Sample and Prune: An Efficient MapReduce Method for Submodular Optimization

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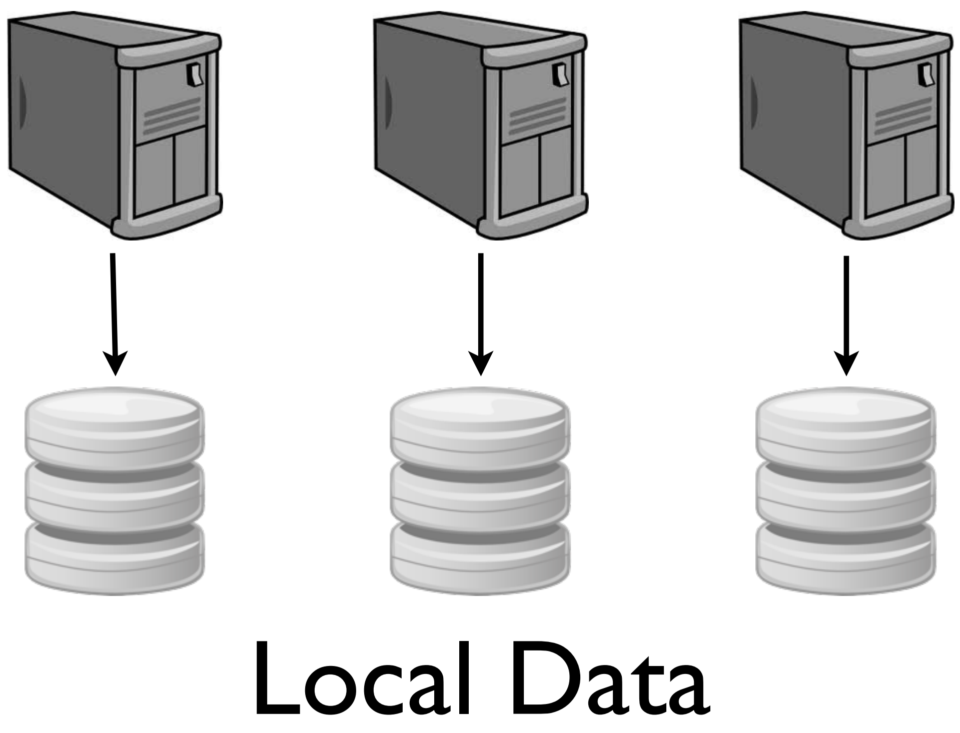
MapReduce Class [Karloff et al.]

- N is the input size
- Sublinear $(N^{1-\epsilon})$ memory on each machine
- Sublinear $(N^{1-\epsilon})$ number of machines
- Mappers/reducers are $\mathrm{poly}(N)$ computable
- $\mathcal{MRC}^0$: algorithms that run in $O(1)$ rounds

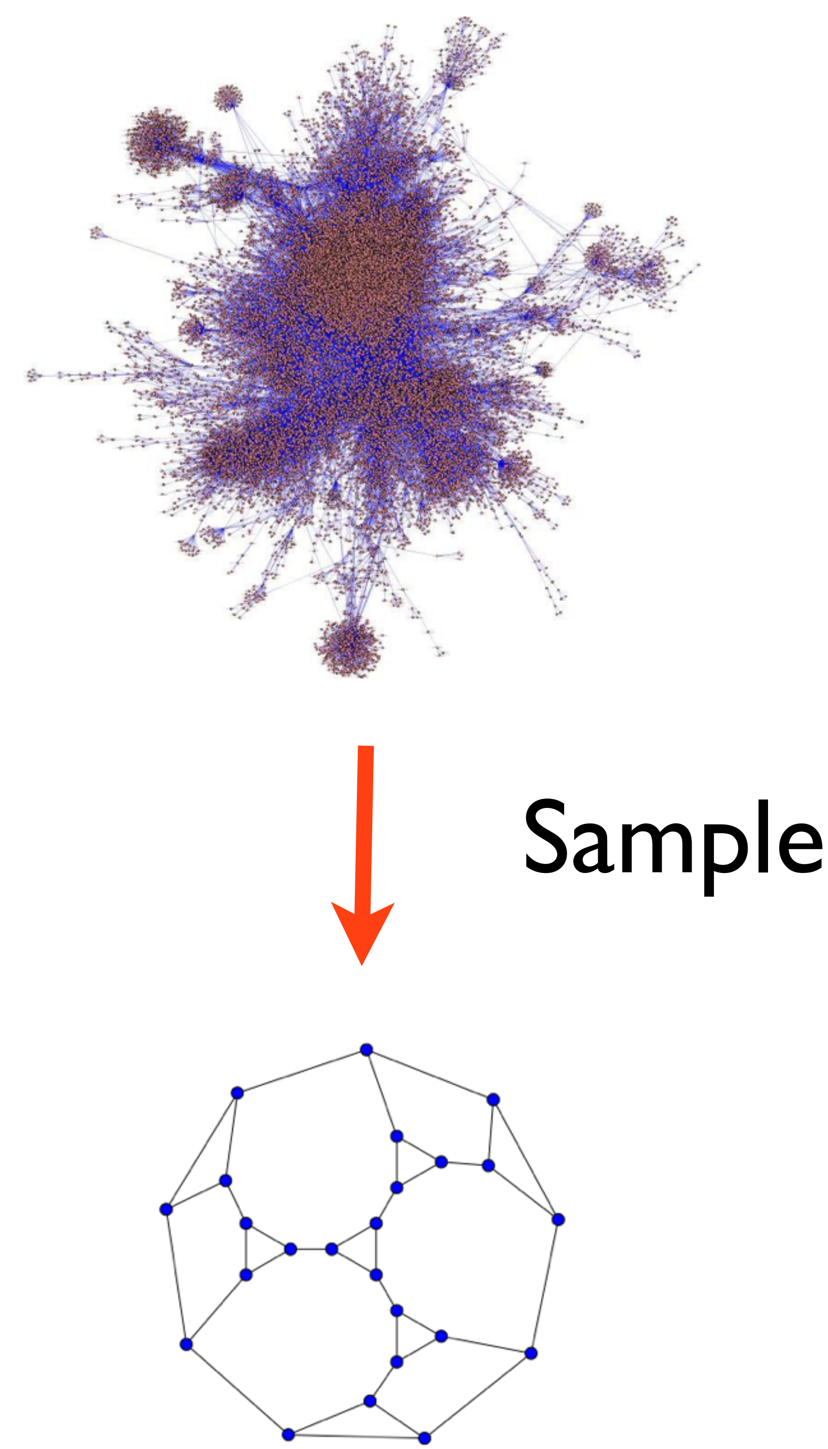
Algorithmic Design in MapReduce

- No one machine can see the entire input
- No communication between machines during a phase
- Total memory is large



Sampling

- Sample the input in parallel
    - Well represent the input space
    - Generally must be done adaptively
- Sample should be small



Monotone Submodular Function Maximization

- Universe of elements U where $|U| = n$
- A function $f : 2^U \to \mathbb{R}$
- Function is submodular and monotone

$\forall Y, X \subseteq U$ where $X \subseteq Y$ and every $x \in U \setminus Y$ we have

$$f(X \cup \{x\}) - f(X) \geq f(Y \cup \{x\}) - f(Y)$$

Monotone Submodular Function Maximization

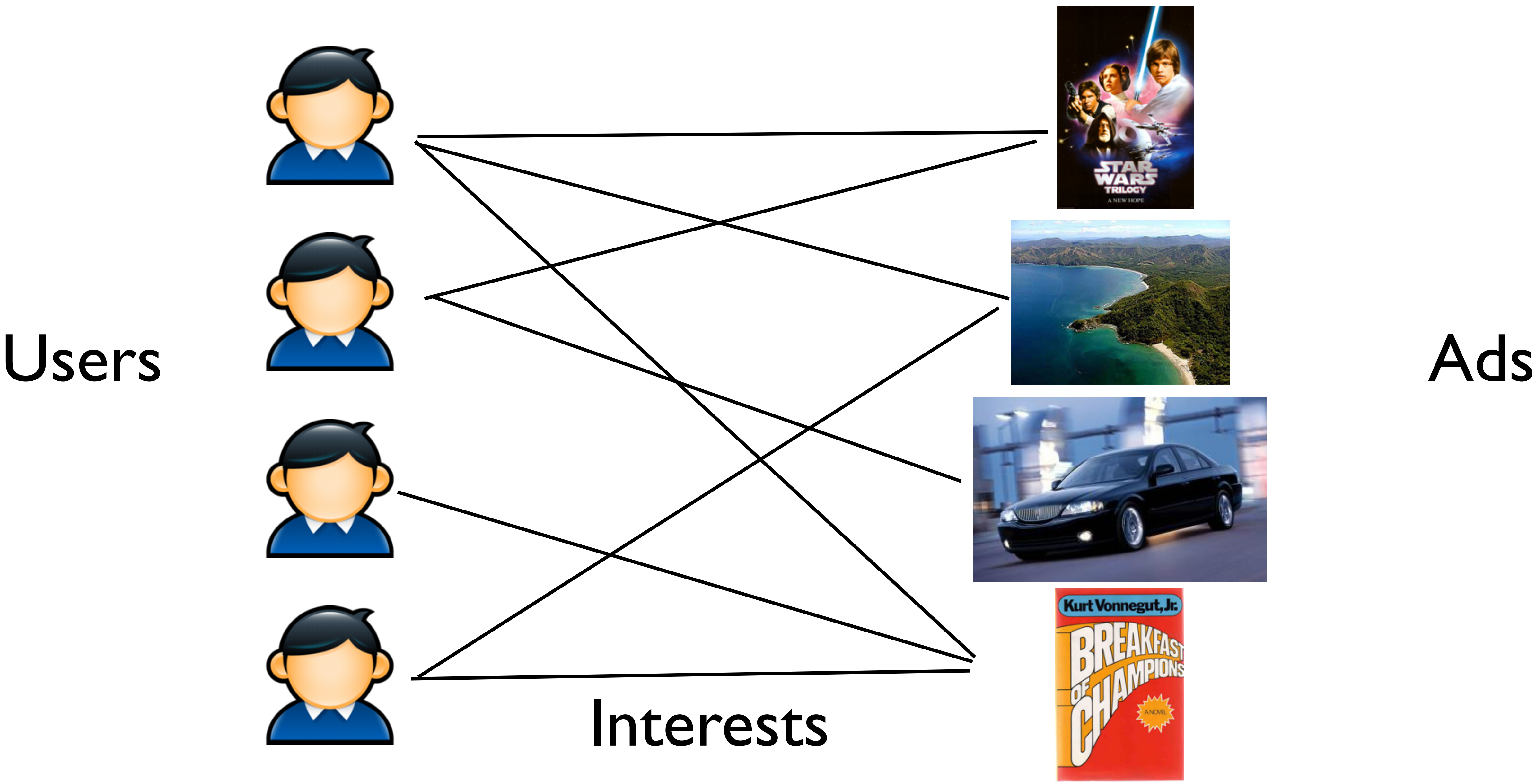
- Find a set S such that

$$\max_S f(S)$$

- Possible constraints (hereditary)
  - Cardinality
  - Matroid (system)
  - Knapsack

- Maximum solution size k
- Memory $\Omega(kn^{\epsilon})$

Maximum Coverage



Submodular Function Maximization

- Maximum submodular coverage
- Minimum spanning tree
- Maximum matching in bipartite graphs
- ...

Greedy Algorithm

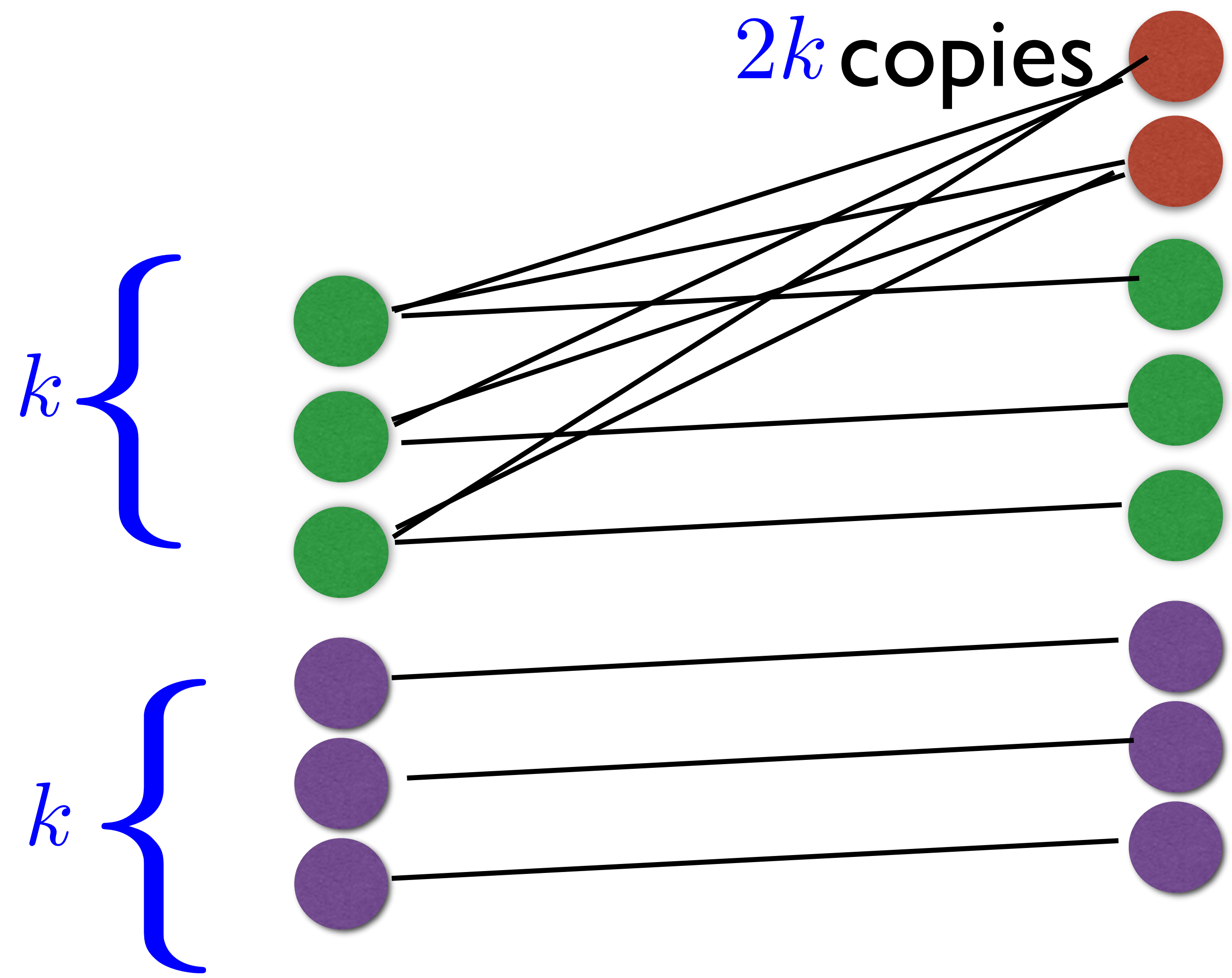
- $Y = \emptyset$
- Add elements sequentially with the most value

$$\max_x f(\{x\} \cup Y)$$

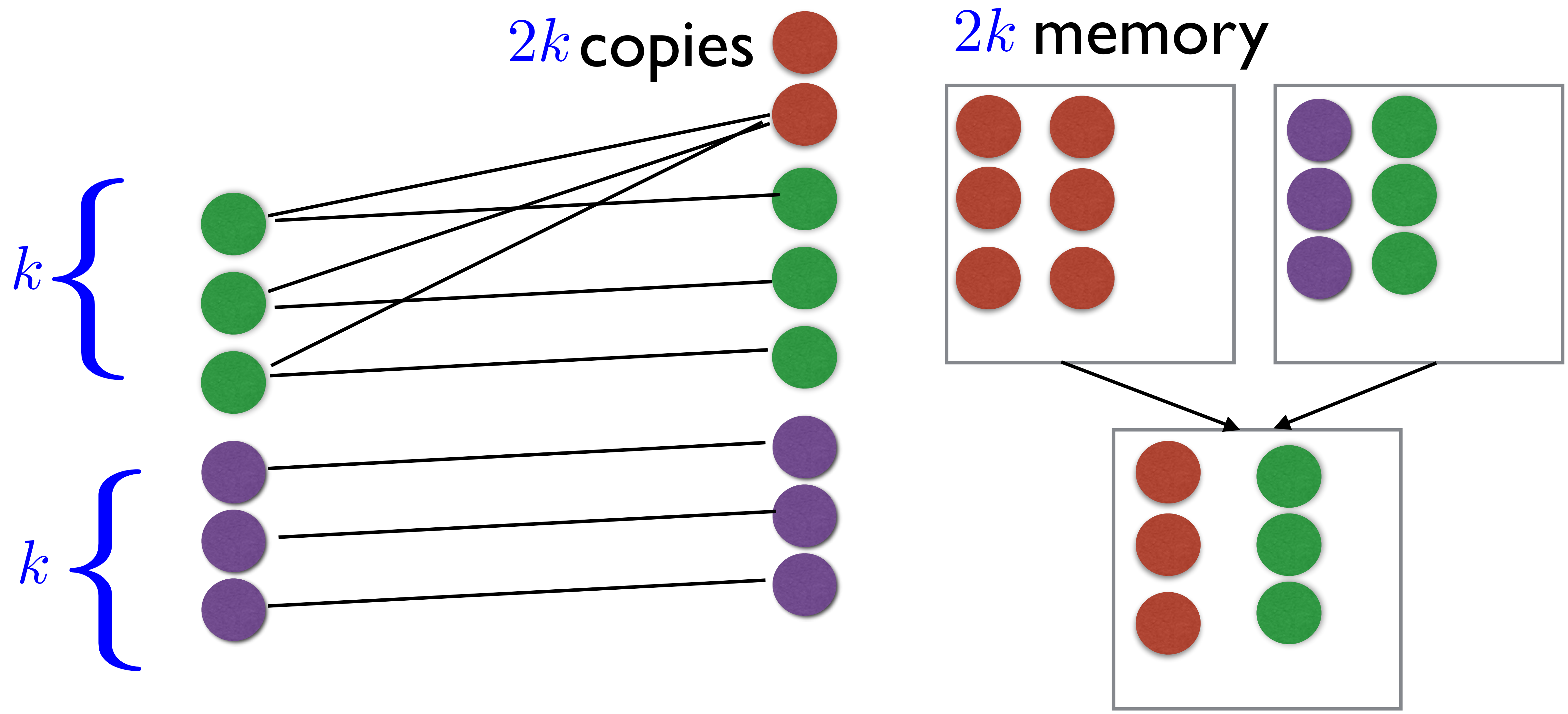
Simulation of the Greedy Algorithm

- A natural approach
  - Partition data across machines
  - U_i data given to machine i
  - Run greedy on each machine to get a set S_i
  - Map all sets to a single machine and run greedy on $\cup_i S_i$

Bad Example



Bad Example



Algorithm's Objective k

Optimal Objective $2k-1$

Greedy in MR [Kumar-M-Vassilvitskii-Vattani]

- v maximum value an element can give
- For submodular objectives, a $O(\log v)$ round algorithm that computes the greedy solution
  - $(1 - \frac{1}{e} - \epsilon)$ -approximation for maximum submodular coverage
  - $\frac{1}{3+\epsilon}$ -approximation for weighted matching

Approximate Greedy Algorithm

- $Y = \emptyset$
- Add elements sequentially with (almost) the most value

$$f(\{x\} \cup Y) - f(Y) \geq \frac{1}{(1+\epsilon')}(f(\{a\} \cup Y) - f(Y)) \quad \forall a \in U$$

Algorithm Overview

- Solution set $Y = \emptyset$
- Find maximum additional value possible

$$v = \max_x f(\{x\})$$

- In ith phase pick elements until

$$\max_x \{f(\{x\} \cup Y) - f(Y)\} < \frac{v}{(1+\epsilon')^i}$$

- Logarithmic rounds:

$$O(\log_{1+\epsilon'}(v))$$

Analysis

- Solution the same as the approximate sequential greedy algorithm's solution
  - Only choose elements that could be chosen by the greedy algorithm

Sampling

- Sample $\tilde{\Theta}(kn^{\epsilon})$ elements to get a set S
- Pick elements of value greater than $\frac{v}{(1+\epsilon')^i}$
- Y_i current solution
- Remove all elements x with value less than $\frac{v}{(1+\epsilon')^i}$
- Recurse until all points are removed

Analysis

- A factor of roughly n^{ϵ} elements removed each iteration
  - $\mathcal{E}_{Y_i}$: Event that the solution is Y_i
  - High value elements would change the solution if sampled

$$\mathbf{Pr}[\mathcal{E}_{Y_i}] \leq (1 - \frac{k}{n^{1-\epsilon}})^{2n^{1-\epsilon}\log n} \leq e^{-2k\log n} = \frac{1}{n^{2k}}$$

Analysis

- Same guarantees as the greedy algorithm
- Number of rounds is $O(\frac{1}{\epsilon}\log_{1+\epsilon'}(v))$
- Memory: $\tilde{\Theta}(kn^{\epsilon})$
- Machines: $\tilde{\Theta}(n^{1-\epsilon})$

Extensions

- Constant round algorithms?
- Tailor the sampling algorithm
  - Ensure only non-important elements are discarded

Extensions

- Constant round algorithms!
  - Optimal algorithm for a modular function subject to one matroid
  - Approximation for cardinality constraint
  - Approximation for d matroids
  - Approximation for d knapsacks

Streaming Algorithm for the Cardinality Constraint

- Select a set S of size k to maximize $f(S)$
- Let OPT denote the optimal solution
- Streaming Algorithm
  - Set $S = \emptyset$
  - Consider elements in any order $e_1, e_2, \ldots, e_n$
  - Add e_i to S if $\quad f(S \cup \{e_i\}) - f(S) \geq \dfrac{OPT}{2k}$

Analysis

- $\frac{1}{2}$-approximation

- Say k elements are added to S
  - Each gave incremental value at least $\frac{OPT}{2k}$
  - Total value is at least $k \cdot \frac{OPT}{2k} = \frac{OPT}{2}$

Analysis

- Say $|S| < k$
- Let S^* denote the optimal solution
- For elements $e \in S^*$ it is the case that

$$f(S \cup \{e\}) - f(S) < \frac{OPT}{2k}$$

- Thus, we have

$$f(S \cup S^*) - f(S) \leq \frac{OPT}{2k}|S^* \setminus S|$$

Analysis

- We know:

$$f(S \cup S^*) - f(S) \leq \frac{OPT}{2k}|S^* \setminus S|$$

- Which implies

$$f(S \cup S^*) - \frac{OPT}{2k}|S^* \setminus S| \leq f(S)$$

- Using monotonicity

$$OPT = f(S^*) \leq f(S \cup S^*)$$

- Putting this together

$$OPT - \frac{OPT}{2} \leq f(S)$$

MapReduce: Sampling

- Set $Y = \emptyset$
- Sample $\tilde{\Theta}(kn^{\epsilon})$ elements
- Pick elements of value greater than $\frac{OPT}{2k}$
- Y_i current solution
- Remove all elements x with value less than $\frac{OPT}{2k}$
- Recurse until all points are removed

Analysis

- A factor of roughly n^ϵ elements removed each iteration
  - $\mathcal{E}_{Y_i}$: Event that the solution is Y_i
  - High value elements would change the solution if sampled

$$\mathbf{Pr}[\mathcal{E}_{Y_i}] \leq (1 - \frac{k}{n^{1-\epsilon}})^{2n^{1-\epsilon}\log n} \leq e^{-2k\log n} = \frac{1}{n^{2k}}$$

Analysis

- $\frac{1}{2}$-approximation
- Number of rounds is $O(\frac{1}{\epsilon})$
- Memory: $\tilde{\Theta}(kn^{\epsilon})$
- Machines: $\tilde{\Theta}(n^{1-\epsilon})$

Abstracting the Idea: Sample-and-Prune

- Let $\mathcal{G}$ be an algorithm which
  - Returns a set of size at most k
  - $\mathcal{G}(A) \subseteq \mathcal{G}(B)$ for all $A \subseteq B$

Abstracting the Idea: Sample-and-Prune

- Sample $\tilde{\Theta}(kn^{\epsilon})$ elements to get a set S
- Set $Y = \mathcal{G}(S)$
- Remove all elements x where

$$x \notin \mathcal{G}(Y \cup \{x\})$$

Abstracting the Idea: Sample-and-Prune

- Key theorem
  - An n^ϵ fraction of the elements are removed with high probability

Application: Maximal Matching

- Maximal matching
    - No two edges incident to the same node
    - No edge can be added to the solution

Application: Maximal Matching

- Sequential algorithm
  - Consider edges in an arbitrary order
  - Add an edge if it is feasible

  - Let this algorithm be $\mathcal{G}$
  - $\mathcal{G}(A) \subseteq \mathcal{G}(B)$ for all $A \subseteq B$
  - Only discards edges incident to chosen edges

Conclusion

- Algorithmic techniques for MR
  - Sampling
  - Greedy algorithms

Thank You!
Questions?