A course in homological algebra: Chap 1

Due on May, 2021

Yuchen Ge

Some Notes

Comments on 1 example (e): Factor through means:

$$\begin{tikzcd} KG \arrow[r] \arrow[d] & End_{\mathbb{Z}}(V,V) . \\ End_K(V,V) \arrow[ur] & \end{tikzcd}$$

Exercises of Part 1

Problem 2

In fact we have the following conclusion:

$$\phi_2, \phi_4 \text{ surjective and } \phi_5 \text{ injective} \implies \phi_3 \text{ surjective.}$$
$$\phi_2, \phi_4 \text{ injective and } \phi_5 \text{ surjective} \implies \phi_3 \text{ injective.}$$

Problem 5

The same argument as in example (e).

Problem 7

Proof. First the induced map is well defined by defining

$$\mu_m : a' + mA' \to \mu(a') + mA \text{ and } \epsilon_m : a + mA \to \epsilon(a) + mA''.$$

$(i) \implies (iii)$: Let $a'' \in A''$ with $ma'' = 0$, since ϵ is surjective, it follows that $a'' = \epsilon(a)$ for some $a \in A$. Then we have that $\epsilon(ma) = 0$, which follows that $\mu(a') = ma$ with $a' = mb'$ for some $b' \in A'$. Then consider $a - \mu(b')$.
$(iii) \implies (i)$: Similar to above, we have:

$$\begin{aligned} \mu(a') = ma &\implies \epsilon \circ \mu(a') = m\epsilon(a) = 0 \\ &\implies \exists\, b \in A\colon\ \epsilon(b) = \epsilon(a), mb = 0. \\ &\implies \epsilon(b-a) = 0 \implies a - b = \mu(b') \text{ for some } b' \in A'. \\ &\implies \mu(a') = ma = m(a-b) = \mu(mb') \implies a' = mb'. \end{aligned}$$

$(ii) \implies (i)$: trivial.
$(i) \implies (ii)$: trivial again. □

Exercises of Part 2

Some notes

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$: If $\phi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$, then we only have to specify what $\phi(1)$ is, because then $\phi(a) = a\phi(1)$ we know what this particular module-homomorphisms look like for all values of $a \in \mathbb{Z}$. So let $\phi(1) = n$, for which

Problem 7 continued on next page...

value of n is this a module-homomorphism? Well, every value works! Since $\phi(a+b) = an+bn = \phi(a)+\phi(b)$, and $r\phi(a) = ran = \phi(ra)$. Since every value of $n \in \mathbb{Z}$ works, we must have

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}.$$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z})$: Let $\phi \in \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z})$, and let $\phi(1) = a$ for $a \in \mathbb{Z}$. We know that $\phi(0) = 0$ so $\phi(0) = \phi(n) = na = 0$, which implies:

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) = \emptyset.$$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m)$: let $\phi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m)$, and let $\phi(1) = a$ for $a \in \mathbb{Z}_n$. It's clear that $\phi(0) = 0$, so $\phi(n) = \phi(0) = n\phi(1) = na = 0$. But this time $na \equiv_m 0$. So m divides an and $an = mk$ for some integer k. Let $d = \gcd(m, n)$ then $a(n'd) = (m'd)k \Rightarrow an' = m'k \Rightarrow a = \frac{m'k}{n'}$. This needs to be an integer so we need to choose k such that this happens. Since $\gcd(m', n') = 1$ we must have that k is a multiple of n', that is

$$k \in \{n', 2n', 3n', \ldots, dn'\}$$

Thus there are $d = \gcd(m, n)$ choices for k and therefore

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d$$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$: Let $\varphi : \mathbb{Q} \to \mathbb{Z}$ suppose that φ is non-trivial. Let n be the smallest positive integer in $\mathrm{im}\,\varphi$. Pick $a/b \in \varphi^{-1}(n)$. Then

$$n = \varphi(a/b) = \varphi(a/2b + a/2b) = \varphi(a/2b) + \varphi(a/2b),$$

so $\varphi(a/2b) = n/2$, but this is impossible because n was the smallest positive integer in the image. Thereby

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = \emptyset.$$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}$.

Problem 1

Consider $\Lambda = Z_n$.

Problem 2

Consider $\Lambda = Z_n$.

Problem 3

Just check that

$$\begin{aligned}
((\lambda_1\lambda_2)\phi)(a) &= \phi(\lambda_1\lambda_2 a) \\
(\lambda_1(\lambda_2\phi))(a) &= (\lambda_2\phi)(\lambda_1 a) = \phi(\lambda_2\lambda_1 a).
\end{aligned}$$

While if Λ is commutative, we can remedy this.

Problem 4

Define $(\phi\lambda)(a) = \phi(\lambda a)$ as in exercise 3. Then we only need to check:

$$(\phi(\lambda_1\lambda_2))(a) = \phi(\lambda_1\lambda_2 a) = (\phi\lambda_1)(\lambda_2 a) = ((\phi\lambda_1)\lambda_2))(a)$$

Problem 7

Proof. Indeed, it suffices to show that for every morphism $f\colon \mathbb{Z}_m \to A''$ of abelian groups there exists a morphism $g\colon \mathbb{Z}_m \to A$ of abelian groups such that $\epsilon g = f$. But f is determined by $a'' = f(1)$ which satisfies $ma'' = 0$. By Exercise 1.7(iii) we can lift a'', i.e. there exists an $a \in A$ with $\epsilon(a) = a''$ and $ma = 0$. We determine g by $g(1) = a$ and we have $\epsilon g = f$ as desired.
The other implication follows along the same lines. ☐

Problem 8

Define $m\phi n(a) := n\phi(am)$ for $m \in \Sigma$ and $n \in \Gamma$.

Exercises of Part 3

Problem 1

Proof. Just consider the following diagram:

$$\begin{tikzcd} \oplus_j A_j \arrow[r] & \prod_j A_j \ . \\ A_j \arrow[u] \arrow[ur] & \end{tikzcd}$$

☐

Problem 3

Proof. We just prove $\mathrm{Im}\{\epsilon, \alpha\} = \mathrm{Ker}\{\alpha'', -\epsilon\}$. First $\alpha''\epsilon - \epsilon'\alpha = 0$ implies $\mathrm{Im}\{\epsilon, \alpha\} \subset \mathrm{Ker}\{\alpha'', -\epsilon\}$.
Then suppose $\alpha''(a'') = \epsilon'(b)$ for some $a'' \in A''$ and $b \in B$. By diagram chasing, we have the existence of such $a \in A$ whose image is (a'', b). ☐

Problem 5

Proof. An application of universal property. ☐

Problem 7

Proof. $(i) \implies (v)$: Consider $<\alpha', \mu>(a', a'') : \alpha'(a') + \mu(a'')$. We prove it's an isomorphism.
Injection: If $\alpha'(a') + \mu(a'') = 0$, then $0 = \alpha''\alpha'(a') = -\alpha''\mu(a'') = -a''$ and $\alpha'(a') = 0$ implies $a' = 0$.
Surjection: For any $a \in A$, suppose $\alpha'(a') + \mu(a'') = a$, then immediately we get $a'' = \alpha''(a)$ and since $\alpha''(a - \mu(a'')) = 0$, $a' = \alpha'^{-1}(a - \mu(a''))$ is well-defined.
$(v) \implies (ii)$: We have a following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[dr, "\iota_1"] & A \arrow[r, "\alpha''"] & A'' . \\ & & A' \oplus A'' \arrow[u, "<\alpha', \mu>"'] & \end{tikzcd}$$

Then we define $\epsilon = \pi_1 \circ < \alpha', \mu >^{-1}$, where π_1 is the canical projective in the first variable.
$(ii) \implies (i)$ can be proved likewise while the isomorphism is constructed as $A \to A' \oplus A''$and proof for (iii) and (iv) is trivial. ☐

Exercises of Part 4

Problem 4

Proof. **Fact:** In PID, a module is projective iff it's free. Consider $\prod_{i \in N} Z$. ☐

Problem 5

Schanuel's Lemma.

Exercises of Part 5

Problem 2

Proof. **Statement**: let $0 \to A \to B \to C \to 0$ be an exact sequence of R modules, where R is a principal ideal domain. Then if B is finitely generated, then A and C are also finitely generated.
A finitely generated module is a module M for which there exists an R-module surjection $R^n \to M$, with n a positive integer.
From there, assuming that B is finitely generated, it follows that any quotient of B is finitely generated. Hence C is finitely generated.
Proving that A is finitely generated amounts to proving that *any submodule of a finitely generated module is finitely generated*.
Fact: *Any submodule of R^n is finitely generated*. Just let M be a finitely generated submodule of R^n, $I_1 := \{r \in R \mid \text{ there exists } s \in R^{n-1} \text{ such that } (r, s) \in M\}$, $M_1 := \{s \in R^{n-1} \mid (0, s) \in M\}$. Prove that I_1 and M_1 are finitely generated (and consequently, I_1 is principal) and $M \simeq I_1 \oplus M_1$, then induct on n.
If $f : R^n \to B$ is a surjection (it exists since B is finitely generated), then since A is a submodule of B, consider the preimage $f^{-1}(A)$ of A by f. It is a submodule of R^n, hence it is finitely generated. Since A is a quotient of $f^{-1}(A)$, it is also finitely generated. ☐

Exercises of Part 7

Problem 7

Proof. We know every abelian group sits inside a direct product of copies of $\mathbb{Q}/\mathbb{Z}$, hence we have an exact sequence

$$0 \to \mathbb{Q} \to \prod_{i\in I}(\mathbb{Q}/\mathbb{Z})_i \to (\prod_{i\in I}(\mathbb{Q}/\mathbb{Z})_i)/\mathbb{Q} \to 0.$$

Now since $\mathbb{Q}$ is divisible, hence injective, the sequence splits. $\square$

Proof. Another proof follows. **Fact:** A torsion-free divisible abelian group is a direct sum of copies of $\mathbb{Q}$. Since G (a torsion-free divisible group) is divisible given any $y \in G$ and $m \in \mathbb{Z}$ there exists a solution x to $mx = y$. Since G is torsion-free this solution is unique. Thus, we can define $\frac{1}{m}y := x$. One then shows that this defines a $\mathbb{Q}$-vector space structure on G, and so $G \cong \mathbb{Q}^{\oplus\lambda}$ for some cardinal λ. Saying that the dimension of G over $\mathbb{Q}$ is finite is just saying that λ is finite, and so we get it.
Then consider the direct product $G = (\mathbb{Q}/\mathbb{Z})^{\mathbb{N}}$; this group is not torsion, because you can take the element

$$x = \left(\frac{1}{n+1} + \mathbb{Z}\right)_{n\in\mathbb{N}}$$

which is not annihilated by any integer. Thus $G/t(G) \neq 0$, where $t(G)$ denotes the torsion subgroup of G. But $G/t(G)$ is a torsion free divisible group, so it is a direct sum of copies of $\mathbb{Q}$. So $\mathbb{Q}$ is a direct summand of $G/t(G)$. Since $t(G)$ is divisible, it splits, so $G = t(G) \oplus H$, where $H \cong G/t(G)$. $\square$