Introduction to Topological Manifold: Chap 5

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Some Notes

exercise 1

Note1: Fix an infinite subset $A \subset \mathbb{Z}$ whose complement $\mathbb{Z} \setminus A$ is also infinite. Construct a topology on $\mathbb{Z}$ in which:
(a) A is open.
(b) Singletons are never open (i.e., $\forall n \in \mathbb{Z}, \{n\}$ is not open).
(c) For any pair of distinct integers m and n, there are disjoint open sets U and V s. t. $m \in U \ \wedge \ n \in V$.
Solution: One very slick way is to let $f : \mathbb{Z} \to \mathbb{Q}$ be a bijection, set $A = f^{-1}[(0,1)]$, and let the topology on $\mathbb{Z}$ be

$$\tau = \{f^{-1}[U] : U \text{ is open in the usual topology on } \mathbb{Q}\} .$$

Note2: If M is a manifold of dimension $n \neq 0$, then M has no isolated points.
Solution: If $p \in M$ is an isolated point, consider $x : U \to \mathbb{R}^n$ a chart where U is open in M and $p \in U$. Since x is a homeomorphism and $\{p\}$ is an open, we have $\{x(p)\}$ is an open in $\mathbb{R}^n$, but this is only possible if $n = 0$, a contradiction.

Note3: Quotient of locally (path-)connected space is locally (path-)connected.
Solution: Recall that **if U is an open set in X (locally connected), then all connected components of U are open sets in X.** Let U be an open set in X and W be a component of U. For any $x \in W$, there is an open connected neighbourhood $W_x \subseteq U$. But since W_x is connected, we have $W_x \subseteq W$. So we have proved that, for any $x \in W$, there is an open neighbourhood W_x such $W_x \subseteq W$. So W is open.
Now let's prove it: take $y \in Y$. Let A be an open subset of Y such that $y \in A$. Let C be the component of A such that $y \in C$. In order to prove that C in an open connected neighbourhood of y in Y, it is enough to prove C is open.
Then we shall prove that C is **open**: since the topology in Y is the quotient topology, we have that $f^{-1}(A)$ is open and that $f^{-1}(C)$ is a union of components of $f^{-1}(A)$. Since X is locally connected, the components of open sets in X are open. So we have

$$f^{-1}(C) \text{ is a union of open sets in X} \implies f^{-1}(C) \text{ is open} \implies C \text{ is open in Y.}$$

The proof can still go through **when locally connected is changed with locally path-connected.**

exercise 2

Note1: Characteristic map is a **closed** map, and hence a **quotient** map.

Note2: Singleton is both an open and a closed 0-cell.

Note3: Can a cell-complex have no $0 - cell$?
Solution: Suppose X is a nonempty cell complex and let n be minimal such that X has an n-cell. If $n > 0$, then this n-cell has an attaching map $S^{n-1} \to X^{n-1}$ where X^{n-1} is the $(n-1)$-skeleton of X. But by minimality of n, $X^{n-1} = \emptyset$. Since S^{n-1} is nonempty, there are no maps $S^{n-1} \to \emptyset$, so this is a contradiction. **So, if X is any nonempty CW-complex, it must have a 0-cell. (Of course, the empty space is a CW-complex with no cells at all!)**

exercise 3

For a CW complex: (a) locally compact $\iff$ locally finite $\iff$ first-countable (b) connected and locally finite $\implies$ countable.

For a CW complex: locally path-connected $\implies$ open components.

For a connected CW complex: locally finite $\iff$ metrizable.

For a metrizable space: Lindelöf $\iff$ second-countable.

exercise 4

Prop 5.7
Solution: We only need to prove that

$$S \subset X \text{ s.t. } \forall e \subset \mathcal{E} \left(S \bigcap X_n \text{ is closed in } X_n. \right) \implies \text{S is closed.}$$

Let e be a n-cell of X, we only need to prove that $S \bigcap \bar{e}$ is closed in $\bar{e}$. Since $\bar{e} \subset X_n$, we have by properties of subspace topology:

$$S \cap X_n \cap \bar{e} {=} S \cap \bar{e} \text{ is closed in } \bar{e}.$$

Cor 5.15
Solution: Apply Thm 5.14 by changing subset with X.

Prop 5.16
Solution: Note: A finite subcomplex is open, closed and compact. Then a sketch:

$$\begin{aligned} &\text{A CW-complex is locally finite.} \\ \Leftrightarrow &\forall x \in X \ \exists \text{ neighbourhood of x be a finite subcomplex.} \\ \Leftrightarrow &\text{A CW-complex is locally compact.} \end{aligned}$$

Prop 5.33
Solution: Here are some basic facts:

Fact 1: Characteristic map is easy-defined for simplicial complex.

Fact 2: Each interior of the simplices of K is disjoint and a regular open n-cell for some n.

Fact 3: The cell decomposition is locally finite.

Thm 5.39
Solution: Since $\forall x \in K$, x is in a simplical of K with vertex denoted by $\{v_0, \ldots, v_k\}$ then we can write

$$x = \sum_{i=0}^{k} x_i v_i \implies f(x) = \sum_{i=0}^{k} x_i f(v_i) + 0 \implies f(x) \text{ is within some complex in L.}$$

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The deduction above claims that we have a unique extension of f_0 to a function f. The left to prove is as follows:

Goal 1: f is continuous (an application of gluing lemma).

Goal 2: $f|_{\sigma \in K}$ agrees with an affine map taking σ **onto** some simplex in L.

The next thing is just some geometric stuff.

Problems of chapter 5

exercise 1

Proof. (a) Suppose D and D' are $\overline{B^n}$ and $\overline{B^m}$. Then every element other than 0 in D can be expressed uniquely in the form λq where $q \in \partial D$ and $\lambda \in (0, 1]$. Define the map $F(\lambda q) = \lambda f(q)$ which is continuous **by problem 2-15** since

$$\left(\lambda_n q_n \to \lambda q\right) \implies \left(F(\lambda_n q_n) \to F(\lambda q)\right).$$

Finally $\lambda q \in Int\ D$ implies $\lambda < 1$ and so $\lambda f(q)$ is an interior point of D' since $f(q)$ is a boundary point and D' is convex.

Now suppose D and D' are arbitrary closed cells with homeomorphisms $g_1 : \overline{\mathbb{B}^n} \to D$ and $g_2 : \overline{\mathbb{B}^m} \to D'$ (**where possibly** $m = n$). then we have $g_2^{-1} \circ f \circ g_1$ is a continuous map between the boundaries of two closed balls and so from the text above it can be extended to a continuous map $F : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^m}$. The mapping $g_2 \circ F \circ g_1^{-1}$ is a continuous map that satisfies the desired claim.

(b) By proposition 5.1, for any compact convex n-cell D, and $p \in IntD$ there is a homeomorphism $g_p : \overline{\mathbb{B}^n} \to D$ where $g_p(0) = p,\ g_p(\mathbb{B}^n) = Int\ D$ and $g_p(S^{n-1}) = \partial D$.

Starting with two arbitrary closed cells D and D', a continuous $f : \partial D \to \partial D'$, and $p \in IntD$ and $q \in IntD'$ there are homeomorphisms $g_p : \overline{\mathbb{B}^n} \to D$ and $g_q : \overline{\mathbb{B}^m} \to D'$ with the above property. $g_q^{-1} \circ f \circ g_p$ is a continuous map from the boundaries of two closed balls so from part 1 of the proof can be extended continuously to $F : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^m}$. The map $g_q \circ F \circ g_p^{-1}$ is continuous, preserves the map f and

$$(g_q \circ F \circ g_p^{-1})(p) = g_q(F(0)) = g_q(0) = q$$

where F, as constructed from part 1, satisfies $F(0) = 0$.

(c) Suppose D and D' are $\overline{B^n}$ and $\overline{B^m}$ with $f : \partial D \to \partial D'$ is a homeomorphism. Consider $F : D \to D'$ namely $F(\lambda q) = \lambda f(q)$ as in (a). It's clear F is continuous, bijective, and maps a compact space into a Hausdorff space. Then by **the closed map lemma** it is a homeomorphism.

For arbitrary closed cells D and D' with homeomorphisms $g_1 : \overline{\mathbb{B}^n} \to D$ and $g_2 : \overline{\mathbb{B}^m} \to D'$, $g_2^{-1} \circ f \circ g_1$ is a homeomorphism between the boundaries of two closed balls, which follows that it can be extended to a homeomorphism F between the balls. Then $g_2 \circ F \circ g_1^{-1}$ is the desired homeomorphism. □

exercise 2

Proof. (a) Instead of a direct construction, we shall construct a nontrivial one. First we have the facts:

Fact 1: $\overline{\mathbb{B}^n}$ is an n-manifold with boundary. (consider $(r, \theta) \to \mathbb{R}^n$)

Fact 2: For $B \subset \mathbb{R}^n$ (closed subset), $\exists$ continous function $f : \mathbb{R}^n \to [0, \infty)$ whose zero-set is B.

Fact 3: For $B \subset \mathbb{H}^n$ (closed subset), $\exists$ continous function $f : \mathbb{H}^n \to [0, \infty)$ whose zero-set is B.

Then let M be an arbitrary n-manifold with boundary and let B be a closed subset of M. Let $\mathcal{U} = (U_\alpha)$ be a cover of M by open subsets homeomorphic to $\mathbb{H}^n$ or $\mathbb{R}^n$, and let (ϕ_α) be a subordinate partition of unity. For each α, from Fact 2 and 3 we yield a continuous function $u_\alpha : U_\alpha \to [0, \infty)$ such that

$$u_\alpha^{-1}(0) = B \bigcap U_\alpha.$$

Problem 2 continued on next page...

Define $f : M \to \mathbb{R}$ by

$$f(x) = \sum_{\alpha} \phi_\alpha(x) u_\alpha(x),$$

where each summand is to be interpreted as zero outside the support of ϕ_α. We have facts:

Fact 1: Each term in the sum is continuous by the gluing lemma.

Fact 2: Finitely many terms are nonzero in a neighbourhood of each point.

Fact 3: By Fact 1 and 2, f is continuous.

So f is exactly zero on B.
Then applying what we proved above: we find $u, v : M \to [0, \infty)$ such that u vanishes on A and v vanishes on B. Let $A = \partial D$ and $B = \{p\}$, then

$$f(x) = \frac{v(x)}{u(x) + v(x)}$$

is what we want.
(b) A construction similar to Problem 5-1. Consider $D = \overline{\mathbb{B}^n}$, we have:

$$\widehat{F}(\lambda p) = (\frac{1}{2}, 1) + \lambda\left(F(p) - (\frac{1}{2}, 1)\right) \quad \text{and} \quad \pi(x) : \mathbb{R}^2 \to \mathbb{R}.$$

Then we consider $F = \pi \circ \widehat{F}$.

□

exercise 3

Proof. (a) A construction similar to Problem 5-1.
(b) Consider the regular ball basis of X.
(c) Fix a point $x_0 \in X$, then

$$\{y : \exists \text{ homeomorphism } F : X \to X \text{ s.t. } y = F(x_0)\ \}$$

is both open and closed, hence is equal to X. Hence X is homogeneous. (By the way, an obvious modification of the proof shows that the analogous result is also true for a differential manifold: its diffeomorphisms act transitively on the manifold) □

exercise 4

Proof. Consider the regular ball basis of X. □

exercise 5

Proof. It's clear {U: $U \bigcap X_\alpha$ is open in X_α} is a topology in X_α. And suppose τ be a topology s.t. $X_\alpha \hookrightarrow X$ is continuous for any α. Then

$$U \subset \tau \implies U \bigcap X_\alpha \text{ is open in } X_\alpha. \implies \tau \subset \{\text{U: } U \bigcap X_\alpha \text{ is open in } X_\alpha\}.$$

□

exercise 7

Proof. We have

$$U \text{ open in Y} \implies f_\alpha^{-1}(U) = f^{-1}(U)\bigcap X_\alpha \text{ open in } X_\alpha \implies f^{-1}(U) \text{ open in X.}$$

□

exercise 9

Proof. Applying two facts:

Fact 1: A disjoint union of locally path-connected spaces is locally path-connected.

Fact 2: A quotient of a locally path-connected space is locally path-connected.

and consider the map

$$\Phi : \bigsqcup_\alpha D_\alpha \to X \text{ is a quotient map.}$$

□

exercise 10

Proof. Let A be a subset of X such that $A \cap K$ is closed in K for all compact subsets $K \subseteq X$. In particular $A \cap \bar{e}$ is closed in $\bar{e}$ for all cells e in the cell decomposition (**due to the fact that $\bar{e}$ is compact, being the image of $\overline{\mathbb{B}^n}$ under the characteristic map of** e), which implies that A is closed in X. □

exercise 12

Proof. **Manifold Structure:** $\mathbb{R}P^n$ has a standard atlas: $\mathcal{A} = \{(U_i, \psi_i)\}_{i=0}^n$ defined as follows:

$$U_j = \{(x^0 : \ldots : x^n) \in \mathbb{R}P^n : x^j \neq 0\}$$
$$\text{with } \psi_j : U_j \to \mathbb{R}^n, (x^0 : \ldots : x^n) \to (\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}).$$

From the manifold structure as subsets we have :

$$\mathbb{R}P^n = \mathbb{R}^n \sqcup \mathbb{R}P^{n-1} = \ldots = \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \ldots \sqcup \mathbb{R}^0$$

Or more consicely:

$$\{(x_0 : x_1 : x_2 : \ldots : x_n)\} = \{x_n \neq 0\} \sqcup \{(x_0 : x_1 : x_2 : \ldots : x_{n-1} : 0)\}$$

CW-complex Structure:
Define

$$\Phi_m : \overline{\mathbb{B}^m} \to S^{m+1} \setminus \{0\} \to \ \mathbb{P}^m$$
$$\text{with } (x_1, x_2, \ldots, x_m) \to (x_1, x_2, \ldots, x_m, \sqrt{1 - \sum_{i=1}^m |x_i|^2}) \to (x_1 : x_2 : \ldots, x_m : \sqrt{1 - \sum_{i=1}^m |x_i|^2}).$$

Then Φ_m is a continuous map with $\Phi_m|_{B^m}$ is a homeomorphism to $\{x_m \neq 0, x_{m+1} = \ldots = x_n = 0\}$.
(**continuous is clear, we only need to prove it's an open map**)

□

exercise 13

Proof. **Manifold Structure:** $\mathbb{C}P^n$ has a standard atlas: $\mathcal{A} = \{(U_i, \psi_i)\}_{i=0}^n$ defined as follows:

$$U_j = \{(z^0 : \ldots : z^n) \in \mathbb{C}P^n : z^j \neq 0\}$$
$$\text{with } \psi_j : U_j \to \mathbb{C}^n, (z^0 : \ldots : z^n) \to (\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}).$$

From the manifold structure as subsets we have :

$$\mathbb{C}P^n = \mathbb{C}^n \sqcup \mathbb{C}P^{n-1} = \ldots = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \ldots \sqcup \mathbb{C}^0$$

Or more consicely:

$$\{(z_0 : z_1 : z_2 : \ldots : z_n)\} = \{z_n \neq 0\} \sqcup \{(z_0 : z_1 : z_2 : \ldots : z_{n-1} : 0)\}$$

CW-complex Structure:
Define

$$\Phi_k : D^{2k} \to CP^k$$
$$\text{with } (y_1, y_2, \ldots, y_{2k}) \to (y_1 + iy_2 : y_3 + iy_4 : \ldots : y_{2k-1} + iy_{2k} : \sqrt{1 - |y|^2}).$$

It's easy to check that Φ_k is continuous (**composition of continuous map**), onto (**consider the last coordinate**), maps the interior homeomorphically onto $\mathbb{C}P^k - \mathbb{C}P^{k-1}$ (**one-to-one is just computational and homeomorphism is from open-mapping-property**) and maps the buoundary onto $\mathbb{C}P^{k-1}$ as a quotient map. CW property is an immediate consequence of local-finiteness.

□

exercise 14

Proof. Consider the simplex of maximal dimension contained in D, if the maximal dimension is 1, the condition is trivial. If the maximal dimension is greater than 1, just apply **Prop 5.1**. □

exercise 15

With a geometric insight it's trivial.