NOTES ON TENSOR PRODUCTS AND THE EXTERIOR ALGEBRA

PATRICK BROSNAN

1. INTRODUCTION

These are a set of notes which attempt to introduce tensor products and exterior algebras in the abstract way: using the various universal properties. They are meant to provide details to the first chapter in the book by Darling on differential forms [Dar94].

Here is a list of other texts that deal with the exterior algebra in one way or another:

(i) Rudin. *Principles of Mathematical Analysis* [Rud76]. Constructs the exterior algebra in a completely *ad hoc* (and elementary) way, and uses it to prove Stokes' theorem.

(ii) Warner. *Foundations of Differentiable Manifolds and Lie Groups* [War83]. Excellent text in my opinion. The part on the tensor algebra is similar to what is in these notes.

(iii) Bourbaki. *Algebra. 1-3* [Bou98]. Similar to Warner but more advanced. My proof of Theorem 55 is modeled on the proof in Bourbaki. One nice thing about this treatment (which you hopefully see from the consequences of Theorem 55) is that it is independent of the existence of determinants. Actually, the exterior algebra is really just a more sophisticated way of packaging the theory of determinants (and minors). Theorem 55 makes it possible to prove that determinants exist without knowing it *a priori*.

On the other hand, Bourbaki is part of a long encyclopedic text which aims to do everything in great generality. So that can make it unsuitable for reading as a textbook.

There is also a book by Spivak called *Calculus on Manifolds*, which covers differential forms and Stokes' theorem [Spi65]. There are a lot of nice things about that book, and I have to admit that I basically learned Stokes' theorem from it along with Rudin's book. However, there is a major mistake in Spivak's text: he never puts any differentiability assumption on chains. The problem is really on page 101 in the second display where he nonsensically defines what the integral of a differential form is over a singular chain. Also Spivak's book works with the exterior algebra in a different way from the other texts listed above (including [Dar94]). In essence, he defines only the dual of the exterior algebra and not the exterior algebra itself. For these, reasons I don't really recommend Spivak.

2. REVIEW

Here I'll review some things from linear algebra and algebra: Math 405 and 403 at UMD. All vector spaces with will real vector spaces in these notes.

2.1. **Maps of sets.** Suppose S and T are sets. I write $\mathrm{Maps}(S,T) := \{f : S \to T : f \text{ is a function}\}$. In other words, $\mathrm{Maps}(S,T)$ is just the set of all functions (or maps) from S to T. Sometimes we write T^S for $\mathrm{Maps}(S,T)$ for short. This is because, if S and T are finite, then the cardinality of $\mathrm{Maps}(S,T)$ is $|\mathrm{Maps}(S,T) = |T|^{|S|}$.

2.2. **Vector space structure.** Suppose now that X is a vector space over $\mathbb{R}$. Then $\mathrm{Maps}(S,X)$ has the structure of a vector space as follows:

2.2.1. *Addition.* Given $f, g \in \text{Maps}(S, X)$, we define

$$(f+g)(s) := f(s) + g(s), \text{ for all } s \in S.$$

2.2.2. *Scalar Multiplication.* Given $f \in \text{Maps}(S, X)$ and $x \in \mathbb{R}$, we define

$$(xf)(s) = xf(x), \text{ for all } s \in S.$$

It is not hard (but just slightly tedious) to verify that, with this structure, $\text{Maps}(S, X)$ is a vector space. The 0 element is the 0-function. That is the function 0 with $0(s) = 0$ for all $s \in S$. I'll skip this.

2.3. **Linear tranformations.** If V and W are vector spaces, then I write $\text{Lin}(V, W)$ for the set of all linear transformations from V to W. It is not hard to see that $\text{Lin}(V, W)$ is a subspace of $\text{Maps}(V, W)$.

2.4. **Free vector space.** Suppose S is a set. We write

$$\text{Free}\, S = \{f \in \mathbb{R}^S : f(s) = 0 \text{ for all but finitely many } s \in S.\}$$

For short, we sometimes write $\mathbb{R}[S]$ for $\text{Free}\, S$. It is easy to see that $\text{Free}\, S$ is a subspace of $\text{Maps}(S, \mathbb{R})$.

For each $s \in S$, write δ_s for the function in $\mathbb{R}^S$ given by

$$\delta_s(t) = \begin{cases} 1, & s = t; \\ 0, & s \neq t. \end{cases}$$

Obviously, $\delta_s \in \text{Free}\, S$.

Claim 1. *The set $\{\delta_s\}_{s \in S}$ is a basis of* $\text{Free}\, S$. *In other words, every $f \in$* $\text{Free}\, S$ *can be written uniquely as $\sum_{s \in S} a_s \delta_s$ for some $a_s \in \mathbb{R}$.*

Proof. Suppose $f \in \text{Free}\, S$. Then the sum $\sum_{s \in S} f(s)\delta_s$ has only finitely many terms. So the sum makes sense. Moreover, it is easy to $f = \sum_{s \in S} f(s)\delta_s$ and that this presentation of f as a linear combination the δ_s's is unique. □

We write $\delta : S \to \text{Free}\, S$ for the map $s \mapsto \delta_s$.

Theorem 2 (Universal Property of $\text{Free}\, S$)**.** *Suppose X is a vector space. For $T \in$* $\text{Maps}(\text{Free}\, S, X)$, *write $\delta^* T : S \to X$ for the map $(\delta^* T)(s) = T(\delta_s)$. Then the map*

$$\text{Lin}(\text{Free}\, S, X) \xrightarrow{\delta^*} \text{Maps}(S, X) \textit{ given by}$$
$$T \mapsto \delta^* T$$

is an isomorphism of vector spaces.

Proof. It's very easy to check that δ^* is a linear transformation of vector spaces. So we leave that to the reader.

To see see that δ^* is 1-1, note that $\delta^* T = 0 \Rightarrow T(\delta_s) = 0$ for all $s \in S$. But that implies that $T(f) = 0$ for all $f \in \text{Free}\, S$ since δ_s is a basis of $\text{Free}\, S$. So $T = 0$.

To see that δ^* is onto, suppose $Q : S \to X$ is a map. For $f \in \text{Free}\, S$, set $T(f) = \sum_{\{s : f(s) \neq 0\}} f(s)Q(s)$. This defines a linear transformation $T \in \text{Lin}(\text{Free}\, S, X)$, and it is clear that $T(\delta_s) = Q(s)$ for all $s \in S$. So $(\delta^* T)(s) = T(\delta_s) = Q(s)$ for all $s \in S$. Therefore, $\delta^* T = Q$. □

Remark 3. Since the map $\delta : S \to \text{Free}\, S$ given by $s \mapsto \delta_s$ in clearly one-one, we sometimes abuse notation and identify each element $s \in S$ with the corresponding element δ_s of $\text{Free}\, S$. In this way, we can think of S as a subset of $\text{Free}\, S$. Then $\text{Free}\, S$ is a real vector space with basis S.

2.5. **Surjective linear maps.** Suppose $T : V \to W$ is a map of vector spaces and X is another vector space. We have a map

$$T^* : \mathrm{Lin}(W, X) \to \mathrm{Lin}(V, X) \text{ given by}$$
$$S \mapsto S \circ T.$$

Claim 4. *The map T^* defined above is a linear transformation.*

Proof. Obvious. Skip it. □

Claim 5. *Suppose $T : V \to W$ is onto. Then $T^* : \mathrm{Lin}(W, X) \to \mathrm{Lin}(V, X)$ in one-one for any vector space X.*

Proof. Suppose $S \in \ker T^*$ and that T is onto. Then $S \circ T = 0$ So $S(T(v)) = 0$ for all $v \in V$. But T is onto, so this implies that $S(w) = 0$ for all $w \in W$. Therefore $S = 0$. □

Remark 6. The converse to Claim 4 also holds, but I won't use it so I won't prove it. (For infinite dimensional vector spaces, I believe it requires the axiom of choice.)

Theorem 7. *Suppose $T \in \mathrm{Lin}(V, W)$ is a linear transformation of vector spaces which is onto with kernel K, and suppose that X is another vector space. Then T^* is one-one with*

$$\mathrm{Im}(T^* : \mathrm{Lin}(W, X) \to \mathrm{Lin}(V, X)) = \{A \in \mathrm{Lin}(V, X) : K \subset \ker A\}.$$

In particular, for every $A \in \mathrm{Lin}(V, X)$ containing K in its kernel, there exists a unique $B \in L(W, X)$ such that $A = B \circ T$.

Proof. We've already seen that T^* is one-one. Suppose $A \in \mathrm{Im}\, T^*$. Then there exists $B \in \mathrm{Lin}(W, X)$ such that $B = A \circ T$. But then, for $k \in K$, $Ak = B(T(k)) = B(0) = 0$. So $K \subset \ker A$.

On the other hand, suppose $A \in \mathrm{Lin}(V, X)$ with $K \subset \ker A$. Write

$$U := \{(Tv, Av) \in W \times X : v \in V\}.$$

It is easy to see that U is a subspace of $W \times X$ as it is just the image of the map $V \to W \times X$ given by $v \mapsto (Tv, Av)$.

I claim that for each $w \in W$ there is a unique $x \in X$ such that $(w, x) \in U$. For uniqueness, suppose $(w, x), (w, x') \in U$. Then there exists $v, v' \in V$ such that $(w, x) = (Tv, Av), (w, x') = (Tv', Av')$. So $w = Tv = Tv'$. Therefore, $v' - v \in \ker T = K$. So $v' - v \in \ker A$ also by our hypotheses. Therefore $x = Av = Av' = x'$.

To show existence, note that T is onto. So, for every $w \in W$, there exists a $v \in V$ with $Tv = w$. Then $(w, Av) = (Tv, Av) \in U$.

Now let $B : W \to X$ be the map sending $w \in W$ to the unique $x \in X$ with $(w, x) \in U$. It is easy to check that B is linear, and it is also more or less obvious that $B \circ T = A$. □

2.6. **Quotients.** Suppose V is a vector space. If S and T are subsets of V, then we write $S + T := \{s + t : s \in S, t \in T\}$. Similarly, for $v \in V$ and $T \subset V$, we write $v + T$ for $\{v\} + T$ when no confusion can arise.

If K is a subspace of V, then a *coset* of K in V is a subset of V of the form $v + K$ with $v \in V$. Sometimes we write $[v]$ for $v + K$. We write

$$V/K := \{[v]\}_{v \in V}$$

for the set of cosets of K. We have a map $\pi_K : V \to V/K$ given by $\pi_K(v) = [v]$. This map is clearly onto. Sometimes I'll write π for π_K when K is fixed.

Proposition 8. *With the above notation, we have $[v] = [w] \Leftrightarrow v - w \in K$.*

Proof. It's easy to see that $v + K = w + K \Leftrightarrow (v - w) + K = K \Leftrightarrow v - w \in K$. □

Theorem 9. *There exists a unique structure of a real vector space on V/K such that the map $\pi : V \to V/K$ is a linear transformation. Moreover, this linear transformation is onto with kernel K.*

Sketch. (Uniqueness) Suppose $\pi : V \to V/K$ is a linear transformation. Then, for $[v], [w] \in V/K$ we have $[v] + [w] = \pi(v) + \pi(w) = \pi(v + w) = [v + w]$. And, for $x \in \mathbb{R}$, we have $x[v] = x\pi(v) = \pi(xv) = [xv]$.

(Existence) We want to define addition and scalar multiplication on V/K by the formulas above in the proof of uniqueness. That is, for $[v], [w] \in V/K$ and $x \in \mathbb{R}$, we want to define

$$[v] + [w] = [v + w];$$
$$x[v] = [xv].$$

The thing to check though is that they are well defined. In other words, we have to show that, if $[v] = [v']$ and $[w] = [w']$, then $[xv] = [xv']$ and $[v + w] = [v' + w']$. This is easy, though: to check the first statement, note that $[v] = [v'] \Rightarrow v' - v \in K \Rightarrow xv' - xv \in K \Rightarrow [xv] = [xv']$. And, similarly, if $[v] = [v']$ and $[w] = [w']$, then $v' - v, w' - w \in K$. So $v' + w' - v - w \in K$. And therefore, $[v + w] = [v' + w']$.

(Vector space Axioms) Now that addition and scalar multiplication are defined, we have to prove that the vector space axioms hold. This is all easy and I'll leave it to the reader. One thing to note that is that the 0 element of V/K is $0 = [0] = 0 + K = k + K = [k]$ for any $k \in K$. From that it's clear that the kernel of π is exactly K. □

2.7. **Duality.** If V is a vector space, we write $V^* = \mathrm{Lin}(V, \mathbb{R})$. Then if $T \in \mathrm{Lin}(V, W)$ we get a map $T^* : W^* \to V^*$ as in §4. The vector space V^* is called the *dual* of V. The map T^* is called the *transpose* or sometimes the *adjoint* of T.

If V has a finite basis $\{v_i\}_{i=1}^n$ then V^* has a particular finite basis called the *dual basis* $\{\lambda_i\}_{i=1}^n$. This basis has the property that $\lambda_j(v_i) = \delta_{ij}$ where δ_{ij} is the Kronecker δ:

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & \text{else}. \end{cases}$$

Sometimes people write v_i^* for λ_i, but this is slightly dangerous notation because v_i^* depends not just on v_i but on the whole basis $\{v_i\}$. Still, it's convenient so everyone does it.

Also, some people write (λ, v) for $\lambda(v)$ when $\lambda \in V^*$ and $v \in V$. In this notation, the transpose of a linear operator $T : V \to W$ satisfies the formula

$$(\lambda, Tv) = (T^*\lambda, v)$$

for $\lambda \in W^*$ and $v \in V$.

2.8. **Double duality.** There is a very nice map $c_V : V \to V^{**}$ defined as follows. For $v \in V$ and $\lambda \in V^*$, $c(v)(\lambda) = \lambda(v)$. If $T : V \to W$ is a linear transformation, then we set $c(T) = T^{**}$. It is then easy to see that $c(\mathrm{id}_V) = \mathrm{id}_{V^{**}}$ and that $c(ST) = c(S)c(T)$ when ST makes sense.

Proposition 10. *Suppose* $\dim V < \infty$. *Then $c_V : V \to V^{**}$ is an isomorphism.*

Sketch. Take a basis $\{v_i\}_{i=1}^n$ for V and a dual basis $\{\lambda_i\}_{i=1}^n$. Then not too hard to easy to see that $\{c(v_i)\}$ is the dual basis of $\{\lambda_i\}$. (This is in most linear algebra books used in 405. For example Hoffman-Kunze or Friedberg-Insel-Spence.) □

Remark 11. Often people tacitly identify V with V^{**} using the isomorphism c_V (when V is finite dimensional). In other words, sometimes people think of them as basically the same thing. This turns out to be not too dangerous and it can also make the notation a lot easier.

3. Tensor Products

3.1. **Multilinear maps.** Suppose $V_1, \dots, V_n$ and X are all vector spaces. A map $T : V_1 \times \cdots \times V_n \to X$ is said to be *multilinear* if it is linear in each variable when the other variables are fixed. In other words, for each i, each $x, y \in \mathbb{R}$ and each set of vectors $v_i, w_i \in V_i$ we have

$$T(v_1, \dots, v_{i-1}, xv_i + yw_i, v_{i+1}, \dots, v_n) = xT(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_n) + yT(v_1, \dots, v_{i-1}, w_i, v_{i+1}, \dots, v_n).$$

We write $\mathrm{Mult}(V_1, \dots, V_n; X)$ for the set of all multilinear maps from $V_1 \times \cdots \times V_n$ to X. It is (very) easy to see that $\mathrm{Mult}(V_1, \dots, V_n; X)$ is a subspace of $\mathrm{Maps}(V_1 \times \cdots \times V_n, X)$.

In the rare cases where it is helpful to emphasize the n, we will write $\mathrm{Mult}_n(V_1, \dots, V_n; X)$ for $\mathrm{Mult}(V_1, \dots, V_n; X)$. We can also call a map in this space an *n-linear map*.

Proposition 12 (Change of target)**.** *Suppose $V_1, \dots, V_n$, X and Y are all vector spaces. Let $T \in \mathrm{Lin}(X, Y)$ and $S \in \mathrm{Mult}(V_1, \dots, V_n; X)$ be linear and bilinear maps respectively. Then $T \circ S \in \mathrm{Mult}(V_1, \dots, V_n; Y)$.*

Proof. This is very easy and left to the reader. □

Lemma 13. *In the situation of Proposition 12, if $S \in \mathrm{Mult}(V_1, \dots, V_n; X)$, then the map*

$$S^* : \mathrm{Lin}(X, Y) \to \mathrm{Mult}(V_1, \dots, V_n; Y)$$

given by $T \mapsto T \circ S$ is a linear transformation.

Proof. This is also very easy. □

Definition 14. Suppose $V_1, \dots, V_n$ are vector spaces. Set $\mathcal{U} = \mathrm{Free}\, V_1 \times \cdots \times V_n$ and let $\mathcal{R}$ be the subspace of $\mathcal{U}$ generated by the all expressions of the form

$$(v_1, \dots, v_{i-1}, xv_i + yw_i, v_{i+1}, \dots v_n) - x(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_n) \\ + y(v_1, \dots, v_{i-1}, w_i, v_{i+1}, \dots, v_n).$$

The *tensor product* $V_1 \otimes \cdots \otimes V_n$ is the quotient space $\mathcal{U}/\mathcal{R}$.

In the definition above, I've used Remark 3 to think of $V_1 \times \cdots \times V_n$ as a basis of $\mathrm{Free}\, V_1 \times \cdots \times V_n$.

The set $\mathcal{U}$ is called the set of *generators* for the tensor product and the set $\mathcal{R}$ is called the set of *relations*.

Write $\pi : \mathcal{U} \to V_1 \otimes \cdots \otimes V_n$ for the quotient map and write $v_1 \otimes \cdots \otimes v_n$ for $\pi(v_1, \dots, v_n)$ in the tensor product. Write $\rho : V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$ for the map taking $(v_1, \dots, v_n)$ to $v_1 \otimes \cdots \otimes v_n$.

Remark 15. An element of the tensor product in the image of ρ is sometimes called a *pure tensor*.

Proposition 16. *The map ρ above is in $\mathrm{Mult}(V_1, \cdots, V_n; V_1 \otimes \cdots \otimes V_n)$.*

Proof. This follows directly from the form of the relations. □

Theorem 17 (Universal property of tensor product)**.** *Suppose $V_1, \dots, V_n$ and X are vector spaces. Write*

$$\rho^* : \mathrm{Lin}(V_1 \otimes \cdots \otimes V_n, X) \to \mathrm{Mult}(V_1 \times \cdots \times V_n; X)$$

for the map taking a $T \in \mathrm{Lin}(V_1 \otimes \cdots \otimes V_n, X)$ to $T \circ \rho$. Then ρ^* is a vector space isomorphism.

Proof. This is the same as in class on Monday (Feb 4, 2019). But I'll write the proof.

We have to show that the map ρ^* is one-one and onto.

One-one. Suppose $\rho^* T = 0$. Then $T(v_1 \otimes \cdots \otimes v_n) = 0$ for all pure tensors $v_1 \otimes \cdots \otimes v_n$. But, since $\mathcal{U}$ is generated by the elements $(v_1, \dots, v_n)$, the pure tensors generate the tensor product. So this implies that $T = 0$.

Onto. Suppose $S \in \mathrm{Mult}(V_1, \dots, V_n; X)$. Since $V_1 \times \cdots \times V_n$ is a basis for $\mathcal{U}$, there is a unique map $\hat{T} : \mathcal{U} \to X$ such that $\hat{T}(v_1, \dots, v_n) = S(v_1, \dots, v_n)$ for all n-tuples $(v_1, \dots, v_n)$ with $v_i \in V_i$. But, since S is linear, it is easy to see that $\mathcal{R} \subset \ker \hat{T}$. Therefore, there exists a $T \in \mathrm{Lin}(\mathcal{U}/\mathcal{R}, X)$ such that $T(v_1 \otimes \cdots \otimes v_n) = S(v_1, \dots, v_n)$. In other words, $T = \rho^* S$. ☐

3.2. **Properties of tensor product.**

Proposition 18. *Suppose $V_1, \dots, V_n$ are vector spaces. Then there is a unique map*

$$V_1 \otimes \cdots \otimes V_n \to V_1 \otimes (V_2 \otimes \cdots (V_{n-1} \otimes V_n))$$

taking the pure tensor $v_1 \otimes \cdots \otimes v_n$ to $v_1 \otimes (v_2 \otimes \cdots (v_{n-1} \otimes v_n))$. This map is an isomorphism.

Proof. Easy using the universal property and induction. Let's skip it. ☐

Lemma 19 (Foil)**.** *Suppose $v_i \in V, w_i \in W$ for $i = 1, 2$ and $a, b, c, d \in \mathbb{R}$. Then*

$$(av_1 + bv_2) \otimes (cw_1 + dw_2) = acv_1 \otimes w_1 + adv_1 \otimes w_2 + bcv_2 \otimes w_1 + bdv_2 \otimes w_2.$$

Proof. Follows from repeatedly using the relation in $\mathcal{R}$. ☐

Proposition 20. *Suppose V and W are vector spaces with bases $\{v_i\}_{i=1}^n$ and $\{w_j\}_{j=1}^m$ respectively. Then $B = \{v_i \otimes w_j\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}$ is a basis of $V \otimes W$.*

Proof. It is easy to see (using the relations in $\mathcal{R}$ and the fact that pure tensors generate $V \otimes W$) that B generates $V \otimes W$. So we just need to show that B is linearly independent.

For this, let $\{\lambda_i\}_{i=1}^n$ and $\{\mu_j\}_{j=1}^m$ denote the dual bases of $\{v_i\}_{i=1}^n$ and $\{w_j\}_{j=1}^m$ respectively. Then, for each pair (k, ℓ) with $1 \le k \le n$ and $1 \le \ell \le m$, let $S_{kl} : V \times W \to \mathbb{R}$ denote the function $(v, w) \mapsto \lambda_k(v)\mu_\ell(w)$. It is easy to see that $S_{k\ell} \in \mathrm{Mult}(V, W; \mathbb{R})$. So, by the universal property, Theorem 17, there is a linear transformation $T_{k\ell} \in \mathrm{Lin}(V \otimes W, \mathbb{R})$ such that $T_{k\ell}(v_i \otimes w_j) = \lambda_k(v_i)\mu_\ell(w_j) = \delta_{ik}\delta_{j\ell}$. (Here δ_{ik} is the Kronecker delta: 1 for $i = k$ and 0 otherwise.)

Now, suppose $r = \sum x_{ij} v_i \otimes w_j = 0$. Then, for each pair (k, ℓ) as above, $x_{k\ell} = T_{k\ell}(r) = 0$. It follows that B is linearly independent. ☐

Here are a few corollaries.

Corollary 21. *We have* $\dim V \otimes W = (\dim V)(\dim W)$ *when* $\dim V, \dim W < \infty$. *In fact, if $V_1, \dots, V_n$ are finite dimensional vector spaces, then*

$$\dim V_1 \otimes \cdots \otimes V_n = \prod_{i=1}^n \dim V_i.$$

Proof. The first statement follows directly from Proposition 20. The second follows from the first and Proposition 18. ☐

Lemma 22. *Suppose $V_1, \dots, V_n$ and $W_1, \dots, W_n$ are vector spaces and $T_i \in \mathrm{Lin}(V_i, W_i)$. Then there is a unique linear map*

$$T_1 \otimes \cdots \otimes T_n : V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots W_n$$

sending the pure tensor $v_1 \otimes \cdots \otimes v_n$ to $T_1 v_1 \otimes \cdots T_n v_n$.

Proof. The map $V_1 \times \cdots \times V_n \to W_1 \otimes \cdots \otimes W_n$ given by $(v_1, \dots, v_n) \mapsto Tv_1 \otimes \cdots T_n v_n$ is easily seen to be in $\mathrm{Mult}(v_1, \dots, v_n; W_1 \otimes \cdots W_n)$. So the result follows from the universal property. ☐

Corollary 23 (Distributive Property). *Suppose V_1, V_2, W_1, W_2 are vector spaces. Then we have vector space isomorphisms*

$$V_1 \otimes V_2 \oplus W_1 \otimes V_2 \xrightarrow{\cong} (V_1 \oplus W_1) \otimes V_2$$
$$V_1 \otimes V_2 \oplus V_1 \otimes W_2 \xrightarrow{\cong} V_1 \otimes (V_2 \oplus W_2).$$

Sketch. To give the first map, let $A : V_1 \to V_1 \oplus W_1$ and $B : W_1 \to V_1 \oplus W_2$ denote the inclusions. Then, using Lemma 22, we get linear transformations $A \otimes \mathrm{id}_{V_2}$ and $B \otimes \mathrm{id}_{V_2}$. The first map in Corollary 23 is just $T := A \otimes \mathrm{id}_{V_2} + B \otimes \mathrm{id}_{V_2}$. If all the vector spaces in question are finite dimensional, then it is easy to check using Proposition 20 that the first map is an isomorphism. Alternatively, one can prove the Corollary without using Proposition 20 by using the universal property to define an inverse of T. (If you do this, then you get an alternate proof of Proposition 20 as a corollary.) □

3.3. **Adjunction.** Write $\mathrm{Bilin}(V, W; X)$ for $\mathrm{Mult}(V, W; X)$. Functions in $\mathrm{Bilin}(V, W; X)$ are called *bilinear maps* from $V \times W$ to X.

Suppose $T \in \mathrm{Lin}(V, \mathrm{Lin}(W, X))$. Define a map $\phi(T) : V \times W \to X$ by setting $\phi(T)(v, w) = (T(v))(w)$. It is easy to see, once you parse through all the parentheses, that $\phi(T) \in \mathrm{Bilin}(V, W; X)$. So we get a map $\phi : \mathrm{Lin}(V, \mathrm{Lin}(W, X)) \to \mathrm{Bilin}(V, W; X)$ sending T to $\phi(T)$. It is also easy to see that this map ϕ is itself a linear transformation.

Proposition 24. *The map $\phi : \mathrm{Lin}(V, \mathrm{Lin}(W, X)) \to \mathrm{Bilin}(V, W; X)$ is a vector space isomorphism.*

Proof. This is also easy. Given an $f \in \mathrm{Bilin}(V, W; X)$, define $\psi(f) \in \mathrm{Lin}(V, \mathrm{Lin}(W, X))$ by setting $(\phi(f)(v))(w) = f(v, w)$. You have to check that $\psi(f)$ actually is in $\mathrm{Lin}(V, \mathrm{Lin}(W, X))$ and that $f \mapsto \psi(f)$ actually is a linear transformation, but that is very straightforward. □

Corollary 25. *We have an isomorphism $\mathrm{Lin}(V, \mathrm{Lin}(W, X)) \cong \mathrm{Lin}(V \otimes W, X)$.*

Proof. By the universal property $\mathrm{Bilin}(V, W; X) \cong \mathrm{Lin}(V \otimes W, X)$. □

Corollary 25 is often called the *adjunction isomorphism* or just *adjunction*. Note that there is a *fixed* isomorphism: the one coming from composition ϕ with the isomorphism coming from the universal property.

Corollary 26. *Suppose that $\dim W < \infty$. Then, for any vector space V, we have an isomorphism $\mathrm{Lin}(V, W) \cong W \otimes V^*$.*

Proof. Adjuntion gives us an isomorphism $\mathrm{Lin}(V \otimes W^*, \mathbb{R}) = \mathrm{Lin}(V, \mathrm{Lin}(W^*, \mathbb{R})) = \mathrm{Lin}(V, W^{**})$. But, if $\dim W < \infty$, double-duality gives us an isomorphisms $c : W \to W^{**}$. □

4. Algebras

4.1. **$\mathbb{R}$-algebras.** An associative $\mathbb{R}$-algebra is an associative ring A equipped with a ring homomorphism $\sigma_A : \mathbb{R} \to A$ such that, for all $a \in A$ and $x \in \mathbb{R}$, $\sigma_A(x)a = a\sigma_A(x)$. (In other words, σ_A maps $\mathbb{R}$ to the center of A.)

A homomorphism from an associative $\mathbb{R}$-algebras (A, σ_A) to another associative $\mathbb{R}$-algebra (B, σ_B) is a ring homomorphism $f : A \to B$ such that $f \circ \sigma_A = \sigma_B$.

For us the only algebras we care about are associative $\mathbb{R}$-algebra. So I'll just call those "algebras" for short. I write $\mathrm{Hom}(A, B)$ or $\mathrm{Hom}_{\mathbb{R}}(A, B)$ for the set of homomorphisms from an algebra A to an algebra B.

Remark 27. Any $\mathbb{R}$-algebra A is itself a real vector space. The scalar multiplication is sends the pair $(x, a) \in \mathbb{R} \times A$ to $\sigma_A(x)a$.

Examples 28. $\mathbb{R}$, $\mathbb{C}$, the algebra $M_n(\mathbb{R})$ of $n \times n$ matrices (with real coefficients) are all examples of associative $\mathbb{R}$-algebras. The ring homomorphisms $\sigma_A : \mathbb{R} \to A$ should be obvious in each case. (It's the identity for $\mathbb{R}$, the inclusion for $\mathbb{C}$ and the map to the scalar diagonal matrices for $M_n(\mathbb{R})$.)

4.2. **Graded algebras.** A *grading* of an algebra is a decomposition $A = \oplus_{i\in\mathbb{N}} A_i$ such that $A_i A_j \subset A_{i+j}$ for all $i, j \in \mathbb{N}$. An element $a \in A$ is called *homogeneous* of degree i (with respect to the grading) if $a \in A_i$. If $a \in A$ is any element, then, by the definition of a direct sum, we can write $a = \sum a_i$ uniquely with $a_i \in A_i$. Then element a_i is called the *degree i homogeneous part of a*.

4.3. **Homogeneous ideals.** A *homogeneous ideal* in a graded algebra $A = \oplus A_i$ is a (two-sided) ideal I which is generated by homogeneous elements.

Remark 29. By *ideal* I always mean two-sided ideals. And two-sided ideals are the only ideals that will appear in these notes.

Lemma 30. *Suppose I is a homogeneous ideal. Set $I_k = I \cap A_k$. Then $I = \oplus I_K$.*

Proof. Suppose $r \in I$. Then we can find finitely many elements $a_k, b_k \in A$ and f_k homogeneous elements of I such that $r = \sum a_k f_k + \sum f_k b_k$. By decomposing the a_k and b_k into their homogenous parts, we can assume that all the a_k and b_k are homogeneous. This shows that any r in I can be written as a sum of $r = \sum r_j$ where $r_j \in I_j$. And the decomposition is unique because $A = \oplus A_j$. So $I = \oplus I_k$. □

Corollary 31. *Suppose $A = \oplus A_k$ is a graded ring and I is a homogenous ideal. Write $(A/I)_k = (A_k + I)/I \cong A_k/(A_k \cap I) = A_k/I_k$. Then $A/I = \oplus(A/I)_k$. In other words, A/I is a graded ring.*

Proof. Let's skip it. □

4.4. **Tensor algebra.** Suppose V is a vector space. Set $T^0 V = \mathbb{R}$ and, for $n > 0$, set $T^n V = V \otimes \cdots \otimes V$ where there are n copies of V tensored together. So $T^2 = V \otimes V$, etc. We have an obvious map

$$\bullet : T^n V \times T^m V \to T^{n+m} V$$

sending $(v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots w_m)$ to $v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots w_m$.

Set $TV = \oplus_{n\geq 0} T^n V$. Then, expanding $*$ out by linearity defines a product on all of TV making TV into a graded associative $\mathbb{R}$-algebra. (The structure of an $\mathbb{R}$-algebra comes from the inclusion of $T^0 V$.

For convenience in the next theorem, I'll write $i_k : T^k V \to TV$ for the inclusion. (But note that I just view $T^k V$ as a subspace of TV.) Also, I drop the $\bullet$ from the notation in writing the multiplication in TV.

Theorem 32 (Universal property of the tensor algebra)**.** *Suppose A is any (associative $\mathbb{R}$-)algebra. Let*

$$i_1^* : \mathrm{Hom}(TV, A) \to \mathrm{Lin}(V, A)$$

be the map sending an algebra homomorphism $f : TV \to A$ to the linear map $f \circ i_1$. Then i_1^* is one-one and onto.

Remark 33. One interesting consequence of Theorem 32 is that $\mathrm{Hom}(TV, A)$ always has the structure of a real vector space (since $\mathrm{Lin}(V, A)$ does).

Proof. **One-one.** Suppose $f, g : TV \to A$ are two algebra homomorphisms with $f(v) = g(v)$ for all $v \in V$. Then, for any pure tensor, $v_1 \otimes \cdots \otimes v_n \in T^n V$, we have $f(v_1 \otimes \cdots \otimes v_n) = f(v_1) \cdots f(v_n) = g(v_1) \cdots g(v_n) = g(v_1 \otimes \cdots \otimes v_n)$. But, since the pure tensors generate $T^n V$ as a vector space for each n, they also generate TV as a vector space. So $f = g$.

Onto. Suppose $S : V \to A$ is a linear transformation. The map $S_n : V^n \to A$ given by $(v_1, \dots, v_n) \mapsto S(v_1)S(v_2)\cdots S(v_n)$ is then easily seen to be multilinear — in $\mathrm{Mult}(V, V, \dots, V; A)$ where V appears n times. So it defines a map $f_n : T^n V \to A$. Expanding out by linearity gives a linear map $f : TV \to A$. It is pretty easy to check that f is an algebra homomorphism. So I leave that to the reader. Then it is obvious that $f(v) = S(v)$ for $v \in V$. In other words, $f \circ i_1 = S$. So we are done. □

Definition 34. Suppose $A = \oplus A_k$ and $B = \oplus B_k$ are graded algebras. A *graded algebra homomorphism* is an algebra homomorphism $\phi : A \to B$ such that $\phi(A_k) \subset B_k$.

Theorem 35. *Suppose $f \in \mathrm{Lin}(V, W)$ where V and W are vector spaces. Then there is a unique algebra homomorphism $T(f) : TV \to TW$ such that, for $v \in V = T^1 V$, $T(f)(v) = f(v)$. Moreover,*

- (i) *$T(f)$ is a graded algebra homomorphism. We write $T^n(f) : T^n V \to T^n W$ for the induced homomorphism on the components;*
- (ii) *$T(\mathrm{id}_V) = \mathrm{id}_{T(V)}$;*
- (iii) *If $g : W \to X$ is another linear transformation, then $T(g \circ f) = T(g) \circ T(f)$.*

Proof. Using the universal property for TV, the composition $V \xrightarrow{f} W \xrightarrow{i_1} TW$ gives rise to a unique algebra homomorphism $T(f)$ such that $T(f)(v) = f(v)$. Since $T(f)$ is an algebra homomorphism, it takes the pure tensor $v_1 \otimes \cdots \otimes v_n \in T^n V$ to the pure tensor $(fv_1) \otimes \cdots \otimes (fv_n) \in T^n W$. So, since $T^n V$ is generated as a vector space by pure tensors, $T(f)[T^n V] \subset T^n W$.

The rest is easy and I leave it to the reader. □

5. Exterior algebras

Definition 36. Suppose V is a vector space. Write I for the ideal in TV generated by the elements $v \otimes v \in T^2 V$. The *exterior algebra* or *Grassman algebra* of V is the quotient algebra $\wedge V := TV/I$. For $\alpha, \beta \in \wedge V$, we write $\alpha \wedge \beta$ for the product.

Remark 37. Since the generators of I are all in $T^2 V$, I is homogeneous. So, if we set $I_n := I \cap T^n V$, we have $I = \oplus_{n \geq 0} I_n$. Moreover, the algebra $\wedge V$ is graded: $\wedge V = \oplus \wedge^n V$ with $\wedge^n V = T^n V / I_n$.

Since I is generated (as in ideal) by elements in degree 2, we have $I_0 = I_1 = 0$. So $\wedge^n V = T^n V$ for $n < 2$. More explicitly, we have $\wedge^0 V = \mathbb{R}$ and $\wedge^1 V = V$.

Note that, since $T^n V$ is generated as a vector space by pure tensors, $\wedge^n V$ is generated by the images of pure tensors under the quotient map. In other, words it is generated by expressions of the form $v_1 \wedge \cdots \wedge v_n$. (Although it is not commonly used terminology, we might call these *pure symbols*.)

In the remark above, it is really more correct to say that $\wedge^0 V$ is isomorphic to $\mathbb{R}$ and $\wedge^1 V$ is isomorphic to V. But it makes it a lot easier to think and write about the exterior algebra if we think of these as being equalities. This is unproblematic, because the remark gives us fixed isomorphisms.

Lemma 38. *Suppose $v, w \in V$. Then, in $\wedge V$, we have*

$$v \wedge w = -(w \wedge v).$$

Proof. Since $u \wedge u = 0$ for any $u \in V$, we have

$$\begin{aligned} v \wedge w &= v \wedge w + (v - w) \wedge (v - w) \\ &= v \wedge w + v \wedge v - v \wedge w - w \wedge v + w \wedge w \\ &= -w \wedge v. \end{aligned}$$

□

Corollary 39. *Suppose $v_1, \dots, v_n \in V$. Then*

(i) *if $v_i = v_j$ for any $i \neq j$, then $v_1 \wedge \cdots \wedge v_n = 0$;*
(ii) *if $\sigma \in S_n$ is a permutation, then*

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \operatorname{sign}(\sigma) v_1 \wedge \cdots \wedge v_n.$$

Here $\operatorname{sign}(\sigma)$ denotes the sign of the permutation σ.

Proof. Exercise in repeatedly applying Lemma 38. Let's skip it. □

As for the tensor product, there are two universal properties for the exterior product: one for $\wedge^n V$ and another for the algebra $\wedge V$. To explain the one for $\wedge^n V$ we need a definition.

Definition 40. Suppose V and X are vectors spaces. We write $\mathrm{Alt}_n(V, X)$ for the subspace of $\mathrm{Mult}_n(V^n; X)$ consisting of n-linear maps $f : V^n \to X$ such $f(v_1, \dots, v_n) = 0$ if there exists an $i < n$ with $v_i = v_{i+1}$.

We have the following Proposition which is analogous to Corollary 39.

Proposition 41. *Suppose $f \in \mathrm{Alt}_n(V, W)$ and $v_1, \dots, v_n \in V$.*

(i) *We have*

$$f(v_1, \dots, v_{i-1}, v_i, v_{i+1}, v_{i+2}, \dots, v_n) = -f(v_1, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \dots, v_n).$$

(ii) *If $v_i = v_j$ for any $i \neq j$, then $f(v_1, \dots, v_n) = 0$.*
(iii) *If $\sigma \in S_n$, then*

$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \operatorname{sign}(\sigma) f(v_1, \dots, v_n).$$

Proof. We just prove (i), the rest is analogous to the proof of Corollary 39. I.e., the rest is an exercise in using (i) and the definition of an alternating map.

$$\begin{aligned} f(v_1, \dots, v_n) &= f(v_1, \dots, v_n) + f(v_1, \dots, v_{i-1}, v_{i+1} - v_i, v_{i+1} - v_i, v_{i+2}, \dots, v_n) \\ &= f(v_1, \dots, v_n) - f(v_1, \dots, v_{i-1}, v_i, v_{i+1}, v_{i+2}, \dots, v_n) - f(v_1, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \dots, v_n) \\ &= - f(v_1, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \dots, v_n). \end{aligned}$$

□

Lemma 42. *The map $j_n : V^n \to \wedge^n V$ sending $(v_1, \dots, v_n)$ to $v_1 \wedge \cdots \wedge v_n$ is in $\mathrm{Alt}_n(V^n, \wedge^n V)$.*

Proof. If $i_n : V^n \to T^n V$ is the map to the tensor algebra from §4.4, then $j_n = \pi \circ i_n$ where $\pi : TV \to \wedge V$ is the quotient map. So j_n is in $\mathrm{Mult}_n(V^n, \wedge^n V)$ since π is linear.

And since $u \wedge u = 0$ for all $u \in V$, $j_n(v_1, \dots, v_n) = 0$ if $v_i = v_{i+1}$ for some i. □

Lemma 43. *Suppose V, X and Y are vector spaces and $T : X \to Y$ is a linear transformation. For $S \in \mathrm{Alt}_n(V, X)$, $T \circ S \in \mathrm{Alt}_n(V, Y)$. Moreover, the resulting map*

$$S^* : \mathrm{Lin}(X, Y) \to \mathrm{Alt}_n(V, Y)$$

sending T to $T \circ S$ is linear.

Proof. Same as Proposition 12 and Lemma 13. □

Theorem 44. *Suppose V and X are vector spaces. Then the map*

$$j_n^* : \mathrm{Lin}(\wedge^n V, X) \to \mathrm{Alt}_n(V, X)$$

given by $T \mapsto T \circ j_n$ is an isomorphism.

Proof. The map j_n^* is one-one because $\wedge^n$ is generated by pure symbols.

To see that j_n^* is onto, suppose $f \in \mathrm{Alt}_n(V, X)$. We have $\mathrm{Alt}_n(V, X) \subset \mathrm{Mult}_n(V^n, X)$. So, by the universal property of tensor products, we can find a map $F : T^n V \to X$ such that $F(v_1 \otimes \cdots \otimes v_n) = f(v_1, \ldots, v_n)$. But, from the fact that f is alternating, it follows that $F(v_1 \otimes \cdots \otimes v_n) = 0$ if $v_i = v_{i+1}$ for some $i < n$. But this easily implies that $I_n \subset \ker F$. So, writing $\pi : T^n V \to \wedge^n V$ for the quotient map, we see that there exists a map $S : \wedge^n V \to X$ such that $F = S \circ \pi$. But then $S \circ j_n = f$. □

We now move to the universal property for algebras.

Lemma 45. *Suppose $\alpha \in \wedge^n V$ and $\beta \in \wedge^m V$. Then $\alpha \wedge \beta = (-1)^{nm} \beta \wedge \alpha$.*

Proof. It is not hard to see that it suffices to prove the Lemma for pure symbols α and β (using the fact that the pure symbols generate $\wedge V$ as a vector space). But for pure symbols, the Lemma follows by repeatedly applying Lemma 38. □

Theorem 46. *Suppose V is a vector space, and A is an algebra. Let*

$$j_1^* : \mathrm{Hom}(\wedge V, A) \to \mathrm{Lin}(V, A)$$

be the map sending sending an algebra homomorphism f to the linear map $f \circ j_1$. Then j_1^* is one-one and its image is exactly the set of linear transformations $S : V \to A$ such that $S(v)^2 = 0$ for all $v \in V$.

Proof. The algebra $\wedge V$ is generated as an algebra over $\mathbb{R}$ by $\wedge^1 V = V$. In other words, every element of $\wedge V$ is a (finite) sum of (finite) products of elements of V. From this, it follows that j_1^* is one-one.

To see that j_1^* has the image claimed, suppose $S \in \mathrm{Lin}(V, A)$ and suppose $S(v)^2 = 0$ for all $v \in V$. Using the universal property of tensor algebras, we see that there exists a (unique) $f \in \mathrm{Hom}(TV, A)$ such that $f(v) = S(v)$ for all $v \in V$. In particular, $f(v)^2 = f(v \otimes v) = 0$ for all $v \in V$. Since f is an algebra homomorphism, this implies that $I \subset \ker f$. So there is a (unique) algebra homorphism $g : \wedge V \to A$ such that $g \circ \pi = f$ (where $\pi : TV \to \wedge V$ is the quotient map). But then we're done. □

The next Proposition is analogous to Theorem 35

Proposition 47. *Suppose $f \in \mathrm{Lin}(V, W)$ where V and W are vector spaces. Then there is a unique algebra homomorphism $\wedge(S) : \wedge V \to \wedge W$ such that $\wedge(f)(v) = f(v)$ for $v \in \wedge^1 V = V$. Moreover,*

(i) *$\wedge(f)$ is a graded algebra homomorphism. We write $\wedge^n(f) : \wedge^n V \to \wedge^n W$ for the induced homomorphism on the components;*
(ii) *$\wedge(\mathrm{id}_V) = \mathrm{id}_{\wedge(V)}$;*
(iii) *If $g : W \to X$ is another linear transformation, then $\wedge(g \circ f) = \wedge(g) \circ \wedge(f)$.*

Proof. The proof is also analogous to the proof of Theorem 35 and I leave it to the reader. □

Example 48. Suppose we V is a one-dimensional vector space with basis $B = \{v\}$. Then $\wedge^p V = 0$ for all $p > 1$. To prove this, note that any pure symbol is of the form $(x_1 v) \wedge \cdots (x_p v) = (\prod_{i=1}^p x_i) v \wedge \cdots \wedge v$ and this is 0 for $p > 1$ since $v \wedge v = 0$. Since the pure symbols generate $\wedge^p V$ as a vector space, it follows that $\wedge^p V = 0$.

On the other hand, we have $\wedge^0 V = \mathbb{R}$ and $\wedge^1 V = V$. So we know $\wedge V$ completely.

6. Tensors, sums and bases of exterior algebras

Suppose V and W are two vector spaces. For $a, b, c, d \in \mathbb{N}$, let's define a map

$$\bullet : (\wedge^a V \otimes \wedge^b W) \times (\wedge^c V \otimes \wedge^d W) \to \wedge^{a+c} V \otimes \wedge^{b+d} W$$

by sending $(\alpha \otimes \beta, \gamma \otimes \delta)$ to $(-1)^{bc}(\alpha \wedge \gamma) \otimes (\beta \wedge \delta)$. If we expand out by linearity, we get a map

$$\bullet : (\wedge V \otimes \wedge W) \times (\wedge V \otimes \wedge W) \to \wedge V \otimes \wedge W.$$

Proposition 49. *The product $\bullet$ defined above makes $(\wedge V) \otimes (\wedge W)$ into an associative algebra $C := C(V, W)$. If we set $C_k = \oplus_{p+q=k} \wedge^p V \otimes \wedge^q W$ then $C = \oplus C_k$ is graded.*

Proof. Associativity seems to be the main thing to check. So I leave it to the reader. □

Remark 50. The definition of the product $\bullet$ on C is similar to what Rudin does to give an *ad hoc* definition of the exterior algebra.

Definition 51. A graded algebra $A = \oplus A_k$ is said to be *graded commutative* if, for $a \in A_p, b \in A_q$, $ab = (-1)^{pq} ba$.

It follows from Lemma 45 that $\wedge V$ is graded commutative, and it is not hard to see using the definition above that $C = \wedge V \otimes \wedge W$ is as well (using the product $\bullet$).

Lemma 52. *Suppose A is a graded commutative (real) algebra, and k is an odd number. Then, for $a \in A_k$, we have $a^2 = 0$.*

Proof. $a^2 = (-1)^{k^2} a^2 = -a^2$. So $2a^2 = 0$. But that implies that $a^2 = 0$. □

I want to show that, in fact, C is isomorphic as an algebra to $\wedge(V \oplus W)$. For this, I need maps back and forth.

Using Proposition 47, the inclusions $i_V : V \to V \oplus W$ and $i_W : W \to V \oplus W$ give us maps $\wedge(i_v) : \wedge V \to \wedge(V \oplus W)$ and $\wedge(i_W) : \wedge W \to \wedge(V \oplus W)$. Then we define a map

$$\bar{f} : \wedge V \times \wedge W \to \wedge(V \oplus W)$$

by setting $\bar{f}(\alpha, \beta) = (\wedge(i_V)(\alpha)) \wedge (\wedge(i_W)(\beta))$. This map is clearly bilinear, so, by the universal property of tensor products, it gives a map

$$f : \wedge V \otimes \wedge W \to \wedge(V \oplus W).$$

If we view V and W as being contained in the factors of $V \oplus W$, we can see the map f is an easier way: we just have $f(\alpha \otimes \beta) = \alpha \wedge \beta$ for $\alpha \in \wedge V$ and $\beta \in \wedge W$.

Proposition 53. *The map $f : \wedge V \otimes \wedge W \to \wedge(V \otimes W)$ is an algebra homomorphism.*

Proof. This is more or less obvious from the description of f given above (although it is slightly annoying to write down in formulas). □

Now we want to come up with a homomorphism back the other way. This is slightly easier. The degree 1 part of $C = \wedge V \otimes \wedge W$ is just $V \otimes \mathbb{R} \oplus \mathbb{R} \otimes W$. So it is isomorphic to $V \oplus W$ in an obvious way (and we will just identify it with $V \oplus \mathbb{R}$. To be super explicit, the map $V \to V \otimes \mathbb{R}$ sending v to $v \otimes 1$ is an isomorphism and similarly the map $w \to 1 \otimes w$ is an isomorphism. Se get a map $\bar{g} : V \oplus W \to C_1 \subset C$ sending (v, w) to $v \otimes 1 + 1 \otimes w$.

Lemma 54. *We have $\bar{g}(v, w)^2 = 0$ for all $(v, w) \in V \oplus W$.*

Proof. Since $\bar{g}(v, w) \in C_1$, this follows from Lemma 52. □

Now, from the universal property of the exterior algebra and Lemma 54, it follows that we get a (unique) algebra homomorphism

$$g : \wedge(V \oplus W) \to C = (\wedge V) \otimes (\wedge W)$$

such that $g(v,w) = \bar{g}(v,w)$ for $(v,w) \in V \oplus W$.

Theorem 55. *The maps f and g above are inverse algebra homomorphisms. Consequently, we have an isomrophism of algebras*

$$\wedge(V \oplus W) \cong (\wedge V) \otimes (\wedge W).$$

Proof. The trick to prove this is to compose the maps and apply the universal properties to see that the compositions are the identity. I'll skip it. □

Corollary 56. *Suppose V has basis $v_1, \ldots, v_n$. For each $I = (i_1, \ldots, i_p)$ with $1 \le i_1 < i_2 < \cdots < i_p \le n$, write $v_I = v_{i_1} \wedge \cdots \wedge v_{i_p}$. Then the v_I as I ranges over all subsets of $\{1, \ldots, n\}$ form a basis for $\wedge V$. Consequently,* $\dim \wedge V = 2^n$ *and* $\dim \wedge^p V = \binom{n}{p}$.

Proof. Let's first prove, by induction on $\dim V$, that $\dim \wedge V = 2^{\dim V}$. We know this when $\dim V = 1$ and it's obvious when $\dim V = 0$. So assume $\dim V > 1$. Then, set $A = \langle v_1, \ldots, v_{n-1} \rangle$ and $B = \langle v_n \rangle$. By induction, $\dim \wedge A = 2^{n-1}$ and $\dim \wedge B = 2$. So, since $V = A \oplus B$,

$$\begin{aligned} \dim \wedge V &= \dim((\wedge A) \otimes (\wedge B)) \\ &= (\dim \wedge A)(\dim \wedge B) = 2^{n-1} \cdot 2 = 2^n. \end{aligned}$$

□

Definition 57. A p-tuple $I = (i_1, \ldots, i_p)$ with $1 \le i_1 < \cdots < i_p \le n$ as above is called a *multi-index* of cardinality p. We write $|I| = p$.

Corollary 58. *Suppose V is an n-dimensional vector space and $f \in \operatorname{End} V$ is a linear transformation from V to itself. Then there is a unique real number $\delta(f)$ such that the map $\wedge^n(f) : \wedge^n(V) \to \wedge^n(V)$ is multiplication by $\delta(f)$. We have $\delta(\mathrm{id}_V) = 1$, and, if g is another element of $\operatorname{End} V$, then $\delta(fg) = \delta(f)\delta(g)$.*

Proof. Since V is n-dimensional, $\wedge^n V$ is 1-dimensional (by Corollary 56). Therefore any map from $\wedge^n V$ to itself is given by multiplication by an (obviously unique) scalar. This proves the existence (and uniqueness) of $\delta(f)$. The rest follows from Proposition 47. □

Theorem 59. *We have (in the context of Corollary 58),*

$$\delta(f) = \det f$$

for $f \in \operatorname{End} V$.

Proof. The scalar $\det : \operatorname{End} V \to \mathbb{R}$ is characterized by two properties

(i) $\det \mathrm{id}_V = 1$;
(ii) det is an alternating multi-linear function of the columns of f (in the matrix representation with respect to a basis of V).

The Property (i) holds by Corollary 58. Property (ii) is easy since the map $V^n \to \wedge^n V$ sending $(v_1, \ldots, v_n)$ to $v_1 \wedge \cdots \wedge v_n$ is multi-linear and alternating. It then follows that, if $v_1, \ldots, v_n$ is an ordered basis of V, then the function taking a linear transformation f to $(fv_1) \wedge \cdots \wedge (fv_n)$ is an alternating, multilinear function of the columns. From that, it follows easily that $\det f = \delta(f)$. □

Proposition 60. *There is a unique linear transformation* $\langle\, ,\, \rangle : \wedge^p V^* \otimes \wedge^p V \to \mathbb{R}$ *sending the tensor* $(\lambda_1 \wedge \cdots \wedge \lambda_p) \otimes (v_1 \wedge \cdots \wedge v_n)$ *to* $[\det \lambda_i(v_j)]_{i,j=1}^p$. *This pairing induces an isomorphism*

$$\phi : \wedge^p(V^*) \to (\wedge^p V)^*.$$

If $\{v_i\}_{i=1}^n$ *is a basis of* V *and* $\{\lambda_i\}$ *is the dual basis then, for* I *and* J *multi-indices of cardinality* p, *we have* $\langle \lambda_I, v_J \rangle = \delta_{IJ}$. *(Here* δ_{IJ} *is* 0 *if* $I \neq J$ *and* 1 *otherwise.)*

Proof. To get the pairing, it suffices (by adjunction), to get a linear map $\phi : \wedge^p V^* \to \mathrm{Lin}(\wedge^p V, \mathbb{R}) = (\wedge^p V)^*$. This we have to do in steps using the various universal properties.

First, suppose $(\lambda_1, \ldots, \lambda_p) \in (V^*)^p$. Define a map $\tilde{\phi}(\lambda_1, \ldots, \lambda_p) : V^p \to \mathbb{R}$ by settting

$$\tilde{\phi}(\lambda_1, \ldots, \lambda_p)(v_1, \ldots v_p) = \det[\lambda_i(v_j)]_{i,j=1}^p.$$

It is easy to see that, holding the λ's fixed, the above expression is alternating and multilinear in the $v_i's$. So, using the universal property for V, we see that $\tilde{\phi}$ gives rise to a map $\hat{\phi}(\lambda_1, \ldots, \lambda_p) \in \mathrm{Lin}(\wedge^p V, \mathbb{R})$.

This then gives us a map $\hat{\phi} : (V^*)^p \to \mathrm{Lin}(\wedge^p V, \mathbb{R})$. It is then easy to see that this map $\hat{\phi}$ is itself alternating and mutltilinear. So, now using the universal property for V^*, we get a map $\phi : \wedge^p V^* \to (\wedge^p V)^*$.

We have $\phi(\lambda_I)(v_J) = \det[\lambda_{i_s}, v_{j_t}])_{s,t=1}^p$ where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_p)$. So, it is not hard to see that $\langle \lambda_I, v_J \rangle = \delta_{IJ}$. This proves that ϕ is an isomorphism. $\square$

Remark 61. When V is finite dimensional, people use this isomorphism ϕ to think of $\wedge^p V^*$ and $(\wedge^p V)^*$ as basically the same thing. This is analogous to the situation with double-duals.

7. Bilinear forms, orientations and the star operator

7.1. **Inner products.** Fix a positive definite inner product $\langle\, ,\, \rangle : V \times V \to \mathbb{R}$ on a finite dimensional vector space V. This is a symmetric bilinear map such that, for each $v \in V$,

$$\|v\|^2 := \langle v, v \rangle \geq 0$$

with strict inequality for all nonzero v. Here *symmetric* means that $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$. As $\langle\, ,\, \rangle$ is a bilinear form, it gives rise (via adjuntion) to a linear transformation $\gamma : V \to V^*$, and it is not hard to see that the positive definiteness implies that γ is an isomorphism.

A good example of a positive definite inner product is just the standard dot product on $\mathbb{R}^n$. In this case, if $\{e_i\}$ is the standard basis and $\{e_i^*\}$ is the dual basis, then we just have $\gamma(e_i) = e_i^*$. Hoffman and Kunze is a decent place to read about positive definite inner products.

So on the exterior algebra side, if we use γ to identify V and V^*, Proposition 60 gives us a linear transformation

$$\langle\, ,\, \rangle : \wedge^p V \otimes \wedge^p V \to \mathbb{R}. \tag{62}$$

Fix an orthonormal basis $\{e_i\}$ for V. (Using Graham-Schmidt, every V has such a basis.) Then it is not hard to see that $\langle e_I, e_J \rangle = \delta_{IJ}$ (where here I and J are multi-indexes. It follows that the pairing (62) is itself a positive definite inner product with orthonormal basis e_I.

7.2. **Orientations.** Now suppose $\dim V = n$. Then $\dim \wedge^n V = 1$. So, under the inner product (62) there are exactly two vectors $\alpha \in \wedge^n V$ with $\|\alpha\| = 1$. If we have a orthonormal basis $e_1, \ldots, e_n$ there are exactly $\pm e_1 \wedge \cdots \wedge e_n$. The choice of one of these vectors Ω of norm 1 is called an *orientation* of V. By reordering our orthonormal basis $\{e_i\}$, we can always arrange that $\Omega = e_1 \wedge \cdots \wedge e_n$. In fact, we call an (ordered) orthonormal basis $v_1, \ldots, v_n$ an *oriented basis* if $v_1 \wedge \cdots \wedge v_n = \Omega$.

The choice of an orientation gives rise to a unique isomorphism $\omega : \wedge^n V \to \mathbb{R}$ taking Ω to 1. We fix this isomorphism. Using it we get a composition

$$\wedge^p V \otimes \wedge^{n-p} V \to \wedge^n V \xrightarrow{\omega} \mathbb{R} \tag{63}$$

where the first map is induced (using the universal property of tensor products) from the (obviously bilinear) wedge product.

Theorem 64. *For each $\lambda \in \wedge^p V$, there exists a unique $*\lambda \in \wedge^{n-p} V$ such that, for any $\mu \in \wedge^{n-p} V$,*

$$\lambda \wedge \mu = \langle *\lambda, \mu \rangle \Omega. \tag{65}$$

Moreover, the resulting map $* : \wedge^p V \to \wedge^{n-p} V$ is linear.

Sketch. This essentially follows from the fact that (62) is positive definite (and thus non-degenerate). □

The operator $*$ is called the *Hodge star operator*. Except for the matter of signs, it is easy to understand in terms of multi-indexes.

Lemma 66. *Suppose $\{e_i\}$ is an oriented orthonormal basis. For any multiindex I, write I' for the* complement. *That is the set of indices in I' is the complement in $\{1, \ldots, n\}$ of the set of indices in I. Then*

$$*e_I = \pm 1 e_{I'}.$$

Proof. Suppose $|I| = p$. Write $*e_I = \sum_{|J|=n-p} a_J e_J$ where each a_J is a real number. For K a multi-index with $|K| = n - p$, we have $e_I \wedge e_K = 0$ unless $K = I'$. So if $K \neq I'$, $a_K = \langle *e_I, e_K \rangle = 0$.

On the other hand, if $K = I'$, we have $e_I \wedge e_{I'} = \epsilon \Omega$ where $\epsilon = \pm 1$. So

$$\begin{aligned} e_I \wedge e_{I'} = \epsilon \Omega &= \langle *e_I, e_{I'} \rangle \Omega \\ &= \langle \sum a_J e_J, e_{I'} \rangle \Omega \\ &= a_{I'} \Omega. \end{aligned}$$

So $\epsilon = a_{I'}$. So it follows that $*e_I = \epsilon e_{I'}$. □

Let's record the last part of the proof.

Corollary 67. *Suppose $\{e_i\}$ is an ordered basis. For a multi-index I, let $e_I \wedge e_{I'} = \epsilon \Omega$. Then $\epsilon = \pm 1$ and*

$$*e_I = \epsilon \Omega.$$

Example 68. This makes it very easy to compute the $*$ operator. Let's do it for $\dim V = 3$. Then $\Omega = e_1 \wedge e_2 \wedge e_3$. For $I = (1)$, $I' = (2, 3)$ and $\epsilon = 1$. For $I = (2)$, $I' = (1, 3)$ and $\epsilon = -1$. And for $I = (3)$, $I' = (12)$ and $\epsilon = 1$. So $*e_1 = e_2 \wedge e_3$, $*e_2 = -e_1 \wedge e_3 = e_3 \wedge e_1$, and $*e_3 = e_1 \wedge e_2$.

References

[Bou98] Nicolas Bourbaki. *Algebra I. Chapters 1–3*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998, pages xxiv+709. ISBN: 3-540-64243-9. Translated from the French, Reprint of the 1989 English translation [MR0979982 (90d:00002)].

[Dar94] R. W. R. Darling. *Differential forms and connections*. Cambridge University Press, Cambridge, 1994, pages x+256. ISBN: 0-521-46800-0. DOI: 10.1017/CBO9780511805110. URL: https://doi.org/10.1017/CBO9780511805110.

[Rud76] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976, pages x+342. International Series in Pure and Applied Mathematics.

[Spi65] Michael Spivak. *Calculus on manifolds. A modern approach to classical theorems of advanced calculus*. W. A. Benjamin, Inc., New York-Amsterdam, 1965, pages xii+144.

[War83] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983, pages ix+272. ISBN: 0-387-90894-3. Corrected reprint of the 1971 edition.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD USA
Email address: pbrosnanmath.umd.edu