Introduction to Topological Manifold: Chap 4

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Some Notes

exercise 1

Note 1: For $A \subseteq B \subseteq C$:
1. If A is a compact subspace of B, then A is a compact subspace of C.
2. If A is a compact subspace of C, then A is a compact subspace of B.

Note 2: Suppose that $f : X \to Y$ is a homeomorphism and U is a subset of X. Show that the restriction $f|_U$ is a homeomorphism from U to $f[U]$
Solution:$f : X \to Y$ continuous, so $f|_A : A \to Y$ continuous (domain restriction), and hence $(f|_A)' : A \to f[A]$ continuous (codomain restriction).
If $g : Y \to X$ is the continuous inverse of A, $g|_{f[A]} \to X$ is continuous (domain restriction) and so $(g|_{f[A]})' : f[A] \to g[f[A]]$ is continuous (codomain restriction).
Domains and codomains match and the two equations still hold:

$$(g|_{f[A]})' \circ (f|_A)' = 1_A$$

and

$$(f|_A)' \circ (g|_{f[A]})' = 1_{f[A]}.$$

So indeed $(f|_A)' : A \to f[A]$ is a homeomorphism as well.

exercise 2

Exercise 4.10

Proof. We are assuming M is connected. The double $D(M)$ is equal to a union of two copies of M that intersect in $\partial M \neq \emptyset$. On the other hand, it is a standard lemma in topology that the union of two connected sets that has a nontrivial intersection is connected. □

Prop 4.10

Proof. By applying the locally euclidean property, we can obtain (path) connected basis for each small open set. Then we take their union and obtain a (path) connected basis basis. □

Problems of Chapter 3

Problem 1

Proof. Suppose $\mathbb{R}^n$ is homeomorphic to $U \subset \mathbb{R}$ which is open and f be such a homeomorphism. Let $y \in U$ and $f(x) = y$. Then $f|_{\mathbb{R}^n \setminus \{y\}}$ is a homeomorphism between $\mathbb{R}^n \setminus \{y\}$ and $U \setminus \{x\}$. However, $\mathbb{R}^n \setminus \{y\}$ is connected but $U \setminus \{x\}$ is not. □

Problem 3

Proof. Suppose p is both an interior and boundary point. Choose coordinate charts (U, ϕ) and (V, ψ) such that $\phi(U)$ is open in $\mathrm{Int}\mathbb{H}$ and $\psi(V)$ is open in $\mathbb{H}$, with $\psi(p) \in \partial\mathbb{H}$. Let $W = U \cap V$; then $\phi(W)$ is homeomorphic to $\psi(V)$. $\phi(W) - \phi(p)$ is not connected. So we would reach a contradiction if $\psi(V) - \psi(p)$ is connected. □

Problem 4

Proof. (a) We should show that it isn't a manifold of any dimension. If we take any point of X other than the origin, it clearly has a neighbourhood homeomorphic to an interval in $\mathbb{R}$, so if X is a manifold, it is a one-dimensional manifold. However the origin can't be locally euclidean of dimension one: suppose it were a 1-manifold and V were a neighbourhood of the origin which is homeomorphic to $\mathbb{R}$. Then removing the origin gives us 4 components in V and 2 components in $\mathbb{R}$. A contradiction!
(b) Show that the origin causes trouble. Assume that C is a topological surface. Open sets $U \subseteq C$ and $V \subseteq \mathbb{R}^2$ exists together with a homeomorphism $\phi : U \to V$ and $(0,0,0) \in U$.
For $(a,b) := \phi((0,0,0)) \in V$, some open ball $B \subset \mathbb{R}^2$ centered at (a,b) exist with $(a,b) \in B \subseteq V$.
Let $W := \phi^{-1}(B)$ and prescribe $\psi : W \to B$ by $w \mapsto \phi(w)$. Then ψ is a homeomorphism. However, it sends the notconnected set $W - \{(0,0,0)\}$ to the connected set $B - \{(a,b)\}$. A contradiction! ☐

Problem 5

Hausdorff and second countable and locally Euclidean are all topological invariance property! Then we only need to prove $\pi : S \times \mathbb{R} \to C$ is a quotient map: to prove it sends saturated open subsets to open subsets, we prove two situations on **whether the image of saturated open subsets contains the origin!**

Problem 6

Proof. (a) If B is uncountable and $<$ a well-order on B, then let $C = \{x \in B : \{y | y < x\} \text{ is uncountable}\}$. Then we have:
(1) If $C \neq \phi$ then let $x_C = \min C$. By definition of C and of x_C, if $y < x_C$ then $\{z | z < y\}$ is countable. Let $Y = \{y : y < x_C\}$ which is uncuntable.
(2) If $C = \phi$, then it's trivial.
(b) **Properties of Long line are considered trivial if imagined as an extension of real line.** ☐

Problem 7

Proof. Lemma: Let $f : X \to Y$ be a continuous map between topological spaces, and let $\mathcal{B}$ be a basis for the topology of X. Let $f(\mathcal{B})$ denote the collection of subsets of Y . Show that $f(\mathcal{B})$ is a basis for the topology of Y if and only if f is surjective and open. ☐

q is open quotient which clearly satisfies the conditions for f in the lemma, then the conclusions for locally connectedness, locally path-connectedness and locally compactness are clear.

Problem 8

Fact1: If X is locally connected, the connected components are open.

Problem 9

Fact1: Every component is a manifold and open.
Fact2: Every open subset of an n-manifold is an n-manifold.

Problem 10

Problem 11

Note that $CX = X \times I/(X \times \{0\})$.

Proof. (a) It suffices to show that there exists a point p to which any point in CX can be connected by a continuous path. In the cone, we can take this 'connecting point' to be the vertex of the cone.
To see this, let (x,t) be a point in CX and consider the map $\lambda : I \to X \times I$ defined by $\lambda(s) = (x, (1-s)t)$. Since both coordinates of λ are continuous, λ itself is continuous (i.e., λ is a continuous path in $X \times I$ that joins the point (x,t) to the point $(x,0)$ (this is still true if $t = 0$.) Let $\pi : X \times I \to CX$ be the quotient map. Since π is continuous, the composition $\pi \circ \lambda$ is also continuous. This composition is thus a continuous path that joins (x,t) to $p = (x,0)$.
Since we've shown than an arbitrary point (x,t) in CX can be joined to p, we can conclude that CX is indeed path-connected.
(b) Assume first that X is locally path connected. It's actually two facts that a finite product of locally (path) connected spaces is locally (path) connected and the image of a locally (path) connected space with a open quotient map is locally (path) connected. Putting these two facts together shows that CX must be locally path connected. (**Note: to prove it's an open mapping, it suffices to prove that the images of basis open sets are open.**)
For the converse, assume CX is locally path connected. It's clear that every point have a basis of path connected neighbourhoods. So take a point $x \in X$, which corresponds to $(x,1) \in CX$. Let U be a neighbourhood of x in X. Then $U \times (\frac{1}{2}, 1]$ is a neighbourhood of x in CX and it doesn't contain the apex of the cone. By assumption, there exists a path-connected neighbourhood V of x below $U \times (1/2, 1]$. Projecting V onto X then gives us a path connected neighbourhood of x in X below U (remember, we are basically in a product space now and **projections are continuous, surjective and open maps**).
Locally connected can be proved in a similar way. □

Problem 12

Proof. Let $X = \bigcup_\alpha O_\alpha$, then

$$S \bigcap O_\alpha = \emptyset \text{ or } O_\alpha.$$

□

Problem 13

(a) First we prove it's connected but not path-connected and locally connected.

Proof. **The topologist's sine curve is connected:** The first method: call the topologist's sine curve T, and let $A = \{(x, \sin 1/x) \in \mathbb{R}^2 \mid x \in \mathbb{R}^+\}$. Then $A \subseteq T \subseteq \overline{A}$. It isn't difficult to show that A is connected (even path connected!) then the conclusion immediately follows.

The second method: if the graph X of the topologist's sine curve were not connected, then there would be disjoint non-empty open sets A, B covering X. Let's assume a point $(x,\ \sin(1/x)) \in B$ for some $x > 0$. Then the whole graph for positive x is contained in B, only leaving the point $(0,0)$ for the set A. But any open set about $(0,0)$ would contain $(1/n\pi,\ \sin(n\pi))$ for large enough $n \in \mathbb{N}$, thus A would intersect B.

Problem 13 continued on next page...

****The topologist's sine curve is not path-connected:**** If $S = \{(0,0)\} \cup \{(x, \sin(1/x)) : 0 < x < 1\}$ and $f = (f_1, f_2) : [0,1] \to S$ is a path with $f(0) = (0,0)$, then $f(t) = (0,0)$ for all t. (Prove by contradiction)

****The topologist's sine curve is not locally connected:**** Let's stick with a particular point on the interval $0 \times [-1,1]$, say $p = (0,0)$. Consider open squares and let $U_\epsilon := (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ be some open square centered at p (where $\epsilon > 0$). Then $U_\epsilon \bigcap \overline{S}$ consists of $0 \times (-\epsilon, \epsilon)$ and the graph of the function $\sin(1/x)$ restricted to the domain $D_\epsilon := \{x \in (0, \epsilon) : |\sin(1/x)| < \epsilon\}$. We should be picturing a bunch of very short curve segments which are almost vertical. We can choose ϵ small enough that D_ϵ does not contain any x such that $\sin(1/x) = 1$.
Now let V be some nonempty open subset of U_ϵ containing p. It contains $U_{\epsilon'}$ for some smaller $\epsilon' > 0$. Then there exists some $x_0 \in (0, \epsilon')$ such that $\sin(1/x_0) = 1$ and $(x_0, \infty) \cap D_{\epsilon'} \neq \emptyset$. It follows that

$$D_{\epsilon'} = \Big(D_{\epsilon'} \cap (0, x_0)\Big) \bigcup \Big(D_{\epsilon'} \cap (x_0, \infty)\Big),$$

which follows that it is disconnected.
We can use this information to prove that $V \cap \overline{S}$ is disconnected. The idea is to look at the intersections of this set with $(-\infty, x_0) \times \mathbb{R}$ and with $(x_0, \infty) \times \mathbb{R}$. Note that neither of these intersections is empty. Secondly, these open sets do indeed cover $V \cap \overline{S}$ since V contains no point whose y-coordinate is 1. So we conclude that $V \cap \overline{S}$ is disconnected.
(b) Answer without proof:
****Component:**** The whole curve.
****Path-component:**** The origin and all other parts. □

Problem 15

A stronger statement of (b) and (d) would be:

(b) If U is any nonempty open subset of G, then the subgroup $\langle U \rangle$ generated by U is both open and closed.
(d) If G is connected, then every nonempty open subset of G generates G.

Proof. (a) This is trivial. Because $U^c = \bigcup_{g : gU \neq U} gU$ is open.
(b) Replace U by $U \cup U^{-1}$ and then observe that

$$\langle U \rangle = \bigcup_{m=1}^{\infty} U^m.$$

It suffices therefore to show that U^m is open for each m, which follows from a simple induction argument and the fact that UV is open if U, V are open.
(c) Here it is important that $1 \in U$. I argue that $1 \in U^m$ for each m. Since U^m is connected (continuous image of Cartesian products of U), $\bigcup_{m=1}^{\infty} U^m$ is connected.
(d) This follows from (b). Because $\langle U \rangle$ is both open and closed, and since is non-empty it must be the whole G (since G is connected.) □

Problem 16

Proof. We just sketch the proof:
$\Rightarrow$: Manifold admits a basis of regular balls.
$\Leftarrow$: It suffices to show X the space is second countable. Let K_n be the covering of X by compact sets, and assume they're all nonempty. For each $x \in K_n$, there is an open neighborhood $U_{x,n}$ which is homeomorphic to an open ball in $\mathbb{R}^m$. Then we have $K_n \subset \bigcup_{x\in K_n} U_{x,n}$. By compactness, there are finitely many $U_{x,n}$ which cover each K_n. By unioning these all together for each n gives us a countable covering for X by open sets which are homeomorphic to open balls in $\mathbb{R}^m$. Each of these sets has a countable basis. The countable union of these sets is also a countable set, so it remains to show that this set is a basis, which is easy. □

Problem 17

Recall that **a coordinate ball** $B \subset M$ is a regular coordinate ball if there is a neighborhood $\hat{B}$ of $\bar{B}$ and a homeomorphism $f : \hat{B} \to B_{\hat{r}}(x)$ that takes B to $B_r(x)$ and $\bar{B}$ to $\overline{B_r(x)}$ for some $\hat{r} > r > 0$ and $x \subset \mathbb{R}^n$.
An n-dimensional manifold with boundary is a second countable Hausdorff space in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$, or to an open subset of $\mathbb{H}^n$.

Proof. With the subspace topology, it's clear that $M \setminus B$ is second countable and hausdorff. Wo only need to prove it's locally euclidean. For $x \in \hat{B} - \bar{B}$, we have

$$f|_{\hat{B}-\bar{B}} : (\hat{B} - \bar{B}) \longrightarrow (B_{\hat{r}}(x) - \overline{B_r(x)}).$$

Consider $x \in M - \hat{B}$. Since M is locally euclidean in the beginning, we have $\phi_x : U_x \to O$, where U_x is some neighbourhood of x and O is an open set in $\mathbb{R}^n$. Then construct:

$$\phi_x|_{U_x\cap(M-\bar{B})} : U_x \cap (M - \bar{B}) \to \phi_x\Big(U_x \cap (M - \bar{B})\Big) \subset O \subset \mathbb{R}^n,$$

which sends a neighbourhood of x to an open set of O.
Finally consider $x \in \bar{B} - B$, we have:

$$f|_{\hat{B}-B} : (\hat{B} - B) \longrightarrow (B_{\hat{r}}(x) - B_r(x)).$$

We can cut the $B_{\hat{r}}(x) - B_r(x)$ into pieces so as to send them homeomorphically to some open sets in $\mathbb{H}^n$. □

Problem 18

(a) Thm 3.79.
(b) Images of M_1 and M_2 are both connected and intersect.
(c) Images of M_1 and M_2 are both compact. (**Fact: finite union of compact sets is compact!**)

Problem 19

Question: If M and N are two n-manifolds, and $B_1 \subset M$ and $B_2 \subset N$ are two open, regular coordinate balls (definition below), the connected sum $M\#N$ is the quotient of the disjoint union $(M - B_1)\bigsqcup(N - B_2)$ by the relation that identifies points on the spherical boundaries of each component via some homeomorphism h. Now I want to show that there are two open sets $U, V \subset M\#N$, such that:

Problem 19 continued on next page. . .

1. $U \cong M - \{p\}$ and $V \cong N - \{q\}$, for some points $p \in M$ and $q \in N$
2. $U \cap V \cong S^{n-1} \times \mathbb{R}$
3. $U \cup V = M \# N$.

1. Sketch: Take a larger coordinate ball D in N, containing B_2, which works because B_2 is regular. $(M - B_1) \coprod (D - B_2)$ is a saturated open set, so its image in $M\#N$ is open. Let U be the image of $(M - B_1) \coprod (D - B_2)$.
Now we have a homeomorphism from $D - B_2$ to $\overline{\mathbb{B}}_t(0) - \{0\}$ (the punctured closed ball). Then map that punctured ball to $\overline{B}_1 - \{p\}$, with the composition denoted by h. And show the map $f : (M - B_1) \coprod (D - B_2) \to M - \{p\}$

$$f(x) = \begin{cases} x & x \in M - B_1 \\ h(x) & x \in D - B_2 \end{cases}$$

is coherent with h and is a quotient map, and then use the uniqueness of quotients to show U is homeomorphic to $M - \{p\}$.
(**Note: we let t=1 actually to avoid some notation issue!**)

Proof. Assume the homeomorphism is $h : \partial B_2 \to \partial B_1$. Now we have a homeomorphism $g : D - B_2 \to \overline{\mathbb{B}}_t(0) - \{0\}$and a homeomorphism $k : \overline{B_1} \to \overline{\mathbb{B}_t(0)}$.
As above mentioned, $k^{-1} \circ g$ is a homeomorphism from $D - B_2$ to $\overline{B}_1 - \{p\}$ (assuming $k(p) = 0$). The problem is what happens on the boundary: we need it to do exactly what h does. **The key is that we can do something between g and k^{-1} that helps with that.**
Thinking just about the boundary, we want to find a map $r : \partial\mathbb{B}_t(0) \to \partial\mathbb{B}_t(0)$, such that

$$k^{-1} \circ r \circ g \equiv h$$

as maps from ∂B_2 to ∂B_1.
Then clearly we have r: $r = k \circ h \circ g^{-1}$. But we need this to be defined on all of $\overline{\mathbb{B}}_t(0) - \{0\}$.
We define $\tilde{r} : \overline{\mathbb{B}}_t(0) - \{0\} \to \overline{\mathbb{B}}_t(0) - \{0\}$ as

$$\tilde{r}(x) = |x| r\left(\frac{x}{|x|}\right)$$

It's easy to check this is a homeomorphism (the inverse is just $|x|r^{-1}\left(\frac{x}{|x|}\right)$). Define $G = k^{-1} \circ \tilde{r} \circ g$, we have a homeomorphism from $D - B_2$ to $\overline{B}_1 - \{p\}$. And more importantly,

$$G(x) = h(x) \text{ for } x \in \partial B_2$$

So we can define $f : (M - B_1) \sqcup (D - B_2) \to M - \{p\}$ as

$$f(x) = \begin{cases} x & x \in M - B_1 \\ G(x) & x \in D - B_2 \end{cases}$$

This map is continuous and surjective and is a quotient map. And with the help of $\tilde{r}$, this map now makes the same identifications as the original quotient map.

2. Sketch: it helps to look at the preimage of $U \cap V$ under the quotient map. There's a nice quotient map, from the "N" and "M" pieces, to $\mathbb{S}^{n-1} \times \mathbb{R}$. (Imagination!)
3. easy! □

Problem 20

Recall that in the topology U is open iff $U = -U$. A space X is said to be **limit point compact** if every infinite subset of X has a limit point in X, and **sequentially compact** if every sequence of points in X has a subsequence that converges to a point in X.

Proof. To prove it's not compact, consider the cover $\{(-n, n) : n \in \mathbb{N}^*\}$.
It's easy to prove limit point compact. □

Problem 21

Prove the basis of each topology generates each other.

Problem 22

Problem 23

Proof. (a) We just need to apply below two facts:

Fact1: compact subspace of hausdorff space is closed.

Fact2: $U \cup (Y - C) = Y - (C - U)$.

(b) Compactness is clear. And hausdorff property comes from the definition of locally compactness.
For example, for $x \in X$ and $y = \infty$, we have
(c) We have

$$\begin{aligned} & \text{A sequence of points in X diverges to infinity.} \\ \Leftrightarrow\ & \text{For every compact set } K \subset X \text{, there are at most finitely many values of i for which } x_i \subset K. \\ \Leftrightarrow\ & \text{For every compact set } K \subset X \text{, there are all but finitely many values of i for which } x_i \subset X^* \setminus K. \\ \Leftrightarrow\ & \text{It converges to } \infty \text{ in } X^*. \end{aligned}$$

(d) and (e) are clear. □

Problem 24

Proof. A sketch:

$\Rightarrow$: Problem 23

$\Leftarrow$: Open subset of locally compact hausdorff space is agian locally hausdorff.

□

Problem 25

A corollary of problem 27.

Problem 26

Problem 27

Proof. **A space in which all compact subsets are closed is called a KC-space.** Clearly Hausdorff spaces have this property.
To avoid ambiguous notation, let us write $f_1 : X_1 \to Y_1$ for the extension of f via $f_1(\infty_X) = \infty_Y$. We then have without any requirements on X, Y.

$$f_1 \text{ is continuous} \Leftrightarrow f^{-1}(K) \text{ is compact for each compact closed } K \subset Y.$$

To prove it, consider $V_1 \subset Y_1$ open. If $V_1 \subset Y$, then $f_1^{-1}(V_1) = f^{-1}(V_1)$ is open in X (since f is continuous), and thus open in X_1. Therefore we have

$$\begin{aligned}
& f_1 \text{ is continuous} \\
\Leftrightarrow\ & f_1^{-1}(V_1) \text{ is open in } X_1 \text{ for all } V_1 = Y_1 \setminus K \text{ with compact closed } K \subset Y. \\
& (\text{ Since we have } f_1^{-1}(Y_1 \setminus K) = X_1 \setminus f_1^{-1}(K) = X_1 \setminus f^{-1}(K).\ ) \\
\Leftrightarrow\ & f^{-1}(K) \text{ is a closed subset of } X. \\
\Leftrightarrow\ & f \text{ is proper.}
\end{aligned}$$

□

Problem 28

Problem 29

Proof. Note that the countable closed set (denoted by C) is complete metric or locally compact Hausdorff with subspace topology. So Baire category theorem applies to C as well.
Note

$$\begin{aligned}
& \text{Point } x \text{ is not an isolated point of } C. \\
\Leftrightarrow\ & \text{Every neighbourhood of x doesn't only intersects C with x.} \\
\Leftrightarrow\ & \text{For subspace C, every neighbourhood of x doesn't only intersects C with x.} \\
\Leftrightarrow\ & \text{For subspace C, every neighbourhood of x intersect with } C \setminus \{x\}. \\
\Leftrightarrow\ & U_x = C \setminus \{x\} \text{ is dense in } C.
\end{aligned}$$

And in both cases C have closed singletons, so U_x is open. So if no point is isolated,

$$\emptyset = \bigcap_{x \in C} U_x$$

contradicts Baires theorem for C. □

Problem 30

Proof. We only need to prove $f|_{A_i}$ is continuosu with A_i covering X.
Since $\{A_\alpha\}$ is a locally finite closed cover of X, we have $\forall x \in X$

$$\exists U_x \text{ and } \{A_{x_i}\}_{i=1}^{n_x} \text{ s.t. } \{A_{x_i}\}_{i=1}^{n_x} \text{ only intersect with } U_x.$$

Then since $f|_{A_{x_i} \bigcap U_x}$ is continuous and the i's are finite, by gluing lemma $f|_{U_x}$ is continuous. By gluing lemma again, f is continuous. □

Problem 33

Proof. $X = \{X_a\}$ and let $\{P_a\}$ be the associated partition of unity, the index set for both is A. Define $A' = \{a | a \in A, P_a \neq 0\}$ and $V_a = P_a^{-1}((0, 1])$ for $a \in A'$, open in M. It is easy to see that $\{V_a | a \in A'\}$ is a locally finite refinement.
First, $\forall x \in X$, $\exists$ b such that $P_b(x) > 0$, which proves that V_a is an open refinement. Second, V_a is locally compact since $V_a \subset suppP_a$ Combing the two facts together, we have $\{V_a\}$ forms a locally finite open refinement. □

Problem 34

Proof. Let $f : M \to \mathbb{R}^k$ be the injective continuous map and let $g : M \to \mathbb{R}$ be an exhaustion function, then the map x to $(f(x), g(x))$, which is proper, injective and continuous.
(it's proper since $g(x) \to \infty$ as $x \to \infty$ in M.) □