AMath 584 / Math 584
Homework #2
Due 11:00pm PDT
Tuesday, October 25, 2016

Name: Yifei Guan

Netid: gyf135

Problem 1. Find (by hand) both the reduced and full SVD of the matrix

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Remember that things should be ordered so that $\sigma_1 \geq \sigma_2$.

Use the SVD to determine the following:

- The best rank-1 approximation to A.
- The 2-norm of A.
- A basis for the linear subspace $\{x \in \mathbb{R}^2 : \|Ax\|_2 = \|A\|_2 \|x\|_2\}$ and the dimension of this space.
- A basis for the orthogonal complement of the range of A.
- A basis for the null space of A^T.

Solution:
Finding the SVD of A consists of calculating the eigenvalues and eigenvectors of AA^T and A^TA. The eigenvectors of A^TA make up the columns of V, the eigenvectors of AA^T make up the columns of U and the singular values in Σ are square roots of eigenvalues from AA^T or A^TA. The singular values are diagonal entries of the Σ matrix and are arranged in descending order. The singular values are always real numbers. Since the matrix A is a real matrix, U and V are also real.

$$\text{Let } B = AA^T = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 12 \\ 0 & 4 & 0 \\ 12 & 0 & 16 \end{bmatrix}$$

$$\text{Let } C = A^TA = \begin{bmatrix} 4 & 0 \\ 0 & 25 \end{bmatrix}$$

To find the eigenvalues and eigenvectors, we can solve $det(B - \lambda I) = 0$ for λ where det denotes the determinant and λ is the eigenvalues of matrix B. Then we find that the eigenvalues of B are $\lambda_1 = 25, \lambda_2 = 4, and \lambda_3 = 0$. Note that λ_1 and λ_2 are the eigenvalues of matrix C. For eigenvalues, we can calculate the eigenvectors respectively. As for λ_1, $BV_{B1} = \lambda_1 V_{B1} \Rightarrow V_{B1} = \begin{bmatrix} 0.6 \\ 0 \\ 0.8 \end{bmatrix}$ and $CV_{C1} = \lambda_1 V_{C1} \Rightarrow V_{C1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. As for λ_2, $BV_{B2} = \lambda_2 V_{B2} \Rightarrow V_{B2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $CV_{C2} = \lambda_2 V_{C2} \Rightarrow V_{C2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

As for λ_3, $BV_{B3} = \lambda_3 V_{B3} = 0 \Rightarrow V_{B3} = \begin{bmatrix} 0.8 \\ 0 \\ -0.6 \end{bmatrix}$.

Now that we have calculated the eigenvalues and eigenvectors. The full SVD of matrix A can be written as:

$$U = \begin{bmatrix} 0.6 & 0 & 0.8 \\ 0 & 1 & 0 \\ 0.8 & 0 & -0.6 \end{bmatrix}, \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Double check that $U\Sigma V^* = A$ (Always needed since the eigenvectors can be of opposite sign).
Since the matrix A is of rank 2, there are only two singular values. The reduce SVD of matrix A can be interpreted as:

$$\hat{U} = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \\ 0.8 & 0 \end{bmatrix}, \hat{\Sigma} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Double check that $\hat{U}\hat{\Sigma}V^* = A$

The best rank-1 approximation is thus $u_1\sigma_1 v_1^* = 5 \begin{bmatrix} 0.6 \\ 0 \\ 0.8 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 0 & 4 \end{bmatrix}$
where u_1 and v_1 is the first column in matrix U and V respectively. And σ_1 is the largest singular value

The 2-norm of A: $\| A \|_2 = \sigma_1 = 5$

To interpret the subspace, let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $Ax = \begin{bmatrix} 3x_2 \\ 2x_1 \\ 4x_2 \end{bmatrix}$. Then the subspace should satisfy $[(3x_2)^2 + (2x_1)^2 + (4x_2)^2]^{\frac{1}{2}} = 5 \times (x_1^2 + x_2^2)^{\frac{1}{2}} \Rightarrow x_1 = 0$ and $x_2 \in R$.
Therefore, the basis of the subspace is $\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ and thus the subspace is just a line. Therefore the dimension of subspace is 1.

The $range(A) = Ax = \begin{bmatrix} 3x_2 \\ 2x_1 \\ 4x_2 \end{bmatrix}$, $\forall x_1, x_2 \in R$. And thus the orthogonal complement is $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $\{y_1, y_2, y_3 \in \mathbb{R} : x^T y = 0\}$. In order to make the equation $x^T y = 0$ true $\forall x_1, x_2 \in \mathbb{R}$, $y_2 = 0$ and $y_3 = -\frac{3}{4}y_1$. Therefore, the orthogonal complement of the range of matrix A is $Y = y_1 \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{4} \end{bmatrix}$. And the basis of it is $\begin{bmatrix} 1 \\ 0 \\ -\frac{3}{4} \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix}$ and let $A^T x = 0 \Rightarrow \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ 3x_1 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Solve the equation, we have $x_2 = 0$ and $x_3 = -\frac{3}{4}x_1$. Therefore, the basis for the null space of A^T is $\begin{bmatrix} 1 \\ 0 \\ -\frac{3}{4} \end{bmatrix}$

In fact, it can be shown that if the subspace is described as the range of a matrix, then the orthogonal complement is the set of vectors orthogonal to the rows of A, which is the nullspace of A^T

Problem 2. Determine by hand the SVD of the matrix

$$A = \begin{bmatrix} 17 & 1 \\ 6 & 18 \end{bmatrix}.$$

Hint: one right singular vector is

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Solution: Based on the fact that the right singular vector is unitary, we can solve it with $v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, which gives:

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$AV = \begin{bmatrix} 17 & 1 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{18}{\sqrt{2}} & \frac{16}{\sqrt{2}} \\ \frac{24}{\sqrt{2}} & -\frac{12}{\sqrt{2}} \end{bmatrix}$$

Notice that $U\Sigma = AV = \begin{bmatrix} \frac{18}{\sqrt{2}} & \frac{16}{\sqrt{2}} \\ \frac{24}{\sqrt{2}} & -\frac{12}{\sqrt{2}} \end{bmatrix}$. Since U is also a unitary matrix, it can be solved that $U = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \frac{30}{\sqrt{2}} & 0 \\ 0 & \frac{20}{\sqrt{2}} \end{bmatrix}$

Problem 3. Exercise 6.1 in Trefethen and Bau. In addition, illustrate with a sketch showing the effect of $A = I - 2P$ on a typical vector v for the case

$$P = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Also explain why a matrix A of this form is sometimes called a "reflector". **Hint:** See also Figure 10.2 in the book and the discussion for the case of a "Householder reflector".

Exercise 6.1:
If P is an orthogonal projector, then I - 2P is unitary.
Solution:

$$(I - 2P)(I - 2P)^* = (I - 2P)(I - 2P^*) = I - 2P - 2P^* + 4PP^*$$

And since for orthogonal projector, $P = P^* = P^2$, the equation above becomes:

$$(I - 2P)(I - 2P)^* = I - 2P - 2P + 4P^2 = I - 4P + 4P = I$$

Therefore, $I - 2P$ is unitary.

Let $P = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and then $A = I - 2P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & -\frac{3}{5} \end{bmatrix}$
As for a general vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, we have $Av = \begin{bmatrix} \frac{3}{5}v_1 - \frac{2}{5}v_2 \\ -\frac{2}{5}v_1 - \frac{3}{5}v_2 \end{bmatrix}$

Problem 4. Exercise 6.3 in Trefethen and Bau. Use the SVD to do this.

Exercise 6.3
Given $A \in \mathbb{C}^{m \times n}$ with $mleqn$, show that A^*A is nonsingular if and only if A has full rank.
Solution:
Perform SVD for matrix A and let $A = U\Sigma V^*$. Then A is of full rank if and only if Σ is of full rank. Since $A^*A = V\Sigma^*\Sigma V^*$, to regard this as an eigenvalue decomposition, A^*A is non-singular if and only if all eigenvalues are non-zero which means the diagonal matrix Σ doesn't have zero on its diagonal and thus it is full rank. Therefore, A^*A is nonsingular if and only if A has full rank.

Problem 5. Exercise 6.4 in Trefethen and Bau.

Exercise 6.4
$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ **Solution:**
$range(A) = < \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} >$
and $range(B) = < \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} >$
(a)From equation 6.13,

$$P_A = A(A^*A)^{-1}A^* = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

And the image under P of the vector $(1,2,3)^*$ is:

$$\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$$

(b) Same approach on B:

$$P_B = B(B^*B)^{-1}B^* = \frac{1}{6} \begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

And the image under P of the vector $(1,2,3)^*$ is:

$$\frac{1}{6} \begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \end{bmatrix}$$

Problem 6. Exercise 7.1 in Trefethen and Bau.

Problem 7. Exercise 7.2 in Trefethen and Bau.

Problem 8. Exercise 8.2 in Trefethen and Bau.

You can write a Python function instead of Matlab if you prefer.

Test your routine on the matrices from Exercise 6.4 of Trefethen and Bau and submit your output.

Test it on other matrices of different sizes to convince yourself it is working properly. You do not need to turn in more output, but submit the code in a file `mgs.m` or `mgs.py` in a way that we can test it on other matrices of our choosing.

Please write readable code!