Introduction to Topological Manifold: Chap 2

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Section 3.1

exercise 1

exercise 3.61:

Proof. Simply observe that if q is continuous and open:

$$\left(U \text{ is open in Y} \Leftrightarrow q^{-1}(U) \text{ is open in Y}\right) \Leftrightarrow \left(\text{it takes saturated open subsets to open subsets}\right).$$

□

exercise 3.62(d):

Proof. Observe: (Note: the first equivalence is from properties of subspace topology.)

$$V \text{ is open in q(U)} \Leftrightarrow V \text{ is open in Y} \Leftrightarrow q^{-1}(V) \text{ is open in X..}$$

□

exercise 3.62(e):

Proof. Just prove that it takes saturated open subsets to open subsets. □

exercise 2

Question: more generally, when the spaces X_ι have the trivial topology for all but finitely many indices, then product topology is equal to box topology?:

Proof. Consider basis of box topology can be generated by product topology. □

Problem 1

Proof. Consider the C_2 and hausdorff property is inhereted by subspace toplogy. We only need to prove that it's locally euclidean of dimension $n-1$, which is obvious since $IntM$ and ∂M are disjoint. □

Problem 2

Proof. Since the closure of A respect to $B \subset X$ is equal to $\bar{A} \bigcap B$, then

$$\begin{aligned} \text{A is dense in X} \Leftrightarrow \bar{A} = X \Leftrightarrow \forall A \subset B \subset X : \text{A is dense in B and B is dense in X} \\ \Leftrightarrow \forall A \subset B \subset X : B = \bar{A} \bigcap B \text{ and } X = \bar{B} \bigcap X \ . \end{aligned}$$

□

Problem 3

Proof. Consider the space $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ with the topology induced by the inclusion $X \subseteq \mathbb{R}$. Pick any non-continuous function $g : X \to \mathbb{R}$, consider the covering

$$\{A_i : i \in \mathbb{N}_0\} \text{ with } A_0 = \{0\} \text{ and } A_n = \{\frac{1}{n}\} \text{ for each } n \in \mathbb{N},$$

this is a countable closed covering of X— and define the functions $f_i = g|_{A_i} : A_i \to \mathbb{R}$, for each $i \in \mathbb{N}_0$. All the f_i are continuous, and there does not exists any continuous function $f : X \to \mathbb{R}$ such that $f|_{A_i} = f_i$ for each $i \in \mathbb{N}_0$. □

Problem 4

Proof. For the unit ball in $\mathbb{R}^n$, consider the map

$$\pi \circ \sigma^{-1} : \mathbb{R}^n \to \mathbb{R}^n :$$

where σ is the stereographic projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ and π is a projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ that **omits some coordinate other than the last**.

Consicely, it is enough to solve this for the closed unit ball $\bar{\mathbb{B}}^n$ in $\mathbb{R}^{n+1}$, since any other closed ball $\bar{B}_r(p)$ in $\mathbb{R}^n$ is homeomorphic to $\bar{\mathbb{B}}^n$ by composition of translation $T : \bar{B}_r(p) \to \bar{B}_r(0)$, defined as $x \mapsto x - p$ together with dilation $D : \bar{B}_r(0) \to \bar{\mathbb{B}}^n$, defined as $x \mapsto \frac{x}{r}$.

As a subspace of $\mathbb{R}^n$, $\bar{\mathbb{B}}^n$ is a second countable Hausdorff space. For any point $p \in \mathbb{B}^n$, the identity map on $\mathbb{B}^n$ serve as the homeomorphism. So we only need to construct homeomorphisms between neighbourhood of points on $\partial\bar{\mathbb{B}}^n = \mathbb{S}^{n-1}$ with open subsets in $\mathbb{H}^n$. To do this we need to consider $\bar{\mathbb{B}}^n$ as a subspace of $\mathbb{R}^{n+1}$.

Consider the stereographic projection from the south pole $\sigma : \mathbb{S}^n \setminus \{S\} \to \mathbb{R}^n$, which is a homeomorphism, defined as

$$\sigma(x_1, \ldots, x_{n+1}) = \frac{(x_1, \ldots, x_n)}{1 + x_{n+1}}.$$

For $i = 1, \ldots, n$, define

$$U_i^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0\}, \quad U_i^- = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i < 0\}$$

be $2n$-many open subsets of $\mathbb{R}^n$, and also for $i = 1, \ldots, n$

$$\widetilde{U}_i^+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i > 0\}, \quad \widetilde{U}_i^- = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i < 0\}$$

are $2n$-many open subsets in $\mathbb{R}^{n+1}$.

Observe that

$$\sigma^{-1}(U_i^\pm) = \mathbb{S}^n \cap \widetilde{U}_i^\pm$$

for each $i = 1, \ldots, n$. That is σ^{-1} map U_i^+ to the open hemisphere of $\mathbb{S}^n$ where $x_i > 0$, and same for U_i^-. Since these hemispheres $\sigma^{-1}(U_i^\pm)$ homeomorphic to open unit ball $\mathbb{B}^n$ via projection map

$$\pi_i : (x_1, \ldots, x_i, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$$

, then by restricting the composition map $\pi_i \circ \sigma^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ to $U_i^\pm \cap \bar{\mathbb{B}}^n$, we obtain the desired homeomorphisms

$$\varphi := \pi_i \circ (\sigma^{-1})|_{U_i^\pm \cap \bar{\mathbb{B}}^n} : U_i^\pm \cap \bar{\mathbb{B}}^n \to \mathbb{H}^n,$$

with domains cover $\partial\bar{\mathbb{B}}^n = \mathbb{S}^{n-1}$. By construction, any $p \in \mathbb{S}^{n-1}$ must contained in one such neighbourhoods, with $\varphi(p) = 0 \in \partial\mathbb{H}^n$ and $\varphi(U_i^\pm \cap \bar{\mathbb{B}}^n)$ is an open half unit ball in $\mathbb{H}^n$. Therefore, $\bar{\mathbb{B}}^n$ is an n-manifold with boundary with manifold boundary is equal to its topological boundary $\mathbb{S}^{n-1}$.

Problem 4 continued on next page...

Note that by similar way we can show that the complement of any open ball is an n-manifold with boundary, with its topological boundary as the manifold boundary. Only this time we use stereographic projection form the north pole. □

Problem 5

Proof. Let $D = \{0\}$ be the one-point discrete space. Let $f : \mathbb{R} \to \mathbb{R} : x \mapsto x$ be the identity map, and let $g : \mathbb{R} \to D : x \mapsto 0$. Both f and g are easily seen to be closed, but $f \times g : \mathbb{R}^2 \to \mathbb{R} \times D$ is not: it maps the graph of $xy = 1$, which is a closed set in $\mathbb{R}^2$, to

$$\Big(\mathbb{R} \setminus \{0\}\Big) \times \{0\} \ ,$$

which is not closed in $\mathbb{R} \times D$: $(0, 0)$ is in its closure. □

Problem 6

Proof. We prove two directions:

Suppose first that Δ is closed in $X \times X$. To show that X is Hausdorff, you must show that if x and y are any two points of X, then there are open sets U and V in X such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Look at the point $p = (x, y) \in X \times X$. Because $x \neq y$, $p \notin \Delta$. This means that p is in the open set $(X \times X) \setminus \Delta$. Thus, there must be a basic open set B in the product topology such that $p \in B \subseteq (X \times X) \setminus \Delta$. Basic open sets in the product topology are open boxes, i.e., sets of the form $U \times V$, where U and V are open in X, so let $B = U \times V$ for such $U, V \subseteq X$. Then the following is clear.

Now suppose that X is Hausdorff. To show that Δ is closed in $X \times X$, we need only show that $(X \times X) \setminus \Delta$ is open. To do this, just take any point $p \in (X \times X) \setminus \Delta$ and show that it has an open neighborhood disjoint from D. First, $p = \langle x, y \rangle$ for some $x, y \in X$, and since $p \notin D$, $x \neq y$. Now use disjoint open neighborhoods of x and y to build a basic open box around p that is disjoint from Δ. □

Problem 7

Proof. First construct $Id_{\mathbb{R}^2}$. Then prove that it sends all basis subsets in $\mathbb{R}^2$ to all basis subsets in $\mathbb{R}_d \times \mathbb{R}$. □

Problem 8

Proof. Just prove it by definition as below:

$$f^{-1}(\bigcup_{i=1}^{\infty}\{(x_1, x_2, \ldots) : x_i > 1 + \frac{1}{i}\}) = \bigcup_{i=1}^{\infty} f^{-1}(\{(x_1, x_2, \ldots) : x_i > 1 + \frac{1}{i}\}) = [1, \infty).$$

□

Problem 9

Proof. Obviously we have:

$$\text{Metrizable} \implies \text{First Countable} \implies \text{There is a sequence of elements of X converging to z.}$$

□

Problem 10

Proof. It's clear from property of subspace topology that

$$f : \bigsqcup_\alpha A_\alpha \to A \text{ continuous} \implies f|_{A_\alpha} : A_\alpha \to A \text{ continuous}$$

Then the inversed direction is clear, from prop 2.19 (**local property of continuous map**). □

Problem 12

Proof. Let (c) be an example: Just prove it's the biggest topology:

$$U \text{ open in } \bigsqcup_\alpha A_\alpha \implies \forall\alpha\ j_\alpha^{-1}(U) = U \bigcap A_\alpha \text{ is open in } A_\alpha.$$

$$\implies U \text{ open in } \bigsqcup_\alpha A_\alpha \textbf{ with respect to the disjoint topology.}$$

□

Problem 13

Proof. Let (a) be an example: If the left inverse exist, denoted by f^{-1}, then

$$f^{-1} \circ f = Id \implies f \text{ is injective and } f^{-1}|_{f(X)} \text{ plays the ROLE of inverse function.}$$

□

Problem 14

Proof. Since $P^n = (\mathbb{R}^{n+1} \backslash \{0\}) / \sim$, then it's hausdorff by Cor 3.58. Let the quotient map denote π. Consider the subsets $U_i \subset R^n$ where $x_i = 1$, then $\left((\pi|_{U_i})^{-1}, U_i \right)$ is a coordinate system!
Then it follows from Prop 3.56 that it's second countable. □

Problem 15

Mimic what we did in Problem 3-14.

Problem 16

It's clear that the quotient map is an open map. Moreover, $\sim$ is not closed in $(\mathbb{R} \times \{0\}) \times (\mathbb{R} \times \{1\})$! By applying corollary 3.58, it's not hausdorff!

Problem 17

A counter example.

Problem 18

(a) Express both spaces as quotients of a disjoint union of intervals. Let X be disjoint union of intervals, then we suppose two canonical maps:

$$\pi_1 : X \to \mathbb{R}/\mathbb{Z} \quad \textit{and} \quad \pi_2 : X \to \text{wedge sum of countably infinitely many circles.}$$

The only thing we need to prove is both π_1 and π_2 are quotient maps. For π_1, it's an open map. For π_2, we see that it's a composition of quotient topolgy.

Problem 19

Notice that the product with inversion $f : G \times G \to G$ defined by $f(x, y) = xy^{-1}$ is continuous.
Therefore, $f^{-1}(\overline{H})$ is closed. Now, notice that $H \times H \subset f^{-1}(\overline{H})$. So, taking closures,

$$\overline{H \times H} \subset f^{-1}(\overline{H}).$$

Now, you just have to show that $\overline{H} \times \overline{H} \subset \overline{H \times H}$, to conclude that

$$f(\overline{H} \times \overline{H}) \subset \overline{H}.$$

Problem 20

Already apply the conclusion in problem 19.

Problem 21

Proof. (a) From passing down the quotient, just consider the following diagram:

$$\begin{tikzcd}
G \arrow[r, "L_g"] \arrow[d, "\pi"] & G \arrow[r, "\pi"] & G/\Gamma \\
G/\Gamma \arrow[rru, "\theta_g"'] & &
\end{tikzcd}$$

(b) A corollary of (a). $\square$

Problem 22

(a) Assume the quotient map be π. If U is open in X, then gU is open in X. Thus

$$GU = \bigcup_{g \in G} gU \text{ is open in X.} \implies \pi(GU) \text{ is open in X/G. (} \pi^{-1}(\pi(GU) = GU \text{)}$$
$$\implies \pi(GU) = \pi(U) \text{ is open in X/G.}$$

(b) COrollary 3.58.

Problem 23

Proof. we have the following diagram:

$$\begin{tikzcd}
G \times G \arrow[rr, "\cdot_G"] \arrow[d, "\pi_{G \times G}"] & & G \arrow[dd, "\pi_G"] \\
(G \times G)/(H \times H) \arrow[d, "f"] & & \\
G/H \times G/H \arrow[rr, "\cdot_{G/H}"] & & G/H
\end{tikzcd} .$$

The map $\cdot_{G/H}$ is continuos if and only if $\cdot_{G/H} \circ f$ is continuos because the function f that maps every $(g_1, g_2)H \times H$ to $(g_1 H, g_2 H)$ is a homeomorphism (oviously could not be an homeomorphism between topological groups because you don't know if they are topological groups).
By property of quotient, $\cdot_{G/H} \circ f$ is continuos if and only if $\cdot_{G/H} \circ f \circ \pi_{G \times G}$ is continuos and you can observe that this map is $\pi_G \circ \cdot_G$ that it is continuos because $\cdot_G$ and π_G are continuos.
So $\cdot_{G/H}$ is a continuos map. We can use a similar way to prove that $\cdot_{G/H}^{-1}$ is a continuos map and so G/H is a topological group. ☐

Problem 24

Consider the function $f : R^n \to [0, 1)$ given by $f(x) = |x|$.