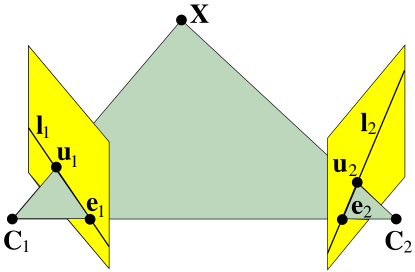
Calibrated cameras: essential matrix 1/2



- Calibration matrix **K** is known, rotation **R** and translation **t** between coordinate systems are unknown.
- Lines $\mathbf{C}_1\mathbf{u}_1$, $\mathbf{C}_2\mathbf{u}_2$, $\mathbf{C}_1\mathbf{C}_2$ lay within the same plane:

$$\mathbf{C}_2\mathbf{u}_2 \cdot [\mathbf{C}_1\mathbf{C}_2 \times \mathbf{C}_1\mathbf{u}_1] = 0$$

Calibrated cameras: essential matrix 2/2

- In the second camera system, the following equation holds if homogeneous coordinates are used:

$$\mathbf{u}_2 \cdot [\mathbf{t} \times \mathbf{R}\mathbf{u}_1] = 0$$

- Using the **essential matrix E** (Longuet-Higgins, 1981):

$$\mathbf{u}_2^T \mathbf{E} \mathbf{u}_1 = 0, \tag{1}$$

where essential matrix is defined as

$$\mathbf{E} \doteq [\mathbf{t}]_\times R \tag{2}$$

  - $[\mathbf{a}]_\times$ is the cross-product matrix:

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} \doteq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Properties of an essential matrix

- The equation $\mathbf{u}_2^T \mathbf{E} \mathbf{u}_1 = 0$ is valid if the 2D coorinates are normalized by $\mathbf{K}$.
  - Normalized camera matrix: $\mathbf{P} \longrightarrow \mathbf{K}^{-1}\mathbf{P} = [\mathbf{R}| - \mathbf{t}]$
  - $\rightarrow$ Normalized coordinates: $\mathbf{u} \longrightarrow \mathbf{K}^{-1}\mathbf{u}$
- Matrix $\mathbf{E} = [\mathbf{t}]_\times R$ has 5 degree of freedom (DoF).
  - $3(\mathbf{R}) + 3(\mathbf{t}) - 1(\lambda)$
  - λ: (scalar unambigity)
- Rank of essential matrix is 2.
  - $\mathbf{E}$ has two equal, non-zero singular value.
- Matrix $\mathbf{E}$ can be decomposed to translation and rotation by SVD.
  - translation is up to an unknown scale
  - sign of $\mathbf{t}$ is also ambiguous

Uncalibrated case: fundamental matrix

- Longuet-Higgins formula in case of **uncalibrated** cameras

$$\mathbf{u}_2^T F \mathbf{u}_1 = 0, \tag{3}$$

where the **fundamental matrix** is defined as

$$\mathbf{F} \doteq \mathbf{K}_2^{-T} \mathbf{E} \mathbf{K}_1^{-1} \tag{4}$$

  - $\mathbf{u}_1$ and $\mathbf{u}_2$ are unnormalized coordinates.

- Matrix **F** has 7 DoF.
- Rank of **F** is 2
  - Epipolar lines intersect each other in the same points
  - $\det \mathbf{F} = 0 \longrightarrow$ **F** cannot be inverted, it is non-singular.
- Epipolar lines: $\mathbf{l}_1 = \mathbf{F}^T \mathbf{u}_2$, $\mathbf{l}_2 = \mathbf{F} \mathbf{u}_1$
- Epipoles: $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$, $\mathbf{F}^T \mathbf{e}_2 = \mathbf{0}^T$

Overview

Estimation of fundamental matrix

- We are given N point correspondences:
  $\{\mathbf{u}_{1i} \leftrightarrow \mathbf{u}_{2i}\}, i = 1, 2, \ldots, N$
  - Degree of freedom for $\mathbf{F}$ is 7 : $\longrightarrow N \geq 7$ required
  - Usually, $N \geq 8$. (Eight-point method)
  - If correspondences are contaminated $\longrightarrow$ robust estimation needed
  - In case of outliers: $N \gg 7$
- Basic equation: $\mathbf{u}_{2i}^T \mathbf{F} \mathbf{u}_{1i} = 0$
- Goal is to find the singular matrix closest to $\mathbf{F}$.

Eight-point method

Input: N point correspondences $\{\mathbf{u}_{1i} \leftrightarrow \mathbf{u}_{2i}\}, N \geq 8$
Output: fundamental matrix $\mathbf{F}$

Algoritmus: *Normalized 8-point method*

1. Data-normalization is separately carried out for the two point set:
   - translation
   - scale
2. Estimating $\hat{\mathbf{F}}'$ for normalized data
   (a) Linear solution by SVD $\longrightarrow \hat{\mathbf{F}}'$
   (b) Then singularity constraint $\det \hat{\mathbf{F}}' = 0$ is forced $\longrightarrow \hat{\mathbf{F}}'$
3. Denormalization
   - $\hat{\mathbf{F}}' \longrightarrow \mathbf{F}$

Data normalization and denormalization

- Goal of **data normalization**: numerical stability
  - **Obligatory step**: non-normalized method is not reliable.
  - Components of coefficient matrix should be in the same order of magnitude.
- Two point-sets are normalized by affine transformations $\mathbf{T}_1$ and $\mathbf{T}_2$.
  - Offset: origin is moved to the center(s) of gravity
  - Scale: average of point distances are scaled to be $\sqrt{2}$.
- **Denormalization:** correction by affine tranformations:

$$\hat{\mathbf{F}} = \mathbf{T}_2^T \hat{\mathbf{F}}' \mathbf{T}_1 \tag{5}$$

Homogeneous linear system to estimate F

- For each point correspondence: $\mathbf{u}_2^T \mathbf{F} \mathbf{u}_1 = 0$, where $\mathbf{u}_k = [u_k, v_k, 1]^T, k = 1, 2$

$\rightarrow$ For element of the fundamental matrix, the following equation is valid:

$$u_2 u_1 f_{11} + u_2 v_1 f_{12} + u_2 f_{13} + v_2 u_1 f_{21} + v_2 v_1 f_{22} + v_2 f_{23} + u_1 f_{31} + v_1 f_{32} + f_{33} = 0$$

- If notation $\mathbf{f} = [f_{11}, f_{12}, \ldots, f_{33}]^T$ is introduced, the equation can be written as a dot product:

$$[u_2 u_1, u_2 v_1, u_2, v_2 u_1, v_2 v_1, v_2, u_1, v_1, 1]\mathbf{f} = 0$$

- For all i: $\{\mathbf{u}_{1i} \leftrightarrow \mathbf{u}_{2i}\}$

$$A\mathbf{f} \doteq \begin{bmatrix} u_{21}u_{11} & u_{21}v_{11} & u_{21} & v_{21}u_{11} & v_{21}v_{11} & v_{21} & u_{11} & v_{11} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{2N}u_{1N} & u_{2N}v_{1N} & u_{2N} & v_{2N}u_{1N} & v_{2N}v_{1N} & v_{2N} & u_{1N} & v_{1N} & 1 \end{bmatrix} \mathbf{f} = \mathbf{0}$$

Sulution as homogeneous linear system of equations

- Estimation is similar to that of **homography**.
- Trivial solution $\mathbf{f} = \mathbf{0}$ has to be excluded.
  - vector $\mathbf{f}$ can be computed up to a scale
  - $\rightarrow$ vector norm is fixed as $\|\mathbf{f}\| = 1$
- If $\operatorname{rank} \mathbf{A} \leq 8$
  - $\operatorname{rank} \mathbf{A} = 8 \longrightarrow$ exact solution: nullvector
  - $\operatorname{rank} \mathbf{A} < 8 \longrightarrow$ solution is linear combination of nullvectors
- For noisy correspondences, $\operatorname{rank} \mathbf{A} = 9$.
  - optimal solution for algebraic error $\|\mathbf{Af}\|$
  - $\|\mathbf{f}\| = 1 \longrightarrow$ minimization of $\|\mathbf{Af}\| / \|\mathbf{f}\|$
  - $\rightarrow$ optimal solution is the eigenvector of $\mathbf{A}^T\mathbf{A}$ corresponding to the smallest eigenvalue
- Solution can also be obtained from SVD of $\mathbf{A}$:
  - $\mathbf{A} = \mathbf{UDV}^T \longrightarrow$ last column (vector) of $\mathbf{V}$.

Singular constraint

- If $\det \mathbf{F} \neq 0$
  - epipolar lines do not intersect each other in epipole.
  - $\rightarrow$ less accurate epipolar geometry $\longrightarrow$ less accurate reconstruction
- Solution of homogeneous linear system does not guarantee singularity: $\det \widehat{\mathbf{F}} \neq 0$.
- Task is to find matrix $\widehat{\mathbf{F}}'$, for which
  - Frobenius norm $\|\widehat{\mathbf{F}} - \widehat{\mathbf{F}}'\|$ is minimal, and
  - $\det \widehat{F}' = 0$
- SVD of $\mathbf{A}$: $\mathbf{A} = \mathbf{UDV}^{\mathrm{T}}$
  - $\mathbf{D} = \mathrm{diag}(\delta_1, \delta_2, \delta_3)$ is the diagonal matrix containing singular values, and $\delta_1 \geq \delta_2 \geq \delta_3$
  - The estimation for closest matrix, fulfilling singularity constraint:

$$\widehat{F}' = \mathbf{U}\,\mathrm{diag}(\delta_1, \delta_2, 0)\mathbf{V}^{\mathrm{T}} \tag{6}$$

Epipoles from fudamental matrix $\mathbf{F}$

- The epipoles are the null-vectors of $\mathbf{F}$ and $\mathbf{F}^T$: $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$, and $\mathbf{F}^T\mathbf{e}_2 = \mathbf{0}$.
- Nullvector can be calculated by e.g. SVD.
- Singularity constraint guarantees that $\mathbf{F}$ has a null-vector
- Singular Value Decomposition: $\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^T$, and then
  - $\mathbf{e}_1$: last column of $\mathbf{V}$.
  - $\mathbf{e}_2$: last column of $\mathbf{U}$.

Limits of eight-point method

- Similar to homography/projective matrix estimation
  - Significant difference: singularity constraint introduces

  $\rightarrow$ Similar benefits/weak points to homography/proj. matrix estimation
- Method is not robust
  - RANSAC-like robustification can be applied.
- There are another solution
  - Seven-point method: determinant constraint is forced to linear combination of null-spaces.

Non-linear methods to estimate F

- Algebraic error
  - It yields initial value(s) for numerical optimization.
- Geometric error
  - line-point distance

$$\epsilon = \frac{\mathbf{x'}^T \mathbf{F} \mathbf{x}}{\left| \mathbf{F} \mathbf{x} \right|_{1:2}}$$

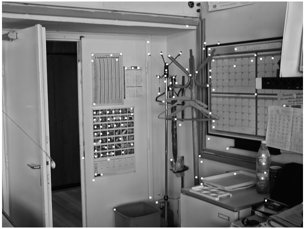
  - Symmetric version

$$\epsilon = \frac{\mathbf{x'}^T \mathbf{F} \mathbf{x}}{\left| \mathbf{F} \mathbf{x} \right|_{1:2}} + \frac{\mathbf{x}^T \mathbf{F}^T \mathbf{x'}}{\left| \mathbf{F}^T \mathbf{x'} \right|_{1:2}}$$

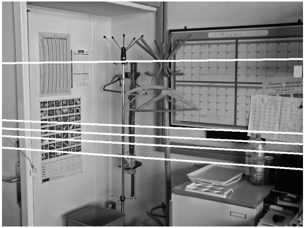
  - where operator $(\mathbf{x})_{1:2}$ denotes the first two coordinates of vector $\mathbf{x}$.
  - Geometric error minimized by numerical techniques.

Estimation of epipolar geometry: 1st example

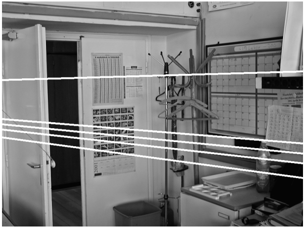
KLT feature points #1



KLT feature points #2



epipolar lines #1



epipolar lines #2

Estimation of epipolar geometry: 2nd example