Plane trees and generalised Chebyshev polynomials

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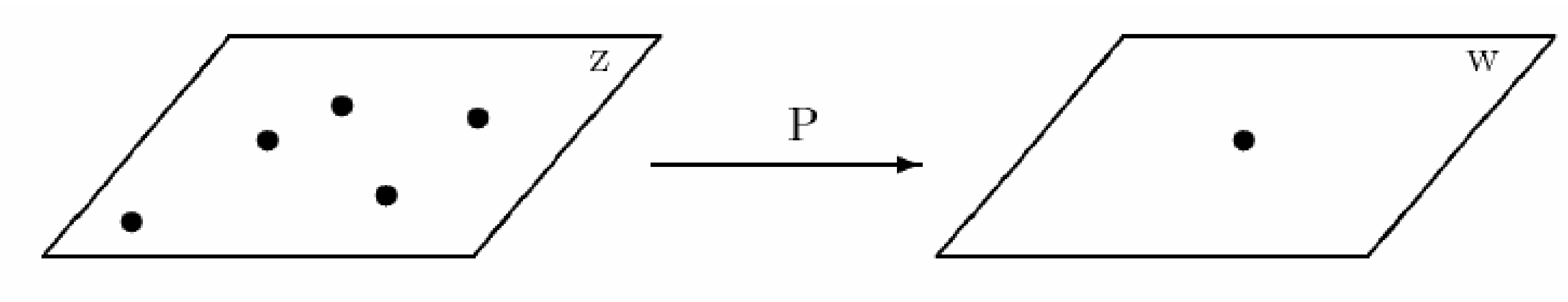
Complex polynomials

- Polynomial $P(z)$ of degree n with complex coefficients maps complex plane into complex plane
- Inverse image of a point ω: $P^{-1}(\omega) = \{z|P(z) = \omega\}$. Usually this set consists of n distinct points (solutions of $P(z) = \omega$), recall the main theorem of algebra.



Critical points and values

- A point z at which $P'(z)=0$ is called a critical point
- A point z at which $P'(z)=0$, $P''(z)=0$, ..., $P^{(k-1)}=0$, $P^{(k)}\neq 0$ is called a critical point of order k
- A value $P(z)$ of the polynomial at a critical point is called a critical value
- Usually $P(z)$ has n-1 critical points and n-1 critical values if n is the degree of $P(z)$
- In some situations by abuse of language we will call some noncritical points critical points of order 1

Degenerate cases

- What is the smallest number of critical values of $P(z)$?
- $P(z) = z^n$ gives us 1 critical point
- $P(z) = a + (b + cz)^n$
- How do polynomials with 2 critical values look like?

Chebyshev polynomials

- $\cos(n\varphi) = T_n(\cos(\varphi))$
- $\cos(2\varphi) = 2\cos^2(\varphi) - 1$

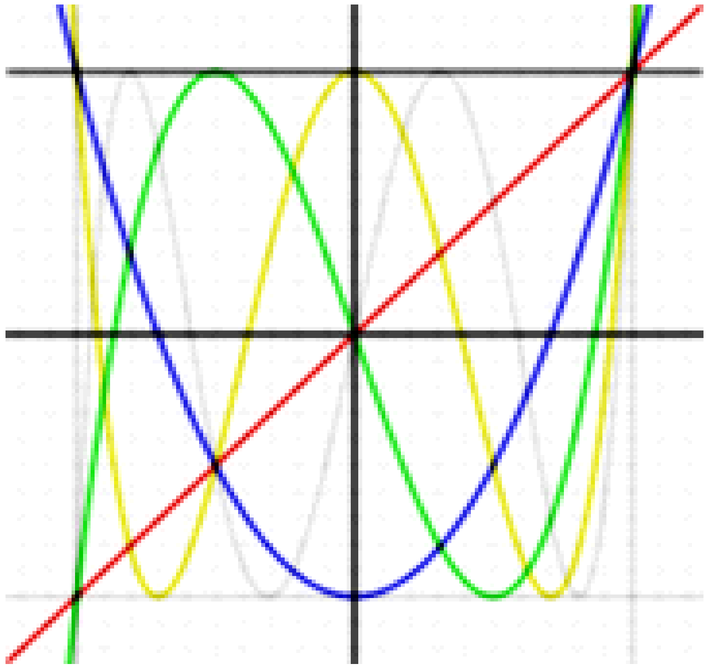
  $\cos(3\varphi) = 4\cos^3(\varphi) - 3\cos(\varphi)$
- $T_0(z) = 1$

  $T_1(z) = z$

  $T_2(z) = 2z^2 - 1$

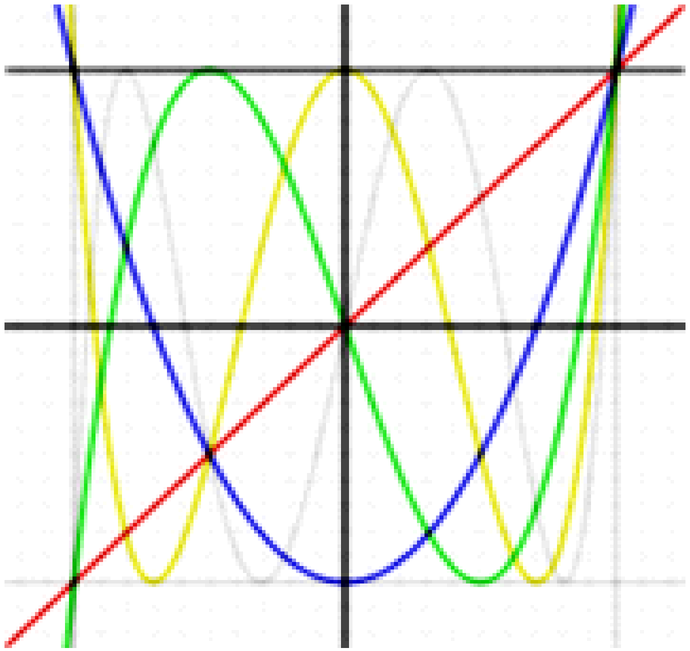
  $T_3(z) = 4z^3 - 3z$

  $T_4(z) = 8z^4 - 8z^2 + 1$
- $T_n(z) = 2zT_{n-1}(z) - T_{n-2}(z)$, for $n \geq 2$



Chebyshev polynomials

- The graph of $T_n(z)$ on the segment [-1, 1] resembles that cos(nφ) on the segment $[-\pi;0]$: all its maxima are equal to 1 and all its minima are equal to -1
- $T_n(z)$ has n-1 critical points, but only 2 critical values: $\omega=\pm 1$



Generalised Chebyshev polynomials

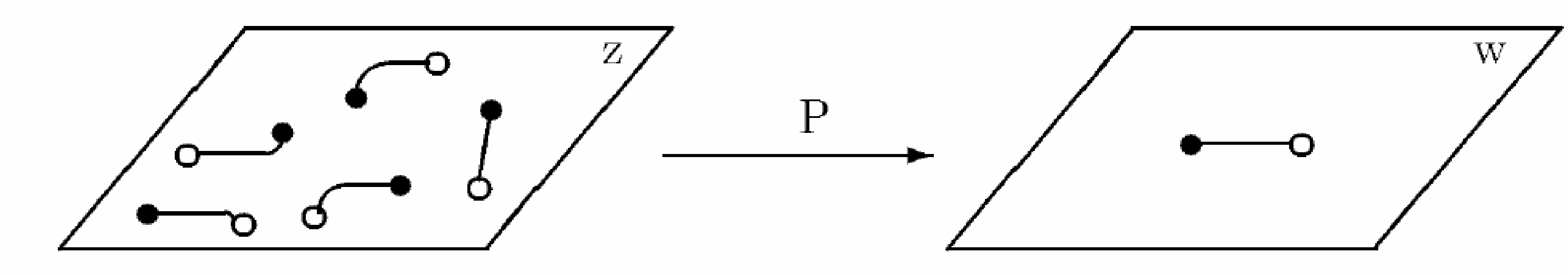
- There are many other polynomials with 2 critical values, they are all called generalised Chebyshev polynomials (GCP)
- In many publications they are also called Shabat polynomials

About Chebyshev

- Pafnuty Lvovich Chebyshev(May 26, 1821 – December 8, 1894)
- One of nine children, he was born in the village of Okatovo, the district of Borovsk, province of Kaluga into the family of landowner Lev Pavlovich Chebyshev. In 1832 the family moved to Moscow.
- In 1837 Chebyshev started the studies of mathematics at the philosophical department of Moscow University and graduated from the university as "the most outstanding candidate".
- In 1847, Chebyshev defended his dissertation "About integration with the help of logarithms" at St Petersburg University. Chebyshev lectured at the university from 1847 to 1882. In 1882 he left the university and completely devoted his life to research.
- Chebyshev is known for his work in the field of probability, statistics and number theory. Chebyshev is considered to be one of the founding fathers of Russian mathematics. Among his students were Aleksandr Lyapunov and Andrey Markov

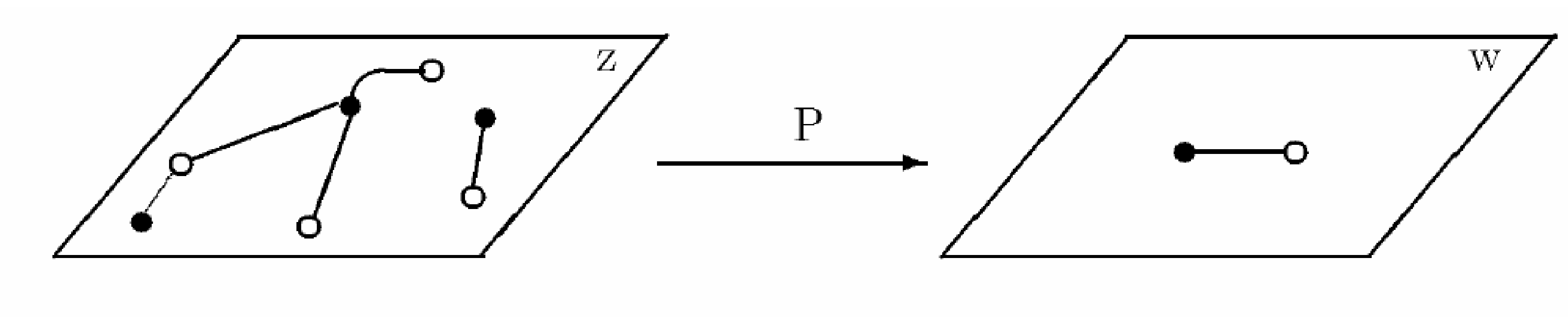
Inverse image of a segment

- Let's study an inverse image of a segment [c_0, c_1] on the complex plane.
- Suppose there are no critical values inside the segment, then inverse image of the segment is a disjoint union of n separate sets on z-plane, each one being homeomorphic to a segment.



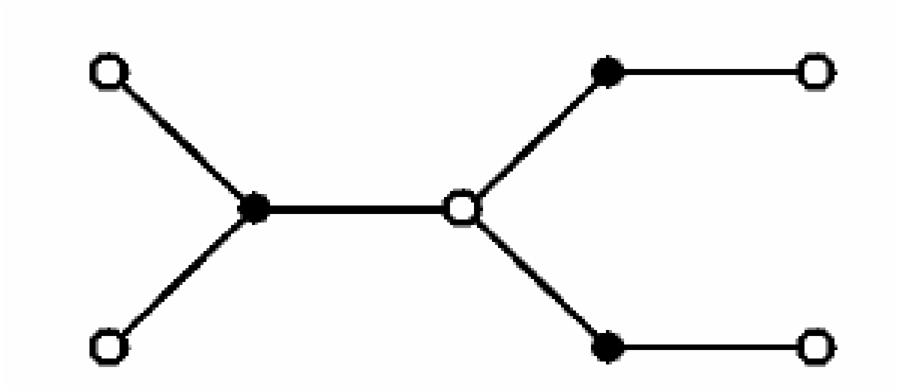
Inverse image of a segment

- Now suppose there are still no critical values inside the segment, but one or more of its ends becomes critical, then some of the "curvilinear segments" are glued one to another
- What happens if $P(z)$ is a GCP and c_0 and c_1 are its (only) critical values?



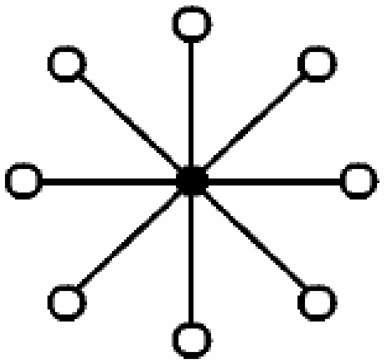
Inverse image of a segment

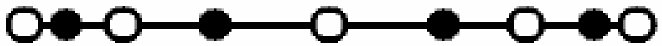
- The answer is that we will see on z-plane a bicolored plane tree



Examples

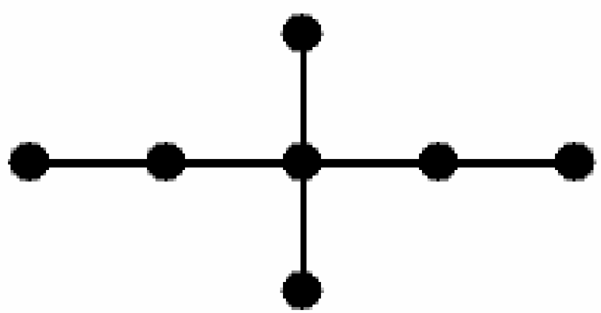
- For $P(z)=z^n$ the inverse image of the segment $[c_0,c_1]=[0,1]$ is a “star-tree”, for $P(z)=T_n(z)$, a Chebyshev polynomial, the inverse image of the segment $[c_0,c_1]=[-1,1]$ is a “chain-tree”





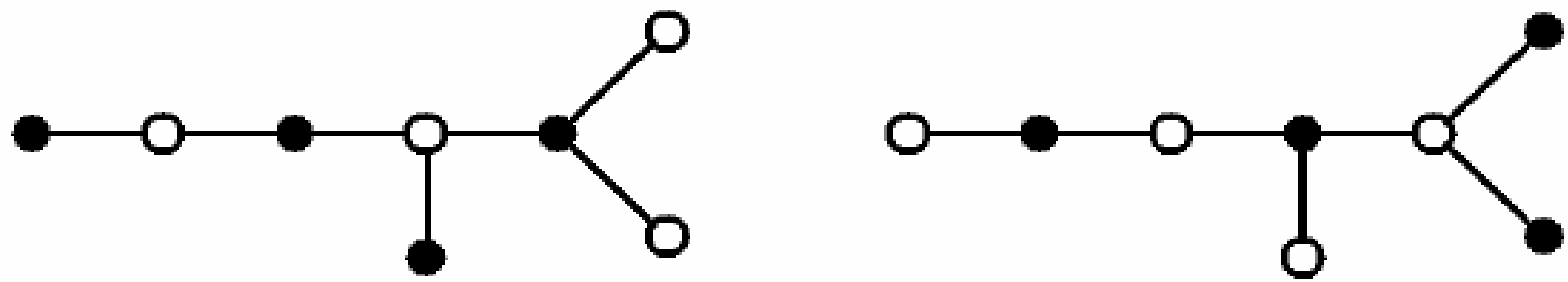
Combinatorial bicolored plane trees

- A tree is a connected graph without circuits
- A plane tree is a tree which is drawn on (or embedded into) the plane
- An isomorphism of plane trees is an isomorphism of trees as graphs which also preserves cyclic order on the adjacent vertices
- An isomorphism class of plane trees is called a combinatorial plane tree



Combinatorial bicolored plane trees

- Any tree has a natural structure of a bipartite graph, so it can be colored in two colors in such way that adjacent vertices have different colors
- A bicolored plane tree is a plane tree with one of the two possible colorings chosen
- Isomorphism of bicoloured plane trees must preserve colors of the corresponding vertices



Set of valencies

- Let n be the number of edges, and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ (resp. $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$) be the sequence of valencies of black (resp. white) vertices in descending order, therefore:

$$\sum_{i=1}^{p} \alpha_i = \sum_{i=1}^{q} \beta_i = n$$

$$p + q = n + 1$$

Set of valencies

- Thus, $<\alpha,\beta>$ is just a pair of partitions of the number n having (together) a total of n+1 parts. We call it the set of valencies of the tree, and say also the tree is of type $<\alpha,\beta>$

Set of valencies

- **Lemma**: For any pair $<\alpha,\beta>$ of partitions of the number n having a total of n+1 parts there exists a bicoloured tree of type $<\alpha,\beta>$
- The lemma above can be proved using induction by the number n
- Is the tree unique?

Linear transformations and equivalent Chebyshev polynomials

- Linear transformations of the z-plane and w-plane do not destroy the property of a polynomial to be of "generalised Chebyshev" type
- Let P(z) and Q(z) be two GCP and let c_0, c_1 and d_0, d_1 be their critical values. We call the two pairs

  $(P, [c_0, c_1])$ and $(Q, [d_0, d_1])$ equivalent, if there exist complex constants A, B, a, b; A, a ≠ 0, such that:

$$Q(z) = AP(az + b) + B$$

$$d_0 = Ac_0 + B$$

$$d_1 = Ac_1 + B$$

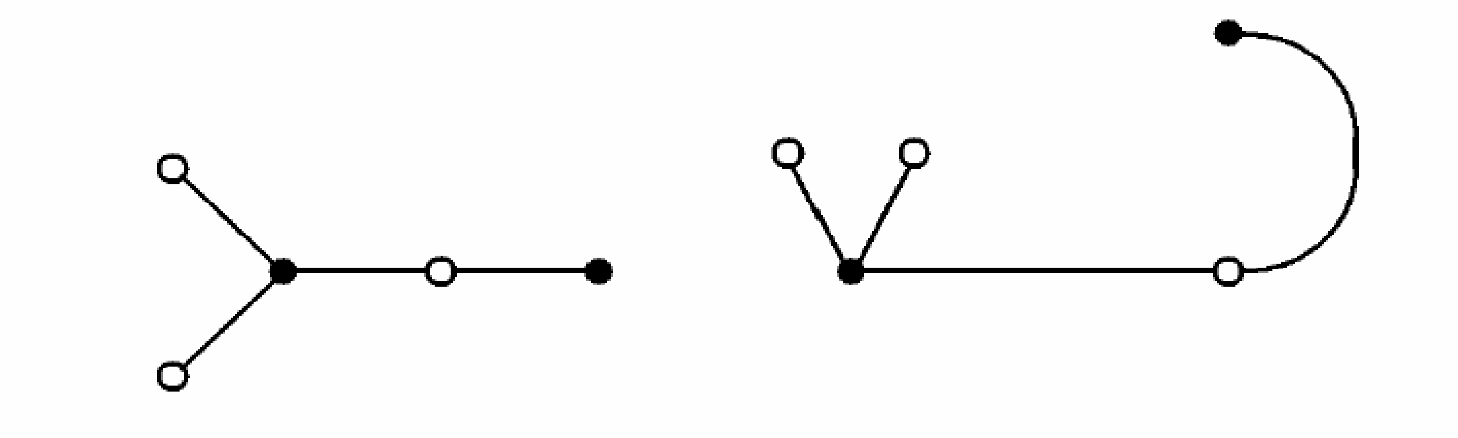
- By abuse of language, we shall say that the polynomials P and Q are equivalent

Main theorem

- Theorem: There is a bijection between the set of (combinatorial bicoloured plane) trees and the set of equivalence classes of generalised Chebyshev polynomials

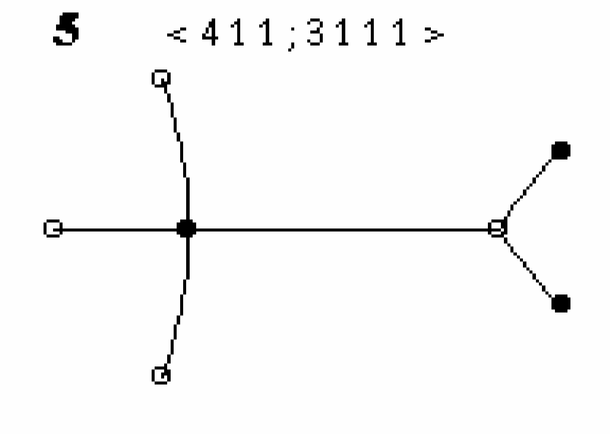
Geometry of plane trees

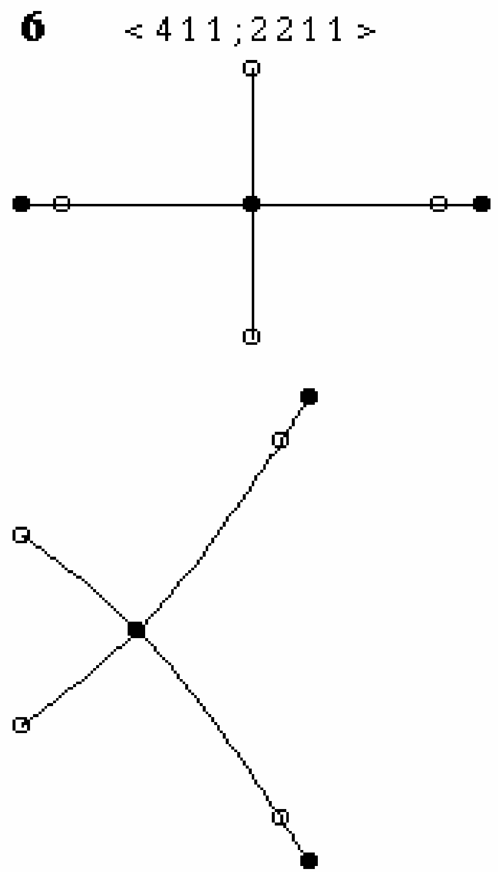
- Consequence from the main theorem: every plane tree has a unique and canonical geometrical form
- Indeed, the linear transformation mentioned above may change the size of the tree and its position on the z-plane, but it doesn't change its geometric form
- Two "geometric forms" of the same combinatorial tree:



Geometry of plane trees

- Excerpt from the catalog of “true” geometric forms:





Calculation of generalised Chebyshev polynomials

- Let the type $<\alpha,\beta>=<\alpha_1,\alpha_2,\ldots,\alpha_p;\beta_1,\beta_2,\ldots,\beta_q>$ of a tree be given, and let us also fix $c_0 = 0$ and $c_1 = 1$. Denote $a_1,a_2,..,a_p$ and $b_1,b_2,..,b_q$ coordinates of the black and white vertices respectively. Then we have:

$$P(z)=\lambda(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\ldots(z-a_P)^{\alpha_P}$$

$$P(z)-1=\lambda(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\ldots(z-b_q)^{\beta_q}$$

Calculation of generalised Chebyshev polynomials

- The equalities between the coefficients provide us with n algebraic equations in n+2 unknowns:

$$\lambda, a_1, a_2, .., a_p, b_1, b_2, .., b_q$$

- Two additional "degrees of freedom" correspond to the possibility of making a linear transformation of the z-plane. We may use it the way that we find convenient.
- The method, taken literally, is far too complicated, even if we use a powerful system of symbolic computations, such as MAPLE. Many improvements of various levels of generality were proposed.

Composition of polynomials

- Let's fix the image segment $[c_0, c_1] = [0,1]$
- Composition of polynomials $R(z) = P(Q(z))$
- $R'(z) = P'(u)Q'(z)$, where $u = Q(z)$

$$R'(z) = 0 \Leftrightarrow P'(u) = 0 \vee Q'(z) = 0$$

$$1) P'(u) = 0 \Rightarrow P(u) \in \{0,1\} \Rightarrow R(z) \in \{0,1\}$$

$$2) Q'(z) = 0 \Rightarrow Q(z) = u \in \{0,1\} \Rightarrow R(z) \in \{P(0), P(1)\}$$

Composition of polynomials

- This can be summed up as theorem: if P and Q are generalised Chebyshev polynomials, and

$$P(0), P(1) \in \{0,1\}$$

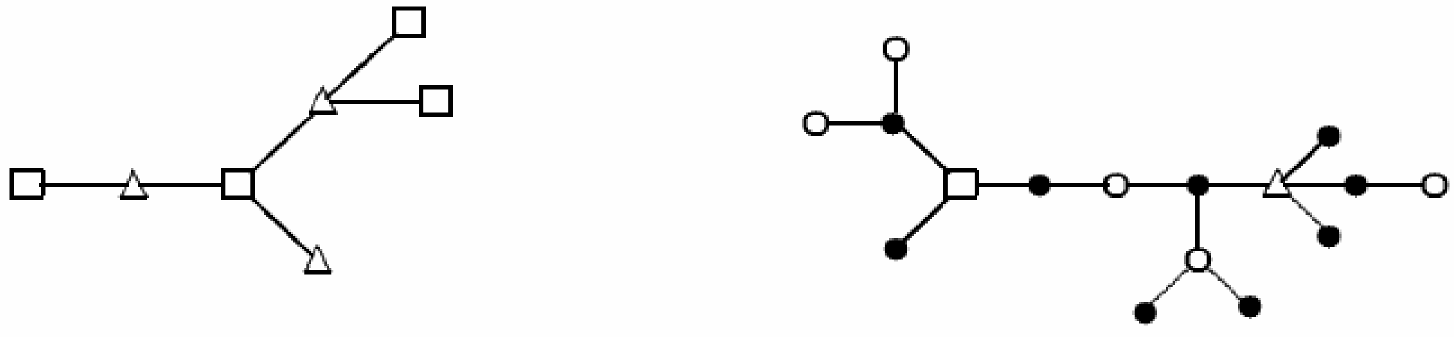
then R(z) = P(Q(z)) is a generalised Chebyshev polynomial.

Composition of trees

- Let's denote by T_P, T_Q and T_R the trees that correspond to the polynomials P, Q and R.
- Our goal is to reconstruct the tree T_R combinatorily from the trees T_P and T_Q or, in other words, to define an operation of the composition of trees.

Composition of trees

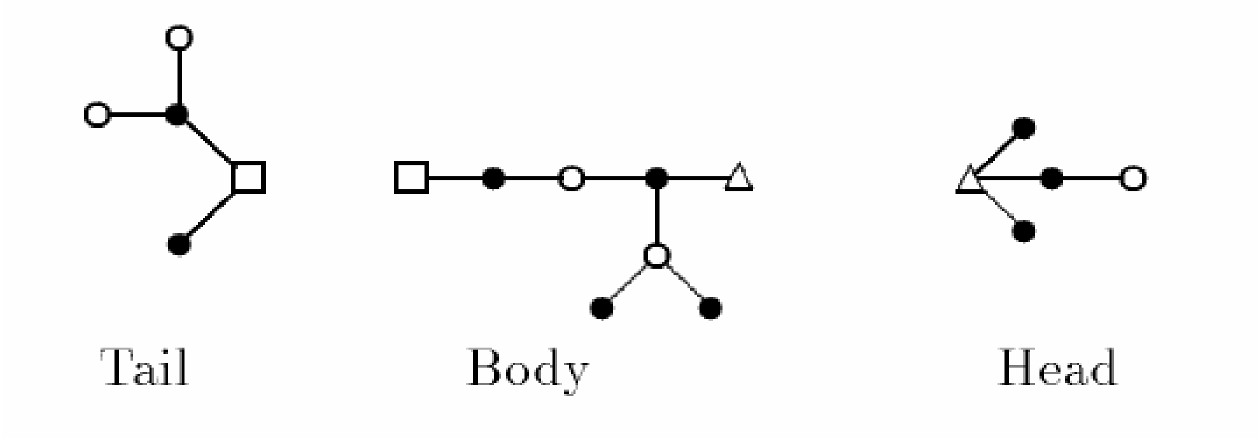
- The condition that P(0) and P(1) are equal to 0 or 1 means that there are two vertices of T_P, that lie at the points u = 0 (marked as square) and u = 1 (marked as triangle).
- The vertices of T_q are pre-images of the “square” and “triangular” points of the plane of variable u. There fore we mark them as square and triangular vertices.



T_Q T_P

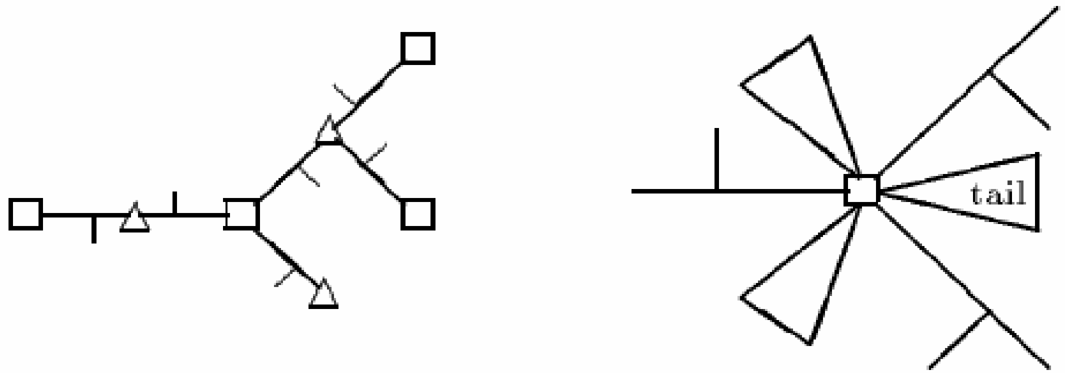
Composition of trees

- In general terms, the procedure of obtaining the tree T_R from T_P and T_Q consists of substituting the tree T_P for every edge of the tree T_Q.
- Decompose T_P into the following parts
  - Spine – the path from the square to the triangle
  - Body – the spine and all the branches attached to it
  - Head – all branches (except the spine) attached to the triangle
  - Tail – all branches (except the spine) attached to the square



Composition of trees

- Composition procedure:
  - Instead of each directed edge of T_Q we put the “body” of the tree T_P, each time respecting its direction
  - Attach to each triangular vertex of T_Q a number of “heads” of T_P equal to the valency of that vertex
  - Do the same thing with the squares and “tails”



Examples

- If we take $Q(z) = z^n$ then we will get a centrally symmetric tree.
- Let us take for the polynomial P a Chebyshev polynomial, but renormalized in such way as to have critical values at 0 and 1 instead of ±1, and the ends of the "chain-tree" at points u = 0 and u = 1:

$$P(u) = \frac{1}{2}(T_n(2u-1)+1)$$

- The composition with such P consists in the subdivision of every edge of the tree T_Q into n parts.

Examples

- Sometimes by looking at a tree it is not at all easy to guess that it is in fact composition of simpler trees: