Divided Differences

Teodora D. Zapryanova
University of Economics - Varna, Bulgaria

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For a function f defined on $[a, b]$ and points $x_0, x_1, ..., x_n, ...$ from $[a, b]$ we define the *first divided difference of* f by

$$[x_i, x_{i+1}]\, f = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \text{ if } \; x_{i+1} \neq x_i \text{ and } \; [x_i, x_i]\, f = f'(x_i),$$

if this derivative exists $i = 0, 1, 2, ...$For $x_0 \neq x_n$ we define the *n-th divided difference of* f by the recurrence relation

$$[x_0, x_1, ..., x_n]\, f = \frac{[x_1, x_2, ..., x_n]\, f - [x_0, x_1, ..., x_{n-1}]\, f}{x_n - x_0} \qquad (1)$$

and if the multiplisity of the point x_0 is $n + 1$

$$[x_0, x_0, ..., x_0]\, f = f^{(n)}(x_0)/n!.$$

Recall that the *differences* of a function f with step h are defined by

$$\begin{aligned}\Delta_h^0 f(x) &:= f(x),\\ \Delta_h f(x) &:= f(x+h) - f(x),\\ \Delta_h^n f(x) &:= \Delta_h \Delta_h^{n-1} f(x).\end{aligned}$$

One proves by induction

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n+k} \left(\begin{array}{c} n \\ k \end{array} \right) f(x+kh)$$

Assertion. If $x_{i+1} - x_i = h$, $i = 1, 2, ..., n$, (the points $x_0, x_1, ..., x_n$ are equidistant) then

$$[x, x+h, ..., x+nh]\, f = \frac{\Delta_h^n f(x)}{h^n . n!}.$$

Proof.

$$\begin{aligned} & [x, x+h, ..., x+nh]\, f \\ = \quad & \frac{[x+h, ..., x+nh]\, f - [x, x+h, ..., x+(n-1)\,h]\, f}{nh} \\ = \quad & \frac{\frac{\Delta_h^{n-1} f(x+h)}{h^{n-1}.(n-1)!} - \frac{\Delta_h^{n-1} f(x)}{h^{n-1}.(n-1)!}}{nh} = \frac{\Delta_h^n f(x)}{h^n . n!}. \end{aligned}$$

Lemma 1. If there exists the n-th derivative of f and $f^{(n)}(x)$ is continuous, then for some $\xi \in [x, x+nh]$ we have

$$\Delta_h^n f(x) = h^n f^{(n)}(\xi).$$

Proof. We know the useful integral representation of $\Delta_h^n f(x)$ by means of derivative of f.

$$\Delta_h^n f(x) = \underbrace{\int_0^h \ldots \int_0^h}_{n} f^{(n)}(x + t_1 + \ldots + t_n)\, dt_1 \ldots dt_n$$

Let

$$m = \min_{u \in [x, x+nh]} f^{(n)}(u), \qquad M = \max_{u \in [x, x+nh]} f^{(n)}(u).$$

Then $m \le f^{(n)}(u) \le M$, $\forall u \in [x, x+nh]$,

$$mh^n \le \underbrace{\int_0^h \dots \int_0^h}_{n} f^{(n)}(x + t_1 + \dots + t_n)\, dt_1 \dots dt_n \le Mh^n, \quad \text{i.e.}$$

$$m \le \frac{1}{h^n} \underbrace{\int_0^h \dots \int_0^h}_{n} f^{(n)}(x + t_1 + \dots + t_n)\, dt_1 \dots dt_n \le M.$$

Hence there exists $\xi \in [x, x+nh]$ such that

$$\frac{1}{h^n} \underbrace{\int_0^h \dots \int_0^h}_{n} f^{(n)}(x + t_1 + \dots + t_n)\, dt_1 \dots dt_n = f^{(n)}(\xi).$$

Therefore $\frac{1}{h^n}\Delta_h^n f(x) = f^{(n)}(\xi)$ for some $\xi \in [x, x+nh]$. The lemma is proved.

Lemma 2. If all x_i are different

$$[x_0, x_1, ..., x_n]\, f$$

$$= \sum_{k=0}^{n} \frac{f(x_k)}{(x_k - x_0)...(x_k - x_{k-1})(x_k - x_{k+1})...(x_k - x_n)}$$

$$= \sum_{k=0}^{n} \frac{f(x_k)}{\omega'_{0,n}(x_k)} \qquad (2)$$

where

$$\omega_{i,k}(x) = (x - x_i)(x - x_{i+1})...(x - x_{i+k}).$$

Proof. Indeed, (2) can be proved by induction on n. For $n = 1$ equality (2) is true:

$$[x_0, x_1]\, f = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0}.$$

Asume that (2) is true for n and prove it for $n + 1$. By the recurrence relation we have

$$[x_0, x_1, ..., x_{n+1}]\, f = \frac{[x_1, x_2, ..., x_{n+1}]\, f - [x_0, x_1, ..., x_n]\, f}{x_{n+1} - x_0}.$$

But by inductive supposition

$$
\begin{aligned}
& [x_1, x_2, ..., x_{n+1}]\, f \\
= \ & \sum_{k=1}^{n+1} \frac{f(x_k)}{(x_k - x_1)...(x_k - x_{k-1})(x_k - x_{k+1})...(x_k - x_{n+1})} \\
= \ & \sum_{k=1}^{n+1} \frac{f(x_k)}{\omega'_{1,n}(x_k)}
\end{aligned}
$$

and

$$
\begin{aligned}
& [x_0, x_1, ..., x_n]\, f \\
= \ & \sum_{k=0}^{n} \frac{f(x_k)}{(x_k - x_0)...(x_k - x_{k-1})(x_k - x_{k+1})...(x_k - x_n)} \\
= \ & \sum_{k=0}^{n} \frac{f(x_k)}{\omega'_{0,n}(x_k)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& [x_0, x_1, \dots, x_{n+1}]\, f \\
= \;& \frac{[x_1, x_2, \dots, x_{n+1}]\, f - [x_0, x_1, \dots, x_n]\, f}{x_{n+1} - x_0} \\
= \;& \frac{1}{x_{n+1} - x_0} \left(\sum_{k=1}^{n+1} \frac{f(x_k)}{\omega'_{1,n}(x_k)} - \sum_{k=0}^{n} \frac{f(x_k)}{\omega'_{0,n}(x_k)} \right) \\
= \;& \frac{1}{x_{n+1} - x_0} \left[\sum_{k=1}^{n} \left(\underbrace{\frac{f(x_k)}{\omega'_{1,n}(x_k)} - \frac{f(x_k)}{\omega'_{0,n}(x_k)}}_{\omega'_{0,n+1}(x_k)} \right) + \frac{f(x_{n+1})}{\omega'_{1,n}(x_{n+1})} - \frac{f(x_0)}{\omega'_{0,n}(x_0)} \right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{x_{n+1}-x_0}\left[\sum_{k=1}^{n}\left(\frac{f(x_k)(x_k-x_0)-f(x_k)(x_k-x_{n+1})}{(x_k-x_0)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_{n+1})}\right)\right. \\
& +\frac{1}{x_{n+1}-x_0}\left(+\frac{f(x_{n+1})}{\omega'_{1,n}(x_{n+1})}-\frac{f(x_0)}{\omega'_{0,n}(x_0)}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{x_{n+1}-x_0}\left[\sum_{k=1}^{n}\frac{f(x_k)(x_{n+1}-x_0)}{\omega'_{0,n+1}(x_k)}\right] \\
& +\frac{1}{x_{n+1}-x_0}\left(\frac{f(x_{n+1})}{\omega'_{1,n}(x_{n+1})}-\frac{f(x_0)}{\omega'_{0,n}(x_0)}\right) \\
= \quad & \sum_{k=1}^{n}\frac{f(x_k)}{\omega'_{0,n+1}(x_k)}+\frac{f(x_{n+1})}{\omega'_{0,n+1}(x_{n+1})}+\frac{f(x_0)}{\omega'_{0,n+1}(x_0)}=\sum_{k=0}^{n+1}\frac{f(x_k)}{\omega'_{0,n+1}(x_k)}.
\end{aligned}
$$

Corollary 1. $[x_0, x_1, ..., x_n]\, f$ is symmetric in $x_0, x_1, ..., x_n$.

Proof. Follows from Lemma 2

$$
\begin{aligned}
& [x_0, x_1, ..., x_n]\, f \\
= \ & \sum_{k=0}^{n} \frac{f(x_k)}{(x_k - x_0)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_n)}.
\end{aligned}
$$

In *Hermite interpolation,* not only the value of the polynomial

$$P_n(x) = a_0 + a_1 x + a_2 x^2/2! + ... + a_n x^n/n!$$

but also the values of some of its successive derivatives are prescribed. Let

$$y_1, y_2, ..., y_p$$

be distinct real *interpolation points,* which are equipped with multiplicities $m_j > 0$, $j = 1, 2, ..., p$, with $m_1 + m_2 + ... + m_p = n + 1$. Let $c_{j,l}$ be given constants. We look for a P_n which satisfies the equations

$$\begin{aligned} P_n^{(l)}(y_j) &= c_{j,l} \\ l &= 0, ..., m_j - 1, \qquad j = 1, 2, ...p \end{aligned} \tag{3}$$

In particular, for some function f we may take $c_{j,l} = f^{(l)}(y_j)$. Then the polynomial $P_n = P_n(f, x)$ *interpolates the function* f.

Sometimes a slightly different point of view is preferable. Let $X : x_0, \ldots, x_n$ be the interpolation points, with possible repetitions. For each j, the multiplicity m_j of x_j is the number of $x_i = x_j$, while l_j is the number of $x_i = x_j$ with $i \leq j$. For example for points

$$x_0, x_1, x_1, x_2, x_3, x_3, x_3$$

$$\begin{aligned} m_0 &= 1,\ l_0 = 1, \qquad m_1 = m_2 = 2,\ l_1 = 1,\ l_2 = 2, \\ m_3 &= 1,\ l_3 = 1, \qquad m_4 = m_5 = m_6 = 3,\ l_4 = 1,\ l_5 = 2,\ l_6 = 3 \end{aligned}$$

Then equations (3) are replaced by

$$P_n^{\left(l_j - 1\right)}(x_j) = c_j,\ j = 0, \ldots, n$$

with properly chosen c_j. In particular, for some function f we may take $c_j = f^{\left(l_j - 1\right)}(x_j)$. The interpolation polinomial $P_n(x) := P_n(f, X)(x) := P_n(f, X; x)$ depends on the $(n+1)$ - tuple $X : x_0, \ldots, x_n$. It can be obtained by means of Newton's method:

Theorem 1. There exist unique constants $A_0, ..., A_n$ for which the polynomials

$$\begin{aligned} P_0(x) &= A_0 \\ P_1(x) &= A_0 + A_1(x - x_0) \\ &\dots \\ P_n(x) &= A_0 + A_1(x - x_0) + ... + A_n(x - x_0)...(x - x_{n-1}) \end{aligned} \tag{4}$$

are the solutions of the Hermite interpolation problems for the sets of interpolation points $X_0 := \{x_0\}, ..., X_n := \{x_0, ..., x_n\}$ and given data $c_0, ..., c_n$.

Proof: We proceed by induction on k. Certainly $P_0(x) := A_0$ is uniquely defined by the condition $P_0(x_0) = c_0$. Suppose that our assertion is true for $P_0, ..., P_{k-1}$ and $X_0, ..., X_{k-1}$. The additional point x_k in X_k is associated with the condition $P_k^{(l_k-1)}(x_k) = c_k$ for the polynomial

$$P_k(x) = P_{k-1}(x) + A_k(x - x_0) ... (x - x_{k-1}).$$

To define A_k uniquely from the condition

$$c_k = P_k^{(l_k-1)}(x_k) = P_{k-1}^{(l_k-1)}(x_k) + A_k((x - x_0) ... (x - x_{k-1}))^{(l_k-1)}|_{x=x_k}$$

we need that the $(l_k - 1)$ - st derivative of $(x - x_0) ... (x - x_{k-1})$ is non-zero at x_k.This defines A_k uniquely.

Indeed, recall that l_k is the number of $x_i = x_k$ with $i \leq k$ and we have $l_k - 1$ points before x_k equal to x_k. Then in the product

$$(x - x_0) \dots (x - x_{k-1}) = (x - x_0) \dots (x - x_s)\,(x - x_k)^{l_k - 1}, \qquad s = k - l_k.$$

The points $x_0, \dots, x_s$ are different from x_k. From Leibniz's formula it follows that the $(l_k - 1)$ - st derivative of $(x - x_0) \dots (x - x_{k-1}) = (x - x_0) \dots (x - x_s)\,(x - x_k)^{l_k - 1}$ is non-zero at x_k. This defines A_k uniquely.The other conditions $P_k^{(l_i - 1)}(x_i) = c_i$, $i = 0, \dots, k-1$ for P_k and $i < k$ are satisfied because they are satisfied for P_{k-1} and because the $(l_i - 1)$ - st derivative of $(x - x_0) \dots (x - x_{k-1})$ is zero at x_i. Certainly for $i < k$ in the product

$$(x - x_0) \dots (x - x_{k-1}) = (x - x_i)^{l_i} Q_1(x).$$

Then from Leibniz's formula it follows that the $(l_i - 1)$ - st derivative of $(x - x_0) \dots (x - x_{k-1}) = (x - x_i)^{l_i} Q_1(x)$ is equal to $(x - x_i) Q_2(x)$ and is zero at x_i.

Extreme cases of Hermite interpolation are Lagrange interpolation, when all x_i are different and Taylor interpolation, when $x_0 = x_1 = ... = x_n$. In the latter case, if $c_k = f^{(k)}(x_0)$, $k = 0, ..., n$, P_n is the Taylor polynomial

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + ... + f^{(n)}(x_0)(x - x_0)^n / n!$$

When all $x_0, ..., x_n$ are different $P_n(f, X; x)$ is called Lagrange interpolation polynomial $L_n(f; x)$.

For it all $l_j = 1, \ \ j = 0, ..., n$ and $L_n(f; x)$ satisfies the equations

$$L_n(f; x_k) = f(x_k), \ \ k = 0, ..., n.$$

The Lagrange interpolating polynomial is given by

$$L_n(f; x) = \sum_{k=0}^{n} f(x_k) \frac{(x - x_0) ... (x - x_{k-1})(x - x_{k+1}) ... (x - x_n)}{(x_k - x_0) ... (x_k - x_{k-1})(x_k - x_{k+1}) ... (x_k - x_n)}.$$

Proposition. For a function f and points $X : x_0, x_1, ..., x_n$ with possible repetition we have

$$[x_{0,}x_{1,}..., x_n] f = A_n$$

where A_n is the coefficient of x^n of the Hermite polynomial $P_n(f, X; x)$ which interpolates f at $x_0, x_1, ..., x_n$. A_n is given by

$$\begin{aligned} P_0(x) &= A_0 \\ P_1(x) &= A_0 + A_1(x - x_0) \\ &\dots \\ P_n(x) &= A_0 + A_1(x - x_0) + ... + A_n(x - x_0)...(x - x_{n-1}) \end{aligned}$$

Proof. For the proof we proceed by induction on n. Certainly for

$$\begin{aligned} P_0(x) &= A_0 \\ P_1(x) &= A_0 + A_1(x - x_0) \end{aligned}$$

in case of different interpolation points x_0, x_1 we have

$$\begin{aligned} A_0 &= f(x_0), \\ f(x_1) &= f(x_0) + A_1(x_1 - x_0) \end{aligned}$$

hence $A_1 = [x_{0,} x_1] f$; we have

$$\begin{aligned} A_0 &= f(x_0), \\ f'(x_0) &= P_1'(x_0) = A_1 \end{aligned}$$

in case $x_0 = x_1$ i.e. $A_1 = [x_{0,} x_0] f$.

Suppose that our assertion is true for $P_0, P_{1,...}, P_{n-1}$ and $X_0, X_{1,...,}X_{n-1}$. Let S, T be polynomials of degree $\leq n-1$, which interpolate f at the points $x_{0,}x_{1,...,}x_{n-1}$ and $x_1, x_2, ..., x_n$ respectively. Then

$$\frac{x - x_0}{x_n - x_0} T(x) + \frac{x_n - x}{x_n - x_0} S(x) \tag{5}$$

is a polynomial of degree $\leq n$, which interpolates f at $x_{0,}x_{1,...,}x_n$ with the required multiplicities. The leading term of (5)

$$\frac{[x_{1,}x_2, ..., x_n]\, f - [x_{0,}x_{1,...,}x_{n-1}]\, f}{x_n - x_0} = [x_0, x_1, ..., x_n]f.$$

is the coefficient A_n of x^n of the polynomial P_n which interpolates f at $x_{0,}x_{1,...,}x_n$.

Let a function f and points $X: x_0, ..., x_n$ be given. Let the polynomial of degree $n \quad P_n(x) = P_n(f, X; x)$ satisfies the conditions $P_n^{(l_j-1)}(x_j) = f^{(l_j-1)}(x_j)$, $j = 0, ..., n$. From Theorem 1

$$P_n(x) = A_0 + A_1(x - x_0) + ... + A_n(x - x_0) ... (x - x_{n-1}).$$

The coefficients A_k are divided differences $[x_0, ..., x_k] f$ for $k = 0, 1, ..., n$. This leads to Newton's formula for the interpolating polynomial:

$$P_n(f, X; x) = \sum_{k=0}^{n} (x - x_0) ... (x - x_{k-1}) [x_0, ..., x_k] f.$$

Lemma. If $f \in C[a,b]$, $a \le x_i \le b$, $i = 0, ..., n$, then

$$[x_0, ..., x_n]\, f = \frac{f^{(n)}(\xi)}{n!}$$

for some $\xi \in [a,b]$.

Proof. If $P_n(x) = P_n(f, X; x)$, then the function $f(x) - P_n(x)$ has $n+1$ roots - at the points $x_0, ..., x_n$, (with corresponding multiplicities m_j , $j = 0, ..., n$). Then by Rolle's theorem, the first derivative $f'(x) - P_n'(x)$ has n roots, $f''(x) - P_n''(x)$ has $n-1$ rootsetc. $f^{(n)}(x) - P_n^{(n)}(x)$ has a root $\xi \in [a,b]$, i.e. $f^{(n)}(\xi) - P_n^{(n)}(\xi) = 0$ for some $\xi \in [a,b]$. But $P_n(f, X; x) = \sum_{k=0}^{n} (x - x_0) ... (x - x_{k-1}) [x_0, ..., x_k]\, f$ and only the last term $(x - x_0) ... (x - x_{n-1}) [x_0, ..., x_n]\, f$ has degree n, the other terms have degree less than n. Since the coefficient in x^n is $[x_0, ..., x_n]\, f$, we obtain, that $\frac{d^n}{dx^n} P_n(x) = n!. [x_0, ..., x_n]\, f \ \ \forall x \in \mathbb{R}$, which shows that $f^{(n)}(\xi) - n! [x_0, ..., x_n]\, f = 0$ for some $\xi \in [a,b]$. The lemma is proved.

Problem 1. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is three times differentiable, then there exists a real number $\xi \in (-1, 1)$ such that $\frac{f'''(\xi)}{6} = \frac{f(1)-f(-1)}{2} - f'(0)$. ($12^{th}$ International Mathematics Competition for University Students -Blagoevgrad, Bulgaria, July 22 - July 28, 2005).

Solution. Let $P_3(x) = P_3(f, X; x)$ be Hermite's polynomial which interpolates f at points $-1, 0, 0, 1$. Then

$$\begin{aligned} P_3(f, X; x) &= A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1) \\ &\quad + A_3(x - x_0)(x - x_1)(x - x_2) \end{aligned}$$

$$\begin{aligned} A_0 &= f(x_0) = f(-1) \\ A_1 &= [x_0, x_1]\, f = [-1, 0]\, f = \frac{f(0) - f(-1)}{0 - (-1)} = f(0) - f(-1) \\ A_2 &= [x_0, x_1, x_2]\, f = [-1, 0, 0]\, f = \frac{[0, 0]f - [-1, 0]f}{0 - (-1)} \\ &= f'(0) - f(0) + f(-1) \\ A_3 &= [x_0, x_1, x_2, x_3]\, f = [-1, 0, 0, 1]\, f \\ &= \frac{[0, 0, 1]\, f - [-1, 0, 0]\, f}{1 - (-1)} \\ &= \frac{1}{2}\left(f(1) - 2f'(0) - f(-1)\right) = \frac{f(1) - f(-1)}{2} - f'(0) \end{aligned}$$

$$
\begin{aligned}
P_3\left(f, X; x\right) &= f(-1) + \left(f(0) - f(-1)\right)(x+1) \\
&\quad + \left(f'(0) - f(0) + f(-1)\right)(x+1)x \\
&\quad + \frac{1}{2}\left(f(1) - 2f'(0) - f(-1)\right)(x+1)x^2
\end{aligned}
$$

It is easy to check that indeed

$$P_3(\pm 1) = f(\pm 1),\ \ P_3(0) = f(0) \text{ and } \ P_3'(0) = f'(0)$$

Apply Rolle's theorem for the function $h(x) = f(x) - P_3(x)$ and its derivatives.

Since

$$h(-1) = h(0) = h(1) = 0,$$

there exists $\eta \in (-1, 0)$ and $\theta \in (0, 1)$ such that

$$h'(\eta) = h'(\theta) = 0.$$

We also have $h'(0) = 0$, so there exists $\rho \in (\eta, 0)$ and $\sigma \in (0, \theta)$ such that

$$h''(\rho) = h''(\sigma) = 0$$

Finally, there exists a $\xi \in (\rho, \sigma) \subset (-1, 1)$ where $h'''(\xi) = 0$.

Then

$$
\begin{aligned}
f'''(\xi) &= P_3'''(\xi) = \left(\left(\frac{f(1) - f(-1)}{2} - f'(0)\right)(x+1)x^2\right)'''(\xi) \\
&= 6\left(\frac{f(1) - f(-1)}{2} - f'(0)\right).
\end{aligned}
$$

Hence there exists a real number $\xi \in (-1, 1)$ such that

$$
\frac{f'''(\xi)}{6} = \frac{f(1) - f(-1)}{2} - f'(0).
$$

Problem 2. Prove that if $f(x) = \frac{1}{x}$ and $0 < x_0 \leq ... \leq x_n$, then

$$[x_0, ..., x_n] f = \frac{(-1)^n}{x_0 ... x_n}. \tag{6}$$

Solution. For $n = 0$,

$$[x_0] f = f(x_0) = \frac{1}{x_0}$$

, for $n = 1$ and $0 < x_0 < x_1$,

$$\begin{aligned}
[x_0, x_1] f &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
&= \frac{1}{x_1 - x_0} \left(\frac{1}{x_1} - \frac{1}{x_0} \right) = \frac{x_0 - x_1}{(x_1 - x_0) x_0 x_1} = -\frac{1}{x_0 x_1}
\end{aligned}$$

when $0 < x_0 = x_1$,

$$[x_0, x_0] f = f'(x_0) = -\frac{1}{x_0^2}.$$

so for $n = 1$, (6) is satisfied. Next, we assume that (6) is true for $(n-1)$ and wish to infer from this assumption that (6) is true for n.

When not all $x_i, i = 0, ..., n$ coincide,

$$
\begin{aligned}
[x_0, ..., x_n]\, f &= \frac{[x_1, ..., x_n]\, f - [x_0, ..., x_{n-1}]\, f}{x_n - x_0} \\
&= (-1)^{n-1} \frac{\frac{1}{x_1 ... x_n} - \frac{1}{x_0 ... x_{n-1}}}{x_n - x_0} \\
&= (-1)^{n-1} \frac{\frac{1}{x_1 ... x_{n-1}}}{x_n - x_0} \left(\frac{1}{x_n} - \frac{1}{x_0} \right) \\
&= (-1)^{n-1} \frac{\frac{1}{x_1 ... x_{n-1}}}{x_n - x_0} \frac{x_0 - x_n}{x_0 x_n} \\
&= (-1)^{n-1} . \frac{(-1)}{x_1 ... x_{n-1}} . \frac{1}{x_0 x_n} = \frac{(-1)^n}{x_0 x_1 ... x_{n-1} x_n}.
\end{aligned}
$$

When $0 < x_0 = ... = x_n$, then

$$[x_0, ..., x_0]\, f = \frac{\left(x^{-1}\right)^{(n)}(x_0)}{n!} = \frac{(-1)\ldots(-n)\, x_0^{-n}}{n!} = \frac{(-1)^n}{x_0^n}.$$

Consequently, by the Principle of Mathematical Induction we infer that the formula holds for all n.

Problem 3. Suppose that $f(x)$ is 3 times differentiable for any real number and set

$$L_h(f,x) = f(x+2h) - 2hf'(x+h) - f(x) - \frac{h^3}{3}f'''(x+h).$$

Prove that $L_h(f,x) = 0$ for any real x and h if and only if $f(x)$ is a polynomial of degree ≤ 4.

Solution. We calculate

$$[x,x,x,x,x+h]\,f = \frac{f(x+h)-f(x)-hf'(x)-\frac{h^2}{2}f''(x)-\frac{h^3}{6}f'''(x)}{h^4},$$

$$\begin{aligned}[x,x,x,x,x-h]\,f &= \frac{f(x-h)-f(x)+hf'(x)-\frac{h^2}{2}f''(x)+\frac{h^3}{6}f'''(x)}{h^4} \\ &= [x-h,x,x,x,x]\,f.\end{aligned}$$

Then

$$\begin{aligned}[x-h,x,x,x,x,x+h]\,f &= \frac{[x,x,x,x,x+h]\,f-[x-h,x,x,x,x]\,f}{2h} \\ &= \frac{f(x+h)-f(x-h)-2hf'(x)-\frac{h^3}{3}f'''(x)}{2h^5}\end{aligned}$$

Therefore

$$\begin{aligned}
& [x, x+h, x+h, x+h, x+h, x+2h]\, f \\
= \quad & \frac{f(x+2h) - f(x) - 2hf'(x+h) - \frac{h^3}{3} f'''(x+h)}{2h^5} = \frac{1}{2h^5} L_h(f, x).
\end{aligned}$$

But by the last Lemma

$$[x, x+h, x+h, x+h, x+h, x+2h]\, f = \frac{f^{(5)}(\xi)}{5!}$$

for some $\xi \in (x, x+2h)$. Hence

$$\frac{1}{2h^5} L_h(f, x) = \frac{f^{(5)}(\xi)}{5!}$$

for some $\xi \in (x, x+2h)$ if the 5-th derivative of f exists. Thus if f is a polynomial of degree ≤ 4 then $f^{(5)}(x) = 0$ for any real x. Therefore $L_h(f, x) = 0$ for any real x and h.

Now we will prove the converse statement: If $L_h(f,x)=0$ then $f(x)$ is a polynomial of degree ≤ 4. Let

$$L_h(f,x)=f(x+2h)-2hf'(x+h)-f(x)-\frac{h^3}{3}f'''(x+h)=0.$$

Then

$$f'''(x+h)=\frac{3}{h^3}\left(f(x+2h)-2hf'(x+h)-f(x)\right)$$

and setting $x:=x-h$ we get

$$f'''(x)=\frac{3}{h^3}\left(f(x+h)-2hf'(x)-f(x-h)\right).$$

When $h = 1$ we obtain

$$f'''(x) = 3\left(f(x+1) - 2f'(x) - f(x-1)\right).$$

As $f(x)$ is 3 times differentiable for any real number this equality shows that $f'''(x)$ is 2 times differentiable for any real number.

Thus we obtain that if $L_h(f, x) = 0$ for any real x and h and $f(x)$ is 3 times differentiable then $f(x)$ is 5 times differentiable and

$$f^{(5)}(x) = (f'''(x))'' = 3\left(f''(x+1) - 2f'''(x) - f''(x-1)\right).$$

Hence

$$f^{(5)}(x) \in C(-\infty, \infty).$$

But

$$0 = \frac{1}{2h^5} L_h(f, x) = \frac{f^{(5)}(\xi)}{5!}$$

for some $\xi \in (x, x + 2h)$.

If we choose $h > 0$ sufficiently small, when $h \to 0$ we get $\xi \to x$ and

$$\lim_{\xi \to x} f^{(5)}(\xi) = f^{(5)}(x).$$

Thus we obtain that $f^{(5)}(x) = 0$ for any real number. Therefore $f(x)$ is a polynomial of degree ≤ 4.

THANKS FOR YOUR ATTENTION