Introduction to Topological Manifold: Chap 2

Due on March, 2021

Yuchen Ge

Section 2.1

Problem 3

Proof. We have the following equation and similar for (b):

$$\begin{aligned}\overline{X-B} &= \bigcap\{E : (X-B) \subset E \text{ and } E \text{ is closed}\}\\ &= X - \bigcup\{E^c : E^c \subset B \text{ and } E^c \text{ is closed}\}\\ &= X - IntB\end{aligned}$$

□

Problem 4

Proof. (a) We have

$$\begin{aligned}\overline{\bigcap_{A\in\mathcal{A}} A} &= \{x : \text{every neighbourhood of x intersects with } \bigcap_{A\in\mathcal{A}} A\}\\ &\subset \{x : \forall A \text{ s.t. every neighbourhood of x intersects with A}\}\\ &= \bigcap_{A\in\mathcal{A}} \overline{A}.\end{aligned}$$

(b) We have

$$\begin{aligned}\overline{\bigcup_{A\in\mathcal{A}} A} &= \{x : \text{every neighbourhood of x intersects with } \bigcup_{A\in\mathcal{A}} A\}\\ &\supset \{x : \exists A \text{ s.t. every neighbourhood of x intersects with A}\}\\ &= \bigcup_{A\in\mathcal{A}} \overline{A},\end{aligned}$$

and the equation holds if in finite case.
(c) (d) is omitted since they are similar cases. □

Problem 6

Proof. (a) Assume B closed. Let $A = f^{-1}(B)$, we have

$$f(\overline{A}) \subset \overline{f(A)} \implies \overline{A} \subset f^{-1}(f(\overline{A})) \subset f^{-1}(B) = A \implies \text{A is closed.}$$

(b) Assume A closed. Let $B = f(A)$, we have

$$B = f(\overline{A}) \supset \overline{f(A)} = \overline{B} \implies \text{B is closed.}$$

(c) (d) is omitted since they are similar cases. □

Problem 7

Proof. p is a limit point of A IFF. every neighbourhood of p contains a point of A except for p. Then it's clear by applying hausdorff property. □

Problem 8

Proof. We only need to prove that any limit point of the set of limit points of A is still in the set of limit points of A. □

Problem 10

Proof. Hint:

$$(\{x \in X : f(x) = g(x)\})^c = \bigcup_{\text{U,V be disjoint opensets}} \left(f^{-1}(U) \bigcap g^{-1}(V)\right).$$

□

Problem 15

Proof. Hint of (a): preimage of the open sets is open.
Hint of (b): order the neighbourhood! □