Introduction to Topological Manifold: Chap 6

Due on May, 2021

Yuchen Ge

Some Notes

Comments on proof of Prop 6.4: The topology of M is already defined as a qotient space which is also the inherited topology from $\mathbb{R}^n$. As **subspace of a quotient space**, It follows from definition that M_0 is discrete, and for k=1,2, M_k is obtained from M_{k-1} by attaching finitely many k-cells.

Problems

Problem 1

Proof. Showing that $\langle a, b, c | abcb \rangle \subset \langle a, b | abab \rangle$.

□

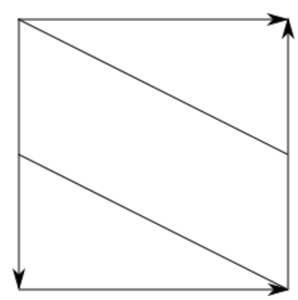
Problem 2

Proof. Imagine that we cut out a vertical rectangle in the middle of the projective plane square. The cutout piece is a Möbius strip.
Now glue first the vertical edges in the right direction of the two remaining pieces and then the parts of edge A and realize that this corresponds to a disc.

□

Problem 3

Proof. This is below a diagram of the Klein bottle, note that the diagonal lines divide it into 2 Möbius strips sharing a boundary: So the answer is yes. □



Problem 2

Another characterization:

Proof. RP^2 is the sphere S^2 where we identify $x \sim -x$. If D is a disc then $RP^2 \setminus D$ is the space $S^2 \setminus (D \cup -D)$ where you identify $x \sim -x$. But $S^2 \setminus (D \cup -D)$ is a cylinder. It's clear that if we identify a cylinder by $x \sim -x$ then we have a Moebius strip. Here is a proof.
We have the CW decomposition $\mathbb{R}P^2 = \mathbb{R}^0 \sqcup \mathbb{R}^1 \sqcup \mathbb{R}^2 = \mathbb{R}P^1 \sqcup \mathbb{R}^2$. We have $\mathbb{R}P^n = \mathbb{R}^n \cup \mathbb{R}P^{n-1}$. To see this consider the embedding $i : \mathbb{R}^n \to \mathbb{R}P^n$ given by $i(x_1, x_2, \ldots, x_n) = [1, x_1, x_2, \ldots, x_n]$. The complement of $i(\mathbb{R}^n)$ in $\mathbb{R}P^n$ is

$$\{[0, x_1, x_2, \ldots, x_n] \mid (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\}\} \cong \mathbb{R}P^{n-1}.$$

Problem 2 continued on next page...

Now we have $\mathbb{R}P^2 = \mathbb{R}^2 \cup \mathbb{R}P^1$, where $\mathbb{R}P^1$ is readily identified with S^1. Immediately construct the desired decomposition

$$\mathbb{R}P^2 = \{[1, r\cos\phi, r\sin\phi] \mid 0 \le r \le 1 \text{ and } \phi \in [0, 2\pi)\} \cup \{[r, \cos\phi, \sin\phi] \mid 0 \le r \le 1 \text{ and } \phi \in [0, 2\pi)\},$$

where the two components are identified with the closed disk D^2 and the moebius band M, with common boundary S^1. ☐

Problem 4

Proof. Hint tells everything. ☐

Problem 5

Proof. Here is a fact:

Fact 1: Boundary of a n-manifold is a $(n-1)$-manifold.

Let M be a compact 2-manifold with boundary. Then ∂M with subspace topology is a 1-manifold. Consider its connected component which is open in ∂M and closed in M. It follows that each connected component is a connected compact 1-manifold, which by theorem 5.27 again follows that it's homeomorphic to S^1. Then we have a homeomorphism (application of coherentness implies homeomorphism):

$$\phi : \partial M = \bigcup_i (\text{connected component of } \partial M) \to \bigsqcup_i S^1$$

Since closed subset of compact space is compact and each component is open in ∂M, I is a finite set. Then we just need to attach sphere to each connected component of ∂M. ☐