2D incompressible Navier–Stokes

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The Navier–Stokes equations for an incompressible fluid of unit density are:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u} + g \hat{\boldsymbol{z}} \,, \tag{1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,. \tag{2}$$

Let us do some simplifications. Assume that the flow is two-dimensional that is it is confined on the (x, y)-plane: $\boldsymbol{u} = (u, v, 0)$. The N–S equations written for each flow component are:

$$\partial_t u + u \partial_x u + v \partial_y u = -\partial_x p + \nu \nabla^2 u \,, \tag{3}$$

$$\partial_t v + u \partial_x v + v \partial_y v = -\partial_y p + \nu \nabla^2 v \,, \tag{4}$$

$$\partial_x u + \partial_y v = 0 \,. \tag{5}$$

For two-dimensional incompressible flow we can define a streamfunction $\psi(x, y, t)$ so that

$$u = -\partial_y \psi \,, \quad v = \partial_x \psi \,. \tag{6}$$

This way Eq. (5) is trivially satisfied by definition so we don't have to worry about it anymore. Now, both Eqs. (3) and (4) involve *only* two fields: the streamfunction $\psi(x, y, t)$ and pressure $p(x, y, t)$.

We can discard pressure if we consider the vorticity of the fluid, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$. Since the flow is two-dimensional the vorticity only has one non-zero component:

$$\boldsymbol{\omega} = (0, 0, \underbrace{\partial_x v - \partial_y u}_{\equiv \zeta}) \,. \tag{7}$$

The z-component of the vorticity ζ is obtained from the streamfunction as $\zeta = \nabla^2 \psi$. By computing: $\partial_x(\text{Eq. (4)}) - \partial_y(\text{Eq. (3)})$ we can show that:

$$\partial_t (\nabla^2 \psi) + u \partial_x \nabla^2 \psi + v \partial_y \nabla^2 \psi = \nu \nabla^4 \psi \,, \tag{8}$$

or equivalently in terms of ζ:

$$\partial_t \zeta + \underbrace{(-\partial_y \nabla^{-2} \zeta)}_{=u} \partial_x \zeta + \underbrace{(\partial_x \nabla^{-2} \zeta)}_{=v} \partial_y \zeta = \nu \nabla^2 \zeta \,. \tag{9}$$

The above equation involves *only* one field, ζ, from which we can recover the flow (u, v)! For the inversion $\psi = \nabla^{-2} \zeta$ we need to specify boundary conditions for our flow fields. The easiest thing is to impose periodic boundary conditions. Then the inversion is rather easy (we will discuss it in class). Equation (9) is what we will solve numerically in class on Wednesday.

Exercise: Derive Eq. (9) from Eqs. (3) and (4).

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