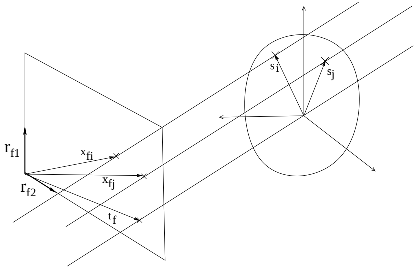
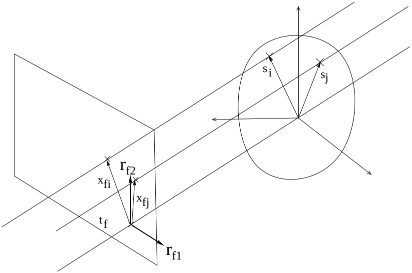
Orthogonal projection



Projection of points

$$\begin{bmatrix} u_{fp} \\ v_{fp} \end{bmatrix} = \begin{bmatrix} \mathbf{r_{f1}^T} \\ \mathbf{r_{f2}^T} \end{bmatrix} \mathbf{s_p} - \mathbf{t_f} \tag{1}$$

Orthogonal projection



Projection: origin is the center of gravity.

$$\begin{bmatrix} u_{fp} \\ v_{fp} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{\mathbf{f1}}^{\mathbf{T}} \\ \mathbf{r}_{\mathbf{f2}}^{\mathbf{T}} \end{bmatrix} \mathbf{s}_{\mathbf{p}} \qquad (2)$$

Outline

Tomasi-Kanade factorization

- Tracked (matched across multi-frames) coordinates are stacked in measurement matrix **W**.
- It can be factorized into two matrices:

$$\mathbf{W} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1P} \\ v_{11} & v_{12} & \cdots & v_{1P} \\ u_{21} & u_{22} & \cdots & u_{2P} \\ v_{21} & v_{22} & \cdots & v_{2P} \\ \vdots & \vdots & \ddots & \vdots \\ u_{F1} & u_{F2} & \cdots & u_{FP} \\ v_{F1} & v_{F2} & \cdots & v_{FP} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{11}^T \\ \mathbf{r}_{12}^T \\ \mathbf{r}_{21}^T \\ \mathbf{r}_{22}^T \\ \vdots \\ \mathbf{r}_{F1}^T \\ \mathbf{r}_{F2}^T \end{bmatrix} \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \dots & \mathbf{s}_P \end{bmatrix}$$

$$\mathbf{W} = \mathbf{MS}$$

Tomasi-Kanade factorization

- As $\mathbf{W} = \mathbf{MS}$, the rank of $\mathbf{W}$ cannot exceed 3 (noiseless-case).
  - Size of $\mathbf{M}$ is $2F \times 3$
  - Size of $\mathbf{S}$ is $3 \times P$
  - Lemma: After factorization, the rank cannot inrease
- Rank reduction of $\mathbf{W}$ by Singular Value Decomposition (SVD)
  - Largest 3 singular values/vectors are kept, other ones are set to zero.
  - $\mathbf{W} = \mathbf{USV^T} \rightarrow \mathbf{W} = \mathbf{U'S'V'^T}$

$$\mathbf{S} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & \dots \\ 0 & \sigma_2 & 0 & 0 & \dots \\ 0 & 0 & \sigma_3 & 0 & \dots \\ 0 & 0 & 0 & \sigma_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \rightarrow \mathbf{S}' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Ambiguity of factorization

- Infinite number of solutions exist:

$$\mathbf{W} = \mathbf{M}\mathbf{S} = \left(\mathbf{M}\mathbf{Q}^{-1}\right)(\mathbf{Q}\mathbf{S}),$$

- where $\mathbf{Q}$ is a 3×3 (affine) matrix.
- $\mathbf{M}_{\mathbf{aff}} = \mathbf{M}\mathbf{Q}^{-1}$: affine motion.
- $\mathbf{S}_{\mathbf{aff}} = \mathbf{Q}\mathbf{S}$ affine structure.
- Constraint to resolve ambiguity: motion vectors $\mathbf{r_i}$ are orthnormal.
  - Camera motion vectors is of length 1.0:

$$\mathbf{r}_{\mathbf{i1}}^{\mathbf{T}}\mathbf{r}_{\mathbf{i1}} = 1$$
$$\mathbf{r}_{\mathbf{i2}}^{\mathbf{T}}\mathbf{r}_{\mathbf{i2}} = 1$$

  - They are perpendicular to each other:

$$\mathbf{r}_{\mathbf{i1}}^{\mathbf{T}}\mathbf{r}_{\mathbf{i2}} = 0$$

Ambiguity removal

- Affine $\rightarrow$ real camera:

$$\begin{array}{rcl} & \mathbf{M_{aff}Q} & = \mathbf{M} \\ \mathbf{M_{aff}} = & \begin{bmatrix} \mathbf{m}_{11}^T\mathbf{Q} \\ \mathbf{m}_{12}^T\mathbf{Q} \\ \vdots \\ \mathbf{m}_{F1}^T\mathbf{Q} \\ \mathbf{m}_{F2}^T\mathbf{Q} \end{bmatrix} & = \begin{bmatrix} \mathbf{r}_{11}^T \\ \mathbf{r}_{12}^T \\ \vdots \\ \mathbf{r}_{F1}^T \\ \mathbf{r}_{F2}^T \end{bmatrix} \end{array}$$

- Constraints for camera vectors:

$$\begin{array}{rcl} \mathbf{r}_{i1}^T\mathbf{r}_{i1} = 1 & \rightarrow & \mathbf{m}_{i1}^T\mathbf{Q}\mathbf{Q}^T\mathbf{m}_{i1} = 1 \\ \mathbf{r}_{i2}^T\mathbf{r}_{i2} = 1 & \rightarrow & \mathbf{m}_{i2}^T\mathbf{Q}\mathbf{Q}^T\mathbf{m}_{i2} = 1 \\ \mathbf{r}_{i1}^T\mathbf{r}_{i2} = 0 & \rightarrow & \mathbf{m}_{i1}^T\mathbf{Q}\mathbf{Q}^T\mathbf{m}_{i2} = 0 \end{array}$$

Estimation of matrix **Q**

- Let us introduce the following notation:

$$\mathbf{L} = \mathbf{Q}\mathbf{Q}^T = \begin{bmatrix} l_1 & l_2 & l_3 \\ l_2 & l_4 & l_5 \\ l_3 & l_5 & l_6 \end{bmatrix}$$

- Important fact: matrix $\mathbf{QQ^T}$ is symmetric
- Constraints can be written in linear form: $\mathbf{A_i l} = \mathbf{b_i}$

$$\mathbf{A_i} = \begin{bmatrix} {m_{i1,x}}^2 & 2m_{i1,x}m_{i1,y} & 2m_{i1,x}m_{i1,z} & {m_{i1,y}}^2 & 2m_{i1,y}m_{i1,z} & {m_{i1,z}}^2 \\ {m_{i2,x}}^2 & 2m_{i2,x}m_{i2,y} & 2m_{i2,x}m_{i2,z} & {m_{i2,y}}^2 & 2m_{i2,y}m_{i2,z} & {m_{i2,z}}^2 \\ m_{i1,x}m_{i2,x} & e_1 & e_2 & m_{i1,y}m_{i2,y} & m_{i1,y}m_{i2,z} + m_{i2,y}m_{i1,z} & m_{i1,z}m_{i2,z} \end{bmatrix}$$

$$\mathbf{l} = [l_1, l_2, l_3, l_4, l_5, l_6]^T \quad \mathbf{b_i} = [1, 1, 0]^T$$

- where $m_{jk,x}$, $m_{jk,y}$ and $m_{jk,z}$ are the coordinates of vector $\mathbf{m_{jk}}$,
- and $e_1 = m_{i1,x}m_{i2,y} + m_{i2,x}m_{i1,y}$, $e_2 = m_{i1,x}m_{i2,z} + m_{i2,x}m_{i2,z}$.

Computation of matrix **Q**

- Constraints can be written in linear form: $\mathbf{Al} = \mathbf{b}$,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_1^T} & \mathbf{A_2^T} & \dots & \mathbf{A_F^T} \end{bmatrix}^T$$

$$\mathbf{b} = [1, 1, 0, 1, 1, 0, \dots, 1, 1, 0]^T$$

- Solution by over-determined inhomogeneous linear system of equations
- Matrix **Q** can be retrieved from **L** by SVD:

$$\mathbf{L} \overset{(SVD)}{=} \mathbf{USU^T}$$

$$\mathbf{Q} = \mathbf{U}\sqrt{\mathbf{S}}$$

Weak-perspective projection

- Modified constraints:
  - motion vectors are perpendicular to each other:

$$\mathbf{r}_{i1}^T \mathbf{r}_{i2} = 0$$

  - Length of vectors are not unit, but equal:

$$\mathbf{r}_{i1}^T \mathbf{r}_{i1} = \mathbf{r}_{i2}^T \mathbf{r}_{i2}$$

- Equations for affine ambuguity, represented by matrix **Q** as follows:

$$\begin{aligned} \mathbf{m}_{i1}^T \mathbf{Q}\mathbf{Q}^T \mathbf{m}_{i1} - \mathbf{m}_{i2}^T \mathbf{Q}\mathbf{Q}^T \mathbf{m}_{i2} &= 0 \\ \mathbf{m}_{i1}^T \mathbf{Q}\mathbf{Q}^T \mathbf{m}_{i2} &= 0 \end{aligned}$$

- Linear, homogeneous system of equations obtained.

Summary of Tomasi-Kanade factorization

1. Tracked points are stacked in measurement matrix $\mathbf{W}$.
2. Origin is moved to the center of gravity, translated coordinates are stacked in matrix $\tilde{\mathbf{W}}$.
3. SVD computed for $\tilde{\mathbf{W}}$: $\tilde{\mathbf{W}} = \mathbf{U}\mathbf{S}\mathbf{V}^T$.
4. Singular elements are replaced by zero, except the first three values in $\mathbf{S}$: $\mathbf{S} \rightarrow \mathbf{S}'$.
5. Affine factorization: $\mathbf{M}_{\mathbf{aff}} = \mathbf{U}\sqrt{\mathbf{S}'}$ and $\mathbf{S}_{\mathbf{aff}} = \sqrt{\mathbf{S}'}\mathbf{V}^{\mathbf{T}}$.
6. Calculation of matrix $\mathbf{Q}$ by metric constraints.
7. Metric factorization: $\mathbf{M} = \mathbf{M}_{\mathbf{aff}}\mathbf{Q}$ and $\mathbf{S} = \mathbf{Q}^{-\mathbf{1}}\mathbf{S}_{\mathbf{aff}}$.

Outline

Multi-view perspective reconstruction

- Three-view geometry
  - Extension of epipolar geometry
  - Relationships can be written for 3D points and lines
  - Trifocal tensor introduced as the extension of the fundamental matrix
  - It has small practical impact
- Perspective Tomasi-Kanade factorization
  - Problem is a perspective auto-calibration
  - Difficulty: projective depths are different for all point/frames
  - Only iterative solutions exist
  - Very complicated
- Viable solution: Concatenation of stereo reconstructions