Visualizing the Wilson Plug

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Overview

Motivation

In 1966, F.W. Wilson published a paper describing a method for modifying a C^∞ vector field on a n-manifold with zero Euler characteristic. This was done in such a way that the minimal sets of the new vector field were a finite collection of $(n-2)$-tori. We investigate this construction, and see how it can be rendered visually by a computer, in a simple 3-dimensional case.

Global definitions

- M^n will refer to a paracompact C^∞ n-manifold, possibly with boundary, with zero Euler characteristic.
- F will refer to a non-singular vector field on M^n.
- $B^n = \{(x_1, x_2, \ldots, x_{n-2}, r, \theta) : |x_i| \leq 1, r \in [0,1], \theta \in [0, 2\pi]\}$

Main Theorem

Let F be a non-singular vector field on M^n with N a submanifold of M^n ($\dim N < n$), to which F is transverse. Then for each integer $k \in [0, n-2]$, there is a C^∞ vector field G on M^n which coincides with F near N and whose only limit sets are a denumerable collection of invariant k-tori, of which at most a finite number are contained in any compact subset of M^n.

Corollary

If M^n is a compact manifold with boundary, and F is a non-singular vector field on M^n transverse to the boundary, then there is a vector field G on M^n, which coincides with F near ∂M^n, and which has a finite collection of invariant $(n-2)$-tori as it's only limit sets.

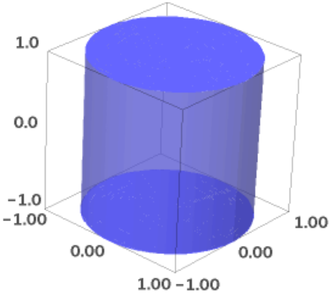
Flow boxes

A *flow box* is a closed n-cell U and a diffeomorphism $h : B^n \to U$, with closed $n-1$ cells βU (the bottom), τU (the top) and σU (the sides), with F transverse inwards on βU, transverse outwards on τU, tangent to σU, and parallel throughout the interior of U.

- h must be transverse to F when restricted to the bottom of B^n.
- Let ψ_t be the flow on U induced by F, and φ_t be the constant flow on B^n.
- There exists a positive c such that $\psi_{ct} \circ h(x) = h \circ \varphi_t(x)$
- We will generally omit h, and focus on the set U.

Flow boxes

If V is a subflow box, with $\beta V \subset \beta U$, $\tau V \subset \tau U$ and $\sigma V \subset$Int U, V is a *shrinkage* of U.



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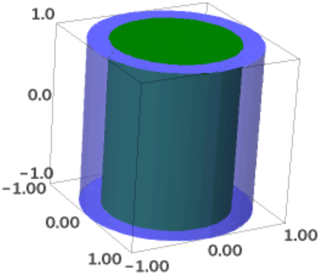

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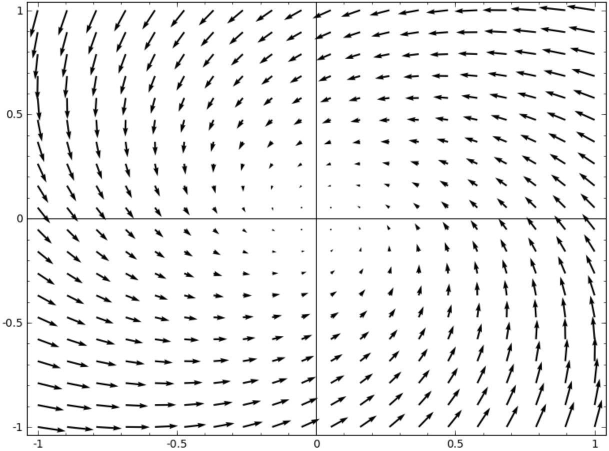

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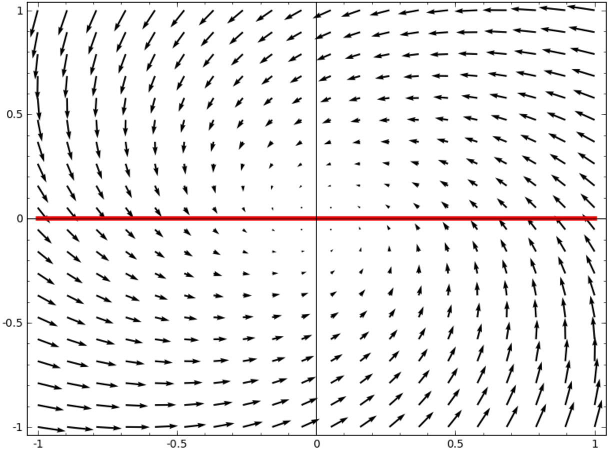

Theorem 1

Let N be a submanifold of M^n (with dimension $< n$, and possibly with boundary), to which F is transverse. Then there exists two families of flow boxes, $\{U_i\}$ and $\{V_i\}$, such that

1. The U_i are disjoint, and do not intersect N.
2. For each i, V_i is a shrinkage of U_i.
3. Each compact subset of M^n intesects only a finite number of the U_i.
4. $\bigcup V_i$ saturates M^n.

Lemma 1.1

If X is compact and $\{U_i\}$ is an open cover of X, then there exists $\epsilon > 0$ such that any family $\{V_i\}$ of open sets with $d_H(X \setminus U_i, X \setminus V_i) < \epsilon$, covers X, where d_H indicates the Hausdorff distance.

Lemma 1.2

If X is σ-compact, and $\{U_i\}$ is a relatively compact open cover of X, then there is a continuous positive function $\varphi : X \to \mathbb{R}$, such that any family of open sets $\{V_i\}$ with $d_H(X \setminus U_i, X \setminus V_i) \leq \min \{\varphi|\overline{U_i}\}$ covers X.

Lemma 1.3

Let U be a flow box for F on M^n, and let N be the union of a finite collection of submanifolds with boundary, each with dimension less than n, to which F is transverse. Let W be a given neighborhood of U. Then there is a finite family of flow boxes in W whose interiors cover U, and whose tops and bottoms are disjoint from one another and from N.

Lemma 1.4

Suppose that $\partial M^n = \emptyset$ and N is a submanifold of M^n, to which F is transverse. Then there is a covering of M^n by the interiors of a family of flow boxes satisfying:

- The tops and bottoms are disjoint from one another and from N.
- Only a finite number intersect any compact subset of M^n.

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If $\partial M^n \neq \emptyset$, sew a collar along the boundary of M^n to create a new manifold, M^*. Let $N = \partial M^n \subset M^*$, and the lemma holds.

Sketch of proof of Theorem 1

- Let $\{W_j\}$ be a family of flow boxes, with tops and bottoms disjoint from each other and from N, and only a finite number of which intersect each compact subset of M^n, as in Lemma 1.4.

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- Choose ϵ_j such that the families $\{U_i\}$ and $\{V_i\}$ are disjoint, where
  - $U_i = \{\varphi_t | x \in \tau W_i \text{ and } |t| \leq \epsilon_j\}$
  - $U_{-i} = \{\varphi_t | x \in \beta W_i \text{ and } |t| \leq \epsilon_j\}$
  - $V_i = \{\varphi_t | x \in \tau Y_i \text{ and } |t| \leq \epsilon_j\}$
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Sketch of proof of Theorem 1

Then the U_i are disjoint and do not intersect N, each V_i is a shrinkage of U_i, only finitely many U_i intersect each compact subset of M^n, and as $\{Y_i\}$ covers M^n and Y_i is saturated by $\tau Y_i \cup \beta Y_i$, $\bigcup V_i$ saturates M^n. Theorem 1 follows.

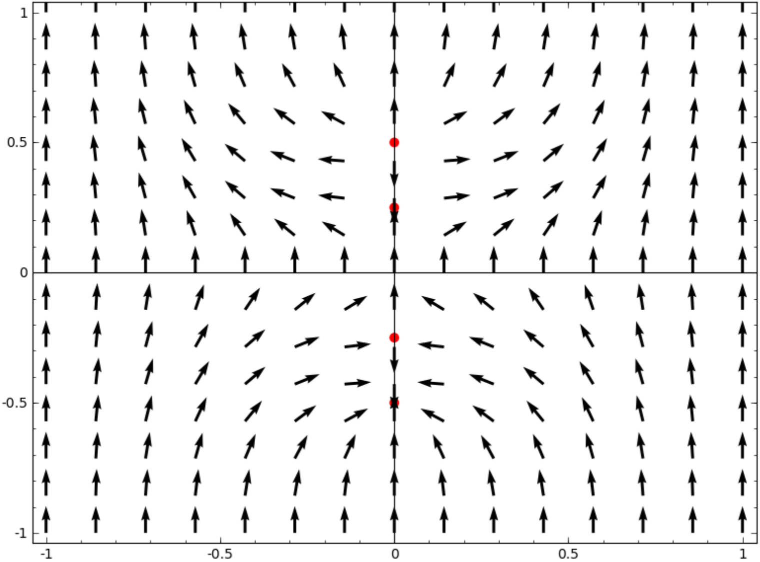
Mirror-image property

- If U is a flow box, let U_0 be a central hyperplane. A vector field F on U has the *mirror-image property* if the flow on the bottom half of U is the negative of the reflection of the flow on the top half of U.
- Let $F(x_1, \ldots, x_n) = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$, then

$$f_1(x_1, \ldots, x_n) = f_1(-x_1, \ldots, x_n)$$

$$f_i(x_1, \ldots, x_n) = -f_i(-x_1, \ldots, x_n) \text{ for } i = 2, \ldots, n.$$

Mirror-image property



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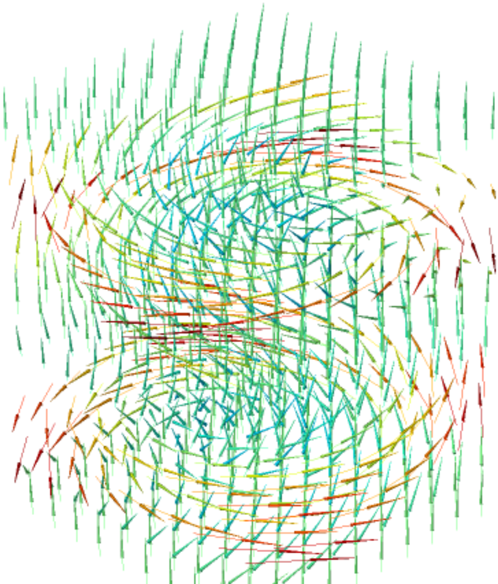


Theorem 2

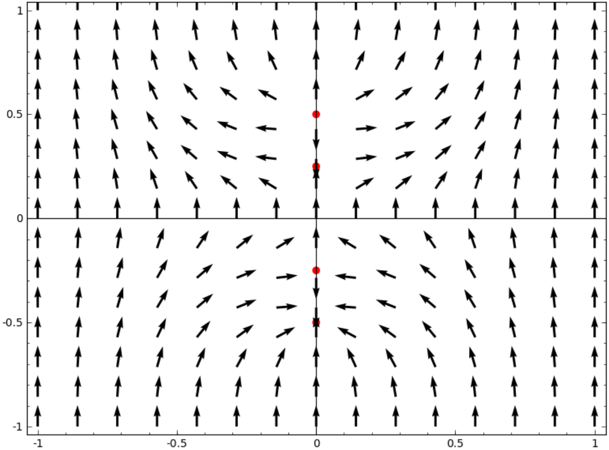
Let U be a flow box of F and V a shrinkage of U. Then for each integer $k \in [0, n-2]$, there is a vector field F^k on U, satisfying

1. F^k coincides with F on a neighborhood of the boundary of U.
2. The only limit sets of F^k are a finite collection of invariant $k-$tori on which the restricted flow is minial.
3. Every trajectory of F^k which intersets $\tau V(\beta V)$ has its α-limit(ω-limit) set in U.
4. F^k satisfies the mirror-image property.

Such a vector field F^k is called a k-annihilator for (U, V).

Special 0-annihilators

An annihilator on an n-dimensional flow box, which has as it's limit sets exactly 4 singular points, with Morse indices $0, 1, n-1$ and n.



Sketch of proof of Theorem 2

Begin with a flow box (U, h), and a shrinkage V of U which is disjoint from the image of the axis of B^n. Construct a special 0-annihilator system on a cross section of B^n homeomorphic to B^{n-1}, where all trajetories passing through are within some ϵ of the sides and the axis of B^n. Denote this F^t. Construct a field F^n, which is zero on a neighborhood of the axis of B^n and the boundary of B^n, which is ± 1 near the singular points of F^t, and which satisfies the mirror image property.

Sketch of proof of Theorem 2

Then $F = F^t + F^n$ satisfies the mirror image property, and each of the 4 singular points of F^t gives one periodic orbit in F. Then F is a special 1-annihilator system on B^n, and it's image under h is a 1-annihilator system on (U, V).

In higher dimensions, assume the existence of a $k-2$-annihilator, and proceed by induction.

Proof of Main Theorem

Let $\{U_i\}$ and $\{V_i\}$ be familes of flow boxes as in Theorem 1. Into each U_i, glue a k-annihilator system for (U_i, V_i), with the new vector field denoted G. If follows from the mirror-image property and the saturation property of $\bigcup V_i$ that any $G-$trajectory eventually enters some V_j positively and some V_l negatively. This trajectory must have it's α and ω limit sets as k-tori in U_j and U_l. Since only a finite number of the U_i intersect each compact subset of M^n, only a finite number of $k-$tori exist as limit sets of G.

Flow on a solid torus

Let F be a non-singular vector field on M^n with N a submanifold of M^n ($\dim N < n$), to which F is transverse.

Flow on a solid torus

- Discretize the torus
- Construct differential equations for the flow
- Render the flow in Sage

$S^1 \times S^1$

$[(3,0,0), (\cos(1/10*\pi)+2, 0, 1/4*\sqrt(5)-1/4), (1/4*\sqrt(5)+9/4, 0, \sin(1/5*\pi)), (\cos(3/10*\pi)+2, 0, 1/4*\sqrt(5)+1/4), (1/4*\sqrt(5)+7/4, 0, \sin(2/5*\pi)), (2,0,1), (-1/4*\sqrt(5)+9/4, 0, \sin(3/5*\pi)), (\cos(7/10*\pi)+2, 0, 1/4*\sqrt(5)+1/4), (-1/4*\sqrt(5)+7/4, 0, \sin(4/5*\pi)), (\cos(9/10*\pi)+2, 0, 1/4*\sqrt(5)-1/4), (1,0,0), (\cos(11/10*\pi)+2, 0, -1/4*\sqrt(5)+1/4), (-1/4*\sqrt(5)+7/4, 0, \sin(6/5*\pi)), (\cos(13/10*\pi)+2, 0, -1/4*\sqrt(5)-1/4), (-1/4*\sqrt(5)+9/4, 0, \sin(7/5*\pi)), (2,0,-1), (1/4*\sqrt(5)+7/4, 0, \sin(8/5*\pi)), (\cos(17/10*\pi)+2, 0, -1/4*\sqrt(5)-1/4), (1/4*\sqrt(5)+9/4, 0, \sin(9/5*\pi)), (\cos(19/10*\pi)+2, 0, -1/4*\sqrt(5)+1/4), (3*\cos(1/10*\pi), 3/4*\sqrt(5)-3/4, 0), ((\cos(1/10*\pi)+2)*\cos(1/10*\pi), 1/4*(\sqrt(5)-1)*(\cos(1/10*\pi)+2), 1/4*\sqrt(5)-1/4), (1/4*(\sqrt(5)+9)*\cos(1/10*\pi), 1/16*(\sqrt(5)+9)*(\sqrt(5)-1), \sin(1/5*\pi)), ((\cos(3/10*\pi)+2)*\cos(1/10*\pi), 1/4*(\sqrt(5)-1)*(\cos(3/10*\pi)+2), 1/4*\sqrt(5)+1/4), (1/4*(\sqrt(5)+7)*\cos(1/10*\pi), 1/16*(\sqrt(5)+7)*(\sqrt(5)-1), \sin(2/5*\pi)), (2*$

Parametric surface vs. Mesh

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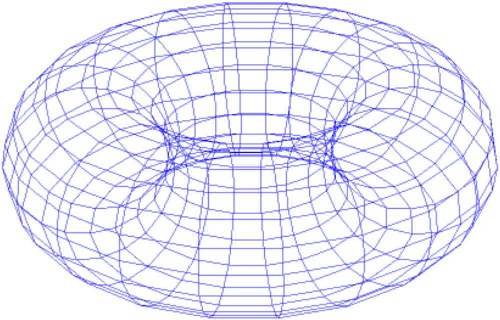

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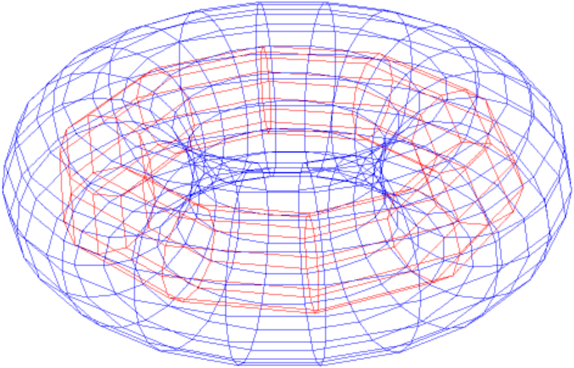

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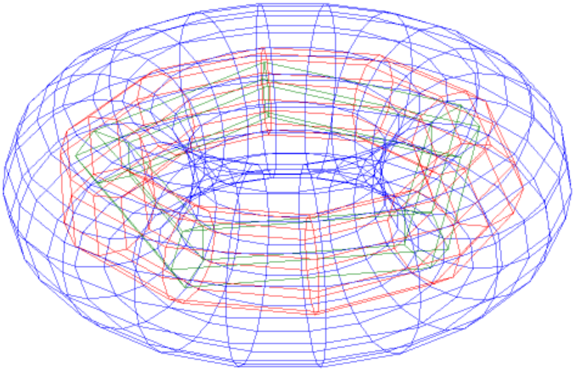

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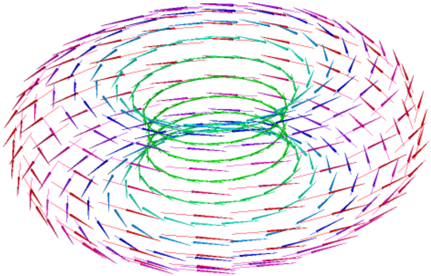

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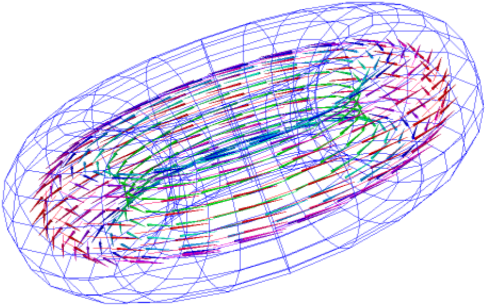

Constructing a finite number of Flow Boxes

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- The set of all V_p covers M^n, and so does a finite set $V_1, \dots, V_m$.
- Since the tops and bottoms are horizontal, let $V_i = [a_i, b_i] \times D^1$.
- Define $U_i = [a_i - c_i, b_i + d_i] \times D^1$
- For appropriate small choices of c_i and d_i, the boxes will have disjoint tops and bottoms.

Modifying the vector field

Then for each integer $k \in [0, n-2]$, there is a C^∞ vector field G on M^n which coincides with F near N and whose only limit sets are a denumerable collection of invariant k-tori, of which at most a finite number are contained in any compact subset of M^n.

Modifying the vector field

- Construct a 0-annihilator on B^2
- Extend the 0-annihilator on B^2 to a 1-annihilator on B^3

0-annihilator on B^2

Starting with $B^2 = [-1, 1]^2$, construct a field with 4 singular points, with appropriate Morse indicies. It's easy to choose (0,-1/2) as a sink, (0,-1/4) and (0,1/4) as saddles, and (0,1/2) as a source.

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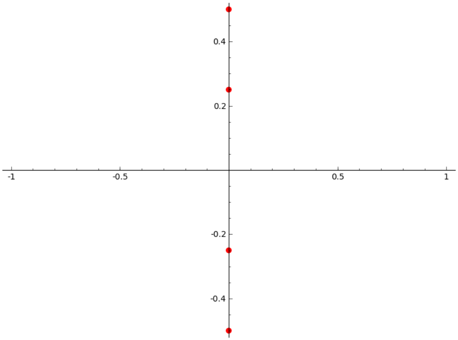

0-annihilator on B^2

A system that works is

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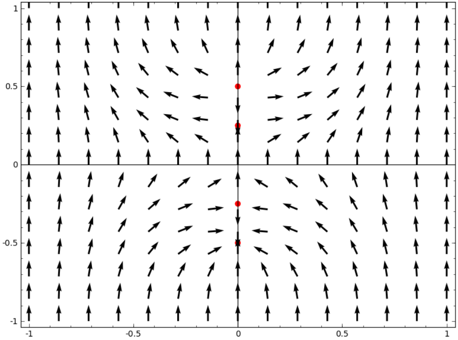


1-annihilator on B^3

Begin with the 0-annihilator on B^2 described above, then construct a new field, which is zero on the boundary and axis of B^3, and ± 1 in the ϵ-neighborhood of the singular points of the 0-annihilator.

1-annihilator on B^3

$$h(x) = \begin{cases} e^{-\frac{1}{1+x^2}} & \text{if} \quad |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$v_1(r) = \begin{cases} h(r-1) & \text{if} \quad r \in [0,1] \\ h(1-r) & \text{otherwise} \end{cases}$$

$$v_2(z) = \begin{cases} h(1-4z) & \text{if} \quad z \in [0,1/4] \\ h(8z-2) & \text{if} \quad z \in [1/4,3/8] \\ 1-h(|3-8z|) & \text{if} \quad z \in [3/8,1/2] \\ h(2z-1) & \text{otherwise} \end{cases}$$

1-annihilator on B^3

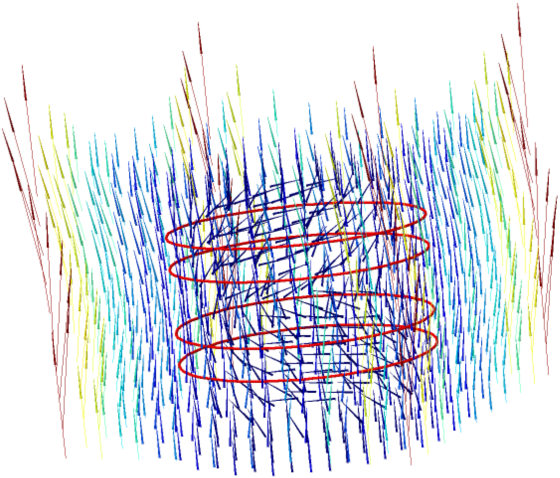
Let Y denote the 0-anihilator on B^2. For our vector field, if $z \geq 0$, we use

$$G = (v_1(r)v_2(z)y + Y(r,z), v_1(r)v_2(z)x + Y(r,z), g(r,z))$$

otherwise, use

$$G = (-v_1(r)v_2(|z|)y + Y(r,z), -v_1(r)v_2(|z|)x + Y(r,z), g(r,z))$$

1-annihilator on B^3



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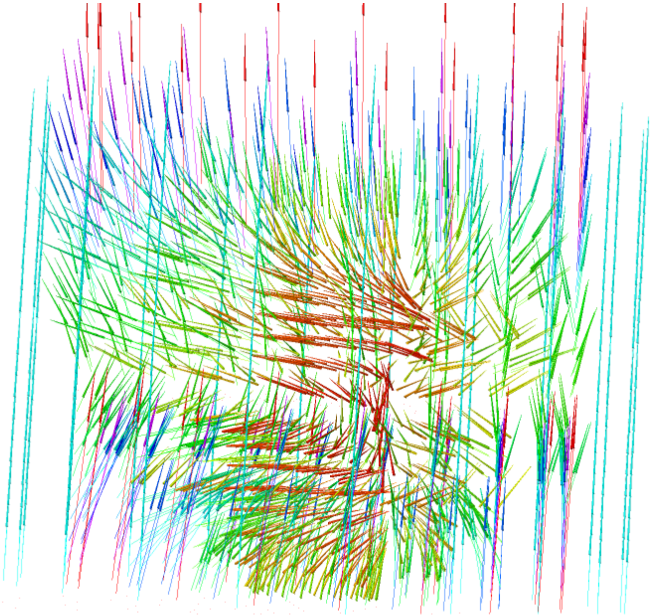

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Questions?