A MEASURED PL FOLIATION OF $\mathbb{R}^3$ WITH ALL LEAVES CONTAINED IN BOUNDED SETS

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Abstract. We present here a construction of a piecewise linear measured-foliation of $\mathbb{R}^3$, with each leaf contained in a bounded set. We construct a sequence of PL approximations of nested tori, and foliate each of these to achieve the bounds on the leaves. A slanted suspension is used to modify these foliations via a flow bordism. The modified foliation is measured. We also prove that this PL measured foliation yields a volume-preserving PL dynamical system.

1. Measured Foliations

We construct here a piecewise-linear, measured 1-foliation of $\mathbb{R}^3$, with all leaves contained in bounded sets. We begin with a PL decomposition of $\mathbb{R}^3$ into nested tori, akin to the smooth construction of [7]. Each of these regions is then foliated with 1-dimensional leaves. The foliations are modified using slanted suspension [8], [9], so that the foliations are measured. Results are then proven that the resulting foliation can be made into a measure-preserving dynamical system.

We begin by defining measured foliations. Assume we have an oriented 1-foliation $\mathcal{F}$ with support M.

A *nice atlas* is a foliation atlas satisfying three conditions [14]:

(1) The covering $\{U_\lambda\}$ is locally finite.
(2) For any φ_λ, the set $\varphi_\lambda(U_\lambda) \subset \mathbb{R}^n$ is an open cube, $(-1,1)^n$.
(3) If φ_{λ_1} and φ_{λ_2} are each chart maps, with $U_{\lambda_1} \cap U_{\lambda_2} \neq \emptyset$, then there exists some φ_{λ_3}, such that $\varphi_{\lambda_3}(U_{\lambda_3})$ is an open cube, U_{λ_3} contains the closure of $U_{\lambda_1} \cup U_{\lambda_2}$, and $\varphi_{\lambda_1} = \varphi_{\lambda_3}|U_{\lambda_1}$.

For an arbitrary foliated manifold, a nice atlas always exists [14]. In a nice atlas, each plaque is diffeomorphic to (-1,1).

To define a measure on a foliation, begin with an n-manifold M and a codimension $n-k$ foliation of M, denoted $\mathcal{F}$, with leaves $\mathcal{L}$. We take a collection of closed disks called *flow boxes* of the form $D_k \times D_{n-k}$, whose interiors cover M. We want each $\mathcal{L}_\alpha$ passing through a flow box to intersect the box in a collection of horizontal disks $D_k \times \{y\}$. We then say the flow box is *transverse* to the foliation.

A *transversal* T is a smooth $(n-k)$-dimensional submanifold which is transverse to each $\mathcal{L}_\alpha$.

We are working only with 1-foliations here, so for all considerations above, $k = 1$. In this case, we can think of flow boxes like charts in an atlas, where the pre-image of each leaf under the chart map has all but one variable constant.

T is *small* if it can be surrounded by a single flow box. A *transverse measure* μ on $\mathcal{F}$ is a function which assigns each small transversal a finite non-negative number[13]. Further, μ must be additive on a union of transversals. This additivity allows us to assign a measure to a transversal which is not small, by assigning it the sum of its measure in all flow boxes which contain it.

If α and β are two small transversals, an *isotopy* is a map $f : \alpha \times [0,1] \to \beta$, such that, for each $t \in [0,1]$, the restriction f_t which maps $x \in \alpha$ to $(x,t) \in \alpha \times [0,1]$ is an embedding. The *track* of the isotopy is the embedding $\hat{f} : \alpha \times [0,1] \to \beta \times [0,1]$, with $(x,t) \mapsto (f(x,t),t)$ [6]. For each $t \in [0,1]$, the isotopy is essentially constructing a path through the embeddings. We will be interested in isotopies where this path is parallel to the leaves of our foliation.

Given two transversals, α, β, a measure μ is *invariant* if, when α and β are isotopic, with isotopy parallel to the foliation, then $\mu(\alpha) = \mu(\beta)$. Then the pair $(\mathcal{F}, \mu)$ is a *measured foliation.* [4]

In some sense we can think of this isotopy as moving one transversal onto another, with each point in the transversal remaining on the same leaf throughout the isotopy. The sets U_λ in the atlas for the foliation serve as flow boxes. To define a measure, take a small transversal α, and let ω be the $n-1$ Euclidean volume form on $\mathbb{R}^n$. Define $\mu(\alpha) = ||\omega(\varphi_\lambda(\alpha))||$ [4]. This measure is locally Lebesgue, and provides convenient coordinates for our calculations.

2. Flow bordisms

We have three types of boundary we wish to consider for a n-manifold M with a 1-foliation $\mathcal{F}$, as defined in [9]. The *transverse boundary* is the portion of the boundary of M where $\mathcal{F}$ is locally modeled by the foliation of $\mathbb{R}^n$ by vertical lines. The *parallel boundary* is the portion of the boundary of M where $\mathcal{F}$ is locally modeled by the foliation of the upper half-space of $\mathbb{R}^n$ by horizontal lines. *Corners* occur at points on the boundary which are neither parallel boundary, nor transverse boundary.

Given a connected, compact manifold P, a *flow bordism* $\mathcal{P}$ is an oriented 1-foliation of P, such that all boundary of P is either transverse, parallel, or corners [9].

Let F_- be the closure of the transverse boundary oriented inwards, and F_+ be the closure of the transverse boundary oriented outwards. We have two additional properties in which we are interested.

- (i) There exists an infinite leaf with an endpoint in F_-
- (ii) There exists a manifold F and two homeomorphisms $\alpha_- : F \to F_-$, $\alpha_+ : F \to F_+$ such that if $\alpha_+(p)$ and $\alpha_-(q)$ are endpoints of a leaf of $\mathcal{P}$, then $p = q$.

If a flow bordism satisfies condition (i), but not condition (ii), it is a *semi-plug*. It is has condition (ii), it has *matched ends*. A flow bordism which satisfies (ii) but not (i) is an *un-plug*. If $\mathcal{P}$ has properties (i) and (ii), it is a *plug* [9].

If $\mathcal{P}$ is a plug, the manifold P is the *support* of P.

If $\mathcal{P}$ has matched ends, then F is the *base* of $\mathcal{P}$. The set of all points in F_- which are endpoints of infinite leaves is the *entry stopped set* of $\mathcal{P}$. Similarly, the points in F_+ which are endpoints of infinite leaves is the *exit stopped set*. Whenever $\mathcal{P}$ has matched ends, the set $S = \alpha_-^{-1}(S_-) = \alpha_+^{-1}(S_+)$, is called the *stopped set*. If S has non-empty interior, then $\mathcal{P}$ *stops content*.

Modification of a foliation using a plug requires the operation of *insertion*. Let $\mathcal{P}$ be a flow bordism, and $\mathcal{X}$ a foliation on some manifold M. An *insertion map* is an embedding of the base of $\mathcal{P}$, which is transverse to $\mathcal{X}$. Denote the embedding map σ. A flow bordism $\mathcal{P}$ is *insertible* if there is an embedding of $\mathcal{P}$ into $\mathbb{R}^n$ which is transverse to vertical lines [9].

In essence, we are looking for the base of $\mathcal{P}$ to be entirely transverse boundary. That will ensure that, when a leaf from $\mathcal{X}$ intersects the base of the flow bordism, that leaf will only intersect a single leaf in the base of the flow bordism. The parallel boundary will guarantee that when a leaf exits the bordism, it will also intersect a single leaf.

The next component we need to consider, is attachability. As per [9], a plug is *attachable* if every leaf in the parallel boundary is finite. Assume we have a plug $\mathcal{P}$, with base F. Consider an open neighborhood of $F \times [0,1]$, denoted $N_{F\times[0,1]}$. Then the embedding $\sigma((F \times [0,1]) \setminus N_{F\times[0,1]})$ will glue the open lip of $N_{F\times[0,1]}$ to the support P of the plug $\mathcal{P}$. This is done via an *attaching map*, denoted $\alpha : N_{F\times[0,1]} \to N_P$, where N_P is a neighborhood of the boundary of P. The attaching map should also satisfy $\alpha(p, 0) = \alpha_-(p)$ and $\alpha(p, 1) = \alpha_+(p)$, where α_- and α_+ are the maps on the closure of the transverse boundary of $\mathcal{P}$ from the definition of a flow bordism.

Since all leaves in the parallel boundary are finite, we are able to do this via a leaf-preserving homeomorphism [9].

We also address the idea of an *untwisted* plug [9]. A plug $\mathcal{P}$ with base F and support P is untwisted if the attaching map extends to a homeomorphism $F \times [0,1] \to P$.

We will construct a sequence of flow bordisms, which are used in our main result. These flow bordisms will always have all leaves in the parallel boundary finite. Furthermore, the open neighborhoods around the embeddings of the base of the flow bordisms will always be foliated with vertical lines. We will therefore have insertibility. All leaves in the parallel boundary will be finite, so we will have attachability. Our attaching maps will extend to homeomorphisms, and our flow bordisms will remain untwisted.

In our constructions, we will only require a flow bordism. That is all that is necessary for our modifications, so it is the only condition that will be checked.

We will construct the foliations to be modified by the flow bordism such that the neighborhood around the space where the bordism is to be inserted is foliated entirely by vertical lines. This will automatically satisfy the condition for insertability from [9]. Careful choices regarding the size of the bordism will prevent any changes in the topology of the original foliated space after insertion.

3. Suspension foliations

Let M be an n-manifold, and f an bijection on M. Take the quotient of $M \times [0,1]$ under the equivalence relation $(f(x),1) \sim (x,0)$. We denote the resulting quotient space $M_{\sim f}$. We have a natural way to get a 1-foliation of this space.

For $x_0 \in M$, define

- $\mathcal{O}_{x_0} = \{\dots, f^{-2}(x_0), f^{-1}(x_0), x_0, f(x_0), f^2(x_0), \dots\}$.
- $L_{x_0} = \{(t,x), t \in [0,1], x \in \mathcal{O}_{x_0}\}$.

Using the orientation induced from [0,1],

$$\mathcal{F} = \cup_{x_0 \in M} L_{x_0}$$

is then an oriented 1-foliation of $M_{\sim f}$.

We assume here that f is a PL-homeomorphism.

A map $g : M \times [0,1] \to M \times [0,1]$ is a pseudo-isotopy if g is the identity when restricted to $M \times \{0\} \cup \partial M \times [0,1]$. Letting f be the restriction of g to $M \times \{1\}$, we say that f is *pseudo-isotopic to the identity.*

Another way to think of this is, if f is a diffeomorphism (or PL homeomorphism) of M, and there exists a diffeomorphism (again, or PL homeomorphism) g of $M \times [0,1]$ such that for all $x \in M$, $g(x,0) = id(x)$ and $g(x,1) = f(x)$.

In some modern usage, the map f is referred to simply as a pseudo-isotopy. We will refer to it here as a map which is pseudo-isotopic to the identity.

We can similarly say that, if g is an isotopy, and f is the restriction of g to $M \times \{1\}$, then f is *isotopic to the identity.*

If f is isotopic or pseudo-isotopic to the identity, then the support of the foliation on $M_{\sim f}$ is homeomorphic to $M \times S^1$ [8]. This follows from the fact that, if f is the identity map, then $M_{\sim f} = M \times S^1$.

Theorem 3.1. *If f is a volume-preserving homeomorphism of M, then the suspension $M_{\sim f}$ is a measured-foliation.*

Proof. Let $f : M \to M$ be volume-preserving, for some measure μ on M. We define a measure μ' on $M_{\sim f}$. Let α be an $n-$dimensional submanifold of $M_{\sim f}$ which is transverse to the foliation $\mathcal{F}$ on $M_{\sim f}$. Let α' be the projection of α along the leaves of the foliation onto M. Let $\mu'(\alpha) = \mu(\alpha')$. Then for two isotopic transversals α and β, $\alpha' = f^n(\beta')$ for some integer n. As f preserves volume, $\mu(\alpha') = \mu(f^n(\beta'))$, and we conclude $\mu'(\alpha) = \mu'(\beta)$. Therefore, the foliation $(M_{\sim f}, \mathcal{F})$ is measured. □

Theorem 3.2. *Given a PL manifold K, the suspension of K under some PL-homeomorphism f is still a PL manifold.*

Proof. Let K be a PL-manifold, and $f : K \to K$ a PL-homeomorphism. K and $f(K)$ are each triangulations of the same polyhedron, so there exists a common subdivision, which we denote L. Triangulate $L \times [0,1]$, take the second barycentric subdivision, and construct the quotient for the suspension. The result is triangulable, □

A property of PL homeomorphisms we will be interested in, is when these maps are measure preserving. To do this, we will need to define a measure on our PL spaces. Let M be a PL n-manifold. A measure is *simplicial* on M if, for some triangulation T of the manifold, the measure is given by the measure of the linear embedding of an m-simplex of T in $\mathbb{R}^m$. A simplicial measure can then be thought of as a volume form which assigns the m-dimensional Euclidean volume to an m-simplex [1]. Any such volume-form will be piecewise-constant on the entire n-manifold M, since it will return a constant volume on each simplex.

Thanks to G. Kuperberg [8], we have a theorem which allows us some flexibility in our choice of PL measures. This is a piecewise-linear analog of the results of Moser [11], Banyaga [2] and Bruveris, et. al, [3].

Theorem 3.3. *Two simplicial measures on a connected, compact, PL-manifold, with the same total volume, are equivalent under a PL homeomorphism. Moreover, any simplicial measure is locally PL-Lebesgue.*

This result is strengthened in Henriques and Pak [5].

Theorem 3.4. *Let M_1 and M_2 be two PL-manifolds, possible with boundary, which are PL-homeomorphic and equipped with piecewise-constant volume-forms ω_1, ω_2. If M_1 and M_2 have equal volume, then there exists a volume-preserving PL-homeomorphism $f : M_1 \to M_2$. That is, the pullback of f gives $f^*(\omega_2) = \omega_1$.*

The main difference here is that we have removed the requirement for the manifold to be compact and connected. The pullback here indicates the PL map, defined by $f^*(\omega_2)(\sigma) = \omega_2(f(\sigma))$, where σ is a simplex in M_2.

Regarding the notion of a locally PL-Lebesgue measure [8], while the simplicial measure may not agree with the Lebesgue measure on the entire PL manifold, when restricted to a single simplex, it will agree with Lebesgue measure.

Given an n-dimensional PL-manifold M and a measure μ on M, a function f defined on M is an *measure-preserving* PL-homeomorphism if there is a triangulation of $f(M)$, whose simplices have the same measure as their pre-image under f.

Lemma 3.5. *Let M be a PL n-manifold and $\mathcal{F}$ a measured 1-foliation of M. If M is retriangulated, $\mathcal{F}$ remains measured.*

Proof. Let μ be a transverse measure on $\mathcal{F}$, and α a transversal in M. Let β be a transversal in M, isotopic to α, so that $\mu(\alpha) = \mu(\beta)$. Retriangulating M does not affect the flow boxes used in computing the transverse measure of the small transversals which make up α and β, so it is possible to keep μ the same. Therefore, under the retriangulation, $\mu(\alpha) = \mu(\beta)$, and the foliation remains measured. □

Lemma 3.6. *Let M be a PL n-manifold, and $\mathcal{F}$ a measured 1-foliation of M. Let $g : M \to M$ be a PL-homeomorphism which preserves simplicial measure on the simplices for some triangulation of M. Then the foliation $g(\mathcal{F})$ is a measured foliation.*

Proof. Let μ be a transverse measure on $\mathcal{F}$, with α and β isotopic transversals. Without loss of generality, assume α and β are small, contained in flow boxes U_α and U_β respectively. Let η be the simplicial measure on M, then $\eta(U_\alpha) = \eta(g)U_\alpha)$, and similarly for U_β. Now consider $g(\alpha)$ and $g(\beta)$. Since g is a PL-homeomorphism, $g(\alpha)$ and $g(\beta)$ are still isotopic. If $g(\alpha)$ and $g(\beta)$ did not possess the same transverse measure, there would exists a retriangulation of M on which areas of simplices would not be preserved under g. We conclude that $g(\mathcal{F})$ remains a measured foliation. □

Corollary 3.7. *A volume-preserving PL flow bordism $\mathcal{P}$ may be inserted into a foliation of $\mathbb{R}^n$, while preserving Lebesgue measure on $\mathbb{R}^n$.*

Proof. Let V represent the total volume of the support P of $\mathcal{P}$. Scale V linearly, so that the total volume of the P is the same as that of the Euclidean volume of the region of $\mathbb{R}^n$ which is being replaced with $\mathcal{P}$. It follows from Theorems 3.3 and 3.4 that the measure on $\mathcal{P}$ may be modified via a PL-homeomorphism to agree with Lebesgue measure. □

4. Slanted suspensions

We will require the use of a *slanted suspension* [8, 9] in the following constructions, which is similar to the suspension foliations discussed above.

As given in [8, 9], for any compact manifold H, let $f : H \times [a, b] \to H \times [a, b]$ be a PL homeomorphism. Fix a real number $l \in (0, 1)$. For each fixed point $x \in H$ and $y \in [a, b]$, let $L_{xy} = \{(x, y + lz, z) : y + lz \in [a, b] \text{ and } z \in [0, 1]\}$. Then $\mathcal{L} = \{L_{xy} : \forall (x, y) \in H \times [a, b]\}$ is a foliation of $H \times [a, b] \times [0, 1]$, with leaves oriented from $H \times [a, b] \times \{0\}$ to $H \times [a, b] \times \{1\}$. The *slanted suspension* of $H \times [a, b]$ with *slant* l is the foliation of the quotient space generated by the equivalence $(f(x, y), 0) \sim (x, y, 1)$, with a foliation $\mathcal{F}$ induced by $\mathcal{L}$. We denote this as $M_{\sim f,l}$.

To use the slanted suspension to construct a PL analog of the earlier constructions, we need that the slanted suspension can yield a measurable foliation.

Theorem 4.1. *Let M be a 2 dimensional PL-manifold, μ a simplicial measure on M, and f a measure-preserving PL-homeomorphism on M. Then the foliation $M_{\sim,f,l}$ given by the slanted suspension of M under f, with slant l, is measured.*

Proof. Let $M_{\sim,f,l}$ be the slanted suspension in the theorem. Let α and β be two isotopic transversals. A transverse measure on the foliation can be given by the extending the measure used on M, as each transversal is isotopic to some subset of M, we assign the transversal the measure of the subset of M to which it is isotopic. Therefore if α is isotopic to β, they are each isotopic to some subset of M, and therefore, the measures of α and β coincide. We conclude that the suspension is measured. $\square$

Just as we have constructed flow bordisms for smooth flows and smooth foliations, we can construct flow bordisms to modify PL-foliations using the slanted suspension. Given an measure-preserving PL-homeomorphism f of a planar region T, which fixes the boundary of T. Depending on our choice of f, it is possible to use the slanted suspension to create a flow bordism, a semi-plug, or a plug.

Theorem 4.2. *Let $M = [a, b] \times [c, d]$, and $f : M \to M$ a PL-homeomorphism. Take the slanted suspension of M with slant l. If for all $p \in \partial M$, $f(p) \neq p - l$, then $M_{\sim,f,l}$ may be made into a flow bordism, via a leaf-preserving map $g : M_{\sim,f,l} \to M_{\sim,f,l}$. Furthermore, if $M_{\sim,f,l}$ is a measured-foliation, g may be chosen such that $g(M_{\sim,f,l})$ remains measured.*

Proof. Assume that $M_{\sim,f,l}$ is as in the theorem. Denote P to be the support of the foliation $M_{\sim,f,l}$. Then $P = [a, b] \times [c, d] \times [0, 1]$, with $[a, b] \times [c, d] \times \{0\} \sim [a, b] \times [c, d] \times \{1\}$. P is then a cylinder with an annular base. We use coordinates (x, y, z), as though we were in the unquotiented space $[a, b] \times [c, d] \times [0, 1]$.

F_- is the subset of P with y coordinate c, and F_+ is the subset of P with y coordinate d.

This is because the definition of the slanted suspension requires that $l > 0$. Any leaf passing through F_- will have an increasing y coordinate following the orientation of the leaf, and any leaf through F_+ will have a decreasing y coordinate following the reversed orientation of the leaf. The non-zero slant prevents any portion of the sets we've designated F_- or F_+ from satisfying the definition of parallel boundary.

To create parallel boundary, we are interested in the components of the boundary for which $x = a$ or $x = b$. This is the boundary component which we have not already designated F_- or F_+.

We will retriangulate these boundary components, in order to define a PL-homeomorphism on P which yields parallel boundary on these components. The condition $f(p) \neq p - l$ is essential to guarantee that there are no circular leaves in the boundary, which would prevent the possibility of parallel boundary.

Begin with a leaf L_0, through $(a, c, 0)$. In the unquotiented space, this leaf will intersect the point $(a, c + l, 1)$, and will continue through the point $(a, f(c + 1), 0)$. In P, this leaf will wrap around the boundary of the cylinder, until identified with the appropriate point. The map we are looking for here is a PL analog of a map which fixes the base of a cylinder, and rotates the top until the leaves are unwound and vertical.

Let $x_0 = c$, $y_0 = c+l$, $x_1 = f(y_0)$, $y_1 = x_1+l$, and so on, with $x_n = f(y_{n-1})$ and $y_n = x_n+l$. Our condition that avoids circular leaves guarantees that this leaf eventually intersects a point with z coordinate d. Use the sequences x_i and y_i to create vertices in the line segment $[c, d]$ and let $\{z_i\}$ be a common subdivision of $[c, d]$. This allows L_0 to move through a sequence of vertices each time it wraps around the boundary of P. In acting on those vertices, we can see this leaf become vertical through a PL-homeomorphism. Since all leaves on this boundary component are parallel to this leaf, this will allow us to create parallel boundary, while preserving F_- and F_+.

Let w be the largest index in $\{z_i\}$. In the case that f is the identity on the boundary, $w = |d - c|/l$. We work in the unquotiented version of P to create the simplices on which g will act. Subdivide $[0, 1]$ and $[a, b]$ each using w evenly spaced vertices.

We will first triangulate $[c, d] \times [0, 1]$, and use this to define g. This is done by adding edges to make w^2 squares, then adding an edge in each square, moving from lower left to upper right. By lower left we mean the vertex closest to $(c, 0)$, and by upper right, the vertex closest to $(d, 1)$.

Label the vertices $v_0, v_1, \ldots, v_{w-1}, v_w, \ldots v_{w^2}$, moving left to right, and from vertices with second coordinate c to the vertices with second coordinate d.

Let g be the map which fixes $v_0, v_1, \ldots, v_{w-1}$, and shift all other vertices to the left (decreasing their z coordinate), accounting for the identified vertices in the slanted suspension. Once all leaf segments with endpoints on $[a, b]\times\{c\}\times[0, 1]$, are vertical, fix vertices $v_0, v_1, \ldots, v_w, v_{w+1}, \ldots, v_{2w-1}$ and repeat the process. This is shown in the case that $w = 1$ in Figures 1-5.

g is applied to the vertices of $\{a\} \times [c,d] \times [0,1]$ and $\{b\} \times [c,d] \times [0,1]$ simultaneously, creating parallel boundary on both components. Since no vertex is mapped to a vertex with a lower y coordinate, the transverse boundary is preserved, as is the orientation of the leaves intersecting the transverse boundary.

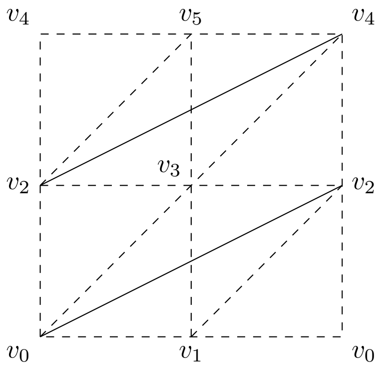

FIGURE 1. Initial triangulation

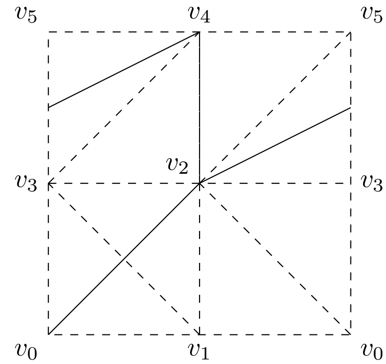

FIGURE 2. Shift $v_2, \ldots, v_5$ one vertex to the left.

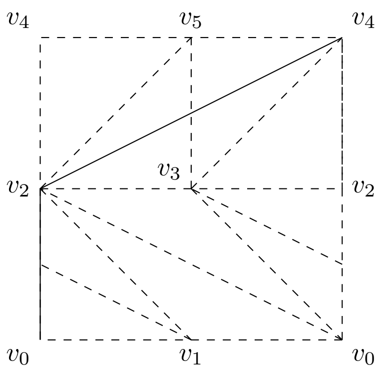

FIGURE 3. Shift $v_2, \ldots, v_5$ one vertex to the left.

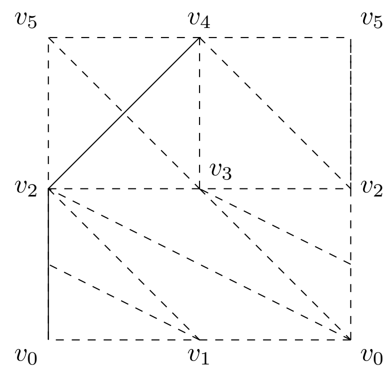

FIGURE 4. Shift v_4 and v_5 one vertex to the left.

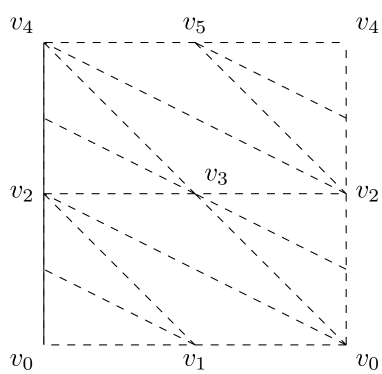

FIGURE 5. Shift v_4 and v_5 one vertex to the left.

Next, we show how g is defined on all of $M_{\sim,f,l}$. We have already subdivided $[a,b]$ into w evenly spaced vertices, and at each of these we triangulate the copy of $[c,d] \times [0,1]$ exactly as before. Add further edges so that each of the squares in $c, d] \times [0,1]$ is now the face of a cube. We can triangulate each face of this cube, and by adding a vertex at the center of the cube, add 3 additional edges connecting opposite vertices of the cube, to fully triangulate the cube.

g shifts all vertices on copies of $[c,d] \times [0,1]$ as before, while fixing the newly added center vertices. The advantage of this triangulation is that g preserves area on all 2-simplices, and since the center vertices of each cube are fixed, the heights of the 3-simplices are also preserved, hence the volume of all simplices in the triangulation are preserved.

All leaves in $\{a,b\} \times [c,d] \times [0,1]$ are now vertical line segments, and so this component of the boundary of $g(M_{\sim,f,l})$ is parallel boundary. Since the slope was positive, no leaves will lie entirely in either $(a,b) \times \{c\} \times [0,1]$, or $(a,b) \times \{c\} \times [0,1]$, so the entire set $(a,b) \times \{c,d\} \times [0,1]$ is transverse boundary. We conclude that $g(M_{\sim,f,l})$ is a flow-bordism.

As g preserves volume on individual simplices, the second portion of the proof, follows immediately from lemmas 3.5 and 3.6.. ☐

5. Results for foliations

Given the correct volume-preserving homeomorphism, we can construct a flow bordism with the correct number of circular orbits and, if necessary, the desired stopped sets. Before moving on to the main result, we present the use of the slanted suspension to construct a PL foliation of $\mathbb{R}^3$ with all leaves contained in bounded sets. This foliation is analogous to the dynamical system created by Jones and Yorke [7]. This will illustrate the use of the slanted suspension to create a flow bordism, and show how the insertion is carried out. Our main proof will then use the same basic construction, but will show how the flow bordism can be made measure-preserving, and what other conditions need to be satisfied to ensure the entire foliation remains measured.

In this construction, we use Dehn surgery to remove a solid torus from the interior of a flow bordism. The bordism arises from a slanted suspension. We are only interested in the trivial Dehn surgery here, which does not change our manifold. We detail this construction in Theorem 5.1. We will construct PL nested tori, with specific choices of triangulation. We then ensure that our use of Theorem 4.2 creates a torus in the interior of our flow bordism whose border shares that same triangulation. This will permit the choice of appropriate meridians to ensure that no Dehn twists occur during the construction.

Theorem 5.1. *There exists a PL 1-foliation of $\mathbb{R}^3$, with each leaf contained in a bounded set.*

Proof. We first nest simplicial approximations of tori (which we call PL tori) together, as per Jones-Yorke [7]. In each torus, there are 16 subdivisions in the triangulation along the major circumference, and 64 subdivisions in the triangulation along the minor circumference. The subdivisions along the major circumference are spaced in intervals of $\pi/8$. Since these simplicial approximations are in $\mathbb{R}^3$, they are always triangulable.[10] Let T_0^{PL} be a PL torus, in the xy-plane, centered at the origin, with minor radius $m_0 = 1$ and major radius $M_0 = 4$. Let T_1^{PL} be a PL torus, in the yz-plane, centered at (0,48,0), with minor radius $m_1 = 10$ and major radius $M_1 = 48$. Rotate the entire torus by an angle of $\pi/16$. This ensures that T_0^{PL} is between vertices on the boundary of T_1^{PL}, as shown in Figure 6. Note that the smaller torus in this figure has all of its vertices contained one one of the subdivided regions of the larger.

Let T_n^{PL} be a PL torus, in the xy-plane if n is even, and the yz-plane if n is odd, centered at (0,(4)12^n,0), with minor radius $m_n = \frac{5}{2}M_{n-1}$ and major radius $M_n = 12M_{n-1}$. Again, rotate by $\pi/16$ after each construction.

This gives a nested collection of PL tori, which union to $\mathbb{R}^3$.

We can foliate all of $\mathbb{R}^3$ using these PL Tori. If we foliate T_n^{PL} by 16-fold simplicial approximations of circles, stopping all leaves which intersect T_{n-1}^{PL} we will have a singular foliation. The idea is to then remove all of these singular points by inserting a sequence of flow bordisms. Note that our rotation of each torus ensures that the segments of a leaf in T_n^{PL} which intersect T_{n-1}^{PL} in a singular point consists entirely of vertical lines.

Referring again to Figure 6, note that all leaves in the segment of T_1^{PL} which contains T_0^{PL} are vertical. Our flow bordism will be constructed so that it is insertible, and therefore embeds transverse to vertical lines. The existing foliation consists of vertical lines, so no modification of leaves is needed during the insertion. Each leaf in the parallel boundary of our flow bordism will be finite, and the parallel boundary of the bordism

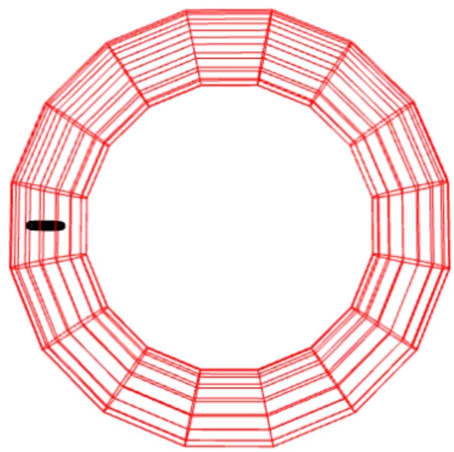

FIGURE 6. T_0^{PL} nested in T_1^{PL}.

will be PL homeomorphic to the product of an annulus and an interval. This careful construction will result in a bordism which is insertible, attachable, and untwisted. It is possible that an twist could be added when removing the torus from the interior of the flow bordism. The Dehn surgery performed will always have slope 1, so the bordism remains untwisted.

As stated above, the current stage of the construction is a singular foliation. To modify this foliation so that it is non-singular, we will construct a slanted suspension of a rectangle. The key is that this rectangle has a square region in the center, which is shifted down by a fixed quantity l, and that we use the same l for our slant in the suspension.

Let $M = [-3, 3] \times [-2, 2]$, with horizontal axis labeled x and vertical axis labeled z, triangulated as in Figure 7. Let f be the function which shifts the vertices (-1,-1), (-1,1), (1,1), and (1,-1) each down by 1/4, fixing all other vertices, as shown in Figure 8.

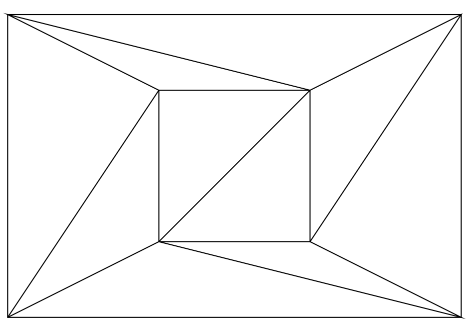

FIGURE 7. M, with simplices indicated

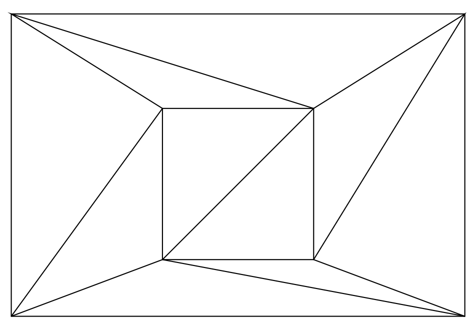

FIGURE 8. $f(M)$, with simplices indicated

Note that this does not preserve the area of each simplex. It is a PL-homeomorphism, so we can use this to create the slanted suspension $M_{\sim,f,1/4}$.

The support of $M_{\sim,f,1/4}$ is a cylinder, with an annular base, which we denote P. In taking the quotient, we have $P = \{(x, y, z) : 1 \leq \sqrt{x^2 + y^2} \leq 7, -2 \leq z \leq 2\}$. This does require a change of coordinates from the notation in the definition of the slanted suspension, but agrees with the coordinates we've given M above. We will proceed from here in the usual Cartesian coordinates.

Visualization of the leaves in $M\sim, f, 1/4$ is done with the return map, as in example 5.16. We look at the points where leaves intersect $M \times \{1\}$ in the unquotiented space. For any point (x, z) in the square at the center of M, $f(x, z) = (x, z - 1/4)$. However, the slant of 1/4, guarantees that there is a leaf connecting $(f(x, z), 0)$ and $(x, z, 1)$. We conclude that all leaves passing through that central square are circular. The square is PL-homeomorphic to S^1, so we have the PL analog of a foliation of a torus by circular leaves at the center of P. This is analogous to the constructions in examples 5.12 and 5.13. We avoid this situation in the parallel boundary, but it is useful to us in the interior of the bordism.

In Figure 9 we show leaves which approach the torus in the center of the bordism, and leaves which pass around the torus.

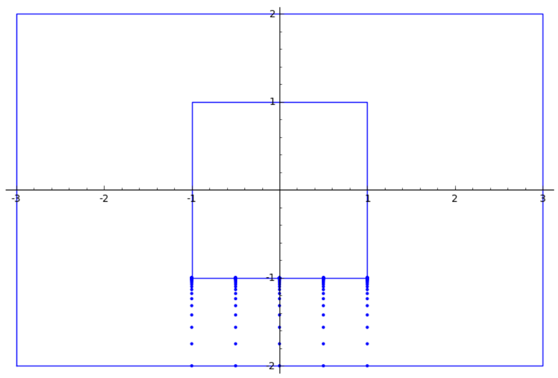

FIGURE 9. Illustration of a leaf approaching the square region in M

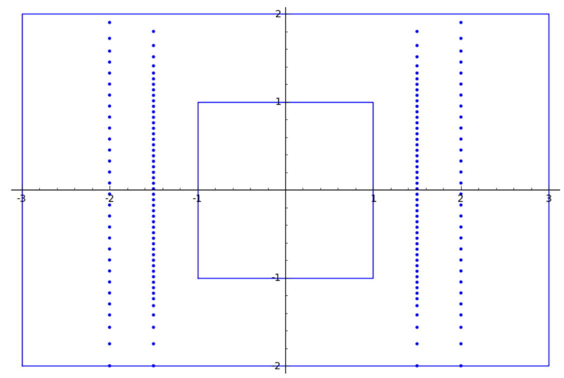

FIGURE 10. Illustration of leaves not approaching the square region in M.

Let $P_0 = P$, with the foliation inherited from $M_{\sim,f,1/4}$. P_n is the support of the slanted suspension obtained by scaling up M so that the square in the center is bounded by vertices $(-m_n, -m_m)$ and (m_n, m_n), taking the slanted suspension with slant $\frac{m_n}{4}$, under the appropriately scaled f.

The sizes of the nested tori are chosen so that there is room to insert each P_n

As constructed, $M_{\sim,f,1/4}$ is not a flow bordism, but this can be remedied by applying the function g of Theorem 5.15. This gives us the parallel boundary we need. The idea from here is to create a sequence of flow bordisms P_n, by scaling P, such that the PL torus in the center of each P_n is not only PL-homeomorphic to T^{PL}_{n-1}, but the exact same size, as given by having coordinates that only differ by a linear translation. The foliation of the boundary of T^{PL}_{n-1} will be by circular leaves, as is the foliation of the PL-torus inside P_n.

Insertion proceeds as follows. First, remove the open torus from the center of P_n, leaving the boundary of a torus, foliated by circular leaves. By our construction, the area in T^{PL}_n into which we want to embed the base of P_n is foliated by vertical leaves. Therefore the insertion can be easily carried out. Furthermore, the parallel boundary of P_n consists of vertical leaves, which have finite length, and so we have an attaching map. Finally, the boundary of T^{PL}_{n-1} is identical to the boundary of the region insider P_n after the open torus is removed. This scaling of the flow bordisms ensure that the subdivisions of the PL-tori above will match with the subdivisions introduced in the application of Theorem 4.2. Remove the boundary of T^{PL}_{n-1} during the insertion map. Then the insertion does not affect the leaves in the interior of T^{PL}_{n-1}.

It is this point when we need to ensure that our meridians match in the Dehn surgery used for the insertion. We have already scaled the PL torus in the center of P_n. In our construction of T^{PL}_{n-1}, we chose a 16 vertices along the major radius and 64 vertices along the minor radius. The use of Theorem 5.15 allows for the same choice of subdivision on the torus in the interior of P_n, and therefore, the boundary of this torus can be chosen to have the same triangulation as T^{PL}_{n-1}. We then have only to choose our meridians. For the meridian on T^{PL}_{n-1}, we can think of each T^{PL}_i as being described by cylindrical coordinates. We use two options here. If i is even, let the coordinates be (r, θ, z) with $x = r\cos\theta$ and $y = r\sin\theta$. If i is odd, use coordinates (r, θ, x), with $z = r\cos\theta$ and $y = r\sin\theta$. Changes in the θ coordinate were already assumed with the rotation by $\pi/16$ in our construction of each T^{PL}_i above. Choose the meridian to be the 64-fold approximation of S_1 with coordinate $\theta = 0$, prior to any rotation of the torus. This clearly bounds a solid disc in T^{PL}_i. For the longitude on the torus in the interior of P_n, choose the 64-fold approximation of S_1 which exists as the boundary of the square with vertices $(-m_n, -m_m)$ and (m_n, m_n), with third coordinate 0, prior to the quotient in the slanted suspension. This clearly also bounds a disc, and so is a meridian.

Since both tori have the same triangulation, we can see that the mapping of one meridian to the other in the insertion of our flow bordism will only affect the trivial Dehn surgery, and that this insertion will not affect the topology of the underlying manifold, that is, $\mathbb{R}^3$.

Each P_n can therefore be inserted around each $T^{PL}n-1$, yielding a foliation of $\mathbb{R}^3$, which has removed all previously indicated singular points, and in which each leaf remains bounded within one of the PL-tori. $\square$

Theorem 5.2. *There exists a PL measured 1-foliation of $\mathbb{R}^3$ with each leaf contained in a bounded set.*

Proof. For this construction, we will start with the same nested PL-tori as in Theorem 5.17, again denoted $\{T_n^{PL} : n \in 0, 1, \ldots\}$. We also begin with the same region $M = [-3,3] \times [-2,2]$ as the starting point for the slanted suspension.

The difference here is that M has a different triangulation, and that the PL-homeomorphism f preserves the area on the 2-simplices in M. We then take the slanted suspension of M, with slant 1/4, which by Theorem 4.2, is a measured foliation.

Next, we apply a PL homeomorphism, as per Theorem 5.15, to the slanted suspension $M_{\sim,f,1/4}$, to get a flow bordism. Next apply Lemmas 3.5 and 3.6 to ensure that the foliation remains measured. The support of the flow bordism is the same cylinder with annular base, which was denoted P in Theorem 5.1.

We can build a sequence of carefully constructed flow bordisms from here. Each is built specifically to be insertible, attachable, and untwisted, with size specific to each of the PL-tori T_n^{PL}. The repeated insertion of these bordisms creates a measured foliation of $\mathbb{R}^3$, with all leaves contained in bounded sets. An application of corollary 3.7 finishes the proof.

Let $M = [-3,3] \times [-2,2]$. Triangulate M, and let $f : M \to M$ be the PL homeomorphism, which preserves area on 2-simplices, and shifts the vertices of the shaded region down 1/4. The vertices are labeled in Figure 11, while the areas of the simplices are shown in Figure 12.

Take the slanted suspension $\mathcal{P} = M_{\sim,f,1/4}$ of M with slant 1/4. Since f preserves area on 2-simplices, $\mathcal{P}$ is a measured foliation, by Theorem 4.1. Denote the support of $\mathcal{P}$ by P. Then $P = \{(x,y,z) : 1 \leq \sqrt{x^2+y^2} \leq 7, -2 \leq z \leq 2\}$, just as in the Theorem 5.1.

The region $[-1,1] \times [-1,1] \times [0,1]$ forms a PL-torus foliated by simplicial approximations of circular leaves when the quotient is applied. This gives a simplicial approximation of a solid torus with major radius 4 and minor radius 1, which is contained in P.

Theorem 4.2 remedies and issues with parallel boundary. Since the slant is $l = 1/4$, the height of M is 4, and f is the identify on the boundary, $w = 16$. We therefore use a 16-fold covering to retriangulate P. Figure 12 shows this applied to $[-2,2] \times [0,1]$ in the unquotiented space. This is analogous to the process described in Theorem 4.2. We will fix each of the vertices with index 0-15 in Figure 12, and shift each of the remaining vertices one to the left. Next, fix all vertices with index 0-31, and repeat the shifting for all higher indexed vertices. As per the details of Theorem 4.2 this process will eventually result in a foliation by vertical leaves, which preserving the area of each simplex in Figure 12.

We can break down P into 3 boundary components. The first, is when $z = -2$. This is our transverse boundary oriented inwards, F_-. The second is when $z = 2$, which is F_+, the transverse boundary oriented outwards. The remaining boundary component, usually referred to as the sides of the cylinder, is not currently parallel boundary. Apply Theorem 4.2 to get remedy this. The support remains P, and we denote the new foliation of P by $\mathcal{S}$. We therefore have that $\mathcal{S}$ is a measured flow bordism, as proven in Theorem 4.2. Note also that all of the circular leaves on the simplicial approximation of the torus contained in P were parallel to the 1-simplices which were only shifted to the left in the application of Theorem 4.2. Therefore, these leaves remain circular.

We can determine the entry stopped set in $\mathcal{P}$, which will have measure zero.

As in Theorem 4.1, all points in the square bounded by (-1,-1) and (1,1) lie on circular leaves. Additionally, the points (0, -3/2) and (0, 13/8) give rise to circular leaves. Their image under f decreases the second coordinate by the same amount as the slant of the suspension.

The entry stopped set F_- is only the simplicial approximation of the circle in P corresponding to the point (0,-2) in M, crossed with [0,1], and with endpoints identified. For any volume form in $\mathbb{R}^3$, the simplicial measure of a circle is zero, hence the entry stopped set has measure zero.

For any other point in M with first coordinate not 0, the first coordinate is changed during the application of f. Therefore the leaf which intersects these points cannot simply have an increasing second coordinate.

They move either left or right while the second coordinate increases, causing the points which intersect $M \times \{1\}$ to move around the center square. This is shown in Figures 13 and 14.

The exit stopped set it determined similarly. The point (0, 13/8) lies on a circular leaf. The leaf through the point (0,2) will approach (0, 13/8), while all other leaves, when traced with the reversed orientation, will avoid the square in the center of M. Therefore F_+ is a circle lying on the component of the boundary of P with $z = 2$. For the same reason as above, the exit stopped set has measure zero.

As stated earlier, we are beginning with the nested PL tori and our scheme for insertion from Theorem 5.1.

To use the flow bordism $\mathcal{S}$ to make this into a non-singular foliation, first remove the interior of the minimal torus from the center of $\mathcal{S}$, and scale it to create a sequence of flow bordisms, $\mathcal{S}_n$, where the removed torus in $\mathcal{S}_n$ is PL-homeomorphic to T_n^{PL}, and the total volume of the removed torus in $\mathcal{S}_\backslash$ is the same as that of T_n^{PL}. The support of $\mathcal{S}$ was retriangulated during the construction of the parallel boundary. In doing so, the region left after removing the interior torus is now bounded by the product of a 16-fold simplicial approximation of S^1 and a 64-fold simplicial approximation of S^1. Lemma 5.9 assured that the foliation $\mathcal{S}$ remained measured during this process.

Once scaled up, the height of the support of each $\mathcal{S}_n$ will be twice the minor radius of the torus contained inside that support. The base remains transverse boundary, and our existing foliation of T_{n+1}^{PL} ensures that all leaves in that neighborhood of T_n^{PL} are already vertical. Therefore the conditions for insertion are satisfied. The conditions for each bordism to be attachable and untwisted are satisfied for the same reasons as in Theorem 5.1. The choice of triangulation for the torus in the interior of $\mathcal{S}_n$ can be chosen just as in Theorem 5.1 This ensures that our Dehn surgery is trivial and does not cause any issues. It is the careful construction of both the bordism, and the foliation on the nested tori which is to be modified that permits this.

Let ω be the simplicial measure on $\mathbb{R}^3$ given by the usual Euclidean volume form. Then each T_n^{PL} has measure equal to its Lebesgue measure. We have constructed $\mathcal{S}_n$ in such a way that the vertices on the torus contained in $\mathcal{S}_n$ are mapped exactly to the vertices on the boundary of T_{n-1}^{PL}. The leaves in the parallel and transverse boundary of $\mathcal{S}_n$ match with the leaves in the existing foliation of the nested PL-tori. The regions where the boundaries meet can be given a common subdivision [12].

Each $\mathcal{S}_n$ is measured, so we let ω_n be the simplicial measure on $\mathcal{S}_n$. Our condition on the scaling of $\mathcal{S}_n$ will give ω and ω_n the same total volume on $\mathcal{S}_n$. In the parlance of Theorem 5.7, let M_1 be the support of $\mathcal{S}_n$, and M_2 be the subset of T_n^{PL} which contains T_{n-1}^{PL} and is PL-homeomorphic to the support of $\mathcal{S}_\backslash$. Then there exists a volume preserving PL-homeomorphism f from M_1 to M_2, which pulls back the volume form ω_n to ω.

Since $\mathcal{S}$ was measured, $\mathcal{S}_n$ is measured. By the argument above, insertion of $\mathcal{S}_n$ preserves the usual Lebesgue measure on $\mathbb{R}^3$. For the portion of each T_n^{PL} which was not modified by insertion, it is foliated by simplicial approximations of circles, and a foliation by circles is measured. Therefore, this constructions is a measured foliation of $\mathbb{R}^3$.

Corollary 5.18 is enough to ensure that the foliation in each $\mathcal{S}_n$ will create a PL dynamical system. To extend this to the rest of T_n^{PL}, we need a whose trajectories are simplicially approximated by the leaves in T_n^{PL}. Take the closure of $T_n^{PL} \setminus \mathcal{S}_n$. This satisfies the conditions of Corollary 5.18, so it also yields a PL dynamical system.

By construction, the foliations on the boundary of $\mathcal{S}_n$ and on $T_n^{PL} \setminus \mathcal{S}_n$ agree, and since our dynamics are only required to be PL, the dynamical systems created both in and out of the flow bordism agree at the switching manifold.

Therefore, the measured foliation yields a measure-preserving dynamical system, and our theorem is satisfied. □

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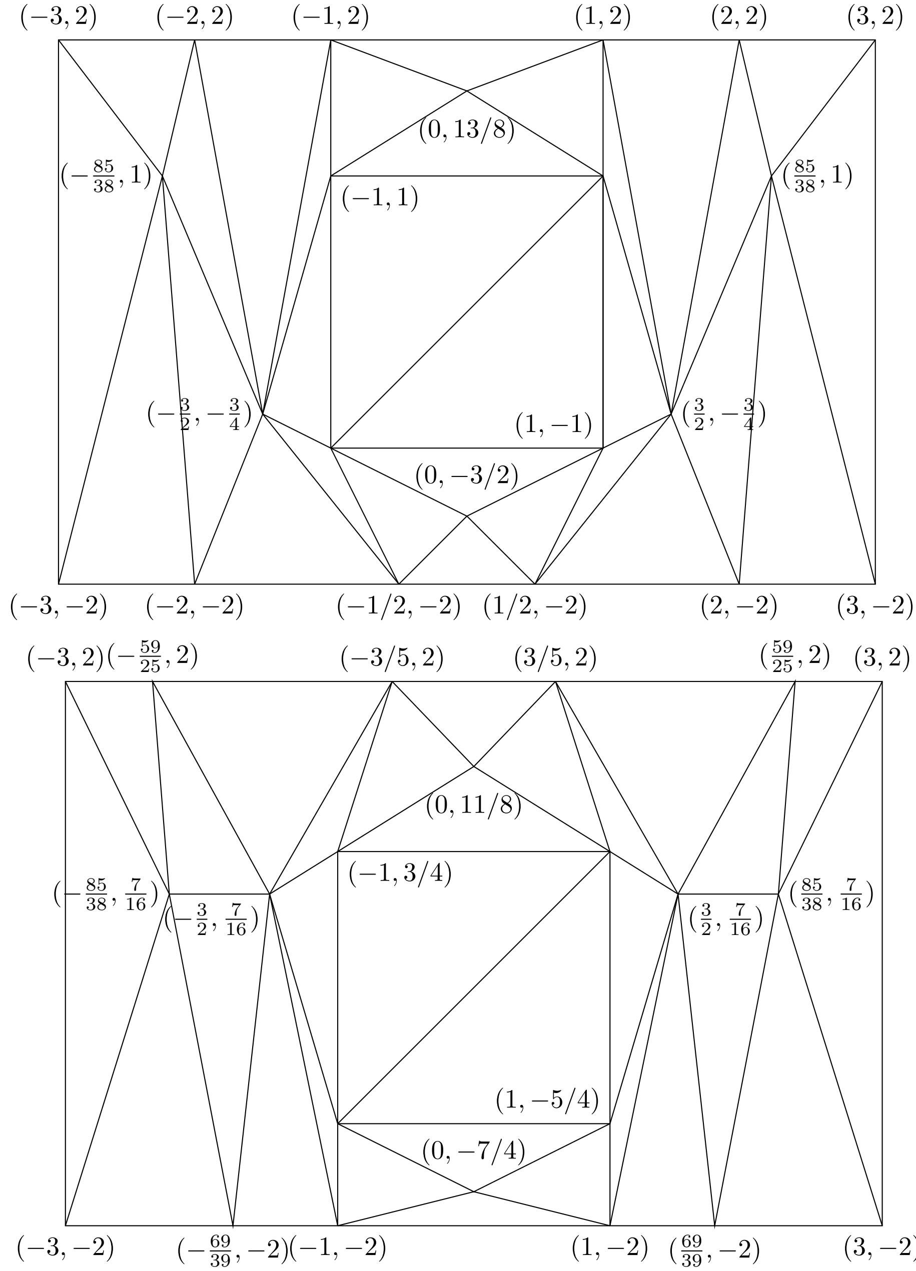
FIGURE 11. M and $f(M)$ with vertices indicated



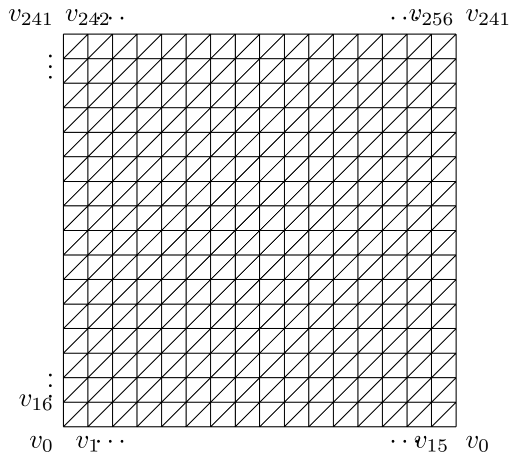


FIGURE 12. Triangulation of a cylinder with labeled vertices. The vertical axis is scaled down by a factor of 4.

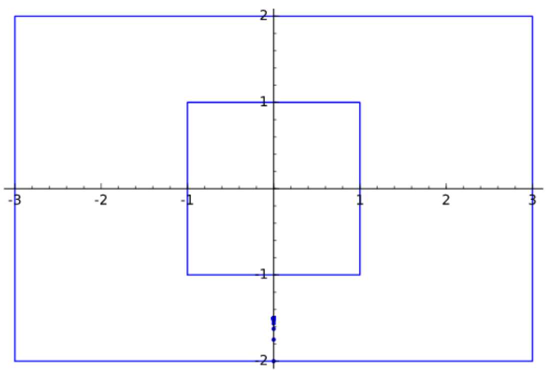


FIGURE 13. Sequence of points in $M \times \{1\}$ arising from a leaf in the entry stopped set.

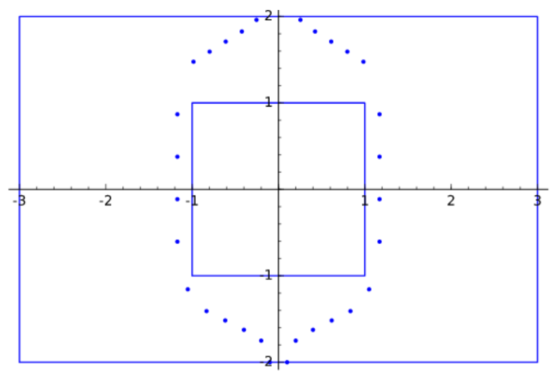


FIGURE 14. Trajectories originating at (0.1,-2) and (-0.1,-2)