Expanders via Local Edge Flips

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Outline

- How can we construct an expander locally?
  - Problem motivation and related works
- A simple distributed protocol
  - The switch and the flip protocols
- A new analysis for the two protocols
  - Obstacles in the analysis and new approach for the problem
- Conclusions and future directions
  - Open problems

How can we construct an expander locally?

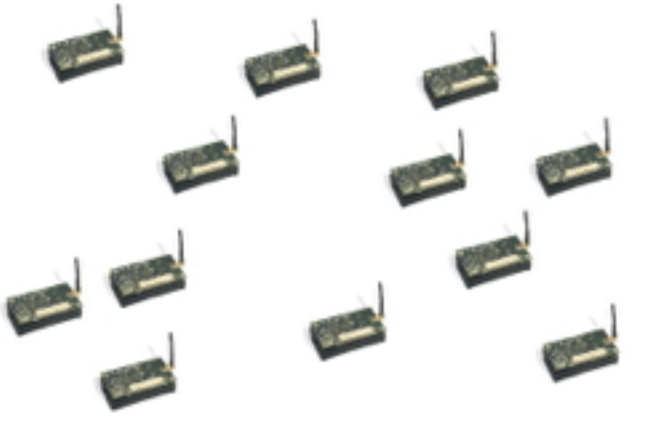
Why is it interesting?

Distributed system

- P2P networks
- Sensor networks
- Asynchronous system

Benefits

- Efficient
- Robust



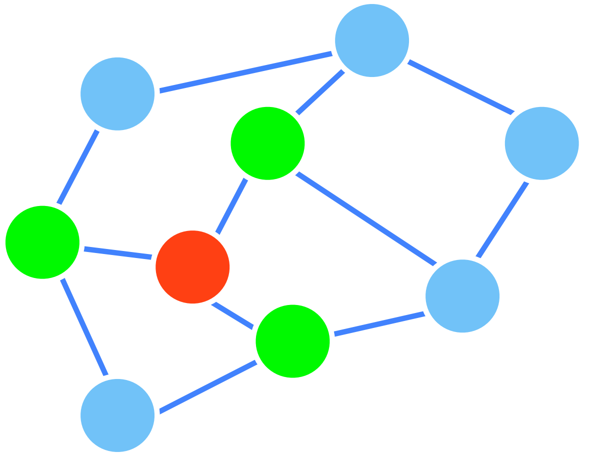
New challenges

- Important to construct quickly good network structure
- Only local communication

Local graph algorithms

Local algorithms

Algorithms based on *local* message passing among nodes



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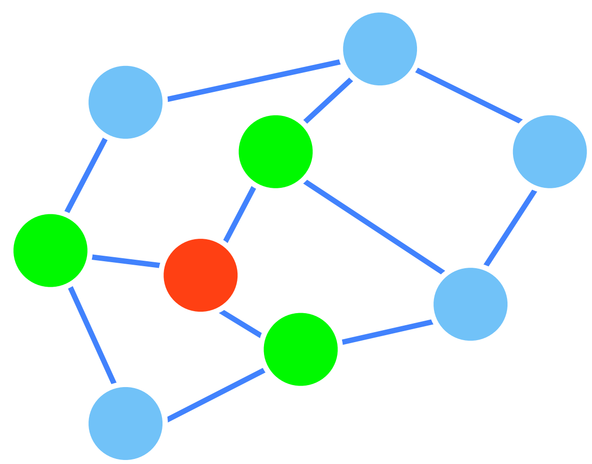

Advantages

Applicable to large scale graphs

Fast, easy to implement in parallel (MapReduce, Hadoop, Pregel...)

Problem

Starting from any connected graph is it possible to construct an expander locally?

Previous work

SKIP+: A Self-Stabilizing Skip Graph.

R. Jacob, A. W. Richa, C. Scheideler, S. Schmid and H. Täubig.

J. ACM 61(6): 36:1-36:26 (2014)

In the Local model it is possible to build an expander locally in $O(\log^2 n)$

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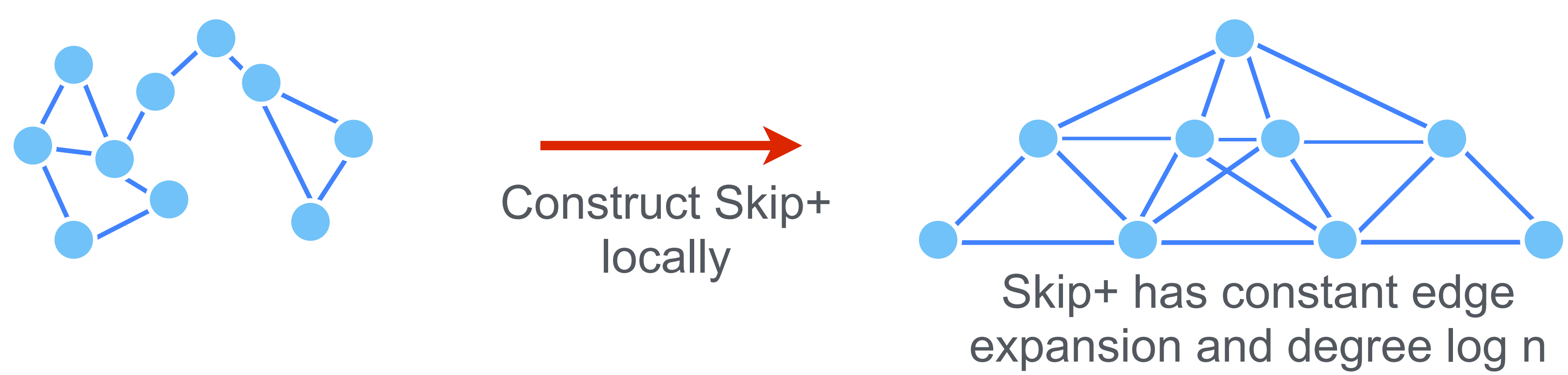

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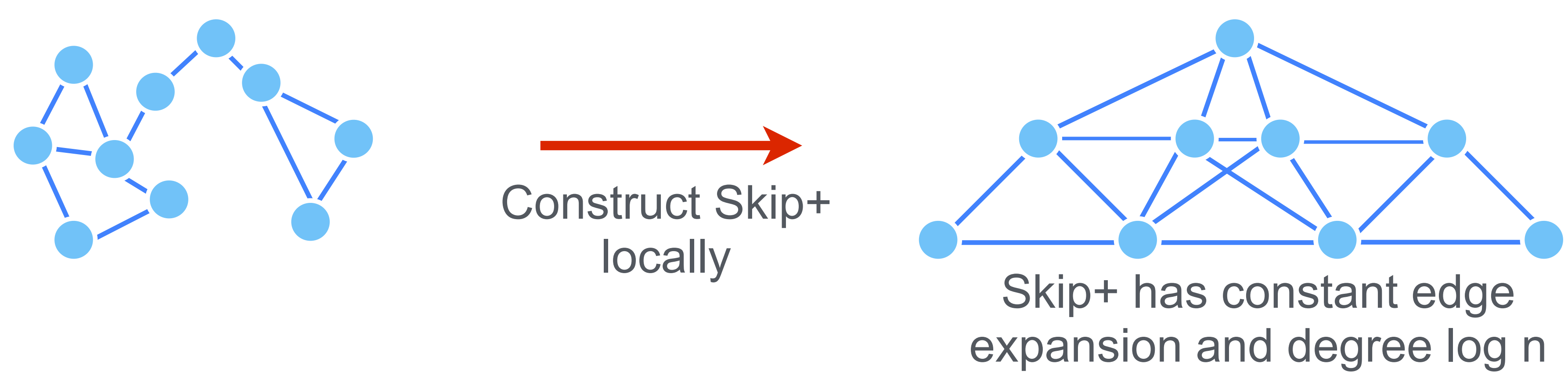


Limitations:

- Using this technique it is not possible to obtain an algebraic expander
- In any round nodes can exchange arbitrary large messages
- Memory needed by a single node in any round is not bounded
- Synchronous model, complex algorithm

Problem

Starting from any connected graph is it possible to define a simple rule to construct an expander locally?

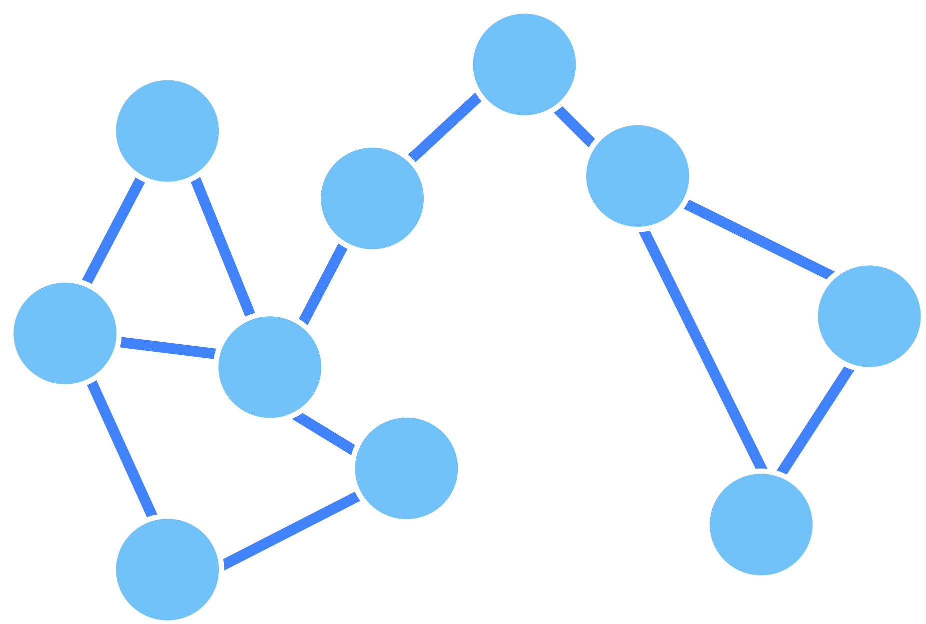
A simple distributed protocol

Switch protocol

[McKay, *Congressus Numerantium* 1981]

A simple protocol:

Pick two edges at random and invert their endpoints



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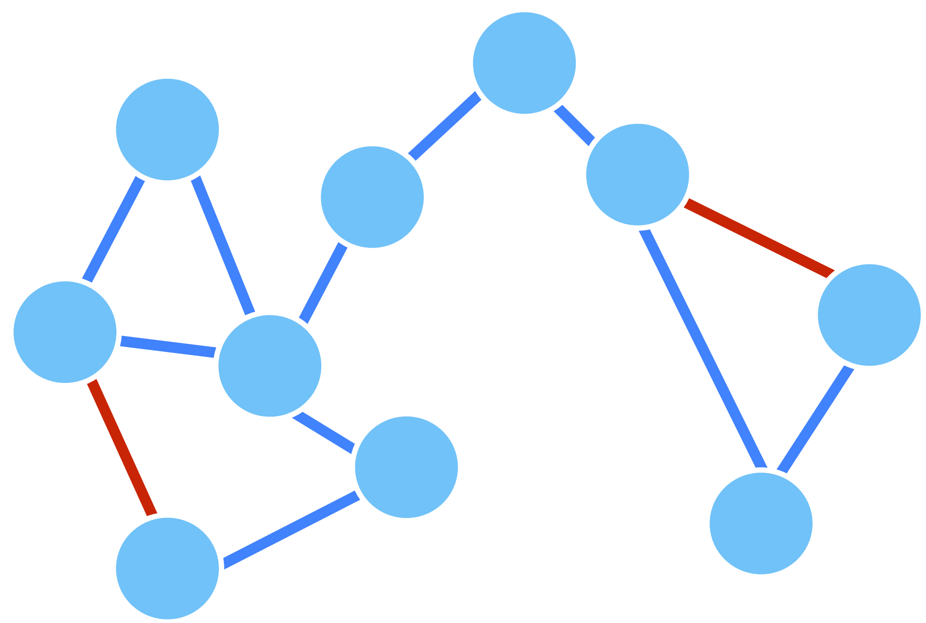

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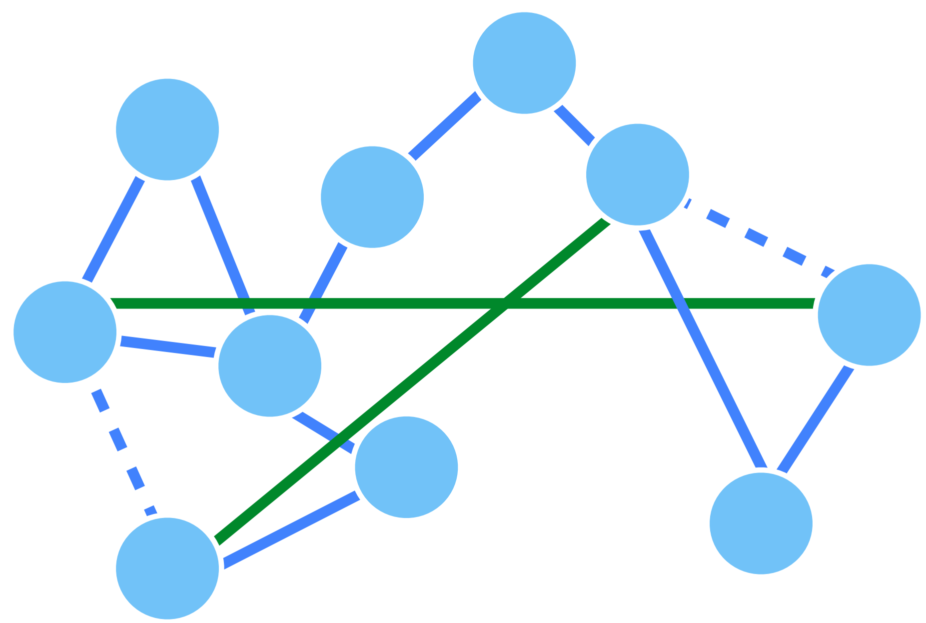

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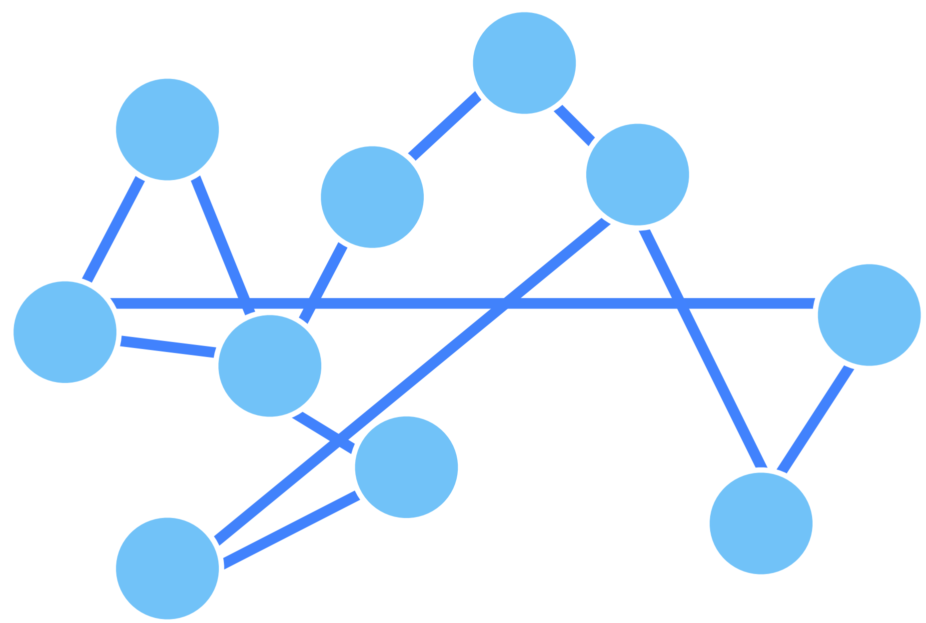
- Pick two edges at random and invert their endpoints
- Creation of parallel edges/self-loops is allowed



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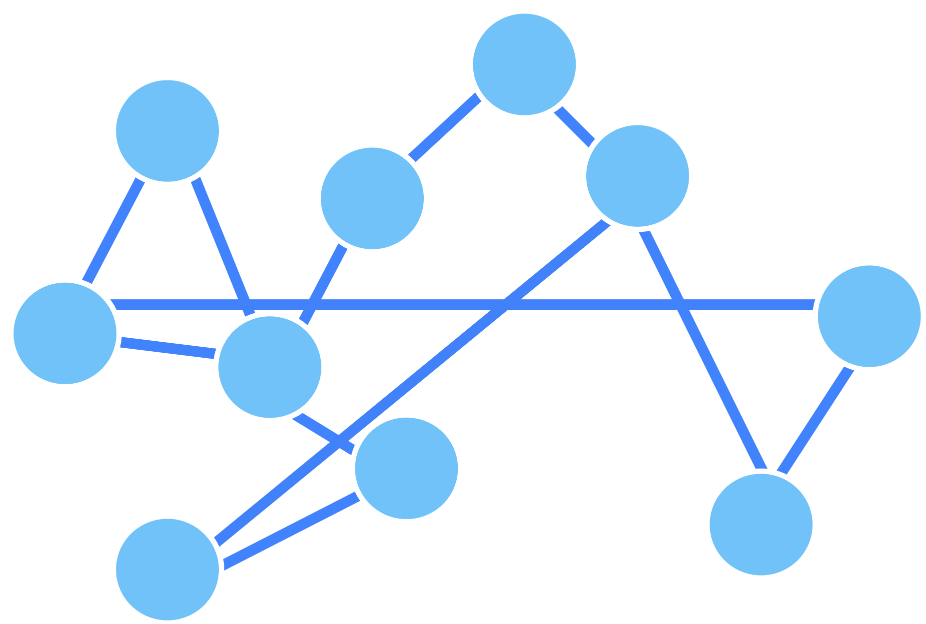


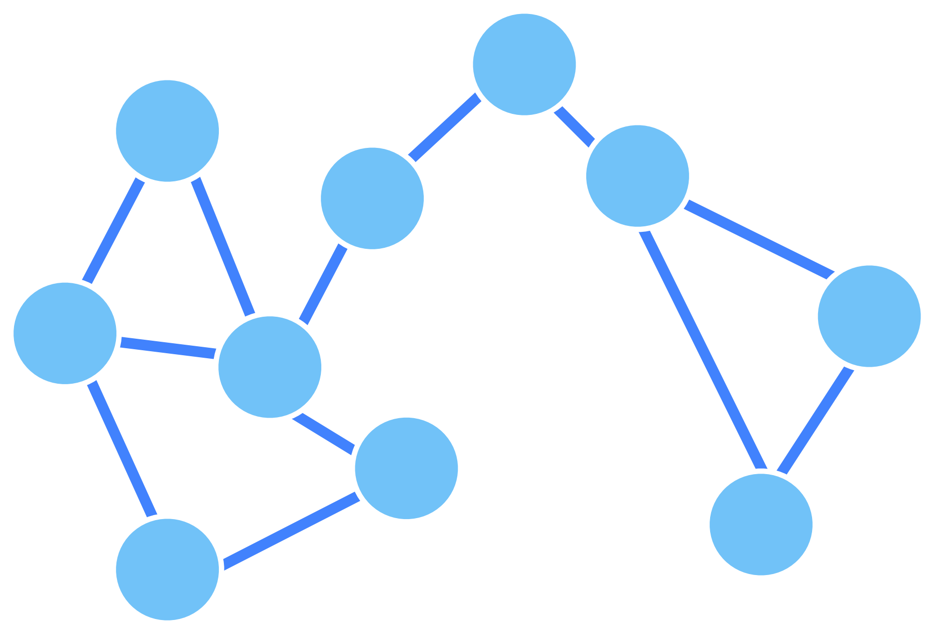
Limitation

- It is not local
- It may disconnect the graph

Flip protocol

[Mahlmann and Schindelhauer, *SPAA* 2005]

Pick a random length 3 path and invert its endpoints



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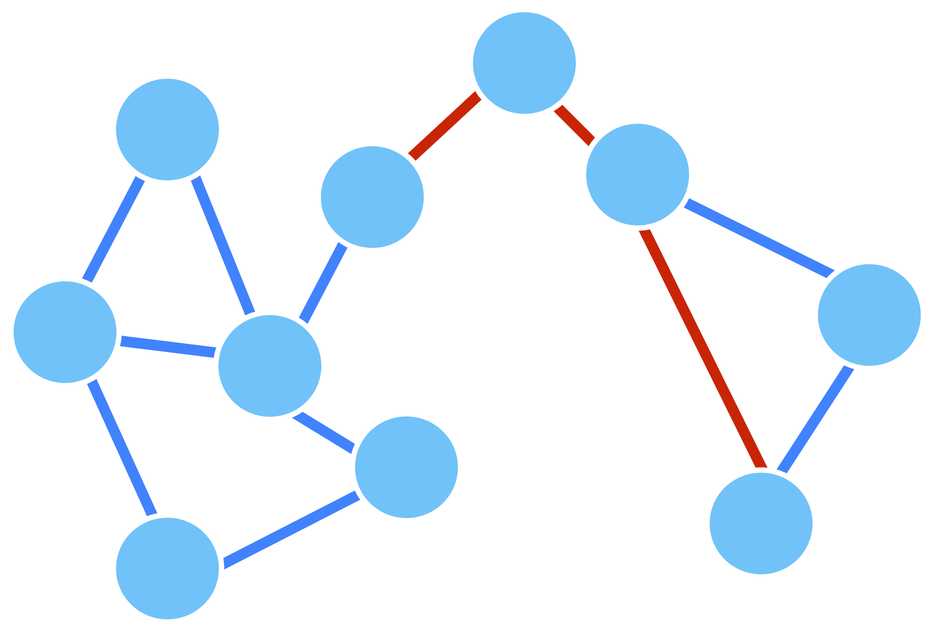

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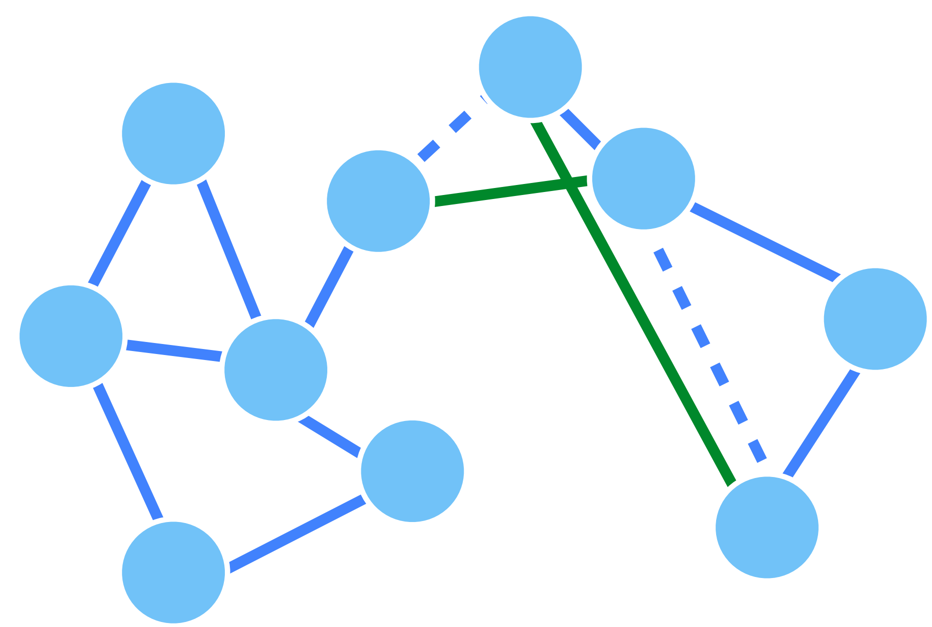

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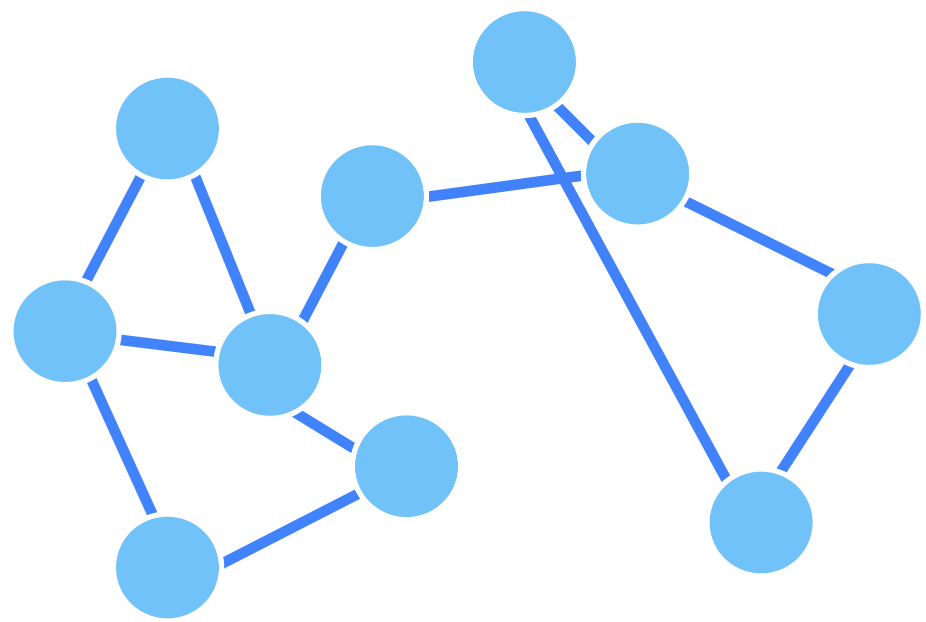

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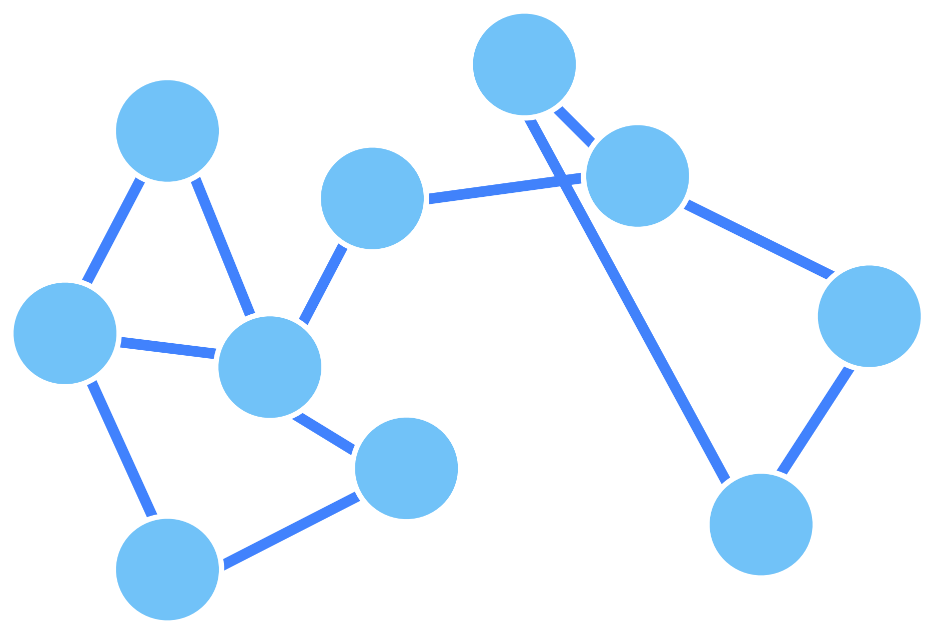

Experimentally it seems to be really fast

What is known about them?

[Cooper, Dyer and Greenhill, *SODA* 2005]

For d-regular graph the switch protocol converges to the configuration model in $\tilde{O}\left(n^8 d^{15}\right)$ steps.

[Greenhill, *SODA* 2015]

For non regular graph with max degree in $O\left(\sqrt{m}\right)$ the switch protocol converges to the configuration model in $\tilde{O}\left(m^{10} d_{max}^{14}\right)$ steps.

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For d-regular graph the flip protocol converges to the configuration model.

[Feder, Guetz, Mihail, and Saberi, *FOCS* 2006]

For d-regular graph the flip protocol converges to the configuration model in $\tilde{O}\left(d^{34} n^{36}\right)$ steps.

[Cooper and Dyer, *PODC* 2009]

For d-regular graph the flip protocol converges to the configuration model in $\tilde{O}\left(d^{23} n^{17}\right)$ steps.

How do they perform in practice?

[Mahlmann and Schindelhauer, *SPAA* 2005]

Experimentally switch and flips protocol transform any graph in an expander very quickly.

Conjectures:

Switch converges on d-regular graph in $O\left(nd\right)$ steps.

Flip converges on d-regular graph in $O\left(nd\log n\right)$ steps.

A new analysis for the two protocols

Results

Starting from any d-regular graph, with $d \in \Omega(\log n)$,

the switch protocol transforms the graph in an algebraic expander in $O\left(nd\right)$ steps.

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Obstacles

Dependencies.

Small cuts may first become smaller and only later increase.

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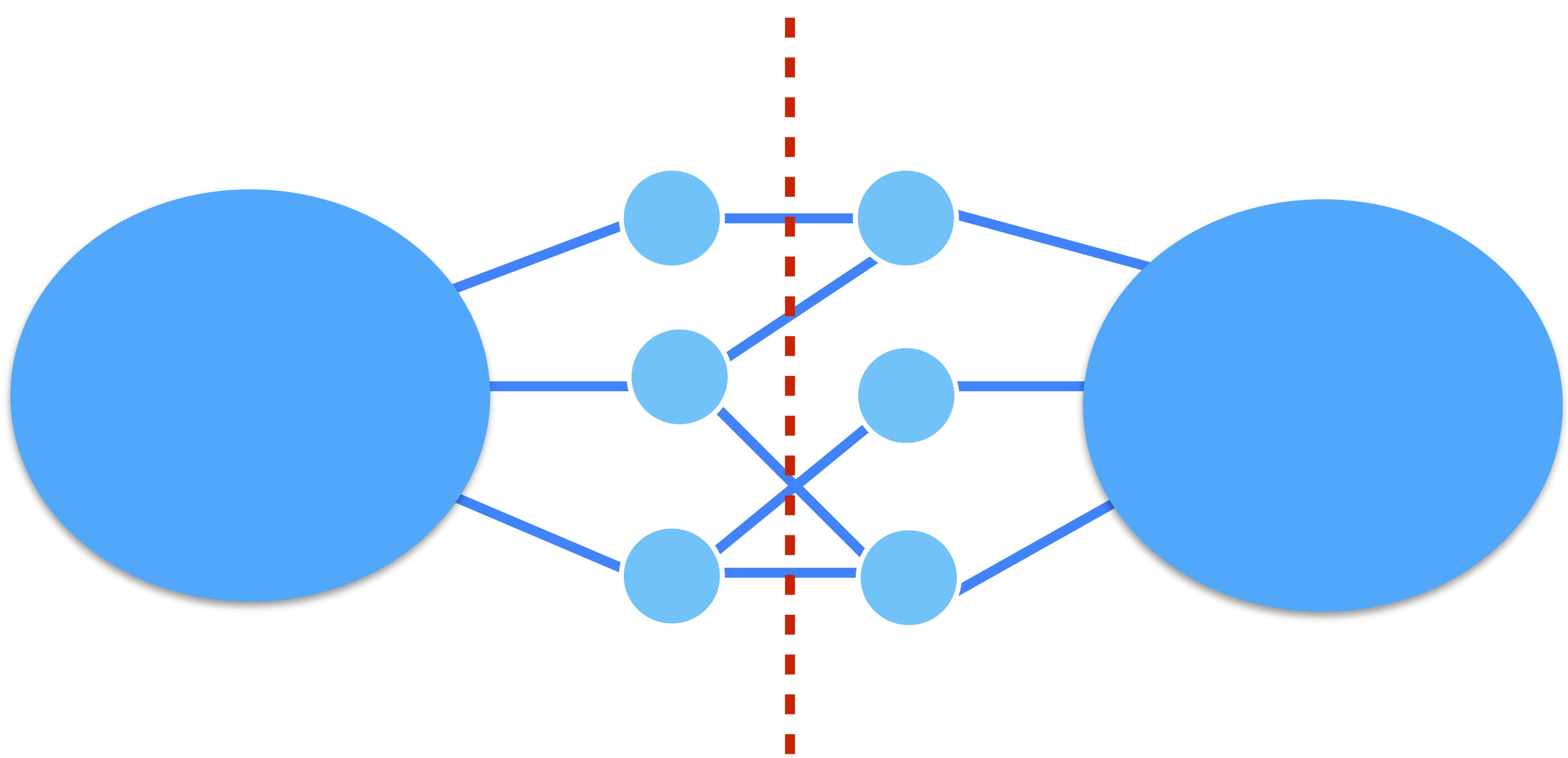
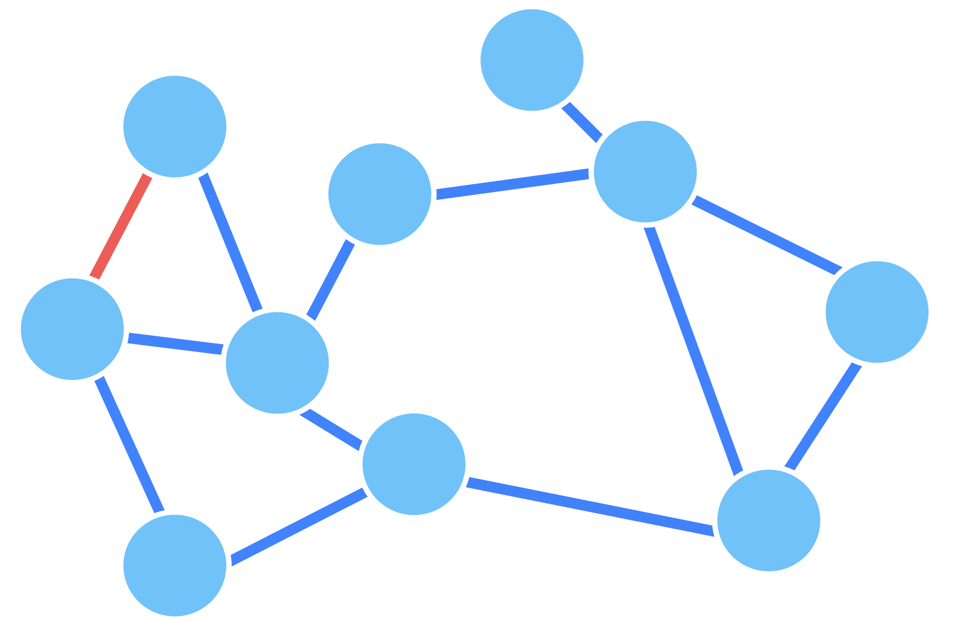


Flip definition

Pick a random edge.



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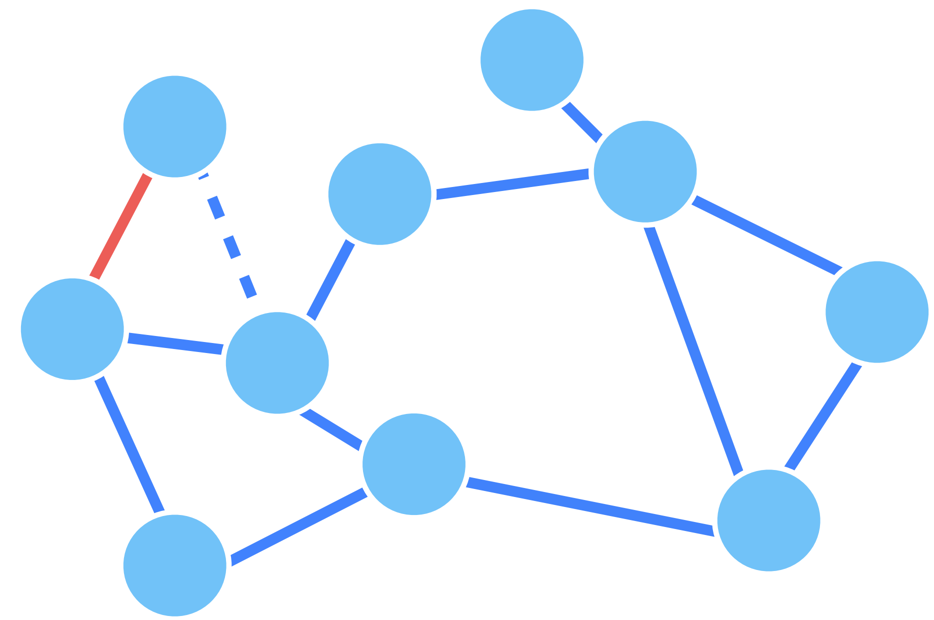

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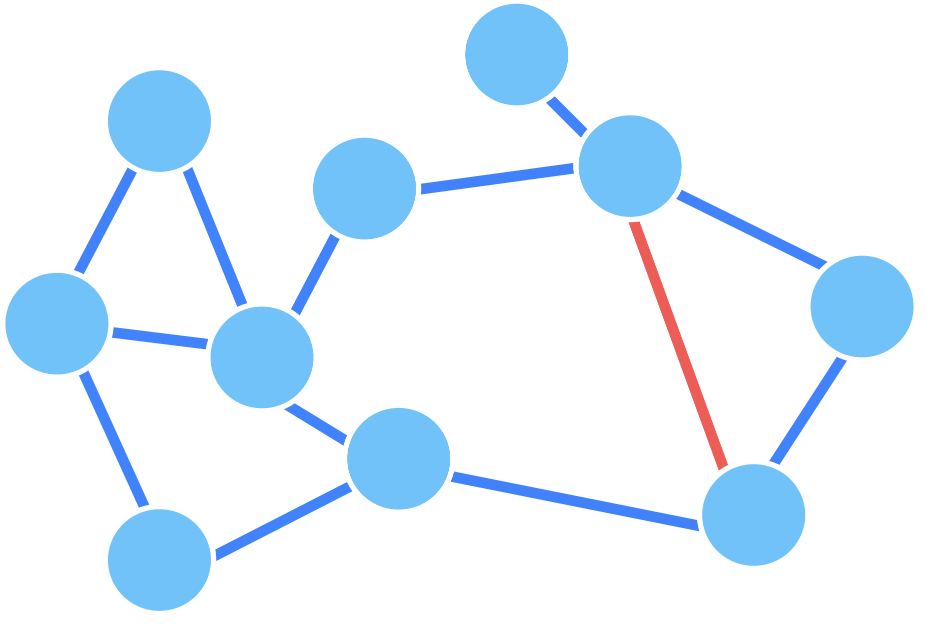

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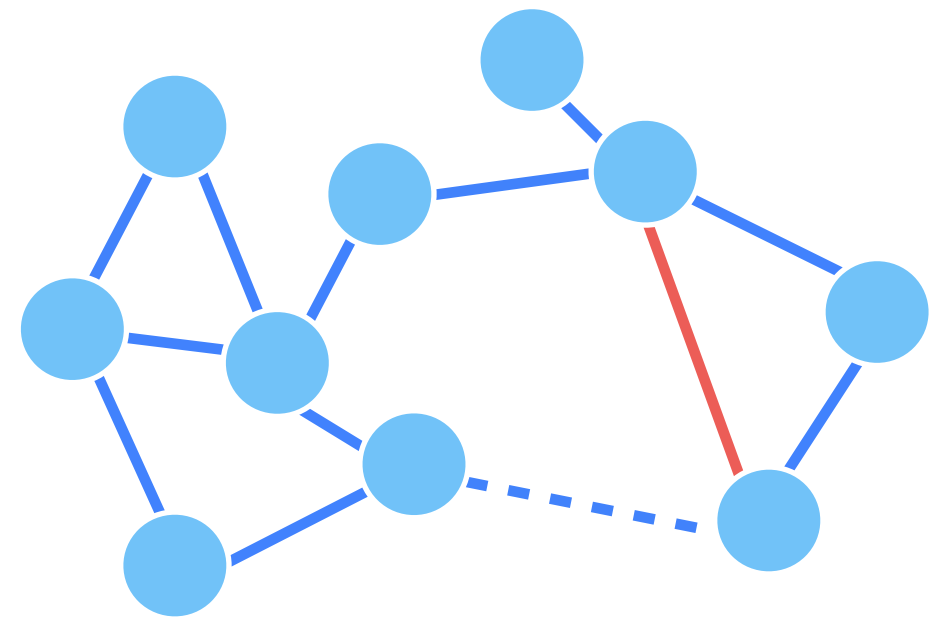

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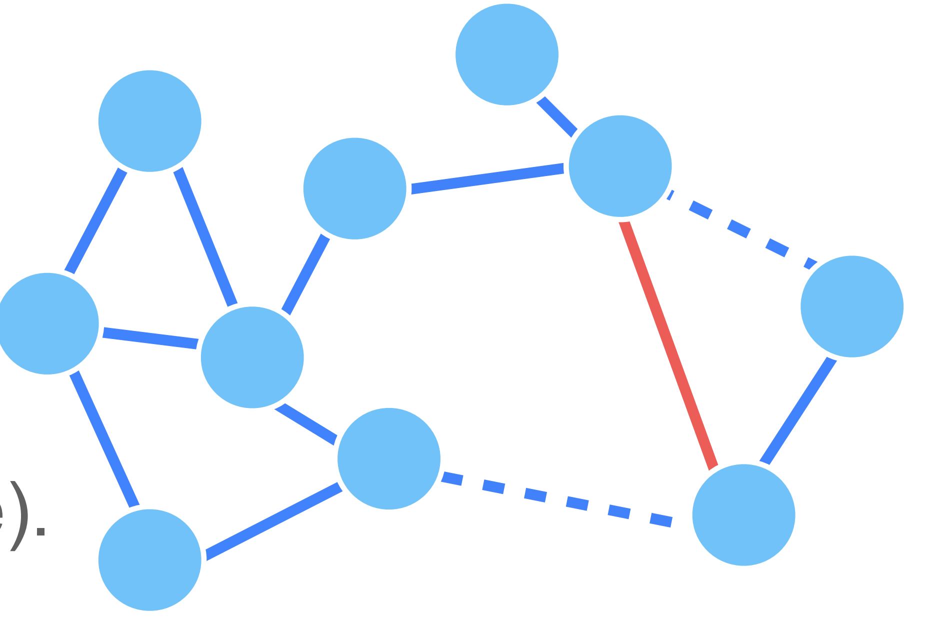

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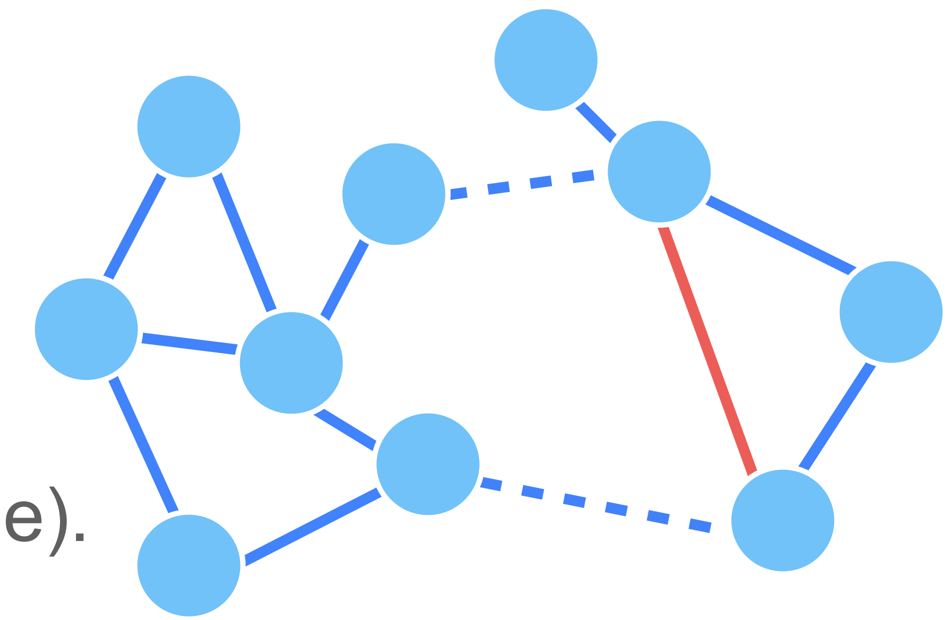

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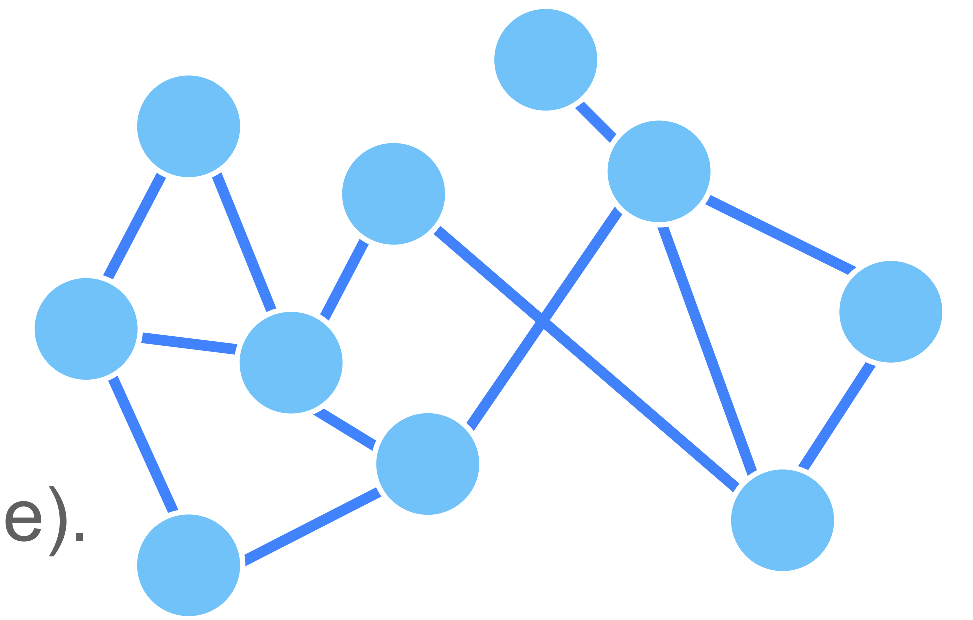
Perform swap.



Expected change of laplacian

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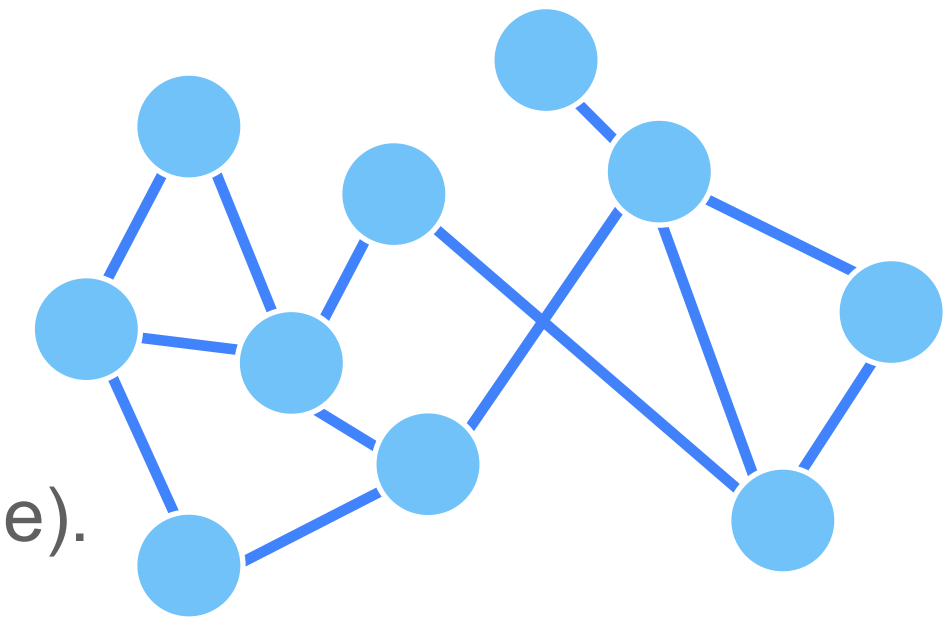

Let $\Delta^{(t)} = L\left(G^{(t+1)}\right) - L\left(G^{(t)}\right)$

$$E\left[\Delta^{(t)} | G^{(t)}\right] = \frac{4}{d^2 n}\left((d+1)L^{(t)} - \left(L^{(t)}\right)^2\right)$$

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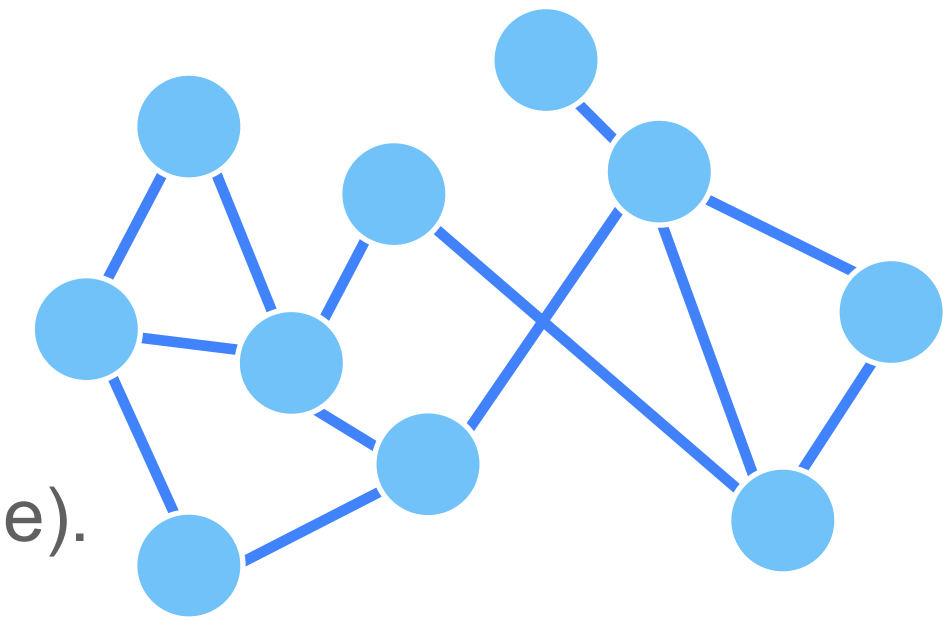

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Nice term. $\left(G^{(t)}\right)^2$ has better expansion.

Potential

Unfortunately we cannot argue directly on the expectation of the matrix after t step.

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We use a classic potential used for matrix concentration:

$$\Phi^{(t)} = \hat{tr}\left(e^{-\frac{20\log n}{d}L^{(t)}}\right)$$

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Note that in order to have $\Phi^{(t)}$ very small all the eigenvalues need to be large.

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We want to show that the potential decreases

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by $e^{-A} = I - A + A^2$

$$= \hat{tr}\left(e^{-\frac{20\log n}{d}L^{(t)}}\left(I - \frac{20\log n}{d}\Delta^{(t)} + \left(\frac{20\log n}{d}\Delta^{(t)}\right)^2\right)\right)$$

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Taking expectation:

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Potential

Using common diagonalization

$$\sum_{1 \le i \le n} e^{-\frac{20 \log n}{d} \lambda_i} \lambda_i (d/2 - \lambda_i)$$

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Two interesting cases:

$$\forall i : \lambda_i \geq \frac{d}{4}$$

$$\sum_{1 \le i \le n} e^{-\frac{20 \log n}{d} \lambda_i} \lambda_i (d/2 - \lambda_i) \in O(n^{-3})$$

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Two interesting cases:

$$\exists i : \lambda_i < \frac{d}{4}$$

We look at:

$$\frac{\sum_{1 \leq i \leq n} e^{-\frac{20 \log n}{d} \lambda_i} \lambda_i (d/2 - \lambda_i)}{\Phi^{(t)}} = \frac{\sum_{1 \leq i \leq n} e^{-\frac{20 \log n}{d} \lambda_i} \lambda_i (d/2 - \lambda_i)}{\sum_{1 \leq i \leq n} e^{-\frac{20 \log n}{d} \lambda_i}}$$

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Potential

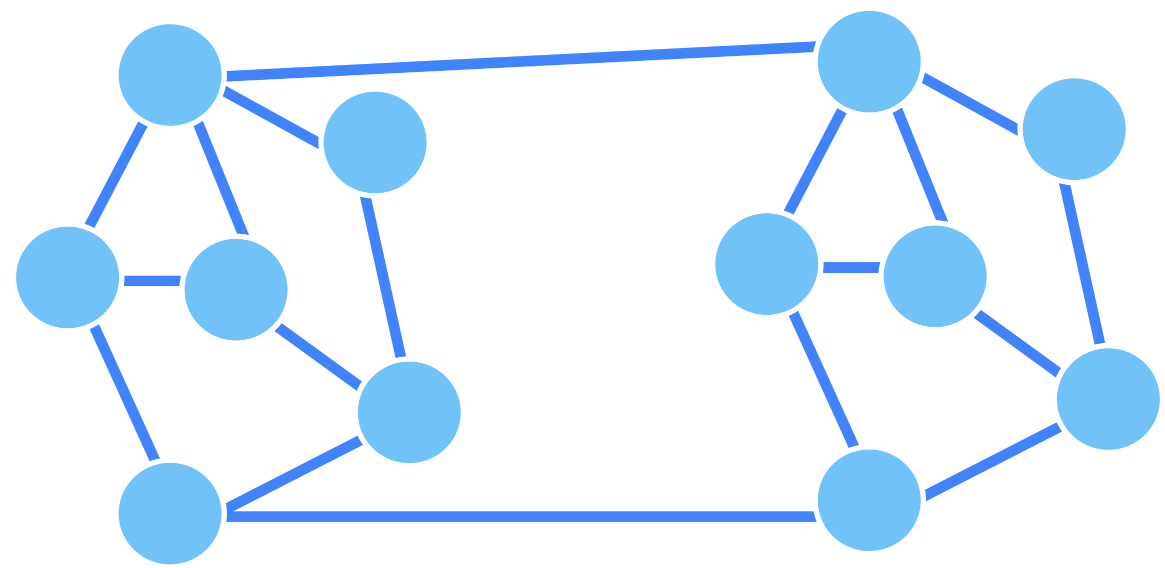
Thus:

$$E\left[\Phi^{(t+1)} | G^{(t)}\right] = \left(1 - \Omega\left(\frac{\sqrt{\log n}}{n^2 d^2}\right)\right)\Phi^{(t)} + O(n^{-3})$$

So in expectation $\Phi^{(t)}$ is in $O(n^{-3})$ after $O(n^2 d^2 \log n)$steps, hence using Markov inequality we get the result.

Limit of our analysis

Expected additive improvement in a round can be $O\left(\frac{1}{n^2 d^2}\right)$



Conclusions and future directions

Conclusions

- New technique to analyze distribute protocol
- New convergence time analysis for flip and switch protocol

Future works

- Improve analysis of the flip
- Study parallelized version of the protocol
- Study node addition or deletion

Thanks!