Planar motion

- A vehicle moves on a planar road.
- It can be rotated and translated.
- Coordinate system fixed to the car, axis Z parallel to the road.
- Two frames of the video yields a stereo problem.
- Vehicle is rotated, due to steering, around axis Y by angle β.
- Translation is in plane XZ: its direction represented by angle α.

$$\mathbf{t} = \begin{bmatrix} t_x \\ 0 \\ t_z \end{bmatrix} = \rho \begin{bmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

Planar motion: essential matrix

- Furthermore

$$\mathbf{t} = \rho \begin{bmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{bmatrix} \quad \rightarrow \quad [\mathbf{t}]_X = \rho \begin{bmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & -\cos\alpha \\ 0 & \cos\alpha & 0 \end{bmatrix}$$

- Then the essential matrix is as follows:

$$\mathbf{E} = [\mathbf{t}]_X \mathbf{R} \sim \begin{bmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & 0 & \sin\alpha\sin\beta - \cos\alpha\cos\beta \\ 0 & \cos\alpha & 0 \end{bmatrix}$$

Planar motion: essential and fundamental matrices

- After applying trigonometric equalities:

$$\mathbf{E} \sim \begin{bmatrix} 0 & -\sin\alpha & 0 \\ \sin(\alpha+\beta) & 0 & -\cos(\alpha+\beta) \\ 0 & \cos\alpha & 0 \end{bmatrix}$$

- If camera intrinsic matrices are the same for the images, and the common matrix is a so-called semi-calibrated one: $\mathbf{K} = diag(f, f, 1)$, then

$$\mathbf{F} = \mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1} \sim \begin{bmatrix} 0 & -\frac{\sin\alpha}{f^2} & 0 \\ \frac{\sin(\alpha+\beta)}{f^2} & 0 & -\frac{\cos(\alpha+\beta)}{f} \\ 0 & \frac{\cos\alpha}{f} & 0 \end{bmatrix}$$

Planar motion: estimation

- Only four out of nine elements in fundamental/essential matrices are nonzero.
  - Essental matrix can be estimated by two point correspondences.
  - Semi-calibrated camera: three correspondences.
- Robustification, e.g. by RANSAC, is fast
- Equation from one correspondence $\mathbf{p}_1 = [u_1, v_1]$, $\mathbf{p}_2 = [u_2, v_2]$ for two angles α and β (calibrated case):

$$\left\langle [v_1, -u_2 v_1, -v_2, v_2 u_1]^T, [\cos\alpha, \sin\alpha, \cos(\alpha+\beta), \sin(\alpha+\beta)]^T \right\rangle = 0$$

- For multiple correspondences, solution can be written as

$$\mathbf{A}_1 \mathbf{v}_1 + \mathbf{A}_2 \mathbf{v}_2 = 0$$

- where $\mathbf{v}_1 = [\cos\alpha, \sin\alpha]^T$ and $\mathbf{v}_2 = [\cos(\alpha+\beta), \sin(\alpha+\beta)]^T$

Planar motion: estimation

- Thus, $\mathbf{v}_1^T\mathbf{v}_1 = \mathbf{v}_2^T\mathbf{v}_2 = 1$.
- Furthermore,

$$\begin{aligned}
\mathbf{A}_1\mathbf{v}_1 + \mathbf{A}_2\mathbf{v}_2 &= 0 && (8)\\
\mathbf{A}_1\mathbf{v}_1 &= -\mathbf{A}_2\mathbf{v}_2 && (9)\\
\mathbf{v}_1 &= -\mathbf{A}_1^\dagger\mathbf{A}_2\mathbf{v}_2 && (10)\\
\mathbf{v}_1^T\mathbf{v}_1 &= \mathbf{v}_2^T\left(\mathbf{A}_1^\dagger\mathbf{A}_2\right)^T\left(\mathbf{A}_1^\dagger\mathbf{A}_2\right)\mathbf{v}_2 = 1 && (11)\\
\mathbf{v}_2^T\mathbf{B}\mathbf{v}_2 &= 1 && (12)
\end{aligned}$$

- If $\mathbf{B} = \left(\mathbf{A}_1^\dagger\mathbf{A}_2\right)^T\left(\mathbf{A}_1^\dagger\mathbf{A}_2\right)$
- Thus, $\mathbf{v}_2$ is given by the intersection of an ellipse and the unit-radius circle as $\mathbf{v}_2\mathbf{B}\mathbf{v}_2 = \mathbf{v}_2^T\mathbf{v}_2 = 1$.

Planar motion: estimation

- Solution is given by Singular Value Decomposition: $\mathbf{B} = \mathbf{U}^T \mathbf{S} \mathbf{U}$.
- Let $\mathbf{r} = [r_x \quad r_y]^T = \mathbf{U}\mathbf{v}_2$.

$$\begin{aligned}
\mathbf{v}_2^T \mathbf{B} \mathbf{v}_2 &= 1 && (13) \\
\mathbf{v}_2^T \mathbf{U}^T \mathbf{S} \mathbf{U} \mathbf{v}_2 &= 1 && (14) \\
\mathbf{r}_2^T \mathbf{S} \mathbf{r}_2 &= 1 && (15) \\
\mathbf{r}_2^T \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \mathbf{r}_2 &= 1 && (16)
\end{aligned}$$

- Therefore, $s_1 r_x^2 + s_2 r_y^2 = 1$
- and $r_x^2 + r_y^2 = 1$

$\rightarrow$ Linear system for r_x^2 and r_y^2. (Four candidate solutions, similarly to general stereo vision.)

- $\mathbf{v}_2 = \mathbf{U}^T \mathbf{r}$ and $\mathbf{v}_1 = -\mathbf{A}_1^\dagger \mathbf{A}_2 \mathbf{v}_2$ gives final solution.