Weibel Homological Algebra: Chap 1

Due on 12, 2021

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Supplements

When are zero morphisms preserved?

Let $\mathcal{A}$ and $\mathcal{B}$ be categories with a zero object. Tt is the case that a morphism is zero if and only if it factors through the zero object (and any such factorisation is unique).
1. Suppose $F : \mathcal{A} \to \mathcal{B}$ preserves zero morphisms. Note that the zero object Z is characterised by the property that $0_Z = \mathrm{id}_Z$; but F preserves 0 and id, so FZ must also be a zero object.
2. Suppose $F : \mathcal{A} \to \mathcal{B}$ preserves the zero object. Then it preserves the factorisations through the zero object, because F preserves composition, so F must also preserve zero morphisms.

Definition of homology in Abelian Category

One of the axioms of an abelian category says that every morphism has a cokernel. The quotient B/A of a monomorphism $f : A \to B$ is simply its cokernel.
A morphism $f : A \to B$ in an abelian category has *four* associated objects and *five* associated morphisms:

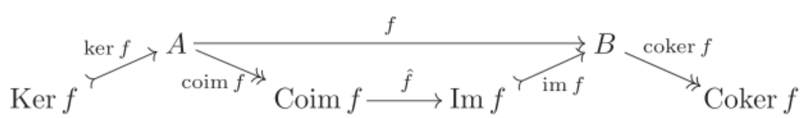

Figure 1: Pic 1

The main axiom of abelian categories states that the canonical morphism $\hat{f} : \mathrm{Coim}\,(f) \to \mathrm{Im}\,(f)$ (uniquely determined by requiring that the diagram be commutative) always is an isomorphism. Now given two morphisms $f : A' \to A$ and $g : A \to A''$ such that $gf = 0$ there are three ways to define the homology of the "complex" $A' \to A \to A''$:
1. $\mathrm{Coker}\,(\mathrm{Im}\,(f) \to \mathrm{Ker}\,(g))$, 2. $\mathrm{Ker}\,(\mathrm{Coker}\,(f) \to \mathrm{Coim}\,(g))$, 3. $\mathrm{Im}\,(\mathrm{Ker}\,(g) \to \mathrm{Coker}\,(f))$.

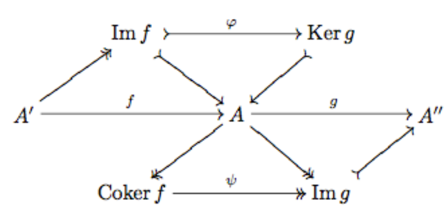

Figure 2: Pic 2

The first of these corresponds to the usual Ker/Im and it is not very hard to show that all three ways give canonically isomorphic objects in an abelian category. It is essential to require the category to be abelian here, the three possibilities are distinct in a general additive category (with kernels and cokernels).

Exercise 1.1.3

Start with a chain complex of vector spaces

$$\cdots \to V_{n+1} \overset{d_{n+1}}{\to} V_n \overset{d_n}{\to} V_{n-1} \to \cdots$$

and looked for a map

$$u_n\colon V_n \to \operatorname{im} d_{n+1} \oplus (\ker d_n/\operatorname{im} d_{n+1}) \oplus \operatorname{im} d_n$$

as suggested by the notation $\{B_n, H_n\}$ in the text of the exercise.
There is only one reasonable map, defined as follows. Choose a basis $\mathcal{B}_1$ for $\operatorname{im} d_{n+1}$ and extend it first to a basis $\mathcal{B}_1 \sqcup \mathcal{B}_2$ for $\ker d_n$, then to a basis $\mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \mathcal{B}_3$ for V_n. As a result, each vector $v \in V_n$ can be uniquely written as $v = v_1 \oplus v_2 \oplus v_3$ where v_i is a linear combination of elements of $\mathcal{B}_i$, $i = 1, 2, 3$. Finally define

$$u_n(v) = v_1 \oplus [v_2] \oplus d_n(v_3)$$

where [ ] denotes the homology class.

Exercise 1.1.4

Elements of $\ker(d_n^*)$ are arrows. Let $i_n : Z_n \to C_n$ be the canonical inclusion. Then $i_n \in Hom(Z_n, C_n)$ and

$$d_n^*(i_n) = d_n \circ i_n = 0.$$

Hence there exists $u \in Hom(Z_n, C_{n+1})$ such that $i_n = d_{n+1} \circ u$ and therefore

$$Z_n \subseteq B_n = im(d_{n+1})$$

Note that this argument is valid to any exact category.
The reciproque is false: consider the complex $0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$ and take $Z_n = \mathbb{Z}_2$.

Exercise 1.1.6

The sequence is

$$0 \to C_1 \xrightarrow{d} C_0.$$

Thus, we simply need to compute $\ker(d)$ with observation that d is essentially acting as the "boundary map", i.e., given any edge e, we have $d(e) = v_i - v_j$, where v_i is the starting point of e and v_j the ending point. Let H_0 be free of rank 1. Finally, for $H_1 = \ker M$, note that it is free as a subgroup of free abelian group. To compute its rank, just use the rank theorem :

$$E = \operatorname{rk} C_1 = \operatorname{rk}\ker M + \operatorname{rk}\operatorname{im} M = \operatorname{rk} H_1 + \operatorname{rk} C_0 - \operatorname{rk} H_0 = \operatorname{rk} H_1 + V - 1$$

Hence $\operatorname{rk} H_1 = E - V + 1$.

Exercise 1.2.5

The general strategy can be interpreted in the following example. Consider the double complex

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & C_{0,2} & \leftarrow & C_{1,2} & \leftarrow & C_{2,2} & \leftarrow & C_{3,2} & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & C_{0,1} & \leftarrow & C_{1,1} & \leftarrow & C_{2,1} & \leftarrow & C_{3,1} & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & C_{0,0} & \leftarrow & C_{1,0} & \leftarrow & C_{2,0} & \leftarrow & C_{3,0} & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows. Let us look at the total complex $T(C)$, and say we concentrate on the diagonal for $n = 3$, that is, we look at

$$T(C)_3 = C_{1,2} \oplus C_{2,1} \oplus C_{3,0}, \tag{1}$$

and say we have some element $x + y + z \in T(C)_3$, with $x \in C_{1,2}$ etc.
We want to show: if $d(x + y + z) = 0$ then there is some $\xi \in T(C)_4$ such that $d\xi = x + y + z$.
What is $d(x + y + z)$? It is

$$d(x + y + z) = d^h x + d^v x + d^h y + d^v y + d^h z + d^v z. \tag{2}$$

Note that then $d(x + y + z) = 0$ is equivalent to the four conditions

$$0 = d^h x \tag{3}$$
$$0 = d^v x + d^h y \tag{4}$$
$$0 = d^v y + d^h z \tag{5}$$
$$0 = d^v z, \tag{6}$$

and the first immediately implies there is some $a \in C_{2,2}$ such that

$$d^h a = x.$$

Then, we plug it into the second condition, to get

$$0 = d^v x + d^h y = d^v d^h a + d^h y = d^h(y - d^v a)$$

so that $y - d^v a = d^h b$ for some $b \in C_{3,1}$. As you may have guessed, the third condition tells us

$$0 = d^v y + d^h z = d^h(z - d^v b)$$

and this would continue on and on, but in this example, the map $d^h : C_{3,0} \to C_{2,0}$ is actually injective, so $z = d^v b$. In our example, the fourth condition doesn't even matter.
Now, $\xi = a + b$ satisfies

$$d\xi = d^h a + (d^v a + d^h b) + d^v b = x + y + z,$$

neato!

Exercise 1.2.8

Self-interpretation: If we regard $f : B \to C$ as a chain complex of objects in $Ch(\mathcal{A})$ concentrated in degree -1 and 0 then applying the sign trick allows us to construct a (homological) double complex D with $D_{p,0} = C_p$, $D_{p,1} = B_p$, $d^h_{p,1} = d_B$, $d^h_{p,0} = d_C$, and $d^v_{p,1} = (-1)^p f_p$. Let C', B' be the chain complexes in $Ch(Ch(\mathcal{A}))$ associated to C, B, concentrated at 0, so $C'_0 = C$ and $B'_0 = B$. Then $B'[-1]_1 = B$. Apply again the sign trick to associate double complexes D(C'), D(B'[-1]) to the chain complexes C' and $B'[-1]$. This fits into a (split) short exact sequence

$$0 \to D(C') \to D \to D(B'[-1]) \to 0$$

. The objects of the resulting total complex are, as desired, $Tot(C)_n = B_{n-1} \oplus C_n$, and the corresponding differential maps are, in matrix form,

$$d_n^{Tot(C)} : B_{n-1} \oplus C_n \to B_{n-2} \oplus C_{n-1} = \begin{bmatrix} d^B_{n-1} & 0 \\ (-1)^{n-1} f_{n-1} & d^C_n \end{bmatrix} \tag{7}$$

Comparing the formula for a mapping cone, we see that one can construct a chain complex $-B$ whose objects are the same as B and whose morphisms are $-d_B$, together with a morphism of chain complexes $f' : -B \to C$ where $f'_n = (-1)^{n+1} f_n$. f' is a morphism of chain complexes as $(-1)^n f_{n-1} \circ (-d_n) = (-1)^{n+1} f_{n-1} d_n = (-1)^{n+1} d_n f_n$ by the assumption that f itself is a morphism of chain complexes. And $-B$ is isomorphic to B by $g : B \to -B$ with $g_n = (-1)^{n+1} id_B$ with $f'g = f$.

Exercise 1.3.3

In an abelian category, consider the five-lemma-like commutative diagram with exact rows (in the category theoretic sense), the left vertical arrow epic, the right vertical arrow monic, and the second and fourth vertical arrows isomorphisms. Then the central vertical arrow is also an isomorphism. We write out the image factorisation of all the horizontal maps.

$$\begin{array}{ccccccccccccccccc} A & \to & I_1 & \to & B & \to & I_2 & \to & C & \to & I_3 & \to & D & \to & I_4 & \to & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \to & I_1' & \to & B' & \to & I_2' & \to & C & \to & I_3' & \to & D' & \to & I_4' & \to & E' \end{array}$$

Then $I_1 \to I_1'$ and $I_4 \to I_4'$ are epic and monomorphic, therefore isomorphic. Then $I_2 \to I_2'$ and $I_3 \to I_3'$ are isomorphic. lemma 1.1 in Hilton tells everything.

Exercise 1.3.5

We have the exact sequences

$$\begin{aligned} &(1) \quad 0 \to \ker f \to C \to \mathrm{im} f \to 0 \\ &(2) \quad 0 \to \mathrm{im} f \to D \to \mathrm{coker} f \to 0 \end{aligned}$$

The long exact sequence of homology coming from (1), along with the assumption that $\ker f$ is acyclic, shows that $C \to \mathrm{im} f$ is a quasi-isomorphism.
The long exact sequence of homology coming from (2), along with the assumption that $\mathrm{coker} f$ is acyclic, shows that the inclusion $\mathrm{im} f \to D$ is a quasi-isomorphism.
Hence the composite $C \to \mathrm{im} f \to D$, which is f, is a quasi-isomorphism.
Counterexample: Considering the following two short exact sequences

$$0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$$

and

$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to 0$$

where the maps from $\mathbb{Z}$ to $\mathbb{Z}$ are identity and other maps are trivial.
Then we can construct a chain complex morphism f between them: the only non-trivial map is the middle one which is identity.
Then $\ker f$ is

$$0 \to 0 \to 0 \to \mathbb{Z} \to 0$$

and coker f is

$$0 \to \mathbb{Z} \to 0 \to 0 \to 0$$

and they are not acyclic. However f is a quasi-isomorphism.

Exercise 1.3.6

$$\alpha^* = \alpha.$$

Note that exactness depends only on the underlying objects, and not on the differentials. It follows that if each sequence at coordinate is exact, then the sequence of total complexes is too, because the direct sum of exact sequences is exact.

Exercise 1.4.1

For the first question: $\ker(d_1)$ might not be free, but as a summand of C_1, it is projective. Now use $\ker(d_1) = \mathrm{im}(d_2)$ and continue the splitting process.
For the second question: the point is that all syzygies of the complex are projective again, and the restriction to f.g. modules is probably only to avoid use of AC.

Exercise 1.4.2

Let $B'_n = sd(C_n)$, $B_n = ds(C_n)$ and $H'_n = (1 - sd - ds)(C_n)$. Im $d_n \cong sd(C_n)$ induces the short exact sequence

$$0 \to Z_n \to C_n \to sd(C_n) \to 0$$

The right split map is just the inclusion $sd(C_n) \hookrightarrow C_n$.
It is also not hard to show that $ds(C_n) = \mathrm{Im}\ d_{n+1} = B_n$. Then we have short exact sequence

$$0 \to ds(C_n) \to Z_n \to Z_n/ds(C_n) \to 0$$

The left split map is ds, so we have $Z_n \cong ds(C_n) \oplus Z_n/ds(C_n)$.
In summary, we have Im $d_{n+1} = ds(C_n)$ and Im $d_n \cong sd(C_n)$. The first splitting short exact sequence implies $\ker d_n = (1 - sd)(C_n)$. Then the second splitting short exact sequence implies $H'_n = (1 - sd - ds)(C_n)$.

Exercise 1.4.3

The only if direction follows from exercise 1.4.2 with a direct construction.

Exercise 1.4.4

Let C_* be the posited chain complex, and let $H_*(C)$ be its homology. By assumption we have chain maps $f : C_* \to H_*(C)$ and $g : H_*(C) \to C_*$ such that gf and fg are homotopic to the identity maps.
Then $dg = 0$ by a commutative diagram. And $1 - gf = ds + sd$ for some R-linear $s : C_* \to C_{*+1}$ because f, g realize a homotopy equivalence. Then:

$$d = d - 0 = d(1 - gf) = d(ds + sd) = 0 + dsd = dsd$$

because $d^2 = 0$.

Exercise 1.5.2/ 1.5.3

$$\begin{tikzcd}
C_{n-1} \oplus C_n \arrow[r, "A"] \arrow[d, "(-s,f)"'] & C_{n-2} \oplus C_{n-1} \arrow[d, "(-s,f)"] \\
D_n \arrow[r, "d"] & D_{n-1}
\end{tikzcd}$$

where A denote

$$A := \begin{pmatrix} -d & \\ -Id & d \end{pmatrix}$$

.

Example 1.6.3

Kernels are problematic: Let R be a non-Noetherian ring. Let I be ideal, which is not finitely generated (this exists because R is not Noetherian). I is the kernel of

$$R \to R/I$$

. Also both R and R/I are finitely generated as R-modules (by 1 actually). Hence the kernel, in the category of finitely generated R-modules, does not always exist.

Exercise 1.6.3

Observe that F preserves monomorphisms: If $i : A \to B$ is a monomorphism, then $0 \to A \xrightarrow{i} B \to \text{coker}(i) \to 0$ is exact, hence also $0 \to F(A) \to F(B) \to F(\text{coker}(i))$ is exact. In particular $F(i)$ is a monomorphism.
Now if $0 \to A \xrightarrow{i} B \xrightarrow{f} C$ is exact, then $0 \to A \xrightarrow{i} B \xrightarrow{f} \text{im}(f) \to 0$ is exact, hence by assumption $0 \to F(A) \to F(B) \to F(\text{im}(f))$ is exact. Since $F(\text{im}(f)) \to F(C)$ is a monomorphism, it follows that also $0 \to F(A) \to F(B) \to F(C)$ is exact.