Parallel Peeling Algorithms

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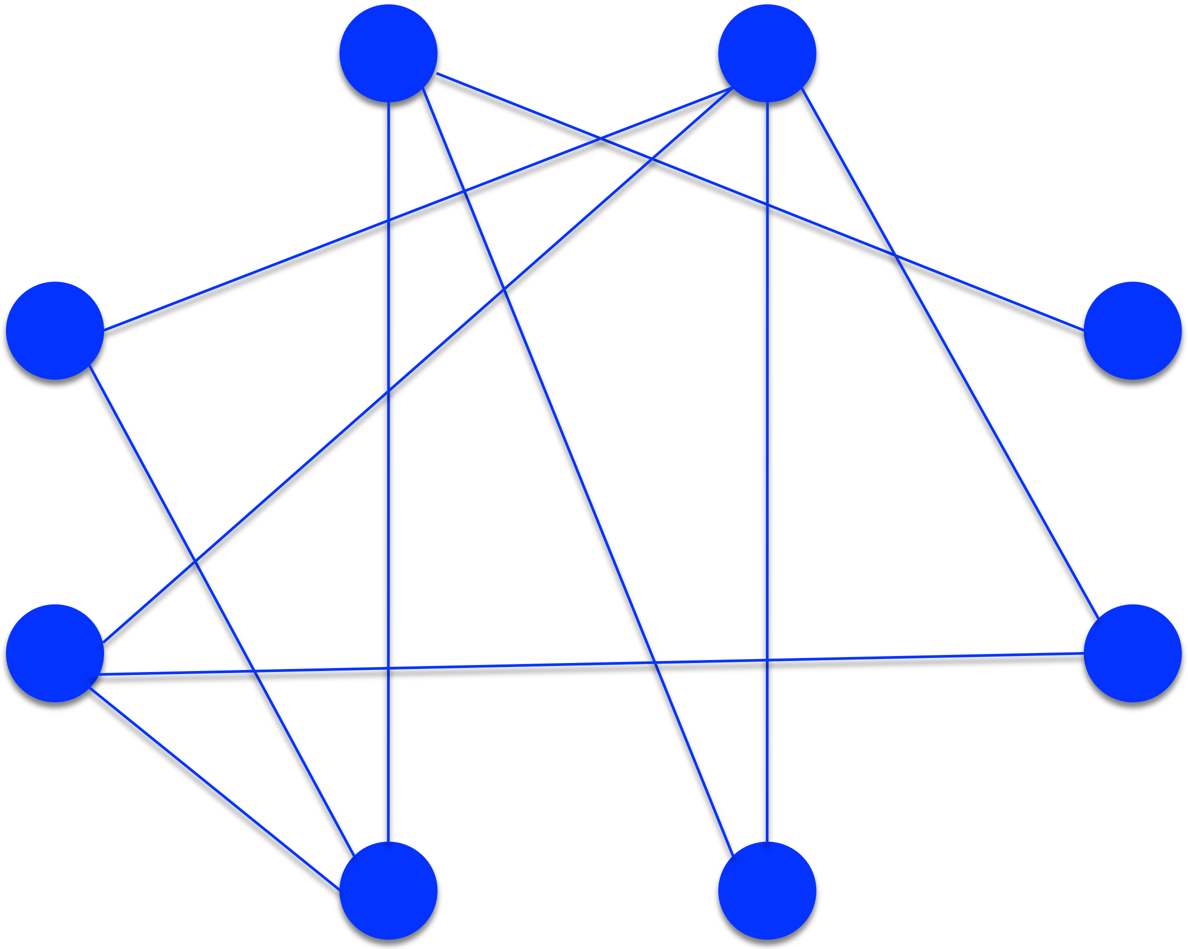
The Peeling Paradigm

- Many important algorithms for a wide variety of problems can be modeled in the same way.
- Start with a (random) hypergraph G.
  - While there exists a node v of degree less than k:
    - Remove v and all incident edges.
- The remaining graph is called the **k-core** of G.
  - k=2 in most applications.
- Typically, the algorithm "succeeds" if the the k-core is empty.
  - To ensure "success", data structure should be designed large enough so that the k-core of G is empty w.h.p.
- Typically yields simple, greedy algorithms running in linear time.

The peeling process when k=2



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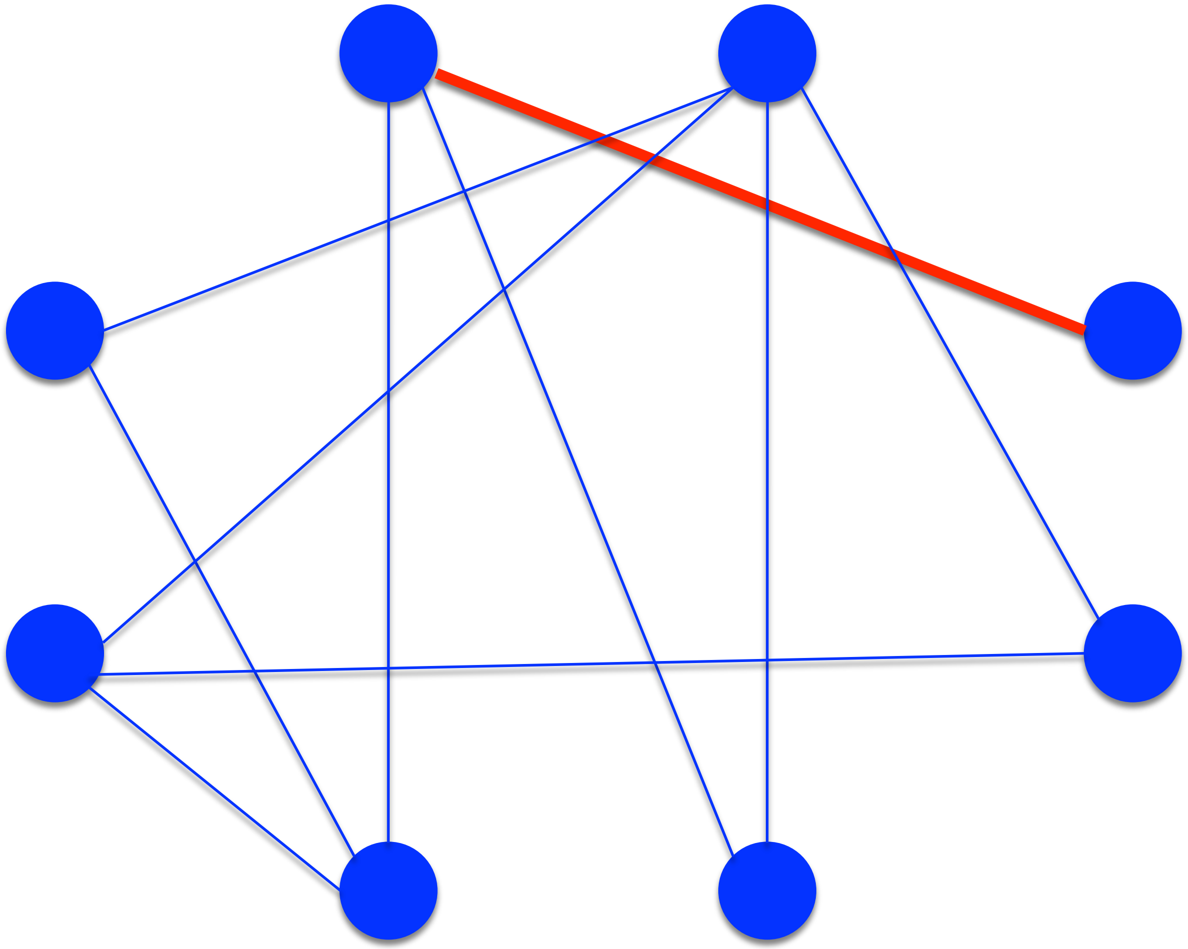

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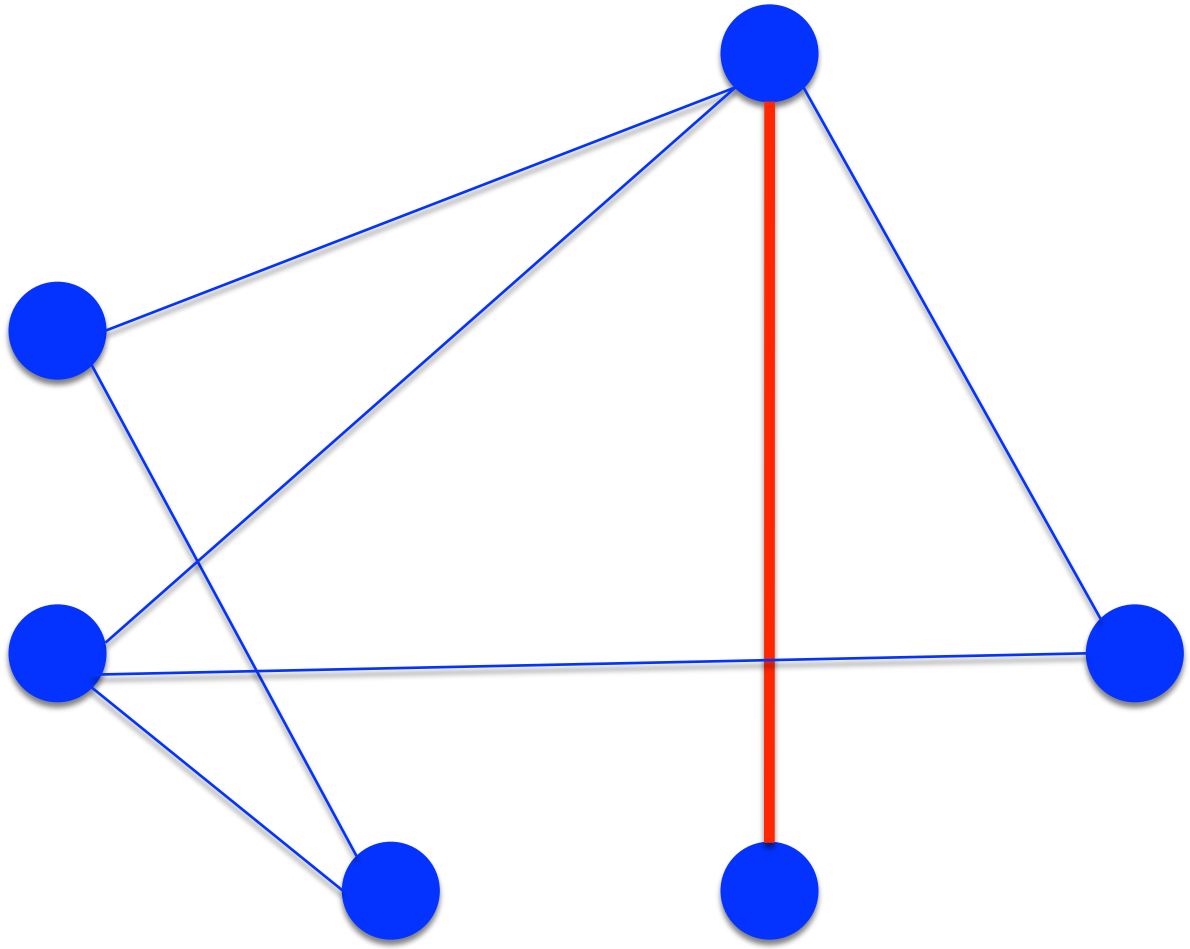

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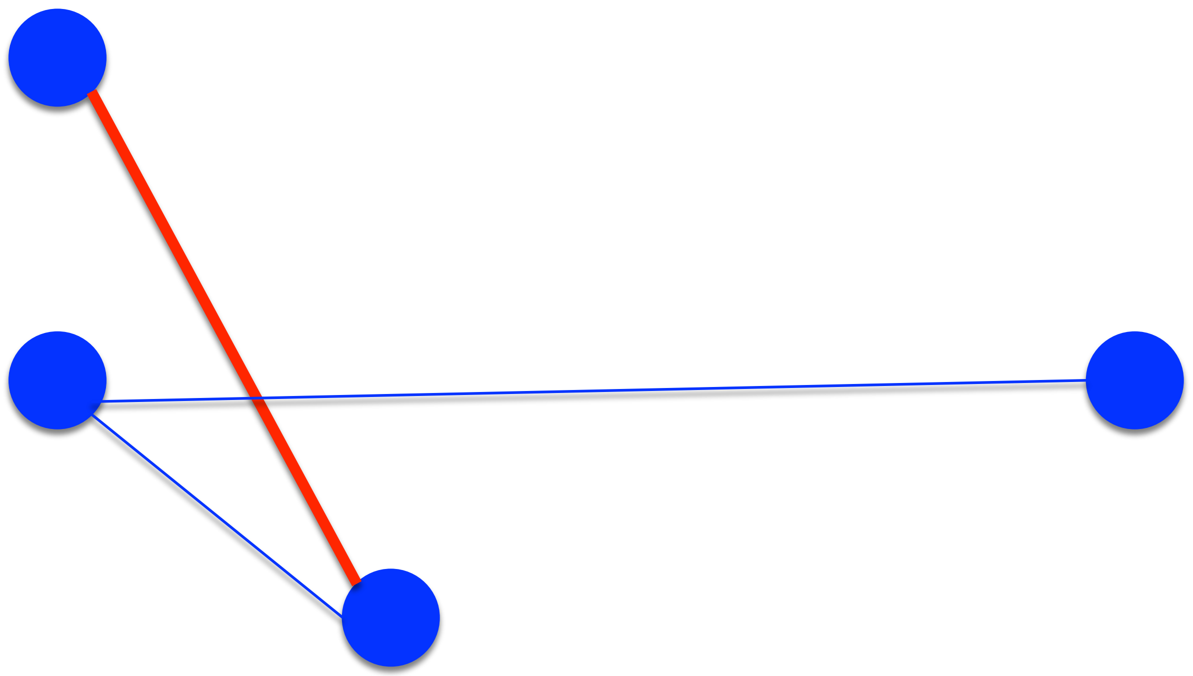

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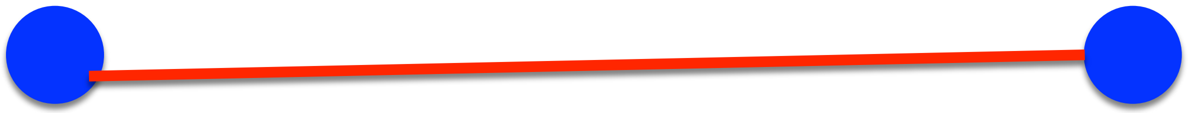


Example Algorithms

Example 1: Sparse Recovery Algorithms

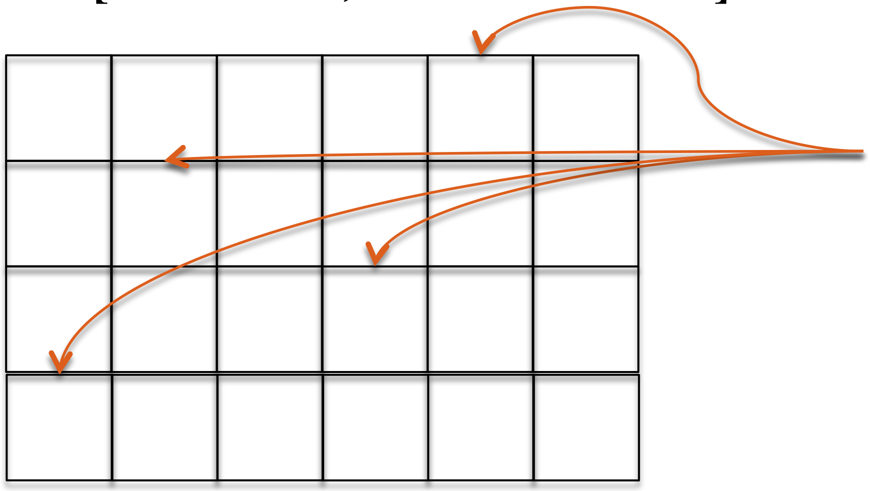
- Consider data streams that insert and delete a lot of items.
  - Flows through a router, people entering/leaving a building.
- Sparse Recovery problem: list all items with non-zero frequency.
- Want listing not at all times, but at "reasonable" or "off-peak" times, when working set size is bounded.
  - If we do M insertions, then M-N deletions, and want a list at the end, we need to list N items.
- Data structure size should be proportional to N, not to M!
  - Proportional to size you want to be able to list, not number of items your system has to handle.
- Central primitive used in more complicated streaming algorithms.
  - E.g. L_0 sampling, which is in turn used to solve problems on dynamic graph streams (see previous talk).

Example 1: Sparse Recovery Algorithms

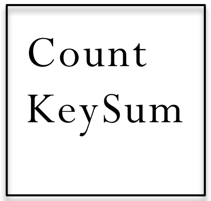
- For simplicity, assume that when listing occurs, no item has frequency more than 1.

Example 1: Sparse Recovery Algorithms

- Sparse Recovery Algorithm: Invertible Bloom Lookup Tables (IBLTs) [Goodrich, Mitzenmacher]



Each stream item hashed to r cells (using r different hash functions)



```
Insert(x):  For each of the j cells that x is hashed to:
            Add key to KeySum
            Increment Count
Delete(x): For each of the j cells x is hashed to:
            Subtract key from keysum
            Decrement Count
```

Listing Algorithm: Peeling

- Call a cell "pure" if its count equals 1.
- While there exists a pure cell:
  - Output x=keySum of the cell.
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- While there exists a pure cell:
  - Output x=keySum of the cell.
  - Call Delete(x) on the IBLT.
- To handle frequencies that are larger than 1, add a checksum field to each cell (details omitted).
- Listing $\Longleftrightarrow$ peeling to 2-core on the hypergraph G where:
  - Cells $\Longleftrightarrow$ vertices of G.
  - Items in IBLT $\Longleftrightarrow$ hyperedges of G.
  - G is r-uniform (each edge has r vertices, one for each cell the item is hashed to).

How Many Cells Does an IBLT Need to Guarantee Successful Listing?

- Consider a random r-uniform hypergraph G with n nodes and m=c*n edges.
  - i.e., each edge has r vertices, chosen uniformly at random from [n] without repetition.
- Known fact: Appearance of a non-empty k-core obeys a sharp threshold.
  - For some constant $c_{k,r}$, when $m < c_{k,r}n$, the k-core is empty with probability 1-o(1).
  - When $m > c_{k,r}n$, the k-core of G is non-empty with probability 1-o(1).
  - Implication: to successfully list a set of size M with probability 1-o(1), the IBLT needs roughly $M/c_{k,r}$ cells.
  - E.g. $c_{2,3} \approx 0.818$, $c_{2,4} \approx 0.772$, $c_{3,3} \approx 1.553$.

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  - E.g. $c_{2,3}$≈0.818, $c_{2,4}$≈0.772, $c_{3,3}$≈1.553.
  - In general:

$$c^*_{k,r} = \min_{x>0} \frac{x}{r(1 - e^{-x}\sum_{j=0}^{k-2}\frac{x^j}{j!})^{r-1}}.$$

Other Examples of Peeling Algorithms

- Low-Density Parity Check Codes for Erasure Channel.
  - [Luby, Mitzenmacher, Shokrollah, Spielman]
- Biff codes (directly use IBLTs).
  - [Mitzenmacher and Varghese]
- k-wise independent hash families with O(1) evaluation time.
  - [Siegel]
- Sparse FFT algorithms.
  - [Hassanieh et al.]
- Cuckoo hashing.
  - [Pagh and Rodler]
- Pure literal rule for computing satisfying assignments of random CNFs.
  - [Franco] [Mitzenmacher] [Molloy] [many others].

Parallel Peeling Algorithms

Our Goal: Parallelize These Peeling Algorithms

- Recall: the aforementioned algorithms are equivalent to peeling a random hypergraph G to its k-core.
- There is a brain dead way to parallelize the peeling process.
    - For each node v in parallel:
        - Check if v has degree less than k.
        - If so, remove v and its incident hyperedges.
- Key question: how many rounds of peeling are required to find the k-core?
- Algorithm is simple, analysis is tricky.

Main Result

- Two behaviors:
  - Parallel peeling completes in O(log log *n*) rounds if the edge density c is "below the threshold" $c_{k,r}$.
  - Parallel peeling requires Ω(log *n*) rounds if the edge density c is "above the threshold" $c_{k,r}$.
- This is great!
  - Most peeling uses the goal is to be *below the threshold*.
  - So "nature" is helping us by making parallelization fast.
  - Implies poly(loglog n) time, O(n poly(loglog n)) work, parallel algorithms for listing elements in an IBLT, decoding LDPC codes, etc.

Precise Upper Bound

Theorem 1. *Let $k, r \geq 2$ with $k + r \geq 5$, and let c be a constant. With probability $1 - o(1)$, the parallel peeling process for the k-core in a random hypergraph $G^r_{n,cn}$ with edge density c and r-ary edges terminates after $\frac{1}{\log((k-1)(r-1))} \log\log n + O(1)$ rounds when $c < c^*_{k,r}$.*

Theorem 2. *Let $k, r \geq 2$ with $k + r \geq 5$, and let c be a constant. With probability $1 - o(1)$, the parallel peeling process for the k-core in a random hypergraph $G^r_{n,cn}$ with edge density c and r-ary edges requires $\frac{1}{\log((k-1)(r-1))} \log\log n - O(1)$ rounds to terminate when $c < c^*_{k,r}$.*

Summary: The right factor in front of the loglog n is 1/(log(*k*-1)(*r*-1)) (tight up to an additive constant).

Lower Bound

Theorem 3. *Let $r \geq 3$ and $k \geq 2$. With probability $1 - o(1)$, the peeling process for the k-core in $G^r_{n,cn}$ terminates after $\Omega(\log n)$ rounds when $c > c^*_{k,r}$,*

Summary: Ω (log n) lower bound matches an earlier O(log n) upper bound due to [Achlioptas and Molloy, 2013].

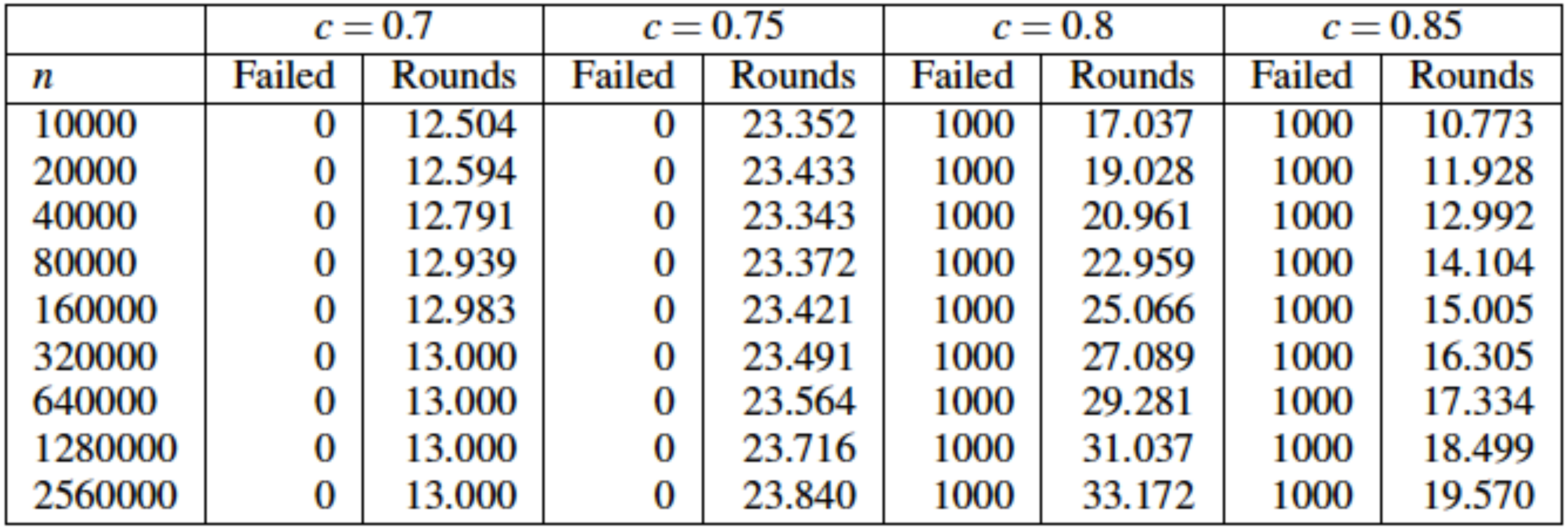
Proof Sketch for Upper Bound

- Let λ_i denote the probability a given vertex v survives i rounds of peeling.
- Claim: $\lambda_{i+1} \leq (C\lambda_i)^{(k-1)(r-1)}$ for some constant C.
  - Suggests $\lambda_i << 1/n$ after about $1/((k-1)(r-1)) * \log\log n$ rounds.
  - A related argument shows that $\lambda_i \leq 1/(2C)$ after $O(1)$ rounds, and after that point the claim implies that λ_i falls doubly-exponentially quickly.

Proof Sketch for Upper Bound

- Let λ_i denote the probability a given vertex *v* survives i rounds of peeling.
- Claim: $\lambda_{i+1} \leq (C\lambda_i)^{(k-1)(r-1)}$ for some constant C.
- **Very** crude sketch of the Claim's plausibility:
  - Node v survives round *i+1* only if it has (at least) *k* incident edges $e_1 \ldots e_k$ that survive round i.
  - Fix a *k*-tuple of edges $e_1 \ldots e_k$ incident to *v*.
  - Assume no node other than *v* appears in more than one of these edges.
  - Then there are *k(r-1)* distinct nodes other than *v* appearing in these edges.
  - The edges all survive round *i* only if **all** *k(r-1)* of these nodes survive round *i*.
  - Let's pretend that the survival of these nodes are independent events.
  - Then the probability all nodes survive round *i* is roughly $\lambda_i^{k(r-1)}$.
  - Finally, union bound over all *k*-tuples of edges incident to *v*.

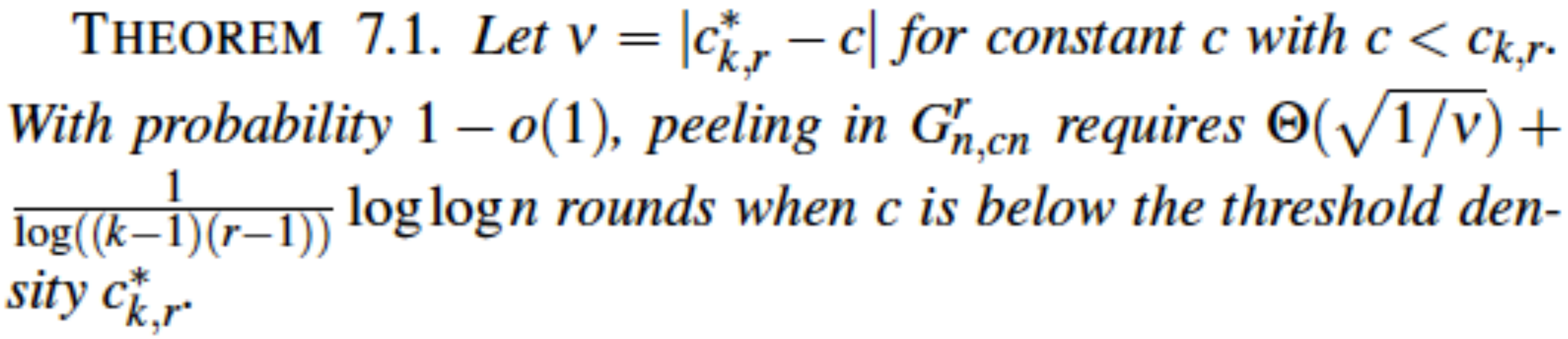
Simulation Results



	$c = 0.7$		$c = 0.75$		$c = 0.8$		$c = 0.85$	
n	Failed	Rounds	Failed	Rounds	Failed	Rounds	Failed	Rounds
10000	0	12.504	0	23.352	1000	17.037	1000	10.773
20000	0	12.594	0	23.433	1000	19.028	1000	11.928
40000	0	12.791	0	23.343	1000	20.961	1000	12.992
80000	0	12.939	0	23.372	1000	22.959	1000	14.104
160000	0	12.983	0	23.421	1000	25.066	1000	15.005
320000	0	13.000	0	23.491	1000	27.089	1000	16.305
640000	0	13.000	0	23.564	1000	29.281	1000	17.334
1280000	0	13.000	0	23.716	1000	31.037	1000	18.499
2560000	0	13.000	0	23.840	1000	33.172	1000	19.570

- Results from simulations of parallel peeling process on random 4-uniform hypergraphs with n nodes and $c*n$ edges using $k = 2$.
- Averaged over 1000 trials.
- Recall that $c_{2,4} \approx 0.772$.

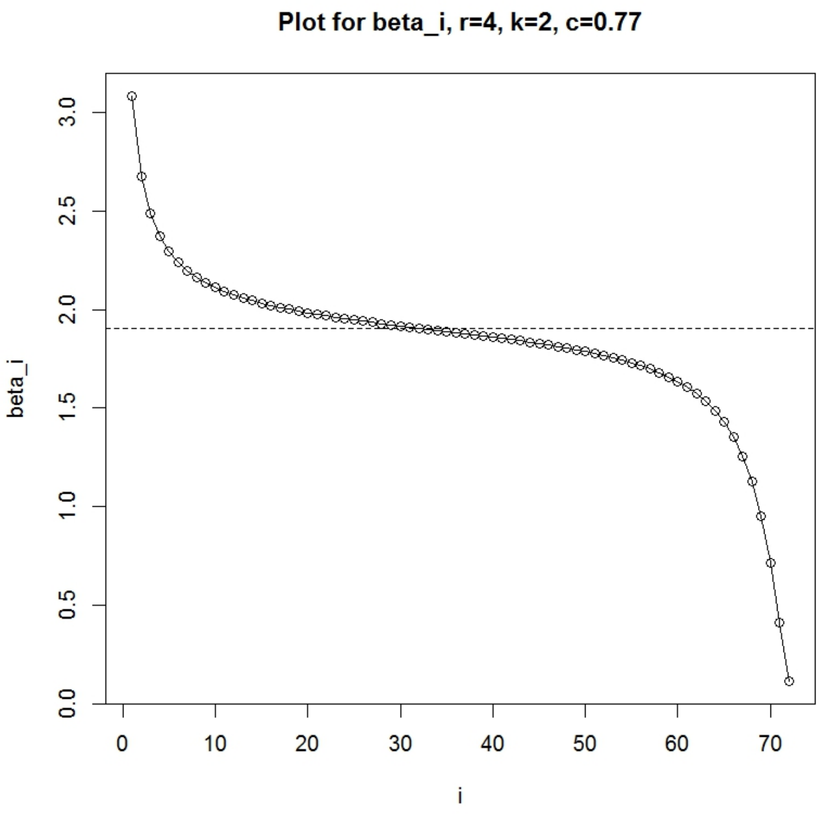
Refined Result: Mind the Gap

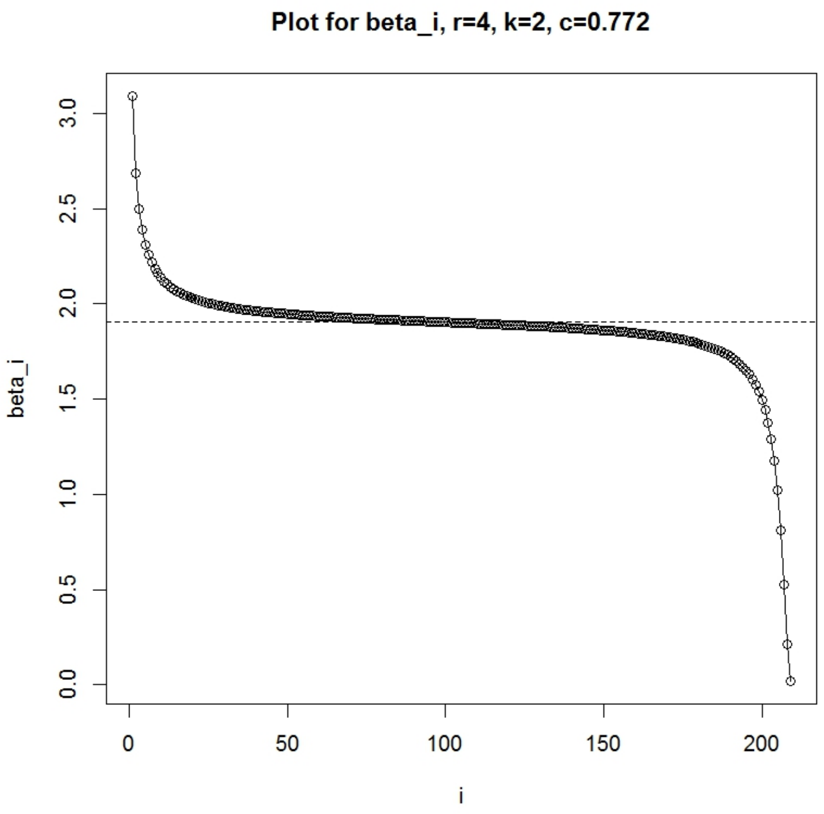


THEOREM 7.1. *Let $\nu = |c^*_{k,r} - c|$ for constant c with $c < c_{k,r}$. With probability $1 - o(1)$, peeling in $G^r_{n,cn}$ requires $\Theta(\sqrt{1/\nu}) + \frac{1}{\log((k-1)(r-1))}\log\log n$ rounds when c is below the threshold density $c^*_{k,r}$.*

Summary: below the threshold, the additive term is $\Theta(1/\sqrt{|\text{gap}|})$.
This can be more important than the $\log \log n$ term if the edge density is close to the threshold!

Refined Simulations: Mind the Gap





Plots show expected progress of the peeling process as a function of the round i, for values of the edge density c approaching the threshold value of $c_{2,4} \approx 0.772$.

Refined Analysis: Mind the Gap

- Analysis shows that peeling process falls into three "stages".
  - First stage: the fraction of surviving nodes falls very quickly as a function of the rounds until it gets close to a certain key value x*.
  - Second stage: $\Theta(1/\sqrt{|gap|})$ rounds are required to go from "close" to x* to "significantly below" x*.
  - Third stage: the analysis of the basic upper bound kicks in, and the fraction of surviving nodes falls doubly-exponentially quickly.

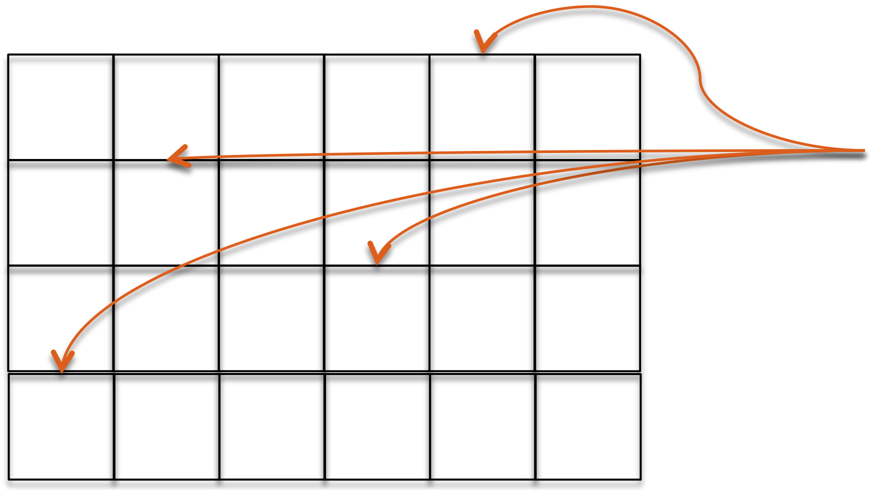
Implementation Issues

GPU Experimental Results

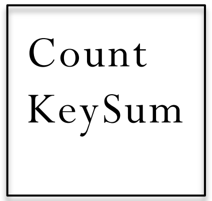
Table Load	No. Table Cells	% Recovered	GPU Recovery Time	Serial Recovery Time	GPU Insert Time	Serial Insert Time
0.75	16.8 million	100%	0.33 s	6.37 s	0.31 s	3.91 s
0.83	16.8 million	50.1%	0.42 s	3.64 s	0.35 s	4.34 s

Table 3: Results of our parallel and serial IBLT implementations with $r = 3$ hash functions. The table load refers to the ratio of the number of items in the IBLT to the number of cells in the IBLT.

Recall: IBLTs



Each stream item hashed to r cells (using r different hash functions)



Insert(x): For each of the j cells that x is hashed to:
Add key to KeySum
Increment Count

Delete(x): For each of the j cells x is hashed to:
Subtract key from keysum
Decrement Count

Recall: IBLT Listing Algorithm

- Call a cell "pure" if its count equals 1.
- While there exists a pure cell:
  - Output x=keySum of the cell.
  - Call Delete(x) on the IBLT.

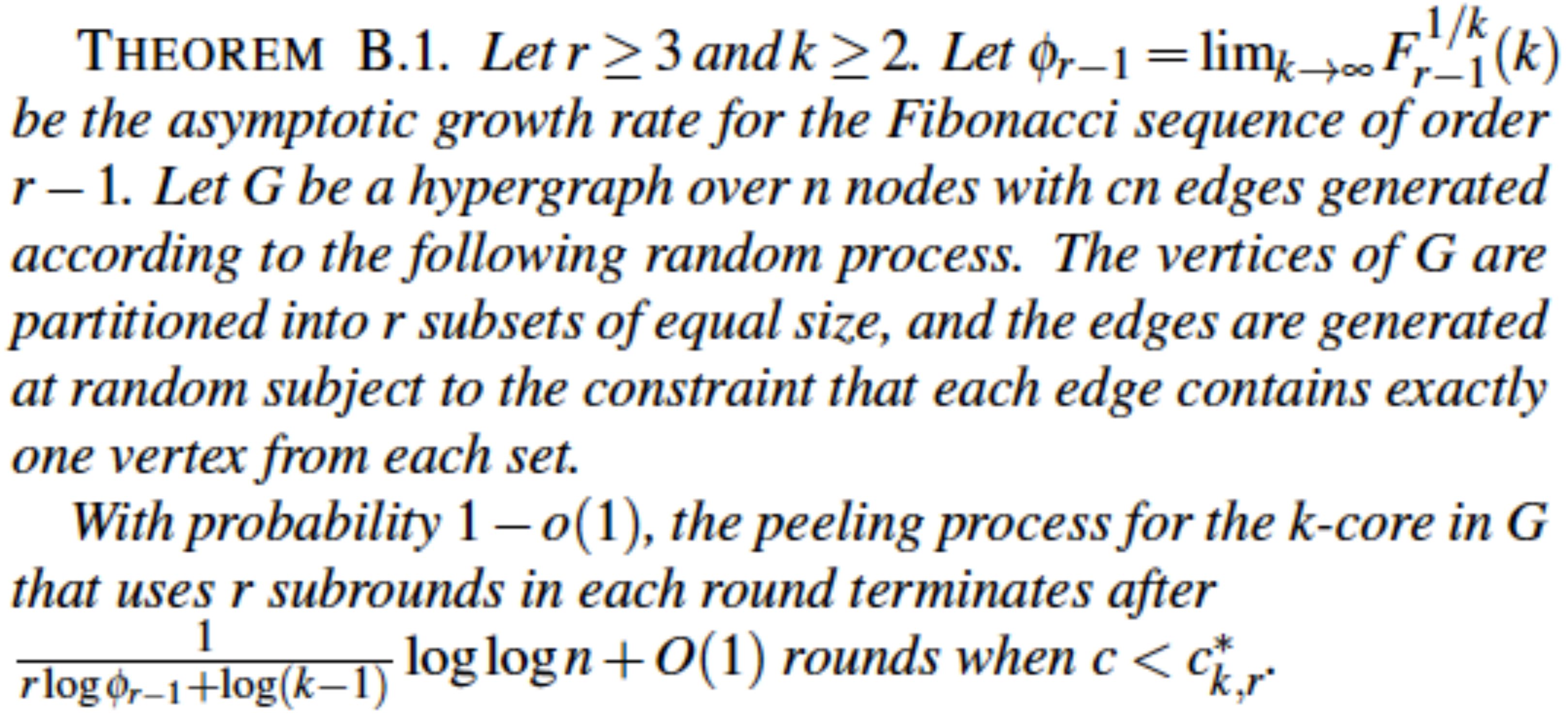
GPU Implementation

- Each cell gets a thread.
- Each cell checks if it is pure.
  - If so, identify the key it contains and remove it from other cells in the IBLT.
  - Do this by subtracting out values in other cells.
- Issue: repeated deletion.
  - Several cells might recover and try to remove the same key in the same round. So a key gets deleted more than once!

Dealing with Repeated Deletion

- To avoid this: use r subtables, such that the *i*th hash function only hashes into subtable *i*.
  - Break the listing algorithm into serial subrounds. In *i*th subround, recover only from the *i*th subtable.
  - Avoids repeated deletions, since each item will be hashed to just 1 cell in each subtable.
  - Leads to interesting variation in the analysis.
- Subrounds increase runtime, since they must happen sequentially.
  - Naively, they may blow up runtime by a factor of r.
  - But we show this does not happen.
    - Gains in one subround can help later subrounds.
    - We show runtime only blows up by a factor of about $\log_2(r-1)$.
- Analysis is similar to Vöcking's *d*-left scheme.
  - Fibonacci numbers show up!

Subround Result



THEOREM B.1. *Let $r \geq 3$ and $k \geq 2$. Let $\phi_{r-1} = \lim_{k\to\infty} F_{r-1}^{1/k}(k)$ be the asymptotic growth rate for the Fibonacci sequence of order $r-1$. Let G be a hypergraph over n nodes with cn edges generated according to the following random process. The vertices of G are partitioned into r subsets of equal size, and the edges are generated at random subject to the constraint that each edge contains exactly one vertex from each set.*

With probability $1-o(1)$, the peeling process for the k-core in G that uses r subrounds in each round terminates after $\frac{1}{r\log\phi_{r-1}+\log(k-1)}\log\log n + O(1)$ rounds when $c < c^*_{k,r}$.

Summary: use of r subtables increase constant factor in front of the log log n, but by much less than a factor or r.

Conclusion

- Peeling gives simple, fast greedy algorithms.
  - Usually linear or quasi-linear total work.
- Particularly well suited for parallelization.
  - Especially when aiming for an empty k-core.
- Implementation leads to interesting variation in the analysis.
  - Subrounds.
- Can analyze dependence on “gap” to the threshold.

Thank you!

Example 1: LDPC Codes for Erasure Channels

$$\begin{tikzcd} \boxed{c_1c_2c_3c_4c_5c_6c_7c_8c_9c_{10}c_{11}c_{12}c_{13}} \arrow[r] & \boxed{\text{Erasure Channel}} \arrow[r] & \boxed{c_1\mathbf{?}c_3c_4c_5\mathbf{?}c_7\mathbf{?}c_9c_{10}c_{11}c_{12}\mathbf{?}} \end{tikzcd}$$

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How does an LDPC code encode an 8-bit message $m_1m_2m_3m_4m_5m_6m_7m_8$?

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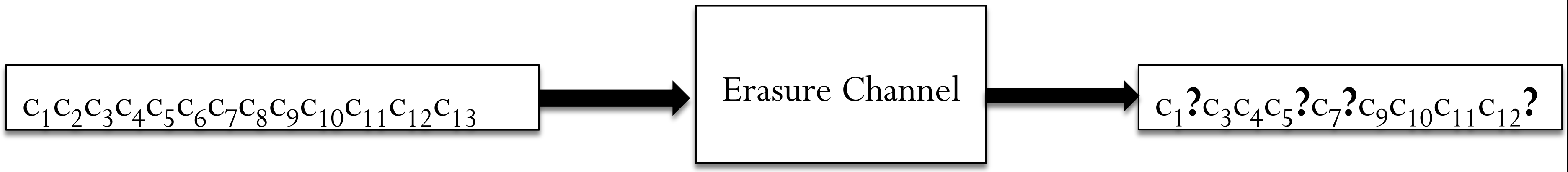

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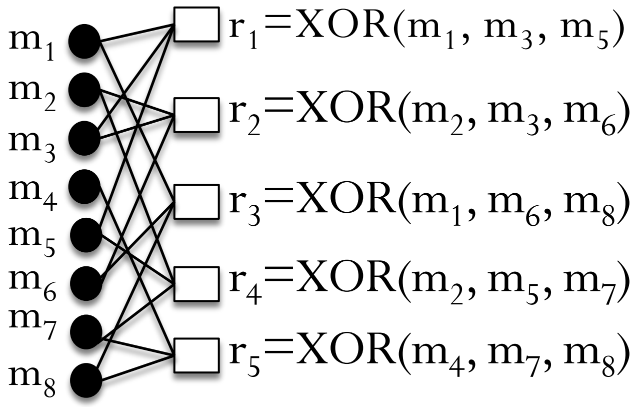

r_1=XOR(m_1, m_3, m_5)

r_2=XOR(m_2, m_3, m_6)

r_3=XOR(m_1, m_6, m_8)

r_4=XOR(m_2, m_5, m_7)

r_5=XOR(m_4, m_7, m_8)

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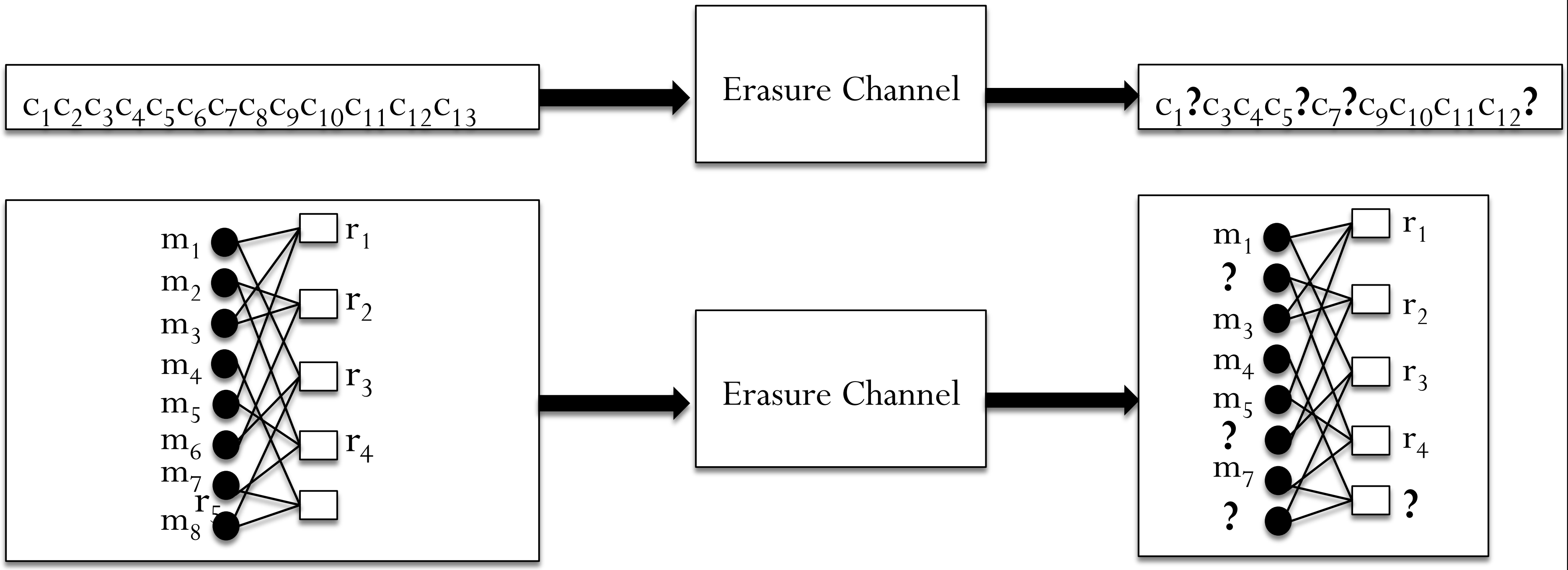

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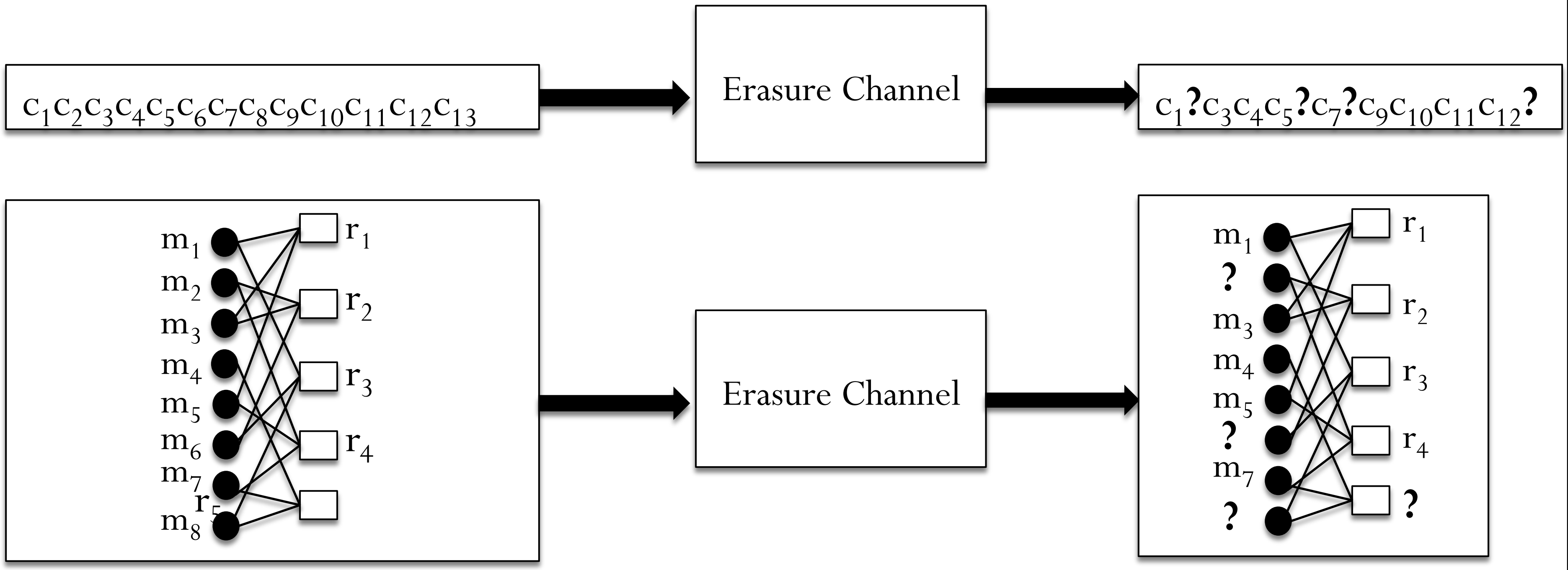

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m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8
r_1 r_2 r_3 r_4 r_5
Erasure Channel
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r_1 r_2 r_3 r_4 ?

Decoding Algorithm:

```
While there exists an un-erased a parity-check bit with exactly one un-erased neighbor:
        Recover the neighbor
```

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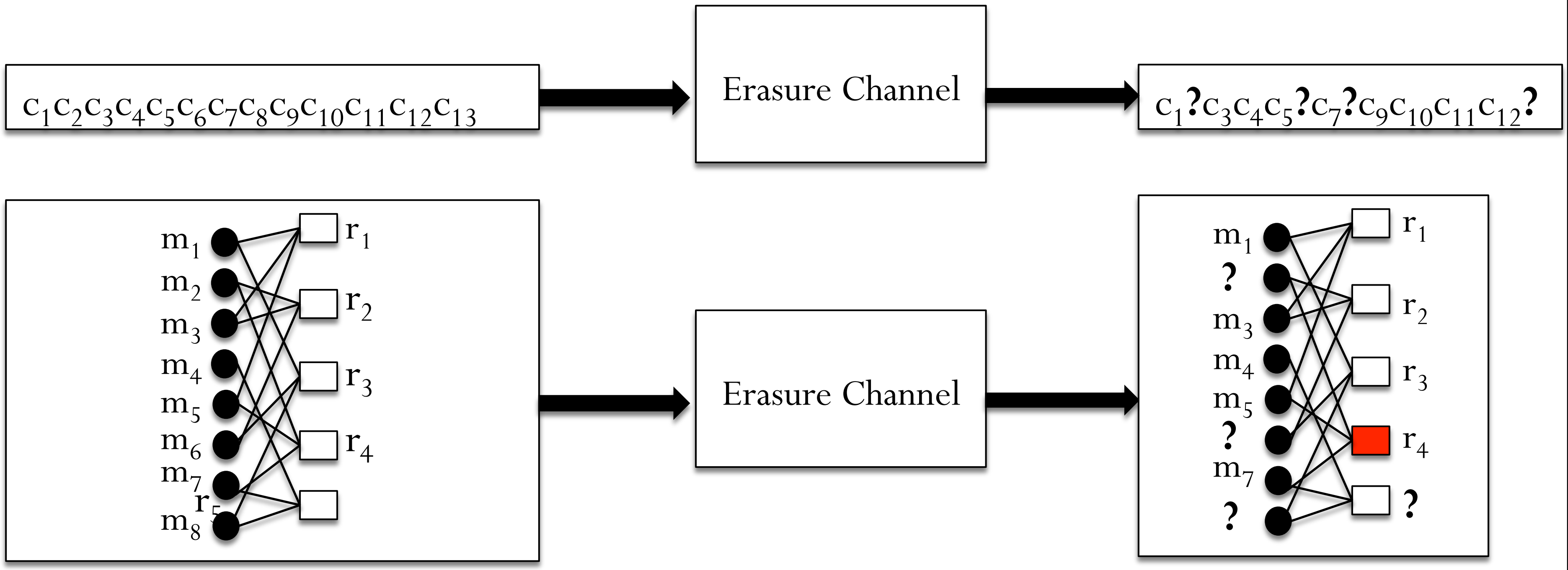

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Erasure Channel
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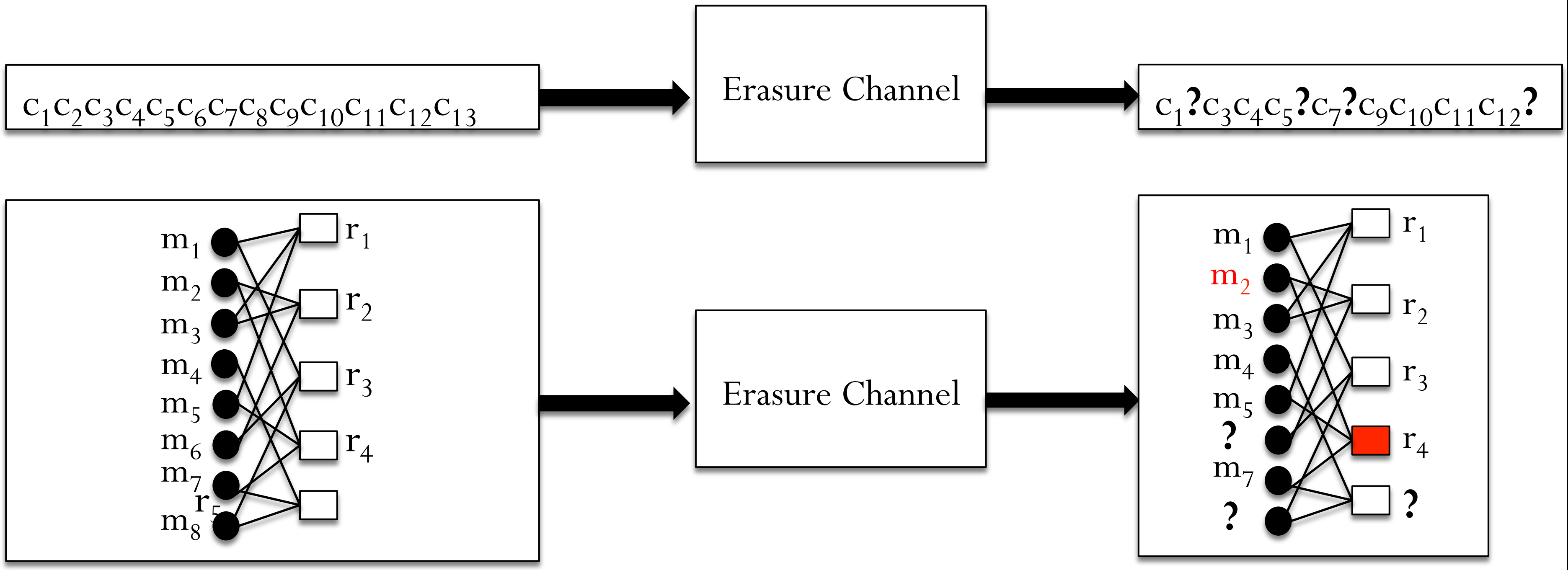

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m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8
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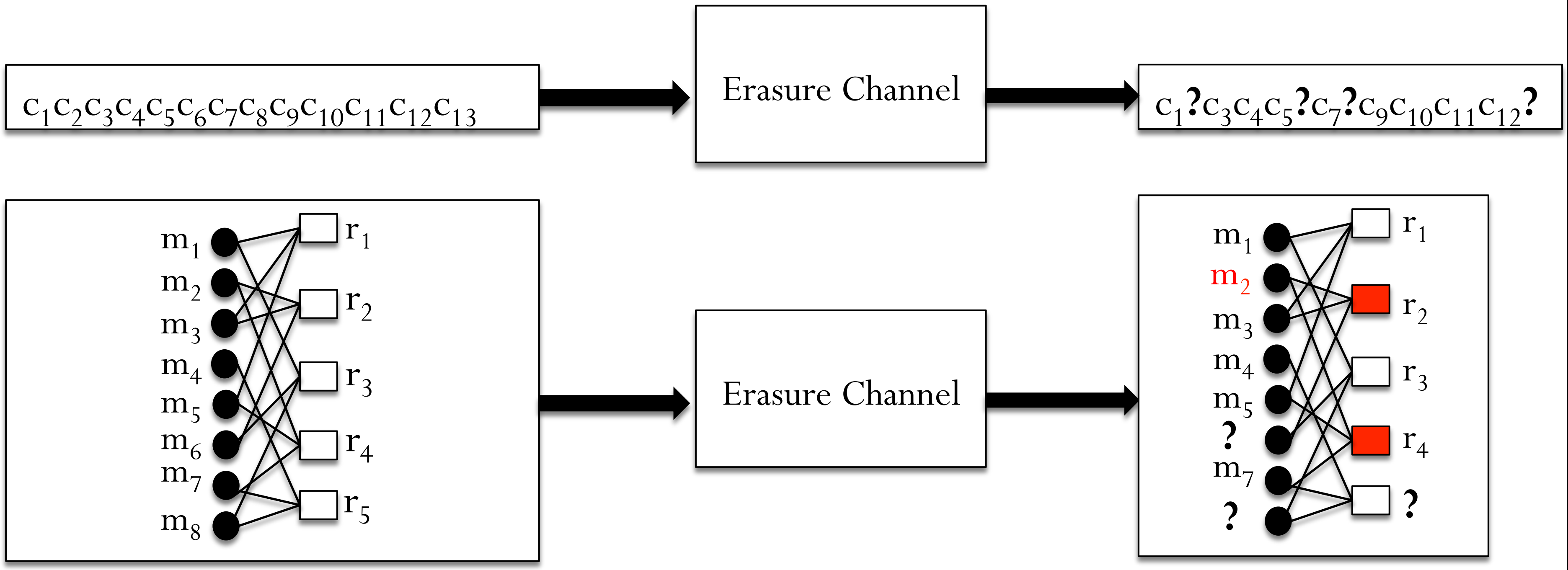
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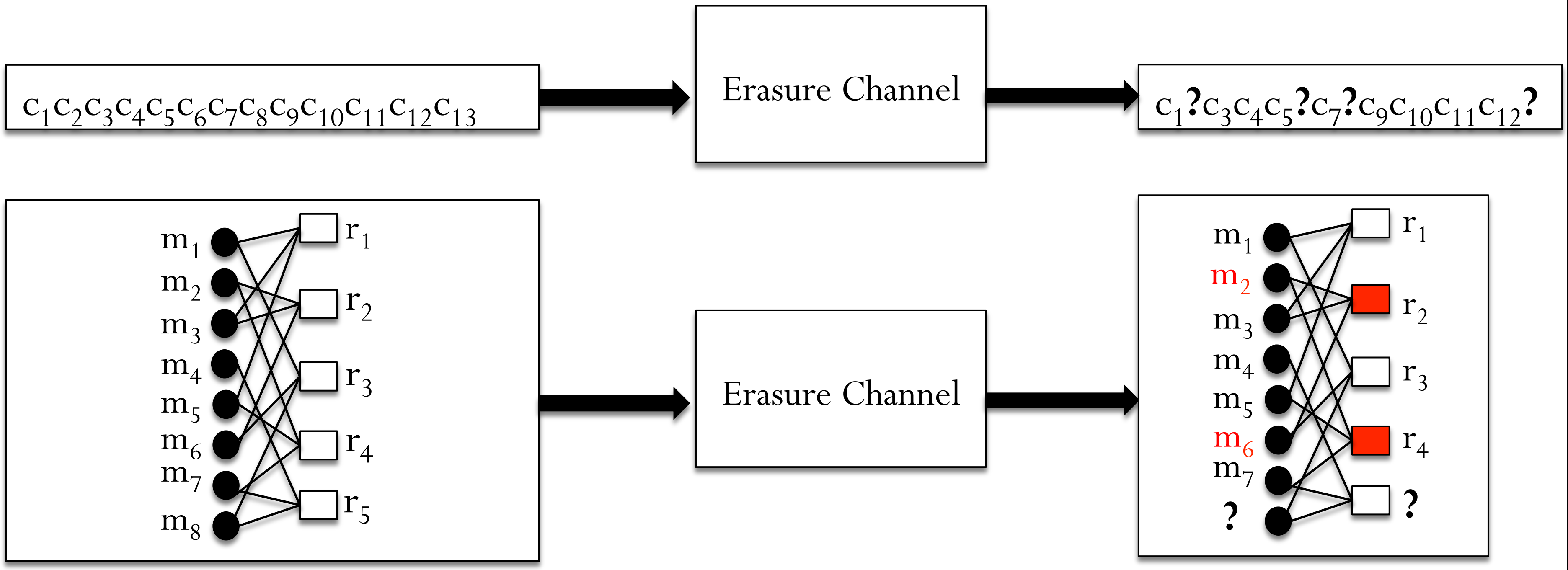
m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8
r_1 r_2 r_3 r_4 r_5
Erasure Channel
m_1 m_2 m_3 m_4 m_5 ? m_7 ?
r_1 r_2 r_3 r_4 ?

Decoding Algorithm:

While there exists an un-erased a parity-check bit with exactly one un-erased neighbor:

    Recover the neighbor

Example 1: LDPC Codes for Erasure Channels

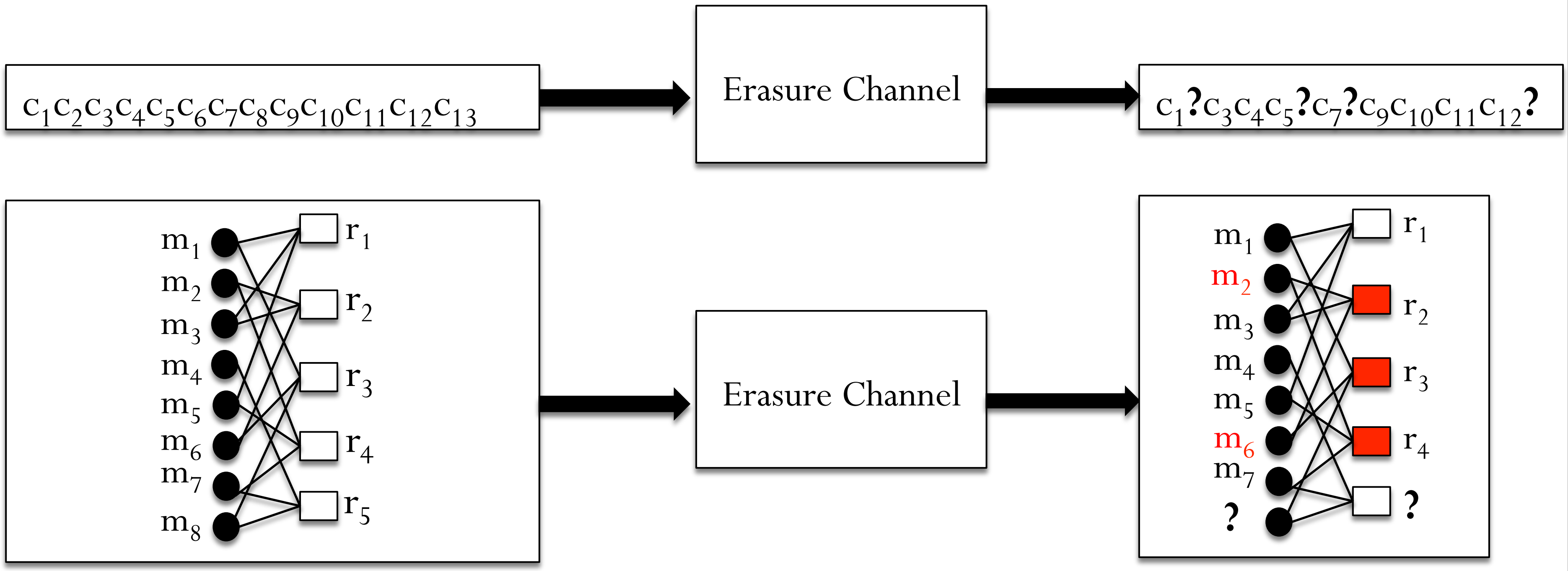

$c_1c_2c_3c_4c_5c_6c_7c_8c_9c_{10}c_{11}c_{12}c_{13}$
Erasure Channel
$c_1?c_3c_4c_5?c_7?c_9c_{10}c_{11}c_{12}?$
m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8
r_1 r_2 r_3 r_4 r_5
Erasure Channel
m_1 m_2 m_3 m_4 m_5 m_6 m_7 ?
r_1 r_2 r_3 r_4 ?

Decoding Algorithm:

While there exists an un-erased a parity-check bit with exactly one un-erased neighbor:

    Recover the neighbor

Example 1: LDPC Codes for Erasure Channels



$c_1c_2c_3c_4c_5c_6c_7c_8c_9c_{10}c_{11}c_{12}c_{13}$
Erasure Channel
$c_1?c_3c_4c_5?c_7?c_9c_{10}c_{11}c_{12}?$
m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8
r_1 r_2 r_3 r_4 r_5
Erasure Channel
m_1 m_2 m_3 m_4 m_5 m_6 m_7 ?
r_1 r_2 r_3 r_4 ?

Decoding Algorithm:

```
While there exists an un-erased a parity-check bit with exactly one un-erased neighbor:
    Recover the neighbor
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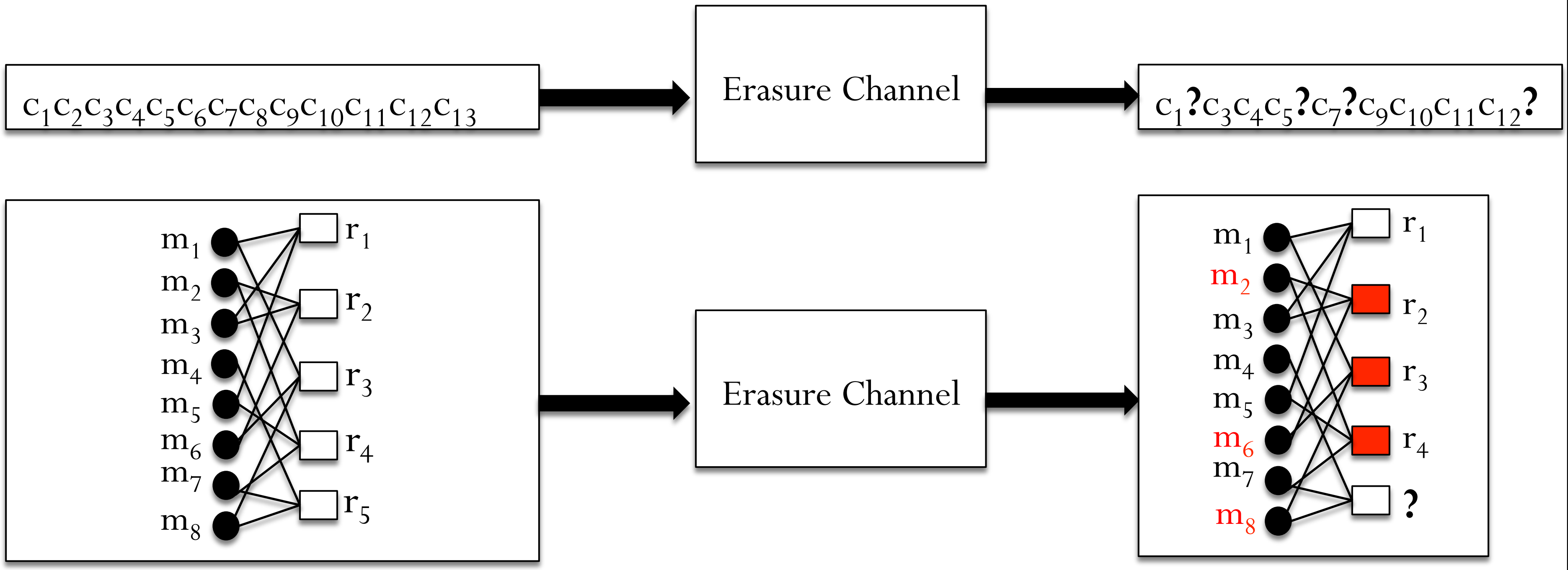

$c_1c_2c_3c_4c_5c_6c_7c_8c_9c_{10}c_{11}c_{12}c_{13}$
Erasure Channel
$c_1?c_3c_4c_5?c_7?c_9c_{10}c_{11}c_{12}?$
m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8
r_1 r_2 r_3 r_4 r_5
Erasure Channel
m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8
r_1 r_2 r_3 r_4 ?

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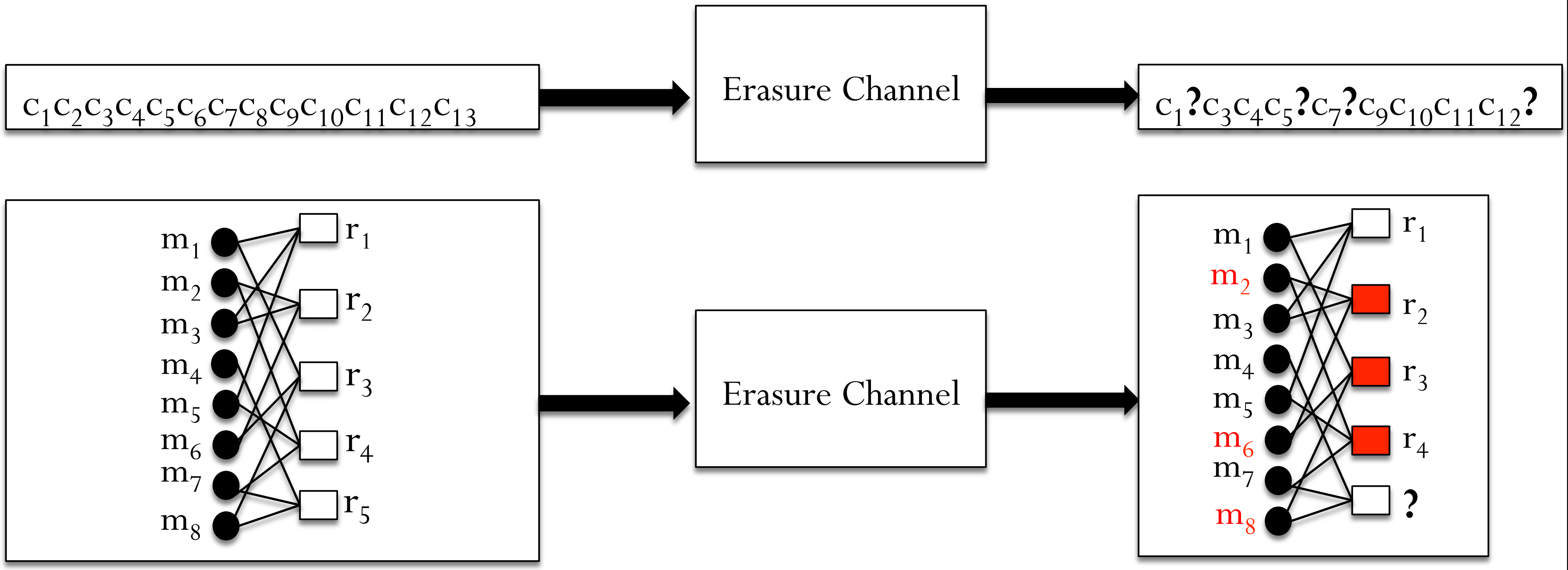
Example 1: LDPC Codes for Erasure Channels


$c_1c_2c_3c_4c_5c_6c_7c_8c_9c_{10}c_{11}c_{12}c_{13}$
Erasure Channel
$c_1?c_3c_4c_5?c_7?c_9c_{10}c_{11}c_{12}?$
m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8
r_1 r_2 r_3 r_4 r_5
Erasure Channel
m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8
r_1 r_2 r_3 r_4 ?

- Decoding $\Longleftrightarrow$ peeling to 2-core on the hypergraph G where:
  - Parity-check bits $\Longleftrightarrow$ vertices of G,
  - Erased message bits $\Longleftrightarrow$ hyperedges of G.
- Yields capacity-achieving codes with linear encoding and decoding time [Luby, Mitzenmacher, Shokrollahi, Spielman]