Real Analysis: Stein Chapter 1

Due on March 23, 2021 at 24:00pm

Professor Lilu Zhao Week 3

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Problem 1

Solution Let C_0 denote the original interval. Let C_k denote the set remaining after the k_{th} stage of construction. (k=1,2,3,...). Let D_k be the complement of C_k in [0,1].

Proof. We prove two statements instead.
1) We first prove: $\forall x \neq y$ s.t. $x, y \in C$, we have a point $z \notin C$ s.t. $x < z < y$. (WLOG, we assume x<y.)
If $x, y \in C$ and $x < y$, then $\exists n \in \mathbb{N}^*$, x,y$\in$ C_n and x,y are in different intervals. It follows from the construction of C that we will **dig a hole** in C_{n+1}. Then it's clear that $\exists z \notin C_{n+1}$ such that $x < z < y$, which follows that $z \notin C = \bigcap_{n=1}^{\infty} C_n$ and $x < z < y$.
2) We then prove: $\forall x \in C\ \forall k \in \mathbb{N}^*\ \exists y_k \in C$ such that $|x - y_k| < \frac{1}{3^k}$.
Let $x \in C$. Then $x \in C_k$ for all k. Hence x is in an interval of C_k. We let the closer boundary point of the interval be y_k. And it's clear that $y_k \in \bigcap_{n=0}^{\infty} C_n$, which follows that

$$|x - y_k| < \frac{1}{3^k} \to 0 \quad (k \to \infty).$$

Thus from 1) and 2) we have C is totally disconnected and has no isolated points. □

Problem 2

Let C_0 denote the original interval. Let C_k denote the set remaining after the k_{th} stage of construction. (k=1,2,3,...). Let D_k be the complement of C_k in [0,1].
(a) we will prove it by directly observing what numbers the procedure removes.

Proof. First we assert that **points in [0,1] have no more than two ternary representations (expansions) with the first terminating and the other not**. For example, we have $\frac{1}{3} = (0.0222\ldots)_3 = (0.1)_3$ and $\frac{2}{3} = (0.1222\ldots)_3 = (0.2)_3$. In problem 2 the attention is only paid to the representation we need.
The first iteration removes $(\frac{1}{3}, \frac{2}{3})$. With the ternary representation

$$\mathrm{D}_1 = [0, (0.0222\ldots)_3] \bigcup [(0.2)_3, (0.222\ldots)_3]$$

, which is exactly the points remained **existing a representation only 0 and 2 in the first place of the ternary representation**. In other words, we remove the open sets of all numbers with 1 in the first place. In the seond iteration, we leave all those points **existing a representation of only 0 and 2 in the second places of the ternary representation**. Continuing the process inductively, we leave all those ponits **existing a representation of only 0 and 2 in all places of the ternary representation**, which proves the statement. □

(b) We leave the proof in three parts.

Proof. 1) well-defined
The only possible concern is that some points may have more than one ternary representations. It's clear that such points have two representations with the first terminating and the other not.
We assume

$$x = \Sigma_{k=1}^{\infty} a_k 3^{-k} = \Sigma_{k=1}^{\infty} \hat{a_k} 3^{-k}$$

with $a_k = \hat{a_k}$ for $k \leq n$; $a_k = \hat{a_k} - 1$ for $k = n+1$; $a_k = 2$ and $\hat{a_k} = 0$ for $k > n+1$. Since $a_k = \hat{a_k} - 1$, it's clear that there's one and only one ternary expansion with 0 and 2 in all places of the ternary representation. WLOG, we assume $x = \Sigma_{k=1}^{\infty} a_k 3^{-k}$. Then

$$F(x) = F(\Sigma_{k=1}^{\infty} a_k 3^{-k}) = \Sigma_{k=1}^{\infty} \frac{a_k}{2^{k+1}}.$$

2) continuous
First, we arbitrarily give $x \in C$ and $\epsilon > 0$. For the $\epsilon > 0$ choose k such that

$$\frac{1}{2^k} < \epsilon$$

. Then we let $\delta = \frac{1}{3^k}$. It's clear that $\forall y \in \{z \in C : |z - x| < \delta\}$. x and y agree on the first k places of the permissable representation, which means that the images of x and y agree on the first k places. It follows that

$$|F(x) - F(y)| < \frac{1}{2^k} < \epsilon.$$

3) determine values on 0 and 1
We have

$$F(0) = F(\Sigma_{k=1}^{\infty} 0 * 3^{-k}) = \Sigma_{k=1}^{\infty} 0 * 2^{-k-1} = 0$$

and

$$F(1) = F(\Sigma_{k=1}^{\infty} 2 * 3^{-k}) = \Sigma_{k=1}^{\infty} 2 * 2^{-k-1} = 1.$$

□

(c) We will prove it through constuction of a preimage.

Proof. Let $x \in [0, 1]$. Choose a binary expansion of x, denoted by $x = (0, a_1 a_2 \ldots a_n)_2$. Then replacing all the 1's in the binary expansion with 2, we get a ternary expansion of a number $y \in [0, 1]$, denoted by $y = (0, \hat{a_1} \hat{a_2} \ldots \hat{a_n})_3$. Thus we have

$$F(y) = \Sigma_{k=1}^{\infty} \frac{\hat{a_k}}{2^{k+1}} = \Sigma_{k=1}^{\infty} \frac{a_k}{2^k} = x.$$

□

(d) We will prove it with the definition of continuity.

Proof. Note that F is increasing on its domain C. Let $G(x) = sup\{F(y) : y \leq x, y \in C\}$. Then G(x)=F(x) for $x \in C$ and G(x) is increasing on [0,1]. We will then prove G is continous on [0,1]:
1) $x \in [0, 1] \bigcap C^c$.
Since C is closed, C^c is open. Then there exist $\delta > 0$ such that $O(x, \delta) \subset C^c$. Then G is constant on the interval $O(x, \delta)$. Thus we have

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall y \in O(x, \delta) \quad |G(y) - G(x)| = 0 < \epsilon.$$

2) $x \in C$.
Assume $\epsilon > 0$. Since F is continuous on C, there exists $\delta > 0$ s.t. $\forall y \in \{z \in C : |z - x| < \delta\}$ we have $|G(y) - G(x)| < \epsilon$.
Arbitrarily choose two points $z_1 \in (x - \delta, x) \bigcap C$ and $z_2 \in (x, x + \delta) \bigcap C$. Then let $\bar{\delta} = min\{x - z_1, z_2 - x\}$..
Then $\forall y \in O(x, \bar{\delta})$:
a) $y \in C$: we have $|G(y) - G(x)| < \epsilon$
b) $y \notin C$: we have either

$$G(x) - \epsilon < G(z_1) \leq G(y) < G(x)$$

or

$$G(x) < G(y) \leq G(z_2) < G(x) + \epsilon.$$

From a) and b) we have:

$$|G(y) - G(x)| < \epsilon \quad \forall y \in O(x, \bar{\delta}).$$

□

Problem 3

Solution Let C_0 denote the original interval. Let C_k denote the set remaining after the k_{th} stage of construction. (k=1,2,3,...). Let D_k be the complement of C_k in [0,1].
(a) We have

$$[0,1]\backslash C_\xi = [0,1]\backslash \bigcap_{k=1}^{\infty} C_k = \bigcup_{k=1}^{\infty}([0,1]\backslash C_k) = \bigcup_{k=1}^{\infty} D_k$$

From the procedure we know the open intervals in D_k are disjoint, thus since $\{D_k\}$ is an increasing sequence and from corollary 3.3 we have

$$m(\bigcup_{k=1}^{\infty} D_k) = lim_{k\to\infty} m(D_k) = \Sigma_{n=0}^{\infty} 2^n \xi (\frac{1-\xi}{2})^n = 1$$

(b) It's clear $C_\xi \subset C_n$ for every n. Thus for every n

$$m^*(C_\xi) < m^*(C_n) = m(C_n) = (1-\xi)^n$$

which immediately follows that $m^*(C_\xi) = 0$ i.e. C_ξ is of exterior measure 0.

Problem 4

Solution Let C_0 denote the original interval. Let C_k denote the set remaining after the k_{th} stage of construction. (k=1,2,3,...)
(a) Since [0,1]$\backslash C_k$ is a union of **disjoint** segments with total length:

$$m([0,1]\backslash C_k) = \Sigma_{j=1}^{k} 2^{j-1} * l_j$$

Thus we have

$$m(C_k) = 1 - m([0,1]\backslash C_k) = 1 - \Sigma_{j=1}^{k} 2^{j-1} * l_j$$

From Corollary 3.3 and since $\bigcap_{k=1}^{\infty} C_k = \hat{C}$, $m(C_0) = 1 < \infty$ and $C_{k+1} \subset C_k$ for every k, we have

$$m(\hat{C}) = lim_{k\to\infty} m(C_k) = 1 - \Sigma_{j=1}^{\infty} 2^{j-1} * l_j$$

which proves the formula.
(b) Let D_k denote the closed interval in C_k which contains x. Let I_{k+1} denote the open interval in $(\hat{C})^c$ such that it's centered in D_k as the construction claims.
Then it's clear that

$$m(I_k) = l_k \leq \frac{1}{2^k} \to 0 \quad (k \to \infty)$$

Arbitrarily choose a point $x_k \in I_k$, then we have

$$|x_k - x| \leq m(D_k) \to 0 \quad (k \to \infty)$$

which immediately follows that $x_k \to x$ and $I_k \to 0$ as $k \to \infty$.
(c) We will prove that every point in $\hat{C}$ is a limit point in $\hat{C}$, which follows that $\hat{C}$ is perfect.
Arbitrarily choose a point x $\in \hat{C} = \bigcap_{k=1}^{\infty} C_k$ and Let D_k denote the closed interval in C_k which contains x. Let x_k be one of the endpoints of D_k. We have

$$|x_k - x| \leq m(D_k) \to 0 \quad (k \to \infty)$$

which follows that x is a limit point of $\hat{C}$.

Problem 4 continued on next page...

Next we will show it contains no open intervals by contradiction. Assume an open interval $J \subset \hat{C}$. Let D$_k$ denote the closed interval in C$_k$ which contains J. Then

$$m(J) \leq m(D_k) \to 0(k \to \infty)$$

a contradiction!
(d) We will construct a contradiction from its **perfect** property.
Assume P=$\{x_i$: $i = 1, 2, \ldots\}$. By **a open neighbourhood of x** we mean a open ball containing x and we will define a closed sequence $\{\bar{V_n}\}$ inductively as follows:
(1) Arbitrarily choose a closed neighbourhood of x to be V_1.
(2) If $V_n \bigcap P \neq \emptyset$, there exist V_{n+1} subject to three properties: $\bar{V}_{n+1} \subset V_n$, $x_n \notin \bar{V}_{n+1}$ and $V_{n+1} \bigcap P \neq \emptyset$. (Attention: Since every point of P is a limit point, the definition is well-defined; since $V_{n+1} \bigcap P \neq \emptyset$, the definition can go down inductively.)
From (1) and (2), we have defined a closed sequence $\{\bar{V_n}\}$. Furthermore we define $K_n = \bar{V_n} \bigcap P$ such that **the intersection of finite elements of the sequence is not empty** (Since $\bigcap_{j=1}^{N} \bar{K}_{i_j} = \bar{K}_{i_N} \neq \emptyset$ if $i_1 \leq i_2 \leq \ldots i_N$). Then we **assert** that $\bigcap_{i=1}^{\infty} L_i \neq \emptyset$. Here is its proof.
We let $G_i = (K_i)^c$. If $\bar{V}_1$ doesn't have any point $x \in \bigcap_{i=2}^{\infty} K_i$, then $\{G_i$: i=1,2,..$\}$ makes an open covering of $\bar{V}_1$. Since K_1 is compact, there exist finite subscripts, $i_1, i_2, \ldots, i_n$ such that

$$K_1 \subset \bigcup_{j=1}^{n} G_{i_j},$$

which means

$$K_1 \subset (\bigcap_{j=1}^{n} K_{i_j})^c.$$

It follows that

$$K_1 \bigcap (\bigcap_{j=1}^{n} K_{i_j}) = \emptyset.$$

A contradiction! Thus we prove the assertion that $\bigcap_{i=1}^{\infty} K_i \neq \emptyset$.
Hence we have $\bigcap_{i=1}^{\infty} K_i \neq \emptyset$. However for every n

$$x_n \notin \bigcap_{k=1}^{\infty} V_k,$$

which follows that

$$\bigcap_{k=1}^{\infty} V_k = \emptyset.$$

A contradiction!

Problem 5

Solution
(a) We will prove it by Corollary 3.3.

Proof. Since x is in the closure of E (denoted by Ē) iff. d(x,E)=0 iff. for every n $\in \mathbb{N}^*$ we have $d(x, E) < \frac{1}{n}$. Thus we have

$$\bar{\mathrm{E}} = \bigcap_{n=1}^{\infty} O_n$$

Since E is compact, we have

$$\mathrm{E} = \bar{\mathrm{E}} = \bigcap_{n=1}^{\infty} O_n$$

Similar to the case stated in Problem 4 (a) above by applying Corollary 3.3 we have

$$m(\mathrm{E}) = lim_{n\to\infty} m(O_n)$$

□

(b) In condition one, we let $\mathrm{E} = \mathbb{Z} \subset \mathbb{R}$. It's immediately m(Z)=0 (which follows from the fact that $\mathbb{Z}$ is a countable set) and $m(O_n) = \infty * \frac{2}{n} = \infty$ for every n.
In condition two we let E=$[0,1] \backslash \bigcap C$ (constuction in problem 4). From (b) we have $\bar{E}$=[0,1]. And it's clear

$$m(\bar{E}) = lim_{n\to\infty} O_n = 1$$

$$m(E) = \Sigma_{j=1}^{\infty} 2^{j-1} * l_j < 1$$

if we assume

$$\Sigma_{j=1}^{\infty} 2^{j-1} * l_j < 1$$

beforehand.

Problem 6

Proof. From the **definition** of exterior measure, we have a **cubical** covering $\{Q_i : i \in \mathbb{N}^*\}$ which covers B_1 with total volume be $v_d + \frac{\epsilon}{r^d}$.
Then we form a linear map ϕ: $x \to rx$ from $\mathbb{R}^d$ to $\mathbb{R}^d$, which takes B_1 to B_r and Q_i to the corresponding one Q_i^*. Thus we have a cubic covering $\{Q_i^* : i \in \mathbb{N}^*\}$ which covers B_r with total volume $r * v_d + \epsilon$, which follows that for all $\epsilon > 0$ the following holds:

$$m(B_r) \le r^d * v_d + \epsilon$$

which follows that

$$m(B_r) \le r^d * v_d$$

Likewise, we can "**retract**" B_r to B_1 by applying ψ: $x \to \frac{x}{r}$ so as to get the inverted inequality, for all $\epsilon > 0$ the following holds:

$$m(B_r) \ge r^d * v_d$$

It follows that $m(B_r) = r * v_d$. □

Problem 7

The procedure of proving the equality between their erterior measure is similar to that of proving problem 6. Next we will show δE is measurable.

Proof. We define a linear map ϕ: $(x_1, x_2, x_3, \ldots, x_d) \to (\delta_1 * x_1, \delta_2 * x_2, \delta_3 * x_3, \ldots, \delta_d * x_d)$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ which is an opening mapping.
From the **definition** of exterior measure, we have a cubic covering $\{Q_i : i \in \mathbb{N}^*\}$ which covers E with total volume to be $m(E) + \frac{\epsilon}{\delta_1 * \delta 2 \ldots * \delta d}$. By applying ϕ we have a rectangle covering $\{Q_i^* : i \in \mathbb{N}^*\}$ (**the equivalence of the definition of exterior measure defined by cubic covering and rectangle covering will be**

shown in problem 15) which covers δE with total volume to be $\delta_1 * \delta 2 \ldots * \delta d * m(E) + \epsilon$. Thus we prove that for all $\epsilon > 0$:

$$m^*(\delta\mathrm{E}) \leq \delta_1 * \delta 2 \ldots * \delta d * m(E) + \epsilon$$

which follows that

$$m^*(\delta\mathrm{E}) \leq \delta_1 * \delta 2 \ldots * \delta d * m(E)$$

Likewise, we can "retract" δE to E so as to get the inverted inequality:

$$m^*(\delta\mathrm{E}) \geq \delta_1 * \delta 2 \ldots * \delta d * m(E)$$

□

The following is to prove δE is measurable.

Proof. First we assume any $\epsilon > 0$.
There exist an open set U = U(ϵ) which contains E such that $m^*(U \backslash \mathrm{E}) < \epsilon$. It follows that $\delta\mathrm{E} \subset \delta\mathrm{U}$ and

$$m^*(\delta\mathrm{U} \backslash \delta\mathrm{E}) = m^*(\delta(\mathrm{U} \backslash \mathrm{E})) = \delta_1 * \delta 2 \ldots * \delta d * m^*(\mathrm{U} \backslash \mathrm{E}) < \delta_1 * \delta 2 \ldots * \delta d * \epsilon$$

Since δU is open (**from open mapping property**), it immediatly follows that δE is measurable. □

The inverted direction which proves E is measurable if δE is measurable can be done by defining $\frac{1}{\delta}E := (\frac{x_1}{\delta_1}, \frac{x_2}{\delta_2}, \ldots, \frac{x_d}{\delta_d})$ and following the prodecure above.

Problem 8

(a) Since linear transformations on finite-dimensional linear vector spaces are always continuous, they map compact sets to compact sets (**which is the property of continuous maps**). Hence, if E is compact, so is $L(E)$.
Assume $E = \bigcup_{n=1}^{\infty} F_n$ where F_n is closed in $\mathbb{R}^n$. (F_n is allowed to be empty)
Since $F = \bigcup_{i_1, 1_2, \ldots, i_n \in \mathbb{Z}} (F \bigcap D_{i_1, 1_2, \ldots, i_n})$ where $D_{i_1, 1_2, \ldots, i_n} = \{x_1, x_2, \ldots, x_n : i_j \leq x_j \leq i_j + 1, j = 1, 2, \ldots, n\}$. (**i.e.** $\mathbb{R}^n$ **is** $\sigma - compact$). Thus we have F_n is a **countable** union of compact sets in $\mathbb{R}^n$, which immediately follows that E is a **countable** union of compact sets in $\mathbb{R}^n$. We write $E = \bigcup_{i=1}^{n} G_i$ where G_i is a compact set. Since we have the equality:

$$\bigcup_{i=1}^{\infty} f(G_i) = f(\bigcup_{i=1}^{\infty} G_i) = f(E)$$

which follows that f(E) is a countable union of compact sets since L sends compact sets to compact sets. Hence, L(E) is a F_σ set.
(b) we let $\|L\| = max\{L(x) : x \in S^{d+1}\}$. Thus we have

$$|L(x) - L(\hat{x})| = |L(x - \hat{x})| \leq \|L\||x - \hat{x}|$$

Thus we let L=M. Since in a cube Q in $\mathbb{R}^d$ the greatest distance between two points is $\sqrt{d}l$, which follows that the greatest distence between two points in $L(Q)$ is no more than $2M\sqrt{d}l$. If $\hat{Q}$ is a cube of side length $4M\sqrt{d}l$ with center x∈ L(Q), the points on the exterior of the cube are all at least $M\sqrt{d}l$ away from x. Thus we have L(Q) $\subset \hat{Q}$ if x∈ L(Q). Since a set of measure 0 has a cubical covering with volume less than ϵ, its image under L has a cubical covering with volume no more than $(4M\sqrt{d}l)^2\epsilon = 16M^2 dl\epsilon$. This implies that L maps a set of measure 0 to a set of measure 0.
Finally, we assume that E is any measurable set. From Corollary 3.5, E = $C \bigcup N$ where C is an F_σ set and N has measure 0. We have just shown that L(C) is also an F_σ set and L(N) also has measure 0. Since we have the relation:

$$L(E) = L(C) \bigcup L(N),$$

L(E) is measurable.

Problem 9

solution Let C_0 denote the original interval. Let C_k denote the set remaining after the k_{th} stage of construction **in problem 4**. (k=1,2,3,...) and $C_\xi = \bigcap_{i=1}^{\infty} C_k$. Let D_k be the complement of C_k in [0,1].

Proof. Let G denote the union of the open intervals removed during the odd steps and G$\hat{G}$ denote the open intervals removed during the even steps. It's clear that $G \bigcap \hat{G} \bigcap C_\xi = [0,1]$. Next in the proof we will show the below two equations

$$\bar{G} = G \bigcup C_\xi \quad ; \quad \partial\bar{G} = C_\xi$$

We shall prove the first equation first. It's clear that $G \bigcup C_\xi$ is closed since $G \bigcup C_\xi = [0,1] \backslash \hat{G}$ where $\hat{G}$ is open in [0,1]. Thus we have $\bar{G} \subset G \bigcup C_\xi$. Meanwhile, assume $x \in C_\xi$. Let D_k denote the closed interval in C_k which contains $x \in C_\xi$. Let I_{k+1} denote the open interval in $(\hat{C})^c$ such that it's centered in D_k as the construction does. Then it's clear that

$$x_k \to x \quad (k \to \infty).$$

Thus we obtain a subsequence such that x_{2k+1} is in G and:

$$x_{2k+1} \to x \quad (k \to \infty).$$

. In other words, all the points in C_ξ are limit points of G. And thus we have

$$G \bigcup C_\xi \subset \bar{G}.$$

Hence $\bar{G} = G \bigcup C_\xi$.
Then we prove the second equation. Since G is open, each point of G is contained in an open interval in G (G is an interior point), which follows that

$$G \bigcap \partial\bar{G} = \emptyset.$$

Similarly we have

$$\hat{G} \bigcap \partial\bar{G} = \emptyset.$$

Hence we have

$$\partial\bar{G} \subset C_\xi.$$

Since we have proved all the points in C_ξ are limit points of G. Similarly by choosing the odd-number subsequence we can prove that all the points in C_ξ are limit points of $\hat{G}$. Thus from the definition of the boundary we have

$$C_\xi \subset \partial\bar{G}.$$

Finally we have $C_\xi = \partial\bar{G}$.
Then for G, an open set in [0,1]:

$$m(\partial\bar{G}) = m(C_\xi) > 0,$$

which is subject to the stated conditions. ☐

Problem 10

We will prove it with the usual mathematical analysis knowledge. Let C_0 denote the original interval. Let C_k denote the set remaining after the k-th stage of construction of cantor-like set detailed in problem 4. (k=1,2,3,...)

Problem 10 continued on next page...

Proof. (a) From the construction it's clear that $0 \leq f_n(x) \leq 1$ and $f_n(x) \geq f_{n+1}(x)$ for every $n \geq 1$ and $x \in [0,1]$ (by induction).
Then we fix a x in [0,1], we have $f_n(x) \geq f_{n+1}(x)$ for every $n \geq 1$. Since $f_n(x)$ is a sequence such that it's monotone (descending) and bounded it's convergent.
(b) We will prove it by contradiction. Assume x=$x_0 \in \hat{C}$ is continous, then from Heine's theorem for any sequence $\{x_n\}$ s.t. $x_n \to x_0$ $(n \to \infty)$, we have $f(x_n) \to 1$ $(n \to \infty)$. Similar to the construction in proof of problem 4(b), Let D_k denote the closed interval in C_k which contains x_0. Let I_{k+1} denote the open interval in $(\hat{C})^c$ such that it's centered in D_k as the construction does.
Then it's clear that

$$m(I_k) = l_k \leq \frac{1}{2^k} \to 0 \quad (k \to \infty)$$

Choose the **centered** point $x_k \in I_k$, then we have

$$|x_k - x_0| \leq m(D_k) \to 0 \quad (k \to \infty)$$

which immediately follows that $x_k \to x_0$ and $f(x_k) = 0$ for every k. A contradiction!

□

Problem 11

A has measure 0, for the same reason as the Cantor set.

Proof. We can construct A as an intersection of some closed sets as Cantor-like set does. The first iterate is the unit interval; the second has a subinterval of length $\frac{1}{10}$ deleted, with segments of lengths $\frac{3}{10}$ and $\frac{6}{10}$ remaining. (The deleted interval corresponds to all numbers with a 4 in the first decimal place.) The next has 9 subintervals of length $\frac{1}{100}$ deleted, corresponding to numbers with a non-4 in the first decimal place and a 4 in the second. Continuing, we get closed sets C_n of length $(\frac{9}{10})^n$, with

$$A = \bigcap_{n=1}^{\infty} \mathrm{C}_n$$

. Clearly A is measurable since each C_n is. Since m(C_n) $\to$ 0 (n$\to \infty$), we have m(A) = 0. □

Problem 12

Solution (a) Consider the boundary of the open disc.

Proof. We will prove it by contradiction. Let Q_1 be one of the open rectangles contained in the disk. Let x be a point in the boundary of Q_1.
Since Q_1 is open, $x \notin Q_1$. Thus, x must be contained in another rectangle, say Q_2. And since Q_2 is open, x is in the interior of Q_2. Thus there exist an open set $\{y : d(x,y) < \epsilon\}$ within Q_2 and it's obvious that $\{y : d(x,y) < \epsilon\}$ must intersect with Q_1, which contradicts the assumption of disjoint union. □

(b) We will prove from two directions.
1) Ω is an open connected rectangle. There exist a covering such that $\Omega = \cup_{i=1}^{n} R_i$ where $R_i \cap R_j = \emptyset$ for $i \neq j$ and R_i are open rectangles.

Proof. Let $R_1 = \Omega$ and $R_i = \emptyset \ \ \forall i \geq 2$. □

2) Ω is open and connected and $\Omega = \cup_{i=1}^{n} R_i$ where $R_i \cup R_j = \emptyset$ for $i \neq j$ and R_i are open rectangles. Then Ω is a rectangle.

Problem 12 continued on next page...

Proof. First we remind ourselves that in $\mathbb{R}^d$ path-connectedness and connectedness are equivalent. (X is connected: $X = A \bigcap B$ where A,B are open sets in X $\Rightarrow$ A=X, B=$\emptyset$ or vice versa.) Following from the condition we let $\Omega = R_1 \bigcup (\bigcup_{i=2}^n R_i)$ where $R_1 \bigcap (\bigcup_{i=2}^n R_i) = \emptyset$ as Ω is connected there are exactly two options:
(i) $R_1 = \Omega$ and $(\bigcup_{i=2}^n R_i) = \emptyset$
(ii) $R_1 = \emptyset$ and $(\bigcup_{i=2}^n R_i) = \Omega$
note that in (i) we have $R_1 = \Omega$ and in (ii) we can separate as before. So by induction we can conclude $\exists k \in \{1, 2, 3, \ldots n\}$ such that $R_k = \Omega$ and $R_i = \emptyset \; \forall i \neq k$, which proves the statement. □

Problem 13

Solution (a) We will prove it with the tool of problem 4 and De Morgan law.

Proof. Let E be a closed set. From proof of problem 5 we have $E = \bigcap_{n=1}^{\infty} O_n$ where O_n denotes the set $\{x : d(x, E) < \frac{1}{n}\}$. Thus we have the conclusion E is the countable intersection of open sets.
Let U be an open set. Since U^c is closed $U^c = \bigcap_{n=1}^{\infty} O_n$ where O_n denotes open sets. Then from De Morgan law:

$$U = \bigcup_{n=1}^{\infty} (O_n)^c$$

which follows that U is the countable union of closed sets. □

(b) We give a construction based on **Baire's theorem**.

Proof. First we give a lemma without proof (**Baire's theorem**): In a complete metric space (X,d): X can't be written as a countable intersection of sets which are open and dense in X.
$\mathbb{Q}$ is the countable union of closed sets (single points) in $\mathbb{R}$. Next we will show it's not the countable intersection of open sets. Since $\bar{\mathbb{Q}} = \mathbb{R}$, then any open set U containing $\mathbb{Q}$ is subject to $\bar{\text{U}} = \mathbb{R}$. Let $V_n = U_n \backslash r_n$ where r_n denotes the n_{th} rational number, then V_n is still dense and open in $\mathbb{R}$. Thus we have

$$\emptyset = \bigcap_{n=1}^{\infty} V_n$$

which contradictst the Baire's theorem. □

(c) Let E={rational points in (0,1) and irrational points in (2,3)}. Next we will prove it's neither a countable union of closed sets nor a countable intersection of open sets.

Proof. First assume E=$\bigcup_i F_i$ where F_i is closed in $\mathbb{R}$. Then

$$(R \backslash \mathbb{Q}) \bigcap (2, 3) = (R \backslash \mathbb{Q}) \bigcap [2, 3] = \bigcup_{i=1}^{\infty} (F_i \bigcap [2, 3])$$

, which means irrational points in (2,3) are a countable union of closed sets. Moreover, since

$$(R \backslash \mathbb{Q}) \bigcap (2, 3) = \bigcup_{i=1}^{\infty} (F_i \bigcap (2, 3)),$$

we have

$$\mathbb{Q} \bigcap (2, 3) = \bigcap_{i=1}^{\infty} ((F_i)^c \bigcap (2, 3)).$$

which means rational points in (2,3) are a countable intersection of open sets, denoted by $\{Q_i\}$. It's clear that Q_i is dense in (2,3) for every i (**as to the subspace topology**). And so is $Q_i \backslash \{r_i\}$ where $\{r_i\}$ is an enumeraion of rational points in (2,3). It's also clear that $Q_i \backslash \{r_i\}$ is dense in (2,3). However,

$$\bigcap_{i=1}^{\infty} Q_i \backslash \{r_i\} = \emptyset$$

contrary to the conclusion of **Baire's theorem**. Similarly we can prove E is not a countable intersection of open sets by **Baire's theorem**. □

Problem 14

(a) We will prove it with definition.

Proof. Assume $E \subset \bigcup_{j=1}^n I_j$. Since $\bar{E}$ is the **smallest** closed set which contains E and $\bigcup_{j=1}^n I_j$ is closed, $\bar{E} \subset \bigcup_{j=1}^n I_j$. Similarly, $\bar{E} \subset \bigcup_{j=1}^n I_j \Rightarrow E \subset \bigcup_{j=1}^n I_j$. Hence, $\bar{E} \subset \bigcup_{j=1}^n I_j \Leftrightarrow E \subset \bigcup_{j=1}^n I_j$.
Thus, we have

$$\{\Sigma_{j=1}^n |I_j| : \bar{E} \subset \bigcup_{j=1}^n I_j\} = \{\Sigma_{j=1}^n |I_j| : E \subset \bigcup_{j=1}^n I_j\}$$

And of course we have:

$$inf\{\Sigma_{j=1}^n |I_j| : \bar{E} \subset \bigcup_{j=1}^n I_j\} = inf\{\Sigma_{j=1}^n |I_j| : E \subset \bigcup_{j=1}^n I_j\}$$

Consequently, $J_*(E) = J_*(\bar{E})$ □

(b) Let E=$\mathbb{Q} \bigcap [0,1]$.

Proof. Since E is countable, $m_*(E)$=0.
Next we will prove $J_*(E)$=1. First since we have $E \subset [0,1]$, $J_*(E) \leq 1$.
Arbitrarily choose a covering of E such that $E \subset \bigcup_{j=1}^n I_j$. We will prove that $\Sigma_{j=1}^n |I_j| \geq 1$, which immediately follows that

$$J_*(E) \geq 1.$$

Assume $I_j = [a_j, b_j]$. WLOG, we let $a_1 = min\{a_i : i = 1, 2, \ldots, n\}$. It's clear that $a_1 \leq 0$. If $b_1 \geq 1$, we immediately have the conclusion. Then suppose $b_1 < 1$, there exist a_j $(j \neq 1)$ such that $a_j \leq b_1$. **By rearranging the intervals**, without loss of generality we change the subscript of a_j to be a_2. Contining the process, we must have $b_j \geq 1$ for some j. It's clear that

$$1 \leq \Sigma_{k=1}^j |I_k|$$

, which follows that

$$1 \leq \Sigma_{k=1}^n |I_k|.$$

□

Problem 15

Solution We will prove it with lemma 1.1. First assume R_j denotes a rectangle and Q_j denotes a cube.

Proof. Similar to soulution to problem 14, we have

$$\{\Sigma_{j=1}^{\infty}|Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j\} \subset \{\Sigma_{j=1}^{\infty}|R_j| : E \subset \bigcup_{j=1}^{\infty} R_j\}.$$

Thus, we have

$$inf\{\Sigma_{j=1}^{\infty}|Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j\} \geq inf\{\Sigma_{j=1}^{\infty}|R_j| : E \subset \bigcup_{j=1}^{\infty} R_j\}.$$

In other words, we have $m_*(E) \geq m_*^R(E)$. Next we prove the inverse direction.
Assume ϵ>0. From the definition of $m_*^R(E)$, there exist a rectangle cover R_j:j=1,2,... over E such that $m_*^R(E) > \Sigma_{j=1}^{\infty}|R_j| - \epsilon$ .Since every rectangle can be expressed by the almost disjoint union of finitely many cubes expect for a space which can be **arbitrarily small**. Next we will prove it in a mathematical way.
For $R_j = [a_1, b_1]\times[a_2, b_2]\times\dots[a_n, b_n]$ with $M = max = \{b_i - a_i : i = 1, 2, \dots, \}$, we let the cube of side length $l = \frac{\epsilon}{2^j \prod_{i=1}^{n}(a_i - b_i)}$. Then we fill the rectangle from its side with the cube one by one each of which is exactly almost disjoint **until we can't fill in a cube any more**, with all the cubes denoted by $\{Q_{jk}$:k=1,2...,$N_j\}$. Since each side length of the rectangle isn't that perfect, we leave a gap between the union of the cubes and the outer rectangle. Then the gap's volume (denoted by V_j) is **at most**

$$\Sigma_{j=1}^{n} l * \prod_{i\neq j}(b_i - a_i) \leq \frac{1}{2^j}\epsilon$$

From lemma 1.1 we have

$$|R_j| = \Sigma_{i=1}^{N_j}|Q_{ji}| + V_j \leq \Sigma_{i=1}^{N_j}|Q_{ji}| + \frac{1}{2^j}\epsilon.$$

Thus we have

$$\Sigma_j|R_j| \leq \Sigma_{j,i}|Q_{ji}| + \Sigma_{j=1}^{\infty}\frac{1}{2^j}\epsilon \leq \Sigma_{i,j}|Q_{ji}| + \epsilon$$

with $\{Q_{ji} : j = 1, 2, \dots; i = 1, 2, \dots, N_j\}$ **almost covers** E except for a space whose outer measure at most $\Sigma_{k=1}^{\infty}|V_k| \leq \epsilon$.
From the property of outer measure:

$$m_*(E) = m_*((\bigcup_{j,i} Q_{ji})\bigcup(E\backslash\bigcup_{j,i} Q_{ji})) \leq \Sigma_{j,i}|Q_{ji}| + \epsilon \leq \Sigma_j|R_j| + \epsilon \leq m_*^R(E) + 2\epsilon$$

Since ϵ is arbitrary, $m_*(E) \leq m_*^R(E)$. □

Problem 16

Solution Let $D_n := \bigcup_{k\geq n} E_k$ and $D := lim_{n\to\infty}\bigcup_{k\geq n} E_k$. (**definition of D is well-defined since** $\{D_n\}$ **is monotone, its superior limit and inferior limit coincide**)
(a) We will prove the statement with an equation.

Proof. we will first show that $E = \bigcap_{n=1}^{\infty}\bigcup_{k\geq n} E_k$.
$x \in E$ iff. x $\in E_k$ for infinitely many k iff. for every n $\in \mathbb{N}^*$ x $\in \bigcup_{k\geq n} E_k$ iff. $x \in \bigcap_{n=1}^{\infty}\bigcup_{k\geq n} E_k$.
Then since the countable intersection of countable unions of measurable sets is still measurable, E is measurable. □

(b) We will prove it again by corollary 3.3.

Problem 16 continued on next page...

Proof. Assume $\epsilon > 0$. There exist n_0 such that $\Sigma_{n=n_0}^{\infty} m(E_n) < \epsilon$ since $\Sigma_{n=1}^{\infty} m(E_n) < \infty$. And we have

$$m(D_{n_0}) \leq \Sigma_{n=n_0}^{\infty} m(E_n) < \epsilon$$

It's clear that $D_n \searrow D$ and $m(D_n) < \infty$ for n_0 (or to be exact for any n $\geq n_0$), thus by applying corollary we have

$$m(D) = lim_{n\to\infty} m(D_n) \leq m(D_{n_0}) < \epsilon$$

Since ϵ is assumed to be any number greater than 0, we have $m(D) = 0$. $\square$