Algorithmic interpretations of fractal dimension

Anastasios Sidiropoulos (The Ohio State University)
Vijay Sridhar (The Ohio State University)

The curse of dimensionality

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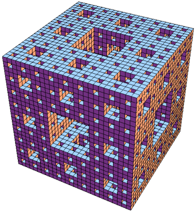
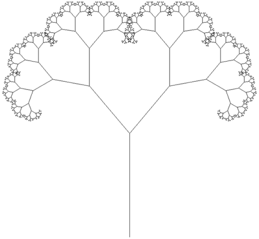
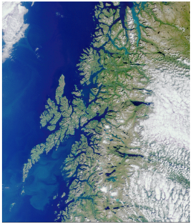
- Geometric problems become harder when dimension increases.
- Several notions of dimension in computational geometry:
  - Euclidean dimension
  - Doubling dimension
  - Rate of growth
  - Highway dimension

How does **fractal dimension** affect algorithmic complexity?

Fractals







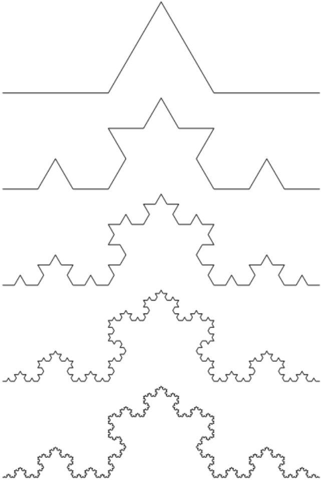
Fractal dimension

Several notions of fractal dimension:

- Hausdorff dimension
- Minkowski dimension
- Box-counting dimension
- ...

Example: Koch curve

length $= \infty$
area $= 0$



Fractal dimension and volume

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Scaling by a factor of $r > 0$ increases the total "volume" by a factor of roughly r^δ.

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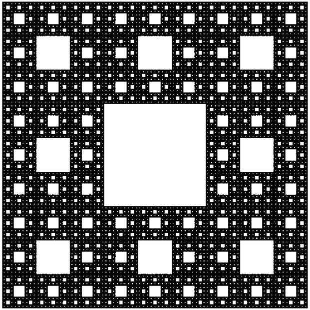
Example: Sierpinski carpet
Scaling by a factor of 3 increases the volume by a factor of 8
$\delta = \log_3 8$



Hausdorff dimension

Let $X \subseteq \mathbb{R}^d$.
δ-dimensional Hausdorff content:

$$C_H^\delta(X) = \inf \left\{ \sum_{i \in I} r_i^\delta : \exists \text{ countable cover of } S \text{ with radii } r_i \right\}$$

Hausdorff dimension:

$$\dim_{\mathsf{H}}(X) = \inf\{\delta \geq 0 : C_H^\delta(X) = 0\}$$

What about discrete sets?

- Most definitions of fractal dimension are *meaningless* for countable sets.
- E.g.

$$\dim_{\mathsf{H}}(\mathbb{Q} \times \mathbb{Q}) = 0$$

and for all $X \subset \mathbb{R}^2$, $Y \subset \mathbb{Q}^2$,

$$\dim_{\mathsf{H}}(X \cup Y) = \dim_{\mathsf{H}}(X)$$

A definition for discrete spaces

Let $M = (X, \rho)$ be a metric space, $|X| = n$.
We define $\dim_{\mathsf{f}}(X) = \delta$ if δ is the infimum number s.t.
for all $\varepsilon > 0$, $R \geq 2\varepsilon$, for all $x \in \mathbb{R}^2$, for all ε-nets N of X

$$|\mathrm{Ball}(x, R) \cap N| = O((R/\varepsilon)^{\delta})$$

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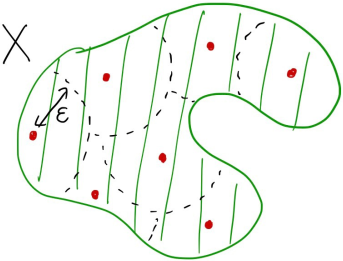
ε**-net:** maximal $N \subseteq X$ s.t. for all $x \neq y \in N$, $\rho(x, y) > \varepsilon$.



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- Discrete Sierpinski carpet:

$$\dim_{\mathsf{f}} \begin{pmatrix} \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & & \circ & \circ & & \circ & \circ & & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & & & & \circ & \circ & \circ \\ \circ & & \circ & & & & \circ & & \circ \\ \circ & \circ & \circ & & & & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & & \circ & \circ & & \circ & \circ & & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{pmatrix} = \log_3 8$$

Relation to other notions of dimension

- Similar to **Minkowski / box-counting dimension**.

$$\dim_{\mathsf{b}}(X) = \lim_{\varepsilon \to 0} \log(I_\varepsilon(X)) / \log(1/\varepsilon)$$

Used in numerical estimations of fractal dimension.
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- **Doubling dimension:** $\dim_{\mathsf{d}}(M) = 2^k$, if any ball of radius R can be covered by k balls of radius $R/2$.
  Fact:

  $$\dim_{\mathsf{d}}(M) = \Theta(\dim_{\mathsf{f}}(M)).$$

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  Fact:

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- **Fractal dimension in percolation theory**: Percolation in $\{1, \ldots, n^{1/d}\}^d$. Largest connected component has size $O(n^\delta)$.

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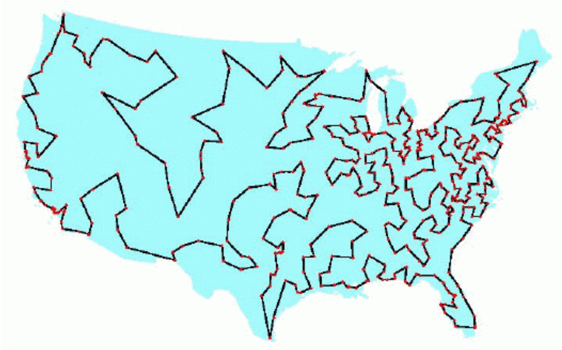

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- $\mathbb{R}^d$: there is no algorithm with running time $2^{O(n^{1-1/d-\varepsilon})}$, assuming ETH [Marx and S. 2014].
- $X \subset \mathbb{R}^{O(1)}$, $\dim_{\mathsf{f}}(X) = \delta > 1$: $2^{O(n^{1-1/\delta} \log n)} n^{O(1)}$ time [S. and Sridhar 2016]

Faster algorithms when $\delta < d$

Why does fractal dimension matter?

NP-hardness of TSP in $\mathbb{R}^2$ [Papadimitriou '77]

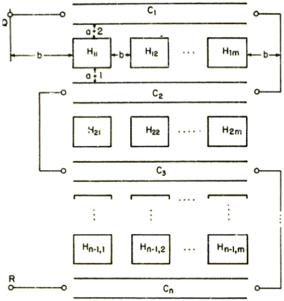

Fig. 4. The skeleton, schematically.

Hard instances have $\dim_{\mathfrak{f}} = 2$.

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Lemma (Smith and Wormald '98)

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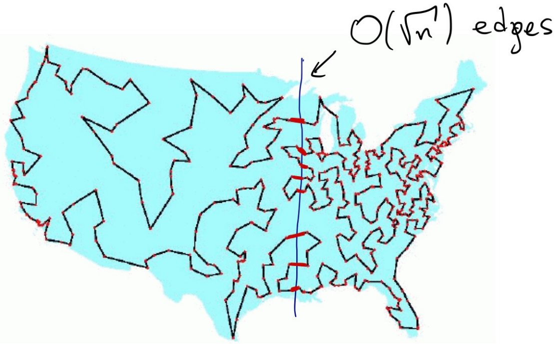
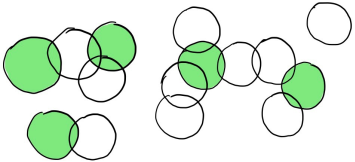


$\dim_{\mathfrak{f}}(X) = \delta > 1 \Rightarrow$ sphere separator of size $O(n^{1-1/\delta})$.

Applications in parameterized problems

k-Independent Set of Unit Balls



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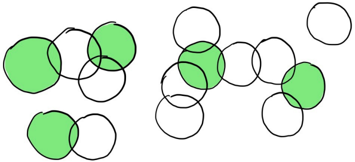

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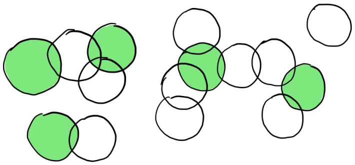

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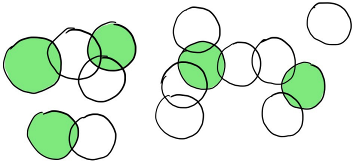

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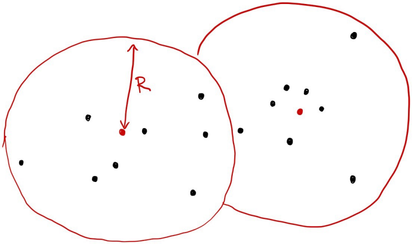
- In $\mathbb{R}^{O(1)}$ if the set of centers has fractal dimension δ [S., Sridhar 2016]:
  - $\delta > 1$: $n^{O(k^{1-1/\delta})+\log n}$ time
  - $\delta \leq 1$: $n^{O(\log n)}$ time

Applications in approximation schemes

R-**Cover**



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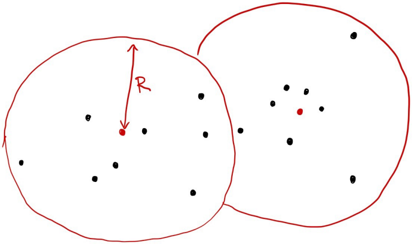

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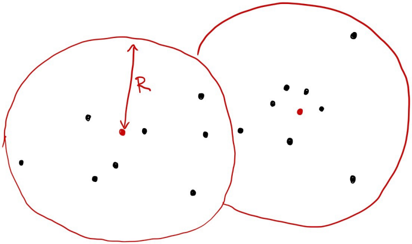

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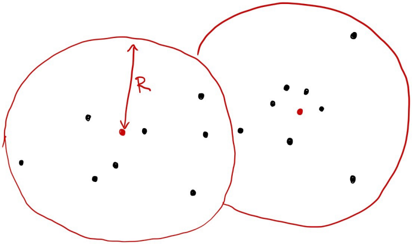

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- Similar result for R-Packing

Spanners

Metric space (X, ρ)
A c-**spanner** is a graph $G = (X, E)$ s.t. for all $x, y \in X$

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Grid minors vs. integer lattices.

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- Many other problems to explore
- What about fractal sets in higher dimensions (e.g. $\mathbb{R}^n$)?