is when $\theta = \frac{\pi}{2}$ and the chosen direction is orthogonal to that of the gradient vector. Therefore, *a scalar function maintains its constant value in the direction perpendicular to that of its gradient vector.* In other words, since the lines, along which the function value remains constant, are called contour lines, the vector $\boldsymbol{\nabla} f(x_0, y_0)$ is normal, or perpendicular, to the contour line of $f(x, y)$ passing through (x_0, y_0).

Example 2.7. *We show that the vector of coefficients* (a, b) *is perpendicular to the straight line* $ax + by = d$*. Indeed, consider function* $f(x, y) = ax + by$*. Its contour lines are described by* $f(x, y) = ax + by = d$*, where* d *is an arbitrary constant. In other words, the equation of a straight line* $ax + by = d$ *is also the equation for a contour line of function* $f(x, y)$*. Therefore, the given straight line is perpendicular to the gradient of* $f(x, y)$*, which is* $\boldsymbol{\nabla} f = (a, b)$*, the coefficient vector.*

The above results are straightforwardly generalised to three dimensions.

Example 2.8. *Show that the vector of coefficients* (a, b, c) *is perpendicular to the plane* $ax + by + cz = d$*.*

Solution. *Consider function* $f(x, y, z) = ax + by + cz$*. By setting* $f(x, y, z) = d$*, where* d *is some constant we see that the given plane is the collection of points in a three-dimensional space for which the value of* $f(x, y, z)$ *remains constant. In other words, the rate of change of* $f(x, y, z)$ *is necessarily 0 along the plane. Therefore, the plane is perpendicular to* $\boldsymbol{\nabla} f = (a, b, c)$*.* ■

More generally, to find a *normal vector to a surface* in a three-dimensional space you need to

(1) rewrite the equation of the surface in an equivalent form $f(x, y, z) = d$, where d is a constant (frequently chosen to be 0 for simplicity);

(2) find $\boldsymbol{\nabla} f(x, y, z)$;

(3) evaluate $\boldsymbol{\nabla} f(x, y, z)$ at the required point (x_0, y_0, z_0) that belongs to the surface (to ensure that the point (x_0, y_0, z_0) is indeed on the surface check if it satisfies the equation $f(x_0, y_0, z_0) = d$).

EXAMPLE **2.9.** *Find the unit normal to the surface given by $x^2+y^2=1+z^2$ at $(x_0,y_0,z_0)=(1,-1,1)$.*

SOLUTION. *First, by direct substitution check that the point $(1,-1,1)$ belongs to the given surface: $1^2+(-1)^2=2=1+1^2$. Then rewrite the equation of the surface in an equivalent form $x^2+y^2-z^2=1$. Use the left-hand side of the equation to define a function $f(x,y,z)=x^2+y^2-z^2$ and find its gradient $\boldsymbol{\nabla} f=(2x,2y,-2z)$. Evaluate the gradient to find a normal vector $\mathbf{N}$ at a given point and compute its magnitude: $\mathbf{N}=(2,-2,-2)$, $||\mathbf{N}||=2\sqrt{3}$. Finally, find the unit normal vector*

$$\mathbf{n}=\frac{\mathbf{N}}{||\mathbf{N}||}=\frac{1}{\sqrt{3}}(1,-1,-1)\,.$$

Note that vector $-\mathbf{n}=-\dfrac{1}{\sqrt{3}}(1,-1,-1)$ is also normal to the surface.

> *At any point a surface has two normals, one in the direction of the gradient of the function level of which the surface represents, the other in the direction opposite to this gradient. One of the normals is called* inward (inner) normal *and the other is* outward (outer) normal. *Which one is which depends on the context of a problem.*

■

EXAMPLE **2.10.** *Find the equation of the tangent plane to the sphere $x^2+y^2+z^2=\dfrac{3}{4}$ at $(x_0,y_0,z_0)=\left(\dfrac{1}{2},-\dfrac{1}{2},\dfrac{1}{2}\right)$.*

SOLUTION. *Check that the given point indeed belongs to a sphere then define function $f=x^2+y^2+z^2$ and find its gradient at $\left(\dfrac{1}{2},-\dfrac{1}{2},\dfrac{1}{2}\right)$. It is $\boldsymbol{\nabla} f=(1,-1,1)$ and is a normal vector to sphere and, consequently, to the tangent plane. Therefore, the equation for the tangent plane should be in the form of $x-y+z=d$. As the plane passes through the point $\left(\dfrac{1}{2},-\dfrac{1}{2},\dfrac{1}{2}\right)$, we obtain $\dfrac{1}{2}-\left(-\dfrac{1}{2}\right)+\dfrac{1}{2}=d$, or $d=\dfrac{3}{2}$. Therefore, the equation for the tangent plane is $x-y+z=\dfrac{3}{2}$.* ■

2.2.7 Potential functions

Definition 2.9 *If for a given vector field* $\mathbf{F}$*, there is a function* φ *such that* $\nabla\varphi = \mathbf{F}$*, then we say that* φ *is a* potential function *of* $\mathbf{F}$ *and* $\mathbf{F}$ *is a* potential vector field.

One of the main reasons why potential functions are introduced is that they enable one to replace the consideration of a multi-component vector fields with that of a single scalar function.[2]

Given a scalar differentiable function $\varphi(x, y, z)$ we can always construct the potential vector field

$$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right) = (F_1, F_2, F_3) = \mathbf{F}\,,$$

but not all vector fields have potential functions.

The following examples demonstrate this.

EXAMPLE **2.11.** *If* $\varphi(x, y) = x^2y$ *then we can use it to generate a potential vector field* $\mathbf{F} = \nabla\varphi = (2xy, x^2)$.

EXAMPLE **2.12.** *Find a potential function for the vector field*

$$\mathbf{F} = (2xy, x^2 - 2y)\,.$$

SOLUTION. *If* φ *is a potential function then we must have*

$$\begin{aligned} \frac{\partial\varphi}{\partial x} &= 2xy\,, \\ \frac{\partial\varphi}{\partial y} &= x^2 - 2y\,. \end{aligned}$$

Integrating the first equation with respect to x *gives* $\varphi = x^2y + g(y)$. *Substituting this result into the second equation leads to*

$$\frac{\partial\varphi}{\partial y} = x^2 + \frac{\mathrm{d}g}{\mathrm{d}y} = x^2 - 2y$$

so that $\frac{\mathrm{d}g}{\mathrm{d}y} = -2y$. *From here we obtain* $g(y) = -y^2 + c$ *so that the potential function for* $\mathbf{F}$ *is* $\varphi = x^2y - y^2 + c$. ■

[2] We will see more applications when we study vector integration, see Module 3.

EXAMPLE **2.13.** *Find a potential function for the vector field* $\mathbf{v} = (y^2, x^2)$.

SOLUTION. *If* φ *is a potential function then we must have*

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= y^2\,, \\ \frac{\partial \varphi}{\partial y} &= x^2\,. \end{aligned}$$

Integrating the first equation with respect to x *gives* $\varphi = xy^2 + g(y)$. *Substituting this result into the second equation leads to*

$$\frac{\partial \varphi}{\partial y} = 2xy + \frac{\mathrm{d}g}{\mathrm{d}y} = x^2\,.$$

Since the terms in the middle of the last equation depend on both x *and* y *while the right-hand side is a function of* x *alone, the equation is inconsistent and cannot be solved. Therefore, a potential function does not exist for the given vector field.* ■

Given a vector field, is there a way of determining whether its potential function exists without going through the complete integration procedure? For two-dimensional fields the following observation is useful.

For any twice differentiable function $\varphi(x, y)$ the result of differentiation is independent of the order in which the differentiation is done, namely,

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x}\,.$$

If in addition $\varphi(x, y)$ is a potential function of a vector field $\mathbf{v} = (v_1, v_2)$ then

$$\frac{\partial \varphi}{\partial x} = v_1 \text{ and } \frac{\partial \varphi}{\partial y} = v_2\,.$$

Therefore, the above equality involving the second mixed derivatives of the potential function can be written as

$$\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x}\,.$$

This is the *necessary condition for the existence of a potential* of a two-dimensional vector field. It is easy to check that in EXAMPLE 2.12

$$\frac{\partial v_1}{\partial y} = 2x = \frac{\partial v_2}{\partial x}$$

so that a potential function could be found. On the other hand, in Example 2.13

$$\frac{\partial v_1}{\partial y} = 2y \neq \frac{\partial v_2}{\partial x} = 2x$$

and the potential function does not exist.

Unfortunately, for three-dimensional vector fields the above simple test cannot be applied. An alternative more general testing procedure will be developed in Section 2.5.2.

Example **2.14.** *Find a potential function for the vector field*

$$\mathbf{F} = (6xy, 3x^2 - 5z, 2z - 5y)$$

if it is known that it exists.

Solution. *If φ is a potential function for $\mathbf{F}$ then*

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= 6xy\,, \\ \frac{\partial \varphi}{\partial y} &= 3x^2 - 5z\,, \\ \frac{\partial \varphi}{\partial z} &= 2z - 5y\,. \end{aligned}$$

Integrating the first equation with respect to x gives $\varphi = 3x^2y + g(y,z)$. Substitute this expression in the second equation and obtain

$$\frac{\partial \varphi}{\partial y} = 3x^2 + \frac{\partial g}{\partial y} = 3x^2 - 5z\,.$$

Thus, $\dfrac{\partial g}{\partial y} = -5z$. Integrating this with respect to y leads to $g(y,z) = -5zy + h(z)$ and $\varphi = 3x^2y - 5zy + h(z)$. Substituting this into the third equation gives

$$\frac{\partial \varphi}{\partial z} = -5y + \frac{\mathrm{d}h}{\mathrm{d}z} = 2z - 5y$$

so that $\dfrac{\mathrm{d}h}{\mathrm{d}z} = 2z$. Integrating with respect to z gives $h(z) = z^2 + c$ and the final answer $\varphi = 3x^2y - 5zy + z^2 + c$.

Alternatively we can integrate the three partial differential equations above with respect to x, y and z, respectively, to obtain three different expressions for φ

$$\begin{aligned} \varphi &= 3x^2y + g(y,z)\,, \\ \varphi &= 3x^2y - 5zy + h(x,z)\,, \\ \varphi &= z^2 - 5zy + k(x,y)\,. \end{aligned}$$

All three expressions for φ must be identical. Therefore, comparing the first two expressions we obtain

$$3x^2y + g(y,z) = 3x^2y - 5zy + h(x,z)\,, \text{ or } g(y,z) = -5zy + h(x,z)\,.$$

Since the left-hand side of the last equation does not depend on x, we must conclude that $h(x,z) = h(z)$ so that $\varphi = 3x^2y - 5zy + h(z)$. Comparing this result with the third expression for φ leads to

$$3x^2y - 5zy + h(z) = z^2 - 5zy + k(x,y)\,, \text{ or } 3x^2y + h(z) = z^2 + k(x,y)\,.$$

This equation is consistent if we choose $h(z) = z^2 + c$ and $k(x,y) = 3x^2 + c$ (we can always add a constant because it can be considered as a function of any variable). Finally, we recover the same final result as before $\varphi = 3x^2y - 5zy + z^2 + c$. ■

2.2.8 Velocity potential

An important application of potential functions is in describing fluid velocity fields. For example, we can easily check that the fluid velocity field $\mathbf{v} = (-x, y)$ has the potential $\varphi = -\frac{1}{2}x^2 + \frac{1}{2}y^2$ since

$$\mathbf{v} = (-x, y) = \boldsymbol{\nabla}\varphi = \boldsymbol{\nabla}\left(-\frac{1}{2}x^2 + \frac{1}{2}y^2\right)\,.$$

The contour lines of this potential function and the corresponding velocity vector field are shown in Figure 2.7. Note the following features of this diagram that are valid in general:

- the velocity field is the gradient of φ and, thus, is normal to the contours of this velocity potential at each point;
- the speed of the fluid (the arrow length) is proportional to the rate of change of the potential (the slope of the surface representing φ) and, thus, inversely proportional to the spacing between the contours;
- the velocity vectors point to higher values of φ;
- the velocity vectors point towards a saddle of a potential function (located at the intersection of two contour lines) along one direction and away from it along the other.

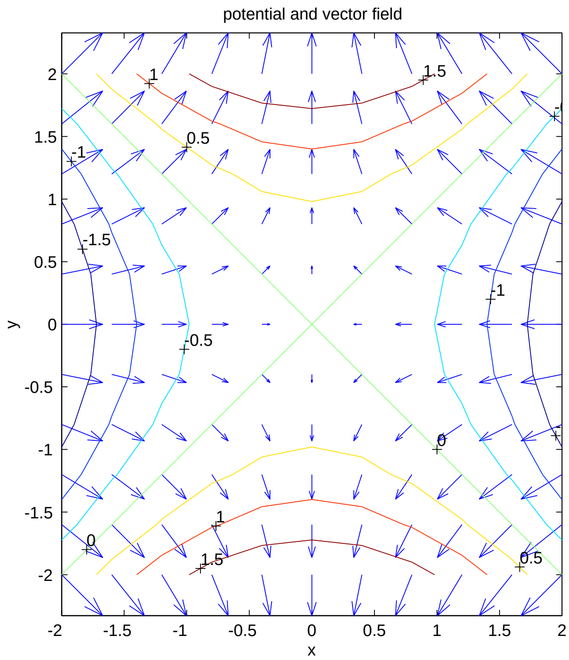
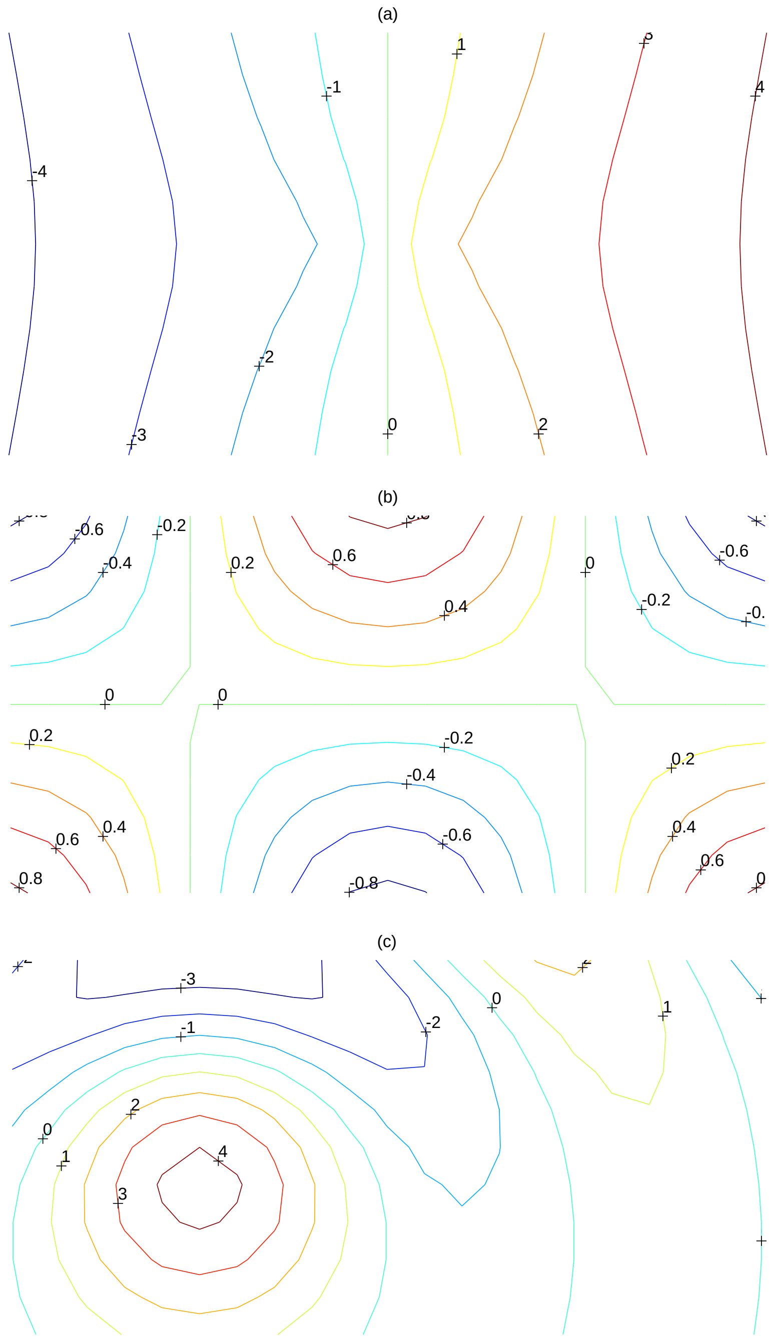


Figure 2.7: Corner flow.

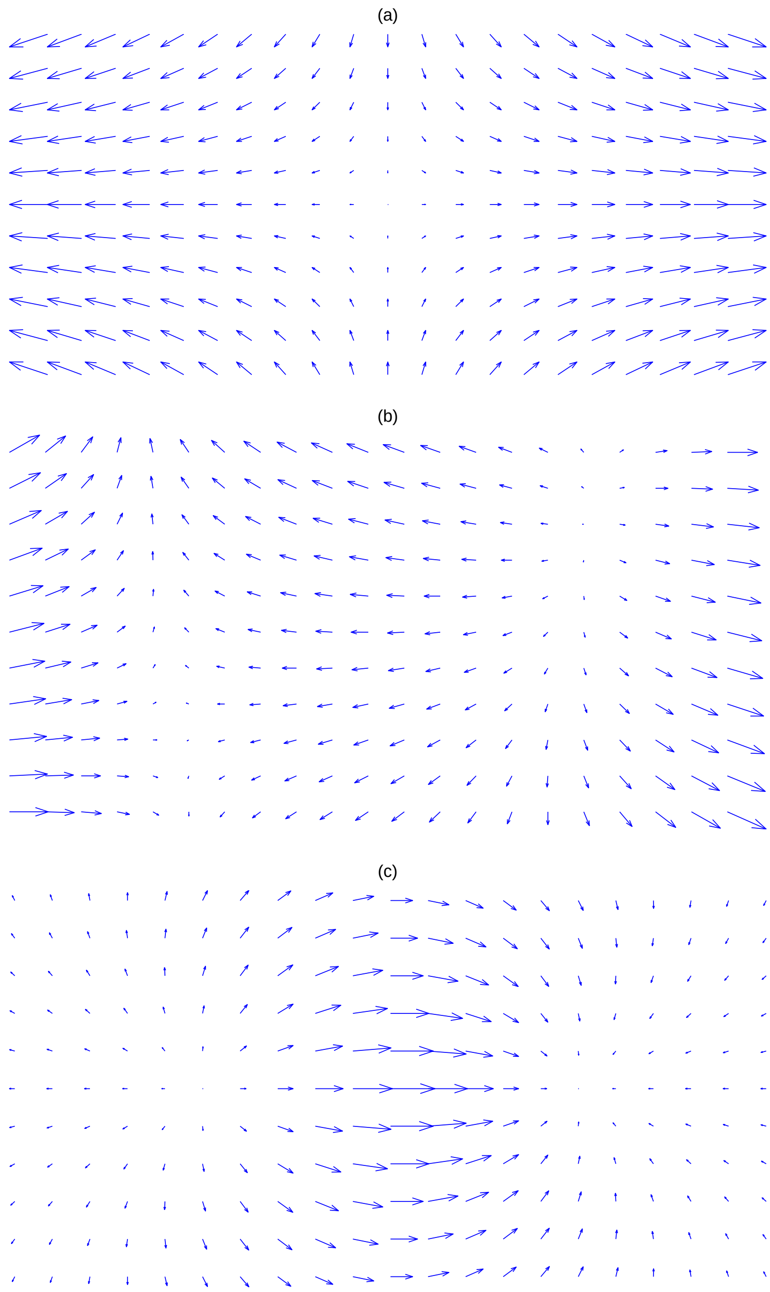
As a point of interest note that the flow depicted in Figure 2.7 is known as a *corner flow*. The name is derived from the fact that the velocity field is such that the fluid particles initially located in one of the quadrants will never penetrate the other quadrants. Therefore, the flow would not change if the $x = 0$ and $y = 0$ lines were replaced with solid walls forming four corners with a common vertex at the origin. A flow of this kind also arises when two water jets of equal strength collide at the origin.

Exercises

Ex. 2.1. For each of the scalar functions with level curves plotted below sketch the vector field of its gradient. Endeavour to get the directions and relative magnitudes of vectors correctly.



Ex. 2.2. Given that the following vector fields are the gradient of a potential function, sketch level curves of the potential and indicate where the potential has a local maximum, minimum or a saddle point.



2.3 The Divergence of a Vector Field

When the operator nabla is used in a dot product with a vector field the result is called *divergence*.

Definition 2.10 *The* divergence *of a vector field* $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = (u, v, w)$ *is defined by*

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(u, v, w) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\,.$$

The divergence operator applies to a vector field and produces a scalar field.

EXAMPLE **2.15.** *Find the divergence of* $\mathbf{v} = (x^2y, x^2 + yz, xyz)$.

SOLUTION.

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(x^2 + yz) + \frac{\partial}{\partial z}(xyz) = 2xy + z + xy = 3xy + z\,.$$

■

EXAMPLE **2.16.** *Find the divergence of* $\mathbf{v} = (3x^2, y^2, 2z^3)$.

SOLUTION.

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}(2z^3) = 6x + 2y + 6z^2\,.$$

■

The divergence of a vector field is a measure of how much a vector field points away from some point P. To see this refer to Figure 2.8. What do we mean by a vector field "pointing away"? It means the average around neighbouring points of the component of $\mathbf{v}$ directed away from the central point P. Here:

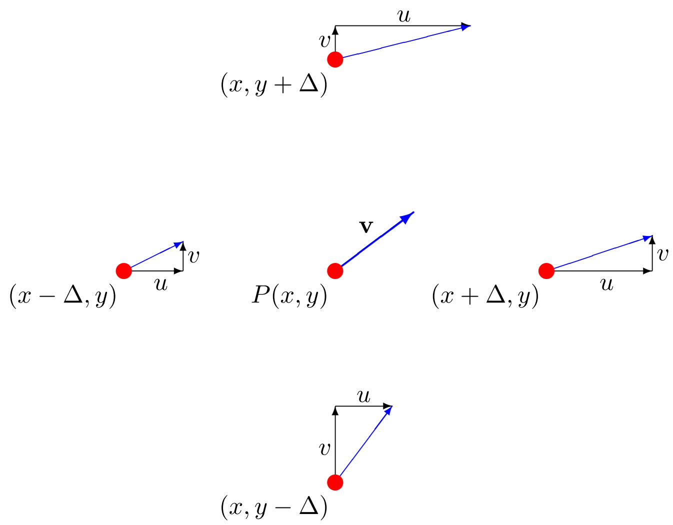


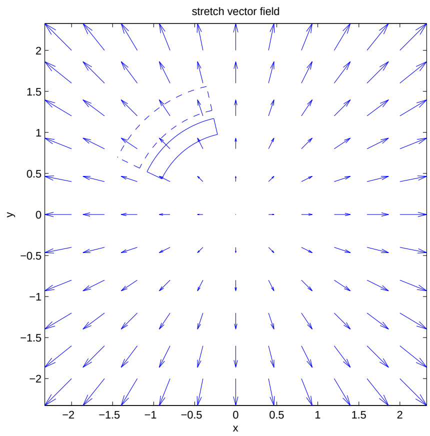
Figure 2.8: The meaning of divergence.

- to the right of P at $(x+\Delta, y)$ the component is $u(x+\Delta, y)$ as the v component is directed neither towards nor away from P;
- above P at $(x, y+\Delta)$ the component is $v(x, y+\Delta)$ as here it is the u component that is neither towards nor away from P;
- to the left of P at $(x-\Delta, y)$ the component is $-u(x-\Delta, y)$ (minus because positive u points towards P);
- below P at $(x, y-\Delta)$ the component is $-v(x, y-\Delta)$ (minus because positive v points towards P).

When the distance Δ between the neighbouring points tends to zero the average of these "pointing away" components is

$$
\begin{aligned}
& \frac{1}{4}\left[u(x+\Delta, y)+v(x, y+\Delta)-u(x-\Delta, y)-v(x, y-\Delta)\right] \\
= \; & \frac{\Delta}{2}\left[\frac{u(x+\Delta, y)-u(x-\Delta x, y)}{2\Delta}+\frac{v(x, y+\Delta)-v(x, y-\Delta)}{2\Delta}\right] \\
\approx \; & \frac{\Delta}{2}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right] \quad \text{by a limit definition of the derivatives} \\
= \; & \frac{\Delta}{2}(\boldsymbol{\nabla}\cdot\mathbf{v}) \quad \text{by Cartesian formula for divergence.}
\end{aligned}
$$

This demonstrates that the divergence of a vector field at any point is proportional to the average of how much the vector field "points away" from the point.

(a) 

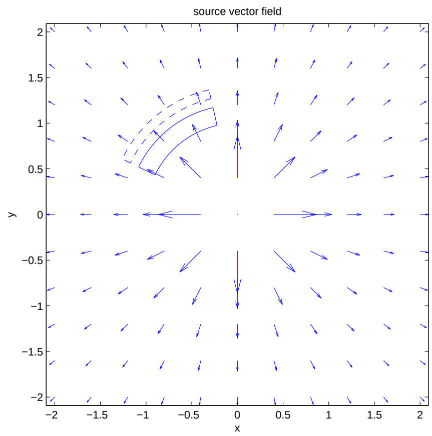
(b) 

Figure 2.9: Examples of (a) non-zero and (b) zero divergence fields.

2.3.1 Divergence in fluids

In the fluids context a non-zero divergence of a velocity field at a point measures the rate, per unit volume, at which the fluid is flowing away ¡from that point:

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \lim_{\Delta V \to 0} \frac{\text{flow out of } \Delta V}{\Delta V}\,,$$

where ΔV is a volume enclosing the point. For example, contrast the vector fields $\mathbf{v} = (x, y)$ and $\mathbf{v} = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right)$ shown in Figure 2.9 (a) and (b), respectively. In the first case the divergence is $\boldsymbol{\nabla}\cdot\mathbf{v} = 2$ everywhere, while in the second the divergence is $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ except at the singularity at the origin. The vector field shown in Figure 2.9 (a) corresponds to a uniform expansion field such as that of a rubber sheet that is being stretched. Examine any point P and the neighbouring arrows: some point towards P, but the arrows pointing away from P are bigger. Thus, there is a net "pointing away" and the divergence is everywhere positive. The vector field shown in Figure 2.9 (b) corresponds to the sort of flow obtained when you turn on a tap and pour water down onto a flat surface: immediately under the tap there is a source of water (the singularity), but everywhere else it just spreads with a velocity that slows further away from the source. Even though the calculations show that the divergence is zero, being a borderline case, it is hard to establish this for sure by visual inspection.

Definition 2.11 *If the divergence of a vector field is zero everywhere then this vector field is called* solenoidal.

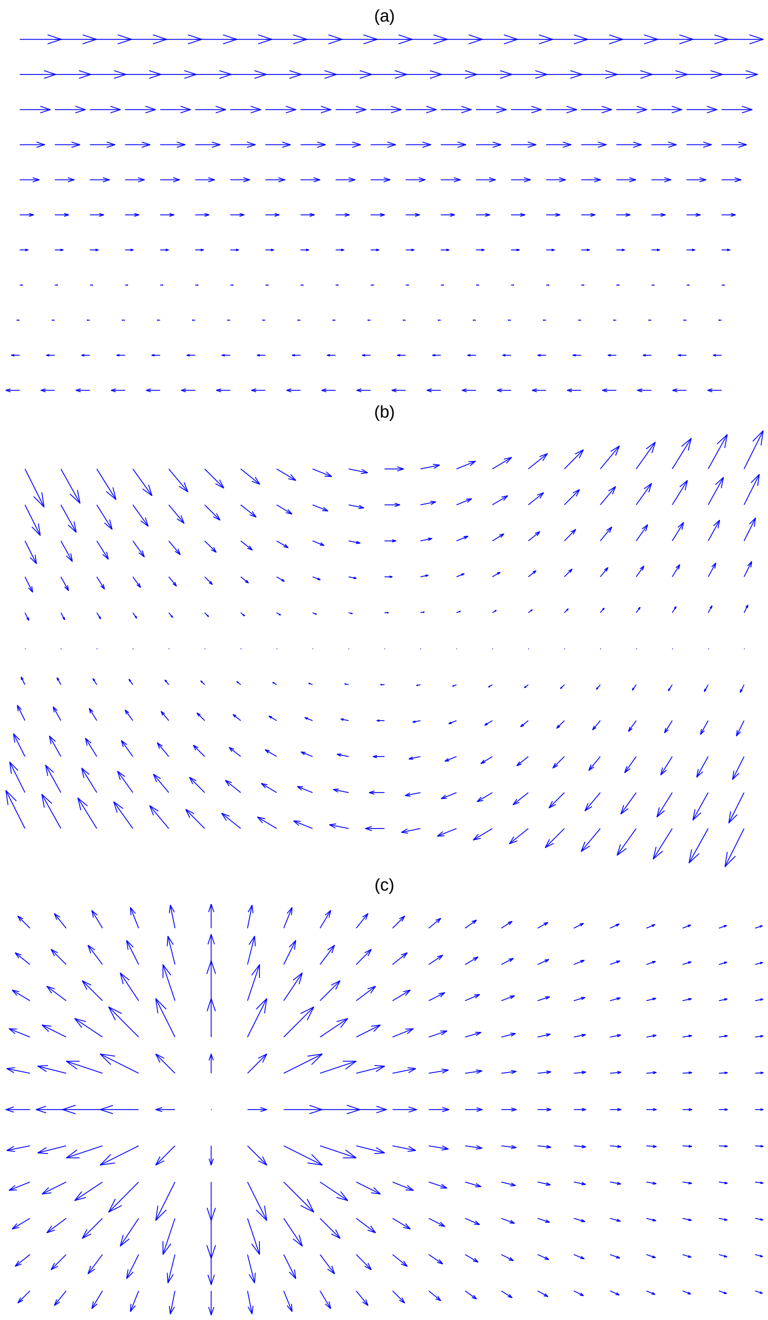
Fluids that can only have solenoidal velocity fields are called *incompressible*. The amount of such fluid entering any region on one side is exactly equal to its amount exiting the region on the other side because such a fluid cannot expand or compress changing its total volume. For example, water is incompressible (very nearly) and so its velocity field is divergence free, that is it satisfies $\nabla\cdot\mathbf{v} = 0$. Air can expand or compress and so its velocity field may have some divergence. However, compressible effects in air only become significant for very fast flows—those which are a significant fraction of the speed of sound. Thus, in many situations of interest the flow of air is also effectively incompressible.

Another graphical way of interpreting divergence can be given through considering a small region with imaginary elastic boundaries which are oriented so that its two opposite sides are parallel to the local velocity vectors and two others are perpendicular to them, see Figure 2.9. In both cases considered above such regions are elements of circular rings with the centre at the origin. Since the boundaries of such elements are "elastic", the region will be deformed by the flowing fluid. In the first of the discussed plots the velocity increases with the distance from the origin so that the outer boundary moves faster and the region stretches in the radial direction. It also stretches circumferentially and, thus, its volume (area) increases. Therefore, the divergence is positive here.

The situation is different for the second plot. Although the region stretches in the circumferential direction, the boundary which is closer to the origin moves faster trying to "catch up" with the opposite side of the region and making it thinner. Thinning of the region in the radial direction and stretching it in the circumferential direction have opposite effects on the volume (area) of the region. Therefore, it is impossible to say from the graph alone which effect would prevail. Only algebraic calculations of the divergence can determine the sign of the divergence in this case or say that it is zero.

Exercises

Ex. 2.3. For the following vector fields, sketch regions where you think the divergence is positive, negative, and approximately zero. Consider a variety of points, and for any one point look at the net effect of vectors in its immediate neighbourhood.



2.4 The Laplacian of a Scalar Field

If ϕ is a scalar field then $\boldsymbol{\nabla}\phi$ is a vector field and its divergence is

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}}\,.$$

This is called the *Laplacian* of a scalar field.

> The Laplacian applies to a scalar field and produces another scalar field.

EXAMPLE **2.17.** *Find $\nabla^2\phi$ given $\phi = 2x^2y - 2y^2x + 4z^2$.*

SOLUTION. *Since* $\dfrac{\partial^2\phi}{\partial x^2} = 4y$, $\dfrac{\partial^2\phi}{\partial y^2} = -4x$ *and* $\dfrac{\partial^2\phi}{\partial z^2} = 8$

$$\nabla^2\phi = 4y - 4x + 8\,.$$

■

2.4.1 The Laplacian in fluids

The Laplacian is a very important operator that has found application in many fields of science and in particular in fluid mechanics. We have already seen that it is sometimes possible to represent a fluid's velocity field using a velocity potential φ: $\mathbf{v} = \boldsymbol{\nabla}\varphi$. If the fluid is incompressible then $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$. Combining these two results we obtain for a velocity potential in incompressible fluid that

$$\nabla^2\phi = 0\,.$$

This is known as *Laplace's equation*. This partial differential equation describes an enormous variety of fluid flows (but not all, see Section 2.5.2!) from ocean waves to flows around submerged bodies and airplanes and other applications where the effects of fluid viscosity can be neglected. This is a rather amazing fact that one simple equation can have such a variety of solutions and a reasonable question is: "What makes the solutions different?". The answer is boundary conditions that are chosen individually for each physical problem. Indeed Laplace's equation belongs to the class of *elliptic equations* that require boundary conditions to be specified at each point along the line or surface surrounding the domain of interest. One of the most important boundary conditions in fluid context is that the fluid cannot flow through a solid surface (this is sometimes called the *no-penetration condition*). Therefore, the velocity component perpendicular to a solid surface must vanish at the surface. We have seen that the velocity components u, v, w in the x, y and z directions are given by the respective partial derivatives of the potential function

$$\mathbf{v} = (u, v, w) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = \boldsymbol{\nabla}\phi\,.$$

However, in applications solid surfaces are not necessarily aligned with the coordinate axes. Therefore, we must be able to determine velocity components in arbitrary directions, and in particular, in the direction normal to a solid wall that is given by a unit vector $\hat{\mathbf{n}}$. This velocity component is given by projecting the velocity vector onto $\hat{\mathbf{n}}$. This is done using a scalar product

$$\mathbf{v}_{\hat{\mathbf{n}}} = \mathbf{v} \cdot \hat{\mathbf{n}} = \boldsymbol{\nabla}\phi \cdot \hat{\mathbf{n}} = D_{\hat{\mathbf{n}}}\phi\,,$$

which is a directional derivative of ϕ in the direction normal to the wall. Therefore, Laplace's equation for a velocity potential has to be supplemented with the boundary condition $D_{\hat{\mathbf{n}}}\phi = 0$ that must be satisfied at each point along the solid boundary. Since the shapes of boundaries vary, so do the solutions of the equation.

2.5 The Curl Operator

We now examine the use of the operator nabla in a vector product resulting in the curl operator.

Definition 2.12 *The* curl *operator* $\boldsymbol{\nabla}\times$ *acts on a vector field* $\mathbf{v} = (v_1, v_2, v_3)$ *and produces another vector field defined by*

$$\boxed{\boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)}\,.$$

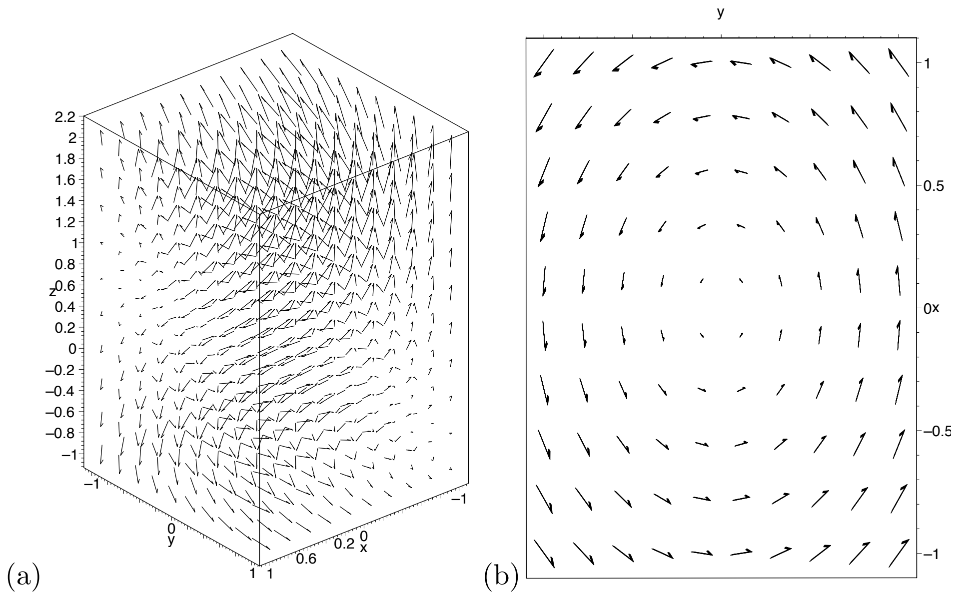


Figure 2.10: Plot of the vector field $\mathbf{v} = (-y, x, z)$ (a) view from a random point and (b) looking directly downwards in the negative z direction.

The curl applies to a vector field and produces another vector field.

The curl of a vector field at a point measures the rotation of this point. Its direction is the direction of the rotation axis. For example, the bird's eye views in Figure 2.10 of the vector field $\mathbf{v} = (-y, x, z)$ reveal that the rotation axis is parallel to the z-axis. We can check this as follows:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = 2\mathbf{k} = (0, 0, 2)\,.$$

Example **2.18.** *Calculate the curl of* $\mathbf{v} = (3x^2y, 5xyz^2, -7xy)$.

Solution. $\dfrac{\partial v_3}{\partial y} = -7x$, $\dfrac{\partial v_2}{\partial z} = 10xyz$, $\dfrac{\partial v_3}{\partial x} = -7y$, $\dfrac{\partial v_1}{\partial z} = 0$, $\dfrac{\partial v_2}{\partial x} = 5yz^2$, $\dfrac{\partial v_1}{\partial y} = 3x^2$ *so that* $\nabla \times \mathbf{v} = (-7x - 10xyz, 7y, 5yz^2 - 3x^2)$. ■

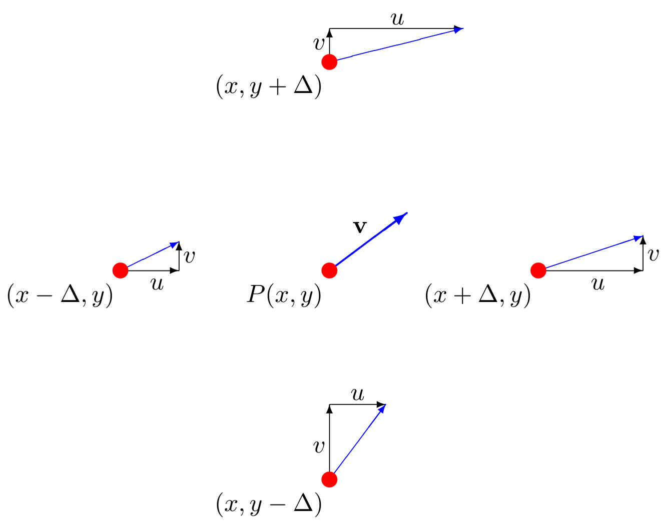


Figure 2.11: The meaning of curl.

If the vector field is two-dimensional $\mathbf{v} = (v_1(x, y), v_2(x, y), 0)$ then its curl is perpendicular to the plane of the field:

$$\boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & 0 \end{vmatrix} = \left(0, 0, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right).$$

2.5.1 Curl in fluids: vorticity

In fluid dynamics, the curl of the velocity field $\mathbf{v}$ is called the *vorticity*, usually denoted by $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$. Sometimes the vorticity is also called the *rotation*. This term gave the name to velocity fields with zero curl: they are called *irrotational fields*. We also call fluid flows for which the vorticity is zero everywhere *irrotational flows*.

Very loosely speaking, the vorticity—the curl of a vector field—at a point P measures the rate at which the vector field "rotates" locally about P. See this via the following argument similar to that used for the divergence. Choose any point $P(x, y)$ and again consider a velocity field $\mathbf{v} = (u, v)$ at four neighbouring points in two dimensions as shown in Figure 2.11. What do we mean by the rate at which a vector field "rotates" about a point P? It is the average over the neighbouring points of the component of $\mathbf{v}$ directed at right angles to the central point P and hence tending to twist the vector field about the pivotal point P. Here:

- to the right at $(x+\Delta, y)$ the anti-clockwise rotational component is $v(x+\Delta, y)$ as the u component is directed neither one way nor the other around P;
- below at $(x, y-\Delta)$ the anti-clockwise component is $u(x, y-\Delta)$ as here it is the v component that is pointing neither one way nor the other around P;
- to the left at $(x-\Delta, y)$ the component is $-v(x-\Delta, y)$ (minus because positive v points clockwise around P);
- above at $(x, y+\Delta)$ the component is $-u(x, y+\Delta)$ (minus because positive u points clockwise around P).

Note that these components all act at a distance Δ away from P, so that the average "rate" they "rotate" about P is given by the above components divided by the length of the rotation arm, namely Δ. Thus, the average rate of rotation caused by these components is

$$\begin{aligned}
&\frac{1}{4\Delta}\left[v(x+\Delta,y)-u(x,y+\Delta)-v(x-\Delta,y)+u(x,y-\Delta)\right]\\
=\ &\frac{1}{2}\left[\frac{v(x+\Delta,y)-v(x-\Delta,y)}{2\Delta}-\frac{u(x,y+\Delta)-u(x,y-\Delta)}{2\Delta}\right]\\
\approx\ &\frac{1}{2}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right] \quad \text{by definition of the derivative}\\
=\ &\frac{1}{2} \text{ of the } \mathbf{k} \text{ component of } \boldsymbol{\nabla}\times\mathbf{v}\,.
\end{aligned}$$

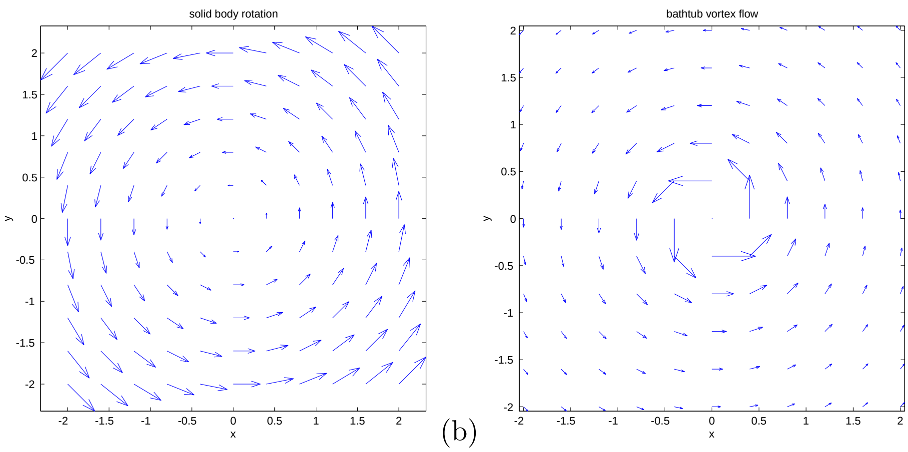
This demonstrates that the curl of a vector field at any point is proportional to the local rate of rotation about that point inherent in the vector field.

Two examples illustrate this. Contrast the velocity field of a *solid body rotation* $\mathbf{v}=(-y,x)$ shown in Figure 2.12 (a) for which

$$\boldsymbol{\nabla}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ -y & x & 0\end{vmatrix}=2\mathbf{k}=(0,0,2)\neq\mathbf{0}$$

with the bathtub vortex velocity field $\mathbf{v}=(-y,x)/(x^2+y^2)$ shown in Figure 2.12 (b) for which

$$\boldsymbol{\nabla}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0\end{vmatrix}=\left(\frac{2}{x^2+y^2}-\frac{2(x^2+y^2)}{(x^2+y^2)^2}\right)\mathbf{k}\,,$$



(a) (b)

Figure 2.12: Examples of (a) rotational (solid body rotation) and (b) irrotational (bathtub vortex) velocity fields.

which is zero everywhere except the origin where the velocity field becomes singular. The first velocity field could be established if you put a glass of water on a turntable and let it rotate for some time. The second is observed every time you pull a plug in the bath: the water drains spinning around the sink hole so that its motion accelerates closer to the drain. In both of these fields fluid particles travel in circles around the origin and it might be a bit of surprise that in the second example the fluid flow turns out to be irrotational.

To resolve this paradox imagine a matchstick floating some distance away from the center of the vortex so that it is initially perpendicular to the local velocity vectors. In both cases the center of the matchstick will be following a circular trajectory i.e. will rotate about the center of the vortex in anti-clockwise direction. However, in solid body rotation the orientation of the stick will change with time, namely, it will rotate in the anti-clockwise direction about its own centre. These two rotations (one about the centre of the vortex, one about the centre of the stick) add up to give the total nonzero curl (computed to be 2). The situation is quite different in the bathtub vortex case: while the rotation about the vortex center is still the same, the speed of the matchstick end that is further away from the vortex center is slower than that of the other end. As a result the matchstick will rotate about its center in the clockwise direction. Since these two rotations occur in the opposite directions, they cancel each other resulting in zero curl.

The somewhat counter-intuitive concept of "irrotational rotation" could be even better demonstrated in another example, a Ferris wheel. It rotates about its horizontal axis, yet people going for a ride do not fall out of their

Figure 2.13: Ferris wheel: irrotational rotation.

chairs. The chairs maintain the same vertical orientation because they rotate in the opposite direction about the axis at which they are attached to the wheel, so that people going in circles do not turn up side down: a perfect example of an irrotational revolving motion.

2.5.2 The existence of potential functions

We are now in a position to formulate the general condition that enables one to determine whether a potential function can be introduced.

Theorem 2.13 *A vector field has a potential function if and only if it is irrotational.*

It is easy to show by direct calculations that if the potential function ϕ exists then the vector field $\boldsymbol{\nabla}\phi$ is irrotational

$$\begin{aligned}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}, \frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}, \frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)=\mathbf{0}\,.\end{aligned}$$

The converse statement that if the vector field is irrotational then it has a potential function is also true, but its proof requires Stokes' theorem that will be introduced in Section 3.2.5.1.

In Section 2.2.7 we saw that the necessary condition for the existence of a potential of a two-dimensional vector field

$$\mathbf{v} = (v_1(x, y), v_2(x, y))$$

the condition

$$\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0$$

must be satisfied. We can see now that this is equivalent to the condition $\boldsymbol{\nabla} \times \mathbf{v} = 0$ written for a two-dimensional field.

Example **2.19.** *Determine whether $\mathbf{v} = (y+z, x+z, x+y)$ has a potential function and, if so, find it.*

Solution.

$$\boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z & x+z & x+y \end{vmatrix} = (1-1, -1+1, 1-1) = (0, 0, 0) = \mathbf{0}\,.$$

The field is irrotational, thus a potential function ϕ for $\mathbf{v}$ exists and $\boldsymbol{\nabla}\phi = \mathbf{v}$. Therefore,

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= y+z\,, \\ \frac{\partial \phi}{\partial y} &= x+z\,, \\ \frac{\partial \phi}{\partial z} &= x+y\,. \end{aligned}$$

By integrating the first equation we obtain $\phi = xy + xz + g(y, z)$. Substitute this solution ϕ into the second equation to get $x + \dfrac{\partial g(y,z)}{\partial y} = x + z$. Hence, $\dfrac{\partial g(y,z)}{\partial y} = z$, $g(y, z) = yz + h(z)$ and $\phi = xy + xz + yz + h(z)$. The substitution into the third equation then leads to $x + y + \dfrac{\mathrm{d}h}{\mathrm{d}z} = x + y$, or $\dfrac{\mathrm{d}h}{\mathrm{d}z} = 0$, with solution $h(z) = C$, where C is a true constant. Therefore,

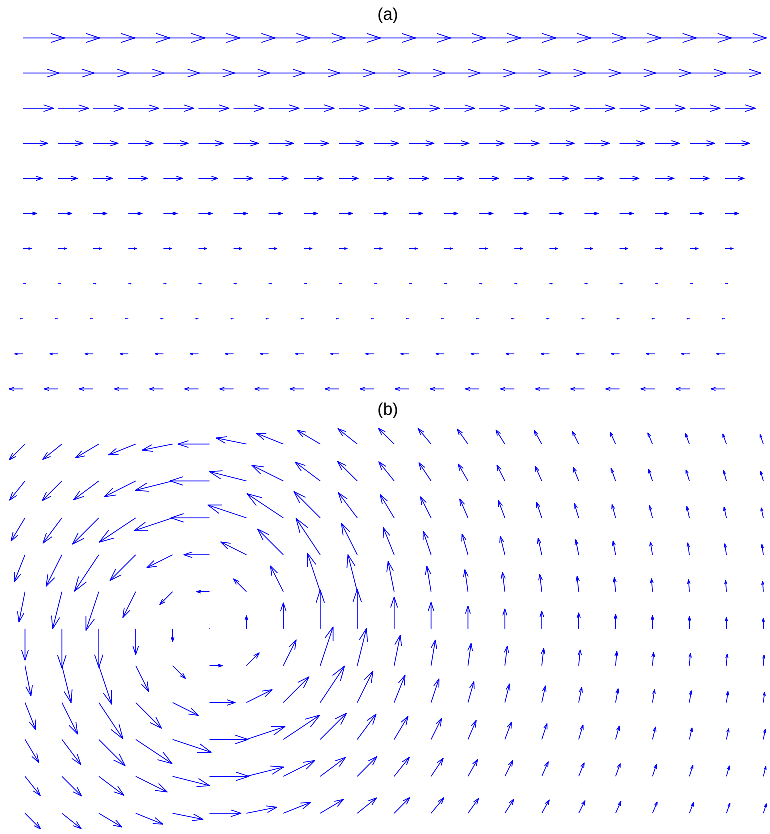
$$\phi = xy + yz + xz + C\,.$$

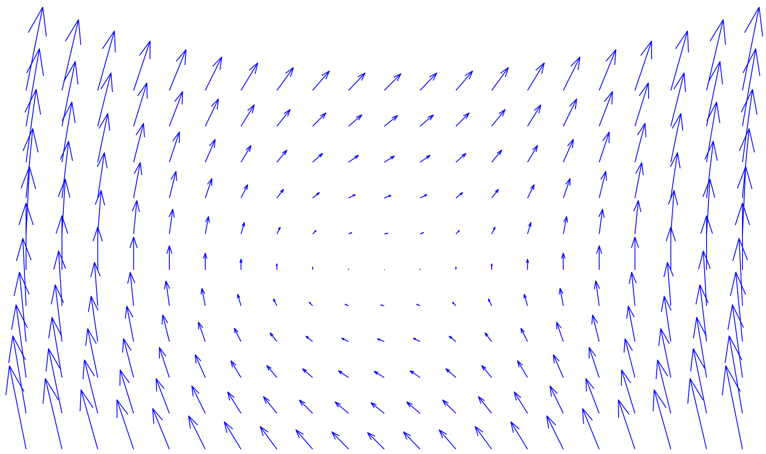
■

The actual value of a constant in the expression for a potential function is not important as potential functions are only used via their gradients.

Exercises

Ex. 2.4. For the following vector fields, indicate regions where you expect the curl to be positive (the local rotation is anti-clockwise), negative (the local rotation is clockwise) or approximately zero (no local rotation).





2.6 Vector Identities Involving Operator Nabla ∇

2.6.1 Curl of gradient

We have shown in Section 2.5.2 that

$$\boxed{\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \mathbf{0}}\,.$$

2.6.2 Gradient of divergence

We have seen in Section 2.4 that applying operator nabla to a scalar field twice leads to a new operator called the Laplacian. Namely, the Laplacian is "divergence of gradient". What happens if we apply these two operators in reverse order, namely, look for "gradient of divergence". We immediately note the asymmetry because the gradient would have to be applied to a scalar field while the divergence can act only on a vector field. This asymmetry means that the operators of gradient and divergence are not *commutative operators* and, unlike in simple multiplication of numbers or functions, swapping the order of operators changes the result completely.

Consider a three-dimensional Cartesian vector field

$$\mathbf{v} = (u(x,y,z), v(x,y,z), w(x,y,z))\,.$$

Its divergence is $\boldsymbol{\nabla}\cdot\mathbf{v} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}$ and

$$\boxed{\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) = \begin{bmatrix} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 v}{\partial x \partial y} + \dfrac{\partial^2 w}{\partial x \partial z} \\ \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 w}{\partial y \partial z} \\ \dfrac{\partial^2 u}{\partial x \partial z} + \dfrac{\partial^2 v}{\partial y \partial z} + \dfrac{\partial^2 w}{\partial z^2} \end{bmatrix}},$$

quite a complicated expression which is completely different from the Laplacian.

2.6.3 Divergence of curl

Use Cartesian definitions of the curl and divergence again to obtain

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{v} &= \boldsymbol{\nabla}\cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \boldsymbol{\nabla}\cdot\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ &= \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 v}{\partial z \partial x} - \frac{\partial^2 u}{\partial z \partial y} = 0\,. \end{aligned}$$

Thus,

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{v} = 0}\,.$$

2.6.4 Other combinations of two operators

We leave it as an exersise to show that

$$\boxed{\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \nabla^2\mathbf{v}},$$

where the last term represents the Laplacian applied to the individual components of vector $\mathbf{v}$:

$$\nabla^2\mathbf{v} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}, \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)\,.$$

It is easy to check that other combinations, namely, $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot$, $\boldsymbol{\nabla}\boldsymbol{\nabla}$, $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\cdot$, $\boldsymbol{\nabla}\boldsymbol{\nabla}\times$ are meaningless because the output of the first operator does not agree with the required input for the second. For example, $\boldsymbol{\nabla}\cdot\mathbf{v}$ produces a scalar field and the second application of $\boldsymbol{\nabla}\cdot$ becomes impossible because it requires a vector field as an input. Therefore, $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot\mathbf{v}$ is not a computable combination.

2.7 Hints to Selected Exercises

Ex. 2.1. Follow the procedure:

- the lines containing the gradient vectors must be orthogonal to the level curves;
- the gradient vectors must point in the direction of increasing contour values as the gradient shows the direction of the maximum increase of the function;
- the length of the gradient vectors must be inversely proportional to the distance between the contour lines because the steeper the slope, the larger the gradient, the closer together the contour lines are.

Ex. 2.2. Follow the procedure:

- draw the contour lines which are orthogonal to the gradient vectors everywhere;
- draw more contours in the regions with longer gradient vectors as the larger the gradient, the steeper the slope, the closer together the contour line are;
- the centre of closed contour lines (point where the gradient vector has zero length) is a maximum of a function if the gradient vectors point towards it because they show the direction of the function's increase, and is the minimum if the vectors point away from the centre;
- the contour lines intersecting at a point where the gradient vector has zero length indicate a saddle point.

Ex. 2.3. Follow the procedure:

- pick a point and draw a (curvilinear) box around it so that its two opposite sides are parallel to the vectors and two others are perpendicular to them;
- imagine that the vectors represent a fluid velocity and estimate the fluid flux in and out of the box (take into account the direction and magnitude of the velocity vectors and the length of box sides orthogonal to the vectors);
- if flux out is greater (smaller) than flux in the divergence is positive (negative) as it shows that the velocity field in average diverges from the central point; if the two fluxes are equal then the divergence is zero.

Ex. 2.4. Follow the procedure:

- Pick a point and imagine its trajectory if it follows the vectors should they represent the velocity of a flowing fluid. Does it look like a segment of a circular path about some centre? If so, note the direction of this rotation.
- Imagine a matchstick the centre of which is "nailed" at a chosen point. Note the velocities of its ends. Would such a matchstick rotate? If so, note the direction of this rotation.
- If both of the above rotations are clockwise then the curl at a chosen point is negative, if both of the above rotations are anti-clockwise then the curl at a chosen point is positive. If the two rotations occur in the opposite directions, try to see which one is faster and make a conclusion regarding the sign of the curl. In the latter case your judgement will be somewhat subjective and the safest way would be to state that the exact balance between the two rotations, and thus the sign of the curl, cannot be determined accurately unless the detailed quantitative information about the vector field is available.

2.8 Review Exercises

Gradient of scalar fields and directional derivatives

Ex. 2.5. Find the vector $\boldsymbol{\nabla}\phi$ for each of the following functions. In each case sketch some of the level curves $\phi =$ const. and show $\boldsymbol{\nabla}\phi$. Endeavour to show the direction and the relative magnitude of the vectors correctly.

(a) $\phi = x - 2y$, (b) $\phi = x^2 - y$, (c) $\phi = 2x - 5y$,
(d) $\phi = x^2 + y^2$, (e) $\phi = 4x^2 - y^2$, (f) $\phi = xy$,
(g) $\phi = 9x^2 + y^2$, (h) $\phi = \frac{x}{y}$, (i) $\phi = \frac{1}{xy}$.

Ex. 2.6. Find the directional derivative of f at point P in the direction of $\mathbf{u}$ and the maximum rate of change of f at P and specify its direction.
(a) $f = 2x^2 - xy$, $P(3,1)$, $\mathbf{u} = (1,1)$;
(b) $f = x^2 - y$, $P(1,-2)$, $\mathbf{u} = (2,1)$;
(c) $f = 4x + 3y$, $P(3,2)$, $\mathbf{u} = (1,2)$;
(d) $f = x^2 + y^2$, $P(1,2)$, $\mathbf{u} = (3,4)$;
(e) $f = e^y \sin x$, $P\left(\frac{\pi}{4}, 2\right)$, $\mathbf{u} = (2,2)$;
(f) $f = y \ln x$, $P(4,2)$, $\mathbf{u} = (1,-2)$.

Ex. 2.7. Find the maximum rate of change of ϕ at the point $(2,1,1)$ for each of the following functions (note that $r^2 = x^2 + y^2 + z^2$):

(a) $\phi = 2x + y - z$, (b) $\phi = x^2 - zy$, (c) $\phi = x^2y - 5yz^2$,
(d) $\phi = x^2 + y^2 + z^2$, (e) $\phi = \frac{1}{r}$, (f) $\phi = r^3$,
(g) $\phi = xyz$, (h) $\phi = \ln(x^2 + y^2 + z^2)$, (i) $\phi = e^{-r}$.

Ex. 2.8. For each of the following curves find and graph two normal vectors at the given point P.
(a) $y = 3x - 4$, $P(1,-1)$; (b) $3x^2 - 2y^2 = 1$, $P(1,1)$.

Ex. 2.9. For each of the following surfaces find a pair of unit normal vectors at the point given P.

(a) $2x + 3y - z = 1$, $P(x,y,z)$;
(b) $x^2 + y^2 - 4z^2 = 1$, $P\left(1,1,-\frac{1}{2}\right)$;
(c) $z = x^2 + y^2$, $P(2,2,8)$.

Ex. 2.10. Find the directional derivative of the function

$$f(x,y) = x^3 - 2xy + y^2$$

at the point $P(2,-3)$ in the direction of the vector $\mathbf{u} = (5,-12)$.

Ex. 2.11. Find the rate of change of the function

$$f(x,y) = x^3 - 2xy + y^2$$

at the point $P(2,1)$ in the direction of the vector $(4,-3)$.

Ex. 2.12. Find the maximum rate of change of the function

$$f(x,y,z) = x^3 - 2x^2y + zy^2$$

at the point $P(2,1,1)$ and also the rate of change of f in the direction of the vectors (a) $\mathbf{u} = (2,-1,2)$ and (b) $\mathbf{u} = (1,1,1)$.

Ex. 2.13. (a) Find the magnitude r of vector $\mathbf{r} = (x,y)$.

(b) Find $\dfrac{\partial r}{\partial x}$ and $\dfrac{\partial r}{\partial y}$.

(c) If $U = \beta \ln r$, find $\boldsymbol{\nabla} U$.

Ex. 2.14. (a) Find the magnitude of vector $\mathbf{r} = (x,y,z)$.

(b) Find $\dfrac{\partial r}{\partial x}$, $\dfrac{\partial r}{\partial y}$ and $\dfrac{\partial r}{\partial z}$.

(c) If $U = -\beta/r$, find $\boldsymbol{\nabla} U$.

Ex. 2.15. The sphere of radius 13 is centred at the origin. Find the equation of the tangent plane to this sphere at the point $P(3,4,12)$.

Potential functions

Ex. 2.16. For the following vector fields determine if they have a potential function and if so find it:

(a) $\mathbf{F} = e^{xy}(y,x)$,

(b) $\mathbf{F} = (3x^2y - y^2, x^3 - 2xy)$,

(c) $\mathbf{F} = (3x^2 + xy, -x^3 - y)$,

(d) $\mathbf{F} = (e^{-x} + y, x + e^y)$,

(e) $\mathbf{F} = (x^2 - y^2, y^2 - 2xy)$,

(f) $\mathbf{F} = (\sin(xy), \cos(xy))$,

(g) $\mathbf{F} = (y^3 - y, 3xy^2 - x)$,

(h) $\mathbf{F} = (y\cos(xy), x\cos(xy))$.

Divergence and curl

Ex. 2.17. For the following vector fields calculate $\boldsymbol{\nabla}\cdot\mathbf{v}$ and $\boldsymbol{\nabla}\times\mathbf{v}$:

(a) $\mathbf{v}=(x,2y,3z)$,

(b) $\mathbf{v}=(3x^2-y^2,x^2-2xy,3z)$,

(c) $\mathbf{v}=(3x^2,y^2,z^2)$,

(d) $\mathbf{v}=(xy,yz,xz)$,

(e) $\mathbf{v}=(yz,-zx,xy,$

(f) $\mathbf{v}=(e^{-x},xe^{-y},xy)$,

(g) $\mathbf{v}=(2xy+3x^2z,x^2-3z^2,x^3-6yz)$,

(h) $\mathbf{v}=(3x^2y+z^2,x^3+6y^2z,2xz+2y^3)$.

Laplacian

Ex. 2.18. Calculate the Laplacian $\nabla^2\phi$ for the following functions:

(a) $\phi=x^2+y^2+z^2$,

(b) $\phi=3x^2y-3xy^2+2z^2$,

(c) $\phi=2x^2yz+3zy^2-2xyz^2$,

(d) $\phi=e^x\sin y-ze^x\cos y$,

(e) $\phi=\dfrac{1}{x^2+y^2+z^2}$,

(f) $\phi=\exp(-r)$, $r=\sqrt{x^2+y^2+z^2}$,

(g) $\phi=x^2y^2z^3$,

(h) $\phi=\dfrac{1}{r}$, $r=\sqrt{x^2+y^2+z^2}$.

2.9 Answers to Selected Review Exercises

Gradient of scalar fields and directional derivatives

Recall that the gradient is the vector $\boldsymbol{\nabla}\phi = \left(\dfrac{\partial \phi}{\partial x}, \dfrac{\partial \phi}{\partial y}, \dfrac{\partial \phi}{\partial z}\right)$.

Ex. 2.5. Only a subset of the sketches is shown below. This should be enough for you to work out how the remaining sketches are drawn. Notice that each arrow is always perpendicular to the curve from which it originates and its length is inversely proportional to the distance between the contour lines.

(a) $\boldsymbol{\nabla}\phi = (1, -2)$, (b) $\boldsymbol{\nabla}\phi = (2x, -1)$, (c) $\boldsymbol{\nabla}\phi = (2, -5)$,
(d) $\boldsymbol{\nabla}\phi = (2x, 2y)$, (e) $\boldsymbol{\nabla}\phi = (8x, -2y)$, (f) $\boldsymbol{\nabla}\phi = (y, x)$,
(g) $\boldsymbol{\nabla}\phi = (18x, 2y)$, (h) $\boldsymbol{\nabla}\phi = \left(\dfrac{1}{y}, -\dfrac{x}{y^2}\right)$, (i) $\boldsymbol{\nabla}\phi = \left(-\dfrac{1}{x^2 y}, -\dfrac{1}{xy^2}\right)$.

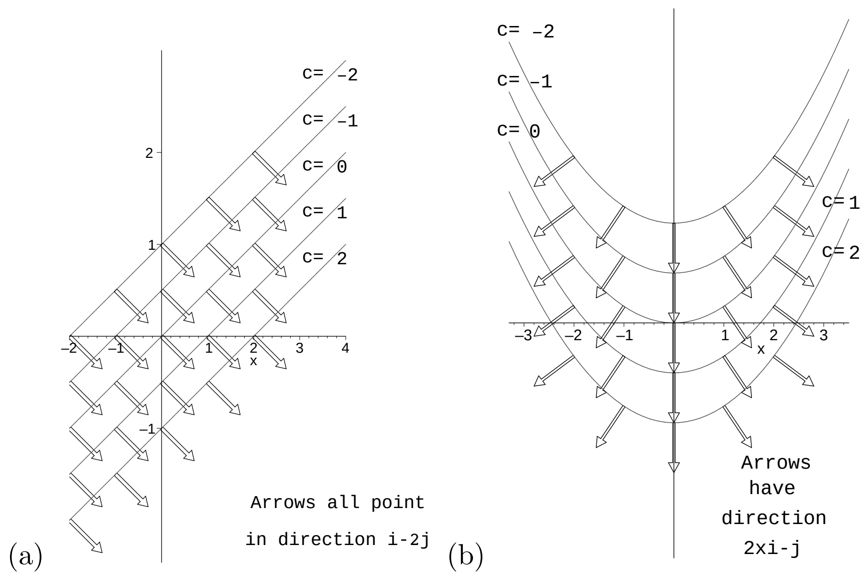


Figure 2.14: Contour plots of ϕ for Exercise 2.5 (a) and (b). Arrows show $\boldsymbol{\nabla}\phi$. In plot (b), the lengths of the arrows have been scaled to unity for clarity so that they represent $\dfrac{\boldsymbol{\nabla}\phi}{||\boldsymbol{\nabla}\phi||}$.

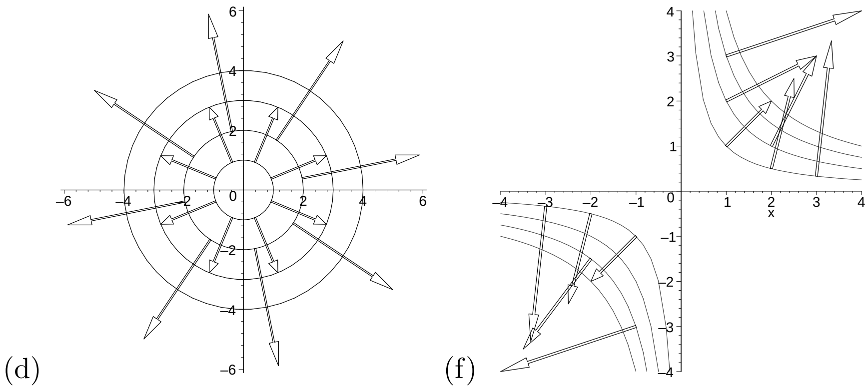


Figure 2.15: Contour plots of $\phi = c$ for Exercise 2.5 (d) (c =1, 4, 9, 16) and (f). In plot (d), the negative values of c are not admissible. Notice that the arrows have length equal to twice the radius of the circle on which they originate (which is $\sqrt{c}$). In plot (f), the curves are shown only for positive values of c. Negative values of c would result in the mirror images in the other two quadrants.

Ex. 2.6.

	$\boldsymbol{\nabla} f$	$\boldsymbol{\nabla} f(P)$	$D_{\mathbf{u}} f$
(a)	$(4x - y, -x)$	$(11, -3)$	$4\sqrt{2}$
(b)	$(2x, -1)$	$(2, -1)$	$\dfrac{3}{\sqrt{5}}$
(c)	$(4, 3)$	$(4, 3)$	$2\sqrt{5}$
(d)	$(2x, 2y)$	$(2, 4)$	$\dfrac{22}{5}$
(e)	$e^y(\cos x, \sin x)$	$e^2\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$	e^2
(f)	$\left(\dfrac{y}{x}, -\ln x\right)$	$\left(\dfrac{1}{2}, -\ln 4\right)$	$\dfrac{1}{\sqrt{5}}\left(\dfrac{1}{2} - 2\ln 4\right)$

The maximum rates of change are:

(a) $\sqrt{130}$ in the direction of $(11, -3)$,

(b) $\sqrt{5}$ in the direction of $(2, -1)$,

(c) 5 in the direction of $(4, 3)$,

(d) $2\sqrt{5}$ in the direction of $(1, 2)$,

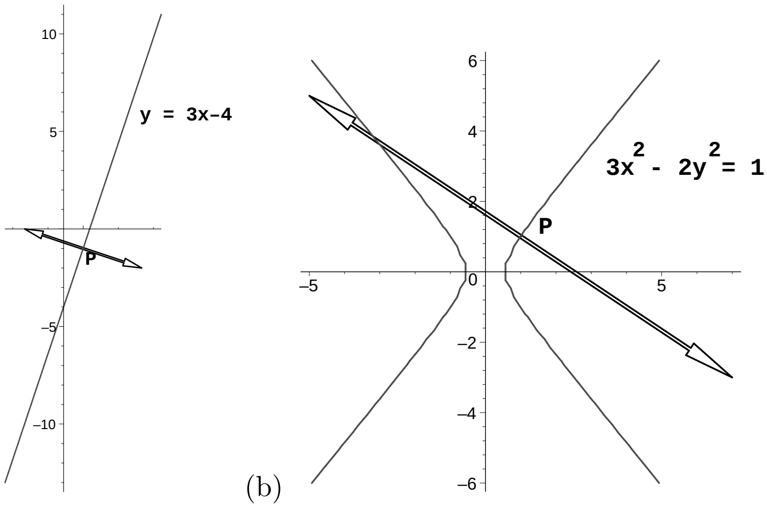
(e) e^2 in the direction of $(1, 1)$,

(f) $\sqrt{\dfrac{1}{4} + \ln^2 4}$ in the direction of $\left(\dfrac{1}{2}, -\ln 4\right)$.

Ex. 2.7.

	$\nabla\phi$	$\nabla\phi(2,1,1)$	$\left.\dfrac{d\phi}{ds}\right\vert_{\max}$
(a)	$(2,1,-1)$	$(2,1,-1)$	$\sqrt{6}$
(b)	$(2x,-z,-y)$,	$(4,-1,-1)$	$\sqrt{18}$
(c)	$(2xy, x^2-5z^2, -10yz)$	$(4,-1,-10)$	$\sqrt{117}$
(d)	$(2x,2y,2z)$	$(4,2,2)$	$\sqrt{24}$
(e)	$-\dfrac{\hat{\mathbf{r}}}{r^2}$,	$-\dfrac{1}{6\sqrt{6}}(2,1,1)$	$\dfrac{1}{6}$
(f)	$3r^2\hat{\mathbf{r}}$,	$3\sqrt{6}(2,1,1)$	18
(g)	(yz,xz,xy)	$(1,2,2)$	3
(h)	$\dfrac{2}{r}\hat{\mathbf{r}}$,	$\dfrac{1}{3}(2,1,1)$	$\dfrac{\sqrt{6}}{3}$
(i)	$-e^{-r}\hat{\mathbf{r}}$,	$-\dfrac{e^{-\sqrt{6}}}{\sqrt{6}}(2,1,1)$	$e^{-\sqrt{6}}$

Ex. 2.8. (a) $\pm(3,-1)$; (b) $\pm(6,-4)$.



(a) (b)

Ex. 2.9. (a) $\pm\dfrac{1}{\sqrt{14}}(2,3,-1)$; (b) $\pm\dfrac{1}{2\sqrt{6}}(2,2,4)$; (c) $\pm\dfrac{1}{\sqrt{33}}(4,4,-1)$.

Ex. 2.10. $\nabla f = (3x^2-2y, 2y-2x)$, $\nabla f(2,-3) = (18,-10)$, $D_{\mathbf{u}}f = \dfrac{210}{13}$.

Ex. 2.11. $\nabla f = (3x^2-2y, 2y-2x)$, $\nabla f(2,1) = (10,-2)$, $D_{\mathbf{u}}f = \dfrac{46}{5}$.

Ex. 2.12. $\nabla f = (3x^2-4xy, 2yz-2x^2, y^2)$, $\nabla f(2,1,1) = (4,-6,1)$, $||\nabla f(2,1,1)|| = \sqrt{53}$. Therefore, (a) $D_{\mathbf{u}}f = \dfrac{16}{3}$ for $\mathbf{u} = (2,-1,2)$ and (b) $D_{\mathbf{u}}f = -\dfrac{1}{\sqrt{3}}$ for $\mathbf{u} = (1,1,1)$.

Ex. 2.13. (a) $\sqrt{x^2+y^2}$, (b) $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, (c) $\dfrac{\beta}{r^2}\mathbf{r}$.

Ex. 2.14. (a) $\sqrt{x^2+y^2+z^2}$; (b) $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$; (c) $\dfrac{\beta}{r^3}\mathbf{r}$.

Ex. 2.15. The sphere has equation $f(x,y,z) = x^2+y^2+z^2-169 = 0$ so that $\boldsymbol{\nabla} f = 2(x,y,z)$. Hence at the point $P(3,4,12)$ the normal to the sphere has the direction given by $(3,4,12)$. Then the tangent plane has equation $3x+4y+12z = c$. The point $(3,4,12)$ is in this plane and so we find c by substitution giving an answer of $3x+4y+12z = 169$.

Potential functions

Ex. 2.16. (a) For the potential function to exist the vector field must be irrotational that is $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0}$. Therefore confirm that this is the case first:

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \left(0,0,\frac{\partial}{\partial x}(xe^{xy}) - \frac{\partial}{\partial y}(ye^{xy})\right)\\ &= (0,0,e^{xy}+xye^{xy}-e^{xy}-xye^{xy}) = \mathbf{0}\,.\end{aligned}$$

Now set $\dfrac{\partial\phi}{\partial x} = ye^{xy}$ and $\dfrac{\partial\phi}{\partial y} = xe^{xy}$ and solve simultaneously for ϕ. To achieve this, we integrate the first equation with respect to x and obtain $\phi = e^{xy}+c(y)$. Next, substitute this result into the second equation to obtain $xe^{xy}+\dfrac{dc}{dy} = xe^{xy}$. This means that $c(y)$ must be constant. Therefore $\phi = e^{xy}+c$. The rest of the problems in this exercise are solved similarly.

(b) $\phi = x^3y - xy^2 + c$.

(c) $\boldsymbol{\nabla}\times\mathbf{F} = (0,0,-x(3x+1)) \neq \mathbf{0}$ so that no potential function exists.

(d) $\phi = e^y - e^{-x} + xy + c$.

(e) $\phi = \dfrac{1}{3}(x^3+y^3) - xy^2 + c$.

(f) $\boldsymbol{\nabla}\times\mathbf{F} = (0,0,-y\sin(xy)+x\cos(xy)) \neq \mathbf{0}$ so that no potential function exists.

(g) $\phi = xy^3 - xy + c$.

(h) $\phi = \sin(xy) + c$.

Divergence and curl

Ex. 2.17.

	$\nabla\cdot\mathbf{v}$	$\nabla\times\mathbf{v}$
(a)	6	$\mathbf{0}$
(b)	$4x+3$	$(0,0,2x)$
(c)	$6x+2y+2z$	$\mathbf{0}$
(d)	$y+z+x$	$(-y,-z,-x)$
(e)	0	$(2x,0,-2z)$
(f)	$-e^{-x}-xe^{-y}$	$(x,-y,e^{-y})$
(g)	$6xz-4y$	$\mathbf{0}$
(h)	$6xy+12yz+2x$	$\mathbf{0}$

Laplacian

Ex. 2.18. (a) 6;

(b) $6y-6x+4$;

(c) $-4xy+6z+4yz$;

(d) 0;

(e) Since $r^2=x^2+y^2+z^2$, $\phi=\frac{1}{r^2}$. According to Section 2.2.4

$$\nabla\phi=\frac{\mathrm{d}\phi}{\mathrm{d}r}\hat{\mathbf{r}}=-2\frac{\hat{\mathbf{r}}}{r^3}=-\frac{2}{r^4}(x,y,z)\,.$$

Then

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial}{\partial x}\left(-2\frac{x}{r^4}\right)=-\frac{2}{r^4}+8\frac{x^2}{r^6}\,.$$

Similarly,

$$\frac{\partial^2\phi}{\partial y^2}=-\frac{2}{r^4}+8\frac{y^2}{r^6}\ \text{and}\ \frac{\partial^2\phi}{\partial z^2}=-\frac{2}{r^4}+8\frac{z^2}{r^6}\,.$$

Adding these results together we obtain

$$\nabla^2\phi=-3\frac{2}{r^4}+8\frac{x^2+y^2+z^2}{r^6}=\frac{2}{r^4}\,.$$

(f) $\dfrac{\partial^2\phi}{\partial x^2}=e^{-r}\left(\frac{x^2}{r^3}+\frac{x^2}{r^2}-\frac{1}{r}\right)$ and similarly to the solution of the previous problem $\nabla^2\phi=e^{-r}\left(1-\dfrac{2}{r}\right)$;

(g) $2z^3(x^2+y^2)+6x^2y^2z$;

(h) $\dfrac{\partial^2\phi}{\partial x^2}=-\dfrac{1}{r^3}+\dfrac{3x^2}{r^5}$ and $\nabla^2\phi=0$.

Module 3

Vector Calculus: Vector Integration

Module contents

3.1 Line Integrals

The simplest example of a *line integral* is the integral in one variable, for example, $\int f(x)\,\mathrm{d}x$. This integral may be thought of as following a straight line which is the x-axis. However, in applications in three-dimensional space one might need to integrate along any curvilinear path in that space. Depending upon the application such integrals effectively sum some property along the path. For example, the length of a sagging telephone wire between two posts is the sum (hence an integral) of all the small nearly straight segments it could be "cut" into. There are two sorts of line integrals: integrals of scalar functions, and integrals of vector functions. They share many of the properties of the ordinary integral and have many uses.

3.1.1 Review of line integrals of scalar functions

Recall from your previous studies that a line integral of a scalar function along the curvilinear *integration path* C is defined as

$$\int_C f(\mathbf{r})\,\mathrm{d}s = \lim_{N\to\infty}\sum_{n=1}^{N} f(\mathbf{r}_n)\Delta s_n\,,$$

where $\mathbf{r}=(x,y,z)$ is the position vector connecting the origin with a point on curve C. Here the curve C is cut into N small segments of length Δs_n and $\mathbf{r}_n$ is a point of C in the nth segment. This definition implies equal parts of C have equal weight in the integral.

To obtain a practical formula for the computation of such a line integral we convert it into an ordinary integral. Let C be the set of points $\mathbf{r}(t)$ for $a\le t\le b$. The variable t is referred to as a parameter and the above description as *parameterisation* leading to the *parametric representation* of the integration path C. Cutting the t interval $[a,b]$ into N pieces of length Δt_n corresponds to cutting C into N pieces of length

$$\Delta s_n = \frac{\mathrm{d}s}{\mathrm{d}t}\Delta t_n = \left\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right\|\Delta t_n\,.$$

The first equality above is a consequence of the small increment formula you studied previously but to see where the second equality comes from consider the following arguments.

Vector $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}=\left(\dfrac{\mathrm{d}x}{\mathrm{d}t},\dfrac{dy}{dt}\right)$ has a slope of

$$\frac{\dfrac{\mathrm{d}y}{\mathrm{d}t}}{\dfrac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\mathrm{d}y}{\mathrm{d}x}\,,$$

which is the slope of the given curve. Therefore, vector $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ is tangent to the considered curve and $\left|\left|\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|\right| \Delta t$ is the distance traveled along its direction. In the limit of small Δt the line segment it parameterises is very close to a straight line and, thus, the distance traveled along the direction given by vector $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ becomes equal to that traveled along the curve itself.

Hence the expression or the sum over C becomes

$$\sum_{n=1}^{N} f(\mathbf{r}_n)\Delta s_n = \sum_{n=1}^{N} f(\mathbf{r}(t_n)) \left|\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|\right| \Delta t_n \approx \int_a^b f(\mathbf{r}(t)) \left|\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|\right| \mathrm{d}t\,.$$

In the limit of $N \to \infty$ (or $\Delta s_n \to 0$) we obtain the expression for evaluating a line integral of a scalar function

$$\boxed{\int_C f(\mathbf{r})\, ds = \int_a^b f(\mathbf{r}(t)) \left|\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|\right| \mathrm{d}t}\,. \tag{3.1}$$

EXAMPLE **3.1.** *Find the line integral of $f = (x^2 + y^2 + z^2)^2$ over a single circular helical coil of radius $R = 2$ in the (x, y) plane and step of $h = 6\pi$ in the z direction.*

SOLUTION. *First, we need to develop an appropriate parameterisation of the curvilinear integration path. Its projection onto the (x, y) plane is a circle*

$$x^2 + y^2 = R^2 = 4\,.$$

On the other hand, the well-known trigonometric identity is

$$\cos^2 t + \sin^2 t = 1\,.$$

If the latter equation is multiplied by 4, it can be identified with the former one by setting

$$x = 2\cos t\,, \quad y = 2\sin t\,.$$

One full revolution would be completed by letting t to vary between 0 and 2π. Over this range of t values the value of z has to increase by h, that is $z = h/(2\pi)t = 3t$, which completes parameterisation of the integration path:

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = (2\cos t, 2\sin t, 3t)\,, \quad 0 \le t \le 2\pi\,.$$

Next, we compute the individual "ingredients" of (3.1) for the given problem:

$$f = (x^2 + y^2 + z^2)^2 = \left(4 + 9t^2\right)^2\,.$$

and

$$\left|\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|\right| = ||(-2\sin t, 2\cos t, 3)|| = \sqrt{13}\,.$$

Thus, we deduce

$$\begin{aligned}\int_C f\,\mathrm{d}s &= \int_0^{2\pi} \left(4+9t^2\right)^2 \sqrt{13}\,\mathrm{d}t \\ &= \sqrt{13}\left[32\pi + 24(2\pi)^3 + \frac{81}{5}(2\pi)^5\right].\end{aligned}$$

■

EXAMPLE **3.2.** *How long is the arc* $\mathbf{r} = \left(t, t^2, \frac{2}{3}t^3\right)$ *connecting* $A(0,0,0)$ *to* $B\left(1,1,\frac{2}{3}\right)$*?*

SOLUTION. *Recall that the length of a curve* C *is given by* $\int_C 1\,\mathrm{d}s$*. Then since* $0 \le t \le 1$ *parameterises the curve from* A *to* B*,*

$$\begin{aligned}\textit{length} &= \int_C 1\,\mathrm{d}s \\ &= \int_0^1 \left|\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|\right| \mathrm{d}t \quad \textit{by evaluation formula} \\ &= \int_0^1 \sqrt{1+4t^2+4t^4}\,\mathrm{d}t \quad \textit{as } \mathbf{r}' = (1, 2t, 2t^2) \\ &= \int_0^1 (1+2t^2)\,\mathrm{d}t \quad \textit{upon simplifying} \\ &= \frac{5}{3}\,.\end{aligned}$$

■

3.1.2 Work integral

The concept of line integration can be generalised to vector functions. One of the most important applications of line integrals of vector functions is the *work integral*. Historically, this term was introduced in the context

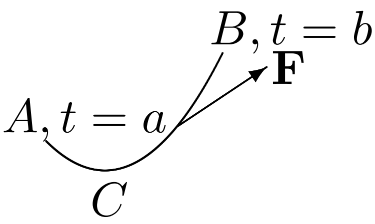


Figure 3.1: Integration path and vector field.

of mechanics. However, nowadays it is used to denote a certain type of integral of vector-valued functions regardless of whether they arise in an actual physical problem.

Suppose a force $\mathbf{F}$ acts on a particle as it moves along a curve C and the position of the particle at any time is given by vector $\mathbf{r}$. The elementary work done by this force over a small particle displacement $\Delta\mathbf{r}_i$ is defined as $\Delta W_i = ||\mathbf{F}||\,||\Delta\mathbf{r}_i||\cos\theta = \mathbf{F}\cdot\Delta\mathbf{r}_i$, where θ is the angle between the direction of particle displacement and the force[1]. Thus, the total work done by $\mathbf{F}$ along path C should be

$$\lim_{\Delta\mathbf{r}_i\to 0}\sum_i \mathbf{F}\cdot\Delta\mathbf{r}_i\,.$$

This limit is called the *work integral* which is a particular case of a line integral of vector field $\mathbf{F}$ along the path C. It is denoted by

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}\,.$$

Note that the dot product $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ returns a scalar function, therefore, computing the work integral is quite similar to that of a line integral of a scalar function reviewed in Section 3.1.1. Namely, since the shape of the particle trajectory can be quite complicated and the force acting on it can change from one point to another, we describe them parametrically. Note that at any time moment t the coordinates of a physical particle are $x(t)$, $y(t)$ and $z(t)$ so that the position vector describing the trajectory curve C (also called the integration path) and the force vector $\mathbf{F}$ at that particular point are

$$\mathbf{r}(t) = (x(t), y(t), z(t)) \quad \text{and} \quad \mathbf{F}(t) = \mathbf{F}(x(t), y(t), z(t))\,,$$

respectively. In other words, time t can always be chosen as a natural parameter in physical problems. After both $\mathbf{r}$ and $\mathbf{F}$ have been expressed as

[1]Recollect from elementary physics that a force that is always perpendicular to the direction of motion does no work.

functions of t we differentiate $\mathbf{r}$ to obtain

$$\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t$$

and, finally, obtain the expression for the work integral in parametric form

$$\boxed{W = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(t)\cdot\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\,\mathrm{d}t},$$

where $t = a$ and $t = b$ are the values of the parameter corresponding to the start and end points of path C, see Figure 3.1. Therefore, the path parameterisation reduces a line integral to an ordinary definite integral. Furthermore, because a vector field and a position vector in general have three components, $\mathbf{F}(t) = (F_1(t), F_2(t), F_3(t))$ and $\mathbf{r}(t) = (x(t), y(t), z(t))$, the expression for the derivative of a position vector becomes $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \left(\dfrac{\mathrm{d}x}{\mathrm{d}t}, \dfrac{\mathrm{d}y}{\mathrm{d}t}, \dfrac{\mathrm{d}z}{\mathrm{d}t}\right)$ and the expression for the integral becomes

$$\boxed{W = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_a^b \left(F_1\frac{\mathrm{d}x}{\mathrm{d}t} + F_2\frac{\mathrm{d}y}{\mathrm{d}t} + F_3\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t}.$$

EXAMPLE **3.3.** *Given $\mathbf{F} = (x^3, xy)$, calculate the work W along path C if*

(a) *C is the parabola $y = x^2$ joining $(0,0)$ and $(1,1)$.*

(b) *C is the line segment $y = x$ joining $(0,0)$ and $(1,1)$.*

SOLUTION. *In both cases, we can treat $x = t$ as the parameter.*

(a) *$y = x^2 = t^2$ so that*

$$\mathbf{F} = (t^3, tt^2) = (t^3, t^3), \quad \mathbf{r} = (t, t^2) \quad \textit{and} \quad \mathrm{d}\mathbf{r} = (1, 2t)\,\mathrm{d}t.$$

Thus,

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^1 (t^3\cdot 1 + t^3\cdot 2t)\,\mathrm{d}t = \frac{13}{20}.$$

(b) *$y = x = t$ so that*

$$\mathbf{F} = (t^3, tt) = (t^3, t^2), \quad \mathbf{r} = (t, t) \quad \textit{and} \quad \mathrm{d}\mathbf{r} = (1, 1)\,\mathrm{d}t.$$

Thus,

$$W = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^1 (t^3\cdot 1 + t^2\cdot 1)\,\mathrm{d}t = \frac{7}{12}.$$

■

Example 3.4. *A force $\mathbf{F} = (xy, y^2)$ acts on a particle moving along a path C which is the part of the parabola $y = x^2$ running from $(0, 0)$ to $(2, 4)$. Calculate the work $W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.*

Solution. *Choosing $x = t$ the path C can be parameterised as $\mathbf{r} = (x, y) = (t, t^2)$ and the expression for the force becomes $\mathbf{F} = (t\,t^2, (t^2)^2) = (t^3, t^4)$. Thus, $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (1, 2t)$ and*

$$W = \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_C (t^3, t^4)\cdot(1, 2t)\,\mathrm{d}t = \int_0^2 (t^3+2t^5)\,\mathrm{d}t = \left.\frac{t^4}{4} + \frac{t^6}{3}\right|_0^2 = \frac{76}{3}\,.$$

■

Example 3.5. *Given the vector field $\mathbf{v} = (6x^2 - 2y, x + z, 12yz)$ calculate the line integral $\int_C \mathbf{v} \cdot \mathrm{d}\mathbf{r}$, where C is*

(a) *the straight line running from the origin $(0, 0, 0)$ to point $(1, 1, 1)$;*

(b) *the part of the twisted cubic running from $(0, 0, 0)$ to $(1, 1, 1)$ and given parametrically as $(x, y, z) = (t, t^2, t^3)$, $0 \le t \le 1$.*

Solution.

(a) *The path C can be described parametrically as $x = t$, $y = t$, $z = t$, $0 \le t \le 1$. Therefore, $\mathbf{r} = (t, t, t)$ and $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (1, 1, 1)$. Then we obtain for the line integral*

$$\begin{aligned}\int_C \mathbf{v}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 (6t^2 - 2t, t + t, 12t^2)\cdot(1, 1, 1)\,\mathrm{d}t \\ &= \int_0^1 (6t^2 - 2t + 2t + 12t^2)\,\mathrm{d}t = 6t^3\big|_0^1 = 6\,.\end{aligned}$$

(b) *In this case $\mathbf{r} = (t, t^2, t^3)$ and $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (1, 2t, 3t^2)$ so that the line integral becomes*

$$\begin{aligned}\int_C \mathbf{v}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 (6t^2 - 2t^2, t + t^3, 12t^2t^3)\cdot(1, 2t, 3t^2)\,\mathrm{d}t \\ &= \int_0^1 (6t^2 - 2t^2 + 2t^2 + 2t^4 + 36t^7)\,\mathrm{d}t \\ &= \int_0^1 (6t^2 + 2t^4 + 36t^7)\,\mathrm{d}t = \left.2t^3 + \frac{2t^5}{5} + \frac{36t^8}{8}\right|_0^1 = \frac{69}{10}\,.\end{aligned}$$

■

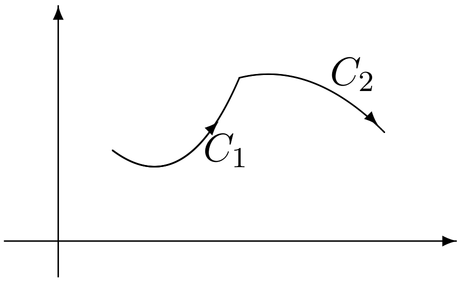


Figure 3.2: Combining two integration paths.

3.1.3 Combining paths

If the end point of path C_1 is the initial point of another path C_2, see Figure 3.2, we can combine them into a single path $C_1 + C_2$ and calculate the line integral along the combined path using

$$\int_{C_1+C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}\,.$$

The converse is also true: in order to compute a line integral one can break the integration path into connected segments, compute the integral for individual segments and add them together to obtain the final result.

Observe also that if C is a path specified by a function $\mathbf{r}(t)$, $a \le t \le b$, then we can obtain the opposite path $-C$ by defining a path $\mathbf{r}^*(t) = \mathbf{r}((a+b)-t)$, $a \le t \le b$. Specifically, we can evaluate the line integral $\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ by simply reversing the limits of integration. Thus, if

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \left(F_1 \frac{\mathrm{d}x}{\mathrm{d}t} + F_2 \frac{\mathrm{d}y}{\mathrm{d}t} + F_3 \frac{\mathrm{d}z}{\mathrm{d}t} \right) \mathrm{d}t\,,$$

then

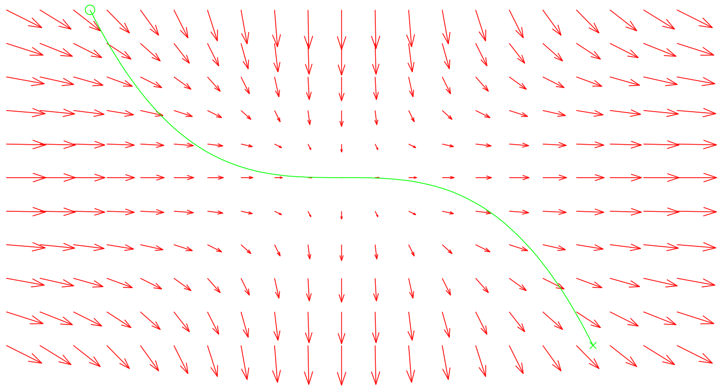
$$\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_b^a \left(F_1 \frac{\mathrm{d}x}{\mathrm{d}t} + F_2 \frac{\mathrm{d}y}{\mathrm{d}t} + F_3 \frac{\mathrm{d}z}{\mathrm{d}t} \right) \mathrm{d}t\,.$$

That is

$$\boxed{\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}}\,.$$

Computations of work integrals can be quite involved algebraically. Therefore, it is always useful to have a qualitative idea of what the answer should be. The fact that the line integral can be broken into individual integrals along consecutive connected line segments suggests a simple test that enables one to determine the sign of the required integral before the detailed calculations are carried out.

EXAMPLE **3.6.** *In the following picture a vector field* $\mathbf{v}$ *is plotted.*

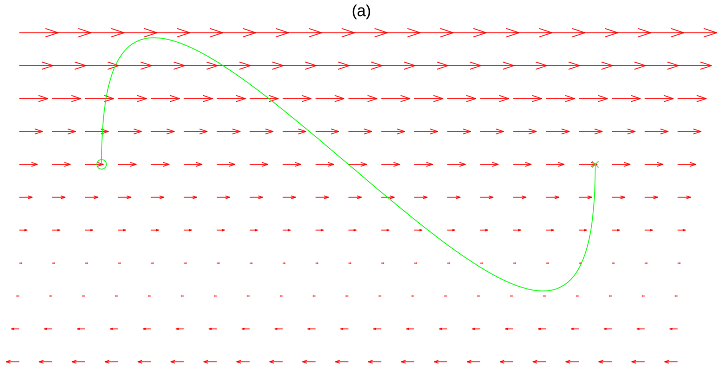


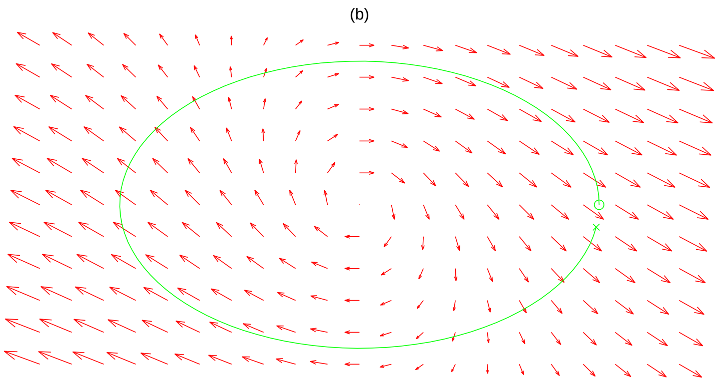
Superimposed is a path of integration C *going from* $\circ$ *to* $\times$*. Notice when following this path in the specified direction that the vector field, although at an angle to the curve, always points, more or less, in the direction of travel along* C*. Thus,* $\mathbf{v} \cdot \mathrm{d}r$ *is always positive (or at most zero when the middle segment of a path is considered, where the vector field is perpendicular to the curve). Therefore, we deduce that the work integral, which is in essence a sum of local values of* $\mathbf{v} \cdot \mathrm{d}r$*, is also positive,* $\int_C \mathbf{v} \cdot \mathrm{d}\mathbf{r} > 0$*. If the direction of integration was reversed, then* $\mathbf{v}$ *would be pointing in, more or less, opposite direction of travel and the work integral would be negative.*

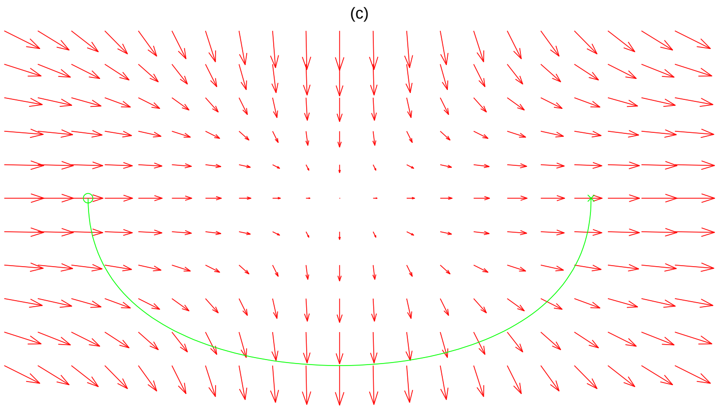
Exercises

EX. 3.1. For the following vector fields and the curves plotted on them, going from $\circ$ to $\times$, estimate whether the work integrals along the curve would be positive, negative, or approximately zero.

(a)







3.1.4 Circulation

Example 3.7. *A particle completes one full revolution in anti-clockwise direction around the origin along the circular path $x^2 + y^2 = R^2$ starting from $(x, y) = (R, 0)$ in a vector field $\mathbf{v} = (-y, x)$*[2]. *Calculate the line integral $\oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{r}$.*

Solution. *The path C can be described by the parametric equations $x = R\cos t$, $y = R\sin t$ with $0 \le t \le 2\pi$. Therefore, $\mathbf{r} = (R\cos t, R\sin t)$ and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (-R\sin t, R\cos t)$. Then the line integral is*

$$\begin{aligned}\oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} (-R\sin t, R\cos t) \cdot (-R\sin t, R\cos t)\,\mathrm{d}t \\ &= R^2 \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,\mathrm{d}t = \int_0^{2\pi} 1\,\mathrm{d}t = 2\pi R^2\,.\end{aligned}$$

■

[2]Recollect from Section 2.5.1 that this corresponds to the velocity field of a solid body rotation with $\boldsymbol{\nabla} \times \mathbf{v} = (0, 0, 2)$.

At a first glance the example above looks very similar to calculations of work integrals that have been considered in Section 3.1.2. The only difference is that the integration path C is closed, that is its start and end points coincide. To emphasise this the integral sign has a little circle in it. This is a standard notation for an integral along a closed path that is also referred to as a *closed contour*.

Definition 3.1 *A work integral of vector field* $\mathbf{v}$ *along a closed path* C *is called the* circulation *of the vector field* $\mathbf{v}$*. It is denoted by* $\Gamma = \oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{r}$*.*

To make things definite we also need to specify the direction along the closed path in which the circulation is computed. Conventionally, the positive integration direction is chosen to be anti-clockwise or, more precisely, when following the closed contour the enclosed region of interest should remain on your left.

EXAMPLE **3.8.** *Reconsider* EXAMPLE *3.7 with the integration path taken to be a circle of radius* R *centred at an arbitrary point* (a, b)*.*

SOLUTION. *A circle of radius* R *centred at the point* (a, b) *may be parameterised as* $\mathbf{r} = (x, y) = (a + R\cos t, b + R\sin t)$ *for* $0 \le t \le 2\pi$*. Then the velocity field evaluated on* C *is* $\mathbf{v} = (-(b+R\sin t), a+R\cos t)$*. Thus,*

$$\begin{aligned}
\Gamma &= \int_0^{2\pi} (-(b + R\sin t), (a + R\cos t)) \cdot (-R\sin t, R\cos t)\,\mathrm{d}t \\
&= \int_0^{2\pi} (bR\sin t + R^2\sin^2 t + aR\cos t + R^2\cos^2 t)\,\mathrm{d}t \\
&= \int_0^{2\pi} (bR\sin t + aR\cos t + R^2)\,\mathrm{d}t \\
&= 2\pi R^2\,.
\end{aligned}$$

This is the same result as in EXAMPLE *3.7: the circulation is just twice the area of the circle enclosed by the circular curve, independent of the centre of the circular path.* ■

The fact that the circulation values computed in EXAMPLE 3.7 and EXAMPLE 3.8 are identical is not a coincidence. Moreover, the factor of 2 that multiplies the area within the circle happens to be equal to $||\boldsymbol{\nabla}\times\mathbf{v}||$ computed for the given vector field in Section 2.5.1. Therefore, we are prompted to conclude that there should be a link between the values of curl and circulation.

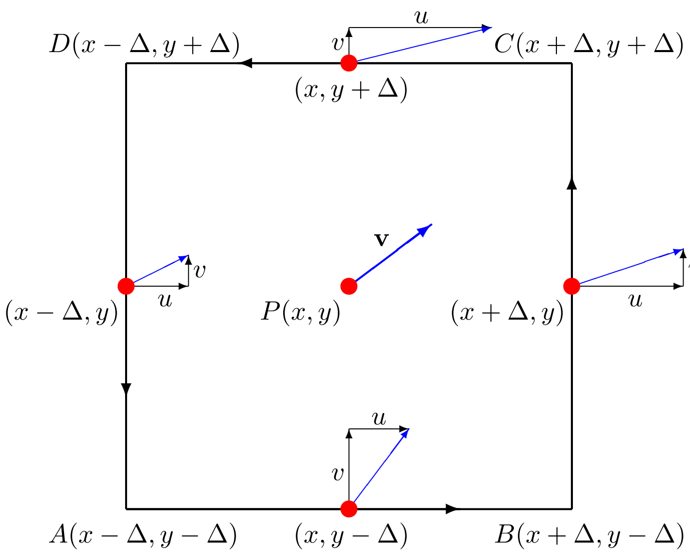


Figure 3.3: Circulation and curl.

To see this consider the circulation of a vector field $\mathbf{v} = (u(x, y), v(x, y))$ around a small rectangular element with vertices at points $A(x-\Delta, y-\Delta)$, $B(x+\Delta, y-\Delta)$, $C(x+\Delta, y+\Delta)$ and $D(x-\Delta, y+\Delta)$, see Figure 3.3. The line integral can be broken into four separate parts:

$$
\begin{aligned}
\int_{AB} \mathbf{v} \cdot \mathrm{d}\mathbf{r} &\approx 2u(x, y-\Delta)\Delta\,, \\
\int_{BC} \mathbf{v} \cdot \mathrm{d}\mathbf{r} &\approx 2v(x+\Delta, y)\Delta\,, \\
\int_{CD} \mathbf{v} \cdot \mathrm{d}\mathbf{r} &\approx -2u(x, y+\Delta)\Delta\,, \\
\int_{DA} \mathbf{v} \cdot \mathrm{d}\mathbf{r} &\approx -2v(x-\Delta, y)\Delta\,.
\end{aligned}
$$

Adding these contributions together we obtain for small Δ

$$
\begin{aligned}
\Gamma &= \oint_{ABCD} \mathbf{v} \cdot \mathrm{d}\mathbf{r} \\
&\approx 2\Delta(v(x+\Delta, y) - v(x-\Delta, y) - u(x, y+\Delta) + u(x, y-\Delta)) \\
&= 4\Delta^2 \left(\frac{v(x+\Delta, y) - v(x-\Delta, y)}{2\Delta} - \frac{u(x, y+\Delta) - u(x, y-\Delta)}{2\Delta} \right) \\
&\approx 4\Delta^2 \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = ||\boldsymbol{\nabla} \times \mathbf{v}|| S\,,
\end{aligned}
$$

where $S = 4\Delta^2$ is the area within the integration path $ABCD$. Thus, we reach an important conclusion:

The curl of a vector field can be considered as its circulation per unit area.

EXAMPLE **3.9.** *Find the circulation of the vector field* $\mathbf{v} = (x^2, -5xy)$ *around the triangular path starting at the origin and running along the* x *axis to the point* $(1, 0)$*, then along the line* $x = 1$ *to the point* $(1, 1)$ *and finally along the line* $y = x$ *back to the origin.*

SOLUTION. *The integration path can be broken into three segments:*

C_1*:* $\mathbf{r} = (x, 0)$ *(since* $y = 0$*),* $\mathrm{d}\mathbf{r} = (1, 0)\,\mathrm{d}x$*,* $0 \le x \le 1$*;*

C_2*:* $\mathbf{r} = (1, y)$ *(since* $x = 1$*),* $\mathrm{d}\mathbf{r} = (0, 1)\,\mathrm{d}y$*,* $0 \le y \le 1$*;*

C_3*:* $\mathbf{r} = (x, x)$ *(since* $y = x$*),* $\mathrm{d}\mathbf{r} = (1, 1)\,\mathrm{d}x$*,* $1 \ge x \ge 0$ *(since this segment is traveled in the direction of decreasing* x *towards the origin).*

Accordingly we obtain three separate line integrals:

$$\begin{aligned}
\int_{C_1} \mathbf{v} \cdot \mathrm{d}\mathbf{r} &= \int_0^1 (x^2, 0) \cdot (1, 0)\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}x = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}, \\
\int_{C_2} \mathbf{v} \cdot \mathrm{d}\mathbf{r} &= \int_0^1 (1, -5y) \cdot (0, 1)\,\mathrm{d}y = \int_0^1 -5y\,\mathrm{d}y = -\left.\frac{5y^2}{2}\right|_0^1 = -\frac{5}{2}, \\
\int_{C_3} \mathbf{v} \cdot \mathrm{d}\mathbf{r} &= \int_1^0 (x^2, -5x^2) \cdot (1, 1)\,\mathrm{d}x = -\int_1^0 4x^2\,\mathrm{d}x = -\left.\frac{4x^3}{3}\right|_1^0 = \frac{4}{3}.
\end{aligned}$$

Therefore,

$$\oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{v} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{v} \cdot \mathrm{d}\mathbf{r} + \int_{C_3} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \frac{1}{3} - \frac{5}{2} + \frac{4}{3} = -\frac{5}{6}.$$

■

3.1.5 Path independence

In EXAMPLE 3.9 we saw that the line integral depends on the integral path: specifically $C_1 + C_2$ have the same end points as C_3 while the line integrals of $\mathbf{v}$ along them are different. But there is a special case when the integral depends *only* on the initial and terminal points of the path.

Definition 3.2 *If the value of the work integral of a vector field* $\mathbf{v}$ *depends only on the initial and terminal points but not on the shape of the integration path, such an integral is called path independent and* $\mathbf{v}$ *is said to be a* conservative vector field.

Note that there is a clear link between the path independence of a work integral and the circulation of a vector field. To see that, refer to Figure 3.4. If a work integral is path independent, then for any two paths C_1 and C_2

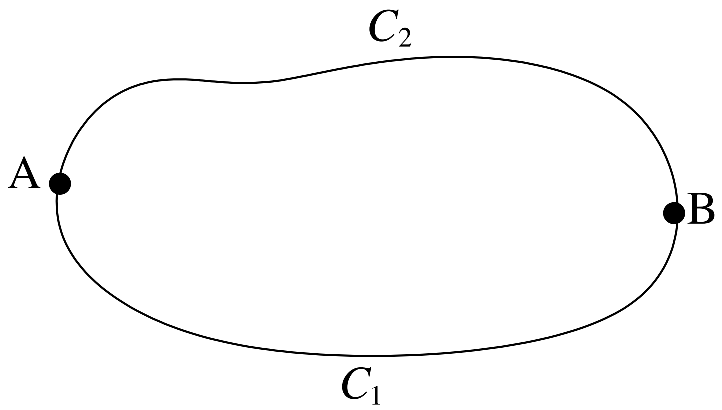


Figure 3.4: Path independence and circulation.

connecting point A and B

$$\int_{C_1} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{v} \cdot \mathrm{d}\mathbf{r}\,.$$

Therefore,

$$\int_{C_1} \mathbf{v} \cdot \mathrm{d}\mathbf{r} - \int_{C_2} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{v} \cdot \mathrm{d}\mathbf{r} + \int_{-C_2} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \oint_{C_1 - C_2} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = 0\,.$$

Conversely, if the circulation of a vector field is zero, then by reverting the above derivation we obtain that the work integrals along arbitrary paths starting and finishing at points A and B are necessarily equal to each other demonstrating the path independence. Physically this means that there are no energy sources or sinks in the system, the energy is conserved that is the energy used to move an object from point A to B is returned to the system when the object is returned to its initial position.

Further, in Section 3.1.4 we demonstrated that the circulation is proportional to the curl of a vector field. Therefore, if the curl of a field is zero, so should be its circulation. We will formalise this observation later in Section 3.2.5 but use it here to formulate the following theorem.

Theorem 3.3 *If* $\mathbf{v}$ *is an irrotational vector field, then it is also conservative if domain of definition of* $\mathbf{v}$ *is simply connected. Furthermore, if* ϕ *is a potential function for* $\mathbf{v}$ *and* C *is any path joining points* P *and* Q*, then*

$$\boxed{\int_C \mathbf{v}\cdot\,\mathrm{d}\mathbf{r} = \phi(Q) - \phi(P)}\,.$$

In the above the concept of a simply connected domain is introduced:

Definition 3.4 *A region is called a* simply connected domain *if any closed contour that belongs to this region can be continuously reduced to a point.*

Loosely speaking, this definition means that there are no singularities or islands in the domain and any elastic ring in it can shrink to a point without encountering any obstacles or singularities. The requirement of a region being simply connected is rather important. For example, it is easy to check by direct evaluation that the circulation is zero around any contour not containing the centre of a bathtub vortex shown in Figure 2.12 (b) but it is nonzero around any contour containing $(x, y) = (0, 0)$. Therefore, work integrals computed for such a field along paths that connect two points and are located on one side of the origin will be the same but they will differ from work integrals computed along curves connecting the same two points but passing on the other side of the origin.

It follows from the **Theorem** 3.3 that a work integral of an irrotational vector field along any closed path in a simply connected domain of definition is necessarily 0.

Example **3.10.** *Given* $\mathbf{v} = (y + z, x + z, x + y)$ *and the path* C *joining* $P(1, 2, 3)$ *and* $Q(2, -1, 0)$*, find* $\int_C \mathbf{v}\cdot\,\mathrm{d}\mathbf{r}$.

Solution. *We showed in* Example *2.19 that* $\boldsymbol{\nabla}\times\mathbf{v} = 0$ *is irrotational so that the potential function* $\phi(x, y, z) = xy+yz+xz+C$ *exists for this vector field. The vector field* $\mathbf{v}$ *does not have any singularities and, thus, according to the above theorem*

$$\int_C \mathbf{v}\cdot\,\mathrm{d}\mathbf{r} = \phi(2, -1, 0) - \phi(1, 2, 3) = (-2+0+0+C) - (2+6+3+C) = -13\,.$$

■

Example 3.10 illustrates one of the most important applications of potential functions: computing work integrals using only the values of the potential functions at the end points of the integration path. This approach requires checking whether the vector field is irrotational and finding the potential function before the integration can be completed, yet these additional steps are frequently easier to complete than to parameterise each specific integration path of interest (parameterisation is typically the most time consuming step in line integration). This example also shows that the knowledge of constant C in the expression for the potential function is not necessary as it cancels out in the integral computations. For this reason this constant is frequently set to 0 and we say that a potential function is defined up to an arbitrary additive constant.

3.1.6 Circulation in fluids

Circulation is intimately related to the *aerodynamic lift* of an aeroplane wing. Since the air flows faster above the wing, due to the *Venturi effect* the pressure above the wing is lower compared to that below it. Hence a net lifting force is generated.

Consider computing the circulation in the air flow around a wing when the wind is blowing from left to right (or, equivalently, when the wing moves to the left), see Figure 3.5. Take the path of integration to be anti-clockwise in the air adjacent to the aerofoil. Along the top part of the wing the air moves faster and the length of the integration path is longer due to the curvature of the wing's upper surface. Thus, there is a large negative contribution to the circulation. Along the bottom surface of the wing the air moves more slowly and the path length is shorter because the wing surface is flat there. Thus, there is a smaller positive contribution to the circulation there. Therefore, a net (negative in Figure 3.5) circulation exists. This non-zero circulation is a direct consequence of the differential air speed above and below the wing and is, consequently, proportional to the lift of the wing. Since we have shown in Section 3.1.4 that the circulation is proportional to curl of a velocity field, we conclude that if $\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}$ everywhere, that is if the velocity field is potential, then no lift can be created. Thus, the flow vorticity is what enables aeroplanes to fly.

The circulation can be computed for each individual cross-section of an aeroplane wing (aerofoil) and the results could be added together to compute the total lift. This process could be thought of as adding values along an imaginary line within the wing as if there was a vortex (like the bathtub vortex) within the wing. Indeed, the first good wing theory suggested by German engineer Ludwig Prandtl was based precisely upon this idea.

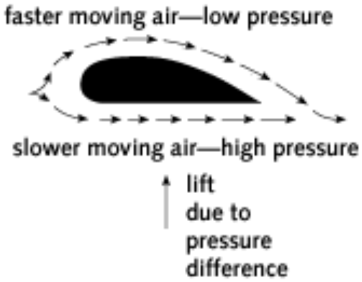


Figure 3.5: Circulation creates lift.

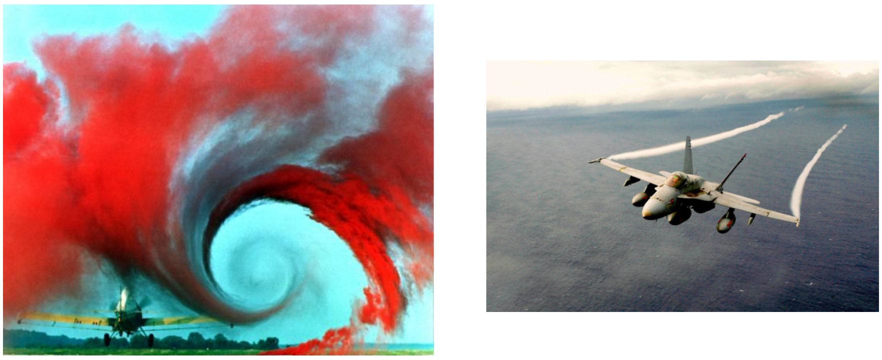


Figure 3.6: Trailing and wingtip vortices created by aeroplane wings.

Prandtl's lifting line theory replaces a wing by a vortex with the same circulation. This theory could show, for example, that drag is minimised with a tapered wing—an elliptical planform being the strict optimum.

This picture of the vorticity "bound" within a wing has some far reaching consequences. Recall the basic vector identity $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) = 0$ which indicates that the flow vorticity is divergence free. This means that the vortices cannot appear or disappear, they either form a closed loop, as in a smoke ring, or stretch to infinity. Thus, the bound vortex within a wing has to go somewhere when it exits from the wing tip! The bound vorticity turns at a right-angle at the wing-tip and trails back behind the wing forming what is known as trailing vortex, see the left image in Figure 3.6. This trailing wingtip vortex can often be seen when a plane flies in humid conditions, see the right image in Figure 3.6. The vorticity is so intense that the water vapour trapped inside a vortex is forced to condense and becomes visible. In principle, the trailing vortices stretch from the wing tips all the way back to the aeroport that the plane took off from; back at the aeroport the two trailing vortices rejoin to form a closed loop. Such vorticity decays very slowly through internal friction and through being disrupted by turbulence.

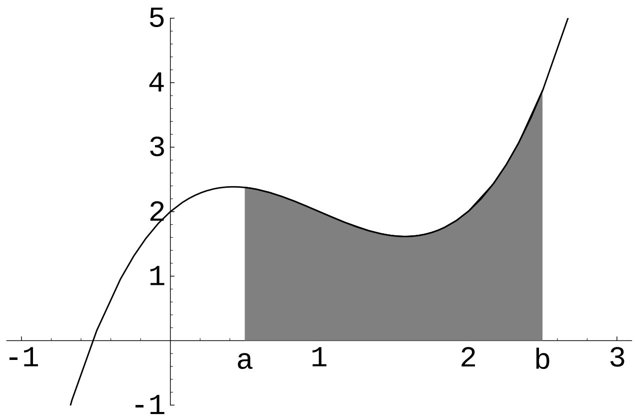


Figure 3.7: Integral is the area under the curve.

These immensely strong vortices at aeroports, strong enough to flip a small plane, place a severe constraint upon the minimum distance between planes that use a runway to take off and land.

3.2 Multiple Integrals and Integral Theorems

3.2.1 Repeated integrals

For a function of one variable $y = f(x)$ the integral $\int_a^b f(x)\,\mathrm{d}x$ is defined as the area under the graph of the curve $y = f(x)$ for x lying between a and b, see Figure 3.7. This area can be calculated by dividing the region under the graph into a large number of thin strips parallel to the y axis and adding them together. However, a more accurate method is to use the *fundamental theorem of calculus*. According to it, the area under the curve is given by

$$\int_a^b f(x)\,\mathrm{d}x = F(x)|_a^b = F(b) - F(a)\,,$$

where $F(x)$ is an anti-derivative of $f(x)$, that is $\dfrac{\mathrm{d}F}{\mathrm{d}x} = f(x)$.

If $z = f(x, y)$ is a function of two variables and R is a rectangular region in the (x, y) plane with $a \le x \le b$ and $c \le y \le d$, then if we fix x, we can form the integral

$$\int_c^d f(x, y)\,\mathrm{d}y = F(x)\,.$$

The resulting function $F(x)$ gives the area of the region below the surface of $z = f(x, y)$ and above the line $\{x = \text{const.},\ z = 0\}$. If we consider the region as being divided into a large number of parallel slices of thickness $\mathrm{d}x$, we obtain the expression for the volume $\mathrm{d}V$ of each of such slices: $\mathrm{d}V \approx F(x)\,\mathrm{d}x$ so that $\dfrac{\mathrm{d}V}{\mathrm{d}x} = F(x)$. Therefore, for the total volume V below the surface and above the rectangular region in the (x, y) plane we obtain

$$V = \int_a^b F(x)\,\mathrm{d}x = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

The integral of such a form is called a *repeated integral*.

Example **3.11.** *Find the volume below the surface of $z = 2x + 4y$ and above the rectangular region in the (x, y) plane bounded by $x = 1$, $x = 3$, $y = 2$ and $y = 6$.*

Solution. *The volume can be written as the repeated integral $V = \int_1^3 \int_2^6 (2x + 4y)\,\mathrm{d}y\,\mathrm{d}x$. It is evaluated by working from the inside out. The integral is written in a strict nested form: the inner differential $\mathrm{d}y$ corresponds to the inner integral $\int_2^6$ in y and the outer differential $\mathrm{d}x$ corresponds to the outer integral $\int_1^3$ in x. Therefore,*

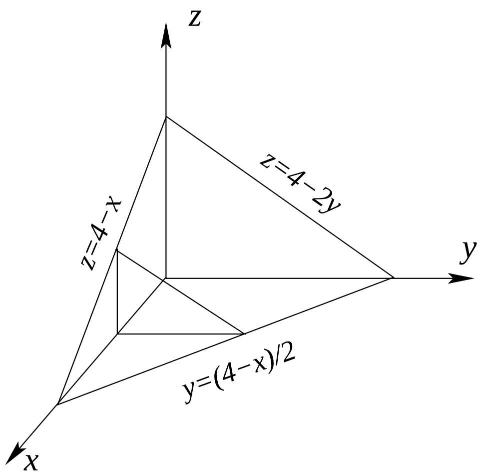
$$\begin{aligned}
V &= \int_1^3 (2xy + 2y^2)\big|_{y=2}^{y=6}\,\mathrm{d}x \\
&= \int_1^3 ((2x \cdot 6) + 2 \cdot 6^2) - (2x \cdot 2 + 2 \cdot 2^2))\,\mathrm{d}x \\
&= \int_1^3 (8x + 64)\,\mathrm{d}x = 4x^2 + 64x\big|_{x=1}^{x=3} = 160\,.
\end{aligned}$$

■

We now consider the case where the region in the plane is not rectangular.

Example **3.12.** *Consider the problem of finding the volume beneath the surface $z = 4 - (x + 2y)$ and above the plane $z = 0$ and bounded by the planes $x = 0$ and $y = 0$, see the diagram below.*

Solution. *First describe the edges of the given surface. To do that substitute the values $x = 0$, $y = 0$ or $z = 0$ into the equation of the surface, see the diagram below. Then to calculate the volume divide it into a large number of triangular slices which are parallel to the (y, z) plane so that x is constant on the slice. The area of the slice varies as x changes between 0 and 4, that is the area is a function of x: $A = A(x)$.*



To calculate it note that any point on a slice the height of the area is given by $z(x,y)$ while y changes between the left edge of the volume base that is from $y=0$ to the right edge given by $y=(4-x)/2$. Then

$$A(x)=\int_0^{(4-x)/2} z\,\mathrm{d}y=\int_0^{(4-x)/2}(4-(x+2y))\,\mathrm{d}y\,.$$

To find the volume we need to "add" all the slices together, which is done by integrating the slice areas in x between $x=0$ that corresponds to the rear face of the volume and $x=4$ that is the coordinate of the front vertex:

$$V=\int_0^4 A(x)\,\mathrm{d}x=\int_0^4\int_0^{(4-x)/2}(4-(x+2y))\,\mathrm{d}y\,\mathrm{d}x\,.$$

Thus, we obtained a double integral written as a repeated integral in y and x.

To be meaningful, repeated integrals must satisfy the following general rules:

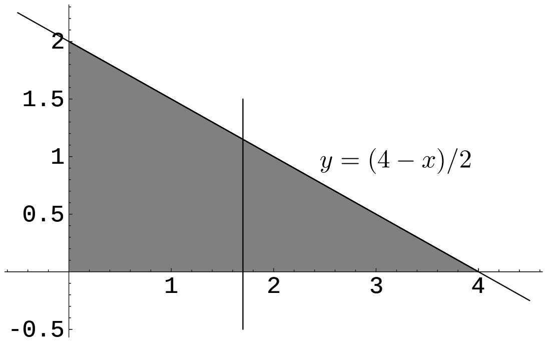
- *the order of integration is strictly dictated by the order of differentials (in the above example $\mathrm{d}y\,\mathrm{d}x$ means that integration in y must be performed first and then followed by integration in x);*

- *the limits of the outermost integral must be constant while the limits of other integrals can be functions only of outer variables, but not of the current integration variable or any inner variables (in the above example the limits of the inner y integral are functions of x but not of y as in the specified integration sequence $\mathrm{d}y\,\mathrm{d}x$ x is outer with respect to y).*

Having checked that both of these properties are satisfied, we can evaluate the repeated integral by working from the inside out:

$$
\begin{aligned}
V &= \int_0^4 (4y-(xy+y^2))|_{y=0}^{y=(4-x)/2}\,\mathrm{d}x \\
&= \int_0^4 (4(4-x)/2-(x(4-x)/2+((4-x)/2)^2)\,\mathrm{d}x \\
&= \int_0^4 \frac{1}{4}\left(16-8x+x^2\right)\,\mathrm{d}x \\
&= \frac{1}{4}\left(16x-4x^2+\frac{x^3}{3}\right)\Bigg|_{x=0}^{x=4} \\
&= \left(4x-x^2+\frac{x^3}{12}\right)\Bigg|_{x=0}^{x=4} = 32-32+\frac{64}{12}=\frac{16}{3}\,.
\end{aligned}
$$

Here we have integrated over a region in the (x, y) plane that is shown in the diagram below. We first integrated with respect to y (by fixing x) and then with respect to x.



Depending on the actual shape of the region it might be convenient to change the order of integration. When this is done one must be careful with the limits of integration making sure that the above two rules are not violated.

If we swap the order of integration from $\mathrm{d}y\,\mathrm{d}x$ *to* $\mathrm{d}x\,\mathrm{d}y$*, that is if we consider slices parallel to the* (x, z) *plane along which* y *remains constant, then the first integration needs to be with respect to* x *from the rear edge* $x = 0$ *up to the front edge given by* $y = (4 - x)/2$*. However, the above rules prohibit to have limits that are functions of the current integration variable (*x *in this case). Therefore, the equation of the edge must be re-written in the equivalent form making the integration variable the subject:* $x = 4 - 2y$*. Subsequently, we will need to add all the slices together using the* y *integration from the left face* $y = 0$ *up to the right vertex with coordinate* $y = 2$*. This results in*

$$V = \int_0^2 \int_0^{4-2y} [4 - (x + 2y)]\,\mathrm{d}x\,\mathrm{d}y\,.$$

It is straightforward to evaluate this integral and obtain the value $V = \dfrac{16}{3}$ *as before. Clearly, changing the order of integration does not change the result.* ■

3.2.2 General double integrals

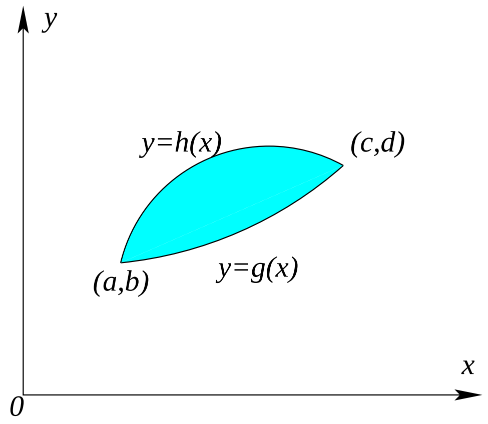
In general a double integral is defined by a region R in the (x, y) plane and a function $z = f(x, y)$ called the *integrand*. The volume beneath the graph of the $z = f(x, y)$ and above the region R can be expressed as

$$V = \iint_R f(x, y)\,\mathrm{d}A = \lim_{\substack{\delta A \to 0 \\ N \to \infty}} \sum_{i=0}^{N} f(x_i, y_i)\delta A\,,$$

where the region R is divided into a large number of small subregions with areas δA_i and (x_i, y_i) is a point in a subregion. In practice the integral is evaluated by expressing the integral as a repeated integral. For the region bounded by the functions $y = g(x)$ and $y = h(x)$ shown in the diagram above we obtain

$$\iint_R f(x, y)\,\mathrm{d}A = \int_a^c \int_{g(x)}^{h(x)} f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_b^d \int_{g^{-1}(y)}^{h^{-1}(y)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

where superscript $^{-1}$ denotes inverse functions. The particular order in which one may prefer to integrate depends on the shape of the region bounded by $y = g(x)$ and $y = h(x)$ and on the integrand $f(x, y)$.



EXAMPLE **3.13.** *Evaluate* $\int\int(x^2+y^2)\,\mathrm{d}A$ *over the region for which* $x \geq 0$, $y \geq 0$ *and* $x+2y \leq 2$. *Change the order of integration and re-evaluate the integral.*

SOLUTION. *First, sketch the integration region* R. *It is a triangle shown in Figure 3.8.*

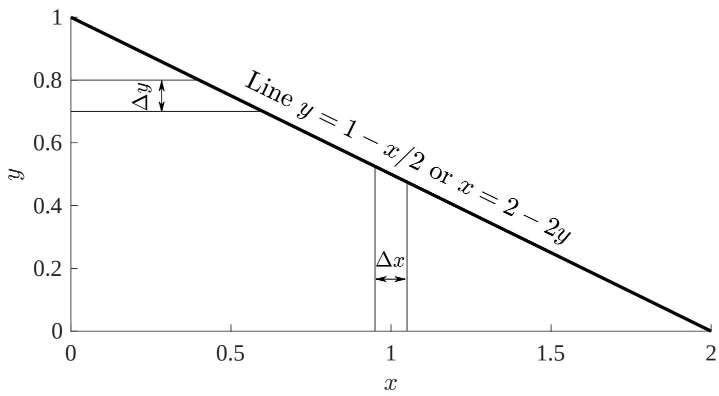


Figure 3.8: Triangular integration region.

Let us integrate first in y *and then in* x. *In this case for each value of* x *ranging from 0 to 2* y *varies from 0 to* $1-\dfrac{x}{2}$.

$$\begin{aligned}\iint_R f(x,y)\,\mathrm{d}A &= \int_0^2\int_0^{1-\frac{x}{2}}(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_0^2\left(x^2y+\frac{1}{3}y^3\right)\Bigg|_{y=0}^{y=1-\frac{x}{2}}\mathrm{d}x\\ &= \int_0^2\left(x^2\left(1-\frac{x}{2}\right)+\frac{1}{3}\left(1-\frac{x}{2}\right)^3\right)\mathrm{d}x=\frac{5}{6}\,.\end{aligned}$$

Alternatively, we can integrate with respect to x first and then with respect to y. Now for each value of y ranging from 0 to 1 x varies from 0 to $2-2y$.

$$\begin{aligned}\iint_R f(x,y)\,\mathrm{d}A &= \int_0^1\int_0^{2-2y}(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1\left(\frac{x^3}{3}+xy^2\right)\Big|_{x=0}^{x=2-2y}\,\mathrm{d}y\\ &= \int_0^1\left(\frac{(2-2y)^3}{3}+(2-2y)y^2\right)\,\mathrm{d}y=\frac{5}{6}\,.\end{aligned}$$

■

EXAMPLE **3.14.** *Change the integration order in*

$$\int_0^2\int_{x^2}^{2x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

SOLUTION. *The integration region is shown in Figure 3.9. Reversing*

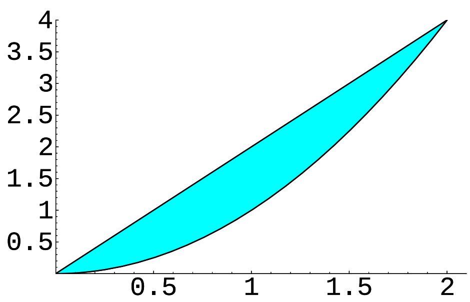


Figure 3.9: Integration region in EXAMPLE 3.14.

the order of integration leads to

$$\int_0^4\int_{\frac{y}{2}}^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

■

EXAMPLE **3.15.** *Evaluate the integral* $\displaystyle\iint_R \frac{\sin y}{y}\,\mathrm{d}A$ *over the region R bounded by the lines $x=0$, $y=\dfrac{\pi}{2}$ and $y=x$ shown in Figure 3.10.*

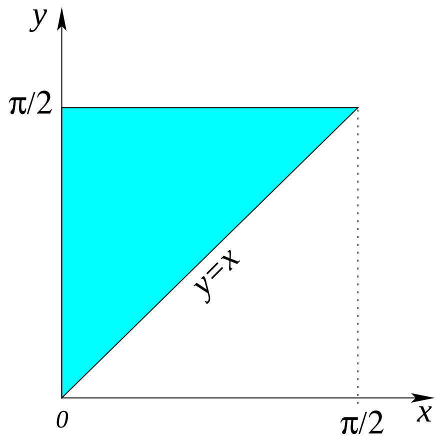


Figure 3.10: Integration region in EXAMPLE 3.15.

SOLUTION. *The given double integral can be written as a repeated integral in two different ways. The integration region can be traversed in y from $y = x$ to $y = \frac{\pi}{2}$ and then in x from $x = 0$ to $x = \frac{\pi}{2}$ or, alternatively, in x from $x = 0$ to $x = y$ and then in y from $y = 0$ to $y = \frac{\pi}{2}$. However, in the former case we would need to evaluate the integral $\int \frac{\sin y}{y}\,\mathrm{d}y$ that cannot be written in elementary functions. Therefore, we consider*

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_0^y \frac{\sin y}{y}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\frac{\pi}{2}} x\,\frac{\sin y}{y}\bigg|_0^y \mathrm{d}y = \int_0^{\frac{\pi}{2}}\left(y\frac{\sin y}{y} - 0\right)\mathrm{d}y \\ &= \int_0^{\frac{\pi}{2}} \sin y\,\mathrm{d}y = 1\,.\end{aligned}$$

■

3.2.3 Double integrals in polar coordinates

Regions with circular boundaries are very common in applications. It is much more natural to describe them using *polar coordinates*.

3.2.3.1 Review of polar coordinates

Recall that Cartesian coordinates (x, y) and polar coordinates (r, θ) are related as

$$x = r\cos\theta\,,\; y = r\sin\theta \quad \text{and} \quad r = \sqrt{x^2+y^2} \geq 0\,,\; -\pi < \theta = \tan^{-1}\left(\frac{y}{x}\right) \leq \pi\,.$$

For example, for the point with polar coordinates $\left(6, \dfrac{3\pi}{4}\right)$ the Cartesian coordinates are

$$x = 6\cos\frac{3\pi}{4} = 6\left(-\frac{\sqrt{2}}{2}\right) = -3\sqrt{2} \text{ and } y = 6\sin\frac{3\pi}{4} = 6\frac{\sqrt{2}}{2} = 3\sqrt{2}\,.$$

The point with Cartesian coordinates $(1,1)$ has polar coordinates $r = \sqrt{2}$ and $\theta = \tan^{-1} = \dfrac{\pi}{4}$. The Cartesian equation $x^2 + y^2 = a^2$ for a circle of radius a simplifies to just $r = a$ in polar coordinates. The polar equation $\theta = \alpha$, where α is a constant, describes a ray emanating from the origin and making an angle α with the x axis, see Figure 3.11.

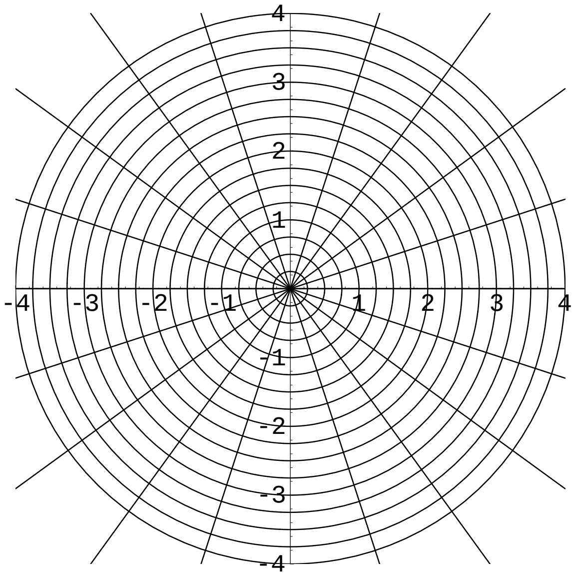


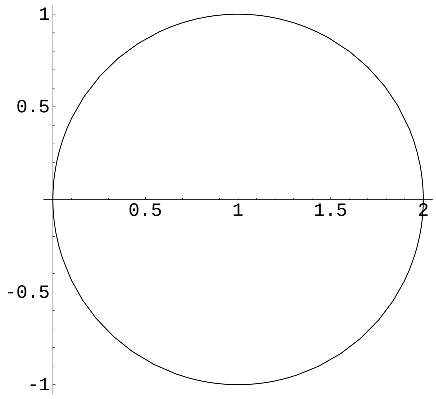
Figure 3.11: Coordinate lines in the polar coordinate system.

EXAMPLE **3.16.** *Transform the Cartesian equation $y = x$ to polar coordinates.*

SOLUTION. *Substitute $x = r\cos\theta$ and $y = r\sin\theta$ to obtain $r\cos\theta = r\sin\theta$, which reduces to $\tan\theta = 1$. This defines two solutions for θ in the range between $-\pi$ and π: $\theta = -\dfrac{3\pi}{4}$ and $\theta = \dfrac{\pi}{4}$. These correspond to two rays emanating from the origin in the opposite directions, one in the first quadrant and the other in the third. Their union, of course, is the straight line $y = x$.* ■

EXAMPLE **3.17.** *Transform the equation $r = 2\cos\theta$ to Cartesian coordinates.*

SOLUTION. *Replacing $\cos\theta$ by $\frac{x}{r}$ we obtain $r = 2\frac{x}{r}$ or $r^2 = 2x$. Replace further r^2 by $x^2 + y^2$ to obtain the equation $x^2 + y^2 = 2x$. Now rearrange the equation and complete the square to obtain $x^2 - 2x + 1 + y^2 = 1$ or $(x-1)^2 + y^2 = 1$. This is recognised as an equation for a circle with centre at $(x, y) = (1, 0)$ and radius 1, see the diagram below.*



- *The top half of the interior of the circle can be described as the region in polar coordinates where r ranges between 0 and $2\cos\theta$ along each ray $\theta = const.$, $0 \le \theta \le \frac{\pi}{2}$.*
- *The whole interior of a circle can be described as the region in polar coordinates where r ranges between 0 and $2\cos\theta$ along each ray $\theta = const.$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.*

■

The following observations may be useful in describing various lines and regions in polar coordinates.

- The straight line $x = a$ has equation $r\cos\theta = a$ in polar coordinates, or
$$r = \frac{a}{\cos\theta} = a\sec\theta\,.$$

- The straight line $y = b$ has equation $r \sin\theta = b$ in polar coordinates, or
$$r = \frac{b}{\sin\theta} = b \csc\theta\,.$$
- The general equation to a straight line in Cartesian coordinates $ax + by = c$ takes the form $ar\cos\theta + br\sin\theta = c$. Using the identity $a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta+\alpha)$, where $\tan\alpha = \dfrac{b}{a}$ we write the equation as $r\sqrt{a^2+b^2}\cos(\theta-\alpha) = c$ or
$$r = \frac{c}{\sqrt{a^2+b^2}}\sec(\theta-\alpha)\,.$$
For example, the straight line with equation $x + y = 1$ has the equation $r = \dfrac{1}{\sqrt{2}}\sec\left(\theta - \dfrac{\pi}{4}\right)$.

3.2.3.2 Evaluating double integrals in polar coordinates

To evaluate double integrals in polar coordinates we first need to transform a small element of area $\mathrm{d}x\,\mathrm{d}y$ in Cartesian coordinates to that in polar. To do that refer to Figure 3.12. A small sectorial area segment located distance r away from the origin has a radial width $\mathrm{d}r$ and a circumferential length $r\,\mathrm{d}\theta$. Since $\mathrm{d}\theta$ is small, the shape of the area element is close to rectangular and can be approximated as $\mathrm{d}r \cdot r\,\mathrm{d}\theta = r\,\mathrm{d}r\,\mathrm{d}\theta$. Therefore, we can write:
$$\iint_R f(x,y)\,\mathrm{d}A = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta\,.$$
The procedure of evaluating double integrals in polar coordinates is summarised below.

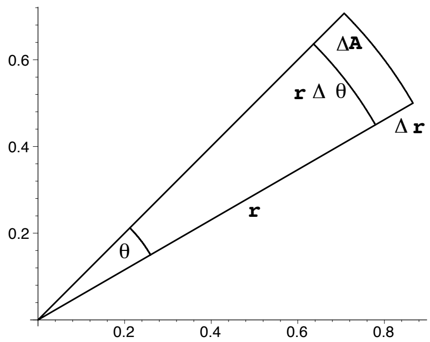


Figure 3.12: Small element of area in polar coordinates.

1. Make the substitutions $x = r\cos\theta$ and $y = r\sin\theta$ in the expression for the integrand $f(x,y)$.
2. Express the area element in polar coordinates $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$.
3. Determine the limits of integration by describing the region in polar coordinates.
4. Choose the order of integration considering the geometry of the integration region and the structure of the integrand $f(r,\theta)$.
5. Evaluate the resulting repeated integral.

Example **3.18.** *Consider an integral $\iint_R f(x,y)\,\mathrm{d}A$, where R is the semi-circular region bounded by the top half of the circle $x^2+y^2=1$ and the x axis shown in Figure 3.13. The integral can be expressed in*

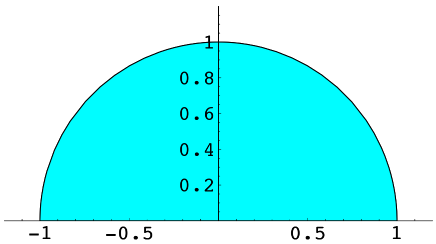


Figure 3.13: Semicircular integration region.

Cartesian coordinates by first integrating with respect to y from $y=0$ to $y=\sqrt{1-x^2}$ and then integrating with respect to x from $x=-1$ to $x=1$, that is

$$\iint_R f(x,y)\,\mathrm{d}A = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

However, this would involve bulky algebraic manipulations involving $\sqrt{1-x^2}$ that comes from the Cartesian description of the boundary of a circular region. To avoid this we transform the integral to polar coordinates where the integration limits become constant: $0\le r\le 1$ and $0\le\theta\le\pi$. Therefore, the integral simplifies to

$$\iint_R f(x,y)\,\mathrm{d}A = \int_0^{\pi}\int_0^{1} f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r\,\mathrm{d}\theta\,.$$

Example **3.19.** *Evaluate* $\iint_R (x^2+y^2)\,\mathrm{d}A$ *over the region* R *shown in Figure 3.13.*

Solution. *We have*

$$\iint_R (x^2+y^2)\,\mathrm{d}A = \int_0^{\pi}\int_0^1 (r^2)r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi} \left.\frac{r^4}{4}\right|_{r=0}^{r=1} \mathrm{d}\theta = \int_0^{\pi} \frac{1}{4}\,\mathrm{d}\theta = \frac{\pi}{4}\,.$$

■

Example **3.20.** *Evaluate the integral* $\displaystyle\iint_R y^2\,\mathrm{d}A$*, where* R *is the quarter of a unit disk in the first quadrant.*

Solution. *This integral could be evaluated in Cartesian coordinates as*

$$\iint_R y^2\,\mathrm{d}A = \int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,\mathrm{d}y\,\mathrm{d}x$$

but this is not easy. Instead we describe the region R *using polar coordinates as*

$$R = \left\{(r,\theta):\quad 0 \le r \le 1\,,\quad 0 \le \theta \le \frac{\pi}{2}\right\}\,.$$

Then

$$\begin{aligned}\iint_R y^2\,\mathrm{d}A &= \int_0^{\frac{\pi}{2}}\int_0^1 (r\sin\theta)^2 r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \left.\frac{1}{4}r^4\sin^2\theta\right|_{r=0}^{r=1} \mathrm{d}\theta \\ &= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2\theta\,\mathrm{d}\theta = \frac{1}{8}\int_0^{\frac{\pi}{2}} (1-\cos(2\theta))\,\mathrm{d}\theta = \frac{\pi}{16}\,.\end{aligned}$$

■

EXAMPLE **3.21.** *Evaluate the integral* $\int_0^2 \int_0^x y \, \mathrm{d}y \, \mathrm{d}x$ *in polar coordinates.*

SOLUTION. *The integration region R here is a triangle bounded by the x axis and the lines $y = x$ and $x = 2$:*

$$R = \left\{ (r, \theta) : \quad 0 \le r \le 2 \sec\theta \,, \quad 0 \le \theta \le \frac{\pi}{4} \right\} .$$

Here we used the fact that the line $x = 2$ in polar coordinates is $r = 2\sec\theta$. Therefore,

$$\int_0^2 \int_0^x y \, \mathrm{d}y \, \mathrm{d}x = \int_0^{\frac{\pi}{4}} \int_0^{2\sec\theta} r \sin\theta \, r \, \mathrm{d}r \, \mathrm{d}\theta = \frac{8}{3} \int_0^{\frac{\pi}{4}} \sec^3\theta \sin\theta \, \mathrm{d}\theta \,.$$

Upon the substitution $u = \cos\theta$ we obtain

$$\int_0^2 \int_0^x y \, \mathrm{d}y \, \mathrm{d}x = -\frac{8}{3} \int_1^{\frac{\sqrt{2}}{2}} u^{-3} \, \mathrm{d}u = \frac{4}{3} \,.$$

■

3.2.4 Surface integrals

In Section 3.1 we saw that an integral along a straight line, say, the Cartesian coordinate axis x, $\int f(x) \, \mathrm{d}x$, can be generalised to a case when the integration path is not a straight line, but rather a curve in space. Such line integrals can be computed for both scalar and vector functions. So far in Section 3.2 we have considered double integrals that are evaluated over "flat" regions in coordinate (x, y) or (r, θ) planes. These can be considered as a generalisation of a straight line integration. The next logical step then is to generalise double integration over planar regions to that over regions that are given by curved surfaces in space. Such surface integrals could be also of two types: surface integrals of scalar and vector functions. The scalar integrals are generally used to determine properties of the surface itself: its area, centre of mass, the total energy accumulated in the elastic surface etc. The vector integrals are typically used to quantify various transport processes through a surface: the flux of fluid flowing across the surface, the flux of momentum carried through it etc. However, before we can look into the evaluation of surface integrals we need to learn how to describe surfaces themselves.

3.2.4.1 Description of surfaces in space

Following the mentioned similarity between the curve and surface integration we endeavor to describe surfaces parametrically. It is clear intuitively that in contrast to line in space two different parameters, say, u and v will be required to describe a surface. Each of these two parameters will define a curve in space provided that the second parametre is kept fixed. The surface then can be viewed as made of an infinite number of such lines. It is also intuitively clear that such a surface will not “disintegrate” only if the two families of individual lines corresponding to each of the two parameters intersect. Think of a surface as of a rug that is made of perpendicular threads: they interweave and this stops the rug from falling apart. As an example, consider an *elliptic cylinder* shown in Figure 3.14. It can be given

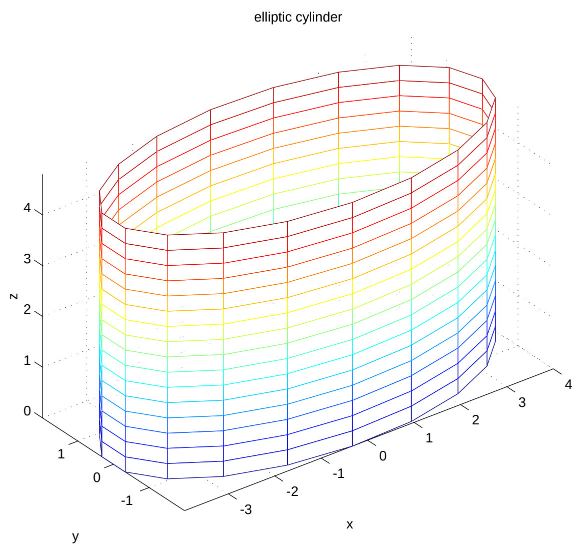


Figure 3.14: Surface of an elliptic cylinder.

parametrically as

$$\begin{aligned}\mathbf{r}(u,v) &= (x(u,v),y(u,v),z(u,v)) = 4\cos u\,\mathbf{i}+2\sin u\,\mathbf{j}+v\mathbf{k}\\ &= (4\cos u, 2\sin u, v)\,.\end{aligned}$$

Here $\mathbf{r}$ is the position vector corresponding to a point on a surface and $-\infty < v < \infty$ and $0 \le u \le 2\pi$ are two parameters. By setting $v = const.$ we obtain

$$\frac{x^2}{16}+\frac{y^2}{4} = \cos^2 u + \sin^2 u = 1 \qquad z = const.\,.$$

This shows that by cutting the surface with the $z = const.$ planes we obtain a set of ellipses. If we now set $u = const.$, then we obtain that both x and y

coordinates will remain constant yet z will be able to vary freely from $-\infty$ to $+\infty$. This will correspond to a set of vertical lines. Composition of such ellipses intersecting with vertical lines is the surface that we call an elliptic cylinder. In fact drawing a multitude of parametric lines is exactly how any plotting software such as MATLAB used to produce Figure 3.14 creates an image of a surface.

Fundamental properties of any surface are its *tangent* and *normal vector*s. We know from Section 3.1 that a derivative with respect to a parameter of a position vector describing points on a curve produces a tangent vector to this curve. Thus, we conclude that a surface represented parametrically as $\mathbf{r}(u,v)$ than two families of tangent vectors

$$\boxed{\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} \text{ and } \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}},$$

can be defined for this surface. These vectors are not parallel because the parametric lines that are tangent to them must intersect.

Definition 3.5 *The plane that contains two non-parallel tangent vectors* $\mathbf{r}_u$ *and* $\mathbf{r}_v$ *to the plane is called the* tangent plane.

Definition 3.6 *The vector* $\mathbf{n}$ *that is perpendicular to the tangent plane of the surface at a given point is called the normal vector to the surface at this point.*

The normal vector can be defined using the definition of a vector product

$$\boxed{\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v}. \tag{3.2}$$

Clearly, vector $-\mathbf{N} = \mathbf{r}_v \times \mathbf{r}_u$ is also a normal to the same surface pointing in the opposite direction. A unit normal is obtained simply by

$$\boxed{\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{N}||}}. \tag{3.3}$$

In the considered example of an elliptic cylinder

$$\begin{aligned}
\mathbf{r}_u &= (-4\sin u, 2\cos u, 0)\,, \quad \mathbf{r}_v = (0,0,1)\,,\\
\mathbf{N} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos u, 4\sin u, 0)\,,\\
\mathbf{n} &= \frac{\mathbf{N}}{||\mathbf{N}||} = \frac{1}{\sqrt{1+3\sin^2 u}}(\cos u, 2\sin u, 0)\,.
\end{aligned}$$

Having defined the main properties of surfaces in space we now proceed to study surface integrals.

3.2.4.2 Surface integrals of scalar functions

Surface integrals are essentially a sum of some property, as expressed by the integrand, over some curved surface, the domain of integration. For example, the total amount of contaminant, such as oil, on the surface of the sea is the integral of the density of the oil over the undulating, even splashing, water's surface.

The basic difficulty is that a curved surface generally slopes and so areas on the surface are magnified by their slope when compared to areas of a convenient reference plane below the surface. For a simple example, consider the plane surface $z = 1 - x - y$ shown in Figure 3.15. Each little patch of

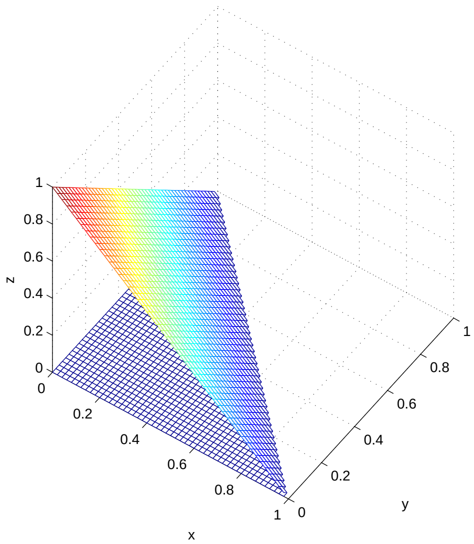


Figure 3.15: Inclined plane.

this sloping surface is bigger in area compared to the corresponding patch in the (x, y) plane below. Exactly the same considerations apply to curved surfaces, but they are more complicated because the slope of the surface varies from point to point. In order to evaluate surface integrals it is crucial to know the surface area accurately.

Recollect from Section 3.1 that for parametrically given curves the length of a tangent vector is approximately given by the derivative of a position vector with respect to a parameter multiplied by the parameter increment. Therfore if S is given parametrically by $\mathbf{r}(u, v)$, then small tangent vectors $\mathbf{r}_u \Delta u$ and $\mathbf{r}_v \Delta v$ form a small parallelogram on the surface whose area can be computed using the definition of a vector product $\Delta A = ||\mathbf{r}_u \times \mathbf{r}_v||\, \Delta u\, \Delta v$. Thus, we define the surface integral of a scalar function G as follows. Cut the surface into N small patches of area ΔA_n. Let a typical value of G on the nth patch be G_n. Then

$$\iint_S G\,\mathrm{d}A = \lim_{N\to\infty}\sum_{n=1}^{N} G_n\,\Delta A_n\,. \tag{3.4}$$

Then equation (3.4) becomes

$$\sum_{n=1}^{N} G_n\,\Delta A_n \approx \sum_{n=1}^{\infty} G_n\,||\mathbf{r}_u\times\mathbf{r}_v||\,\Delta u_n\,\Delta v_n \approx \iint_R G[\mathbf{r}(u,v)]\,||\mathbf{r}_u\times\mathbf{r}_v||\,\mathrm{d}u\,\mathrm{d}v$$

by definition of a double integral. Thus, surface integrals are *evaluated* as the double integral

$$\boxed{\iint_S G\,\mathrm{d}A = \iint_R G(\mathbf{r}(u,v))\,||\mathbf{r}_u\times\mathbf{r}_v||\,\mathrm{d}u\,\mathrm{d}v}\,, \tag{3.5}$$

where R is the region of the parameter plane corresponding to surface S.

Consider the special case where the surface S is in the plane $z=0$ so that its parameterisation is $\mathbf{r}=(x(u,v),y(u,v))$ for some domain R of the (u,v) plane. Then

$$\mathbf{r}_u\times\mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_u & y_u & 0\\ x_v & y_v & 0\end{vmatrix} = (0,0,x_uy_v - x_vy_u)\,.$$

Thus, (3.5) reduces to

$$\boxed{\iint_S G\,\mathrm{d}A = \iint_R G(x(u,v),y(u,v))\,|x_uy_v - x_vy_u|\,\mathrm{d}u\,\mathrm{d}v}\,.$$

Since in this particular case the number of coordinates (x and y) is equal to the number of parameters (u and v), the parameterisation of a flat surface is equivalent to a transformation of coordinates. The combination

$$J = x_uy_v - x_vy_u$$

is called the *Jacobian* of a transformation. In particular, for polar coordinates considered in Section 3.2.3.2 $u\equiv r$, $v\equiv\theta$ and

$$J = r\cos\theta^2 + r\sin^2\theta = r\,,$$

which is the additional factor appearing in the integrand when Cartesian coordinates are transformed to polar.

3.2.4.3 Surface integrals of vector functions: flux

The most important surface integral of a vector function is the so called *flux integral*. Imagine a vector field $\mathbf{F}$ that is arbitrarily oriented with respect to surface S. If such a vector field describes a flow of some quantity, then it can only cross the boundary given by the specified surface if it has the component $F_n \neq 0$ normal to the surface. To compute the total flux one than needs to evaluate $\int_S F_n \,\mathrm{d}A$.[3] Since $F_n = \mathbf{F}\cdot\mathbf{n}$, where $\mathbf{n}$ is the unit normal to the surface, using (3.2) and (3.3) for parametric surfaces we obtain that the flux Φ is given by

$$\Phi = \int_S F_n \,\mathrm{d}A = \iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,\mathrm{d}u\,\mathrm{d}v\,. \qquad (3.6)$$

Sometimes the flux integral is written as $\Phi = \iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{A}$. The differential $\mathrm{d}\mathbf{A} = \mathbf{n}\;\mathrm{d}A$ is often termed the differential *vector area*. It has the magnitude $\mathrm{d}A$ equal to the area of an element and direction $\mathbf{n}$ normal to the surface of the element.

EXAMPLE **3.22.** *Calculate the flux in the positive x direction of the vector field $\mathbf{F} = (x^2y, yz, 3x^3z)$ through the rectangular region S in the plane $x = 2$ bounded by $y = 0$, $y = 1$, $z = 0$ and $z = 3$.*

SOLUTION. *In this example the required unit normal vector to the surface is constant and is equal to $\mathbf{i} = (1,0,0)$. Hence,*

$$\Phi = \iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A = \iint_S (x^2y, yz, 3x^3z)\cdot(1,0,0)\,\mathrm{d}A = \iint_S x^2y\,\mathrm{d}A\,.$$

In the plane region $x = 2$ and $\mathrm{d}A = \mathrm{d}y\,\mathrm{d}z$. Therefore,

$$\Phi = \int_0^3\int_0^1 4y\,\mathrm{d}y\,\mathrm{d}z = \int_0^3 2y^2|_{y=0}^{y=1}\,\mathrm{d}z = \int_0^3 2\,\mathrm{d}z = 2z|_0^3 = 6\,.$$

■

[3] This is somewhat similar to a work integral. However, there only the component that is tangent to a curve is accounted for.

EXAMPLE **3.23.** *Calculate the outward flux of the vector field* $\mathbf{F} = (x^2y, yz, 3x^3z)$ *over the surface of the unit cube with faces located at* $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ *and* $z = 1$.

SOLUTION. *There are six square surfaces to consider.*

- *The back square in the plane* $x = 0$. *Here* $\mathbf{n} = (-1, 0, 0)$ *and* $\mathrm{d}A = \mathrm{d}y\,\mathrm{d}z$.
- *The front square in the plane* $x = 1$. *Here* $\mathbf{n} = (1, 0, 0)$ *and* $\mathrm{d}A = \mathrm{d}y\,\mathrm{d}z$.
- *The left side square in the plane* $y = 0$. *Here* $\mathbf{n} = (0, -1, 0)$ *and* $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}z$.
- *The right side square in the plane* $y = 1$. *Here* $\mathbf{n} = (0, 1, 0)$ *and* $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}z$.
- *The bottom square in the plane* $z = 0$. *Here* $\mathbf{n} = (0, 0, -1)$ *and* $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$.
- *The top square in the plane* $z = 1$. *Here* $\mathbf{n} = (0, 0, 1)$ *and* $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$.

Calculate the corresponding fluxes over each face:

$$
\begin{aligned}
\Phi_1 &= \int_0^1\int_0^1 (x^2y, yz, 3x^3z)\cdot(-1,0,0)\,\mathrm{d}y\,\mathrm{d}z \\
&= -\int_0^1\int_0^1 x^2y\,\mathrm{d}y\,\mathrm{d}z = 0 \qquad \text{since } x = 0\,; \\
\Phi_2 &= \int_0^1\int_0^1 (x^2y, yz, 3x^3z)\cdot(1,0,0)\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^1\int_0^1 x^2y\,\mathrm{d}y\,\mathrm{d}z = \int_0^1\int_0^1 y\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{2} \qquad \text{since } x = 1\,; \\
\Phi_3 &= \int_0^1\int_0^1 (x^2y, yz, 3x^3z)\cdot(0,-1,0)\,\mathrm{d}x\,\mathrm{d}z \\
&= -\int_0^1\int_0^1 yz\,\mathrm{d}x\,\mathrm{d}z = 0 \qquad \text{since } y = 0\,; \\
\Phi_4 &= \int_0^1\int_0^1 (x^2y, yz, 3x^3z)\cdot(0,1,0)\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_0^1\int_0^1 z\,\mathrm{d}x\,\mathrm{d}z = \frac{1}{2} \qquad \text{since } y = 1\,; \\
\Phi_5 &= \int_0^1\int_0^1 (x^2y, yz, 3x^3z)\cdot(0,0,-1)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1\int_0^1 3x^3z\,\mathrm{d}x\,\mathrm{d}y = 0 \qquad \text{since } z = 0\,;
\end{aligned}
$$

$$\begin{aligned}\Phi_6 &= \int_0^1\int_0^1 (x^2y, yz, 3x^3z)\cdot(0,0,1)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1\int_0^1 3x^3\,\mathrm{d}x\,\mathrm{d}y = \frac{3}{4} \quad \textit{since } z=1\,.\end{aligned}$$

The total flux over the surface of the cube then is

$$\Phi = \Phi_1+\Phi_2+\Phi_3+\Phi_4+\Phi_5+\Phi_6 = 0+\frac{1}{2}+0+\frac{1}{2}+0+\frac{3}{4}=\frac{7}{4}\,.$$

■

EXAMPLE **3.24.** *Calculate outflux of* $\mathbf{F}=(2x,2y,z^2)$ *through the surface of section of the elliptic cylinder section considered in Section 3.2.4.1 and contained between planes* $z=0$ *and* $z=4$.

SOLUTION. *We use parametric representation of the surface of the cylinder to write* $\mathbf{F}=(8\cos u, 4\sin u, v^2)$. *Subsequently, we need to decide on the direction of the normal vector we have to choose given that both* $\mathbf{N}=(2\cos u, 4\sin u, 0)$ *and* $-\mathbf{N}$ *are perpendicular to the surface. Since we are interested in finding the outflux, the outer normal to the surface of the cylinder (pointing from inside out) needs to be selected. The easiest way to decide then is to evaluate* $\mathbf{N}$ *and* $-\mathbf{N}$ *at some point. For example,* $u=0$ *corresponds to* $(x,y)=(4,0)$, *the rightmost point on the surface of a cylinder in the* (x,y) *plane. At this point* $\mathbf{N}=(2,0,0)$, *that is this vector is directed "from inside out" and should be selected. Finally, noting that* $0\le v=z\le 4$ *we write*

$$\begin{aligned}\Phi &= \int_0^4\int_0^{2\pi}(8\cos u, 4\sin u, v^2)\cdot(2\cos u, 4\sin u, 0)\,\mathrm{d}u\,\mathrm{d}v\\ &= 16\int_0^4\int_0^{2\pi}\mathrm{d}u\,\mathrm{d}v = 16\cdot(2\pi)\cdot 4 = 128\pi\,.\end{aligned}$$

■

3.2.4.4 Fluxes in fluids

If ρ is the density of a fluid flowing with velocity field $\mathbf{v}$, then $\iint_S(\rho\mathbf{v})\cdot\mathrm{d}\mathbf{A}$ is the total mass flux of fluid material per unit time across the surface S. Similarly, since ρu is the density of momentum in the x-direction, and $\mathbf{v}$ is the velocity with which it is being carried, then $\iint_S(\rho u\mathbf{v})\cdot\mathrm{d}\mathbf{A}$ denotes the rate at which x-momentum is being carried across the surface S.

3.2.5 Stokes' theorem

The following theorem attributed to Stokes establishes a further link between surface and line integrals, namely, between the flux and circulation of vector fields.

Theorem 3.7 *Let S be an orientable smooth surface that is bounded by a* simple closed piecewise smooth curve *C and let $\mathbf{v}$ be a vector field with continuous first derivatives. Then*

$$\boxed{\iint_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathbf{n} \, \mathrm{d}A = \oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{r}},$$

where $\mathbf{n}$ is a unit normal to the surface and the direction of line integration is according to the right hand rule *that is if the right hand fingers point in the direction of travel, then the thumb shows the direction of the normal to the surface.*

We need to define new terms appearing in the formulation of Stokes' theorem before we can use it in practice.

Definition 3.8 *A curve is closed if its initial and terminal points coincide.*

Definition 3.9 *A curve is simple if it does not have self-intersections.*

For example, the digit "0" can be considered as a smooth simple closed curve, digit "3" is simple and piecewise smooth (as it has one cusp singularity in the middle) but not closed, while "8" is closed and smooth but not simple as it has a self-intersection.

Definition 3.10 *A surface is called orientable if a small disk with a fixed normal cannot be moved continuously along the surface and returned to the original location with the normal pointing in the opposite direction.*

Examples of non-orientable surfaces include Möbius strip, Klein bottle and Roman surface.

Definition 3.11 *A surface is called smooth if its normal depends continuously on the point on the surface.*

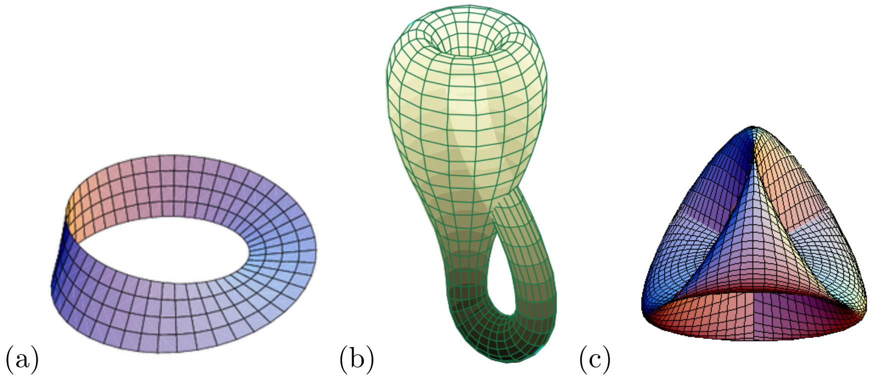


Figure 3.16: Examples of non-orientable surfaces for which Stokes' theorem is invalid: (a) Möbius strip, (b) Klein bottle, (c) Roman surface.

A cone is a non-smooth surface at its vertex.

The following example demonstrates how the use of Stokes' theorem can significantly simplify computations of the circulation integrals.

EXAMPLE **3.25.** *Calculate the circulation* Γ *of the vector* $\mathbf{v} = \left(x^3, \frac{1}{y}, z^2\right)$ *along the closed path* C *consisting of the part of the curve* $y = 9 - x^2$ *lying above the* x*-axis and the segment of the* x *axis between* $x = -3$ *and* $x = 3$.

SOLUTION. *We can use Stokes' theorem to write*

$$\int_C \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \iint_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathbf{n}\, \mathrm{d}A\,,$$

where S *is the region in the* (x, y) *plane enclosed by curve* C. *Since*

$$\boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^3 & \dfrac{1}{y} & z^2 \end{vmatrix} = \mathbf{0}$$

we immediately conclude that the integral $\Gamma = 0$ *without any further computations.* ■

EXAMPLE **3.26.** *Calculate the circulation* Γ *of the vector* $\mathbf{v} = (z, xy, x)$ *around the closed path* C *consisting of the circular segment*

$$x^2 + y^2 = a^2\,,\ 0 \le x \le a\,,\ 0 \le y \le a$$

in the first quadrant and the parts of the x and y axes between 0 and a.

SOLUTION. *According to Stokes' theorem*

$$\Gamma = \int_C \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \iint_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathbf{n} \, \mathrm{d}A \,,$$

where S is the quarter of a circle enclosed by curve C. We find that

$$\boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & xy & x \end{vmatrix} = (0, 0, y) \,.$$

Taking $\mathbf{n} = (0, 0, 1)$ we use the right-hand rule to define the line integration direction as anti-clockwise. Further, using polar coordinates we transform the integral as

$$\Gamma = \iint_S (0, 0, y) \cdot (0, 0, 1) \, \mathrm{d}A = \iint_S y \, \mathrm{d}A = \int_0^{\frac{\pi}{2}} \int_0^a (r \sin\theta) r \, \mathrm{d}r \, \mathrm{d}\theta = \frac{a^3}{3} \,.$$

■

3.2.5.1 Irrotational fields must have a scalar potential

In Section 2.5.2 we showed that if the scalar potential ϕ exists, then the corresponding vector field $\mathbf{v} = \boldsymbol{\nabla}\phi$ is necessarily irrotational. Below we will prove that the converse is also true: if a vector field is irrotational , that is $\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}$, then the scalar potential must exist. Let $\mathbf{v} = (v_1, v_2, v_3)$.

- Pick some reference point $P(x_0, y_0, z_0)$. Then for any point $Q(x, y, z)$ in the domain construct two different paths C_1 and C_2 from P to Q and define the corresponding functions

$$\phi_1(x, y, z) = \int_{C_1} \mathbf{v} \cdot \mathrm{d}\mathbf{r} \quad \text{and} \quad \phi_2(x, y, z) = \int_{C_2} \mathbf{v} \cdot \mathrm{d}\mathbf{r} \,.$$

- Let a third path C_3 be a combination of C_1 from P to Q and $-C_2$ from Q to P. Therefore, C_3 is a closed curve starting and finishing at

P. Then let S be any smooth surface with C_3 as its edge and apply Stokes' theorem to obtain

$$\begin{aligned}\int_{C_1} \mathbf{v}\cdot\,\mathrm{d}\mathbf{r} - \int_{C_2} \mathbf{v}\cdot\,\mathrm{d}\mathbf{r} &= \oint_{C_3} \mathbf{v}\cdot\,\mathrm{d}\mathbf{r} \\ &= \iint_S (\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathbf{n}\,\mathrm{d}A \quad \text{by Stokes' theorem} \\ &= 0 \quad \text{as } \boldsymbol{\nabla}\times\mathbf{v} = \mathbf{0}\,.\end{aligned}$$

Thus, $\int_{C_1} \mathbf{v}\cdot\,\mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{v}\cdot\,\mathrm{d}\mathbf{r}$ for all curves with the same end points and hence $\phi_1(x,y,z) = \phi_2(x,y,z) = \phi(x,y,z)$ is independent of the path that is $\phi(x,y,z)$ is unique and its value depends only on the end point Q.

- Now show that ϕ is a scalar potential for $\mathbf{v}$. Consider $\frac{\partial\phi}{\partial x}$. By its definition

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \lim_{h\to 0}\frac{1}{h}\left[\phi(x+h,y,z)-\phi(x,y,z)\right] \\ &= \lim_{h\to 0}\frac{1}{h}\left[\int_P^{(x+h,y,z)} \mathbf{v}\cdot\,\mathrm{d}\mathbf{r} - \int_P^{(x,y,z)} \mathbf{v}\cdot\,\mathrm{d}\mathbf{r}\right] \\ &= \lim_{h\to 0}\frac{1}{h}\int_{(x,y,z)}^{(x+h,y,z)} \mathbf{v}\cdot\,\mathrm{d}\mathbf{r} \\ &= \lim_{h\to 0}\frac{1}{h}(v_1,v_2,v_3)\cdot(0,0,h) = v_1\,.\end{aligned}$$

Similarly we obtain that $\dfrac{\partial\phi}{\partial y} = v_2$, $\dfrac{\partial\phi}{\partial z} = v_3$ so that $\mathbf{v} = \boldsymbol{\nabla}\phi$ and ϕ is indeed the scalar potential for $\mathbf{v}$.

3.2.5.2 Green's theorem—a particular case of Stokes' theorem

When the surface under consideration in Stokes' theorem is restricted to a plane then the theorem is also known as *Green's theorem* in a plane. In this case the vector field has only two components, say, $\mathbf{v} = (v_1(x,y), v_2(x,y))$ and $\boldsymbol{\nabla}\times\mathbf{v} = \left(0, 0, \dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_1}{\partial y}\right)$. The unit normal to the (x,y) plane is $\mathbf{n} = (0,0,1)$ so that

$$(\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathbf{n} = \left(0,0,\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\cdot(0,0,1) = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\,.$$

Since the position vector $\mathbf{r} = (x,y)$ in the plane has only two components, the path element becomes $\mathrm{d}\mathbf{r} = (\,\mathrm{d}x,\,\mathrm{d}y)$ and then

$$\mathbf{v}\cdot\,\mathrm{d}\mathbf{r} = (v_1,v_2)\cdot(\,\mathrm{d}x,\,\mathrm{d}y) = v_1\,\mathrm{d}x + v_2\,\mathrm{d}y\,.$$

Substituting these expressions into the integral equation of Stokes' theorem we obtain

$$\boxed{\iint_S \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \mathrm{d}A = \oint_C (v_1 \,\mathrm{d}x + v_2 \,\mathrm{d}y)}.$$

In particular, if $\dfrac{\partial v_2}{\partial x} = \dfrac{\partial v_1}{\partial y}$, then the two-dimensional vector field is irrotational and its scalar potential exists as we have already established in Section 2.2.7.

3.2.5.3 Stokes' theorem in fluids

The integral equality stated in Stokes' theorem becomes very intuitive if put in the context of fluids. Indeed, let $\mathbf{v}$ denote the fluid velocity. Then the integral in the left-hand side is recognised as a flux of fluid vorticity through the given surface in the direction normal to the surface while the right-hand side gives fluid's circulation along the boundary curve. As we learned in Section 2.5 curl (and vorticity) represents a local "microscopic" rotation and the surface integral appearing in Stokes' theorem sums the rotations created by these microscopic vortices. Since by the conditions of the theorem the vector field is differentiable, it must be continuous that is the intensity of nearby vortices and the direction of their rotation should be nearly the same. However, as seen from Figure 3.17, in this case the

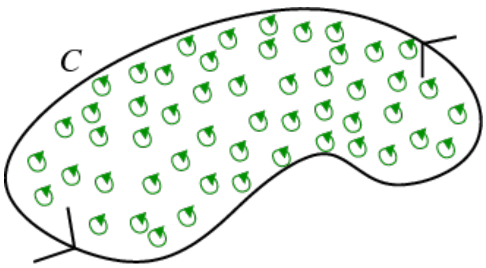
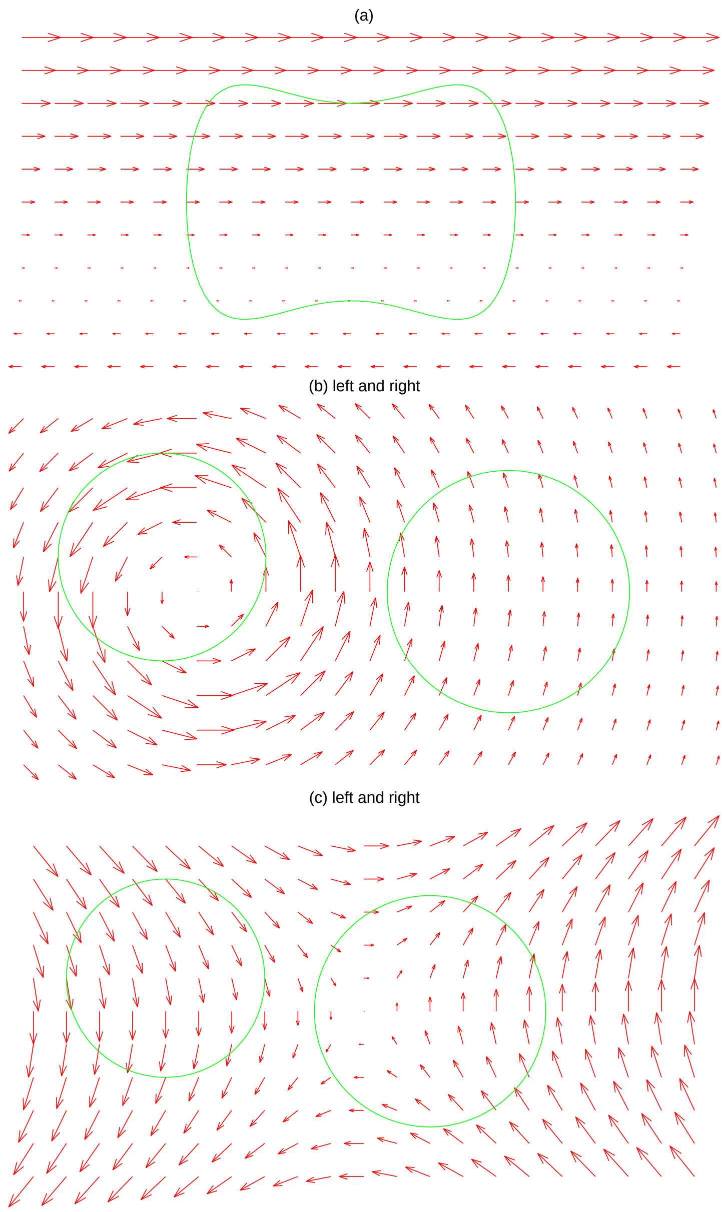


Figure 3.17: Illustration of Stokes' theorem.

neighbouring vortices induce opposite velocities between them that cancel each other inside the surface. Such a cancellation, of course, does not occur near the boundary of a surface as the vortices are found only on one side of it and all microscopic vortices induce the velocity generally in the same direction. This creaties a macroscopic motion along the boundary that as we know from Section 3.1.4 is the circulation represented by the integral in the right-hand side. This once again justifies why an early theory of an aeroplane wing replaced it with vortices distributed across the aerofoil.

Exercises

Ex. 3.2. Consider the fluid velocity fields and the closed curves C directed anti-clockwise below. Estimate whether the net fluid circulation $\oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{r}$ is positive, negative or approximately zero. Use Stokes' theorem to conclude whether these estimates agree with your identification of regions of positive, negative and zero vorticity that you made in Ex. 2.4.



3.2.6 Volume integrals

A further natural generalisation of muti-dimensional integrals is *volume integrals*. Here we will consider the most frequently used of them, the volume integral of a scalar function. You can think of it as the task of finding a mass of a three-dimensional object whose density is not uniform in space. Defined in a similar manner to other integrals we have introduced so far, a volume integral of a scalar function $f(x, y, z)$ over a region T in space

$$\iiint_T f(x,y,z)\,\mathrm{d}V = \lim_{\Delta V \to 0} \sum_i f(x_i, y_i, z_i)\Delta V_i\,,$$

where the point (x_i, y_i, z_i) is inside the small volume element ΔV_i of T. As with the evaluation of a surface integral, volume integrals may be converted to repeated triple integrals. If the region V has the description

$$T = \{(x,y,z): \quad a_1 \le x \le a_2, b_1(x) \le y \le b_2(x), c_1(x,y) \le z \le c_2(x,y)\}\,,$$

then

$$\iiint_T f\,\mathrm{d}V = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \int_{c_1(x,y)}^{c_2(x,y)} f\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,.$$

The main technical difficulty in evaluating volume integrals is the correct choice of integration limits given the specified order of integration. The rules specified in Section 3.2.1 for double integrals apply in the case of triple integrals.

EXAMPLE **3.27.** *Find the mass of the body in the first octant confined by surfaces $y = 1 - x^2$ and $z = x$, see Figure 3.18 if the density of the material is given by $\rho = 4z$.*

SOLUTION. *We start with choosing the order of integration, say, $\iiint_T 4z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Next we need to choose the appropriate limits of integration. It is easier to start from the outer integral because its limits must be constant and range from the minimum to maximum values of the outer integration variable, z in this case: $0 \le z \le 1$. For each fixed value of z the integration area is similar to the base of the object in shape but is smaller in size, see Figure 3.18 and y varies from its minimum value of 0 to its maximum value computed for the intersection the inclined top plane $x = z$ and the curved side surface $y = 1 - x^2 = 1 - z^2$. Thus, $0 \le y \le 1 - z^2$. Note that the expression for the y limit depends on the outer variable z but not the inner variable x. Finally, the x integration is performed from the minimum value of $x = z$ that is at the edge of the horizontal section that belongs to the*

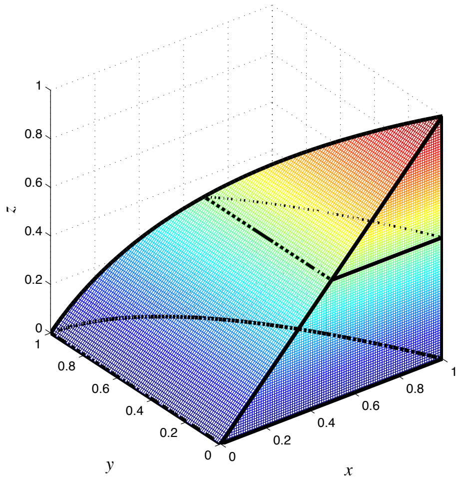


Figure 3.18: Curvilinear volume.

inclined plane to the maximum value at the curved surface which is $x = \sqrt{1-y}$. *Note that the limits of the inner integral depends on all outer variables. Thus,*

$$\begin{aligned}\iiint_T 4z\,\mathrm{d}V &= 4\int_0^1\int_0^{1-z^2}\int_z^{\sqrt{1-y}} z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &= 4\int_0^1\int_0^{1-z^2} z(\sqrt{1-y}-z)\,\mathrm{d}y\,\mathrm{d}z\end{aligned}$$

$$\begin{aligned}&= -4\int_0^1\left(\frac{2z}{3}(1-y)^{\frac{3}{2}}\Big|_0^{1-z^2} + z^2y|_0^{1-z^2}\right)\mathrm{d}z\\ &= -4\int_0^1\left(\frac{2z^4}{3}-\frac{2z}{3}+z^2-z^4\right)\mathrm{d}z\\ &= -4\left(\frac{2}{15}-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}\right)=\frac{4}{15}\,.\end{aligned}$$

■

Exercises

Ex. 3.3. Re-evaluate the integral in Example 3.27 using the opposite order of integration $\iiint_T 4z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ and observe the reduction in algebraic complexity. Trace this fact back to simpler expressions for the integration limits.

3.2.7 Ostrogradsky-Gauss' Divergence theorem

As we have seen multidimensional integration frequently can be quite involved algebraically, especially if the integration region has a complicated shape. Stokes' theorem considered in Section 3.2.5 provided means of reducing the surface integral to a line integral. It turns out that in some cases three-dimensional integration can be linked to a two-dimensional surface integral. This is of practical interest because one of the two kinds of integral is often simpler than the other.

Theorem 3.12 *Let T be a closed* bounded region *in space with a* piecewise smooth orientable boundary surface S *and* $\mathbf{F}(x, y, z)$ *be a vector function that is continuous and has continuous first partial derivatives in some domain containing T. Then*

$$\boxed{\iiint_T \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A},$$

where $\mathbf{n}$ *is the unit normal to the surface.*

While we take this theorem without proof we need to explain various terms used in its formulation in order to understand in what practical situations it can be used.

Definition 3.13 *A region is called closed if all its boundary points are considered to be a part of it.*

Definition 3.14 *A region is called bounded if it can be enclosed in a sphere of a finite radius.*

Definition 3.15 *A surface is called piecewise smooth if it consists of the finite number of smooth segments.*

A sphere is a smooth surface while a cube has a piecewise smooth surface.

The following examples demonstrate that in some cases converting a surface flux integral to a volume integral using the Divergence theorem significantly reduces the amount of algebraic manipulations required to compute a surface integral. This typically occurs when the differentiation of a given vector field leads to simpler expressions.

EXAMPLE **3.28.** *Verify the Divergence theorem for the vector field and unit cube of* EXAMPLE *3.23.*

SOLUTION. *By evaluating the flux integrals over the six cube faces in* EXAMPLE *3.23 we have found that the total flux through the surface of the cube is* $\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A = \frac{7}{4}$. *We now calculate* $\iiint_T \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V$, *where*

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{F} &= \boldsymbol{\nabla}\cdot(x^2y, yz, 3x^3z) = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(3x^3z) \\ &= 2xy + z + 3x^3\,.\end{aligned}$$

$$\begin{aligned}\iiint_T \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V &= \int_0^1\int_0^1\int_0^1 (2xy + z + 3x^3)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1\int_0^1 \left(2xyz + \frac{z^2}{2} + 3x^3z\right)\Bigg|_{z=0}^{z=1} \mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1\int_0^1 \left(2xy + \frac{1}{2} + 3x^3\right)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 \left(xy^2 + \frac{1}{2}y + 3x^3y\right)\Bigg|_{y=0}^{y=1} \mathrm{d}x \\ &= \int_0^1 \left(x + \frac{1}{2} + 3x^3\right)\,\mathrm{d}x = \left(\frac{x^2}{2} + \frac{1}{2}x + 3\frac{x^4}{4}\right)\Bigg|_{x=0}^{x=1} \\ &= \frac{7}{4}\,.\end{aligned}$$

This verifies Ostrogradsky-Gauss' Divergence theorem for the surface of a cube. ■

EXAMPLE **3.29.** *Apply the Divergence theorem to evaluate the surface integral* $\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A$*, where* $\mathbf{F} = (x^2, -(2x-1)y, 4z)$ *and* S *is the surface of the cone* $x^2+y^2 \le z^2$ *and* $0 \le z \le 2$*, see Figure 3.19.*

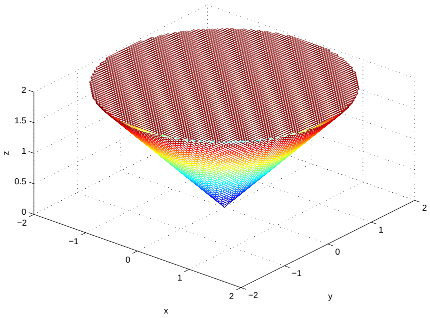


Figure 3.19: The surface of the cone $x^2+y^2 \le z^2$, $0 \le z \le 2$.

SOLUTION.

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A &= \iiint_T \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V \quad \textit{(by the Divergence theorem)}\\
&= \iiint_T (2x-(2x-1)+4)\,\mathrm{d}V \quad \textit{(computing the divergence)}\\
&= \iiint_T 5\,\mathrm{d}V \quad \textit{(simplifying)}\\
&= 5\times(\textit{the volume } V)\\
&= 5\frac{1}{3}2\pi 2^2 \quad \textit{using } \frac{1}{3}\textit{(height)(base area)}\\
&= \frac{40}{3}\pi\,.
\end{aligned}$$

■

EXAMPLE **3.30.** *Calculate the flux of the vector field* $\mathbf{F} = (yz, xz, xy)$ *over the entire surface of the rectangular box with faces at* $x=0$, $x=2$, $y=0$, $y=3$, $z=0$ *and* $z=4$.

SOLUTION. *To calculate the flux as a surface integral we would have to look at each of the six faces and compute the integrals separately as we did in* EXAMPLE 3.23. *However, it is much quicker to transform the surface integral to the volume integral using the Divergence theorem. Taking into account that* $\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$ *we immediately obtain*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A = \iiint_T \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \iiint_T 0\,\mathrm{d}V = 0\,.$$

■

EXAMPLE **3.31.** *Establish the geometrical meaning of the flux integral of a position vector* $\mathbf{r} = (x, y, z)$ *over an arbitrary closed surface satisfying the requirements of the Divergence theorem.*

SOLUTION. $\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{r} = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 1 + 1 + 1 = 3$. *Therefore,*

$$\iint_S \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}A = \iiint_T \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{r}\,\mathrm{d}V = \iiint_T 3\,\mathrm{d}V = 3V\,.$$

We obtain a remarkable result that the volume of an arbitrary region in space whose surface satisfies the requirements of the Divergence theorem is given by

$$\boxed{V = \frac{1}{3}\iint_S \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}A}\,.$$

■

EXAMPLE **3.32.** *Calculate the flux of the vector function* $\mathbf{r} = (x, y, z)$ *over the rectangular box surface defined in* EXAMPLE *3.30.*

SOLUTION. *The volume of the considered rectangular box is* $V = 2 \times 3 \times 4 = 24$. *Therefore, using the result of* EXAMPLE *3.31 we immediately obtain that*

$$\iint_S \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}A = 3V = 72\,.$$

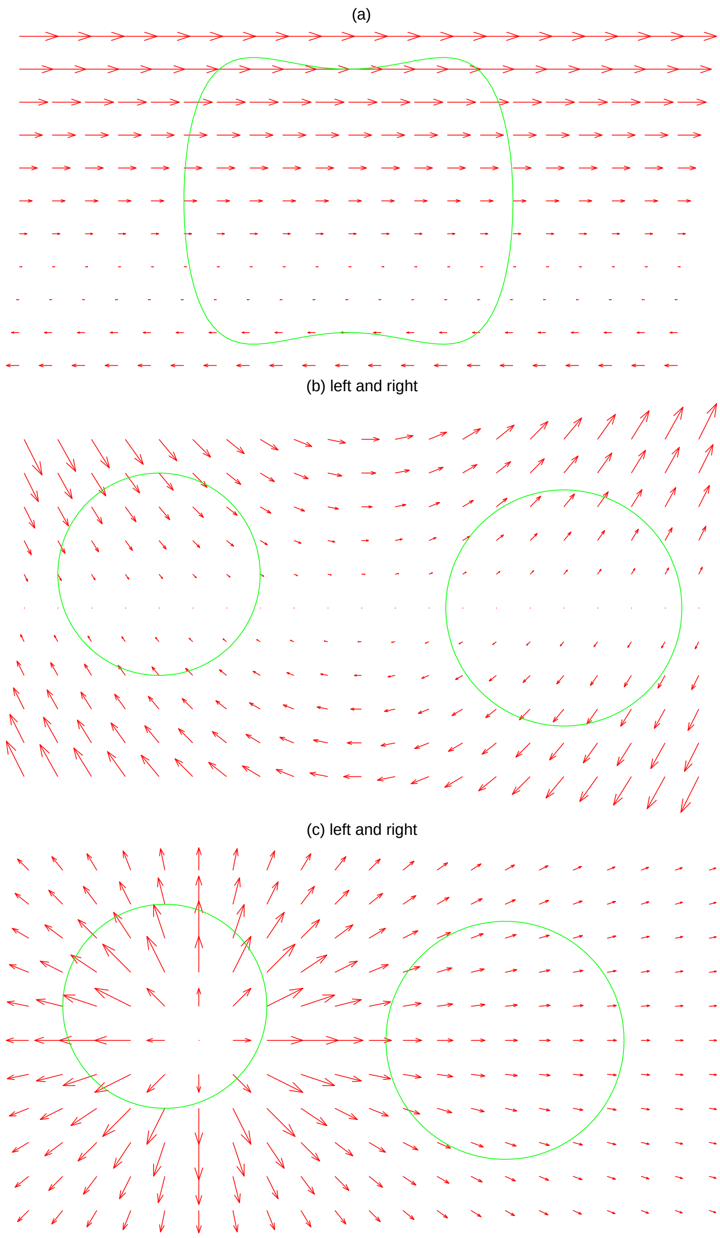
■

The physical meaning of Ostrogradsky-Gauss' Divergence theorem is that the total flux of a vector field through a surface enclosing a region in space is equal to the cumulative intensity of a distributed source (given by the divergence) inside this region.

Observe this in the following exercises.

Exercises

Ex. 3.4. Consider the vector fields below. The shown curves are two-dimensional cuts through closed cylindrical surfaces. Estimate whether the net flux *out* of the enclosed region, $\iint_S \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}A$, is positive, negative or approximately zero. Use Ostrogradsky-Gauss' Divergence theorem to conclude whether these estimates agree with your identification of regions of positive, negative and zero divergence that you made in Ex. 2.3.



Ex. 3.5. You should not only be able to use Stokes' and Ostrogradsky-Gauss' Divergence theorems to transform integrals but also to choose what methods are most appropriate to use in different circumstances. For each of the following integrals outline two different ways of analytically evaluating them: you may need to consider transformations based on any of Stokes' theorem, Ostrogradsky-Gauss' Divergence theorem, or scalar potentials. Evaluate each of the integrals via the "quickest" method of your choice.

(a) $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$, where $\mathbf{F} = (-z, xy, x^2)$ and the curve C goes from $(0,1,2)$ to $(1,0,2)$ along the intersection of the plane $x+y=1$ and the surface $z = xy+2$.

(b) $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$, where $\mathbf{F} = (2xy+z^3, x^2, 3xz^2)$ and the curve C goes from $(1,0,1)$ to $(3,1,2)$ along the intersection of the plane $x-2y=1$ and the parabolic cylinder $z = 1+y^2$.

(c) $\iint_S \mathbf{G}\cdot\mathbf{n}\,\mathrm{d}A$, where $\mathbf{G} = (2y, -z, 2y-1)$ and S is the parabolic bowl $z = x^2+y^2$ for $0 \le z \le 4$ (take the normal to S to point away from the z axis).

(d) $\iint_S (0, y\cos^2 x + y^3, z(\sin^2 x - 3y^2))\cdot\mathbf{n}\,\mathrm{d}A$, where S is the surface of the sphere $x^2+y^2+z^2 = 1$ (with the normal to S pointing outward).

(e) $\iint_S (2z, 0, x-y-z)\cdot\mathbf{n}\,\mathrm{d}A$, where S is the plane triangle with vertices $(1,0,0)$, $(0,2,0)$ and $(0,0,1)$ (with the normal to S pointing upward).

(f) $\iint_S \mathbf{B}\cdot\mathbf{n}\,\mathrm{d}A$, where $\mathbf{B} = (x^2+2x, -2xy, y+z)$ and S is the spherical surface $x^2+y^2+z^2 = 4$ (with the normal to S pointing outward).

3.2.7.1 Divergence theorem and fluid continuity

The Divergence theorem in the fluids context represents a mathematical formalism which is essentially a statement of fundamental physical conservation laws: conservation of mass, energy and momentum. Here we demonstrate this in the example of mass conservation that leads to the so-called *continuity equation*.

Let $\rho(x,y,z,t)$ be the density of (compressible) fluid that in general depends on spatial coordinates x, y, z and time t. Then the mass of fluid contained within an arbitrary fixed spatial region T is $M = \iiint_T \rho\,\mathrm{d}V$. Its rate of change is $\dfrac{\mathrm{d}M}{\mathrm{d}t}$. According to the fundamental mass conservation principle the mass cannot disappear or appear from nowhere, the amount of material that it quantifies can only change if there is a net mass flux through the surface S bounding the region:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\Phi\,,$$

where as we know from Section 3.2.4.4 that

$$\Phi = \iint_S (\rho\mathbf{v}) \cdot \mathbf{n}\,\mathrm{d}A\,.$$

Here $\mathbf{v}(x, u, z, t)$ is the fluid velocity. Remember that by definition the flux through the surface is positive if the vector field (fluid velocity here) has a component in the direction of the outer normal to the surface that is if the fluid leaves the region. In this case the mass conservation principle dictates that $\dfrac{\mathrm{d}M}{\mathrm{d}t} < 0$. Therefore, the minus sign appears in the mass balance equation above. According to the Divergence theorem we can write

$$\Phi = \iiint_T \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\mathbf{v})\,\mathrm{d}V$$

and since region T is fixed and does not move or deform in time we can swap the order of the time differentiation and the spatial integration to obtain

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_T \rho\,\mathrm{d}V = \iiint_T \frac{\partial\rho}{\partial t}\,\mathrm{d}V$$

and, consequently,

$$\iiint_T \frac{\partial\rho}{\partial t}\,\mathrm{d}V = -\iiint_T \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\mathbf{v})\,\mathrm{d}V\,.$$

The integrals in the left- and right-hand sides of the balance equation can be combined so that

$$\iiint_T \left(\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\mathbf{v})\right)\,\mathrm{d}V = 0\,.$$

Since region T is chosen arbitrarily and yet the integral is always equal to zero, we must conclude that the integrand itself must be zero:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\mathbf{v}) = 0\,.$$

This is known as the *continuity equation for compressible fluid* such as air. The density of many fluids such as water can be considered constant under normal conditions. Such fluids are referred to as incompressible and the *continuity equation for incompressible fluids* becomes

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v} = 0\,.$$

Thus, the continuity equation is the mathematical formulation of the physical mass conservation principle and must be satisfied by all physical velocity fields corresponding to realistic fluid flows.

Similar considerations can be applied to the momentum and energy conservation principles to obtain Euler, Navier-Stokes and thermal energy equations that along with the continuity equation describe most of fluid motions.

3.3 Hints to Selected Exercises

Ex. 3.1. Follow the procedure:

- Select a small element of the curve at its start. Decompose the nearest vector into two components: tangent and normal to the curve. If the tangential component is in the direction of motion along the curve, then the local contribution of this component to the integral is positive, if it is against it, then the contribution negative, and if the field vectors are perpendicular to the curve, then the contribution is zero. The magnitude of the contribution is proportional to the magnitude of the tangential component of the field.
- Repeat the above procedure for the rest of the elements along the curve. Keep in mind the sign and estimated magnitude of all local contributions. Then deduce whether their sum is positive, negative or zero. The answer required is about the total integral and not about the local contributions so that you do not have to report the details of the above deduction.

Ex. 3.4. Follow the procedure:

- Select a small piece of boundary and see whether the velocity vectors cross it from inside out or vise versa. In the former case the local contribution to the overall flux out of the volume bounded by the surface is positive, in the latter it is negative.
- Estimate the magnitude of this local flux. Remember that it is proportional to (a) the size of the boundary piece your chose and (b) to the magnitude of the velocity component locally normal to the boundary.
- Repeat for other boundary elements along the complete boundary.
- Estimate whether the sum of all local flux contributions is negative, positive or zero. According to the Divergence theorem the sign of the velocity divergence will be that of the total flux through the complete boundary: positive if more fluid exits the volume than enters it and negative otherwise.

In particular, the divergence is zero in plot (a) as all fluid entering the volume on one side exits it on the other and the divergence is negative (positive) in the left (right) circle in plot (b) as more fluid enters (exits) the volume than exits (enters) it.

Ex. 3.5. (a) $-\dfrac{5}{2}$; (b) 32; (c) 4π; (d) $\dfrac{4\pi}{3}$; (e) 0; (f) 32π.

3.4 Review Exercises

Double Integrals

Ex. 3.6. Evaluate

(a) $\displaystyle\int_1^2\int_0^3 (x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x$, (b) $\displaystyle\int_0^1\int_x^{x^2} (x^2+3y+2)\,\mathrm{d}y\,\mathrm{d}x$,

(c) $\displaystyle\int_1^2\int_y^{y^2} \mathrm{d}x\,\mathrm{d}y$, (d) $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{1+\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$,

(e) $\displaystyle\int_{-2}^2\int_0^{\frac{1}{2}\sqrt{4-x^2}} (x+2)\,\mathrm{d}y\,\mathrm{d}x$, (f) $\displaystyle\int_0^1\int_0^1 e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta$.

Ex. 3.7. Sketch the region of integration and evaluate the following integrals:

(a) $\displaystyle\iint_R \frac{\mathrm{d}A}{xy}$, where R is the region bounded by $x=1$, $y=4$, $x=3$ and $y=1$;

(b) $\displaystyle\iint_R (y+3xy)\,\mathrm{d}A$, where R is the region bounded by $x=0$, $y=2$ and $x=y$;

(c) $\displaystyle\iint_R (x^2+y^2)\,\mathrm{d}A$, where R is the region bounded by $x=0$, $x=3$, $y=0$, $y=x+1$;

(d) $\displaystyle\iint_R (x^2+y^2)\,\mathrm{d}A$, where R is the region bounded by the circle $x^2+y^2=a^2$.

Ex. 3.8. Sketch the region of integration but do not evaluate the following integrals:

(a) $\displaystyle\int_0^1\int_{x^2}^{x} \sin(x+y)\,\mathrm{d}y\,\mathrm{d}x$, (b) $\displaystyle\int_0^1\int_y^{\sqrt{y}} \exp(xy)\,\mathrm{d}x\,\mathrm{d}y$,

(c) $\displaystyle\int_0^{\ln 8}\int_{e^x}^{8} \cos(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x$, (d) $\displaystyle\int_0^{\pi}\int_0^{\sin x} x^2\exp(xy)\,\mathrm{d}y\,\mathrm{d}x$.

Ex. 3.9. Evaluate the following double integrals and then re-evaluate them by reversing the order of integration (verify that the same result is obtained in each case):

(a) $\displaystyle\int_0^2 \int_1^{e^x} \mathrm{d}y\,\mathrm{d}x$; (b) $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \mathrm{d}x\,\mathrm{d}y$;

(c) $\displaystyle\int_0^{\sqrt{2}} \int_{-\sqrt{4-2x^2}}^{\sqrt{4-2x^2}} x\,\mathrm{d}y\,\mathrm{d}x$; (d) $\displaystyle\int_{-2}^1 \int_{x^2+4x}^{3x+2} \mathrm{d}y\,\mathrm{d}x$.

Ex. 3.10. Note that the following integrals cannot be written in terms of standard elementary functions and, thus, evaluate them by reversing the order of integration:

(a) $\displaystyle\int_0^2 \int_y^2 e^{x^2}\,\mathrm{d}x\,\mathrm{d}y$, (b) $\displaystyle\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,\mathrm{d}y\,\mathrm{d}x$.

Ex. 3.11. Evaluate the following integrals by transforming them to polar form:

(a) $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$, (b) $\displaystyle\int_0^2 \int_0^x y\,\mathrm{d}y\,\mathrm{d}x$,

(c) $\displaystyle\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-(x^2+y^2))\,\mathrm{d}y\,\mathrm{d}x$, (d) $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} x^2\,\mathrm{d}y\,\mathrm{d}x$,

(e) $\displaystyle\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (9-\sqrt{x^2+y^2})\,\mathrm{d}y\,\mathrm{d}x$, (f) $\displaystyle\int_{\frac{1}{\sqrt{2}}}^1 \int_{\sqrt{1-x^2}}^x \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$,

(g) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,\mathrm{d}y\,\mathrm{d}x$, (h) $\displaystyle\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,\mathrm{d}y\,\mathrm{d}x$.

Line integrals, scalar potential and Stokes's theorem

Ex. 3.12. Check if the following vector fields have a scalar potential and, if yes, find it.

(a) $\mathbf{F} = (y-2x, x)$, (b) $\mathbf{F} = (4xyz, 2x^2z+3, 2x^2y)$,
(c) $\mathbf{F} = (yz, xz, xy)$, (d) $\mathbf{F} = (2x-z^3, 2yz, y^2-3xz^2)$.

Ex. 3.13. For the following vector fields $\mathbf{F}$ show that they are irrotational, then find a potential function for $\mathbf{F}$ and evaluate the line integral $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$, where C is a path running from the origin to the point indicated.

(a) $\mathbf{F} = (y^2 + 2xz^2, 2xy, 2x^2z)$, $P(2,1,3)$;

(b) $\mathbf{F} = (x + y^2 + 4z, 2xy - 3y - z, 4x - y + 2z)$, $P(3,-2,3)$;

(c) $\mathbf{F} = (4xyz, 2x^2z + 3, 2x^2y)$, $P(2,2,3)$;

(d) $\mathbf{F} = (6xy - 4yz, 3x^2 - 4xz + z^3, 3z^2y - 4xy + 1)$, $P(3,-1,2)$.

Ex. 3.14. C is the triangular path consisting of the straight line segments running from $(0,0)$ to $(1,0)$ to $(1,1)$ and back to $(0,0)$. Evaluate $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ (i) as a line integral and (ii) using Stokes's theorem if

(a) $\mathbf{F} = (y, x^2)$, (b) $\mathbf{F} = (y^3 - y, 3xy^2 - x)$,
(c) $\mathbf{F} = (3x^2y - y^2, x^3 - 2xy)$, (d) $\mathbf{F} = (x^2y + y^2, x^3 + 4xy)$.

Ex. 3.15. C is the square path consisting of the straight line segments running from $(1,1)$ to $(2,1)$ to $(2,2)$ to $(1,2)$ and back to $(1,1)$. Evaluate $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ (i) as a line integral and (ii) using Stokes's theorem if

(a) $\mathbf{F} = (y, x^2)$, (b) $\mathbf{F} = (y^3 - y, 3xy^2 - x)$,
(c) $\mathbf{F} = (3x^2y - y^2, x^3 - 2xy)$, (d) $\mathbf{F} = (x^2y + y^2, x^3 + 4xy)$.

Ex. 3.16. C is the anti-clockwise circular path $x^2 + y^2 = 4$ starting and ending at $(0,2)$. Evaluate the line integrals $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ if
(a) $\mathbf{F} = (y, -x)$, (b) $\mathbf{F} = (xy, xy)$, (c) $\mathbf{F} = (3x, -2y)$.

Triple integrals

Ex. 3.17. The moment of inertia about the yz plane, I_{yz}, of a solid body T of density $\rho(x,y,z)$ is given by $I_{yz} = \iiint_T x^2\rho(x,y,z)\,\mathrm{d}V$. Find I_{yz} for the cuboid $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$ with density $\rho(x,y,z) = xy^2$.

Ex. 3.18. Find $I = \int_0^1 \int_0^{1-x} \int_0^x xy^2z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

Ostrogradsky-Gauss Divergence theorem

Ex. 3.19. Use the Divergence theorem to evaluate the integral $\iint_S \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}A$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Hint: the Cartesian volume element $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ becomes $r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$ in spherical coordinates and $r = 1$, $0 \le \phi < 2\pi$, $0 \le \theta < \pi$ parameterise the full surface of the sphere.

Ex. 3.20. Using the Divergence theorem, find $\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A$, where the region of integration T is the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ and $\mathbf{F} = (x, y, z)$.

Ex. 3.21. Using the Divergence theorem, find $\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A$, where the region of integration T is the rectangular parallelepiped $0 \le x \le 1$, $0 \le y \le 3$, $0 \le z \le 2$ and $\mathbf{F} = (x^2, z, y)$.

3.5 Answers to Selected Review Exercises

Double Integrals

Ex. 3.6.

(a) $$\int_1^2 \int_0^3 (x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x = \int_1^2 \left(x^2y + \frac{1}{3}y^3\right)\Big|_{y=0}^{y=3}\mathrm{d}x$$
$$= \int_1^2 (3x^2+9)\,\mathrm{d}x = (x^3+9x)|_1^2 = 16\,,$$

(b) $$\int_0^1 \int_x^{x^2} (x^2+3y+2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left(x^2y + \frac{3}{2}y^2 + 2y\right)\Big|_{y=x}^{y=x^2}\mathrm{d}x$$
$$= \int_0^1 \left(\frac{5}{2}x^4 - x^3 + \frac{1}{2}x^2 - 2x\right)\mathrm{d}x = -\frac{7}{12}\,,$$

(c) $$\int_1^2 \int_y^{y^2} \mathrm{d}x\,\mathrm{d}y = \int_1^2 x|_{x=y}^{x=y^2}\,\mathrm{d}y = \int_1^2 (y^2-y)\,\mathrm{d}y = \frac{5}{6}\,,$$

(d) $$\int_0^{\frac{\pi}{2}} \int_0^{1+\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}r^2\Big|_{r=0}^{r=1+\cos\theta}\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta = 1+\frac{3\pi}{8}\,,$$

(e) $$\int_{-2}^2 \int_0^{\frac{1}{2}\sqrt{4-x^2}} (x+2)\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\int_{-2}^2 (x+2)\sqrt{4-x^2}\,\mathrm{d}x$$
(use substitution $x = 2\sin\theta$)
$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)\cos^2\theta\,\mathrm{d}\theta = 2\pi\,,$$

(f) $$\int_0^1 \int_0^1 e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta = -\frac{1}{2}\int_0^1 e^{-r^2}|_{r=0}^{r=1}\,\mathrm{d}\theta = \frac{1}{2}\int_0^1 (1-e^{-1})\,\mathrm{d}\theta$$
$$= \frac{1}{2}(1-e^{-1})\,.$$

Ex. 3.7. The sketches are very simple and are not given here. The integrals become:

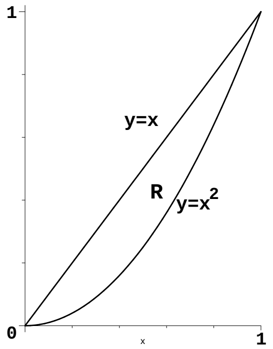
(a) $$\int_1^3 \int_1^4 \frac{1}{xy}\,\mathrm{d}y\,\mathrm{d}x = \ln 3 \cdot \ln 4;$$

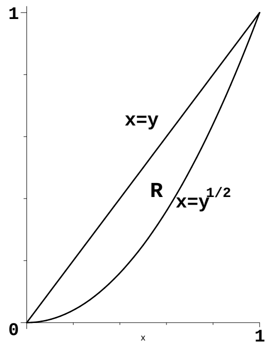
(b) $$\int_0^2 \int_0^y (y+3xy)\,\mathrm{d}x\,\mathrm{d}y = \frac{26}{3};$$

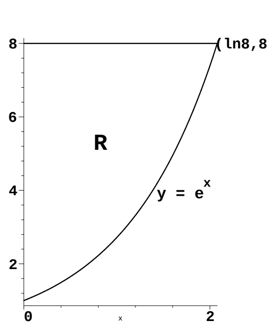
(c) $$\int_0^3 \int_0^{x+1} (x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x = \frac{101}{2};$$

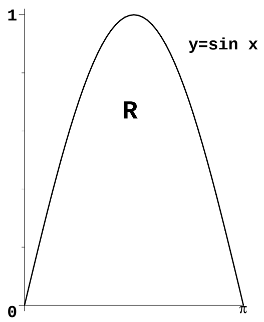
(d) it is more natural to use polar coordinates here, since the region is a circle about the origin so that $\displaystyle\int_0^{2\pi}\int_0^a r^2 r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi a^4}{2}$.

Ex. 3.8.

(a) 

(b) 

(c) 

(d) 

Ex. 3.9.

$$\text{(a)}\quad \int_0^2\int_1^{e^x}\mathrm{d}y\,\mathrm{d}x = \int_1^{e^2}\int_{\ln y}^2\mathrm{d}x\,\mathrm{d}y = e^2-3;$$

$$\text{(b)}\quad \int_0^1\int_{\sqrt{y}}^1\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^{x^2}\mathrm{d}y\,\mathrm{d}x = \frac{1}{3};$$

$$\text{(c)}\quad \int_0^{\sqrt{2}}\int_{-\sqrt{4-2x^2}}^{\sqrt{4-2x^2}} x\,\mathrm{d}y\,\mathrm{d}x = \int_{-2}^2\int_0^{\sqrt{2-\frac{y^2}{2}}} x\,\mathrm{d}x\,\mathrm{d}y = \frac{8}{3};$$

$$\text{(d)}\quad \int_{-2}^1\int_{x^2+4x}^{3x+2}\mathrm{d}y\,\mathrm{d}x = \int_{-4}^5\int_{\frac{1}{3}(2-y)}^{-2+\sqrt{y^2+4}}\mathrm{d}x\,\mathrm{d}y = \frac{9}{2}.$$

Ex. 3.10.

$$\text{(a)}\quad \int_0^2\int_y^2 e^{x^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^2\int_0^x e^{x^2}\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}(e^4-1),$$

$$\text{(b)}\quad \int_0^\pi\int_x^\pi \frac{\sin y}{y}\,\mathrm{d}y\,\mathrm{d}x = \int_0^\pi\int_0^y \frac{\sin y}{y}\,\mathrm{d}x\,\mathrm{d}y = 2.$$

Ex. 3.11.

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi}{8}$;

(b) $\displaystyle\int_0^{\frac{\pi}{4}} \int_0^{2\sec\theta} r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta = \frac{4}{3}$;

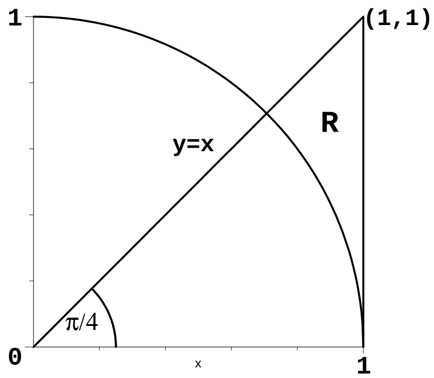
(c) $\displaystyle\int_0^{2\pi} \int_0^2 (4r - r^3) \,\mathrm{d}r\,\mathrm{d}\theta = 8\pi$.

(d) The integration region is the top half of a unit circle centred at $(1,0)$. This means that θ varies from 0 to $\frac{\pi}{2}$ while r ranges from 0 to the value given by solving the equation $y = \sqrt{2x - x^2}$. After squaring both sides, this equation transforms in polars to $r^2\sin^2\theta = 2r\cos\theta - r^2\cos^2\theta$ or $r = 2\cos\theta$. Therefore,

$$\int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3\cos^2\theta \,\mathrm{d}r\,\mathrm{d}\theta = \frac{5\pi}{8}\,.$$

(e) $\displaystyle\int_0^{2\pi} \int_0^3 (9r - r^2) \,\mathrm{d}r\,\mathrm{d}\theta = 63\pi$.

(f) The integration region is shown below.



After performing the r integration we obtain $\displaystyle\frac{1}{3}\int_0^{\frac{\pi}{4}} (\sec^3\theta - 1)\,\mathrm{d}\theta$. The -1 part integrates to give $-\dfrac{\pi}{12}$. The remaining integral is $\displaystyle\frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}\theta}{\cos^3\theta}$. Make a substitution $u = \tan\theta$ to rewrite the integral as $\displaystyle\frac{1}{3}\int_0^1 \sqrt{1+u^2}\,\mathrm{d}u$. Make another substitution $u = \sinh v$ and note that $v = 0$ when $u = 0$ and $v = \ln(1+\sqrt{2})$ when

$u = 1$ to obtain

$$\begin{aligned}
\frac{1}{3}\int_0^1 \sqrt{1+u^2}\,\mathrm{d}u &= \frac{1}{3}\int_0^{\ln(1+\sqrt{2})} \cosh^2 v\,\mathrm{d}v \\
&= \frac{1}{6}\int_0^{\ln(1+\sqrt{2})} (1+\cosh(2v))\,\mathrm{d}v \\
&= \frac{1}{6}\left(v + \frac{1}{2}\sinh(2v)\right)\Big|_0^{\ln(1+\sqrt{2})} .
\end{aligned}$$

Thus,

$$\int_0^{\frac{\pi}{4}} \int_1^{\sec\theta} r^2\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{6}\sqrt{2} + \frac{1}{6}\ln(1+\sqrt{2}) - \frac{\pi}{12} .$$

(g) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^a re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi}{4}(1-e^{-a^2})$;

(h) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^\infty re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi}{4}$.

Line integrals, scalar potential and Stokes's theorem

Ex. 3.12. In all cases $\nabla \times F = \mathbf{0}$, thus scalar potential exist.

(a) $\dfrac{\partial\phi}{\partial x} = y - 2x$, $\dfrac{\partial\phi}{\partial y} = x$, thus $\phi = xy - x^2 + C$.

(b) $\dfrac{\partial\phi}{\partial x} = 4xyz$, $\dfrac{\partial\phi}{\partial y} = 2x^2z+3$, $\dfrac{\partial\phi}{\partial z} = 2x^2y$, thus $\phi = 2x^2yz+3y+C$.

(c) $\dfrac{\partial\phi}{\partial x} = yz$, $\dfrac{\partial\phi}{\partial y} = xz$, $\dfrac{\partial\phi}{\partial z} = xy$, thus $\phi = xyz + C$.

(d) $\dfrac{\partial\phi}{\partial x} = 2x - z^3$, $\dfrac{\partial\phi}{\partial y} = 2yz$, $\dfrac{\partial\phi}{\partial z} = y^2 - 3xz^2$, thus $\phi = x^2 + y^2z - xz^3 + C$.

Ex. 3.13. In all four cases, the irrotationality is proved by showing that $\nabla \times \mathbf{F} = 0$. Once the potential functions ϕ are the path independent line integrals are calculated as $\int_O^P \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \phi(P) - \phi(O)$.

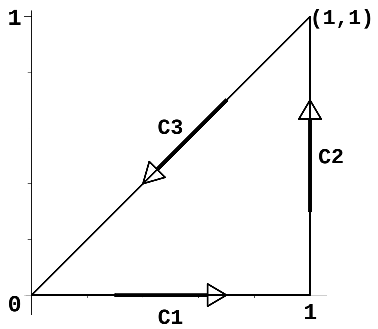
(a) $\phi = xy^2 + x^2z^2 + C$;
$$\int_O^P \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \phi(2,1,3) - \phi(0,0,0) = 38.$$

(b) $\phi = \frac{1}{2}x^2 + xy^2 + 4xz - \frac{3}{2}y^2 - yz + z^2 + C$;
$$\int_O^P \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \phi(3,-2,3) - \phi(0,0,0) = \frac{123}{2}.$$

(c) $\phi = 2x^2yz + 3y + C$;
$\displaystyle\int_O^P \mathbf{F}\cdot d\mathbf{r} = \phi(2,2,3) - \phi(0,0,0) = 54.$

(d) $\phi = 3x^2y - 4xyz + yz^3 + z$;
$\displaystyle\int_O^P \mathbf{F}\cdot d\mathbf{r} = \phi(3,-1,2) - \phi(0,0,0) = -9.$

Ex. 3.14. The integration path is as follows:



It can be parameterised as $\mathbf{r} = (t,0)$ and $d\mathbf{r} = (1,0)\,dt$ along C1, $\mathbf{r} = (1,t)$ and $d\mathbf{r} = (0,1)\,dt$ along C2 and $\mathbf{r} = (1-t, 1-t)$ and $d\mathbf{r} = (-1,1)\,dt$ along C3, where $0 \le t \le 1$. However, it is more convenient to replace C3 with $-$C4 along which $\mathbf{r} = (t,t)$ and $d\mathbf{r} = (1,1)\,dt$. The unit normal to the triangle enclosed by C is $\mathbf{n} = (0,0,1)$.

(a) $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 0\,dt + \int_0^1 1\,dt - \int_0^1 (t+t^2)\,dt = \frac{1}{6}$. Using Stokes's theorem: $\boldsymbol{\nabla}\times\mathbf{F} = (0,0,2x-1)$ and $\displaystyle\int_0^1\int_0^x (2x-1)\,dy\,dx = \frac{1}{6}$.

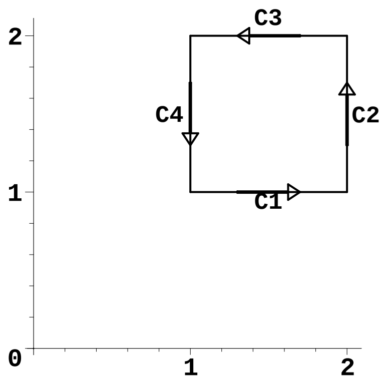
(b) $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 0\,dt + \int_0^1 (3t^2-1)\,dt - \int_0^1 (4t^3-2t)\,dt = 0$. Using Stokes's theorem: $\boldsymbol{\nabla}\times\mathbf{F} = 0$ and so is the integral.

(c) $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 0\,dt + \int_0^1 (1-2t)dt - \int_0^1 (4t^3-3t^2)\,dt = 0$. Using Stokes's theorem: $\boldsymbol{\nabla}\times\mathbf{F} = 0$ and so is the integral.

(d) $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 0\,dt + \int_0^1 (1+4t)\,dt - \int_0^1 (2t^3+5t^2)\,dt = \frac{5}{6}$. Using Stokes's theorem: $\boldsymbol{\nabla}\times\mathbf{F} = (0,0,2x^2+2y)$ and $\displaystyle\int_0^1\int_0^x (2x^2+2y)\,dy\,dx = \frac{5}{6}$.

Ex. 3.15. The integration path is as follows:

It can be parameterised as $\mathbf{r} = (t,1)$ and $d\mathbf{r} = (1,0)\,dt$ along C1, $\mathbf{r} = (2,t)$ and $d\mathbf{r} = (0,1)\,dt$ along C2, $\mathbf{r} = (3-t, 2)$ and $d\mathbf{r} = (-1,0)\,dt$



along C3 and $\mathbf{r} = (1, 3-t)$ and $\mathrm{d}\mathbf{r} = (0, -1)\,\mathrm{d}t$ along C4, where $1 \le t \le 2$. The unit normal to the square enclosed by C is $\mathbf{n} = (0, 0, 1)$.

(a) $\displaystyle\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_1^2 1\,\mathrm{d}t + \int_1^2 4\,\mathrm{d}t - \int_1^2 2\,\mathrm{d}t - \int_1^2 \mathrm{d}t = 2$. Using Stokes's theorem: $\boldsymbol{\nabla}\times\mathbf{F} = (0, 0, 2x-1)$ and $\displaystyle\int_1^2\int_1^2 (2x-1)\,\mathrm{d}y\,\mathrm{d}x = 2$.

(b) $\displaystyle\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_1^2 0\,\mathrm{d}t + \int_1^2 (6t^2 - 2)\,\mathrm{d}t - \int_1^2 6\,\mathrm{d}t - \int_1^2 (3(3-t)^2 - 1)\,\mathrm{d}t = 0$. Using Stokes's theorem: $\boldsymbol{\nabla}\times\mathbf{F} = 0$ and so is the integral.

(c) $\displaystyle\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_1^2 (3t^2 - 1)\,\mathrm{d}t + \int_1^2 (8-4t)\,\mathrm{d}t - \int_1^2 (6(3-t)^2 - 4)\,\mathrm{d}t - \int_1^2 (1 - 2(3-t))\,\mathrm{d}t = 0$. Using Stokes's theorem: $\boldsymbol{\nabla}\times\mathbf{F} = 0$ and so is the integral.

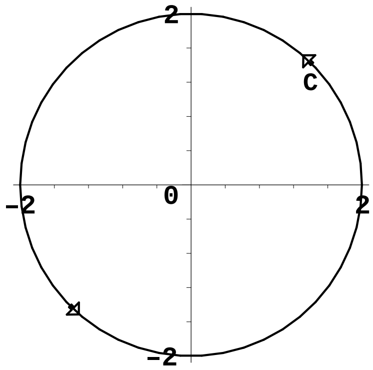
(d) $\displaystyle\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_1^2 (t^2 + 1)\,\mathrm{d}t + \int_1^2 (8+8t)\,\mathrm{d}t - \int_2^1 (2(3-t)^2 + 4)\,\mathrm{d}t - \int_1^2 (1 + 4(3-t))\,\mathrm{d}t = \frac{23}{3}$. Using Stokes's theorem $\boldsymbol{\nabla}\times\mathbf{F} = (0, 0, 2x^2 + 2y)$ and $\displaystyle\int_1^2\int_1^2 (2x^2 + 2y)\,\mathrm{d}y\,\mathrm{d}x = \frac{23}{3}$.

Ex. 3.16. The integration path is as follows:

It can be parameterised as $\mathbf{r} = (2\cos\theta, 2\sin\theta)$ and $\mathrm{d}\mathbf{r} = (-2\sin\theta, 2\cos\theta)\,\mathrm{d}\theta$, $0 \le \theta \le 2\pi$.

(a) $\displaystyle\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} -4\,\mathrm{d}\theta = -8\pi$.

(b) $\displaystyle\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 8\int_0^{2\pi} (\cos^2\theta\sin\theta - \sin^2\theta\cos\theta)\,\mathrm{d}\theta = 0$.



(c) $\displaystyle\oint_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = 4\int_0^{2\pi}\sin(2\theta)\,\mathrm{d}\theta = 0.$

Triple integrals

Ex. 3.17. 2.

Ex. 3.18. $I = \dfrac{1}{2}\displaystyle\int_0^1\int_0^{1-x} x^3y^2\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{6}\int_0^1 x^3(1-x)^3\,\mathrm{d}x = \frac{1}{840}.$

Ostrogradsky-Gauss Divergence theorem

Ex. 3.19.

$$\begin{aligned}\iint_S \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}A &= \iiint_T \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{r}\,\mathrm{d}V = \iiint_T 3\,\mathrm{d}V \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 3r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi\end{aligned}$$

or, alternatively, three times the volume of the sphere.

Ex. 3.20. $\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F} = 3$ and we need to evaluate the integral

$$\int_0^1\int_0^1\int_0^1 3\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 3\,.$$

Ex. 3.21. $\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F} = 2x$ and we need to evaluate the integral

$$\int_0^1\int_0^3\int_0^2 2x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 6\,.$$

Module 4

Introduction to Complex Analysis

Module contents

To proceed with the studies of this module you are expected to be fluent with the previously studied material reviewed in Appendix A.4.

4.1 Functions of a Complex Variable

In this section we will generalise the familiar concept of a real-valued function to that of a *function of a complex variable* $z = x + \mathrm{i}y$, $w = f(z)$. By that we understand that the argument of a function is now a complex number. In general this will result in the function value that is also complex.

Example 4.1.

(a) $w = (1 + 2\mathrm{i})z$,

(b) $w = (1 + 2\mathrm{i})z + (3 + 4\mathrm{i})$,

(c) $w = z^2$,

(d) $w = \dfrac{1}{z}$,

(e) $w = \dfrac{2z + 3}{z - 1}$,

(f) $w = e^z$.

It is usual to write $w = u + \mathrm{i}v$, so that u is the real part of w and v is the imaginary part. These are functions of x and y, i.e. $u = u(x, y)$, $v = v(x, y)$. We find expressions for the corresponding u and v for functions given in Example 4.1 below.

Example 4.2.

(a) *If* $w = (1 + 2\mathrm{i})z$, *then* $w = u + \mathrm{i}v = (1 + 2\mathrm{i})z$ *so that* $u + \mathrm{i}v = (1+2\mathrm{i})(x+y\mathrm{i}) = x+y\mathrm{i}+2x\mathrm{i}-2y = (x-2y)+\mathrm{i}(2x-y)$. *Equating the real and imaginary parts we obtain* $u = x-2y$ *and* $v = y+2x$.

(b) *If* $w = (1 + 2\mathrm{i})z + (3 + 4\mathrm{i})$, *then* $w = u + \mathrm{i} = x - 2y + 3 + \mathrm{i}(y + 4)$ *so that* $u = x - 2y + 3$ *and* $v = y + 4$.

(c) *If* $w = z^2$, *then* $w = u + \mathrm{i}v = z^2 = (x + y\mathrm{i})^2 = x^2 - y^2 + 2xy\mathrm{i}$ *so that equating the real and imaginary parts we obtain* $u = x^2 - y^2$ *and* $v = 2xy$.

(d) *If* $w=\dfrac{1}{z}$, *then* $w=u+\mathrm{i}v=\dfrac{1}{z}=\dfrac{1}{x+y\mathrm{i}}\dfrac{x-y\mathrm{i}}{x-y\mathrm{i}}=\dfrac{x-y\mathrm{i}}{x^2+y^2}$ *so that* $u=\dfrac{x}{x^2+y^2}$ and $v=\dfrac{-y}{x^2+y^2}$.

(e) *If* $w=\dfrac{2z+3}{z-1}=\dfrac{(2x+3+2\mathrm{i}y)(x-1-\mathrm{i}y)}{(x-1+\mathrm{i}y)(x-1-\mathrm{i}y)}$ $=\dfrac{(2x+3+2\mathrm{i}y)(x-1-\mathrm{i}y)}{(x-1)^2+y^2}=\dfrac{(2x+3)(x-1)+2y^2-5\mathrm{i}y}{(x-1)^2+y^2}$ *so that* $u=\dfrac{(2x+3)(x-1)+2y^2}{(x-1)^2+y^2}$ and $v=\dfrac{-5y}{(x-1)^2+y^2}$.

(f) *If* $w=e^z$, *then* $u+\mathrm{i}v=e^{x+\mathrm{i}y}=e^xe^{\mathrm{i}y}=e^x(\cos y+\mathrm{i}\sin y)$ *so that* $u=e^x\cos y$ *and* $v=e^x\sin y$.

4.1.1 Differentiable functions of a complex variable

The derivative of a function of a real variable $f(x)$ is defined by

$$\frac{\mathrm{d}f}{\mathrm{d}x}=f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}\,.$$

We can apply this to a function of a complex variable $w=f(z)$

$$\frac{\mathrm{d}w}{\mathrm{d}z}=f'(z)=\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}\,.$$

For example, if $f(z)=z^2$ then

$$f'(z)=\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}=\frac{(z+\delta z)^2-z^2}{\delta z}=\lim_{\delta z\to 0}(2z+\delta z)=2z\,.$$

All results for differentiation of *polynomial functions* carry over for functions of a complex variable and so do the *product rule*, the *quotient rule* and the *chain rule*.

However, not all functions of a complex variable are differentiable. An example of a non-differentiable function is $f(z)=\bar{z}$. Indeed, when evaluating the limit

$$f'(z)=\lim_{\delta z\to 0}\frac{\overline{(z+\delta z)}-\bar{z}}{\delta z}=\frac{\overline{\delta z}}{\delta z}$$

we can let δz approach 0 in various ways. If we choose $\delta z=\delta x$ (keeping y fixed) we have $\overline{\delta z}=\delta x$ so that $\dfrac{\overline{\delta z}}{\delta z}=1$. On the other hand, if we let $\delta z=\mathrm{i}\delta y$ (keeping x fixed) then we have $\overline{\delta z}=-\mathrm{i}\delta y$ so that $\dfrac{\overline{\delta z}}{\delta z}=-1$. Thus we obtain different results for the limit depending on how we approach the value $\delta z=0$. This means that the limit does not exist i.e. function $f(z)=\bar{z}$ is not differentiable and $f'(z)$ does not exist.

4.1.2 Cauchy-Riemann conditions

Suppose $w = f(z) = f(x + \mathrm{i}y)$ is a function of a complex variable and that $\mathrm{Re}(w) = u$ and $\mathrm{Im}(w) = v$. For example, if $w = z^2$ then we have $w = (x + y\mathrm{i})^2 = x^2 - y^2 + 2xy\mathrm{i}$ so that $u = x^2 - y^2$ and $v = 2xy$.

Let $f(z)$ be differentiable. We can consider the limit $\lim\limits_{\delta z\to 0} \dfrac{f(z+\delta z) - f(z)}{\delta z}$ as in Section 4.1.1 by letting $\delta z = \delta x$ and $\delta z = \mathrm{i}\delta y$. In the former case we obtain the expression

$$\begin{aligned}
&\lim_{\delta z\to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \\
&\qquad = \lim_{\delta x\to 0} \frac{u(x+\delta x, y) + \mathrm{i}v(x+\delta x, y) - (u(x,y) + \mathrm{i}v(x,y)}{\delta x}\,, \\
&f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\,.
\end{aligned}$$

In the second case we get

$$\begin{aligned}
&\lim_{\delta z\to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \\
&\qquad = \lim_{\delta y\to 0} \frac{u(x, y+\delta y) + \mathrm{i}v(x, y+\delta y) - (u(x,y) + \mathrm{i}v(x,y)}{\mathrm{i}\delta y} \\
&\qquad = \frac{1}{\mathrm{i}}\left(\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y}\right) = -\mathrm{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\,, \\
&f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}\,.
\end{aligned}$$

For the function to be differentiable the two expressions must be equal:

$$\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}\,,$$

so that by equating the real and imaginary parts we obtain the pair of equations

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}\,. \tag{4.1}$$

These are the *Cauchy-Riemann conditions*. The real and imaginary parts of a function of a complex variable must satisfy these equations if it is differentiable.

Definition 4.1 (analytic function) *A function satisfying Cauchy-Riemann conditions is called analytic.*

Example **4.3.** *Show that the function $f(z) = z^2$ is analytic everywhere.*

Solution. *We have $w = z^2 = x^2 - y^2 + 2xy\mathrm{i}$ so that $u = x^2 - y^2$ and $v = 2xy$. Then $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial v}{\partial y} = 2x$, so that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$. Also $\dfrac{\partial u}{\partial y} = -2y$ and $\dfrac{\partial v}{\partial x} = 2y$ so that $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. Thus u and v satisfy the Cauchy-Riemann conditions. We can also obtain an expression for $f'(z) = \dfrac{\partial u}{\partial x} + \mathrm{i}\dfrac{\partial v}{\partial x} = 2x + 2y\mathrm{i} = 2z$, exactly as expected from the case of functions of a real variable.* ■

Example **4.4.** *Show that the function $f(z) = \dfrac{1}{z}$ is analytic everywhere except at the origin and find an expression for $f'(z)$.*

Solution. *We have $w = \dfrac{1}{z} = \dfrac{x - y\mathrm{i}}{x^2 + y^2}$ so that $u = \dfrac{x}{x^2 + y^2}$ and $v = -\dfrac{y}{x^2 + y^2}$. Then*

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{x^2 + y^2 - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}\,,\\
\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{-y}{x^2 + y^2}\right] = -\frac{x^2 + y^2 - y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}\,.
\end{aligned}
$$

Thus $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$. Next,

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{x}{x^2 + y^2}\right] = \frac{\partial}{\partial y}\left[x(x^2 + y^2)^{-1}\right] = -x(x^2 + y^2)^{-2}(2y)\\
&= -\frac{2xy}{(x^2 + y^2)^2}\,,\\
\frac{\partial v}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{-y}{x^2 + y^2}\right] = \frac{\partial}{\partial x}\left[-y(x^2 + y^2)^{-1}\right] = y(x^2 + y^2)^{-2}(2x)\\
&= \frac{2xy}{(x^2 + y^2)^2} = -\frac{\partial u}{\partial y}\,.
\end{aligned}
$$

Again we see that the Cauchy-Riemann conditions are satisfied and we have

$$
f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{2xy\mathrm{i}}{(x^2 + y^2)^2}\,.
$$

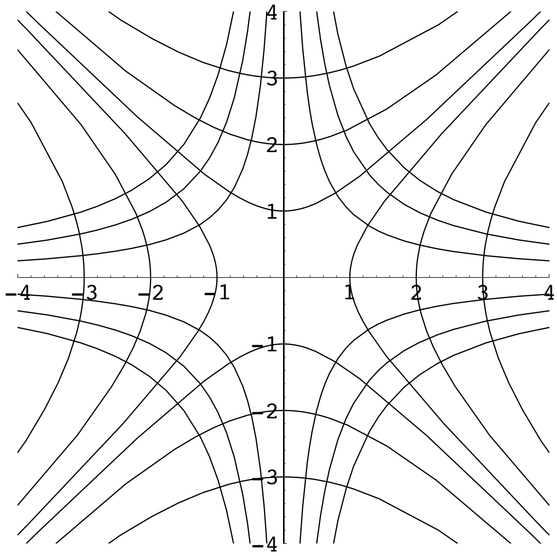


Figure 4.1: The curves $x^2 - y^2 =$ const. and $xy =$ const.

It is not obvious what function of z this is, but we do know that $f'(z) \equiv f'(x + y\mathrm{i})$ so that if we set $y = 0$ we obtain $f'(z) = f'(x)$ i.e.

$$f'(x + 0\mathrm{i}) = f'(x) = \frac{0^2 - x^2}{(x^2 + 0^2)^2} + \frac{2x0\mathrm{i}}{(x^2 + 0^2)^2} = \frac{-x^2}{x^4} = -\frac{1}{x^2}\,.$$

Thus $f'(x) = -\dfrac{1}{x^2}$ *and* $f'(z) = -\dfrac{1}{z^2}$. ■

If $f(z) = u(x, y) + \mathrm{i}v(x, y)$ is analytic then the curves $u(x, y) =$ const. and $v(x, y) =$ const. are orthogonal to each other.

For example, for the function $f(z) = z^2$ we have $u = x^2 - y^2$ and $v = 2xy$. The families of curves $x^2 - y^2 =$ const. and $2xy =$ const. are mutually orthogonal, see Figure 4.1. To see this in general, suppose that $f(z) = u(x, y) + \mathrm{i}v(x, y)$ is analytic. Consider the families of level curves $u(x, y) =$const. and $v(x, y) =$const. As was discussed in Section 2.2.6 the direction perpendicular to level curves is given by the direction of gradient. Therefore, to demonstrate the orthogonality of the level curves we need to show that $\nabla u \cdot \nabla v = 0$: if the gradient directions are perpendicular, so are the directions of tangents to the level curves. The orthogonality of the gradient directions is easily established by taking into account Cauchy-Riemann conditions (4.1):

$$\begin{aligned}\nabla u \cdot \nabla v &= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}\right) \\ &= -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = 0\,.\end{aligned}$$

4.1.3 Analytic functions in fluids

Inspect Figure 4.1 to realise that we have actually seen it already in a different context: compare it with Figure 2.7. Lines $\mathrm{Re}(z^2) = u(x, y) = x^2 - y^2$ =const. are the equipotential lines for a vector field representing a fluid flow in a corner. Comparison of the two figures also confirms that the velocity vectors that are given by $\mathbf{v} = \nabla u$ are tangent to the lines $\mathrm{Im}(z^2) = xy$ =const. These lines are called *streamlines*: a fluid flows along them without ever crossing them. Thus both u and v functions in the general representation of a function of complex variable have a straightforward physical meaning. In the context of two-dimensional fluid flows (and general potential vector fields) $f(z) = u(x, y) + \mathrm{i}v(x, y)$ is *complex potential*, where $u(x, y)$ is a potential of a velocity vector field and $v(x, y)$ is referred to as a *stream function*: its level curves are streamlines.

4.1.4 Harmonic functions

Suppose u and v satisfy the Cauchy-Riemann conditions $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. Then differentiating the first of them with respect to x gives $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y}$ and differentiating the second one with respect to y produces $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial y \partial x}$. However, $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$. Therefore, we must have $\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2}$ or, equivalently,

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}\,.$$

This equation is known as *Laplace's equation* (after a French mathematician, physicist and astronomer Pierre-Simon, marquis de Laplace (1749–1827)).

Definition 4.2 (harmonic function) *Functions satisfying Laplace's equation are called harmonic.*

We have just shown that the real part of an analytic function is harmonic. In the same way we can show that the imaginary part of an analytic function is harmonic.

Suppose u and v are harmonic functions satisfying the Cauchy-Riemann conditions. Then function $f(z)$ defined by $f(z) = u + \mathrm{i}v$ will be an analytic function of z.

If we are given a harmonic function u we can use the Cauchy-Riemann equations to find a *conjugate harmonic function* and construct an analytic function $f(z) = u + \mathrm{i}v$.

EXAMPLE **4.5.** *Show that the function $u = x^3 - 3xy^2 + 4xy$ is harmonic. Find a conjugate harmonic function of u and express $f = u + \mathrm{i}v$ as a function of z.*

SOLUTION. *We have $\dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 + 4y$ and $\dfrac{\partial^2 u}{\partial x^2} = 6x$, while $\dfrac{\partial u}{\partial y} = -6xy + 4x$ and $\dfrac{\partial^2 u}{\partial y^2} = -6x$. Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and u is harmonic.*

To find a conjugate harmonic function v we must have $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2 + 4y$. Integrating this with respect to y gives $v = 3x^2y - y^3 + 2y^2 + g(x)$. To determine g we differentiate this result with respect to x to obtain $\dfrac{\partial v}{\partial x} = 6xy + \dfrac{\mathrm{d}g}{\mathrm{d}x} = -\dfrac{\partial u}{\partial y} = -(-6xy + 4x)$. Thus $\dfrac{\mathrm{d}g}{\mathrm{d}x} = -4x$ and $g(x) = -2x^2 + c$. Then the conjugate harmonic function of u is

$$v = 3x^2y - y^3 + 2y^2 - 2x^2 + c$$

and

$$f(x + \mathrm{i}y) = u + \mathrm{i}v = x^3 - 3xy^2 + 4xy + (3x^2y - y^3 + 2y^2 - 2x^2 + c)\mathrm{i}\,.$$

While this answer is correct it would not be convenient to use in any further computations due to its algebraic complexity. Therefore, it is preferable to rewrite it in terms of a single complex variable z rather than have it as a function of two real variables x and y. Note that whenever $y = 0$, $z = x + \mathrm{i}0 = x$. Thus by setting $y = 0$ in the above expression we obtain $f(x) = x^3 - 2x^2\mathrm{i} + \mathrm{i}c$, which is valid along the real axis in the complex plane. Then to obtain the expression valid for all complex numbers we simply need to replace x with z: $f(z) = z^3 - 2z^2\mathrm{i} + \mathrm{i}c$, where c is an arbitrary real constant. This is a much more compact form of the answer. ■

4.1.5 Exercises

Ex. 4.1. Find the real and imaginary parts of the functions

1. $w = (3-2\mathrm{i})z + (5+7\mathrm{i})$ 2. $w = (3+4\mathrm{i})z^2$ 3. $w = \dfrac{z-1}{z}$
4. $w = \dfrac{z-1}{z}$ 5. $w = z^3$ 6. $w = (2+5\mathrm{i})z^3$
7. $w = z^4$ 8. $w = \dfrac{1}{z-1}$ 9. $w = \dfrac{z^2+3}{z-1}$
10. $w = \mathrm{i}e^z$ 11. $w = (1+\mathrm{i})e^z$ 12. $w = ze^z$

Ex. 4.2. Given

$$\cos\theta = \frac{1}{2}\left(e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}\right) \quad \text{and} \quad \sin\theta = \frac{1}{2\mathrm{i}}\left(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}\right)$$

replace θ by $\mathrm{i}\theta$ to show that

$$\cos(\mathrm{i}\theta) = \cosh\theta \quad \text{and} \quad \sin(\mathrm{i}\theta) = \mathrm{i}\sinh\theta\,.$$

Then use the addition formulae $\cos(A+B) = \cos A\cos B - \sin A\sin B$ and $\sin(A+B) = \sin A\cos B + \sin B\cos A$ to show that

$$\boxed{\cos z = \cos(x+y\mathrm{i}) = \cos x\cosh y - \mathrm{i}\sin x\sinh y}$$

and

$$\boxed{\sin z = \sin(x+y\mathrm{i}) = \cosh y\sin x + \mathrm{i}\cos x\sinh y}\,.$$

Ex. 4.3. Show that function $w = z^3 + (2+\mathrm{i})z$ is analytic and find an expression for its derivative.

Ex. 4.4. Show that the function $f(z) = \dfrac{z^2+\mathrm{i}}{z}$ is analytic and find an expression for its derivative.

Ex. 4.5. Use the Cauchy-Riemann conditions to specify the region, where the function $f(z) = \dfrac{z}{z-\mathrm{i}}$ is differentiable, and to determine its derivative $f'(z)$ as a function of z, indicating where this derivative exists. Verify the result of using the differentiation of a quotient rule.xs

Ex. 4.6. Check whether function $f(z) = z\bar{z}$ is analytic.

Ex. 4.7. Check whether function $f(z) = \mathrm{i}\bar{z}$ satisfies the Cauchy-Riemann conditions.

Ex. 4.8. Show that function $f(z) = e^z$ satisfies the Cauchy-Riemann conditions.

Ex. 4.9. Show that function $f(z) = ze^z$ satisfies the Cauchy-Riemann conditions.

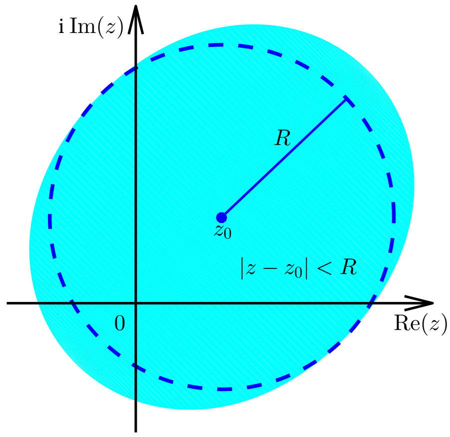


Figure 4.2: Region of convergence (cyan), disk of convergence (interior of a dashed circle), radius of convergence R and the centre of expansion z_0 of a series.

Ex. 4.10. Show that the following functions u are harmonic, find their conjugate harmonic functions v and express $f = u + iv$ as a function of z.

(a) $u = 4x^2 - 6yx - 4y^2$,
(b) $u = 3x^3 - 6yx^2 - 9y^2x + 2y^3$,
(c) $u = x^3 + 3x^2 - 3y^2x - 3y^2$,
(d) $u = x^2 - 4xy - 3x^2y - y^2 + y^3$,
(e) $u = e^x x \cos y - e^x y \sin y$,
(f) $u = e^{-x}(x \sin y - y \cos y)$.

4.2 Series

4.2.1 Taylor series

From your previous studies you should know that any function $f(x)$ of a real argument x that can be differentiated at point x_0 infinitely many times can be represented by *Taylor series* (after a British mathematician Brook Taylor (1685–1731)) as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots .$$

It can be shown that such an expansion can be generalised to the case of a function $f(z)$ of a complex variable z provided that $f(z)$ can be differentiated that is if $f(z)$ satisfies Cauchy-Riemann conditions and is analytic function in a region containing point z_0. Indeed, let $f(z) = f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$ satisfy (4.1). Then $f'(z)$ exists and is independent on the direction of differentiation in the complex z plane. In particular, it can be explicitly differentiated along the real axis x to obtain

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = u_1(x, y) + \mathrm{i}v_1(x, y)\,.$$

Considering function $f'(z)$ defined above we can show that it also satisfies the Cauchy-Riemann conditions:

$$\begin{aligned}
\frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y} &= \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = \frac{\partial 0}{\partial x} = 0\,,\\
\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} &= \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\partial 0}{\partial x} = 0\,.
\end{aligned}$$

Therefore, $f'(z)$ can be differentiated to obtain $f''(z)$. Clearly, repeating the above steps for $f''(z)$ we can show that $f'''(z)$ also exists and so on. We arrive at a very powerful conclusion:

If function $f(z)$ is analytic that is it satisfies the Cauchy-Riemann conditions, it can be differentiated infinitely many times.

Therefore, we can generalise Taylor series representation of a real function to the case of a function of a complex variable:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \frac{f'''(z_0)}{3!}(z - z_0)^3 + \cdots$$

or

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n\,,$$

where $f^{(n)}(z_0) \equiv \left.\frac{\mathrm{d}^n f}{\mathrm{d}z^n}\right|_{z=z_0}$. The point z_0, see Figure 4.2, is called the *centre of expansion* or expansion point. A particular case is $z_0 = 0$ that is known as *Maclaurin series* (after a Scottish mathematician Colin Maclaurin (1698–1746))

$$f(z) = f(0) + f'(0)(z) + \frac{f''(0)}{2}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots \quad \text{or} \quad f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n\,.$$

Truncated Taylor series containing only a few first terms are often used to approximate the behaviour of a function in the vicinity of z_0. The accuracy of such an approximation depends on the total number of the retained terms (i.e. on the length of the *partial sum of a series*) and the *convergence of the series* (see Section 4.2.2), i.e. the existence of the finite limit of a sequence of partial sums of a particular series. In the case of complex-valued series its convergence at point z in general depends not only on the distance from the expansion point z_0 but also on the location of z relative to z_0.

Definition 4.3 *The multitude of all points z in a complex plane for which the series converges is called the* region of convergence*.*

It is shown schematically in cyan in Figure 4.2. The exact shape of such a region is not always possible to establish. For this reason the concepts of a *disk of convergence* and a *radius of convergence* are introduced.

Definition 4.4 *A non-negative real number R is called the radius of convergence of a series if this series converges for all points $|z - z_0| < R$. A disk with the radius R and the center at z_0 is called the disk of convergence.*

The disc and radius of convergence are shown in Figure 4.2. The disc of convergence is the largest disk centered at the expansion point that can be inscribed into the convergence region.

Definition 4.5 *Points z satisfying $|z - z_0| = R$, where z_0 is the center of expansion and R is the radius of convergence is called the* boundary of convergence*.*

A series can (a) converge at all points along the boundary of convergence, (b) diverge at all points along the boundary of convergence, or (c) converge at some and diverge at other points along the boundary of convergence.

What exactly happens at the boundary of convergence requires a separate mathematical analysis that will be left outside the scope of this module. The fact that the boundaries of convergence is left out will be graphically illustrated by showing them by dashed lines as in Figure 4.2.

EXAMPLE **4.6.** *Find the Taylor series expansion for the function* $f(z) = e^z$ *about* $z_0 = 0$.

SOLUTION. *First we need to show that function* e^z *is analytic at point* $z = 0$. *Since*

$$f(z) = e^z = e^{x+\mathrm{i}y} = e^x e^{\mathrm{i}y} = \underbrace{e^x \cos y}_{u(x,y)} + \mathrm{i}\, \underbrace{e^x \sin y}_{v(x,y)},$$

Cauchy-Riemann conditions

$$\begin{aligned} \frac{\partial u}{\partial x} &= \quad e^x \cos y &&= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -e^x \sin y &&= -\frac{\partial v}{\partial x} \end{aligned}$$

are satisfied and the given function is indeed analytic. Therefore, Taylor series expansion exists and

$$\begin{aligned} f(z) &= e^z, & f(0) &= e^0 = 1, \\ f'(z) &= e^z, & f'(0) &= e^0 = 1, \\ f''(z) &= e^z, & f''(0) &= e^0 = 1, \\ f^{(n)}(z) &= e^z, & f^{(n)}(0) &= e^0 = 1. \end{aligned}$$

Thus the series expansion of e^z *about 0 is*

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \frac{1}{5!}z^5 + \cdots + \frac{1}{n!}z^n + \cdots .$$

In particular, choosing $z = \mathrm{i}y$ *we obtain*

$$\begin{aligned} e^{\mathrm{i}y} &= 1 + \mathrm{i}y - \frac{1}{2!}y^2 - \frac{\mathrm{i}}{3!}y^3 + \frac{1}{4!}y^4 + \frac{\mathrm{i}}{5!}y^5 + \cdots + \frac{1}{n!}(\mathrm{i}y)^n + \cdots \\ &= \underbrace{1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \cdots}_{\cos y} + \mathrm{i} \underbrace{\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 + \cdots\right)}_{\sin y} \\ &= \cos y + \mathrm{i} \sin y, \end{aligned}$$

where the known Taylor series expansions of real functions $\cos y$ *and* $\sin y$ *are recognised and the famous* Euler's formulae

$$\boxed{e^{\mathrm{i}y} = \cos y + \mathrm{i} \sin y \quad \text{and} \quad e^{\mathrm{i}\pi} + 1 = 0}$$

is obtained. The latter is known as the most elegant formula in mathematics. It combines 5 fundamental mathematical constants: e, i, π, *1 and 0.* ■

Example 4.7. *Find Taylor series expansion for the function* $f(z) = \ln(1+z)$ *about* $z_0 = 0$.

Solution. *Since* $f(z) = \ln(1+x+\mathrm{i}y) = \ln(re^{\mathrm{i}\theta}) = \ln r + \mathrm{i}\theta$, *where* $r = \sqrt{(1+x)^2+y^2}$ *and* $\theta = \tan^{-1}\dfrac{y}{1+x}$ *we can write*

$$f(z) = \underbrace{\frac{1}{2}\ln((1+x)^2+y^2)}_{u(x,y)} + \mathrm{i}\underbrace{\tan^{-1}\frac{y}{1+x}}_{v(x,y)} .$$

Then Cauchy-Riemann conditions

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1+x}{(1+x)^2+y^2} &&= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= \frac{y}{(1+x)^2+y^2} &&= -\frac{\partial v}{\partial x}, \end{aligned}$$

are satisfied, the function is analytic, the Taylor series expansion exists and

$$\begin{aligned} f(z) &= \ln(1+z), & f(0) &= 0, \\ f'(z) &= \frac{1}{1+z} = (1+z)^{-1}, & f'(0) &= 1, \\ f''(z) &= -\frac{1}{(1+z)^2} = -(1+z)^{-2}, & f''(0) &= -1, \\ f'''(z) &= \frac{2}{(1+z)^3} = 2(1+z)^{-3}, & f'''(0) &= 2, \end{aligned}$$

$$f^{(iv)}(z) = -\frac{3\cdot 2}{(1+z)^4} = -3\cdot 2(1+z)^{-4}, \qquad f^{(iv)}(0) = -3\cdot 2,$$

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}$$

$$f^{(n+1)}(z) = \frac{(-1)^n n!}{(1+z)^{n+1}} = (-1)^n n!(1+z)^{-(n+1)}, \quad f^{(n+1)}(0) = (-1)^n n!.$$

Substituting these into the Taylor series we obtain

$$\begin{aligned} f(z) = \ln(1+z) =& f(0) + f'(0)(z) + \frac{f''(0)}{2}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots \\ =& z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots . \end{aligned}$$

■

Example 4.8. *Find the Taylor series expansion for the function $f(z) = \sqrt{z}$ about $z_0 = 1$.*

Solution. *Let $z = x + \mathrm{i}y = re^{\mathrm{i}\theta}$, where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$. Then*

$$\sqrt{z} = \sqrt{r}e^{\mathrm{i}\frac{\theta}{2}} = \underbrace{\sqrt[4]{x^2+y^2}\cos\frac{\theta}{2}}_{u(x,y)} + \mathrm{i}\underbrace{\sqrt[4]{x^2+y^2}\sin\frac{\theta}{2}}_{v(x,y)}.$$

Then Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{x\cos\frac{\theta}{2} + y\sin\frac{\theta}{2}}{2(x^2+y^2)^{\frac{3}{4}}} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = \frac{y\cos\frac{\theta}{2} - x\sin\frac{\theta}{2}}{2(x^2+y^2)^{\frac{3}{4}}} = -\frac{\partial v}{\partial x},$$

are satisfied, the function is analytic at $z_0 = 1$, Taylor series expansion about this point exists[1] *and*

$$\begin{aligned} f(z) &= z^{\frac{1}{2}}, & f(1) &= 1, \\ f'(z) &= \frac{1}{2}z^{-\frac{1}{2}}, & f'(1) &= \frac{1}{2}, \\ f''(z) &= \frac{1}{2}\cdot\left(-\frac{1}{2}\right)z^{-\frac{3}{2}}, & f''(1) &= -\frac{1}{4}, \\ f'''(z) &= \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)z^{-\frac{5}{2}}, & f'''(1) &= \frac{3}{8}. \end{aligned}$$

The general expansion about $z_0 = 1$ is

$$f(z) = f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f'''(1)}{3!}(z-1)^3 + \cdots.$$

Therefore,

$$\begin{aligned} f(z) = z^{\frac{1}{2}} &= 1 + \frac{1}{2}(z-1) + \frac{-\frac{1}{4}}{2!}(z-1)^2 + \frac{\frac{3}{8}}{3!}(z-1)^3 + \cdots, \\ &= 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \frac{1}{16}(z-1)^3 + \cdots. \end{aligned}$$

■

[1] Note that the partial derivatives of this function are discontinuous at $z = 0$ and thus the function is not analytic at the origin and cannot be expanded in Taylor series about it.

4.2.2 Power series

All Taylor series considered in the previous section can be generally written as

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots \text{ or } f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n\,,$$

where a_n are constant coefficients. More generally, such series are called *power series*. They are series of increasing non-negative integer powers of $z - z_0$. The coefficients of power series can be obtained using Taylor series expansion and, similarly to Taylor series, truncated power series are used to approximate a function in the vicinity of point z_0. The *ratio test* is often used to determine the radius of convergence of a power series

$$\boxed{R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|}\,.$$

Then the series converges to a function it represents within the disk of convergence $|z - z_0| < R$.

Note that a formal substitution $z - z_0 = \dfrac{1}{u}$ converts a series of non-negative integer powers to a series of non-positive integer powers

$$f(u) = a_0 + a_1u^{-1} + a_2u^{-2} + a_3u^{-3} + \cdots \text{ or } f(u) = \sum_{n=0}^{\infty} a_n u^{-n}\,.$$

Therefore, power series are series that contain either non-negative or non-positive powers, but not their mixture.

EXAMPLE **4.9.** *The real* geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad \textit{for any real } |x| < 1$$

can be generalised to a complex case:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n \quad \textit{valid for } |z| < 1\,.$$

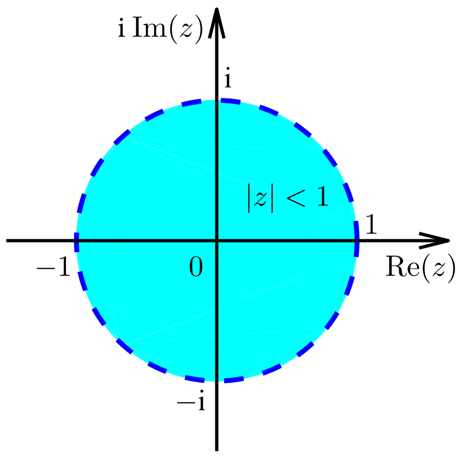


Figure 4.3: The disk of convergence with the unit radius and the center at the origin for the geometric series.

Here the radius of convergence is found using the ratio test and the fact that all series coefficients are equal to 1. The power series converges for any z with $|z| < 1$, see Figure 4.3, e.g. if $z = \dfrac{1}{2}$ we have

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{1}{1 - \frac{1}{2}} = 2\,.$$

If $z = 1$ the series becomes $1 + 1 + 1 + \cdots$, which is clearly divergent. For $z = -1$ the partial sums of the series will oscillate between 1 and 0 depending on whether odd or even number of terms is retained. Once again this demonstrates that the geometric series does not converge at this point at the boundary of convergence even though the value of the function it represents can be easily computed by a direct substitution: $\frac{1}{1-(-1)} = \dfrac{1}{2}$.

We can replace z by $-z$ and obtain the series

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n z^n \quad \text{valid for } |z| < 1\,. \tag{4.2}$$

Again the interval of convergence is found using the ratio test taking into account that the magnitudes of all series coefficients is equal to 1.

If we integrate the above equation term-by-term and apply the ratio test we obtain

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} z^n \quad \text{valid for } |z| < 1\,.$$

We can substitute other expressions, as long as their magnitude remains smaller than 1, into the series.

EXAMPLE **4.10.** *If we replace z by z^2 in (4.2), we obtain the expansion*

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots .$$

Term-by-term integration of this equation gives

$$\arctan z = \tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots .$$

EXAMPLE **4.11.** *The* binomial expansion

$$(1+z)^n = 1+nz+\frac{n(n-1)}{2!}z^2+\frac{n(n-1)(n-2)}{3!}z^3+\cdots \quad \textit{valid for } |z|<1.$$

is valid for any n. Its coefficients are determined using Taylor series expansion about $z_0 = 0$ and the radius of convergence is found using the ratio test:

$$R = \lim_{m\to\infty}\left|\frac{a_m}{a_{m+1}}\right| = \lim_{m\to\infty}\left|\frac{m+1}{n-m}\cdot\frac{n-m+1}{m}\right| = 1\,,$$

where

$$a_m = \frac{n(n-1)(n-2)\cdots(n-m+1)}{m!}$$

and

$$a_{m+1} = \frac{n(n-1)(n-2)\cdots(n-m)}{(m+1)!}\,.$$

In particular, the geometric series is a binomial series with $n = -1$.

If $n = \frac{1}{2}$ we have

$$\begin{aligned}(1+z)^{\frac{1}{2}} &= \sqrt{1+z} = 1 + \frac{1}{2}z + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2!}z^2 + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}z^3 + \cdots \\ &= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \cdots .\end{aligned}$$

If $n = -2$ then

$$\frac{1}{(1+z)^2} = (1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots .$$

For any non-negative integer n the binomial series terminates as a_{n+1} and all further terms become zero. Since the total number of terms in this case is finite regardless of the value of z for non-negative integer n, binomial series converges for any z.

This can be checked by a direct multiplication. For example, for $n = 4$ we obtain

$$(1+z)^4 = 1+4z+\frac{4\cdot 3}{2!}z^2+\frac{4\cdot 3\cdot 2}{3!}z^3+\frac{4\cdot 3\cdot 2}{4!}z^4 = 1+4z+6z^2+4z^3+z^4\,.$$

We can obtain multiple series expansions for functions such as $(a+z)^n$, where a is not necessarily 1.

Example **4.12.** *Consider various expansions for* $\dfrac{1}{2+z}$.

(a) *We can write*

$$\begin{aligned}\frac{1}{2+z} &= \frac{1}{2\left(1+\frac{z}{2}\right)}\\ &= \frac{1}{2}\cdot\left(1+\frac{z}{2}\right)^{-1}\\ &= \frac{1}{2}\cdot\left(1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\left(\frac{z}{3}\right)^3\cdots\right)\,.\end{aligned}$$

In the above we effectively used a binomial (geometric) series expansion for $(1+u)^{-1}$ with $u = \dfrac{z}{2}$. This series will be convergent if $|u| = \left|\dfrac{z}{2}\right| < 1$ i.e. if $|z| < 2$, see Figure 4.4.

(b) *We can obtain a different expansion by writing*

$$\begin{aligned}\frac{1}{2+z} &= \frac{1}{(1+(1+z))}\\ &= 1-(1+z)+(1+z)^2-(1+z)^3+(1+z)^4-\cdots\,.\end{aligned}$$

This gives a power series expansion in terms of $u = 1+z$ about the point $z = -1$. The series will converge for $|u| = |z+1| < 1$, see Figure 4.5. Thus we obtain different series for the same function by expanding about different points.

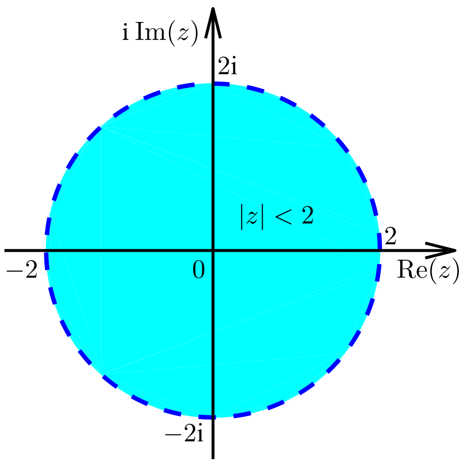


Figure 4.4: The disk of convergence of radius $R = 2$ centered at the origin for $f(z) = \dfrac{1}{2} - \dfrac{z}{4} + \dfrac{z^2}{8} - \dfrac{z^3}{16} \cdots$.

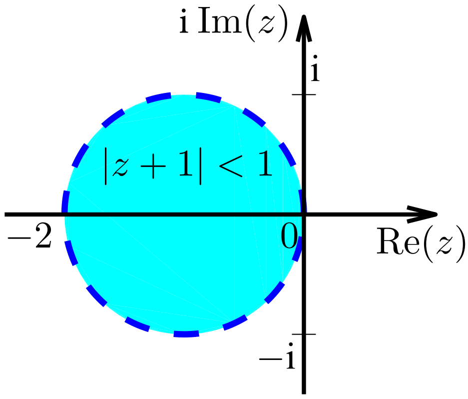


Figure 4.5: The disk of convergence of unit radius centered at $z_0 = -1$ for $f(z) = 1 - (1+z) + (1+z)^2 - (1+z)^3 + (1+z)^4 - \cdots$.

(c) *We can also expand in a series in powers of* $u = \dfrac{1}{z}$.

$$
\begin{aligned}
\frac{1}{2+z} &= \frac{1}{z}\frac{1}{\left(1+\frac{2}{z}\right)} \\
&= \frac{1}{z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \left(\frac{2}{z}\right)^4 - \cdots\right) \\
&= \frac{1}{z} - \frac{2}{z^2} + \frac{4}{z^3} - \frac{8}{z^4} + \cdots .
\end{aligned}
$$

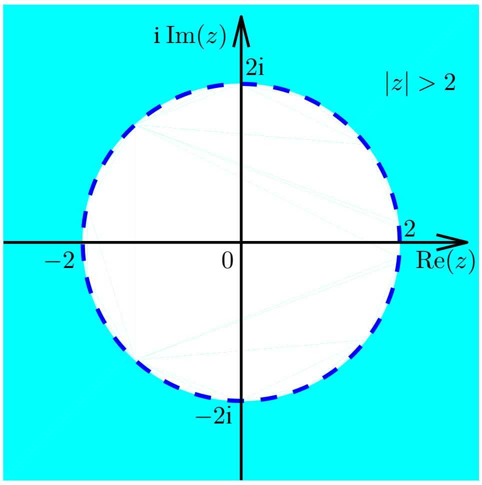


Figure 4.6: The disk of convergence for $f(z) = \frac{1}{z} - \frac{2}{z^2} + \frac{4}{z^3} - \frac{8}{z^4} + \cdots$.

This series will converge for $|u| = \left|\frac{2}{z}\right| < 1$ *i.e. for* $|z| > 2$*, see Figure 4.6.*

Example **4.13.** *Find power series expansions for the function* $\frac{1}{z-3}$ *for the following regions in the form indicated:*

(a) $|z| < 3, \quad \sum_{n=0}^{\infty} a_n z^n,$

(b) $|z-2| < 1, \quad \sum_{n=0}^{\infty} a_n (z-2)^n,$

(c) $|z| > 3, \quad \sum_{n=0}^{\infty} \frac{a_n}{z^n}.$

Solution. *In each case we use the expansion*

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots\,, \quad |u| < 1\,.$$

(a)

$$\frac{1}{z-3} = -\frac{1}{3}\frac{1}{1-\frac{z}{3}}$$
$$= -\frac{1}{3}\left(1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \left(\frac{z}{3}\right)^4 + \cdots\right)\,.$$

Here we have taken $u = \dfrac{z}{3}$ *so that the expansion is valid for* $\left|\dfrac{z}{3}\right| < 1$ *i.e. for* $|z| < 3$.

(b) $\dfrac{1}{z-3} = \dfrac{1}{(z-2)-1} = -\dfrac{1}{1-(z-2)}$.
Upon setting $u = z - 2$ *we obtain*

$$\frac{1}{z-3} = -\left(1 + (z-2) + (z-2)^2 + (z-2)^3 + \cdots\right) .$$

This expansion is valid for $|z-2| < 1$.

(c) $\dfrac{1}{z-3} = \dfrac{1}{z\left(1-\frac{3}{z}\right)} = \dfrac{1}{z}\left(1-\dfrac{3}{z}\right)^{-1}$.
Upon setting $u = \dfrac{3}{z}$ *we obtain*

$$\begin{aligned}\frac{1}{z-3} &= \frac{1}{z}\left(1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \cdots\right)\\ &= \frac{1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + \frac{27}{z^4} + \cdots .\end{aligned}$$

This expansion is valid for $\left|\dfrac{3}{z}\right| < 1$ *i.e. for* $|z| > 3$.

4.2.3 Laurent series

Taylor series contain only non-negative powers while power series can include either non-negative or non-positive powers. The more general type of series that can contain a mixture of positive and negative powers are known as *Laurent series* (after a French mathematician and military officer Pierre Alphonse Laurent (1813–1854)). They have the following form:

$$\begin{aligned} f(z) &= \cdots + \frac{c_{-3}}{(z-z_0)^3} + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0} \\ &\quad + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + c_3(z-z_0)^3 + \cdots \qquad (4.3) \\ &= \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n . \end{aligned}$$

Example **4.14.** *Find a Laurent series expansion of $f(z) = \dfrac{1}{z^2(1+z)}$ about (a) $z = 0$ and (b) $z = -1$.*

Solution.

(a) *Note that at $z = 0$ the given function contains division by 0 and thus cannot be evaluated, let alone differentiated, and is not analytic. Therefore, its Taylor series does not exist. However, a part of the function not involving division by z^2 is analytic and can be expanded in the geometric series*

$$(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$$

valid for $|z| < 1$. Thus we have

$$\begin{aligned} \frac{1}{z^2(1+z)} &= \frac{1}{z^2}(1 - z + z^2 - z^3 + \cdots) \\ &= \frac{1}{z^2} - \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots \end{aligned}$$

This series contains both positive and negative powers of z and thus it is not a power series. All of its terms apart from the first two are non-negative powers of z. Since the number of negative-power terms is finite, so is their sum for any value of z except $z = 0$. Therefore, the obtained Laurent series converges for the same range of z values as the above geometric series excluding $z = 0$ i.e. for $0 < |z| < 1$, see Figure 4.7.

(b) *To obtain a series about $z = -1$ we write*

$$\frac{1}{z^2} = ((z+1) - 1)^{-2} = (1 - (z+1))^{-2}\,.$$

Next we set $z + 1 = u$ and note that $u \to 0$ as $z \to -1$. Then from the binomial expansion

$$(1-u)^{-2} = 1 + 2u + 3u^2 + 4u^3 + \cdots$$

we have

$$(1 - (z+1))^{-2} = 1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + \cdots$$

which is valid for $|u| = |z+1| < 1$. Hence

$$\begin{aligned} \frac{1}{z^2(1+z)} &= \frac{1}{1+z}(1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + \cdots) \\ &= \frac{1}{1+z} + 2 + 3(z+1) + 4(z+1)^2 + \cdots\,. \end{aligned}$$

Similarly to the previous case we have to exclude $|u| = |z+1| = 0$ to conclude that this series converges for $0 < |1+z| < 1$, see Figure 4.8.

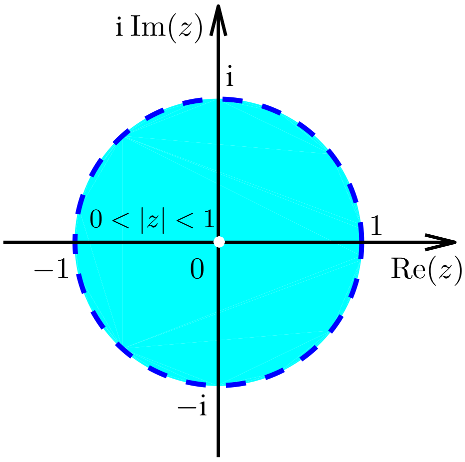


Figure 4.7: The region of convergence of the Laurent series of $f(z) = \dfrac{1}{z^2(1+z)}$ about the origin. Note that the origin $z_0 = 0$ is excluded.

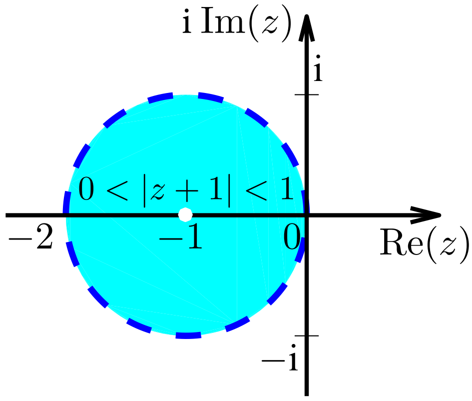


Figure 4.8: The region of convergence of the Laurent series of $f(z) = \dfrac{1}{z^2(1+z)}$ about $z = -1$. Note that the point $z_0 = -1$ is excluded.

■

Laurent series appear when one expands a function that has a singularity at the center of expansion. Thus in contrast to Taylor and power series that represent analytic functions, Laurent series represent the behaviour of singular functions (see Section 4.3).

EXAMPLE **4.15.** *Find all possible series expansions of* $f(z) = \dfrac{1}{(z+1)(z+3)}$ *about the origin* $z_0 = 0$ *and determine the type of the obtained series.*

SOLUTION. *Represent function* $f(z)$ *in terms of partial fractions first:*

$$f(z) = \frac{1}{2}\frac{1}{1+z} - \frac{1}{2}\frac{1}{3+z}.$$

We see that the individual fractions become underfined at $z_1 = -1$ *and* $z_2 = -3$. *Given that a series cannot possibly converge to an undefined value we conclude that these two points must belong to the convergence boundaries* $|z - z_0| = |z| = |z_1 - z_0| = |z_1| = 1$ *and* $|z - z_0| = |z| = |z_2 - z_0| = |z_2| = 3$. *These are circles with radii 1 and 3, respectively centered at the origin, see Figure 4.9. Therefore, these two circles are the convergence boundaries that cut the complex plane into three regions: (a)* $|z| < 1$, *(b)* $1 < |z| < 3$ *and (c)* $|z| > 3$.

(a) *When* $|z| < 1$ *we can use geometric series expansion for both partial fractions (a substitution* $u = \frac{z}{3}$ *is required for the second partial fraction to proceed) after writing*

$$\begin{aligned} f(z) &= \frac{1}{2}\frac{1}{1+z} - \frac{1}{6}\frac{1}{1+\frac{z}{3}} \\ &= \frac{1}{2}\left(1 - z + z^2 - z^3 + \cdots\right) \\ &\quad -\frac{1}{6}\left(1 - \frac{1}{3}z + \frac{1}{9}z^2 - \frac{1}{27}z^3 + \cdots\right) \\ &= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots. \end{aligned}$$

Since all powers of z *in the above expansion are non-negative, it is a power series.*

(b) *For* $1 < |z| < 3$ *we write the partial fractions as*

$$f(z) = \frac{1}{2z}\frac{1}{1+\frac{1}{z}} - \frac{1}{6}\frac{1}{1+\frac{z}{3}}$$

and then use substitutions $u = \frac{1}{z}$ *and* $u = \frac{z}{3}$, $|u| < 1$ *and the geometric series expansion to obtain*

$$\frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{6}\left(1 - \frac{1}{3}z + \frac{1}{9}z^2 - \frac{1}{27}z^3 + \cdots\right).$$

The first series is convergent for $1 < |z|$, *the second for* $|z| < 3$. *Therefore, the combined series converges in the ring shown in Figure 4.9. and we obtain*

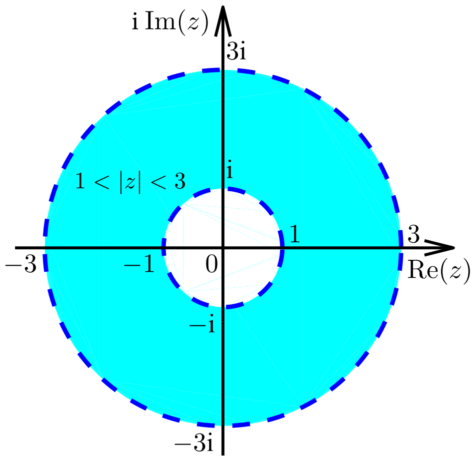


Figure 4.9: The region of convergence for the Laurent series (4.4).

$$f(z) = \cdots + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{1}{18}z - \frac{1}{54}z^2 + \cdots . \qquad (4.4)$$

Since both negative and positive powers are present, this is the Laurent series.

(c) When $|z| > 3$ we write

$$f(z) = \frac{1}{2z}\frac{1}{1+\frac{1}{z}} - \frac{1}{2z}\frac{1}{1+\frac{3}{z}} .$$

The geometric series expansion using the substitutions $u = \frac{1}{z}$ and $u = \frac{3}{z}$, $|u| < 1$ then leads to

$$f(z) = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right) .$$

Finally,

$$f(z) = \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \cdots .$$

Since only non-positive powers are present, this is a power series.

■

4.2.4 Exercises

Ex. 4.11. Find the Laurent series expansion for $f(z) = \dfrac{1}{z(z+1)^2}$ about (a) $z = 0$ and (b) $z = -1$. Specify the region of convergence in each case.

Ex. 4.12. Find the series expansions for $f(z) = \dfrac{1}{(z-1)(2-z)}$ valid in the regions (a) $|z| < 1$, (b) $1 < |z| < 2$ and (c) $2 < |z|$ and specify their types.

Ex. 4.13. Consider $f(z) = \dfrac{1}{z^2 - 7z + 10}$. Identify the regions cumulatively covering the complete complex z plane, for which series expansions of this function can be developed, find the corresponding series and state their types.

4.3 Singularities and Residues

Definition 4.6 (regular point) *A point is called a regular point of a function if the function is analytic there.*

Definition 4.7 (singular point) *A point is called a singular point of a function, or singularity, if it is not a regular point. We say that the function is singular at such point.*

Example **4.16.**

(a) *The function* $f(z) = \dfrac{1}{z}$ *is singular at* $z = 0$.

(b) *The function* $f(z) = \dfrac{1}{(z+\mathrm{i})^3}$ *is singular at* $z = -\mathrm{i}$.

(c) *The function* $f(z) = \dfrac{\sin z}{z}$ *is undefined at* $z = 0$ *and so appears to be singular. However,* $f(z)$ *has a Taylor series expansion about* $z = 0$

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} \cdots$$

obtained by multiplying the Taylor series for $\sin z$ *by* $\dfrac{1}{z}$. *Defining* $f(0) = 1$ *we obtain a function which is not singular at* $z = 0$. *This is an example of* a removable singularity.

Definition 4.8 (principal part) *If a function has a Laurent series expansion (4.3) we call the terms involving negative powers*

$$\cdots + \frac{c_{-3}}{(z-z_0)^3} + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0}$$

the principal part.

Definition 4.9 (essential singularity) *If the principal part of the Laurent series contains infinitely many terms point z_0 is called essential singularity.*

Definition 4.10 (pole of order m) *If there are m terms in the principal part (i.e. $c_{-n} = 0$ for $n > m$), then z_0 is called a pole of order m.*

Definition 4.11 (simple pole) *A pole of order one is called a simple pole.*

Definition 4.12 (residue) *The coefficient c_{-1} of the term $\dfrac{1}{z - z_0}$ is called the residue of the function $f(z)$ at z_0. It is denoted by $\operatorname{Res}(f(z), z_0)$.*

EXAMPLE **4.17.**

(a) *The function $f(z) = \dfrac{1}{z}$ has a pole of order 1 at $z = 0$.*

(b) *The function $f(z) = \dfrac{1}{(z+3)^2} + \dfrac{1}{z+3} + (z+3)^2$ has a pole of order 2 at $z = -3$.*

(c) *The function $f(z) = \dfrac{2\mathrm{i}}{z^2+1} = \dfrac{1}{z-\mathrm{i}} - \dfrac{1}{z+\mathrm{i}}$ has a pole of order one at $z = -\mathrm{i}$ and at $z = \mathrm{i}$.*

(d) *The function $f(z) = e^{\frac{1}{z}} = 1 + \dfrac{1}{z} + \dfrac{1}{2!z^2} + \dfrac{1}{3!z^3} + \cdots$ has an essential singularity at $z = 0$.*

If a function has a *simple pole* at z_0 then its Laurent series expansion about z_0 will be

$$f(z) = \frac{c_{-1}}{z - z_0} + c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + c_3(z - z_0)^3 + \cdots .$$

Multiplying both sides by $(z - z_0)$ we obtain

$$(z - z_0)f(z) = c_{-1} + c_0(z - z_0) + c_1(z - z_0)^2 + c_2(z - z_0)^3 + c_3(z - z_0)^4 + \cdots .$$

This is the Taylor series expansion of the function $(z - z_0)f(z)$ about z_0, and we can see that the residue of $f(z)$ at z_0 is given by the limit

$$\operatorname{Res}(f(z), z_0) = c_{-1} = \lim_{z \to z_0} (z - z_0)f(z) .$$

This result holds for simple poles.

EXAMPLE **4.18.** *Find the residue of* $f(z) = \dfrac{2z}{(z^2+1)(2z-1)}$ *at* $z = -\mathrm{i}$, $z = \dfrac{1}{2}$.

SOLUTION. *Function* $f(z)$ *has simple poles at* $z = -\mathrm{i}$, $z = \mathrm{i}$ *and* $z = \dfrac{1}{2}$. *Residue of* $f(z)$ *at* $z = -\mathrm{i}$ *is*

$$\begin{aligned}\lim_{z\to -\mathrm{i}}(z+\mathrm{i})f(z) &= \lim_{z\to -\mathrm{i}}(z+\mathrm{i})\frac{2z}{(z^2+1)(2z-1)}\\ &= \lim_{z\to -\mathrm{i}}\frac{2z}{(z-\mathrm{i})(2z-1)} = \frac{2(-\mathrm{i})}{(-\mathrm{i}-\mathrm{i})(2(-\mathrm{i})-1)}\\ &= \frac{1}{-1-2\mathrm{i}} = \frac{-1+2\mathrm{i}}{5}\,.\end{aligned}$$

For the case $z = \dfrac{1}{2}$ *we write*

$$f(z) = \frac{2z}{(z^2+1)(2z-1)} = \frac{2z}{(z^2+1)2(z-\frac{1}{2})} = \frac{z}{(z^2+1)\left(z-\frac{1}{2}\right)}\,.$$

Then

$$\begin{aligned}\operatorname{Res}\left(f(z),\frac{1}{2}\right) &= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z}{\left(z-\frac{1}{2}\right)(z^2+1)} = \lim_{z\to\frac{1}{2}}\frac{z}{z^2+1}\\ &= \frac{1/2}{(1/2)^2+1} = \frac{2}{5}\,.\end{aligned}$$

■

If a function has a pole of degree 2 at z_0 then its Laurent series expansion about z_0 will be

$$f(z) = \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots\,.$$

Multiplying both sides by $(z-z_0)^2$ we obtain

$$(z-z_0)^2 f(z) = c_{-2} + c_1(z-z_0) + c_0(z-z_0)^2 + c_1(z-z_0)^3 + c_2(z-z_0)^4 + \cdots\,.$$

However, taking the limit of $z \to z_0$ will define c_{-2}. To find c_{-1} one more step is required: we first differentiate the above

$$\frac{\mathrm{d}}{\mathrm{d}z}(z-z_0)^2 f(z) = c_1 + 2c_0(z-z_0) + 3c_1(z-z_0)^2 + 4c_2(z-z_0)^3 + \cdots$$

and then take the limit to obtain

$$\operatorname{Res}(f(z), z_0) = c_{-1} = \lim_{z\to z_0} \frac{\mathrm{d}}{\mathrm{d}z}(z-z_0)^2 f(z)\,.$$

Similarly, it can be derived that for a pole of order m the formula for the residue is

$$\boxed{c_{-1} = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left[(z-z_0)^m f(z)\right]}\,.$$

Example 4.19. *Find the residue of* $f(z) = \dfrac{e^z}{(z+1)^2}$ *at* $z=-1$.

Solution. $z=-1$ *is a pole of order 2 so that*

$$c_{-1} = \lim_{z\to -1}\frac{\mathrm{d}}{\mathrm{d}z}(z+1)^2\frac{e^z}{(z+1)^2} = \lim_{z\to -1}\frac{\mathrm{d}}{\mathrm{d}z}e^z = e^{-1}\,.$$

■

Example 4.20. *Find the residue of* $f(z) = \dfrac{z^5}{(z+1)^4}$ *at* $z=-1$.

Solution. $z=-1$ *is a pole of order 4 so that*

$$\begin{aligned} c_{-1} &= \lim_{z\to -1}\frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}z^3}(z+1)^4\frac{z^5}{(z+1)^4} = \lim_{z\to -1}\frac{1}{6}\frac{\mathrm{d}^3}{\mathrm{d}z^3}z^5 \\ &= \lim_{z\to -1}\frac{1}{6}\cdot 5\cdot 4\cdot 3\cdot z^2 = 10\,. \end{aligned}$$

■

4.3.1 Exercises

Ex. 4.14. Determine the residues of the following functions at each pole.

(a) $\dfrac{2z+1}{z^2-z-2}$, (b) $\dfrac{1}{z^2(1-z)}$, (c) $\dfrac{3z^2+2}{(z-1)(z^2+9)}$,

(d) $\dfrac{z+1}{(z-1)^2(z+3)}$, (e) $\dfrac{z^6+4z^4+z^3+1}{(z-1)^5}$, (f) $\dfrac{4z+3}{z^3+3z^2+2z}$.

Ex. 4.15. Calculate the residues at the poles indicated

(a) $\dfrac{\cos z}{z}$, $z=0$; (b) $\dfrac{z}{\sin z}$, $z=\pi$; (c) $\dfrac{1}{(1+z^2)^2}$, $z=\mathrm{i}$.

4.4 Contour Integration

A *path* or a *curve in the complex plane* also referred to as *contour* can be described parametrically using a function $z(t) = x(t) + \mathrm{i}y(t)$ of a real parameter t.

EXAMPLE **4.21.**

- *The function* $z(t) = t+\mathrm{i}t^2$, $-1 \le t \le 1$ *describes a parabolic path* C_1 *running from point* $-1+\mathrm{i}$ *through the origin to the point* $1+\mathrm{i}$, *see* Figure 4.10(*a*).
- *The function* $z(t) = e^{\mathrm{i}t} = \cos t + \mathrm{i}\sin t$, $0 \le t < 2\pi$, $(x(t) = \cos t$, $y(t) = \sin t)$ *describes a circle* C_2 *of radius 1 centered at the origin. This is called the* unit circle, *see* Figure 4.10(*b*).
- *The function* $z(t) = t+\mathrm{i}t$, $0 \le t \le 2$ *describes a straight line path* C_3 *running from the origin to the point* $2+2\mathrm{i}$, *see* Figure 4.10(*c*).

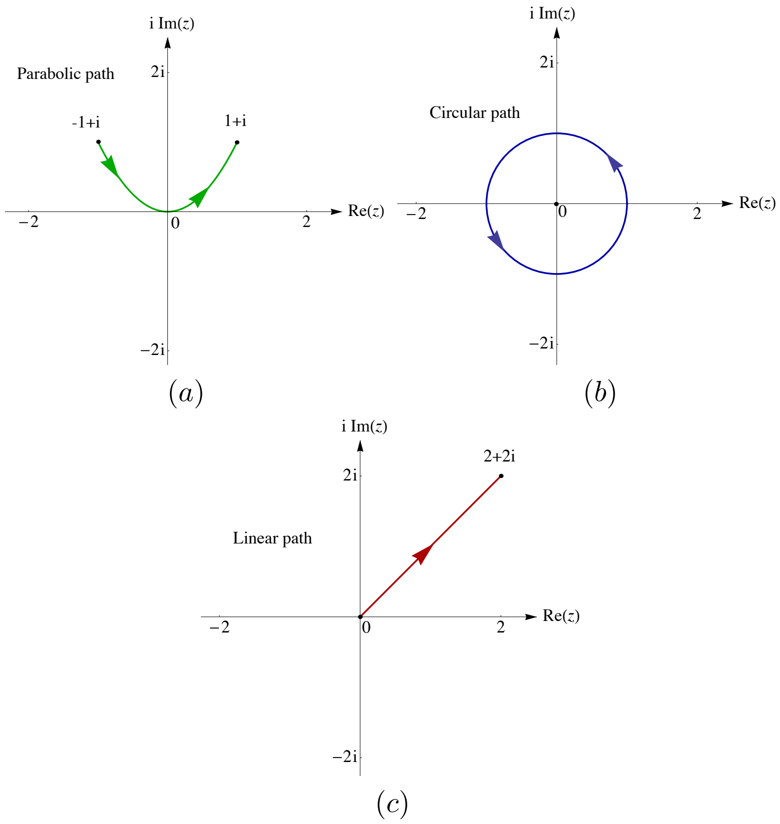


Figure 4.10: The integration paths discussed in EXAMPLE 4.21: parabolic, circular and linear.

If C is a path described by $z(t)$, $a \le t \le b$, where $z(t)$ is a function of the real variable t, then we can define the contour integral of any function $f(z)$ along the path by

$$\boxed{\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t))\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t}\,.$$

Example **4.22.** *If $f(z) = z^2$ and C_1 is the parabolic path described in* Example 4.21*, calculate $\displaystyle\int_{C_1} f(z)\,\mathrm{d}z$.*

Solution. *On C_1 we have $z = t + \mathrm{i}t^2$ so that $\dfrac{\mathrm{d}z}{\mathrm{d}t} = 1 + 2t\mathrm{i}$. Then*

$$\begin{aligned}
\int_{C_1} f(z)\,\mathrm{d}z &= \int_{-1}^{1} \left(t + \mathrm{i}t^2)^2(1 + 2t\mathrm{i}\right)\,\mathrm{d}t \\
&= \int_{-1}^{1} (t^2 + 2t^3\mathrm{i} - t^4)(1 + 2t\mathrm{i})\,\mathrm{d}t \\
&= \int_{-1}^{1} (t^2 - 5t^4 - 2t^5 + 4t^3\mathrm{i})\,\mathrm{d}t \\
&= -\frac{4}{3}\,.
\end{aligned}$$

■

Definition 4.13 (closed path) *A path is closed if the start and end points are the same.*

The integral around a closed path C is denoted by $\displaystyle\oint_C f(z)\,\mathrm{d}z$.

Example **4.23.** *If $f(z) = z^2$ and C_2 is the circular path described in* Example 4.21*, calculate $\displaystyle\oint_{C_2} f(z)\,\mathrm{d}z$.*

Solution. *On C_2 we have $z = e^{\mathrm{i}t}$ so that $\dfrac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{i}e^{\mathrm{i}t}$. Then*

$$\begin{aligned}
\oint_{C_2} f(z)\,\mathrm{d}z &= \int_{C_2} z^2\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t = \int_0^{2\pi} \left(e^{\mathrm{i}t}\right)^2 \mathrm{i}e^{\mathrm{i}t}\,\mathrm{d}t \\
&= \int_0^{2\pi} \mathrm{i}e^{3\mathrm{i}t}\,\mathrm{d}t = \left.\frac{e^{3\mathrm{i}t}}{3}\right|_0^{2\pi} = \frac{e^{6\pi\mathrm{i}}}{3} - \frac{e^0}{3} = 0\,.
\end{aligned}$$

This result is not a coincidence as will be shown in Section 4.4.1. ■

4.4.1 Cauchy's theorem

Let us generalise the result of Example 4.23 by considering $f(z) = (z-z_0)^n$.

Example **4.24.** *Evaluate the integral* $\oint_C (z - z_0)^n \,\mathrm{d}z$*, where* C *is the circular path* $z(t) = z_0 + Re^{\mathrm{i}t}$ *with the centre at* z_0*,* $0 \le t \le 2\pi$ *and* $n \ne -1$*.*

Solution. *Introduce a new variable* $u = z - z_0$*. Then* $u(t) = Re^{\mathrm{i}t}$*,* $\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{i}Re^{\mathrm{i}t}$ *i.e.* $\mathrm{d}u = \mathrm{d}z = \mathrm{i}Re^{\mathrm{i}t}\,\mathrm{d}t$*, the circular contour* C *centered at* z_0 *transforms into a circular path* C_u *of the same radius but centred at* $u_0 = 0$ *and we have*

$$\begin{aligned}
\oint_C (z - z_0)^n \,\mathrm{d}z &= \oint_{C_u} u^n \,\mathrm{d}u = \int_0^{2\pi} R^n e^{n\mathrm{i}t} \mathrm{i}Re^{\mathrm{i}t}\,\mathrm{d}t \\
&= R^{n+1} \int_0^{2\pi} \mathrm{i}e^{(n+1)\mathrm{i}t}\,\mathrm{d}t = R^{n+1} \left.\frac{\mathrm{i}e^{(n+1)\mathrm{i}t}}{(n+1)\mathrm{i}}\right|_0^{2\pi} \\
&= \frac{R^{n+1}}{n+1}\left(e^{(n+1)\mathrm{i}2\pi} - e^0\right) = 0\,.
\end{aligned}$$

This result is valid for all integers n *except* $n = -1$ *i.e.* $n = \cdots, -3, -2, 0, 1, 2, 3, \cdots$. ■

Note that any function $f(z)$ that is analytic inside a circle of radius R centred at $z = z_0$ can be represented by a power series $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ inside this circle. Therefore, the above example demonstrates that the integral over a closed circular path centered at $z = z_0$ of *any* function that is analytic in that circle is zero. This is already quite a remarkable fact yet it can be formulated in a much more general form for a closed contour of an arbitrary shape:

Theorem 4.14 *If* $f(z)$ *is analytic in a region then*

$$\oint_C f(z)\,\mathrm{d}z = 0$$

for any closed path in this region.

This means that the value of any integral of a function along a path in the region, where the function is analytic, depends only on the end points of the path.

We have already encountered such a situation in the context of Vector Calculus) in Section 3.1.5. Such a result should not come as a complete surprise because we have established in Section 4.1.2 that the real part of an analytic function of a complex variable is equivalent to a potential of some vector field and the work integral, which is similar to the contour integral considered here, of such a vector field is path independent. This leads to a significant simplification of integrals of analytic complex valued functions as Example 4.25 demonstrates.

Example 4.25. *Consider the integral in Example 4.22 along an arbitrary path connecting points $-1+\mathrm{i}$ and $1+\mathrm{i}$.*

Solution.

$$\begin{aligned} \int_{C_2} f(z)\,\mathrm{d}z &= \int_{-1+\mathrm{i}}^{1+\mathrm{i}} z^2\,\mathrm{d}z = \left[\frac{z^3}{3}\right]_{-1+\mathrm{i}}^{1+\mathrm{i}} \\ &= \frac{1}{3}((1+\mathrm{i})^3 - (-1+\mathrm{i})^3) = -\frac{4}{3}\,. \end{aligned}$$

As you can see, nowhere in this example did we use the knowledge of any specific path. Thus the path can be chosen arbitrarily, yet the result of the integration remains the same as long as the start and end points do not change. This is because the function $f(z) = z^2$ is analytic everywhere. ■

Thus, if the function of a complex variable is analytic, it can be integrated by following the standard integration rules introduced for real valued functions without worrying about choosing a particular integration path and describing it parametrically.

EXAMPLE **4.26.** *Evaluate the integral* $\displaystyle\oint_C \frac{\mathrm{d}z}{z-z_0}$*, where* C *is the circular path travelled counter-clockwise:* $z(t) = z_0 + R e^{\mathrm{i}t}$, $0 \le t \le 2\pi$.

SOLUTION. *Similar to* EXAMPLE 4.24 *we introduce a substitution* $z - z_0 = u$. *Then* $u(t) = Re^{\mathrm{i}t}$, $\dfrac{\mathrm{d}u}{\mathrm{d}t} = \mathrm{i}Re^{\mathrm{i}t}$ *i.e.* $\mathrm{d}u = \mathrm{i}Re^{\mathrm{i}t}\,\mathrm{d}t$ *and*

$$\oint_C \frac{\mathrm{d}z}{z-z_0} = \oint_{C_u} \frac{\mathrm{d}u}{u} = \int_0^{2\pi} \frac{\mathrm{i}Re^{\mathrm{i}t}}{Re^{\mathrm{i}t}}\,\mathrm{d}t = \int_0^{2\pi} \mathrm{i}\,\mathrm{d}t = 2\pi\mathrm{i}\,.$$

This result does not contradict Cauchy's theorem because any region containing the path C *also contains* $z = z_0$*, where* $\dfrac{1}{z-z_0}$ *is not analytic. The same result holds for any closed path enclosing* z_0. *Again, this result is not a coincidence but rather a consequence of general theorem formulated in* Section 4.4.2. ■

4.4.2 Cauchy's integral formula

Assume that a function $f(z)$ is analytic inside a circle centred at z_0: $|z - z_0| < R$ and that it can be expanded in Taylor series at z_0:

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots .$$

As was shown in EXAMPLE 4.24, the integral of each term in the right-hand side of the above expansion along a closed circular path $C: \ |z - z_0| = R$ is zero.

Let us divide the function by $z - z_0$ to obtain

$$\frac{f(z)}{z-z_0} = \frac{f(z_0)}{z-z_0} + f'(z_0) + \frac{f''(z_0)}{2!}(z-z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-1} + \cdots .$$

The expansion in the right-hand side contains the mixture of positive and negative powers of $z - z_0$ and thus is a Laurent series. It starts with power of -1. Therefore, function $\dfrac{f(z)}{z-z_0}$ has a simple pole at $z = z_0$. Let us integrate the series term-by-term along the circular contour travelled counter-clockwise again. Then using the results of EXAMPLE 4.24 and EXAMPLE 4.26 we obtain

$$\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z = 2\pi\mathrm{i} f(z_0) = 2\pi\mathrm{i}c_{-1}\,,$$

where $c_{-1} = f(z_0)$ is the residue of the function.

More generally, dividing the function by $(z - z_0)^{n+1}$ we obtain

$$\frac{f(z)}{(z-z_0)^{n+1}} = \frac{f(z_0)}{(z-z_0)^{n+1}} + \frac{f'(z_0)}{(z-z_0)^{n}} + \frac{f''(z_0)}{2!(z-z_0)^{n-1}} + \cdots + \frac{f^{(n)}(z_0)}{n!(z-z_0)} + \cdots$$

the integration of which produces

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = 2\pi\mathrm{i}\frac{f^{(n)}(z_0)}{n!} = 2\pi\mathrm{i}c_{-1}\,,$$

where $c_{-1} = \dfrac{f^{(n)}(z_0)}{n!}$, that is the value of the integral is completely determined by the residue of the integrand function. These results are generalised to the case of an arbitrary closed contour in the following theorem known as *Cauchy's integral formula*.

Theorem 4.15 *If $f(z)$ is analytic in a region enclosed by C then for any point z_0 in the region*

$$\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z = 2\pi\mathrm{i}f(z_0)$$

and more generally

$$\boxed{\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = \frac{2\pi\mathrm{i}}{n!}f^{(n)}(z_0)}\,.$$

Before applying this formula we need to decide what "the region enclosed by C" is. By this we mean the region which remains on the left as we follow the path C. For example, if we follow the circle with the center at the origin counter-clockwise then the enclosed region contains the interior of the circle that includes its center. However, if we move clockwise, then the enclosed region will be outside the circle. It will not include the circle center and will extend to infinity.

Example **4.27.** *Evaluate the integral* $I = \oint_{|z|=2} \dfrac{\cos z}{(z+5)z}\,\mathrm{d}z$.

Solution. *The origin $z_0 = 0$ is the only singular point inside the specified integration contour. The integrand has a simple pole there corresponding to $n = 0$ in Cauchy's integral formula. Then according to it, $I = 2\pi\mathrm{i}f(0)$, where $f(z) = \dfrac{\cos z}{z+5}$, and $I = \dfrac{2}{5}\pi\mathrm{i}$.* ■

EXAMPLE **4.28.** *Evaluate the integral* $I = \displaystyle\oint_C \frac{\cos(2z)}{\left(z - \frac{\pi}{2}\right)^5}\,\mathrm{d}z$*, where* C *is the unit circle centered at* $z_0 = \dfrac{\pi}{2}$.

SOLUTION. *According to Cauchy's integral formula*

$$I = \frac{2\pi \mathrm{i}}{n!} f^{(n)}(z_0)\,,$$

where $f(z) = \cos(2z)$ *and* $n = 4$. *Then* $f^{(4)}(z_0) = 16\cos\pi = -16$, $n! = 24$ *and* $I = -\frac{4}{3}\pi\mathrm{i}$. ■

4.4.3 Cauchy's residue theorem

While Cauchy's integral formula offers a straightforward way of evaluating integrals along contours enclosing regions containing a single pole singularity, it cannot be used if multiple singularities are present within the contour. The following theorem offers a robust alternative in this case.

Theorem 4.16 *If* $f(z)$ *is analytic in a region bounded by a path* C *except for a finite number of singularities at* $z_1, z_2, \cdots, z_n$ *then*

$$\boxed{\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{n}\mathrm{Res}(f(z); z_k)}\,.$$

This allows one to calculate integrals around closed contours by just calculating residues for the singularities that are enclosed by the given contour. For example, Figure 4.11 shows a contour, where only two of the three singularities at z_1 and z_3, but not at z_2 need to be included in the calculation.

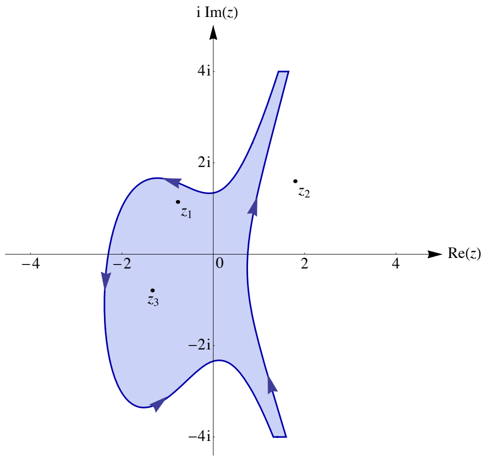


Figure 4.11: The integration contour encloses points z_1 and z_3, but not z_2.

Example **4.29.** *Evaluate* $\oint_C \frac{e^z}{(z+1)^2}\,\mathrm{d}z$*, where* C *is a closed path enclosing the point* $z=-1$ *and travelled anti-clockwise.*

Solution. *We found earlier that* $\operatorname{Res}\left(\frac{e^z}{(z+1)^2}; -1\right) = e^{-1}$*. Thus*

$$\oint_C \frac{e^z}{(z+1)^2}\,\mathrm{d}z = 2\pi \mathrm{i} e^{-1}\,.$$

■

Example **4.30.** *Evaluate* $I = \oint_C \frac{z^3-z^2+z-1}{z^3+4z}\,\mathrm{d}z$*, where* C *is (a) the circle* $|z| = 1$ *or (b) the circle* $|z| = 4$*. Subsequently, reverse the direction of integration along* $|z| = 4$ *and re-evaluate the integral.*

Solution. *The integrand function* $f(z) = \frac{z^3-z^2+z-1}{z^3+4z}$ *has simple poles at* $z=0$*,* $z=-2\mathrm{i}$ *and* $z=2\mathrm{i}$*. We thus have:*

$\operatorname{Res}(f(z))$ *at* $z=0$ *is* $\lim_{z\to 0} z\frac{z^3-z^2+z-1}{z(z^2+4)} = -\frac{1}{4}$,

$\operatorname{Res}(f(z))$ *at* $z=2\mathrm{i}$ *is* $\lim_{z\to 0}(z-2\mathrm{i})\frac{z^3-z^2+z-1}{z(z-2\mathrm{i})(z+2\mathrm{i})} = -\frac{3}{8}+\frac{3}{4}\mathrm{i}$,

$\operatorname{Res}(f(z))$ *at* $z=-2\mathrm{i}$ *is* $\lim_{z\to 0}(z+2\mathrm{i})\frac{z^3-z^2+z-1}{z(z-2\mathrm{i})(z+2\mathrm{i})} = -\frac{3}{8}-\frac{3}{4}\mathrm{i}$.

(a) For the contour $|z| = 1$ travelled in the anticlockwise direction, the only pole within the contour is $z = 0$ so that by Cauchy's residue theorem

$$\oint_C \frac{z^3 - z^2 + z - 1}{z^3 + 4z}\,\mathrm{d}z = 2\pi\mathrm{i}\left(-\frac{1}{4}\right) = -\frac{1}{2}\pi\mathrm{i}\,.$$

(b) For the contour $|z| = 4$, all of the poles are within the contour if it is travelled anti-clockwise. Hence by the residue theorem

$$\oint_C \frac{z^3 - z^2 + z - 1}{z(z^2 + 4)}\,\mathrm{d}z = 2\pi\mathrm{i}\left(-\frac{1}{4} - \frac{3}{8} + \frac{3}{4}\mathrm{i} - \frac{3}{8} - \frac{3}{4}\mathrm{i}\right) = -2\pi\mathrm{i}\,.$$

If the direction of integration is changed to clockwise, we expect that the value of the integral will remain the same but its sign will change to the opposite. However, in this case the region enclosed by the contour will change to the exterior of the contour $|z| = 4$ and the integrand function $f(z)$ does not appear to have any singularities there. Therefore, we expect that the sum of residues in the exterior of the $|z| = 4$ will be zero and, according to Cauchy's residue theorem, so should be the integral, which is a clear contradiction. To resolve it, introduce the transformation $u = \dfrac{1}{z}$. It maps the exterior of $|z| = 4$ in the z complex plane to the interior of $u = \dfrac{1}{4}$ in the u complex plane and clockwise integration in the z plane changes to anticlockwise in the u plane. Therefore, in the u plane we need to consider residues at singular points satisfying $|u| < \dfrac{1}{4}$. From $\dfrac{\mathrm{d}u}{dz} = -\dfrac{1}{z^2} = -u^2$ we obtain $\mathrm{d}z = -\dfrac{\mathrm{d}u}{u^2}$ so that the integral I becomes

$$I_1 = \oint_{|u|=\frac{1}{4}} \frac{u^3 - u^2 + u - 1}{(1 + 4u^2)u^2}\,\mathrm{d}u\,.$$

The integrand function has pole singularities at $u = 0$ and $u = \pm\frac{\mathrm{i}}{2}$, but only $u = 0$ is inside the circle $u = \dfrac{1}{4}$. The partial fraction decomposition leads to

$$\frac{u^3 - u^2 + u - 1}{(1 + 4u^2)u^2} = -\frac{1}{u^2} + \frac{1}{u} + 3\frac{1 - u}{1 + 4u^2}\,.$$

The last term on the right-hand side is non-singular at $u = 0$ and thus will result in power series with non-negative powers. Since we only need c_{-1}, which is the residue of the function, these are of no interest to us and thus we conclude that $\mathrm{Res}(f(u), 0) = c_{-1} = 1$. Therefore, indeed $I_1 = 2\pi\mathrm{i} = -I$ as expected. The perceived contradiction was *due to the fact that we could not see explicitly the simple pole singularity the integrand $f(z)\,\mathrm{d}z$ has at infinity. Its existence only became evident upon applying the transformation $z = \dfrac{1}{u}$.* ■

4.4.4 Evaluation of real integrals

The Caushy's residue theorem can be used to evaluate certain real integrals by choosing suitable contours. We consider two types of integrals below.

Integrals of the form $\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x$. Consider a closed contour C consisting of a straight-line path C_1 running from $-R$ to R along the real x axis and a semi-circular path C_2 of radius R centred at the origin and located in the upper half of the complex plane. Along the path C_1 we have $z = x$ and $\mathrm{d}z = \mathrm{d}x$. Introduce a function $f(z)$ of a complex variable such that $f(z) = F(x)$ along C_1. Then

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{-R}^{R} F(x)\,\mathrm{d}x\,.$$

Along the path C_2, $z = Re^{\mathrm{i}\theta}$, $\mathrm{d}z = \mathrm{i}Re^{\mathrm{i}\theta}\,\mathrm{d}\theta$ and

$$\int_{C_2} f(z)\,\mathrm{d}z = \int_0^{\pi} f(Re^{\mathrm{i}\theta})\mathrm{i}Re^{\mathrm{i}\theta}\,\mathrm{d}\theta\,.$$

Then

$$\begin{aligned}\oint_C f(z)\,\mathrm{d}z &= \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,dz \\ &= \int_{-R}^{R} F(x)\,\mathrm{d}x + \int_0^{\pi} f(Re^{\mathrm{i}\theta})\mathrm{i}Re^{\mathrm{i}\theta}\,\mathrm{d}\theta\,.\end{aligned}$$

By the residue theorem this is equal to the sum of the residues enclosed by the path C.

For many functions $f(z)$ that decay quickly at large values of $|z|$ the integral along the semi-circular path C_2 will become zero as $R \to \infty$. In particular, this is true for any rational function with a denominator of degree at least two higher than that of the numerator. In this case $\lim\limits_{|z|\to\infty} f(z) = \dfrac{a}{z^n}$, where $n \geq 2$. For example, if

$$f(z) = \frac{10z^3 + 10z}{5z^5 + 4}\,, \quad \text{then} \quad \lim_{|z|\to\infty} = \frac{2}{z^2}\,.$$

Letting $z = Re^{i\theta}$ we obtain

$$\lim_{R\to\infty}\left(f(z)\,\mathrm{d}z\right) = \lim_{R\to\infty}\left(\frac{a}{R^n e^{in\theta}}\mathrm{i}Re^{\mathrm{i}\theta}\,\mathrm{d}\theta\right) = \lim_{R\to\infty}\frac{\mathrm{i}ae^{\mathrm{i}(1-n)\theta}}{R^{n-1}} = 0$$

if $n-1 \geq 1$ or $n \geq 2$. Subsequently,

$$\lim_{R\to\infty}\int_{C_2} f(Re^{\mathrm{i}\theta})\mathrm{i}Re^{\mathrm{i}\theta}\,\mathrm{d}\theta = \int_{C_2} 0\,\mathrm{d}\theta = 0\,.$$

Therefore, in this limit we obtain

$$\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x = 2\pi\mathrm{i}\times(\text{sum of residues enclosed by } C).$$

Example 4.31. *Evaluate* $\displaystyle\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,\mathrm{d}x$.

Solution. *Let* $f(z) = \dfrac{1}{z^2+1}$. *This is a rational function with the numerator of degree 0 and the denominator of degree 2. Thus the integral of this function will vanish at large values of the argument. Function* $f(z)$ *has poles at points where* $z^2+1=0$ *i.e. at* $z=\pm\mathrm{i}$. *The pole* $z=\mathrm{i}$ *is the only simple pole enclosed by path* C. *The residue of* $f(z)$ *at* $z=\mathrm{i}$ *is*

$$\mathrm{Res}(f(z);\mathrm{i}) = \lim_{z\to\mathrm{i}}(z-\mathrm{i})f(z) = \lim_{z\to\mathrm{i}}\frac{z-\mathrm{i}}{z^2+1} = \lim_{z\to\mathrm{i}}\frac{z-\mathrm{i}}{(z-\mathrm{i})(z+\mathrm{i})} = -\frac{\mathrm{i}}{2}\,.$$

Now by the residue theorem

$$\oint_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\frac{-\mathrm{i}}{2} = \pi.$$

That is

$$\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,\mathrm{d}x = \pi\,.$$

As a simple check note that the result is real and positive. This is indeed expected since the original integral involves a real positively defined function and obtaining non-positive or complex answer would indicate an error. Of course, this integral can be evaluated without invoking complex variables because it has a standard anti-derivative:

$$\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,\mathrm{d}x = \tan^{-1}x|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi\,,$$

but this is not always the case as the following examples demonstrate.

■

Example 4.32. *Evaluate* $\displaystyle\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,\mathrm{d}x.$

Solution. *The anti-derivative of $F(x)$ is not known. Thus we can only evaluate this integral by invoking complex variables. Let $f(z) = \dfrac{z^2}{z^4+1}$. It is a rational function with the denominator of degree 4 and numerator of degree 2. Thus its integral vanishes at large values of the argument. Function $f(z)$ has simple poles at points, where $z^4+1=0$ i.e. $z = e^{\frac{\pi}{4}\mathrm{i}}$, $z = e^{\frac{3\pi}{4}\mathrm{i}}$, $z = e^{\frac{5\pi}{4}\mathrm{i}}$ and $z = e^{\frac{7\pi}{4}\mathrm{i}}$. Consider the closed contour C that consists of a straight line along the x axis and a semi-circle above it. Then the only poles within the contour are $z = e^{\frac{\pi}{4}\mathrm{i}}$ and $z = e^{\frac{3\pi}{4}\mathrm{i}}$. The residues at these poles are*

$$\operatorname{Res}\left(f(z), e^{\frac{\pi}{4}\mathrm{i}}\right) = \frac{\sqrt{2}}{8}(1-\mathrm{i}) \quad \text{and} \quad \operatorname{Res}\left(f(z); e^{\frac{3\pi}{4}\mathrm{i}}\right) = -\frac{\sqrt{2}}{8}(1+\mathrm{i})$$

(finding these residuals is left as an exercise). Therefore,

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,\mathrm{d}x = 2\pi\mathrm{i}\frac{\sqrt{2}}{8}(1-\mathrm{i}-1-\mathrm{i}) = \frac{\sqrt{2}}{2}\pi\,.$$

Note that even though the residues are complex the obtained final answer is real and positive as expected since the original integral involves a real and non-negative function. ■

Example 4.33. *Evaluate* $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,\mathrm{d}x.$

Solution. *Let $f(z) = \dfrac{1}{(z^2+4)^2}$. This is a rational function. Its numerator has degree 0 and denominator has degree 4. Thus its integral vanishes at large values of the argument. Function $f(z)$ has poles at points where $(z^2+4)^2 = 0$ i.e. at $z = \pm 2\mathrm{i}$. These are poles of order two. We use the same contour as in the previous example, so we only need calculate the residue at $z = 2\mathrm{i}$. Note that $(z^2+4)^2 = (z-2\mathrm{i})^2(z+2\mathrm{i})^2$ so that both poles are of order two. Then*

$$\begin{aligned}
\operatorname{Res}(f(z); 2\mathrm{i}) &= \lim_{z\to 2\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z} \frac{(z-2\mathrm{i})^2}{(z-2\mathrm{i})^2(z+2\mathrm{i})^2} \\
&= \lim_{z\to 2\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{(z+2\mathrm{i})^2} = \lim_{z\to 2\mathrm{i}} \frac{-2}{(z+2\mathrm{i})^3} \\
&= \lim_{z\to 2\mathrm{i}} \frac{-2}{(4\mathrm{i})^3} = -\frac{\mathrm{i}}{32}\,.
\end{aligned}$$

Finally,

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,\mathrm{d}x = 2\pi\mathrm{i}\left(-\frac{\mathrm{i}}{32}\right) = \frac{\pi}{16}\,.$$

Again the result is real and positive as expected for a real positive integrand. ■

Integrals of the form $\int_0^{2\pi} g(\cos\theta, \sin\theta)\,\mathrm{d}\theta$. The idea is to let

$$z = e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta\,,$$

i.e. let z be a point on a unit circle. The position of such a point is fully described by angle θ alone. Then $\frac{1}{z} = z^{-1} = e^{-\mathrm{i}\theta} = \cos\theta - \mathrm{i}\sin\theta$ and we have

$$\boxed{\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{and} \quad \sin\theta = \frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right)\,,\ |z| = 1}\,. \tag{4.5}$$

Also $\frac{\mathrm{d}z}{\mathrm{d}\theta} = \mathrm{i}e^{\mathrm{i}\theta} = \mathrm{i}z$ i.e.

$$\boxed{\mathrm{d}\theta = -\mathrm{i}e^{-\mathrm{i}\theta}\,\mathrm{d}z = -\mathrm{i}\frac{\mathrm{d}z}{z}\,,\ |z| = 1}\,. \tag{4.6}$$

This then allows us to write the original integral in the form

$$\oint_{|z|=1} f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \times (\text{sum of residues inside the unit circle } |z| = 1).$$

Example **4.34.** *Evaluate* $\int_0^{2\pi} \frac{1}{3+\cos\theta}\,\mathrm{d}\theta$.

Solution. *Using (4.5) we obtain*

$$3 + \cos\theta = \frac{z^2 + 6z + 1}{2z}$$

and then using (4.6) we re-write the original integral as

$$\int_0^{2\pi} \frac{1}{3+\cos\theta}\,\mathrm{d}\theta = -\mathrm{i}\oint_{|z|=1} f(z)\,\mathrm{d}z\,,$$

where $f(z) = \dfrac{2}{z^2+6z+1}$ *(make sure that you do not forget the factor* $-\mathrm{i}$ *appearing in front of the transformed integral!). Solving a quadratic equation* $z^2+6z+1 = (z+3-2\sqrt{2})(z+3+2\sqrt{2}) = 0$ *we obtain that the* $f(z)$ *has two simple poles at* $z_1 = -3+2\sqrt{2} \approx -0.2$ *and* $z_2 = -3-2\sqrt{2} \approx -5.8$*. Only* z_1 *is inside the unit circle and thus only its residue needs to be computed:*

$$\begin{aligned}\mathrm{Res}(f(z); -3+2\sqrt{2}) &= \lim_{z\to-3+2\sqrt{2}} \frac{2(z-3+2\sqrt{2})}{(z+3-2\sqrt{2})(z+3+2\sqrt{2})} \\ &= \lim_{z\to-3+2\sqrt{2}} \frac{2}{(z+3+2\sqrt{2})} = \frac{1}{2\sqrt{2}}\,.\end{aligned}$$

Finally,

$$\int_0^{2\pi} \frac{1}{3+\cos\theta}\,\mathrm{d}\theta = -\mathrm{i}\oint_{|z|=1} f(z)\,\mathrm{d}z = -\mathrm{i}\cdot 2\pi\mathrm{i}\cdot\frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{2}\pi\,.$$

As a check point note that $3+\cos\theta > 0$ *for all* θ *and thus the integrand function is real and positive and so should be the resulting integral. The answer we obtained is indeed real and positive.* ■

4.4.5 Exercises

Ex. 4.16. Evaluate the integral $\displaystyle\oint_C \frac{2z}{z^2+1}\,\mathrm{d}z$, where C is

(a) the circle $|z| = \dfrac{1}{2}$, (b) the circle $|z| = 2$.

Ex. 4.17. Evaluate the integral $\displaystyle\oint_C \frac{z^2+3\mathrm{i}z-2}{z^3+9z}\,\mathrm{d}z$, where C is

(a) the circle $|z| = 1$, (b) the circle $|z| = 4$.

Ex. 4.18. Evaluate the integral $\displaystyle\oint_C \frac{\mathrm{d}z}{z^2(1+z^2)^2}$, where C is the circle $|z| = 2$.

Ex. 4.19. Evaluate the integral $\oint_C \frac{1}{2z^2-5z+2}dz$, where C is the unit circle $|z| = 1$.

Ex. 4.20. Evaluate the integral $\displaystyle\oint_C \frac{3z^2+2}{(z-1)(z^2+4)}\,\mathrm{d}z$, where C is

(a) the circle $|z-2| = 2$, (b) the circle $|z| = 4$.

Ex. 4.21. Evaluate the integral $\displaystyle\oint_C \frac{4-3z}{z^2-z}\,\mathrm{d}z$, where C is

(a) the circle $|z| = 2$,

(b) the circle $|z| = \dfrac{1}{2}$,

(c) the circle $|z-1| = \dfrac{1}{2}$,

(d) the circle $|z-3| = 1$.

Ex. 4.22. Use contour integration to evaluate the following real integrals

$$\text{(a)} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+2x+5}\,, \quad \text{(b)} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)^2}\,,$$
$$\text{(c)} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+x+1}\,, \quad \text{(d)} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)(x^2+4)}\,.$$

Ex. 4.23. Evaluate the following trigonometric integrals by transforming them to contour integrals

$$\text{(a)} \int_{0}^{2\pi} \frac{\mathrm{d}\theta}{2+\cos\theta}\,, \quad \text{(b)} \int_{0}^{2\pi} \frac{4\,\mathrm{d}\theta}{5+4\cos\theta}\,,$$
$$\text{(c)} \int_{0}^{2\pi} \frac{\mathrm{d}\theta}{2+\sin\theta}\,, \quad \text{(d)} \int_{0}^{2\pi} \frac{\mathrm{d}\theta}{5-4\cos\theta}\,.$$

4.5 Answers to Selected Exercises

Ex. 4.10.
(a) $f(z) = (4+3\mathrm{i})z^2$, (b) $f(z) = (3+2\mathrm{i})z^3$, (c) $f(z) = z^3 + 3z^2$,
(d) $f(z) = \mathrm{i}z^3 + (1+2\mathrm{i})z^2$, (e) $f(z) = ze^z$.

Ex. 4.11.

(a) $\dfrac{1}{z} - 2 + 3z - 4z^2 + \cdots$, $0 < |z| < 1$;

(b) $-\dfrac{1}{(z+1)^2} - \dfrac{1}{z+1} - 1 - (z+1) - (z+1)^2 \cdots$, $0 < |z+1| < 1$.

Ex. 4.12.

(a) Power series $-\dfrac{1}{2} - \dfrac{3}{4}z - \dfrac{7}{8}z^2 - \cdots$;

(b) Laurent series $\cdots + \dfrac{1}{z^3} + \dfrac{1}{z^2} + \dfrac{1}{z} + \dfrac{1}{2} + \dfrac{z}{4} + \dfrac{z^2}{8} + \dfrac{z^3}{16} + \cdots$;

(c) Power series $-\dfrac{1}{z^2} - \dfrac{3}{z^3} - \dfrac{7}{z^4} - \cdots$.

Ex. 4.14.

(a) Simple poles at $z = -1, 2$; residues are $\dfrac{1}{3}$ and $\dfrac{5}{3}$, respectively.

(b) Simple pole at $z = 1$, second order pole at $z = 0$; residues are -1 and 1, respectively.

(c) Simple poles at $z = 1, 3\mathrm{i}, -3\mathrm{i}$; residues are $\dfrac{1}{2}$, $\dfrac{5}{12}(3-\mathrm{i})$, $\dfrac{5}{12}(3+\mathrm{i})$, respectively.

(d) Simple pole at $z = -3$, second order pole at $z = 1$; residues are $-\dfrac{1}{8}$ and $\dfrac{1}{8}$, respectively.

(e) Pole of order 5 at $z = 1$; residue is 19.

(f) Simple poles at $z = 0, -2, -1$; residues are $\dfrac{3}{2}$, $-\dfrac{5}{2}$ and 1, respectively.

Ex. 4.15. (a) 1, (b) $-\pi$, (c) $-\dfrac{\mathrm{i}}{4}$.

Ex. 4.16. (a) 0, (b) $4\pi\mathrm{i}$.

Ex. 4.17. (a) $-\dfrac{4}{9}\pi\mathrm{i}$, (b) $2\pi\mathrm{i}$.

Ex. 4.18. 0.

Ex. 4.20. (a) $\pi\mathrm{i}$, (b) $6\pi\mathrm{i}$.

Ex. 4.21. (a) $-6\pi\mathrm{i}$, (b) $-8\pi\mathrm{i}$, (c) $2\pi\mathrm{i}$, (d) 0.

Ex. 4.22. (a) $\dfrac{\pi}{2}$, (b) $\dfrac{\pi}{2}$, (c) $\dfrac{2\pi\sqrt{3}}{3}$, (d) $\dfrac{\pi}{6}$.

Ex. 4.23. (a) $\dfrac{2\pi\sqrt{3}}{3}$, (b) $\dfrac{8\pi}{3}$, (c) $\dfrac{2\pi\sqrt{3}}{3}$, (d) $\dfrac{2\pi}{3}$.

Appendix A

Revision of Previously Studied Material

A.1 Systems of Linear Equations: Technical Review

A.1.1 Determinants and Cramer's rule

EXAMPLE **A.1.** *Solve the system of equations*

$$\begin{cases} 2x + 3y &= 8 \\ 4x + 5y &= 2 \end{cases}$$

SOLUTION.

$$x = \frac{\begin{vmatrix} 8 & 3 \\ 2 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 2 & 8 \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}}$$

so that

$$x = \frac{40 - 6}{10 - 12} = \frac{34}{-2} = -17\,, \quad y = \frac{4 - 32}{-2} = 14\,.$$

■

Cramer's rule generalizes to systems of n equations in n unknowns. In the case of three equations in three unknowns we have a system

$$\begin{cases} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{cases}$$

The solution can be written in terms of 3×3 determinants

$$x = \frac{\Delta_1}{\Delta}\,, \quad y = \frac{\Delta_2}{\Delta}\,, \quad z = \frac{\Delta_3}{\Delta}\,,$$

where we have the four 3×3 determinants

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\,, \qquad \Delta_1 = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}\,,$$

$$\Delta_2 = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}\,, \qquad \Delta_3 = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}$$

and the rule for expanding a 3×3 determinant is as follows

$$\begin{aligned} \Delta &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\,. \end{aligned}$$

This is Cramer's rule for a system of three linear equations in three unknowns. It can only be used if $\Delta \neq 0$ and is quite time-consuming.

A.1.2 Gaussian elimination

A more efficient and preferable approach to solving systems of linear equations is the method of *Gaussian elimination*. It has the merit of being applicable to systems of equations where the determinant is zero; moreover it can be applied to systems of equations to produce solutions in cases where the number of equations and of unknowns are not the same. We begin by giving an illustration of the process for a system of three equations in three unknowns.

EXAMPLE **A.2.** *Solve the following system of linear equations:*

$$\begin{cases} 3x_1 + 4x_2 - 2x_3 &= 3 \\ x_1 + 2x_2 + x_3 &= 5 \\ -x_1 - x_2 + 3x_3 &= 4 \end{cases}$$

SOLUTION. *We start with writing down the* augmented coefficient matrix *that combines the standard coefficient matrix and the vector of right-hand sides*[1]

$$\left[\begin{array}{rrr|r} 3 & 4 & -2 & 3 \\ 1 & 2 & 1 & 5 \\ -1 & -1 & 3 & 4 \end{array}\right]$$

Step 1: *Interchange row 2 and row 1 (to get 1 on the diagonal), then subtract 3 times the new top row from the new second row and add the new top row to the bottom row:*

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 5 \\ 3 & 4 & -2 & 3 \\ -1 & -1 & 3 & 4 \end{array}\right] \Longrightarrow \begin{array}{l} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 5 \\ 0 & -2 & -5 & -12 \\ 0 & 1 & 4 & 9 \end{array}\right] \end{array} \begin{array}{l} \\ r_2 - 3r_1 \\ r_3 + r_1 \end{array}$$

Step 2: *interchange rows 2 and 3 (to get 1 on the diagonal), then add 2 times the new second row to the new bottom row:*

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 5 \\ 0 & 1 & 4 & 9 \\ 0 & -2 & -5 & -12 \end{array}\right] \Longrightarrow \left[\begin{array}{rrr|r} 1 & 2 & 1 & 5 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 3 & 6 \end{array}\right] \begin{array}{l} \\ \\ r_3 + 2r_2 \end{array}$$

The elimination is complete. Each tableau above corresponds to a set of equations equivalent to the original one. In this example the successive stages are:

$$\left\{\begin{array}{rrrcr} 3x_1 & +4x_2 & -2x_3 & = & 3 \\ x_1 & +2x_2 & +x_3 & = & 5 \\ -x_1 & -x_2 & +3x_3 & = & 4 \end{array}\right. .$$

Then

$$\left\{\begin{array}{rrrcr} x_1 & +2x_2 & +x_3 & = & 5 \\ & -2x_2 & -5x_3 & = & -12 \\ & x_2 & +4x_3 & = & 9 \end{array}\right.$$

and finally

$$\left\{\begin{array}{rrrcr} x_1 & +2x_2 & +x_3 & = & 5 \\ & x_2 & +4x_3 & = & 9 \\ & & 3x_3 & = & 6 \end{array}\right. .$$

The final augmented matrix in its triangular form will enable us to determine the solution one variable at a time, starting at the bottom and substituting values successively. In this example we obtain

$$3x_3 = 6 \text{ so that } x_3 = 2$$

[1]The augmented coefficient matrix is also frequently referred to as *tableau* in this context.

and substituting this value for x_3 in the second equation gives

$$-2x_2 - 5 \cdot 2 = -12 \text{ so that } x_2 = (12 - 10)/2 = 1$$

and then substituting 2 for x_3 and 1 for x_2 in the first equation gives

$$x_1 + 2 \cdot 1 + 2 = 5 \text{ so that } x_1 = 5 - 2 - 2 = 1\,.$$

That is, the final solution is $x_1 = x_2 = 1$, $x_3 = 2$. ■

General procedure for a system of m equations in n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + & \cdots & +a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + & \cdots & +a_{2n}x_n & = & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + & \cdots & +a_{mn}x_n & = & b_m \end{cases}$$

is to write down the coefficients of the system in an augmented coefficient matrix (a rectangular array also called tableau) with the right-hand sides b_1, b_2, …, b_m in an additional column as

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Then on this tableau we perform the following operations for each column beginning with the first one:

1. Examine the elements in the column that are on or below the main diagonal. At least one of these must be non-zero. Interchange rows if necessary until there is a non-zero number on the principal diagonal (for hand calculation having a unity on this diagonal, if that is possible, leads to the easiest procedure).

2. If the column number is j, rows 1 to j are unchanged. For all lower rows, we add or subtract a multiple of row j, choosing the multiplier so as to make the element in column j become 0. From row i $(i > j)$ we subtract $\dfrac{a_{ij}}{a_{jj}} \times \text{row}\, j$. Column j will then be zero below the main diagonal.

There is nothing to be done for the final column. The final tableau that results from this process will be the *upper triangular matrix*[2] augmented with an extra column $\mathbf{c}$:

$$\left[\begin{array}{cccc|c} t_{11} & t_{12} & \cdots & t_{1n} & c_1 \\ 0 & t_{22} & \cdots & t_{2n} & c_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & t_{mn} & c_m \end{array}\right]$$

We can now solve the corresponding system of equations. If t_{mn} is the only non-zero term in the last line then that line is is equivalent to the equation $t_{mn}x_m = c_m$. Then we can solve for x_m and substitute the result into the equation corresponding to the second last line and continue working back through the system of equations. The process of Gaussian elimination brings the matrix of coefficients into the *row echelon form*. It is recognised by the following pattern. Look along each row and find the first non-zero element in the row. If each of these is to the right of all the ones above it, the matrix is in the row echelon form (if a row has only zeroes, there must be no non-zero elements below it).

Example **A.3.** *Each of the following is in the row echelon form (the first non-zero elements are marked in* **bold***):*

$$\begin{bmatrix} \mathbf{2} & 7 & 1 \\ 0 & \mathbf{3} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{2} & 7 & 1 & 4 \\ 0 & \mathbf{3} & 0 & 2 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{2} & 7 & 1 \\ 0 & \mathbf{3} & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{bmatrix}.$$

Example **A.4.** *The following are* not *in the row echelon form:*

$$\begin{bmatrix} \mathbf{2} & 7 & 1 \\ 0 & \mathbf{3} & 0 \\ 0 & \mathbf{1} & 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{2} & 7 & 1 & 4 \\ 0 & 0 & \mathbf{3} & 2 \\ 0 & \mathbf{1} & 0 & 0 \end{bmatrix}. \quad \begin{bmatrix} \mathbf{2} & 7 & 1 \\ 0 & \mathbf{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}.$$

Note: any upper triangular matrix is in the row echelon form.

[2] The term *upper triangular* means that all of the elements below the main diagonal are zero.

Exercises

Ex. A.1. Use Gaussian elimination for each of the following sets of equations to decide whether the set is consistent and find all solutions of the system. Where there are infinitely many solutions, give the answer in parametric and vector forms.

(a) $\left\{\begin{array}{rcl} x+2y+2z &=& 3 \\ 2x+2y-2z &=& 1 \\ 3x+4y &=& 3 \end{array}\right.$ (b) $\left\{\begin{array}{rcl} a+2b-c &=& 1 \\ 2a+3b+2c &=& 3 \\ a+b+3c &=& 4 \end{array}\right.$

(c) $\left\{\begin{array}{rcr} x-y &=& -1 \\ x+2y &=& 9 \\ 3x+3y &=& 17 \end{array}\right.$ (d) $\begin{array}{rcl} x_1+x_2+2x_3 &=& 1 \\ x_1-x_2-2x_3 &=& 5 \end{array}$

(e) $\left\{\begin{array}{rcl} 3x_1+2x_2-2x_3 &=& 5 \\ x_1+x_2+3x_3 &=& 4 \end{array}\right.$ (f) $\left\{\begin{array}{rcl} a+2b+c &=& 0 \\ a-3b &=& 0 \\ 4a+3c &=& 0 \end{array}\right.$

(g) $\left\{\begin{array}{rcl} a+b+c &=& 0 \\ a-3b &=& 0 \\ 4a+5c &=& 0 \end{array}\right.$ (h) $\left\{\begin{array}{rcl} r+2s-t &=& 1 \\ 4r+3s+t &=& 3 \\ 2r-s+3t &=& 2 \end{array}\right.$

(i) $\left\{\begin{array}{rcl} 2x+y &=& 4 \\ 3x-y &=& 1 \\ x+4y &=& 9 \\ -x+3y &=& 5 \end{array}\right.$ (j) $\left\{\begin{array}{rcl} 2x+y &=& 4 \\ 3x-y &=& 1 \\ 5x+4y &=& 9 \end{array}\right.$

(k) $\left\{\begin{array}{rcl} 2x-y+3z &=& 4 \\ x+4y+z &=& 6 \\ 3x-y-2z &=& 0 \end{array}\right.$ (l) $\left\{\begin{array}{rcl} x_1+2x_2-x_3+x_4 &=& 1 \\ 3x_1-2x_2+x_3+2x_4 &=& 4 \\ x_1+9x_2-5x_3+2x_4 &=& 4 \end{array}\right.$

(m) $\left\{\begin{array}{rcl} x_1+2x_2+x_3 &=& 1 \\ 2x_1-x_2+x_3+x_4 &=& 0 \\ 3x_1+x_2-x_3-x_4 &=& 1 \\ 4x_3+2x_4 &=& 1 \end{array}\right.$ (n) $\left\{\begin{array}{rcl} x_1+2x_2+x_3 &=& 1 \\ 2x_1-x_2+x_3+x_4 &=& 0 \\ 3x_1+x_2-x_3-x_4 &=& 0 \\ -x_2+x_3+x_4 &=& 0 \end{array}\right.$

(o) $\left\{\begin{array}{rcl} x_1+2x_2+x_3 &=& 1 \\ 2x_1-x_2+x_3+x_4 &=& 0 \\ 3x_1+x_2-x_3-x_4 &=& 1 \\ x_2+4x_3+2x_4 &=& 1 \end{array}\right.$ (p) $\left\{\begin{array}{rcl} x_1+2x_2+x_3 &=& 1 \\ 2x_1+4x_2+x_3+x_4 &=& 0 \\ 3x_1+6x_2+2x_3-x_4 &=& 1 \\ x_3-x_4 &=& 2 \end{array}\right.$

A.1.3 Geometrical meaning of linear equations in three dimensions

An equation such as $x_1 + 2x_2 + x_3 = 5$ can be interpreted geometrically as a plane in the three-dimensional (x_1, x_2, x_3) space. A pair of such equations describe a pair of planes. Generally they will intersect along a line. If we have three equations the three planes will generally intersect at a point. However, there are exceptions. For example, it is possible that the planes are parallel and do not intersect. The system of equations in this case has no solution. Three planes can also intersect along the same straight line as in the following example.

EXAMPLE **A.5.** *Solve the following system of linear equations and interprete the answer geometrically*

$$\begin{cases} x_1 + 2x_2 + 3x_3 & = & 7 \\ x_1 + 4x_2 - x_3 & = & 5 \\ x_1 + 3x_2 + x_3 & = & 6 \end{cases}.$$

SOLUTION. *The augmented matrix is reduced as follows*

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 1 & 4 & -1 & 5 \\ 1 & 3 & 1 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 2 & -4 & -2 \\ 0 & 1 & -2 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The three equations have been reduced to the pair of equations

$$\begin{cases} x_1 & +2x_2 & +3x_3 & = & 7 \\ & x_2 & -2x_3 & = & -1 \end{cases}.$$

We can write down the solution in terms of a new parameter t by setting $x_3 = t$ then substituting into the second equation we obtain $x_2 = -1 + 2t$. Substituting into the first equation gives $x_1 = 7 - 2(-1 + 2t) - 3t = 9 - 7t$. That is the solution is the straight line with the parametric equations

$$\begin{cases} x_1 & = & 9 - 7t \\ x_2 & = & -1 + 2t \\ x_3 & = & t \end{cases} \text{ or in the vector form } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}.$$

The three planes described by the given linear equations intersect along this line. ■

When more than three unknowns are present in a system of equations its solution can be interpreted as an intersection of *hyperplanes* in high-dimensional space. The following example considers the intersection of hyperplanes in a four-dimensional space.

EXAMPLE **A.6.** *Find the intersection of the following hyper-planes*

$$\begin{cases} x_1 & +3x_2 & +x_3 & -x_4 & = & 8 \\ 2x_1 & +x_2 & +2x_3 & +x_4 & = & 16 \\ x_1 & -3x_2 & +x_3 & +3x_4 & = & 10 \\ x_1 & +2x_2 & +x_3 & -x_4 & = & 5 \end{cases}$$

SOLUTION. *Write down the tableau for the system of equations and follow the steps of Gaussian elimination*

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & -1 & 8 \\ 2 & 1 & 2 & 1 & 16 \\ 1 & -3 & 1 & 3 & 10 \\ 1 & 2 & 1 & -1 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & -1 & 8 \\ 0 & -5 & 0 & 3 & 0 \\ 0 & -6 & 0 & 4 & 2 \\ 0 & -1 & 0 & 0 & -3 \end{array}\right] \begin{array}{l} \\ r_2 - 2r_1 \\ r_3 - r_1 \\ r_4 - r_1 \end{array}.$$

This is rearranged to

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & -1 & 8 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & -5 & 0 & 3 & 0 \\ 0 & -6 & 0 & 4 & 2 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & -1 & 8 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 15 \\ 0 & 0 & 0 & 4 & 20 \end{array}\right] \begin{array}{l} \\ \\ r_3 + 5r_2 \\ r_4 + 6r_2 \end{array}$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & -1 & 8 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The elimination is now complete: the tableau is in the row echelon form. It corresponds to the following system of equations

$$\begin{cases} x_1 & +3x_2 & +x_3 & -x_4 & = & 8 \\ & x_2 & & & = & 3 \\ & & & x_4 & = & 5 \\ & & & 0 & = & 0 \end{cases}.$$

We can now obtain the general solution. From the third equation we have $x_4 = 5$, and from the second equation we have $x_2 = 3$. Substituting these into the first equation gives $x_1 + 3(3) + x_3 - 5 = 8$. If we now set $x_3 = t$ we obtain $x_1 = 4 - t$. The complete solution is $x_1 = 4 - t$, $x_2 = 3$, $x_3 = t$, $x_4 = 5$, which can be written in the vector form as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4-t \\ 3 \\ t \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Note also that this an example of a system of equations where the determinant is zero and the system has infinitely many solutions representing points along a line that is the intersection of four hyperplanes. The method of introducing a new parameter, or more than one parameter, can be used generally if we have more unknowns than equations in the final array. ■

EXAMPLE **A.7.** *The final array after Gaussian elimination is*

$$\left[\begin{array}{cccr|c} 1 & 2 & 1 & -2 & 6 \\ 0 & 1 & 2 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Write down the general solution to the system.

SOLUTION. *The array corresponds to the system of equations*

$$\begin{cases} x_1 & +2x_2 & +x_3 & -2x_4 & = & 6 \\ & x_2 & +2x_3 & +x_4 & = & 9 \end{cases}.$$

We introduce two parameters: $x_3 = s$, $x_4 = t$. *Then from the last equation we obtain* $x_2 = 9 - 2s - t$. *We now substitute this into the first equation to obtain* $x_1 + 2(9 - 2s - t) + s - 2t = 6$ *so that* $x_1 = -12 + 3s + 4t$. *The general solution then can be written in the vector form*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -12 + 3s + 4t \\ 9 - 2s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -12 \\ 9 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

■

EXAMPLE **A.8.** *The final array after Gaussian elimination is*

$$\left[\begin{array}{cccr|c} 1 & 2 & 1 & -2 & 6 \\ 0 & 1 & 2 & 1 & 9 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right].$$

Write down the general solution to the system.

SOLUTION. *The array corresponds to the system of equations*

$$\begin{cases} x_1 & +2x_2 & +x_3 & -2x_4 & = & 6 \\ & x_2 & +2x_3 & +x_4 & = & 9 \\ & & & 0 & = & 2 \end{cases}.$$

This last equation is contradictory, hence the system has no solution.

■

A.2 Answers to Selected Review Exercises

Ex. A.1. (a) inconsistent, no solution;

(b) inconsistent, no solution;

(c) $x = \frac{7}{3}$, $y = \frac{10}{3}$;

(d) $x_1 = 3$, $x_2 = -2 - 2t$, $x_3 = t$;

(e) $x_1 = 3 - 8t$, $x_2 = 7 - 11t$, $x_3 = t$;

(f) $a = 0$, $b = 0$, $c = 0$;

(g) $a = 0$, $b = 0$, $c = 0$;

(h) inconsistent, no solution;

(i) $x = 1$, $y = 2$;

(j) inconsistent, no solution;

(k) $x = 1$, $y = 1$, $z = 1$;

(l) $x_1 = \frac{5}{4} - \frac{3}{4}t$, $x_2 = -4$, $x_3 = \frac{1}{4}t - \frac{31}{4}$, $x_4 = t$;

(m) $x_1 = \frac{1}{5}$, $x_2 = -\frac{1}{10}$, $x_3 = 1$, $x_4 = -\frac{3}{2}$;

(n) $x_1 = 0$, $x_4 = t$, $x_2 = \frac{1}{3} + \frac{1}{3}t$, $x_3 = \frac{1}{3} - \frac{2}{3}t$;

(o) $x_1 = \frac{1}{5}$, $x_2 = -\frac{1}{5}$, $x_3 = \frac{6}{5}$, $x_4 = -\frac{9}{5}$;

(p) $x_1 = -1 - 2t$, $x_2 = t$, $x_3 = 2$, $x_4 = 0$.

A.3 Vectors and Matrices: Technical Review

A.3.1 Main definitions

Having seen a few simple examples of using matrices and vectors we now introduce their formal definitions.

Definition A.1 Vector *is a one-dimensional array of numbers called* vector elements.

We will distinguish between *row vectors* $\mathbf{r} = [x_1\ x_2\ \cdots\ x_n]$ and *column vectors* $\mathbf{c} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$. Column and row vectors are related via the operation of *transposition*: $\mathbf{r} = \mathbf{c}^T$, $\mathbf{c} = \mathbf{r}^T$.

Definition A.2 Matrix *is a two-dimensional rectangular array of numbers called* matrix elements.

The matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is said to be an $m \times n$ matrix as it has m rows and n columns (m and n are some positive integer numbers). The element a_{ij} is located in the i-th row and in the j-th column. The rows can be described as $\mathbf{r}_1(\mathbf{A}) = [a_{11}\, a_{12}\, \cdots\, a_{1n}]$, $\mathbf{r}_2(\mathbf{A}) = [a_{21}\, a_{22}\, \cdots\, a_{2n}]$, $\cdots$, $\mathbf{r}_m(\mathbf{A}) = [a_{m1}\, a_{m2}\, \cdots\, a_{mn}]$. These are row vectors. The columns are

$$\mathbf{c}_1(\mathbf{A}) = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2(\mathbf{A}) = \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{bmatrix}, \quad \ldots, \quad \mathbf{c}_n(\mathbf{A}) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix}.$$

These are column vectors. Thus, we can write a matrix as being made up of row vectors or column vectors:

$$\mathbf{A} = \begin{bmatrix} \mathbf{r}_1(\mathbf{A}) \\ \mathbf{r}_2(\mathbf{A}) \\ \cdots \\ \mathbf{r}_m(\mathbf{A}) \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1(\mathbf{A}) & \mathbf{c}_2(\mathbf{A}) & \cdots & \mathbf{c}_n(\mathbf{A}) \end{bmatrix}.$$

It is easy to see that vectors can be considered as a special case matrices: a row vector is a $1 \times n$ matrix and a column vector is a $m \times 1$ matrix for some positive integer numbers m and n. A scalar can be defined in a similar way.

Definition A.3 *A* 1×1 *matrix (a single number) is called a* scalar.

Definition A.4 *The square of the* magnitude of a vector, *or of the* length of a vector, *is defined as the sum of squares of its elements*

$$\boxed{||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}}.$$

Definition A.5 *Vector* $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{||\mathbf{v}||}$ *with a unit magnitude is called the* unit vector *in the direction of* $\mathbf{v}$.

Example **A.9.** *If* $\mathbf{v} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$ *then* $||\mathbf{v}|| = \sqrt{36+4+9} = 7$ *and*

$$\hat{\mathbf{v}} = \frac{1}{7}\begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{6}{7} \\ -\frac{2}{7} \\ \frac{3}{7} \end{bmatrix}.$$

Definition A.6 *The two vectors are equal if all their respective elements are equal.*

It follows then that the equality of two n-element vectors can be interpreted as a set of n algebraic equations for their coordinates. For example, if $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ then the equation $\mathbf{u} = \mathbf{v}$ is precisely the same as the pair of equations $a = c$ and $b = d$.

A.3.2 Basic operations with vectors

Vector addition and multiplication by a scalar. These operations are illustrated below for two-element vectors, but they are valid for vectors with an arbitrary number of elements.

1. *Vector addition/commutativity*: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
   e.g. if $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ then
   $\mathbf{u} + \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix} = \begin{bmatrix} c+a \\ d+b \end{bmatrix} = \mathbf{v} + \mathbf{u}$.
2. *Associativity*: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. *Multiplication by a scalar*: if $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ then for any real number k,
   $k\mathbf{u} = k\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ka \\ kb \end{bmatrix}$.
4. *Vector negation*: the negative of $\mathbf{u}$ is $-\mathbf{u} = -1\mathbf{u}$. It is the vector of the same magnitude but in the opposite direction and
   $\mathbf{u} + (-\mathbf{u}) = \mathbf{u} - \mathbf{u} = \mathbf{0}$.
5. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
6. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

7. $(k_1 + k_2)\mathbf{u} = k_1\mathbf{u} + k_2\mathbf{u}$

The operations involving multiplication of vectors are reviewed next.

Definition A.7 *If* $\mathbf{r}$ *is a* $1 \times n$ *row vector and* $\mathbf{c}$ *is a* $n \times 1$ *column vector then the product between them is defined as*

$$\mathbf{r} \cdot \mathbf{c} = [r_1 \; r_2 \; \cdots \; r_n] \cdot \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n \,.$$

EXAMPLE **A.10.** *If* $\mathbf{r} = [2 \;\; -3 \;\; 4 \;\; 5]$ *and* $\mathbf{c} = \begin{bmatrix} 4 \\ -1 \\ -3 \\ 2 \end{bmatrix}$ *then*

$$\mathbf{r} \cdot \mathbf{c} = 2 \times 4 + (-3) \times (-1) + 4 \times -3 + 5 \times 2 = 9 \,.$$

Note: the order of multiplication is important (you can multiply a row by a column, but not vice versa).

Scalar (dot) product. Multiplication of row $\mathbf{r}$ by a column $\mathbf{c}$ can be viewed as a standard *scalar (dot) product* between vectors $\mathbf{r}$ and $\mathbf{c}$.

Definition A.8 *The* dot product *between two* n*-element vectors* $\mathbf{r} = [r_1, r_2, \ldots, r_n]$ *and* $\mathbf{c} = [c_1, c_2, \ldots, c_n]$ *is defined as*

$$\boxed{\mathbf{r} \cdot \mathbf{c} = r_1c_1 + r_2c_2 + \cdots + r_nc_n} \,.$$

Vector (cross) product

Definition A.9 *The* vector product *of two three-element vectors* $\mathbf{a} = [a_1, a_2, a_3]$ *and* $\mathbf{b} = [b_1, b_2, b_3]$ *is defined as*

$$\boxed{\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}} \,,$$

where $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are the coordinate vectors to be introduced in the next section.*

A.3.3 Geometrical meaning of vectors and vector operations

Recall that the position vector of a point P with coordinates (x, y) in the plane is given by $\mathbf{r} = \overrightarrow{OP} = x\mathbf{i}+y\mathbf{j}$. Here O denotes the origin of a coordinate system, $\mathbf{i}$ and $\mathbf{j}$ are the *coordinate vectors* in the direction of the x and y axes, respectively.

Definition A.10 *A vector starting at the origin and ending at a given point is called the* position vector*.*

In the above planar example it has *vector components* $x\mathbf{i}$ and $y\mathbf{j}$ and *vector coordinates* x and y. The coordinates x and y are identical to vector elements defined in the previous section. They are scalar factors multiplying the coordinate vectors. The coordinate vectors themselves can be written as

$$\boxed{\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}.$$

Therefore

$$\mathbf{r} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Similarly, for vectors in three dimensional space we have

$$\boxed{\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}},$$

and the position vector of a point is written as

$$\mathbf{r} = \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

with

$$\boxed{||\mathbf{r}|| = \sqrt{x^2 + y^2 + z^2}}.$$

The above examples demonstrate that the multiplication by a scalar stretches or compresses the length of a vector and can change its direction to opposite, but it cannot rotate a vector.

Alternative definition of the dot product

Definition A.11 *The* dot product *between two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is defined as*

$$\boxed{\mathbf{a}\cdot\mathbf{b} = ||a||\,||b||\cos\theta},$$

where θ *is the angle between* $\mathbf{a}$ *and* $\mathbf{b}$.

If the vectors are co-directed then $\theta = 0$ and their dot product is equal to the product of their magnitudes. Thus, the dot product of a vector with itself is equal to the square of its magnitude, or

$$\boxed{||\mathbf{a}|| = \sqrt{\mathbf{a}\cdot\mathbf{a}}}.$$

Note: for any vector $\mathbf{a}$, $||\mathbf{a}|| \geq 0$, and $||\mathbf{a}|| = 0$ if and only if $\mathbf{a} = \mathbf{0}$, that is $a_i = 0$ for each i.

Definition A.12 *Vector* $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ *is called* zero vector.

Note: zero vector is the only vector with zero magnitude.

Definition A.13 *Two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are said to be* orthogonal vectors *if* $\mathbf{a}\cdot\mathbf{b} = 0$.

EXAMPLE **A.11.** *If* $\mathbf{a} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ *then*

$\mathbf{a}\cdot\mathbf{b} = 3\times 2 + (-1)\times 2 + 4\times(-1) = 0$ *so that vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal,* $\mathbf{a}\perp\mathbf{b}$.

If the vectors are *orthogonal*, then $\theta = \dfrac{\pi}{2}$ because $\mathbf{a} \cdot \mathbf{b} = ||a||\,||b|| \cos \dfrac{\pi}{2} = 0$. By definition, the coordinate vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are chosen to be mutually orthogonal. Therefore the dot products of the unit position vector $\hat{\mathbf{r}}$ with the unit coordinate vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ result in

$$\hat{\mathbf{r}} \cdot \mathbf{i} = \frac{x}{||r||} = \cos\alpha\,, \quad \hat{\mathbf{r}} \cdot \mathbf{j} = \frac{y}{||r||} = \cos\beta\,, \quad \hat{\mathbf{r}} \cdot \mathbf{k} = \frac{z}{||r||} = \cos\gamma\,,$$

so that $\hat{\mathbf{r}} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k} = (\cos\alpha, \cos\beta, \cos\gamma)$, where α, β and γ are the angles between $\hat{\mathbf{r}}$ and the x, y and z axes, respectively. We call $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ the *direction cosines* of $\hat{\mathbf{r}}$. Since $\hat{\mathbf{r}}$ is a unit vector, by Pythagoras theorem its magnitude is given by

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1\,.$$

The direction cosines of an arbitrary vector are the direction cosines of the corresponding unit vector.

EXAMPLE **A.12.** *Find the direction cosines of the vector* $\mathbf{a} = [3, -2, 6]$.

SOLUTION. *The magnitude of the given vector is*

$$||\mathbf{a}|| = \sqrt{9 + 4 + 36} = 7$$

and the corresponding unit vector is $\hat{\mathbf{a}} = \dfrac{1}{7}[3, -2, 6]$. *Then the direction cosines are*

$$\cos\alpha = \frac{3}{7}\,, \quad \cos\beta = -\frac{2}{7}\,, \quad \cos\gamma = \frac{6}{7}\,.$$

From this we calculate the corresponding angles

$$\alpha = \cos^{-1}\frac{3}{7} \approx 1.1279\,, \ \beta = \cos^{-1}\left(-\frac{2}{7}\right) \approx 1.8606\,, \ \gamma = \cos^{-1}\frac{6}{7} \approx 0.5411\,.$$

■

EXAMPLE **A.13.** *If* $\mathbf{a} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 4 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$ *then*

$$\mathbf{a} \cdot \mathbf{b} = 3 \times 1 + 1 \times (-3) + (-2) \times 2 + 4 \times 5 = 16\,.$$

The magnitudes of these vectors are

$$||\mathbf{a}|| = \sqrt{3^2 + 1^2 + (-2)^2 + 4^2} = \sqrt{30}$$

and

$$||\mathbf{b}|| = \sqrt{1^2 + (-3)^2 + 2^2 + 5^2} = \sqrt{39}\,.$$

Vectors

$$\begin{bmatrix} \frac{3}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ \frac{-2}{\sqrt{30}} \\ \frac{4}{\sqrt{30}} \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \frac{1}{\sqrt{39}} \\ \frac{-3}{\sqrt{39}} \\ \frac{2}{\sqrt{39}} \\ \frac{5}{\sqrt{39}} \end{bmatrix}$$

are the unit vectors in the direction of **a** *and* **b**, *respectively.*

Exercises

Ex. A.2. Find the dot product of the vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Ex. A.3. Let $\mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 7 \\ -3 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 5 \\ -2 \\ 8 \\ 1 \end{bmatrix}$.

(a) Find
(i) $\mathbf{v} - \mathbf{w}$, (ii) $2\mathbf{u} + 7\mathbf{v}$, (iii) $-\mathbf{u} + (\mathbf{v} - 4\mathbf{w})$, (iv) $6\mathbf{u} - 3\mathbf{v}$,
(v) $-\mathbf{v} - \mathbf{w}$, (vi) $(6\mathbf{v} - \mathbf{w}) - (4\mathbf{u} + \mathbf{v})$,
(vii) $||\mathbf{u}||$, $||\mathbf{v}||$, $||\mathbf{u}|| + ||\mathbf{v}||$, $||\mathbf{u} + \mathbf{v}||$.

(b) Calculate the dot products $\mathbf{v} \cdot \mathbf{w}$, $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{v} \cdot \mathbf{w}$.

Ex. A.4. Show that the vectors $\mathbf{c} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{d} = \begin{bmatrix} 2 \\ -3 \\ -2 \\ 0 \end{bmatrix}$ are orthogonal.

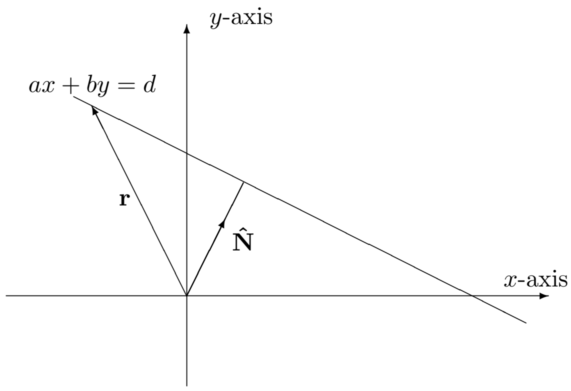


Figure A.1: Perpendicular vector to a line.

Further examples

EXAMPLE **A.14.** *Write an equation of a straight line in a vector form.*

SOLUTION. *Note that the expression $ax + by + cz$ can be factorized using scalar product as*

$$ax + by + cz = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = (a, b, c) \cdot (x, y, z)\,,$$

which in the case of two dimensional vectors reduces to

$$ax + by = (a\mathbf{i} + b\mathbf{j}) \cdot (x\mathbf{i} + y\mathbf{j}) = (a, b) \cdot (x, y)\,.$$

This observation allows us to write the equation $ax + by = d$ of a straight line in the (x, y) plane as $\mathbf{N} \cdot \mathbf{r} = d$, where $\mathbf{N} = a\mathbf{i} + b\mathbf{j} = (a, b)$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = (x, y)$. We can also rewrite it in the equivalent form as $||\mathbf{N}||\hat{\mathbf{N}} \cdot \mathbf{r} = d$ or $\hat{\mathbf{N}} \cdot \mathbf{r} = \dfrac{d}{||\mathbf{N}||} = \dfrac{d}{\sqrt{a^2 + b^2}}$. Recollect from your previous studies that the vector of coefficients $\mathbf{N}$ is perpendicular to the line $ax + by = d$ and that the dot product of a unit vector $\hat{\mathbf{N}}$ with vector $\mathbf{r}$ gives the projection of $\mathbf{r}$ onto the direction of $\hat{\mathbf{N}}$. Therefore we conclude that $\dfrac{|d|}{\sqrt{a^2 + b^2}}$ is the perpendicular distance from the line to the origin. ■

EXAMPLE **A.15.** *In three dimensions a general equation of a plane $ax + by + cz = d$ becomes in vector form $\mathbf{N} \cdot \mathbf{r} = d$, where $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = (a, b, c)$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x, y, z)$. Similarly to* *the discussion above recollect from your previous studies that* $\mathbf{N} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k} = (a, b, c)$ *is the* normal to the plane $ax+by+cz = d$, *and the perpendicular distance from the plane to the origin is* $\dfrac{|d|}{\sqrt{a^2+b^2+c^2}}$.

Alternative definition of the cross product

Definition A.14 *The* vector product *of two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is defined as*

$$\boxed{\mathbf{a} \times \mathbf{b} = ||a||\,||b||\sin\theta\ \mathbf{n}},$$

where $\mathbf{n}$ *is the unit vector orthogonal to* $\mathbf{a}$ *and* $\mathbf{b}$ *in the direction given by the right-hand rule and* θ *is the angle between the vectors.*

It has the following properties:

1. The vector product of any two parallel vectors is a zero vector.
2. For any vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
3. For coordinate vectors of the right coordinate system and $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.
4. The vector $\mathbf{a} \times \mathbf{b}$ has the magnitude equal to the area of the parallelogram spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$.

A.3.4 Matrices

Further definitions

We have already indicated that the determinant of a 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is given by } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(\mathbf{A}) = ad - bc$$

and the determinant of a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

is given by

$$\det(\mathbf{A}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Definition A.15 *A square matrix is said to be* diagonal *if the only nonzero elements that it contains lie on the main diagonal.*

EXAMPLE **A.16.** $\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Definition A.16 *A diagonal matrix is a* unit matrix *if all of the elements on the main diagonal are 1.*

EXAMPLE **A.17.** $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

If $\mathbf{A}$ is a square matrix with the same dimensions as $\mathbf{I}$, then $\mathbf{AI} = \mathbf{A}$.

Definition A.17 *The* transpose of a matrix $\mathbf{A}$, *denoted* $\mathbf{A}^T$ *is the matrix obtained by interchanging rows and columns. Thus,* $r_1(\mathbf{A}^T) = (c_1(\mathbf{A}))^T$, $r_2(\mathbf{A}^T) = (c_2(\mathbf{A}))^T$ *and so on.*

We could also express this as $c_1(\mathbf{A}^T) = (r_1(\mathbf{A}))^T$, $c_2(\mathbf{A}^T) = (r_2(\mathbf{A}))^T$ and so on.

If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{A}^T$ is an $n \times m$ matrix.

EXAMPLE **A.18.** *If* $\mathbf{A} = \begin{bmatrix} 3 & 2 & -3 \\ -5 & 7 & 1 \end{bmatrix}$ *then* $\mathbf{A}^T = \begin{bmatrix} 3 & -5 \\ 2 & 7 \\ -3 & 1 \end{bmatrix}$.

Matrix operations

Matrix addition. If two matrices have the same dimensions then they can be added by adding their corresponding elements.

EXAMPLE **A.19.**

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 1 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 1 & 7 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & -2 \end{bmatrix} - \begin{bmatrix} -1 & 3 & 0 \\ -4 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 7 & -1 & 0 \end{bmatrix}.$$

Multiplication by a scalar. Any matrix can be multiplied by a scalar quantity.

EXAMPLE **A.20.**

$$3\begin{bmatrix} 2 & -1 & -1 \\ 7 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -3 & -3 \\ 21 & -3 & 0 \end{bmatrix}.$$

Matrix multiplication

Definition A.18 *For any $m \times p$ matrix $\mathbf{A}$ and $p \times n$ matrix $\mathbf{B}$,*

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix}$$

the product of the matrices $\mathbf{AB}$ is defined by

$$\mathbf{AB} = \begin{bmatrix} r_1(\mathbf{A}) \cdot c_1(\mathbf{B}) & r_1(\mathbf{A}) \cdot c_2(\mathbf{B}) & \cdots & r_1(\mathbf{A}) \cdot c_n(\mathbf{B}) \\ r_2(\mathbf{A}) \cdot c_1(\mathbf{B}) & r_2(\mathbf{A}) \cdot c_2(\mathbf{B}) & \cdots & r_2(\mathbf{A}) \cdot c_n(\mathbf{B}) \\ \cdots & \cdots & \cdots & \cdots \\ r_m(\mathbf{A}) \cdot c_1(\mathbf{B}) & r_m(\mathbf{A}) \cdot c_2(\mathbf{B}) & \cdots & r_m(\mathbf{A}) \cdot c_n(\mathbf{B}) \end{bmatrix}.$$

The product of an $m \times p$ matrix with a $p \times n$ matrix gives an $m \times n$ matrix.

EXAMPLE **A.21.** *If* $\mathbf{A} = \begin{bmatrix} 3 & 2 & -3 \\ -5 & 7 & 1 \end{bmatrix}$ *and* $\mathbf{B} = \begin{bmatrix} 2 & 6 \\ 4 & 0 \\ -8 & 2 \end{bmatrix}$ *then we can form the products* $\mathbf{AB}$ *and* $\mathbf{BA}$*. They are*

$$\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} 3 & 2 & -3 \\ -5 & 7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 4 & 0 \\ -8 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 3 \times 2 + 2 \times 4 + (-3) \times (-8) & 3 \times 6 + 2 \times 0 + (-3) \times 2 \\ (-5) \times 2 + 7 \times 4 + 1 \times (-8) & (-5) \times 6 + 7 \times 0 + 1 \times 2 \end{bmatrix} \\
&= \begin{bmatrix} 38 & 12 \\ 10 & -28 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{BA} &= \begin{bmatrix} 2 & 6 \\ 4 & 0 \\ -8 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & -3 \\ -5 & 7 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \times 3 + 6 \times (-5) & 2 \times 2 + 6 \times 7 & 2 \times (-3) + 6 \times 1 \\ 4 \times 3 + 0 \times (-5) & 4 \times 2 + 0 \times 7 & 4 \times (-3) + 0 \times 1 \\ (-8) \times 3 + 2 \times (-5) & (-8) \times 2 + 2 \times 7 & (-8) \times (-3) + 2 \times 1 \end{bmatrix} \\
&= \begin{bmatrix} -24 & 46 & -12 \\ 12 & 8 & -12 \\ -34 & -2 & 26 \end{bmatrix}.
\end{aligned}$$

Note: this example illustrates that for matrices, generally $\mathbf{BA} \neq \mathbf{AB}$.

Transpose of a matrix product. If the matrix product $\mathbf{AB}$ is well defined then

$$\boxed{(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T}.$$

Determinant of a matrix product. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size then

$$\boxed{\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})}.$$

Matrix inversion

Definition A.19 *A square matrix* $\mathbf{A}^{-1}$ *is the inverse of the square matrix* $\mathbf{A}$ *if* $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

It can be shown that there is only one inverse. The inverse of a square matrix can be found using the following procedure.

1. Construct for each position (i, j) of the matrix $\mathbf{A}$ the *minor* M_{ij}, which is the determinant obtained by deleting the i-th row and j-th column of the matrix.
2. For each minor M_{ij}, compute a *cofactor* given by $\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$.
3. Construct the matrix $\mathbf{C}$ of cofactors and find its transpose $\mathbf{C}^T$.
4. If $\det(\mathbf{A}) \neq 0$ then

$$\boxed{\mathbf{A}^{-1} = \frac{\mathbf{C}^T}{\det \mathbf{A}}}\,.$$

Note: in the particular case of a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the matrix of cofactors is $\mathbf{C} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ so that $\mathbf{C}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and

$$\boxed{\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}\,.$$

EXAMPLE **A.22.** *Find the inverse of* $\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 5 & 4 & 2 \\ 7 & 3 & 6 \end{bmatrix}$.

SOLUTION.

1. *Calculate the determinant of* $\mathbf{A}$*, e.g. by expanding along the first row:*

$$\det(\mathbf{A}) = 3 \times 18 - 2 \times 16 + (-1) \times (-13) = 35\,.$$

Since the determinant is not zero, the matrix is invertible. If the determinant happened to be zero, the matrix would not be invertible and no further steps would be necessary.

2. *Compute the minors:*
$M_{11} = \begin{vmatrix} 4 & 2 \\ 3 & 6 \end{vmatrix} = 18,\ M_{12} = \begin{vmatrix} 5 & 2 \\ 7 & 6 \end{vmatrix} = 16,\ M_{13} = \begin{vmatrix} 5 & 4 \\ 7 & 3 \end{vmatrix} = -13,$
$M_{21} = \begin{vmatrix} 2 & -1 \\ 3 & 6 \end{vmatrix} = 15,\ M_{22} = \begin{vmatrix} 3 & -1 \\ 7 & 6 \end{vmatrix} = 25,\ M_{23} = \begin{vmatrix} 3 & 2 \\ 7 & 3 \end{vmatrix} = -5,$
$M_{31} = \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = 8,\ M_{32} = \begin{vmatrix} 3 & -1 \\ 5 & 2 \end{vmatrix} = 11,\ M_{33} = \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = 2.$
3. *Compute the corresponding cofactors:*
$C_{11} = 18,\ C_{12} = -16,\ C_{13} = -13,$
$C_{21} = -15,\ C_{22} = 25,\ C_{23} = 5,$
$C_{31} = 8,\ C_{32} = -11,\ C_{23} = 2.$
4. *Construct* $\mathbf{C}^T$: $\begin{bmatrix} 18 & -15 & 8 \\ -16 & 25 & -11 \\ -13 & 5 & 2 \end{bmatrix}$ *and write down the final answer*

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^T}{\det(\mathbf{A})} = \frac{1}{35}\begin{bmatrix} 18 & -15 & 8 \\ -16 & 25 & -11 \\ -13 & 5 & 2 \end{bmatrix}.$$

■

Inverse of a matrix product. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size then

$$\boxed{(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}}.$$

A.4 Complex Numbers

A.4.1 Introduction

The equation $x^2+1=0$ is not satisfied by any real number. For this reason mathematicians have extended the set of real numbers by introducing a new quantity, commonly written as i, but here written as i, which does satisfy this equation. The defining property of i is

$$\mathrm{i}^2+1=0\,, \quad \text{that is} \quad \mathrm{i}^2=-1\,.$$

Given this new quantity, referred to as *imaginary unity*, one can develop the algebra of quantities involving i by applying all of the usual rules of algebra with the single additional rule $\mathrm{i}^2=-1$.

Definition A.20 *Any number z of the form $z=x+\mathrm{i}y$, where x and y are real numbers, is called a complex number. The quantities x and y are called the real part and imaginary part of the complex number, respectively. If the real part of a complex number is zero, this number is called imaginary number. If the imaginary part of the complex number is zero, this number is called real number.*

We write $\mathrm{Re}(z)=x$, $\mathrm{Im}(z)=y$ or $\Re(z)=x$, $\Im(z)=y$.

Example A.23.

$$\begin{aligned}
z_1 &= -2+3\mathrm{i}, & Re(z_1) &= -2, & Im(z_1) &= 3\\
z_2 &= -5.324-\mathrm{i}, & Re(z_2) &= -5.324, & Im(z_2) &= -1\\
z_3 &= \sqrt{3}+\frac{4}{7}\mathrm{i}, & Re(z_3) &= \sqrt{3}, & Im(z_3) &= \frac{4}{7}\\
z_4 &= 8e+\pi\mathrm{i}, & Re(z_4) &= 8e, & Im(z_4) &= \pi\\
z_5 &= 457.92+106.1\mathrm{i}, & Re(z_5) &= 457.92, & Im(z_5) &= 106.1
\end{aligned}$$

Definition A.21 (Equality of complex numbers) *Two complex numbers $z_1=x_1+\mathrm{i}y_1$ and $z_2=x_2+\mathrm{i}y_2$ are equal if and only if* $\mathrm{Re}(z_1)=\mathrm{Re}(z_2)$ *and* $\mathrm{Im}(z_1)=\mathrm{Im}(z_2)$*, i.e. $z_1=z_2$ if and only if $x_1=x_2$ and $y_1=y_2$.*

EXAMPLE **A.24.** *Solve the equation $3z = 6 - 9\mathrm{i}$.*

> SOLUTION. *Letting $z = x + \mathrm{i}y$ we have $3(x + \mathrm{i}y) = 6 - 9\mathrm{i}$ so that $3x + \mathrm{i}3y = 6 - 9\mathrm{i}$. Equating real and imaginary parts gives $3x = 6$ and $y = -9$. Thus $x = 2$, $y = -3$ and $z = 2 - 3\mathrm{i}$.* ■

A.4.2 Arithmetic operations with complex numbers

To state the rules for operating with complex numbers we will use the convention that z is a complex number such that $z = x + \mathrm{i}y$.

A.4.2.1 Addition and Subtraction

$$\begin{aligned} z_1 + z_2 &= (x_1 + \mathrm{i}y_1) + (x_2 + \mathrm{i}y_2) = (x_1 + x_2) + \mathrm{i}(y_1 + y_2)\,, \\ z_1 - z_2 &= (x_1 + \mathrm{i}y_1) - (x_2 + \mathrm{i}y_2) = (x_1 - x_2) + \mathrm{i}(y_1 - y_2)\,. \end{aligned}$$

EXAMPLE **A.25.**

(a) $(2 - 5\mathrm{i}) + (-4 + \mathrm{i}) = (2 - 4) + \mathrm{i}(-5 + 1) = -2 - 4\mathrm{i}$,

(b) $(-3 + 0.7\mathrm{i}) - (4 - 2\mathrm{i}) = (-3 - 4) + \mathrm{i}(0.7 - (-2)) = -7 + 2.7\mathrm{i}$.

A.4.2.2 Multiplication

The important new element here is $\mathrm{i}^2 = -1$. When we apply this to multiplication of complex numbers we obtain

$$\begin{aligned} z_1 z_2 &= (x_1 + \mathrm{i}y_1) \cdot (x_2 + \mathrm{i}y_2) = x_1 x_2 + \mathrm{i}x_1 y_2 + \mathrm{i}y_1 x_2 + \mathrm{i}^2 y_1 y_2 \\ &= x_1 x_2 - y_1 y_2 + \mathrm{i}(x_1 y_2 + x_2 y_1)\,. \end{aligned}$$

EXAMPLE **A.26.**

(a) $(3 + 2\mathrm{i})(2 - 5\mathrm{i}) = 6 - 10\mathrm{i}^2 + \mathrm{i}(4 - 15) = 6 + 10 - 11\mathrm{i} = 16 - 11\mathrm{i}$,

(b) $(\sqrt{2} + \mathrm{i})(\sqrt{2} - \mathrm{i}) = (\sqrt{2})^2 - \mathrm{i}^2 + \mathrm{i}(\sqrt{2} - \sqrt{2}) = 2 + 1 = 3$.

Note that we can compute the successive powers of i:

$$\begin{aligned}
\mathrm{i}^0 &= 1\,,\\
\mathrm{i}^1 &= \mathrm{i}\cdot\mathrm{i}^0 = \mathrm{i}\cdot 1 = \mathrm{i}\,,\\
\mathrm{i}^2 &= \mathrm{i}\cdot\mathrm{i}^1 = \mathrm{i}\cdot\mathrm{i} = -1\,,\\
\mathrm{i}^3 &= \mathrm{i}\cdot\mathrm{i}^2 = \mathrm{i}(-1) = -\mathrm{i}\,,\\
\mathrm{i}^4 &= \mathrm{i}\cdot\mathrm{i}^3 = \mathrm{i}(-\mathrm{i}) = 1\,,\\
\mathrm{i}^5 &= \mathrm{i}\cdot\mathrm{i}^4 = \mathrm{i}\cdot 1 = \mathrm{i}\,,
\end{aligned}$$

and so on. Note also that

$$(x+\mathrm{i}y)(x-\mathrm{i}y) = x^2 - \mathrm{i}^2y^2 = x^2+y^2\,.$$

From this we have such remarkable results as

$$(3+2\mathrm{i})(3-2\mathrm{i}) = 3^2+2^2 = 9+4 = 13\,.$$

Thus the number 13 can be factorised if we use complex numbers.

A.4.2.3 Conjugation

Definition A.22 *If $z = x+\mathrm{i}y$, then the number $\bar z = x-\mathrm{i}y$ is called the complex conjugate of z.*

For example, $\overline{3-2\mathrm{i}} = 3+2\mathrm{i}$. We always have $z\bar z = x^2+y^2$, i.e. the product of a number and its conjugate is a real number. Note that sometimes the conjugate of z is denoted by the asterisk z^*, i.e. $(x+\mathrm{i}y)^* = x-\mathrm{i}y$. It is clear that $(z^*)^* = z$.

A.4.2.4 Division

To express $\dfrac{z_1}{z_2} = \dfrac{x_1+\mathrm{i}y_1}{x_2+\mathrm{i}y_2}$ in the form $z = x+\mathrm{i}y$ we multiply numerator and denominator by $\bar z_2$:

$$\begin{aligned}
\frac{x_1+\mathrm{i}y_1}{x_2+\mathrm{i}y_2} &= \frac{(x_1+\mathrm{i}y_1)\,(x_2-\mathrm{i}y_2)}{(x_2+\mathrm{i}y_2)\,(x_2-\mathrm{i}y_2)}\\
&= \frac{(x_1x_2+y_1y_2)+\mathrm{i}(x_1y_2-x_2y_1)}{x_2^2+y_2^2}\,.
\end{aligned}$$

Example **A.27.**

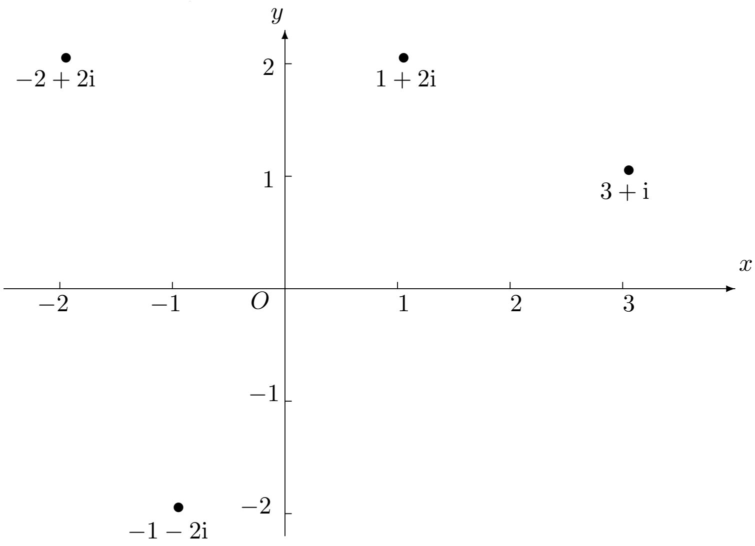
(a) $\dfrac{5+3\mathrm{i}}{4-\mathrm{i}} = \dfrac{5+3\mathrm{i}}{4-\mathrm{i}}\cdot\dfrac{4+\mathrm{i}}{4+\mathrm{i}} = \dfrac{20+5\mathrm{i}+12\mathrm{i}+3\mathrm{i}^2}{4^2+1^2} = 1+\mathrm{i},$

(b) $\dfrac{1}{2-3\mathrm{i}} = \dfrac{1}{2-3\mathrm{i}}\cdot\dfrac{2+3\mathrm{i}}{2+3\mathrm{i}} = \dfrac{2+3\mathrm{i}}{2^2+3^2} = \dfrac{2}{13}+\dfrac{3}{13}\mathrm{i}.$

A.4.3 Geometrical representation of complex numbers

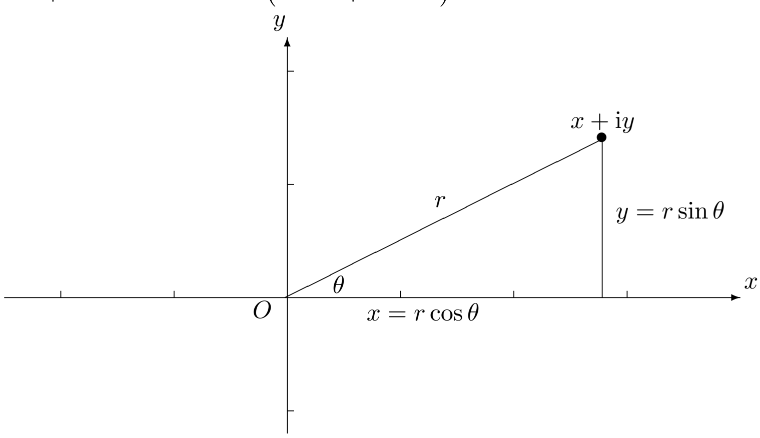
Cartesian form

A complex number $z = x + \mathrm{i}y$ can be represented by the point P with co-ordinates (x, y) in the *Cartesian plane*. Alternatively, we can think of the complex number as the vector $\overrightarrow{OP}$ from the origin O to P. We then call the x-axis the *real axis* and the y-axis the *imaginary axis*. A representation of complex numbers in this way is called an *Argand diagram*. The Cartesian plane is referred to as the *complex plane* when used to represent the set of complex numbers. Addition of complex numbers corresponds to addition of the corresponding vectors.



Polar form

Given a complex number $z = x + \mathrm{i}y$ we can convert it to *polar co-ordinates* in the plane using the relations $x = r\cos\theta$ and $y = r\sin\theta$. Then we have $z = r\cos\theta + \mathrm{i}r\sin\theta$ or $z = r(\cos\theta + \mathrm{i}\sin\theta)$.



The quantity $r = \sqrt{x^2+y^2}$ is called the *absolute value*, the *magnitude* or *modulus* of z. It is written as $|z|$,

$$r = |z| = |x + \mathrm{i}y| = \sqrt{x^2+y^2}\,.$$

Note the relation $z\bar{z} = |z|^2 = r^2$.

Example A.28.

(a) $|3+4\mathrm{i}| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$,
(b) $|1-\mathrm{i}| = \sqrt{1+1} = \sqrt{2}$,
(c) $|-1+\sqrt{3}\mathrm{i}| = \sqrt{1+3} = \sqrt{4} = 2$.

A.4.4 Exponential form of complex numbers

It is convenient to use *Euler's formula*

$$\boxed{e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta\,, \quad \text{where } \theta \text{ is in radians,}}$$

to represent complex numbers as it enables one to significantly simplify operations with complex numbers using the known properties of the exponential function. To demonstrate the validity of this formula consider the *Taylor series* for the functions $\sin x$, $\cos x$ and e^x (review this topic that you studied previously)

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots\,, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\,, \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\,. \end{aligned}$$

If we replace x by $\mathrm{i}\theta$ in the series for e^x, we obtain

$$e^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \frac{(\mathrm{i}\theta)^5}{5!} + \frac{(\mathrm{i}\theta)^6}{6!} + \frac{(\mathrm{i}\theta)^7}{7!} + \dots\,.$$

We can re-arrange this separating the real and imaginary parts to obtain the required result

$$\begin{aligned} e^{\mathrm{i}\theta} &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \dots + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^5}{5!} - \frac{(\mathrm{i}\theta)^7}{7!} + \dots \\ &= \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \dots}_{\cos\theta} + \mathrm{i}\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)}_{\sin\theta}. \end{aligned}$$

Note several important properties of the function $e^{\mathrm{i}\theta}$ that follow directly from its definition:

$$
\begin{aligned}
e^{\mathrm{i}0} &= \cos 0 + \mathrm{i}\sin 0 = 1\,,\\
e^{\mathrm{i}\frac{\pi}{2}} &= \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2} = \mathrm{i}\,,\\
e^{-\mathrm{i}\frac{\pi}{2}} &= \cos\frac{-\pi}{2} + \mathrm{i}\sin\frac{-\pi}{2} = -\mathrm{i}\,,\\
e^{\mathrm{i}\pi} &= \cos\pi + \mathrm{i}\sin\pi = -1\,,\\
\left|e^{\mathrm{i}\theta}\right| &= \cos^2\theta + \sin^2\theta = 1\,,\\
e^{\mathrm{i}\theta}e^{\mathrm{i}\phi} &= e^{\mathrm{i}\theta+\mathrm{i}\phi} = e^{\mathrm{i}(\theta+\phi)}\,,\\
e^{z} &= e^{x+\mathrm{i}y} = e^{x}e^{\mathrm{i}y} = e^{x}(\cos y + \mathrm{i}\sin y)\,,\\
|e^{z}| &= e^{\mathrm{Re}(z)}\,.
\end{aligned}
$$

Any complex number can be now represented in the *complex exponential form*

$$\boxed{z = r(\cos\theta + \mathrm{i}\sin\theta) = |z|e^{\mathrm{i}\theta} \equiv |z|\exp(\mathrm{i}\theta)}\,,$$

where $r = |z|$ is the *magnitude of a complex number* and θ is the *argument of a complex number* or *phase of a complex number*.

Example A.29. *Express $z = e^{3-\mathrm{i}}$ in decimal Cartesian form.*

Solution.

$$z = e^{3}e^{-\mathrm{i}} = e^{3}(\cos(-1) + \mathrm{i}\sin(-1)) \approx 10.85226 - 16.90140\mathrm{i}\,.$$

■

It is important to realize that there are infinitely many possible values for θ given that if $z = r(\cos\theta + \mathrm{i}\sin\theta)$ then

$$z = r(\cos(\theta + 2n\pi) + \mathrm{i}\sin(\theta + 2n\pi))$$

for any integer n.

For this reason we define the principal argument of z.

Definition A.23 (principal argument) *If a complex number is written in the complex exponential form as $z = re^{\mathrm{i}\theta}$, where r is positive and θ satisfies the condition $-\pi < \theta \le \pi$, then the value of θ is called the* principal argument *of z. It is denoted by* $\arg(z)$.

EXAMPLE **A.30.** *Express $1+\mathrm{i}$ in the complex exponential form.*

SOLUTION. $|1+\mathrm{i}| = \sqrt{2}$ *and* $\arg(1+\mathrm{i}) = \arctan(1) = \dfrac{\pi}{4}$ *so that* $1+\mathrm{i} = \sqrt{2}e^{\frac{\pi}{4}\mathrm{i}}$. ■

EXAMPLE **A.31.** *Express $2\sqrt{3}-2\mathrm{i}$ in complex exponential form.*

SOLUTION. $|2\sqrt{3}-2\mathrm{i}| = \sqrt{(2\sqrt{3})^2+2^2} = \sqrt{16} = 4$. *To calculate the argument note that $2\sqrt{3}-2\mathrm{i}$ lies in the fourth quadrant so that* $\arg(2\sqrt{3}-2\mathrm{i}) = -\arctan\dfrac{2}{2\sqrt{3}} = -\dfrac{\pi}{6}$. *Thus* $2\sqrt{3}-2\mathrm{i} = 4e^{\frac{-\pi}{6}\mathrm{i}}$. ■

In general, for $z = x + \mathrm{i}y$ we have

$$\boxed{\begin{aligned}
\arg(z) &= \arctan\frac{y}{x} \quad \text{if} \quad x > 0\,,\\
\arg(z) &= \arctan\frac{y}{x} + \pi \quad \text{if} \quad x < 0\,,\ y \ge 0\,,\\
\arg(z) &= \arctan\frac{y}{x} - \pi \quad \text{if} \quad x < 0\,,\ y < 0\,.
\end{aligned}}$$

EXAMPLE **A.32.** *Express $-1-\mathrm{i}$ in complex exponential form.*

SOLUTION. $|-1-\mathrm{i}| = \sqrt{2}$ *and* $\arg(-1-\mathrm{i}) = \arctan\left(\dfrac{-1}{-1}\right) - \pi = \arctan(1) - \pi = \dfrac{\pi}{4} - \pi = -\dfrac{3\pi}{4}$ *so that* $-1-\mathrm{i} = \sqrt{2}e^{-\frac{3\pi}{4}\mathrm{i}}$ *(compare this with* EXAMPLE *A.30 of the complex exponential form of* $1+\mathrm{i}$*).* ■

EXAMPLE **A.33.** *Express* $3e^{\frac{2\pi}{3}}$ *in Cartesian form.*

SOLUTION.

$$3e^{\frac{2\pi}{3}} = 3\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}\mathrm{i}}{2}\right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i}\,.$$

■

EXAMPLE **A.34.** *Express* $z = -2 - 3\mathrm{i}$ *in complex exponential form.*

SOLUTION. *We have* $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$ *and*

$$\arg(z) = \arctan\left(\frac{-3}{-2}\right) - \pi = \arctan\left(\frac{3}{2}\right) - \pi \approx -2.1588$$

since $\mathrm{Re}(z) < 0$ *and* $\mathrm{Im}(z) < 0$. *Thus* $z \approx \sqrt{13}\exp(-2.1588\mathrm{i})$. ■

Multiplication in complex exponential form

To multiply *any* two complex numbers z_1 and z_2 we can convert them into complex exponential form $z_1 = r_1 e^{\mathrm{i}\theta_1}$ and $z_2 = r_2^{\mathrm{i}\theta_2}$ and then we multiply them as follows

$$\boxed{z_1 z_2 = r_1 e^{\mathrm{i}\theta_1} \cdot r_2 e^{\mathrm{i}\theta_2} = r_1 r_2 e^{\mathrm{i}(\theta_1+\theta_2)}}\,.$$

Multiplication rule: in order to multiply two complex numbers written in a complex exponential form *multiply their magnitudes and add their arguments.* If necessary, subtract a multiple of 2π to get the principal argument of the product.

EXAMPLE **A.35.** *Let* $z_1 = 3\mathrm{i}$ *and* $z_2 = 4 + 4\mathrm{i}$. *Then in Cartesian form we have* $z = z_1 z_2 = 3\mathrm{i}(4 + 4\mathrm{i}) = 12\mathrm{i} + 12\mathrm{i}^2 = -12 + 12\mathrm{i}$. *In complex exponential form we obtain* $|z_1| = 3$, $\arg(z_1) = \frac{\pi}{2}$, $|z_2| = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ *and* $\arg(z_2) = \frac{\pi}{4}$ *so that* $z_1 = 3e^{\frac{\pi}{2}\mathrm{i}}$ *and* $z_2 = 4\sqrt{2}e^{\frac{\pi}{4}\mathrm{i}}$. *Therefore,*

$$\begin{aligned} z_1 z_2 &= 3e^{\mathrm{i}\frac{\pi}{2}} \cdot 4\sqrt{2}e^{\frac{\pi}{4}\mathrm{i}} = 12\sqrt{2}e^{\frac{3\pi}{4}\mathrm{i}} = 12\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right) \\ &= 12\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}\right) = -12 + 12\mathrm{i}\,. \end{aligned}$$

Division in complex exponential form

Similar to multiplication of complex numbers for their division we can use complex exponential representation to write

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{e^{\mathrm{i}\theta_1}}{e^{\mathrm{i}\theta_2}} = \frac{r_1}{r_2}e^{\mathrm{i}(\theta_1-\theta_2)}}\,.$$

Division rule: in order to divide complex numbers written in a complex exponential form *divide their magnitudes and subtract their arguments.*

A.4.4.1 Power of a complex number

Let $z = re^{\mathrm{i}\theta}$ be a complex number. Then

$$\boxed{z^n = (re^{\mathrm{i}\theta})^n = r^n e^{\mathrm{i}n\theta}}\,.$$

Thus,

Power rule: in order to raise a complex number z expressed in a polar form to the power n *raise the magnitude of z to the power n and multiply the argument by n.* To obtain the principal argument it may be necessary to subtract a multiple of 2π.

EXAMPLE **A.36.** *Use complex exponential form to find* $(1+\mathrm{i})^{10}$.

SOLUTION. *We have* $|z| = \sqrt{1+1} = \sqrt{2}$ *and* $\arg(z) = \dfrac{\pi}{4}$. *Thus* $z = \sqrt{2}e^{\frac{\pi}{4}\mathrm{i}}$ *and*

$$z^{10} = (\sqrt{2})^{10}e^{\frac{10\pi}{4}\mathrm{i}} = 2^5 e^{\frac{5\pi}{2}\mathrm{i}} = 32e^{\frac{\pi}{2}\mathrm{i}} = 32\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right) = 32\mathrm{i}\,.$$

■

Example **A.37.** *Use complex exponential form to find* $(-\sqrt{3}+\mathrm{i})^{12}$.

Solution. *We have* $|z| = \sqrt{3+1} = 2$ *and* $\arg(z) = \dfrac{5\pi}{6}$. *Thus* $z = 2e^{\frac{5\pi}{6}\mathrm{i}}$ *and*

$$\begin{aligned} z^{12} &= 2^{12}e^{\left(12\times\frac{5\pi}{6}\right)\mathrm{i}} = 4096e^{10\pi\mathrm{i}} \\ &= 4096e^{0\mathrm{i}} = 4096(\cos 0 + \mathrm{i}\sin 0) = 4096\,. \end{aligned}$$

■

The direct corollary of a the above definitions is

Theorem A.24 (De Moivre's theorem) *For any integer* n,

$$\boxed{(\cos\theta + \mathrm{i}\sin\theta)^n = \left(e^{\mathrm{i}\theta}\right)^n = e^{\mathrm{i}n\theta} = \cos(n\theta) + \mathrm{i}\sin(n\theta)}\,.$$

Example **A.38.** *Use De Moivre's theorem to obtain expressions for* $\sin(3\theta)$ *and* $\cos(3\theta)$.

Solution. *We have*

$$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos(3\theta) + \mathrm{i}\sin(3\theta)\,.$$

Expanding the left-hand side we obtain

$$(\cos^3\theta - 3\cos\theta\sin^2\theta) + \mathrm{i}(3\sin\theta\cos^2\theta - \sin^3\theta) = \cos(3\theta) + \mathrm{i}\sin(3\theta)\,.$$

By equating the real and imaginary parts obtain

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$$

and

$$\sin(3\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta\,.$$

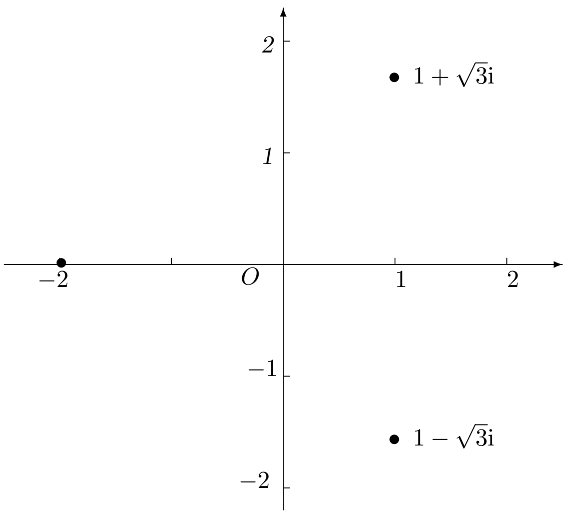
■

A.4.4.2 Roots of complex numbers

Definition A.25 *The n-th root of a complex number z, denoted by $z^{\frac{1}{n}}$ or $\sqrt[n]{z}$, is a number which, when raised to the power n gives z.*

We consider finding cube roots first: $w = z^{\frac{1}{3}}$ iff $w^3 = z$. To find an expression for $z^{\frac{1}{3}}$ given z we write z in a polar form $z = re^{\mathrm{i}\theta}$. Then we assume that $w = \rho e^{\mathrm{i}\phi}$ satisfies $w = z^{\frac{1}{3}}$. Thus $w^3 = z$ so that $\left(\rho e^{\mathrm{i}\phi}\right)^3 = re^{\mathrm{i}\theta}$. Threfore we must have $\rho^3 e^{3\mathrm{i}\phi} = re^{\mathrm{i}\theta}$. Note also that $\rho^3 e^{3\mathrm{i}\phi} = re^{(\theta+2\pi)\mathrm{i}}$, $\rho^3 e^{3\mathrm{i}\phi} = re^{(\theta+4\pi)\mathrm{i}}$, $\rho^3 e^{3\mathrm{i}\phi} = re^{(\theta+6\pi)\mathrm{i}}$ and so on. Therefore, we can have $\rho^3 = r$ so that $\rho = r^{\frac{1}{3}}$ and $3\phi = \theta$ so that $\phi = \frac{\theta}{3}$. Similarly, we can require that $3\phi = \theta + 2\pi$ so that $\phi = \dfrac{\theta}{3} + \dfrac{2\pi}{3}$ and $3\phi = \theta + 4\pi$ so that $\phi = \dfrac{\theta}{3} + \dfrac{4\pi}{3}$. However, if we require that $3\phi = \theta + 6\pi$, then $\phi = \dfrac{\theta}{3} + 2\pi$, but this would be identical to the first argument we found above. Thus there are exactly *three distinct* complex numbers which when raised to the power of three give $z = re^{\mathrm{i}\theta}$. These are $r^{\frac{1}{3}} e^{\mathrm{i}\frac{\theta}{3}}$, $r^{\frac{1}{3}} e^{\mathrm{i}\frac{\theta+2\pi}{3}}$ and $r^{\frac{1}{3}} e^{\mathrm{i}\frac{\theta+4\pi}{3}}$.

EXAMPLE **A.39.** *Find the cube roots of -8.*



SOLUTION. *We express -8 in polar form as $-8 = 8e^{\mathrm{i}\pi}$. A number $w = \rho e^{\mathrm{i}\phi}$ will be a cube root of-8 if $\rho^3 e^{3\mathrm{i}\phi} = 8e^{\mathrm{i}\pi}$. The equation $\rho^3 = 8$ has one solution (since ρ is a real number), but the equation $e^{3\mathrm{i}\phi} = e^{\mathrm{i}\pi}$ will have three distinct solutions. It will be satisfied by the values ϕ such that $3\phi = \pi$, 3π and 5π, i.e. $\phi = \dfrac{\pi}{3}$, $\frac{3\pi}{3}$ and $\dfrac{5\pi}{3}$. Thus the cube roots of -8 are $2e^{\frac{\pi}{3}\mathrm{i}}$, $2e^{\mathrm{i}\pi}$ and $2e^{\frac{5\pi}{3}\mathrm{i}}$ i.e.*

$$2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right), \quad 2(\cos\pi + \mathrm{i}\sin\pi), \quad 2\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right), \text{ or}$$

$$2\left(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right), \quad -2, \quad 2\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right), \text{ or}$$

$$1+\mathrm{i}\sqrt{3}, \; -2, \; -1+\mathrm{i}\sqrt{3}.$$

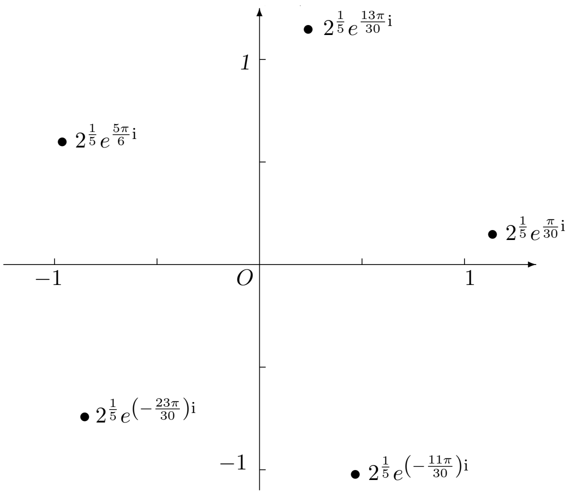
If we plot them on an Argand diagram, these roots are equally spaced on the circle of radius 2 centred at the origin. ■

The same method can be used for finding the n-th root of any complex number.

Roots rule: there are exactly n complex n-th roots for any number $z = re^{\mathrm{i}\theta}$. They are given by

$$\boxed{w_i = r^{\frac{1}{n}} e^{\frac{\theta+2i\pi}{n)\mathrm{i}}}, \quad i = 0, 1, 2, \ldots, n-1}.$$

EXAMPLE **A.40.** *Find the five fifth roots of* $\sqrt{3}+\mathrm{i}$.



SOLUTION. $\left|\sqrt{3}+\mathrm{i}\right| = 2$ *and* $\arg\left(\sqrt{3}+\mathrm{i}\right) = \dfrac{\pi}{6}$ *so that we can write*

$$\sqrt{3}+\mathrm{i} = 2e^{\frac{\pi}{6}\mathrm{i}}, \; 2e^{\frac{13\pi}{6}\mathrm{i}}, \; 2e^{\frac{25\pi}{6}\mathrm{i}}, \; 2e^{\frac{37\pi}{6}\mathrm{i}}, \; 2e^{\frac{49\pi}{6}\mathrm{i}}.$$

Then

$$(\sqrt{3}+\mathrm{i})^{\frac{1}{5}} = 2^{\frac{1}{5}}e^{\frac{\pi}{30}\mathrm{i}}, \; 2^{\frac{1}{5}}e^{\frac{13\pi}{30}\mathrm{i}}, \; 2^{\frac{1}{5}}e^{\frac{25\pi}{30}\mathrm{i}}, \; 2^{\frac{1}{5}}e^{\frac{37\pi}{30}\mathrm{i}}, \; 2^{\frac{1}{5}}e^{\frac{49\pi}{30}\mathrm{i}}.$$

We can re-write these using principal arguments as

$$2^{\frac{1}{5}}e^{\frac{\pi}{30}\mathrm{i}}, \; 2^{\frac{1}{5}}e^{\frac{13\pi}{30}\mathrm{i}}, \; 2^{\frac{1}{5}}e^{\frac{5\pi}{6}\mathrm{i}}, \; 2^{\frac{1}{5}}e^{\frac{-23\pi}{30}\mathrm{i}}, \; 2^{\frac{1}{5}}e^{\frac{-11\pi}{30}\mathrm{i}}.$$

These can be plotted on an Argand diagram as shown above. ■

A.4.5 Exercises

Ex. A.5. Calculate

(i) $|2-3\mathrm{i}|$, (ii) $\left|\dfrac{1}{2-3\mathrm{i}}\right|$, (iii) $\left|\dfrac{3+4\mathrm{i}}{1+2\mathrm{i}}\right|$,

(iv) $|x-2+(y-3)\mathrm{i}|$, (v) $|\cos t+\sin t\mathrm{i}|$, (vi) $|\cos t-\sin t\mathrm{i}|$,

(vii) $|2\cos t-3\sin t\mathrm{i}|$, (viii) $\arg(1-\mathrm{i})$, (ix) $\arg(-1-\mathrm{i})$,

(x) $\arg(3-4\mathrm{i})$, (xi) $\arg\left(\dfrac{1}{3-4\mathrm{i}}\right)$, (xii) $\arg\left((3-4\mathrm{i})^2\right)$.

Ex. A.6. Taking $z=x+y\mathrm{i}$ and $w=u+\mathrm{i}v$ show that

(i) $|z^2|=|z|^2$, (ii) $\left|\dfrac{1}{z}\right|=\dfrac{1}{|z|}$, (iii) $\left(\dfrac{1}{w}\right)^*=\dfrac{1}{w^*}$,

(iv) $|wz|=|w||z|$, (v) $(wz)^*=w^*z^*$.

Appendix B

Useful formulae

B.1 Vectors and Matrices

Vectors

1. If $\mathbf{r} = \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then $||\mathbf{r}|| = \sqrt{x^2 + y^2 + z^2}$
2. $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}|| \cos\theta$
3. $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\mathbf{b} = [a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$
4. $||\mathbf{a}|| = \sqrt{\mathbf{a} \cdot \mathbf{a}}, \qquad \hat{\mathbf{a}} = \frac{\mathbf{a}}{||\mathbf{a}||}$
5. $\mathbf{a} \times \mathbf{b} = ab \sin\theta \ \mathbf{n}$
6. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

Matrices

1. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
2. $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
3. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$
4. Matrix inverse:
   (a) $C_{ij} = (-1)^{i+j}M_{ij}, \qquad \mathbf{A}^{-1} = \dfrac{\mathbf{C}^T}{\det \mathbf{A}}$
   (b) $\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
   (c) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
5. Eigenvalues: $|\mathbf{A} - \lambda\mathbf{I}| = 0$
6. Cayley-Hamilton theorem for 2×2 matrix:

$$\begin{gathered}
\mathbf{A}^2 - \mathrm{Tr}(\mathbf{A})\mathbf{A} + \det(\mathbf{A})\mathbf{I} = \mathbf{0} \\
\mathbf{A}^m = \alpha_1\mathbf{A} + \alpha_0\mathbf{I} \\
\mathbf{A}^{-1} = \frac{\mathrm{Tr}(\mathbf{A})\mathbf{I} - \mathbf{A}}{\det \mathbf{A}} \\
\exp(\mathbf{A}0) = \mathbf{I} = \alpha_0(0)\mathbf{I} + \alpha_1(0)\mathbf{A} \text{ thus } \alpha_0(0) = 1 \text{ and } a_1(0) = 0
\end{gathered}$$

For $n \times n$ matrix

$$\mathbf{A}^m = \alpha_{n-1}\mathbf{A}^{n-1} + \alpha_{m-2}\mathbf{A}^{n-2} + \cdots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$$

B.2 Vectors Calculus

Vector differentiation

$$\begin{gathered}
(c\mathbf{v})' = c\mathbf{v}', \quad (\mathbf{u}+\mathbf{v})' = \mathbf{u}' + \mathbf{v}', \\
(\mathbf{u}\cdot\mathbf{v})' = \mathbf{u}'\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}', \quad (\mathbf{u}\times\mathbf{v})' = \mathbf{u}'\times\mathbf{v} + \mathbf{u}\times\mathbf{v}'
\end{gathered}$$

Differential operators

$$
\begin{aligned}
\boldsymbol{\nabla} f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \\
D_{\hat{\mathbf{a}}} f &= \boldsymbol{\nabla} f \cdot \hat{\mathbf{a}}, \\
\boldsymbol{\nabla} f(r) &= \frac{\mathrm{d} f}{\mathrm{d} r} \boldsymbol{\nabla} r = \frac{\mathrm{d} f}{\mathrm{d} r} \hat{\mathbf{r}}, \quad r = \sqrt{x^2 + y^2 + z^2}, \\
\boldsymbol{\nabla} \cdot \mathbf{v} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \\
\nabla^2 \phi &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}, \\
\boldsymbol{\nabla} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right).
\end{aligned}
$$

Vector identities

$$
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \phi = \mathbf{0}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{v} = 0,
$$

$$
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{v} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \nabla^2 \mathbf{v}, \quad \nabla^2 \mathbf{v} = \begin{bmatrix} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} \\ \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2} \\ \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} \end{bmatrix}
$$

Line integration

$$
\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_a^b \mathbf{F}(t) \cdot \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} \, \mathrm{d}t \\
&= \int_a^b \left(F_1 \frac{\mathrm{d}x}{\mathrm{d}t} + F_2 \frac{\mathrm{d}y}{\mathrm{d}t} + F_3 \frac{\mathrm{d}z}{\mathrm{d}t}\right) \mathrm{d}t, \\
\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}, \\
\int_{PQ} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \phi(Q) - \phi(P) \text{ if } \boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0} \text{ and } \boldsymbol{\nabla} \phi = \mathbf{F}
\end{aligned}
$$

Parametric lines and surfaces in space

$$
\begin{aligned}
\mathbf{r} &= \mathbf{r}(t): \text{ the tangent to a curve is } \mathbf{r}_t = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, \\
\mathbf{r} &= \mathbf{r}(u, v): \text{ the tangents to a surface are } \mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \ \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} \\
&\qquad \text{and the normals are } \mathbf{N}_1 = \mathbf{r}_u \times \mathbf{r}_v = -\mathbf{r}_v \times \mathbf{r}_u = -\mathbf{N}_2.
\end{aligned}
$$

Surface integrals

$$
\begin{aligned}
\int_S G\,\mathrm{d}A &= \iint_R G[\mathbf{r}(u,v)]\,||\mathbf{r}_u \times \mathbf{r}_v||\,\mathrm{d}u\,\mathrm{d}v \\
\int_S G\,\mathrm{d}A &= \iint_R G(x(u,v),y(u,v))\,(x_u y_v - x_v y_u)\,\mathrm{d}u\,\mathrm{d}v \\
\int_S F_n\,\mathrm{d}A &= \iint_S \mathbf{F}\cdot\,\mathrm{d}\mathbf{A} = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}
$$

Integral theorems

Stokes':

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F})\cdot\mathbf{n}\,\mathrm{d}A = \oint_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r}$$

Green's:

$$\iint_S \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\,\mathrm{d}A = \oint_C (v_1\,\mathrm{d}x + v_2\,\mathrm{d}y)\,, \quad \mathbf{v} = (v_1(x,y), v_2(x,y))$$

Ostrogradsky-Gauss':

$$\iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,\mathrm{d}V = \iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A$$

$$V = \frac{1}{3}\iint_S \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}A$$

B.3 Complex Analysis

The straight line with equation $ax + by = c$ has the complex variable form $\bar{\alpha}z + \alpha\bar{z} = 2c$, where $\alpha = a + b\mathrm{i}$.

The circle with equation $(x-a)^2 + (y-b)^2 = r^2$ has the complex variable form $z\bar{z} - (\bar{\alpha}z + \alpha\bar{z}) + \bar{\alpha}\alpha = r^2$, where $\alpha = a + b\mathrm{i}$ (the centre of the circle is the coefficient of $\bar{z}$).

Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Harmonic functions: $u(x,y)$ is harmonic if $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Binomial expansion

$$(1+u)^n = 1+nu+\frac{n(n-1)}{2!}u^2+\frac{n(n-1)(n-2)}{3!}u^3+\frac{n(n-1)(n-2)(n-3)}{4!}u^4+\cdots\,,$$

valid for $|u| < 1$.

Geometric series

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 + \cdots\,, \quad \text{valid for } |u| < 1\,.$$

Cauchy's theorem: if $f(z)$ is analytic in a region containing the simply closed curve C, then $\oint_C f(z)\,\mathrm{d}z = 0$.

Cauchy's integral formula: if $f(z)$ is analytic in a region containing the simply closed curve C, then

$$\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z = 2\pi\mathrm{i} f(z_0) \quad \text{and} \quad \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = \frac{2\pi\mathrm{i}}{n!} f^{(n)}(z_0)\,.$$

Residue theorem: if $f(z)$ is analytic in a region containing the simply closed curve C except for a finite number of poles, then

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \times (\text{sum of residues of } f(z) \text{ at the poles inside } C)\,.$$

If $f(z)$ has a **simple pole** at $z = z_0$, then the residue of $f(z)$ at $z = z_0$ is

$$c_{-1} = \lim_{z\to z_0} (z-z_0) f(z)\,.$$

If $f(z)$ has a **pole of order 2** at $z = z_0$, then the residue of $f(z)$ at $z = z_0$ is

$$c_{-1} = \lim_{z\to z_0} \frac{\mathrm{d}}{\mathrm{d}z}(z-z_0)^2 f(z)\,.$$

If $f(z)$ has a **pole of order** m at $z = z_0$, then the residue of $f(z)$ at $z = z_0$ is

$$c_{-1} = \frac{1}{(m-1)!}\lim_{z\to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}(z-z_0)^m f(z)\,.$$

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