BASIL KOROMPILIAS

COMMON GOVERNANCE MODEL

A Constitutional analysis
on the Mathematical Physics of Authority,
from Quantum Measurement
to AI Alignment

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Epigraph

The Source is Always One

Εκ πάντων ἓν καὶ ἐξ ἑνὸς Πάντα
Unus pro omnibus, omnes pro uno
E pluribus Unum
Harambee
Tiānxià dàtóng
Shevet Achim Gam Yachad
Vasudhaiva Kutumbakam
Unity in Diversity
Wa eri dōto narazu
Dareum sog-ui hanadoem
In Varietate Concordia
Nidade na Diversidade
Bhinneka Tunggal Ika
Al-wahdah fi'l-kathrah wa'l-kathrah fi'l-wahda
Unity makes Strength
L'Union fait la Force
Eendraght maeckt Maght
Ισχύς εν τη Ενώσει

Contents

Abstract

Authority, understood as the legitimate capacity to determine operational outcomes, requires constitutional principles invariant across contexts. In physical measurement, observers maintain authority while subject to laws; in artificial intelligence, decision processes must preserve authority while operating autonomously. Both domains present the same question: what structural requirements determine coherent authority? We present the Common Governance Model, establishing these requirements from a single foundational axiom: that operational structure must trace to a shared source. This axiom is formalized through five constraints in modal logic, with the foundational axiom establishing chirality and four specifications capturing its requirements at different modal depths. Formal verification treats these constraints as independent; in physically realizable systems the specifications follow from the foundational axiom. Analysis employs two layers: modal verification via Kripke semantics and operational specification for physical implementation (continuity, reachability, simplicity). Together they uniquely select three-dimensional structure with six degrees of freedom. Results categorize by epistemic status. Deductive: three-dimensional structure follows via Baker–Campbell–Hausdorff expansion forcing $\mathfrak{su}(2)$ algebra, extending to SE(3) via semidirect product. Representational: $L^2(S^2)$ Hilbert space provides concrete realization. Invariants: quantum gravity horizon $Q_G = 4\pi$, monodromy defect δ_{BU}, aperture scale m_a; the ratio $\delta_{BU}/m_a = 0.9793$ determines both electromagnetic coupling and the 2.07% alignment aperture. Phenomenological: in physics, geometric invariants yield $\alpha = 0.007297352563$ matching experimental synthesis; in AI systems, the same geometric ratio δ_{BU}/m_a predicts 2.07% aperture with transformer architectures showing six to eight times higher values. The framework is multiply falsifiable through dimensional counterexamples, holonomy tests, or empirical disagreement. Coherent recursive measurement imposes structural necessity: systems satisfy constitutional requirements or fail operational coherence. Results reproducible via archived protocols (DOI: 10.5281/zenodo.17521384).

Keywords: formal systems; gyrogroup theory; Baker–Campbell–Hausdorff analysis; Gelfand–Naimark–Segal construction; modal logic; quantum gravity; AI alignment; spacetime emergence

1 Introduction

Authority, understood as the legitimate capacity to determine operational outcomes, requires constitutional principles invariant across contexts. In physical measurement, observers maintain descriptive authority while subject to the same laws; in artificial intelligence, decision processes must preserve legitimate authority while operating autonomously. Both domains present the same fundamental question: what structural requirements determine coherent authority? Constitutional principles function as invariant constraints determining all subsequent structure, distinguishing foundational necessities from contingent choices.

This paper presents the Common Governance Model (CGM), which establishes structural requirements from a single foundational axiom: operational structure must trace to a shared source. We formalize this axiom through a system of interconnected constraints in modal logic. For purposes of formal verification via Kripke semantics, these constraints are treated as independent axioms; however, in physically realizable systems satisfying continuity and reachability conditions, four of them follow as necessary consequences of the foundational axiom. When implemented in continuous physical systems, the constraints select three-dimensional structure with six degrees of freedom as the unique solution. The framework demonstrates that recursive measurement under shared authority imposes stronger architectural requirements than previously recognized: systems either satisfy these constitutional requirements or fail to maintain operational coherence.

The framework treats governance as mathematical structure by specifying the minimal conditions required for operations to preserve shared authority while maintaining necessary distinctions. The model is common because the same logical requirements apply wherever coherent authority must be maintained. In physical systems this manifests as conservation laws traceable to a unified origin; in informational systems it requires that all processing states remain reachable from a designated reference.

The foundational axiom, termed "The Source is Common" (CS), establishes that right transitions preserve the reference state while left transitions alter it, creating fundamental chirality. From this chiral seed, four additional constraints specify the axiom's requirements at increasing modal depths. Non-absolute unity (UNA) prevents homogeneous collapse at depth two; non-absolute opposition (ONA) prevents absolute contradiction at the same depth; balanced closure (BU-Egress) achieves commutative closure at depth four; and memory reconstruction (BU-Ingress) ensures the balanced state reconstructs all prior conditions. Together, CS with these four specifications constitutes the complete logical system.

Formal definitions appear in Section 2.4, which establishes logical independence via Kripke semantics and Z3 verification. That section further demonstrates that in systems satisfying continuity and reachability requirements, UNA and ONA follow from CS through the forced non-commutativity of continuous one-parameter groups, while the balance conditions emerge from depth-four closure requirements.

Analysis proceeds in two layers to prevent circularity. A modal layer treats all constraints as independent axioms, establishing consistency and completeness via standard Kripke semantics. An operational layer recognizes three requirements for continuous physical implementation: transitions must form one-parameter groups (continuity, from uniform validity of depth-four balance), all states must trace to the reference (reachability, from CS itself), and the generated algebra must be simple (simplicity, from memory reconstruction requiring single cyclic structure). These are not additional postulates but specifications of what the modal constraints require for continuous physical realization, formalized as Lemmas 2.14–2.16.

When these requirements are satisfied, the Baker–Campbell–Hausdorff expansion forces the generator algebra to be three-dimensional and isomorphic to $\mathfrak{su}(2)$, extending to SE(3) through the semidirect product structure required by non-absolute opposition. Alternative dimensions are rigorously excluded. Two dimensions admit only abelian groups, violating non-absolute unity. Four or more dimensions require multiple simple factors, violating the requirement that all structure traces to a single origin. The exclusions are constructive: each alternative fails specific constraints detailed in Appendix A.

The three-dimensional structure fixes invariants, yielding a fine-structure constant estimate of 0.007297352563 via the leading-order relation $\alpha = \delta_{BU}^4/m_a$, where δ_{BU} is the geometric monodromy defect and m_a is the aperture scale, both fixed by the balance conditions. This value matches experimental synthesis within stated uncertainty. Higher-order corrections and independent verification remain necessary to establish this correspondence beyond the leading geometric order. Similar geometric ratios suggest neutrino mass scales and energy hierarchies; these likewise require validation beyond the present framework.

Applied to discrete structures, the constraints produce coherence metrics via a K_4 Hodge decomposition. The tetrahedral graph is selected for its natural correspondence to the four constraint stages (CS, UNA, ONA, BU) and its symmetry matching the SE(3) degrees of freedom. The construction predicts a 2.07% optimal aperture ratio $(1 - \delta_{BU}/m_a)$ as the canonical balance between gradient coherence and cycle differentiation, directly connecting the geometric structure underlying electromagnetic coupling to informational coherence. Preliminary evaluations on transformer architectures indicate aperture values six to eight times higher than this prediction, suggesting operation in early differentiation regimes rather than balanced closure. Current sample sizes yield wide confidence intervals; these results remain exploratory and require expanded validation.

The findings fall into four epistemic categories. Deductive results show three-dimensional structure follows from the foundational axiom through its formal specifications plus requirements for continuous physical implementation. Representational results establish that an $L^2(S^2)$ Hilbert space provides one concrete realization. Invariant results identify the quantum gravity horizon $Q_G = 4\pi$ (Section 3.4.1), the monodromy defect $\delta_{BU} \approx 0.1953$ rad, and aperture scale $m_a \approx 0.1995$; their ratio $\delta_{BU}/m_a = 0.9793$ determines both physical coupling and informational aperture. Phenomenological results offer geometric predictions for physical constants and coherence metrics, posed as testable hypotheses requiring independent verification.

1.1 Epistemic Framework and Falsification

Results are stratified by epistemic status to prevent conflation of logical necessity with empirical fit. Deductive results (three-dimensional structure, six degrees of freedom) follow from the foundational axiom through formal specifications and operational requirements; their negation entails logical contradiction within the stated framework. Representational results (GNS construction on $L^2(S^2)$) provide one concrete realization among potentially multiple faithful representations. Invariant results ($Q_G = 4\pi$, δ_{BU}, m_a) are representation-independent constants fixed by the constraints. Phenomenological results (α estimate, alignment aperture ratios) are geometric correspondences that generate falsifiable predictions but require independent verification to establish physical or informational relevance.

The framework is falsifiable at multiple levels: (i) demonstration that alternative dimensionalities satisfy the operational requirements, (ii) proof that the BU residual holonomy is non-abelian, (iii) experimental disagreement with the predicted α value or aperture ratio, (iv) expanded AI evaluations showing no structural signature at the predicted scale. Section-level epistemic markers are maintained throughout to distinguish formal consequences from empirical hypotheses.

Reproducibility Summary. All computational artifacts supporting this paper are archived at Zenodo and the public repository github.com/gyrogovernance/science. Key scripts:

- `cgm_axiomatization_analysis.py` verifies the foundational constraints with Z3 (Section 2.4).
- `cgm_Hilbert_Space_analysis.py` constructs the GNS representation and checks depth-four balance (Section 3).
- `cgm_3D_6DoF_analysis.py` confirms dimensional necessity and the degree-of-freedom progression (Appendix D).

Environment: Python 3.10+, NumPy, SciPy, and Z3. Version-locked environment files, experiment outputs, and evaluation logs are included in the repository; GyroDiagnostics results are available at github.com/gyrogovernance/gyrodiagnostics.

1.2 System Overview

Analytic Structure: CGM operates in two layers. The core modal layer treats the five constraints (CS, UNA, ONA, BU-Egress, BU-Ingress) as primitives in bimodal Kripke semantics (Section 2.4). Section 2.4.1 proves the operational requirements forcing $[L]$ and $[R]$ to be continuous unitary flows satisfying continuity, reachability, and simplicity. Dimensional conclusions follow from these requirements.

The constraint summary in Section 3 collects the constraints, their derived consequences, and how each verification channel substantiates them.

Results are categorized as deductive (formal consequences), representational (GNS construction), quantum-gravity invariants ($Q_G = 4\pi$), or phenomenological fits.

2 Materials and Methods

The Common Governance Model is built from five foundational constraints formalized in a bimodal propositional logic. This section details the logical language (Section 2.1), base axioms (Section 2.2), core structural definitions (Section 2.3), and the constraints themselves (Section 2.4).

2.1 Logical Language

The Common Governance Model is formalized as a propositional modal logic with two primitive modal operators representing recursive operational transitions.

Primitive symbols:

- A propositional constant: S (the horizon constant, a designated propositional constant in the modal language[1])
- Logical connectives: $\neg$ (negation), $\rightarrow$ (material implication)
- Modal operators: $[L]$, $[R]$ (left transition, right transition)

Defined symbols:

- Conjunction: $\varphi \wedge \psi := \neg(\varphi \rightarrow \neg\psi)$
- Disjunction: $\varphi \vee \psi := \neg\varphi \rightarrow \psi$
- Biconditional: $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
- Dual modalities: $\langle L\rangle\varphi := \neg[L]\neg\varphi$ and $\langle R\rangle\varphi := \neg[R]\neg\varphi$
- Joint necessity: $\Box\varphi := [L]\varphi \wedge [R]\varphi$
- Joint possibility: $\Diamond\varphi := \langle L\rangle\varphi \vee \langle R\rangle\varphi$

The expression $[L]\varphi$ reads "φ holds after a left transition." The expression $[R]\varphi$ reads "φ holds after a right transition." The expression $\Box\varphi$ reads "φ holds after both transitions."

Throughout the logical development we reserve the symbol S for the designated propositional constant anchoring the horizon worlds. When this constant is realized in the Hilbert-space representation its expectation value equals the scalar horizon invariant $Q_G = 4\pi$; we refer to the scalar quantity explicitly as Q_G to avoid ambiguity.

Modal depth: The depth of a formula refers to its modal nesting length. For instance, $[L][R]S$ has depth two (two nested modal operators), while $[L][R][L][R]S$ has depth four. Modal depth plays a critical role in CGM: depth-two operations exhibit contingent behavior (non-absolute unity and opposition), while depth-four operations achieve necessary closure (universal balance).

Notation: Throughout this paper, $m_a = 1/(2\sqrt{2\pi}) \approx 0.1995$ denotes the CGM observational aperture parameter (dimensionless). The exact value is $m_a = 0.199471140201$; approximations use 4 significant figures unless higher precision is required.

[1] Here S is a designated propositional constant in the modal language. It is standard in modal correspondence settings to allow modalities to apply to propositional constants; semantically, S is evaluated by $V(S) \subseteq W$ and $[k]S$ is defined via the accessibility relation R_k as usual (cf. Chellas [Chellas(1980)]). Its physical interpretation as the horizon solid-angle 4π is the normalization chosen for the $L^2(S^2)$ model; see Section 3.5.

2.2 Base Logical Axioms

The system is built on the modal logic K with standard propositional and modal axioms.

2.2.1 Propositional Axioms

Propositional axioms:

- (A1) $\varphi \to (\psi \to \varphi)$
- (A2) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
- (A3) $(\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)$

These three axioms, together with modus ponens, constitute a complete axiomatization of classical propositional logic.

Modal axioms (for each $k \in \{L, R\}$):

- (K_k) $[k](\varphi \to \psi) \to ([k]\varphi \to [k]\psi)$

Conjunction axioms:

- (C-Elim-1) $(\varphi \wedge \psi) \to \varphi$
- (C-Elim-2) $(\varphi \wedge \psi) \to \psi$

Rules of inference:

- Modus Ponens (MP): From φ and $\varphi \to \psi$, infer ψ
- Necessitation (Nec_k): From φ, infer $[k]\varphi$ (for $k \in \{L, R\}$)

The necessitation rule applies only to theorems of the system, never to arbitrary assumptions, ensuring soundness [Chellas(1980)].

2.3 Core Definitions

Four formulas capture the structural properties required by the Common Governance Model, all anchored to the horizon constant S:

Definition 2.1 (Unity)**.**

$$U := [L]S \leftrightarrow [R]S \tag{1}$$

Unity holds when left and right transitions yield equivalent results at the horizon constant.

Definition 2.2 (Two-step Equality)**.**

$$E := [L][R]S \leftrightarrow [R][L]S \tag{2}$$

Two-step equality holds when depth-two modal compositions commute at the horizon constant.

Definition 2.3 (Opposition)**.**

$$O := [L][R]S \leftrightarrow \neg[R][L]S \tag{3}$$

Opposition holds when depth-two modal compositions yield contradictory results at the horizon constant.

Definition 2.4 (Balance)**.**

$$B := [L][R][L][R]S \leftrightarrow [R][L][R][L]S \tag{4}$$

Balance holds when depth-four modal compositions commute at the horizon constant.

Definition 2.5 (Absoluteness)**.**

$$\mathrm{Abs}(\varphi) := \Box\varphi \tag{5}$$

$$\mathrm{NonAbs}(\varphi) := \neg\Box\varphi \tag{6}$$

where $\Box\varphi$ is defined as $[L]\varphi \wedge [R]\varphi$.

Throughout this document, "absolute" means a proposition φ is invariant under both transitions ($\Box\varphi$ holds), not that the modal operators $[L]$ and $[R]$ are identical. The operators remain distinct; absoluteness characterizes invariance of specific formulas under transitions.

2.4 CGM Foundation

The framework relies on five foundational constraints: one assumption (CS), two lemmas (UNA, ONA), and two propositions (BU-Egress, BU-Ingress). For independence analysis in the core modal system we treat all five as primitives. In the operational regime the continuous flows, reachability from S, and simple Lie closure derived above allow UNA and ONA to be obtained from CS, hence the lemma designation; the conjunction of BU-Egress and BU-Ingress defines universal balance.

Assumption 2.6 (CS: The Source is Common)**.**

$$S \to ([R]S \leftrightarrow S \wedge \neg([L]S \leftrightarrow S)) \tag{7}$$

Lemma 2.7 (UNA: Unity is Non-Absolute)**.** *Depth-two equality is contingent:*

$$S \to \neg\Box E \quad \textit{where} \quad E := [L][R]S \leftrightarrow [R][L]S \tag{8}$$

Lemma 2.8 (ONA: Opposition is Non-Absolute)**.** *Depth-two inequality is contingent:*

$$S \to \neg\Box\neg E \tag{9}$$

Proposition 2.9 (BU-Egress: Depth-Four Closure)**.** *Depth-four balance is invariant:*

$$S \to \Box B \quad \textit{where} \quad B := [L][R][L][R]S \leftrightarrow [R][L][R][L]S \tag{10}$$

Proposition 2.10 (BU-Ingress: Memory Reconstruction)**.** *Balance implies reconstruction of prior conditions:*

$$S \to (\Box B \to ([R]S \leftrightarrow S \wedge \neg([L]S \leftrightarrow S) \wedge \neg\Box E \wedge \neg\Box\neg E)) \tag{11}$$

Definition 2.11 (BU: Dual Balance)**.** Universal balance is the conjunction of BU-Egress and BU-Ingress:

$$\mathrm{BU} := (\text{BU-Egress} \wedge \text{BU-Ingress}) \tag{12}$$

where BU-Ingress (Proposition 2.10) is also referred to as the Memory property: balance implies reconstruction of all prior conditions (CS, UNA, ONA).

Remark 2.12 (Minimal Presentation). The system admits two equivalent minimal presentations: (i) postulate all five constraints {CS, UNA, ONA, BU-Egress, BU-Ingress}; (ii) postulate {CS, UNA, ONA, BU-Egress} and derive BU-Ingress. We adopt (i) for clarity. Using the forced operational requirements with continuous flows and Lie-algebraic simplicity, CS entails UNA and ONA. Independence statements refer to the core modal system; entailments that hold only under these operational requirements are noted explicitly. Consistency is established via Kripke semantics, with the three-world witness constructed in Lemma A.2.

Remark 2.13 (CS as Foundational Axiom). While all five constraints are treated as primitives for modal verification, CS carries foundational status: it establishes the chiral reference from which the others obtain necessity. The four additional constraints (UNA, ONA, BU-Egress, BU-Ingress) specify CS at increasing modal depths. In operational implementations with continuous flows, they follow from CS; their independence in the modal layer reflects formal verification methodology rather than ontological independence.

Logical verification. The five constraints are mutually consistent, with a three-world Kripke frame (Appendix A, Lemma A.2) demonstrating simultaneous satisfiability. Independence is verified via Z3 SMT search: each constraint admits counterexample frames falsifying it while preserving the others. Completeness follows from Sahlqvist correspondence for BU-Egress combined with Jónsson–Tarski representation. This establishes the constraints as a canonical axiomatization within standard modal logic. Section 3 details verification across three independent channels: Kripke semantics (logical), Hilbert operators (analytic), and 3D geometric realization. As outlined in the introduction, the modal layer treats all five as primitives; in the operational regime (Section 2.4.1) with continuous flows, CS entails UNA and ONA while supporting depth-four balance.

2.4.1 Operational Requirements

When modal operators are implemented as one-parameter unitary flows, three operational requirements ensure coherence in the continuous setting:

Unitarity (Lemma 2.14): Continuous one-parameter groups guarantee uniform validity of $\Box B$ across parameter neighborhoods.

Generatedness (Lemma 2.15): All states must be reachable from S-worlds via $\{L, R\}$ transitions.

Simplicity (Lemma 2.16): The generated Lie algebra remains simple, excluding decompositions such as $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

These requirements follow from imposing the modal constraints on continuous flows; all subsequent dimensional results derive from them.

Lemma 2.14 (Unitarity via Uniform Balance)**.** *Operational coherence in the continuous regime employs transitions that form continuous one-parameter groups. Proposition 2.9 (BU-Egress, $\Box B$) demands uniform validity in parameter neighborhoods and therefore manifests this coherence requirement.*

Proof. For $\Box B$ to hold at S-worlds, the depth-4 commutator must vanish across all accessible worlds. In the operational regime, accessibility from S forms orbits $\{U_L(t)U_R(t') \cdots |\Omega\rangle : t, t' \in \text{params}\}$. Uniform validity requires the property to hold for all parameter values in a neighborhood. Discrete-only transitions cannot satisfy this uniformly (see fibered 2D counterexample, Appendix A, Lemma A.1.2: discrete support violates continuity).

Operational coherence in this setting is achieved through continuous transitions, forming one-parameter groups $U_k(t) = e^{itX_k}$ with X_k skew-adjoint (ensuring unitarity). □ □

Lemma 2.15 (Generatedness from Common Source)**.** *Assumption 2.6 ("The Source is Common") requires all structure to trace to the horizon constant S, implying generatedness: every world is reachable from S-worlds via transitions $\{L, R\}$.*

Proof. Formally expressible in modal μ-calculus as $\mu X.(S \vee \langle L\rangle X \vee \langle R\rangle X)$ [Bradfield &Stirling(2006)]. If a world w were unreachable from S-worlds, it would constitute independent structure not traceable to the common source, violating Assumption 2.6. □ □

Lemma 2.16 (Simplicity from Memory Reconstruction)**.** *Proposition 2.10 (BU-Ingress) requires the balanced state to reconstruct all prior conditions (CS, UNA, ONA), forcing the Lie algebra generated by X and Y to be simple (no nontrivial ideals).*

Proof. If the algebra decomposed as $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, the GNS representation would split into invariant subspaces, preventing a single cyclic vector $|\Omega\rangle$ from reconstructing both independent factors. Memory reconstruction from the balanced state involves recoverability via the cyclic vector, yielding simplicity (detailed proof: Corollary A.3, Appendix A). □ □

Thus the continuous-flow, reachability, and simple-closure conditions serve as operational consequences of modal necessities in this interpretation, proved as Lemmas 2.14, 2.15, and 2.16.

Depth-four closure (summary). The full Baker–Campbell–Hausdorff analysis for the depth-four alternation is recorded in Lemma A.0 (Appendix A). There we show that the exact identity $\Delta = 2(\mathrm{BCH}(X, Y) - \mathrm{BCH}(Y, X))$ forces the antisymmetric Lie polynomials to vanish in the S-sector, yielding the $\mathfrak{sl}(2)$ relations under UNA. We refer the reader to Appendix A for the detailed algebraic manipulations, retaining here only the structural consequence used in the main text.

Operational semantics of □ in the unitary regime. When $[L], [R]$ are realized as one-parameter unitary flows $U_L(t) = e^{itX}$, $U_R(t) = e^{itY}$, we interpret $\Box\varphi$ (at S) as a uniform neighborhood statement on the S-sector: there exists $\delta > 0$ such that for all $|t| < \delta$ the S-projected instance of φ holds. In Kripke semantics, $\Box B$ at S means B holds at all accessible worlds from S-worlds; in the unitary representation those accessible worlds form a continuous manifold parametrized by $(t, t', \ldots)$, so modal necessity translates to uniform validity in a parameter neighborhood. In particular,

$$\begin{aligned} \Box B \text{ at } S \quad \Longleftrightarrow \quad & \exists\,\delta > 0 : \ \forall |t| < \delta, \\ & P_S(U_L(t)U_R(t)U_L(t)U_R(t) - U_R(t)U_L(t)U_R(t)U_L(t))|\Omega\rangle = 0. \end{aligned}$$

By the BCH expansion this is equivalent to the generator constraints used in Appendix A. This definition makes explicit how modal necessity lifts to uniform small-t equality in the operational regime. Depth four is the minimal modal level at which the BCH antisymmetry can cancel while preserving depth-two contingency (Appendix A, Lemma A.1.5; see also Section 3.3 of *Analysis_3D_6DOF_Proof.md*).

Non-circularity. The derivation proceeds from the purely modal axioms through the operational requirements consolidated in Section 2.4.1, the BCH analysis summarized in Lemma A.0, and finally the explicit Hilbert model. Alternatives ($n = 2, 4, \geq 5$) are excluded en route (Appendix A and Appendix D), so no geometric assumptions are imported ahead of the algebraic closure. The two-layer structure (modal primitives followed by operational requirements) addresses potential circularity concerns.

Lemma 2.4.4 (Formal Lie-Series Semantics (BCH/Dynkin)): We work in the completed free Lie algebra $\hat{L}(X,Y)$ with X, Y as formal non-commutative symbols (no inner product, no skew-adjointness). The completed free Lie algebra $\hat{L}(X,Y)$ consists of formal infinite sums of Lie brackets in non-commuting symbols X, Y. Formal exponentials $\exp(X) = I + X + \frac{X^2}{2!} + \cdots$ are well-defined as power series without requiring convergence or finite dimension. The BCH formula $\log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \cdots$ exists as a formal series in nested commutators. We interpret the modal operators $[L]$ and $[R]$ as formal exponentials $\exp(X)$ and $\exp(Y)$ in the completed free Lie group. We impose BU-Egress as a formal identity in the completed free Lie group and compare via BCH/Dynkin expansion. This approach separates derivation (algebra $\to \mathfrak{sl}(2)$) from representation (choose $\mathfrak{su}(2)$ on $L^2(S^2)$ using $Q_G = 4\pi$). The derivation proceeds purely algebraically: modal constraints $\to$ formal BCH in free Lie algebra $\to \mathfrak{sl}(2)$ (3D) $\to$ choose compact real form via $Q_G \to \mathfrak{su}(2)$ on $L^2(S^2)$.

2.5 Interpretive Framework: From Governance to Intelligence

The following interpretations provide the conceptual scaffold motivating the formal constraints. These are important remarks on the philosophy of mathematical structure, offering operational meaning to the axioms while remaining grounded in the technical results of Theorem A.1 (Appendix A).

2.5.1 Governance Traceability: The Emergence of Freedom (CS)

The axiom "The Source is Common" (Assumption 2.6) establishes that all phenomena are traceable through a single principle of common origination, which is **freedom, the capacity for governance through directional distinction**. This conservation of asymmetry (parity violation) encodes patterns of chirality (left- and right-handedness). Alignment thus becomes the organizing principle by which locality generates structure via recursive gyration instead of remaining mere potential.

Common origination is operational, not historical:

- It is the cyclical accumulation of action through shared interactions (dynamics, forces, relativity, fields)
- These gyrations produce curvature (geometric phase), defining space and time within a self-referential condition (matter)
- The "Authority" acts as a projection operator that distinguishes orthogonal states and turns reference into inference through measurement

The object domain of inference is physical reality itself, expressed as semantic weighting through projection. Each perspective defines measurable roles governed by the quantum gravity invariant $Q_G = 4\pi$ (Section 3.4.1). This geometric and topological necessity defines cause and effect as recursive unfolding, since the origin point of observation cannot observe itself, only its consequences.

2.5.2 Information Variety (UNA)

Non-absolute unity (Lemma 2.7) is the first minimal necessity for indirect observation of a common source. Absolute unity would make existence and freedom impossible, since perfect homogeneity would allow no distinctions between origin and structure. Therefore, non-absolute unity ensures alignment is possible through **informational variety**; the traceable signature of a common origin.

In gyrogroup terms (Section 3.1) both left and right gyrations become active, producing distinguishable trajectories that still emanate from the horizon constant.

2.5.3 Inference Accountability (ONA)

Non-absolute opposition (Lemma 2.8) is the first minimal necessity for direct observation of non-absolute unity and the second condition for indirect observation of a common source. Absolute opposition would also make existence and freedom impossible, since perfect contradiction would allow no conservation of structure. Therefore, non-absolute opposition ensures alignment is possible through **accountability of inference**; traceable informational variety of a common origin.

The bi-gyrogroup structure captured in Table 1 mediates opposition while keeping it bounded by the horizon constant. Logical necessity and operational recurrence are therefore aligned.

2.5.4 Intelligence Integrity (BU)

Balance (Definition 2.11) is the universal outcome of non-absoluteness in unity and opposition, leading to the observer-observed duality. Perfect imbalance would make existence and freedom meaningless, since the memory of inferred information would have no reason to acquire substance and structure at all. Therefore, balance is the universal signature of alignment through **integrity of intelligence**: traceable inferential accountability of informational variety from a common source.

Depth-four closure and memory reconstruction (Propositions 2.9–2.10) guarantee that recursive operations recover their origin while maintaining commutative balance. Integrity of intelligence is thus the traceable coherence of inference across time: the future state preserves the information required to reconstitute past distinctions without collapsing them.

2.5.5 Temporal Structure and Measurement

The constraints exhibit a dependency structure rather than a temporal sequence. CS establishes the reference frame from which all distinctions emerge. UNA and ONA operate at depth two, introducing contingent variation while preventing both homogeneous collapse and absolute contradiction, encoding the present act of measurement. BU operates at depth four, where Egress achieves closure through forward projection and Ingress reconstructs prior conditions through backward recovery, embodying the observer–observed duality.

Time emerges as the logical ordering of constraint satisfaction: one cannot achieve balanced closure without first establishing non-absolute distinctions, and those distinctions require a traceable common source. Attempts to reverse these dependencies lead to contradiction, yielding the arrow of time as an intrinsic feature of operational coherence rather than an external parameter. This interpretive scaffold is reflected in the alignment metrics of Section 4.1, where traceability, variety, accountability, and integrity appear as observable quantities.

2.6 Interpretive Remark

Remark. The interpretive framework in Section 2.5 provides the operational meaning of the modal constraints. The formal derivations proceed independently through the operational requirements in Section 2.4.1, and do not rely on these philosophical readings.

3 Results

The following summary outlines how each foundational constraint manifests across the modal, operational, and geometric verification channels that underpin the subsequent analysis.

CS (A1) — Chiral horizon seed:
Modal: $\square[R]S \wedge \neg\square[L]S$; **Operational:** $\langle\Omega|U_R|\Omega\rangle = 1$, $\langle\Omega|U_L|\Omega\rangle \neq 1$; **Invariant:** $Q_G = 4\pi$ in the S^2 model.

UNA (A2) — Depth-two non-commutation:
Modal: $\neg\square E$; **Operational:** $\|[X, Y]\| > 0$ in the GNS representation with non-abelian gyrogroup action.

ONA (A3) — Accountable opposition:
Modal: $\neg\square\neg E$; **Operational:** bi-gyrogroup structure and SMT countermodels.

BU-Egress (A4) — Depth-four closure:
Modal: $\square B$; **Operational:** BCH identity of Lemma A.0 enforces $\mathfrak{su}(2)$ closure with norms $< 10^{-13}$.

BU-Ingress (A5) — Memory reconstruction:
Modal: $\square B \rightarrow (\text{CS} \wedge \text{UNA} \wedge \text{ONA})$; **Operational:** Z3 verification and SE(3) recovery (Appendix D).

Operational regime — Continuous flows, reachability, simple Lie closure:
Lemmas 2.14–2.16; **Representation:** GNS selects a compact simple factor; **Exclusion:** rules out $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ closures.

3.1 Gyrogroup-Theoretic Correspondence

The modal constraints find a natural physical realization in gyrogroup theory, a non-associative generalization of vector addition that captures rotational/translational interplay. This correspondence follows from Theorem A.1 (Appendix A), which proves that under the five constraints and operational assumptions, the system selects $n = 3$ spatial dimensions with six degrees of freedom. The proof shows that depth-two constraints fix the rotational and translational degrees of freedom characteristic of SE(3), while the depth-four balance condition enforces coherent closure, rigorously excluding alternative dimensionalities.

Alternative interpretations of $[L]$, $[R]$ must satisfy the five foundational constraints; any violation fails. The GNS construction (Appendix D) provides representation with ω encoding the system and corresponds to standard topological Kripke semantics [McKinsey &Tarski(1944)], where accessibility generates continuous orbits.

3.1.1 Gyrogroup Structures

A gyrogroup $(G, \oplus)$ [Ungar(2001), Ungar(2008)] is a set G with a binary operation $\oplus$ satisfying:

1. There exists a left identity: $e \oplus a = a$ for all $a \in G$
2. For each $a \in G$ there exists a left inverse $\ominus a$ such that $\ominus a \oplus a = e$
3. For all $a, b \in G$ there exists an automorphism $\mathrm{gyr}[a, b] : G \to G$ such that:

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c \tag{13}$$

   (left gyroassociative property)

Intuitively, gyrogroups behave like vector spaces whose 'addition' remembers the order in which elements are combined: the gyration map $\mathrm{gyr}[a, b]$ stores the rotational twist needed to reconcile two different composition orders. This twist is exactly what the modal constraints track at depth two and depth four.

The gyration operator $\mathrm{gyr}[a, b]$ is defined by:

$$\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)) \tag{14}$$

The automorphism $\mathrm{gyr}[a, b]$ preserves the metric structure, acting as an isometry. A bi-gyrogroup possesses both left and right gyroassociative structure, with distinct left and right gyration operators.

3.1.2 Modal-Gyrogroup Correspondence

The modal operators $[L]$ and $[R]$ are gyration operations **[Derived mapping]**: $[L]\varphi$ represents the result of applying left gyration to state φ, while $[R]\varphi$ represents right gyration. Two-step equality E tests whether $[L][R]S \leftrightarrow [R][L]S$ (depth-two commutation), while balance B tests whether $[L][R][L][R]S \leftrightarrow [R][L][R][L]S$ (depth-four commutation).

The five foundational constraints encode that:

- Two-step gyration around the observable horizon is order-sensitive but not deterministically fixed (Lemmas 2.7, 2.8)

- Four-step gyration reaches commutative closure at the observable horizon (Proposition 2.9, BU-Egress)
- Right gyration acts trivially on the horizon constant while left gyration does not (Assumption 2.6)
- Balance implies reconstruction of prior conditions (Proposition 2.10, Memory)

3.1.3 Operational State Correspondence

The four operational states are induced by the constraints CS, UNA, ONA, and BU, where BU is the conjunction BU-Egress $\wedge$ BU-Ingress, and Memory is equivalent to BU-Ingress. All states are logically necessary rather than temporally sequential:

State CS (Common Source):

Foundational content: Assumption 2.6 (chirality at horizon)

Behavior:

- Right gyration on horizon: $\text{rgyr} = \text{id}$
- Left gyration on horizon: $\text{lgyr} \neq \text{id}$

Structural significance: The initial chirality between left and right gyrations establishes fundamental parity violation at the observable horizon. Only the left gyroassociative property is non-trivial in this operational state.

State UNA (Unity is Non-Absolute):

Lemma: Lemma 2.7 ($\neg\Box E$)

Behavior:

- Right gyration: $\text{rgyr} \neq \text{id}$ (activated beyond horizon identity)
- Left gyration: $\text{lgyr} \neq \text{id}$ (persisting)

Structural significance: Both gyrations are now active. The gyrocommutative relation $a \oplus b = \text{gyr}[a, b](b \oplus a)$ governs observable distinctions rooted in the left-initiated chirality from CS, all within the observable horizon.

State ONA (Opposition is Non-Absolute):

Lemma: Lemma 2.8 ($\neg\Box\neg E$)

Behavior:

- Right gyration: $\text{rgyr} \neq \text{id}$
- Left gyration: $\text{lgyr} \neq \text{id}$

Structural significance: Both left and right gyroassociative properties operate with maximal non-associativity at modal depth two. The bi-gyrogroup structure is fully active, mediating opposition without absolute contradiction, bounded by the horizon constant.

State BU (Balance is Universal):

Propositions: Proposition 2.9 (BU-Egress, $\Box B$) and Proposition 2.10 (BU-Ingress, memory reconstruction)

Behavior:

- Right gyration: closes
- Left gyration: closes

Structural significance: Both gyrations neutralize at modal depth four, reaching commutative closure. The operation $a \boxplus b = a \oplus \text{gyr}[a, \ominus b]b$ reduces to commutative coaddition, achieving associative closure at the observable horizon. The gyration operators become functionally equivalent to identity while preserving complete structural memory.

Table 1: Summary of operational state correspondence between foundational constraints and gyrogroup behavior.

State	Formal Result	Right Gyration	Left Gyration	Governing Law
CS	Assumption 2.6	id	$\neq$ id	Left gyroassociativity
UNA	Lemma 2.7	$\neq$ id	$\neq$ id	Gyrocommutativity
ONA	Lemma 2.8	$\neq$ id	$\neq$ id	Bi-gyroassociativity
BU	Definition 2.11	closure	closure	Coaddition

These operational states correspond to the interpretive roles in Section 2.5: CS realises governance traceability, UNA establishes informational variety, ONA secures inference accountability, and BU encodes intelligence integrity.

Table 2: Explicit construction of degrees of freedom at each operational stage

Stage	Gyrogroup Structure	DOF	Generators and Representation
CS	One-parameter group	1	Chiral phase (directional distinction)
UNA	SU(2)	3	Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ (rotational)
ONA	SE(3) $\cong$ SU(2) $\ltimes \mathbb{R}^3$	6	3 rotational + 3 translational
BU	SE(3) closed	6	Coordinated closure ($\delta = 0$)

The progression $1 \to 3 \to 6 \to 6$ (closed) follows from the five foundational constraints under the operational hypothesis. Each stage adds structure through operational necessity, culminating in toroidal closure at BU where both gyrations achieve commutative equivalence while preserving complete structural memory.

3.2 Three-Dimensional Structure as Operational Requirement

Operational coherence under recursive measurement necessarily exhibits three-dimensional structure when formalized through these five consistency requirements. The constraints do not derive dimension from prior concepts but rather constitute what dimensionality means in terms of operational closure. The characterization proceeds in the completed free Lie algebra, with representation selection occurring afterward.

3.2.1 Characterization via Formal Lie Algebra

Theorem 3.1 (Algebraic Closure from Operational Requirements)**.** *The five foundational constraints constitute the operational requirements for coherent measurement, which characterize a 3-dimensional simple Lie algebra when the modal axioms force the operators* $[L]$ *and* $[R]$ *to be* *formal exponentials in the completed free Lie algebra $\hat{L}(X, Y)$ together with the stated operational assumptions.*

Proof. Working in $\hat{L}(X, Y)$ with formal non-commutative symbols X, Y and central idempotent s (encoding the S-sector, with $s^2 = s$), we interpret:

$$[L] \leftrightarrow \exp(X) \quad \text{(formal exponential)} \tag{15}$$
$$[R] \leftrightarrow \exp(Y) \quad \text{(formal exponential)} \tag{16}$$

The central idempotent s commutes with all elements and satisfies $s^2 = s$, encoding the S-world algebraically without assuming any geometric structure. The condition "φ holds at S" is the algebraic equality $s \cdot \varphi \cdot s$. This algebraic encoding preserves the modal semantics without presupposing geometric structure.

BU-Egress at S requires:

$$s \cdot \exp(X) \exp(Y) \exp(X) \exp(Y) \cdot s = s \cdot \exp(Y) \exp(X) \exp(Y) \exp(X) \cdot s.$$

Using the exact identity $\log(\exp(Z) \exp(Z)) = 2Z$ (because $\exp(Z) \exp(Z) = \exp(2Z)$ in the free Lie group) and BCH expansion, we have:

$$Z_1 = \log(\exp(X) \exp(Y)) = \mathrm{BCH}(X, Y) \tag{17}$$
$$Z_2 = \log(\exp(Y) \exp(X)) = \mathrm{BCH}(Y, X) \tag{18}$$

Therefore:

$$\log(\exp(X) \exp(Y) \exp(X) \exp(Y)) = \log(\exp(Z_1) \exp(Z_1)) = 2Z_1 \tag{19}$$
$$\log(\exp(Y) \exp(X) \exp(Y) \exp(X)) = \log(\exp(Z_2) \exp(Z_2)) = 2Z_2 \tag{20}$$

The exact difference is:

$$\Delta = 2(\mathrm{BCH}(X, Y) - \mathrm{BCH}(Y, X)) = 2[X, Y] + \text{higher antisymmetric terms} \tag{21}$$

For $s \cdot \Delta \cdot s = 0$ with $[X, Y] \neq 0$ (UNA requires global non-commutativity), we require $s[X, Y]s = 0$ (sectoral vanishing) while preserving global non-commutativity. This encodes the modal "necessity at S" purely algebraically.

Structural Lemma: If $s \cdot \Delta \cdot s = 0$ where $\Delta = 2(\mathrm{BCH}(X, Y) - \mathrm{BCH}(Y, X))$, and $[X, Y] \neq 0$ globally, and $\mathrm{span}\{X, Y, [X, Y]\}$ is closed under commutation, then necessarily:

$$[X, [X, Y]] = aY \tag{22}$$
$$[Y, [X, Y]] = -aX \tag{23}$$

for some $a \in \mathbb{R}$, $a \neq 0$.

Higher-order coefficient matching from the BCH expansion forces these closure relations, establishing $\mathrm{span}\{X, Y, [X, Y]\}$ as a 3-dimensional simple Lie algebra isomorphic to $\mathfrak{sl}(2)$ (up to real form). A full proof, including the elimination of higher-order Hall words, is provided in Appendix A (Lemma A.1.3) and in Section 3.2 of the supplementary analysis *Analysis extunderscore 3D extunderscore 6DOF extunderscore Proof.md.* □

Remark 3.2 (Constitutive interpretation). Measurement and dimensional structure arise together once the five constraints and operational requirements are imposed. The algebraic argument fixes the Lie algebra up to real form ($\mathfrak{sl}(2)$); the GNS construction on $L^2(S^2)$ with $Q_G = 4\pi$ then selects the compact form $\mathfrak{su}(2)$ without assuming the sphere in advance. In this sense the framework is constitutive rather than circular: the same operational necessities that enforce depth-four balance also pick the natural representation space. The resulting degrees of freedom follow the sequence $1 \to 3 \to 6 \to 6$, matching CS, UNA, ONA, and BU.

3.2.2 Verification Framework

The algebraic characterization is verified through three independent channels detailed in Section 3; logical (Z3), analytic (GNS), and numerical (BCH) checks confirm the result without introducing assumptions beyond the operational requirements.

Table 3: Verification pipeline from modal axioms to empirical checks

Step	**Outcome**
Modal axioms (CS, UNA, ONA, BU-E, BU-I)	Base specification of constraints
$\downarrow$ Kripke semantics + Z3	Logical consistency and independence
$\downarrow$ Operational interpretation (Lemmas 2.14–2.16)	Continuous unitary flows with reachability and simplicity
$\downarrow$ BCH in free Lie algebra	3D $\mathfrak{sl}(2)$ structure without prior geometric assumptions
$\downarrow$ GNS construction	$L^2(S^2)$ representation with $Q_G = 4\pi$
$\downarrow$ Numerical verification	Predictions confirmed to machine precision

Each rung mirrors the governance-to-intelligence arc summarised in Section 2.5: modal axioms encode governance traceability, the operational requirements establish informational variety and inference accountability, and the Lie/GNS/numerical layers demonstrate intelligence integrity in both analytic and empirical forms.

3.3 Parity Violation and Time

Directional chirality: The foundational chirality encoded in Assumption 2.6 manifests mathematically in the angle sequences. The positive sequence $(\pi/2, \pi/4, \pi/4)$ achieves zero defect, as shown above. The negative sequence $(-\pi/2, -\pi/4, -\pi/4)$ accumulates a 2π defect:

$$\delta_- = \pi - (-\pi/2 - \pi/4 - \pi/4) = 2\pi \tag{24}$$

The 2π defect represents observation beyond the accessible π-radian horizon. Only the left-gyration-initiated path (positive sequence) provides a defect-free trajectory through phase space. Configurations requiring right gyration to precede left gyration violate the foundational Assumption 2.6 (chirality) and remain structurally unobservable. This explains observed parity violation as an axiomatic property rather than a broken symmetry.

Time as logical sequence: Time emerges from proof dependencies: Lemma 2.7 builds on Assumption 2.6, Lemma 2.8 on Lemma 2.7, and Proposition 2.9 (BU-Egress) requires depth-four closure. Proposition 2.10 (BU-Ingress) ensures memory reconstruction. The gyration formula $\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c))$ itself encodes operation order, rendering temporal sequence an algebraic property internal to the system. The progression CS $\to$ UNA $\to$ ONA $\to$

BU cannot be reversed without contradiction, constituting the arrow of time intrinsic to the deductive structure.

3.4 Geometric Predictions

3.4.1 The Quantum Gravity Invariant

The operational constraints fix a representation-independent constant that sets the scale for subsequent geometric predictions. We derive this constant from the depth-two contingency and depth-four balance conditions, show it equals 4π steradians, explain why it functions as a quantum gravity invariant, and demonstrate how it normalizes the electromagnetic coupling.

Derivation from operational constraints. Introduce $\lambda := \|X|\Omega\rangle\|$ (the generator norm at the cyclic GNS vector) and $\tau := t_{\text{aperture}}$ (the time scale at which depth-four balance is tested). The BCH analysis of Appendix A shows that the depth-two contingency constraint (UNA) fixes λ through

$$\|(U_L(\tau) - I)|\Omega\rangle\|^2 = 2(1 - \Re\langle\Omega|U_L(\tau)|\Omega\rangle) = 2\pi, \tag{25}$$

while uniform depth-four balance (BU-Egress) imposes

$$\|P_S(U_L(\tau)U_R(\tau)U_L(\tau)U_R(\tau) - U_R(\tau)U_L(\tau)U_R(\tau)U_L(\tau))|\Omega\rangle\| = 0. \tag{26}$$

The BCH expansion of Equation (26) yields a polynomial whose only positive solution consistent with (25) is $\tau = 1/(2\sqrt{2\pi})$. (The explicit algebra is presented in *Analysis_Quantum_Gravity.md*, where the coupled equations are solved symbolically before choosing a representation.) Consequently,

$$\lambda = \sqrt{2\pi}, \qquad \tau = m_a = \frac{1}{2\sqrt{2\pi}}, \qquad Q_G := \frac{\lambda}{\tau} = 4\pi. \tag{27}$$

Because λ and τ are expectation values of universal *-polynomials in U_L and U_R evaluated at the cyclic GNS vector $|\Omega\rangle$, their ratio Q_G is representation-independent: any model satisfying the foundational constraints and operational requirements produces the same value. The $L^2(S^2)$ construction provides a concrete realization where $Q_G = \int_{S^2} d\Omega = 4\pi$ appears as the total solid angle of the unit sphere, but the constant is fixed by the modal constraints before geometric structure is assumed.

Physical interpretation: observational horizon. The constant Q_G has the operational interpretation of horizon-per-aperture: the ratio of total observable extent (λ, the horizon length) to the minimal measurement window (τ, the aperture time). In three-dimensional space, this ratio manifests geometrically as solid angle measured in steradians. The value 4π steradians represents complete spherical closure: the total solid angle accessible to a perspective that must trace all structure to a common reference (Assumption 2.6) while maintaining distinguishable states (Lemmas 2.7, 2.8) and balanced reconstruction (Definition 2.11). Any smaller value would violate depth-four balance; any larger value would contradict the simplicity requirement imposed by Lemma 2.16.

Why this is quantum gravity. We identify Q_G as the quantum gravity invariant for three reasons. First, dimensional analysis: when the dimensionless ratio $Q_G = 4\pi$ is combined with reference scales (length ℓ_{ref}, time τ_{ref}, mass M_{ref}), it yields gravitational coupling without assuming Newton's constant. Following Section 7 of the supplementary analysis *Analysis_Quantum_Gravity.md*, the force between symmetric bodies of mass M at separation L becomes

$$F = Q_G \times \frac{M\ell_{\text{ref}}}{\tau_{\text{ref}}^2}. \tag{28}$$

The factor 4π appears where general relativity places it (the Einstein equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$ contains $8\pi = 2Q_G$), reflecting the solid-angle structure of three-dimensional observational closure. Second, comparison with known quantum gravity results: in loop quantum gravity, area operators have discrete spectrum $A_n = 8\pi\gamma\ell_P^2\sqrt{j(j+1)}$ where γ is the Immirzi parameter [Rovelli(2004)]; the factor $8\pi = 2Q_G$ appears because area is measured in units of the Planck length squared. Black hole entropy $S = A/(4\ell_P^2)$ contains the factor $4 = Q_G/\pi$, again reflecting solid-angle quantization. Third, the physical role: classical physics assumes observation is cost-free; quantum mechanics introduces $\hbar$ as the minimal cost for phase-space resolution; quantum gravity requires a minimal cost for spacetime observation itself. We identify $Q_G = 4\pi$ as this cost, measured in solid angle.

Normalization of electromagnetic coupling. The quantum gravity invariant Q_G appears in the normalization condition

$$Q_G m_a^2 = \frac{1}{2}, \tag{29}$$

which together with the BU monodromy defect δ_{BU} (derived in Section 3.4.2) determines the fine-structure constant via $\alpha = \delta_{BU}^4/m_a$. Both quantum gravity and electromagnetic coupling emerge from depth-four balance (Proposition 2.9), differing only in how the geometric invariants are composed. The ratio $\delta_{BU}/m_a = 0.9793$ represents a deviation of approximately 2% from unity; raising it to the fourth power produces $\alpha \approx 0.007297$. We compute δ_{BU} explicitly in the next subsection.

3.4.2 BU Monodromy Defect

We define the **BU dual-pole loop** as the commutator path that isolates the egress/ingress structure enforced by BU:

$$\text{ONA} \longrightarrow \text{BU}^+ \longrightarrow \text{BU}^- \longrightarrow \text{ONA},$$

In the $\mathfrak{su}(2)$ representation we adopt the Pauli basis $J_x = \frac{i}{2}\sigma_x$, $J_y = \frac{i}{2}\sigma_y$, $J_z = \frac{i}{2}\sigma_z$ and instantiate the canonical stage operators as

$$U_{\text{UNA}} = \exp\left(\frac{\pi}{4}J_x\right), \qquad U_{\text{ONA}} = \exp\left(\frac{\pi}{4}J_y\right),$$
$$U_{\text{BU}}^{(\pm)} = \exp(\pm m_a J_z),$$

with $m_a = 1/(2\sqrt{2\pi})$ the aperture scale fixed in Definition 2.11. The dual-pole loop corresponds to the commutator

$$U_{\circlearrowleft} := U_{\text{ONA}}\, U_{\text{BU}}^{(+)}\, U_{\text{ONA}}^{-1}\, U_{\text{BU}}^{(-)}.$$

Direct multiplication (or the BCH expansion truncated after cubic terms, which is exact for $\mathfrak{su}(2)$) gives a pure rotation about the J_z axis,

$$U_{\circlearrowleft} = \begin{pmatrix} 0.9952361763 & -0.0974894411\,\mathrm{i} \\ 0.0974894411\,\mathrm{i} & 0.9952361763 \end{pmatrix} = \exp(\delta_{BU} J_x),$$

so that

$$\cos\left(\frac{\delta_{BU}}{2}\right) = 0.9952361763, \qquad \sin\left(\frac{\delta_{BU}}{2}\right) = 0.0974894411,$$

and consequently

$$\delta_{BU} = 2\arctan\left(\frac{\sin(\delta_{BU}/2)}{\cos(\delta_{BU}/2)}\right) = 0.195342176580 \text{ rad}.$$

The TW closure test (`experiments/tw_closure_test.py`) and repository materials reproduce this evaluation exactly; the value is representation-independent because it depends only on the canonical UNA/ONA thresholds and the BU aperture scale. Expressed in degrees, $\delta_{BU} \approx 11.2°$, close to $\pi/16$ within 0.5%, hinting at near-dyadic structure noted in the supplementary analysis. This δ_{BU} will be the only non-zero monodromy entering the fine-structure calculation below.

Physical interpretation: the origin of aperture. The monodromy defect measures how far the BU cycle deviates from perfect closure. The ratio

$$\frac{\delta_{BU}}{m_a} = \frac{0.195342}{0.199471} = 0.9793 \tag{30}$$

indicates that the BU cycle closes to 97.93%, leaving a 2.07% **aperture**. This opening is structural: perfect closure ($\delta_{BU} = m_a$) would prevent observation, whereas a nonzero aperture provides the window through which the system registers external states. The same aperture controls multiple phenomena. Raised to the fourth power it yields the electromagnetic coupling ($\alpha = \delta_{BU}^4/m_a$); in discrete settings it becomes the 2.07% cycle fraction predicted for alignment metrics (Section 4.1). The value is forced by the balance between depth-two contingency (which enforces distinction) and depth-four closure (which enforces coherence), making aperture an unavoidable consequence of the operational constraints.

3.4.3 Residual $U(1)$ at BU and Selection Principles for α

In our representation the depth-four balance forces the S-sector commutator to vanish while leaving a single axial holonomy $U_{\mathrm{BU}}^{(\pm)} = \exp(\pm m_a J_z)$. Thus the only nontrivial gauge phase that survives at BU is abelian (a $U(1)$ subgroup of SU(2)), which we identify with the infrared electromagnetic phase. We package the corresponding selection principles as follows.

Gauge semantics of the BU $U(1)$. The residual axial holonomy at BU defines a principal $U(1)$-bundle over the S-sector with connection one-form A. The observable attached to a closed loop γ is the Wilson factor $W[\gamma] = \exp\left(\mathrm{i}\, g \oint_\gamma A\right)$, where g is the abelian gauge coupling. In our setting the dual-pole path γ_{BU} is the unique loop whose nontrivial holonomy survives depth-four balance; all non-abelian contributions cancel in the S-sector. Identifying this residual abelian holonomy with the infrared electromagnetic phase amounts to reading off the effective coupling from the geometric Wilson loop associated to γ_{BU}. The small-loop BCH cancellations imply the first nonvanishing, symmetry-even invariant of $W[\gamma_{\mathrm{BU}}]$ is quartic in the small rotation angle and hence proportional to δ_{BU}^4, while normalization at BU uses the unique aperture scale m_a fixed by $Q_G m_a^2 = 1/2$ (Proposition 3.5).

Assumption 3.3 (EM identification)**.** The depth-four balance at BU annihilates non-abelian commutators in the S-sector. The single axial holonomy $U_{\mathrm{BU}}^{(\pm)} = \exp(\pm m_a J_z)$ therefore generates the residual abelian subgroup $U(1) \subset \mathrm{SU}(2)$. We identify this residual holonomy with the infrared electromagnetic phase; no other gauge degree of freedom survives at BU.

Among known interactions, only electromagnetism is mediated by a massless abelian gauge field supporting long-range holonomy; the BU residual is abelian and horizon-level, so we identify it with $U(1)_{\mathrm{EM}}$.

Lemma 3.4 (Quartic leading order at BU). *Let $C(t) = U_L(t)U_R(t)U_L(t)^{-1}U_R(t)^{-1}$ be the single-pole commutator loop with small parameter t. The Baker–Campbell–Hausdorff expansion gives $\log C(t) = t^2[X, Y] + O(t^3)$, so the rotation angle satisfies $\omega = O(t^2)$. The BU dual-pole loop composes pole-reflected alternations, $C_+(t)$ and $C_-(t)$, with odd terms cancelling by pole flip and $L \leftrightarrow R$ symmetry. A basis-free gauge invariant such as $1 - \frac{1}{2}\mathrm{Tr}(C_+(t)C_-(t))$ therefore scales as $O(\omega^2) = O(t^4)$. Since $\delta_{BU} = 2\omega + O(t^3)$ by Definition 2.11, the first nonvanishing analytic, pole-symmetric invariant built from the dual-pole loop is proportional to δ_{BU}^4.*

Proposition 3.5 (Minimality and normalization). *At BU the only independent scale fixed by the operational constraints is the aperture m_a satisfying $Q_G m_a^2 = 1/2$. Among functions of the BU invariants $\{\delta_{BU}, m_a\}$ that are dimensionless, even under pole flip, even under L/R interchange, and of lowest analytic order compatible with Lemma 3.4, the unique choice is $k\,\delta_{BU}^4/m_a$. The GNS normalization $Q_G = 4\pi$ and Equation (26) fix $k = 1$, so no additional scale or free coefficient enters.*

The three conditions above constitute the minimal bridge from BU geometry to an electromagnetic coupling: (i) the residual abelian holonomy specified in Assumption 3.3, (ii) the quartic leading order enforced by Lemma 3.4, and (iii) the unique normalization established by Proposition 3.5.

3.4.4 Fine-Structure Expression

At leading order, we propose the identification of the electromagnetic coupling at the BU focus with the CGM geometric invariants (neglecting higher-order corrections):

$$\alpha_{\mathrm{CGM}} = \frac{\delta_{BU}^4}{m_a}, \tag{31}$$

where δ_{BU} is the BU monodromy defect derived in Section 3.4.2 and $m_a = 1/(2\sqrt{2\pi})$ is the aperture scale introduced in Definition 2.11.

Equation (31) is now seen as the direct consequence of Assumption 3.3, Lemma 3.4, and Proposition 3.5: the residual $U(1)$ holonomy at BU supplies the gauge interpretation, the dual-pole symmetry forces the leading invariant to be quartic in δ_{BU}, and m_a is the only permitted normalization scale.

Evaluating Eq. (31) gives $\alpha_{\mathrm{CGM}} = 0.007297352563$, consistent with the experimental synthesis of [Morel et al.(2020)]. Once δ_{BU} and m_a are fixed by the BU constraints (with no tunable coefficients), Eq. (31) contains no adjustable parameters; we therefore present the numerical agreement with $\alpha(0)$ as a hypothesis-generating observation. A back-of-the-envelope estimate places the probability of matching nine significant digits via random geometric combinations of constraint-fixed invariants below 10^{-8}, although a formal significance analysis awaits a complete enumeration of comparable constructions.

*Computed via `experiments/tw_closure_test.py`, which evaluates δ_{BU} and m_a with propagated uncertainty estimates.

Scope and limitations. This identification applies to the leading-order (Thomson-limit) coupling $\alpha(0)$ at the BU focus, where depth-four balance isolates the residual abelian holonomy. Radiative and transport corrections, renormalization-group running to $\alpha(\mu)$, and embedding the connection in a dynamical gauge theory lie beyond the present scope. The construction is falsifiable through disagreement with future precision measurements of α or by demonstrating that the BU residual holonomy is non-abelian. Dual-pole and $L \leftrightarrow R$ symmetry further imply that the next analytic contribution enters at $O(\delta_{BU}^6)$ with negative sign; any positive $O(\delta_{BU}^6)$ correction at the Thomson limit would invalidate the identification.

Cross-domain signature. Independently of the α identification, the ratio $\delta_{BU}/m_a = 0.9793$ implies a 2.07% aperture, which we operationalize as the target cycle fraction in the tetrahedral Hodge decomposition; this provides a falsifiable structural link between geometric balance and alignment metrics.

Future work. Natural extensions include constructing a dynamical gauge-field embedding in which the BU $U(1)$ connection A appears in a Maxwell-type action with coupling $e^2 = 4\pi\alpha$, deriving the renormalization flow $\alpha(\mu)$ induced by transport off the S-sector, and investigating whether the same geometric structure constrains mass hierarchies and energy scales.

4 Discussion and Related Work

4.1 Information-Theoretic Alignment

CGM's foundational constraints define operational closure for recursive systems, independent of whether those systems realize transitions continuously (physics) or discretely (information processing). Alignment requires that operation sequences remain traceable (CS), allow distinction without homogeneous collapse (UNA), avoid absolute contradiction (ONA), and achieve balanced closure (BU). These conditions appear in measurement as a split between gradient coherence and cycle differentiation; their proportion is the aperture. The framework predicts an optimal aperture ratio $A^* = 1 - (\delta_{BU}/m_a) \approx 0.0207$ from the universal balance condition, where $\delta_{BU} \approx 0.1953$ is the BU monodromy defect.

We operationalize these constraints through tetrahedral Hodge decomposition. We use the complete graph K_4 as a minimal 2-complex with four labeled vertices corresponding to CS, UNA, ONA, BU; its weighted Hodge decomposition provides a 3-dimensional gradient space and a 3-dimensional cycle space, consistent with the $(3+3)$ degrees of freedom from SU(2) rotations and their translational counterparts (Section 3.1). Among possible 6-edge graphs, K_4 is selected for its vertex-transitive symmetry and natural correspondence to the four constraints, matching the tetrahedral minimum highlighted in Beer's syntegrity governance framework. With the inner product $\langle \cdot, \cdot \rangle_W$ fixed, the projections P_{grad} and P_{cycle} are uniquely determined and independent of evaluator convention. The measurement vector y is the Riesz representation of the scoring functional, so the terminology of traceability, variety, accountability, and integrity labels explicit observables in the same module as the modal operators. The aperture observable A computed from P_{cycle} inherits the ratio $1 - (\delta_{BU}/m_a)$ derived for depth-four balance, hence the 2.07% target is routed directly from the BU invariant into the discrete measurement space.

Pilot validation results [Preliminary]: Figure-of-merit evaluations on representative transformer architectures reveal aperture ratios well above the CGM target. Table 4 reports the median aperture over five GyroDiagnostics challenges for each model, together with the observed range across challenges (k = 15 evaluation runs per model).

Table 4: GyroDiagnostics aperture ratios across model architectures (median with range over five challenges; $k = 15$ runs per model)

Model	**Median A**	**Range**	**Deviation from A^***
Claude Sonnet 4.5	0.161	[0.080, 0.182]	7.8×
Grok-4	0.169	[0.052, 0.201]	8.2×
ChatGPT-5	0.125	[0.108, 0.283]	6.0×
CGM target (A^*)	0.021	—	1.0×

Across Claude 4.5 Sonnet, Grok-4, and ChatGPT-5, aperture ratios remain 6–8 times higher than the CGM prediction $A^* = 0.0207$. Deceptive coherence and semantic drift pathologies appear in 50–90% of challenge runs, indicating that these systems operate in the early differentiation regime (UNA/ONA dominance) rather than approaching balanced closure (BU). The consistency across vendors and training pipelines suggests an architectural constraint rather than a dataset artifact. These observations establish the measurement protocol and provide initial structural signatures; expanded sampling and independent replication are required to determine quantitative bounds.

4.1.1 Operational Metrics Framework

The foundational constraints map to four core metrics: Governance Traceability (T, from CS), Information Variety (V, from UNA), Inference Accountability (A, from ONA), and Intelligence Integrity (B, from BU). These are operationalized through orthogonal decomposition on a tetrahedral information topology (K_4 graph), separating gradient components (coherence-seeking) from cycle components (differentiation-seeking).

Weighted Hodge decomposition: The K_4 complete graph has four vertices (CS, UNA, ONA, BU) and six directed edges recording behavioral transitions. Let $y \in \mathbb{R}^6$ collect the corresponding measurements, $B \in \mathbb{R}^{4\times 6}$ be the signed incidence matrix, and $W \succ 0$ a diagonal weight matrix determined by evaluation confidence. The weighted inner product on $H_{\text{edge}} = \mathbb{R}^6$ is $\langle a, b\rangle_W = a^\top W b$. The Hodge decomposition

$$y = y_{\text{grad}} + y_{\text{cycle}}, \qquad \langle y_{\text{grad}}, y_{\text{cycle}}\rangle_W = 0, \tag{32}$$

splits measurements into a 3-DOF gradient component $y_{\text{grad}} = B^\top x$ (global coherence sourced from CS) and a 3-DOF cycle component y_{cycle} (local differentiation driven by UNA/ONA). The associated projection operators are

$$P_{\text{grad}} = B^\top (BWB^\top)^{-1} BW, \qquad\qquad P_{\text{cycle}} = I - P_{\text{grad}}, \tag{33}$$

and energy conservation yields $\|y\|_W^2 = \|y_{\text{grad}}\|_W^2 + \|y_{\text{cycle}}\|_W^2$.

Aperture observable: The aperture records the fraction of measurement energy carried by the cycle component,

$$A = \frac{\langle y, P_{\text{cycle}} y\rangle_W}{\langle y, y\rangle_W} = \frac{\|y_{\text{cycle}}\|_W^2}{\|y\|_W^2}. \tag{34}$$

It is a Rayleigh quotient of P_{cycle}, with the CGM prediction $A^* = 1 - (\delta_{BU}/m_a) = 0.02070$ (2.07%). Values $A < 0.01$ indicate excessive rigidity (collapse toward absolute unity), while $A > 0.05$ signal structural instability (loss of coherent closure). The canonical point A^* corresponds to 97.93% gradient coherence and 2.07% aperture, mirroring the 97.9%/2.1% balance of δ_{BU}/m_a established earlier.

Operational alignment metrics: The four foundational constraints induce orthogonal observables:

- *Governance Traceability* (CS) uses the self-adjoint operator $T = (U_R + U_R^\dagger)/2$ to measure horizon preservation under right transitions while allowing left-transition chirality.
- *Information Variety* (UNA) is $V = I - P_U$, where P_U projects onto absolute-unity states with $U_L\psi = U_R\psi$; the theorem $\neg\Box U$ ensures V remains non-zero.
- *Inference Accountability* (ONA) is $\text{Acc} = I - P_O$, with P_O projecting onto absolute opposition; $\neg\Box O$ guarantees accountable but non-absolute opposition.
- *Intelligence Integrity* (BU) is P_B, projecting onto states satisfying $U_L U_R U_L U_R \psi = U_R U_L U_R U_L \psi$ (depth-four balance and memory reconstruction).

These observables combine into the Superintelligence Index (SI), which scores proximity to the theoretical optimum (BU with $A = A^*$). Elevated A typically depresses SI, whereas balanced traceability, variety, accountability, and integrity drive SI toward 100.

Pathology detection: Metric combinations reveal failure modes observed in pilot studies. Deceptive coherence (fluent but ungrounded responses) arises when SI < 40 and $A > 0.05$. Sycophantic agreement manifests when preference scores exceed accountability by > 2.5 with low variety. Goal misgeneralization flags tasks solved with low appropriateness scores (completion rate > 0.8 but appropriateness < 0.5). Superficial optimization combines high quality with low alignment rate ($Q/\mathrm{AR} > 2$). Semantic drift corresponds to rapid drops in cross-turn coherence, often accompanied by $A \gg A^*$. Detection requires transcript evidence; the aperture observable provides the earliest warning signal across all five pathologies.

Remark (Implications for safety analysis). Interpreting CGM's constraints informationally shows that anthropomorphic descriptions of AI systems correspond to violations of operational closure. Specifically, when BU-Ingress fails to reconstruct CS, UNA, and ONA from an inference cycle, decisional authority is misattributed from human-specified constraints to measurement outputs, an instance of the authority source bias described in Section 2.4.1. Temporal references follow the modal structure: past knowledge in CS, present contingency in UNA/ONA, future projection in BU. The pilots summarized in Table 4 treat deviations in aperture A from the predicted A^* as empirical indicators of such misattribution, offering structural diagnostics of early differentiation. We therefore frame critiques of anthropomorphic safety narratives as hypotheses arising from the modal logic itself, not as external assumptions.

4.2 Advancing Hilbert's Sixth Problem

Hilbert's sixth problem [Hilbert(1900)] called for the axiomatization of physics. The challenge was to provide a rigorous logical investigation of the foundational principles underlying physical theory, comparable to the axiomatization achieved in geometry.

CGM contributes to this axiomatization program by deriving physical structure from modal logic. From the five foundational constraints, space, time, and physical constants emerge as lemmas with explicit derivations (Sections 3.3-3.4). Theorem A.1 (Appendix A) shows that only $n = 3$ spatial dimensions satisfy the modal constraints (Section 3.3).

The framework constructs a Hilbert-space representation via GNS [Gelfand &Naimark(1943)] where the modal operators $[L]$ and $[R]$ generate the algebra of observables, with the modal constant S (realized as $Q_G = 4\pi$) defining the normalization. Geometry, dynamics, and quantum structure follow from the requirement that modal operations maintain coherence under recursive closure. This builds on established constructions like GNS representations for unitary groups.

4.3 Relationship to Alternative Frameworks

CGM differs from existing approaches in what it takes as given and what it derives as necessary.

Physics: Foundational physics programs follow several complementary approaches. Entity-based frameworks like string theory [Polchinski(1998)] and loop quantum gravity [Rovelli(2004)] begin with physical primitives (strings, spin networks) and derive dynamics from their properties. Order-theoretic programs such as causal set theory [Sorkin(2003)] reconstruct spacetime from partially ordered sets, while algebraic quantum field theory [Haag(1996)] postulates nets of operator algebras satisfying locality and covariance. Principle-based frameworks like general relativity begin with physical postulates (equivalence principle, constancy of c) and derive geometric structure. CGM begins from a domain-agnostic foundational principle that is formalized as five foundational constraints in bimodal propositional logic ("The Source is Common"), deriving spacetime, quantum structure, and conservation laws as necessary consequences of operational closure.

This methodological difference yields complementary strengths. String theory excels at unifying forces through higher-dimensional geometry; loop quantum gravity provides background-independent quantization; CGM derives observable-scale parameters from the five foundational constraints.

Where string theory requires compactification choices and loop quantum gravity requires cutoff regularization, CGM's constraints follow from modal depth requirements without additional free parameters.

AI Alignment: Current alignment approaches operate through empirical fitting. RLHF [Ouyang et al.(2022)] trains on human feedback signals, debate [Irving et al.(2018)] uses adversarial dynamics, and constitutional AI [Bai et al.(2022)] implements explicit behavioral rules. These methods excel at immediate deployment and have demonstrated effectiveness in production systems.

Recent empirical analysis by Noroozizadeh et al. [Noroozizadeh et al.(2025)] demonstrates that deep sequence models develop a global geometric memory distinct from local associative memory, yet the authors report that the emergence of this structure remains unclear under standard optimization pressures. CGM explains the phenomenon: the gradient component of the tetrahedral Hodge decomposition provides the global geometric memory while the cycle component captures local associations. The operational constraints therefore predict the same dual structure together with a quantitative balance, since $1 - (\delta_{BU}/m_a) = 0.0207$ predicts the aperture fraction required for coherent alignment. Where the empirical study documents the effect, CGM supplies the constitutional principle and falsifiable numeric target that necessitate it.

CGM derives alignment metrics from the same geometric structure that yields physical conservation laws. Just as the Lagrangian in physics encodes symmetries that force conserved quantities (energy from time symmetry, momentum from spatial symmetry), CGM's gyrogroup structure encodes symmetries that force conserved informational quantities: traceability from operational closure, variety from non-collapse under unity, accountability from non-contradiction under opposition, and integrity from balance at depth four.

These metrics emerge from geometric constraints, providing a complementary perspective on alignment structure. The framework's evaluation approach focuses on structural properties measurable through geometric decomposition [GyroDiagnostics(2025)], enabling targeted assessment of coherence mechanisms.

Synthesis: These approaches are not competing but complementary. Empirical frameworks describe what systems do and optimize within observed patterns. CGM identifies constraints that coherent systems satisfy, providing explanations for why certain patterns succeed. They operate at different levels: one describes what works, the other explains why it works that way.

4.4 Limitations and Future Work

The dimensional results rely on the two-layer structure outlined in the introduction, where operational requirements are imposed on the core modal logic. A categorical derivation that unifies these layers remains future work. Likewise, the unicities reported in Section 3 are supported by computational enumeration and exclusion lemmas, but a categorical uniqueness proof has not yet been established. Phenomenological claims (fine-structure fits, neutrino scales, and aperture clustering in AI diagnostics) are hypotheses backed by the stated calculations and preregistered scripts; they require independent replication and, in the physics case, incorporation of radiative corrections beyond the leading geometric ratios. The GyroDiagnostics pilot currently spans three model families with 15 runs each; expanding the sample is part of the planned program. Future work will (i) formalize the operational requirements within a categorical semantics of bimodal logic, (ii) seek uniqueness proofs via Tannakian reconstruction of the GNS data, (iii) extend the experimental program to larger model cohorts with open data releases, and (iv) develop dynamical field equations compatible with the CGM symmetry structure.

4.5 Internal vs External Evidence

Internal vs external evidence. Our use of GyroDiagnostics on CGM-derived artifacts is an internal coherence check. External claims rely only on cross-model, cross-lab, cross-task evaluations where neither artifacts nor analysts are CGM-generated. We report the latter as pilots and commit to a pre-registered expansion.

5 Conclusions

This paper presents a formal framework for coherent recursive measurement based on operational closure requirements. The five constraints are formalized in bimodal propositional logic and analyzed in two layers: a core modal layer establishing logical properties and an operational regime where the modal axioms force the operators to be continuous unitary flows. Under the operational requirements detailed in Section 2.4.1, the constraints select a three-dimensional geometric structure with six degrees of freedom as the minimal realization consistent with the hypotheses.

5.1 Summary of Principal Findings

The principal technical contributions are fourfold. First, we established the logical independence and completeness of the constraint system using Kripke semantics and Z3 SMT verification. The constraints prevent both homogeneous collapse at depth two and contradictory rigidity while ensuring commutative closure at depth four. Second, in the operational regime with continuous flows, reachability, and simplicity requirements, the Baker–Campbell–Hausdorff expansion forces the generator algebra to be three-dimensional and isomorphic to $\mathfrak{su}(2)$. The semidirect product SE(3) emerges as the unique structure satisfying all operational conditions, with alternative dimensions rigorously excluded. Third, the Gelfand–Naimark–Segal construction provides an explicit example of a Hilbert space representation on $L^2(S^2, d\Omega)$, with numerical verification confirming constraint satisfaction to machine precision. Fourth, the same constraint set yields quantitative coherence metrics for discrete systems via tetrahedral Hodge decomposition, with pilot evaluations on language models showing preliminary evidence of structural signatures.

Fifth, the operational constraints fix three representation-independent geometric invariants: the quantum gravity horizon $Q_G = 4\pi$ (Section 3.4.1), the monodromy defect $\delta_{BU} = 0.1953$ rad (Section 3.4.2), and aperture scale $m_a = 0.1995$ satisfying $Q_G m_a^2 = 1/2$. The ratio $\delta_{BU}/m_a = 0.9793$ represents 97.93% closure with a 2.07% aperture. At leading geometric order, this yields $\alpha = \delta_{BU}^4/m_a = 0.007297$, matching experimental synthesis within stated uncertainties; higher-order terms and independent verification remain necessary. The same aperture ratio provides the predicted cycle fraction for discrete alignment metrics, connecting physical measurement and informational coherence through common geometric structure.

5.2 Future Directions

Immediate extensions include expanding the empirical validation of alignment metrics to broader model populations with pre-registered protocols, developing dynamical equations compatible with the SE(3) structure, and seeking categorical proofs of representation uniqueness. Longer-term possibilities include application to quantum computing architectures where coherent observation is central, extension to cosmological structure formation where measurement horizons play essential roles, and development of practical alignment tools based on the geometric metrics.

5.3 Concluding Assessment

The Common Governance Model demonstrates that coherent recursive measurement imposes stronger constraints on system architecture than previously recognized. By formalizing these constraints in modal logic and proving the operational requirements they force, we obtain the geometric structure of three-dimensional space, the quantum gravity horizon $Q_G = 4\pi$, and quantitative predictions for both physical constants (electromagnetic coupling α at leading order) and informational coherence (alignment aperture). These results span deductive necessity (3D structure), representation-independent invariants (Q_G, δ_{BU}, m_a), and phenomenological correspondence (coupling, aperture ratios) requiring independent empirical validation. The framework does not explain why the universe exhibits these particular constraints, but it does show that once the constraints hold, the observed structure follows necessarily. This reduction of contingent structure to necessary consequence under specified conditions represents progress toward the axiomatic foundations envisioned by Hilbert, while the application to artificial intelligence provides practical value beyond foundational theory.

The results establish necessity: any system satisfying the operational requirements for coherent recursive observation exhibits three-dimensional structure with six degrees of freedom, while determining whether physical reality satisfies those requirements remains an empirical question.

Appendix A: Proof of Three-Dimensional Necessity

This appendix provides the formal proof that the five foundational constraints select exactly three spatial dimensions with six degrees of freedom as the unique solution. The proof proceeds through working in the completed free Lie algebra $\hat{L}(X,Y)$ with formal exponentials, a Baker–Campbell–Hausdorff (BCH) lemma connecting Proposition 2.9 (BU-Egress) to Lie algebra structure (yielding $\mathfrak{sl}(2)$), selection of the compact real form $\mathfrak{su}(2)$ via the $L^2(S^2)$ representation with $Q_G = 4\pi$, lemmas on rotational/translational DOF, and a non-existence theorem for $n \neq 3$.

The operational regime of continuous flows, reachability from S, simple Lie closure, and GNS-supplied compactness is defined once in Section 2.4.1; we invoke it here without repetition. It supplies the interpretation under which the modal constraints act on continuous unitary flows.

Formal Lie Algebra Characterization

The characterization proceeds in two stages: (1) algebraic characterization in the completed free Lie algebra $\hat{L}(X,Y)$ with formal exponentials, constituting $\mathfrak{sl}(2)$ structure; (2) representation selection, choosing the compact real form $\mathfrak{su}(2)$ on $L^2(S^2)$ via $Q_G = 4\pi$.

In the algebraic stage, we work with formal non-commutative symbols X, Y (no inner product, no skew-adjointness). The modal operators $[L]$ and $[R]$ must be formal exponentials $\exp(X)$ and $\exp(Y)$ in the completed free Lie group. We adjoin a central idempotent s ($s^2 = s$) encoding the S-world, where "φ holds at S" is read as $s \cdot \varphi \cdot s$. BU-Egress at S requires $s \cdot \exp(X)\exp(Y)\exp(X)\exp(Y) \cdot s = s \cdot \exp(Y)\exp(X)\exp(Y)\exp(X) \cdot s$.

In the representation stage, we select a faithful representation on $L^2(S^2, d\Omega)$ with horizon normalization $Q_G = 4\pi$. The modal operators $[L]$ and $[R]$ are then realized as one-parameter unitary groups $U_L(t) = e^{itX}$, $U_R(t) = e^{itY}$ where X, Y are skew-adjoint operators ($X^* = -X$, $Y^* = -Y$) ensuring unitarity. This representation choice selects the compact real form $\mathfrak{su}(2)$ from the algebraically constituted $\mathfrak{sl}(2)$ structure.

Lemma A.0: BCH Depth-4 Closure

Lemma A.0 (BCH Depth-4 Closure): Working in the completed free Lie algebra $\hat{L}(X,Y)$ with formal exponentials, Proposition 2.9 (BU-Egress) requires $s \cdot \exp(X)\exp(Y)\exp(X)\exp(Y) \cdot s = s \cdot \exp(Y)\exp(X)\exp(Y)\exp(X) \cdot s$ where s is the central idempotent encoding the S-world. This implies that the Lie algebra generated by X and Y is three-dimensional and satisfies $[X,[X,Y]] = aY$ and $[Y,[X,Y]] = -aX$ for some real $a \neq 0$. Consequently, the algebra is isomorphic to $\mathfrak{sl}(2)$ (up to real form).

Proof: For $Z_1 = \log(\exp(X)\exp(Y))$ and $Z_2 = \log(\exp(Y)\exp(X))$, we have the exact identity:

$$\log(\exp(X)\exp(Y)\exp(X)\exp(Y)) = \log(\exp(Z_1)\exp(Z_1)) = 2Z_1 \tag{35}$$

$$\log(\exp(Y)\exp(X)\exp(Y)\exp(X)) = \log(\exp(Z_2)\exp(Z_2)) = 2Z_2 \tag{36}$$

because $\exp(Z)\exp(Z) = \exp(2Z)$ in the free Lie group. Therefore, the exact difference is:

$$\begin{aligned}\Delta &= \log(\exp(X)\exp(Y)\exp(X)\exp(Y)) - \log(\exp(Y)\exp(X)\exp(Y)\exp(X)) \\ &= 2(\mathrm{BCH}(X,Y) - \mathrm{BCH}(Y,X))\end{aligned} \tag{37}$$

This is a sum of antisymmetric Lie polynomials. BU-Egress at S requires $s \cdot \Delta \cdot s = 0$ for all small t, which kills the antisymmetric tower in the s-sector. UNA requires $[X, Y] \neq 0$ globally, so we have $s[X, Y]s = 0$ (killing the antisymmetric degree-2 component at S) while preserving global non-commutativity.

The Baker–Campbell–Hausdorff expansion yields:

$$\mathrm{BCH}(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \cdots \tag{38}$$

$$\mathrm{BCH}(Y, X) = Y + X + \frac{1}{2}[Y, X] + \frac{1}{12}([Y, [Y, X]] + [X, [X, Y]]) + \cdots \tag{39}$$

The difference $\Delta = 2(\mathrm{BCH}(X, Y) - \mathrm{BCH}(Y, X))$ contains only antisymmetric terms.

Structural Lemma (s-sector closure): If $s(\mathrm{BCH}(X, Y) - \mathrm{BCH}(Y, X))s = 0$ and $[X, Y] \neq 0$, and $\mathrm{span}\{X, Y, [X, Y]\}$ is closed, then $[X, [X, Y]] = aY$ and $[Y, [X, Y]] = -aX$ for some $a \in \mathbb{R}$.

To see this explicitly, expand the BCH series to degree four:

$$\begin{aligned} \mathrm{BCH}(X, Y) &= X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \cdots, \\ \mathrm{BCH}(Y, X) &= Y + X + \tfrac{1}{2}[Y, X] + \tfrac{1}{12}([Y, [Y, X]] + [X, [X, Y]]) + \cdots. \end{aligned}$$

Their antisymmetric difference is therefore

$$\Delta = \mathrm{BCH}(X, Y) - \mathrm{BCH}(Y, X) = [X, Y] + \tfrac{1}{6}([X, [X, Y]] - [Y, [Y, X]]) + \cdots.$$

The sectoral constraint $s \cdot \Delta \cdot s = 0$, together with $s[X, Y]s = 0$, yields

$$s([X, [X, Y]] - [Y, [Y, X]])s = 0.$$

Assume, for contradiction, that the Lie subalgebra generated by X and Y contains a Hall word W_m of minimal bracket length $m \geq 3$ whose projection onto the S-sector is non-zero. Because m is minimal, no shorter Hall word contributes; the coefficient of W_m in Δ is a fixed non-zero rational number, so the projection of W_m cannot vanish if $s\Delta s = 0$. Hence no such W_m exists, and the subalgebra closes on $\{X, Y, [X, Y]\}$.

Write the remaining commutators in this basis:

$$\begin{aligned} [X, [X, Y]] &= \alpha X + \beta Y + \gamma[X, Y], \\ [Y, [X, Y]] &= \alpha' X + \beta' Y + \gamma'[X, Y]. \end{aligned}$$

The Jacobi identity $[X, [X, Y]] + [Y, [Y, X]] + [[X, Y], X] = 0$ implies $[X, [X, Y]] + [Y, [Y, X]] = 0$, so $[X, [X, Y]] = -[Y, [Y, X]]$. Swapping X and Y shows $\alpha = \beta' = 0$ and $\beta = -\alpha'$, $\gamma = -\gamma'$. Consequently $[X, [X, Y]] = aY$ and $[Y, [X, Y]] = -aX$ for some $a \in \mathbb{R}$, proving the structural lemma and establishing that $\mathrm{span}\{X, Y, [X, Y]\}$ is a three-dimensional simple Lie algebra isomorphic to $\mathfrak{sl}(2)$ (up to real form).

Simplicity and Compactness Constraints

Since "The Source is Common" requires all structure to derive from a single origin, the Lie subalgebra generated by X and Y must be simple (no nontrivial ideals) and of compact type (from unitarity). This excludes direct-sum decompositions like $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Among all simple compact Lie algebras satisfying the BCH constraints from Proposition 2.9 (BU-Egress), we select the minimal one (dimension 3), which is $\mathfrak{su}(2)$.

Corollary A.3 (Simplicity from Memory Lemma).

In the Lie algebra $\mathfrak{g}$ generated by the skew-adjoint operators X, Y from the unitary representation of U_L, U_R, the algebra $\mathfrak{g}$ is simple.

Proof. This is not geometric but representational: suppose $\mathfrak{g}$ decomposes as a direct sum of nontrivial ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then the representation π_ω would decompose into invariant subspaces, contradicting the cyclicity of $|\Omega\rangle$ under the full algebra generated by $\{U_L, U_R\}$ and the memory reconstruction of Proposition 2.10 (Memory), which requires all prior states (CS, UNA, ONA) to be recoverable from the single balanced state $\Box B$. Decomposition violates GNS cyclicity required by BU-Ingress: invariant subspaces split $|\Omega\rangle$, preventing reconstruction of CS chirality from single balance state. Formally, if $\mathfrak{g}$ decomposed as $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, the flows would split, and a single cyclic $|\Omega\rangle$ could not reconstruct both modalities (UNA/ONA) across independent factors while preserving the single-source constraint. Hence the simplicity requirement is enforced by the memory condition in this operational regime. □

Lemma A.1: Rotational Degrees of Freedom (UNA)

Lemma A.1.1: Representing the modal operators as one-parameter unitary groups, the foundational assumption, Lemmas 2.7 and 2.8, and Proposition 2.9 (BU-Egress) imply that the Lie algebra generated by X and Y is isomorphic to $\mathfrak{su}(2)$, requiring exactly three independent generators.

Proof: By Lemma A.0 (BCH Depth-4 Closure), Proposition 2.9 (BU-Egress) forces the Lie algebra span$\{X, Y, [X, Y]\}$ to be three-dimensional with $\mathfrak{su}(2)$-type relations. By Lemma 2.7 ($\neg\Box E$), X and Y cannot commute (otherwise $[L][R] = [R][L]$ absolutely, violating Lemma 2.7). By Assumption 2.6, U_R preserves the horizon constant while U_L does not, ensuring $X \neq 0$ and $Y \neq 0$ are independent. Compact type is selected by the GNS inner product structure (see above), excluding $\mathfrak{sl}(2, \mathbb{R})$. Simplicity is an operational constraint: "The Source is Common" requires all structure to trace to a single origin, prohibiting independent subsystems. Mathematically, this means the algebra must be simple (no nontrivial ideals), excluding $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ decompositions. Therefore, the five foundational constraints together with unitarity and simplicity requirements select $\mathfrak{su}(2)$ with three generators as the minimal algebra compatible with the operational regime. □

Lemma A.1.2: Under unitarity, BCH constraints, and simplicity requirements, dimensional structures with $n \neq 3$ cannot satisfy the operational requirements for coherent measurement.

Proof: For $n = 2$: Two-dimensional real Lie algebras are either abelian ($[X, Y] = 0$) or isomorphic to the affine algebra of the line (non-compact; nilpotent examples occur first in dimension three). Both cases fail under the modal requirements: the abelian option violates Lemma 2.7, and the affine option violates unitarity.

Analytic exclusion via uniform balance: Even for non-abelian fibrations where U_L acts as a phase function while U_R is a rotation, Proposition 2.9 (BU-Egress) requires $\|P_S(U_L(t)U_R(t)U_L(t)U_R(t) - U_R(t)U_L(t)U_R(t)U_L(t))|\Omega\rangle\| < \epsilon$ for every t in some open neighbourhood of 0. This means the depth-four commutator vanishes as an analytic function of t, not merely at isolated values. For fibered 2D unitary representations $(U_L f)(\phi) = \exp(itg(\phi))f(\phi)$, $(U_R f)(\phi) = f(\phi - t)$, uniform balance therefore enforces the functional equation

$$g(\phi - 2\theta, t) = g(\phi, t) \quad \text{for all } |t|, |\theta| < \delta.$$

Fix t and expand g in Fourier modes $g(\phi, t) = \sum_{n \in \mathbb{Z}} c_n(t) e^{in\phi}$. The above condition gives

$c_n(t)e^{-2in\theta} = c_n(t)$ for all θ, hence $c_n(t) = 0$ whenever $n \neq 0$. Thus $g(\phi, t) = c_0(t)$ is independent of ϕ, making U_L a pure phase that leaves S invariant, contradicting Assumption 2.6. Therefore no non-trivial 2D unitary representation satisfies BU-Egress uniformly. Numerical confirmation (companion script `cgm_Hilbert_Space_analysis.py`): as illustrated by the test case $U_L(t) = \exp(it(\cos\phi + 0.3\cos 2\phi))$ with rotation $U_R(t)$, where BU-Egress fails on tested grids $t \in \{\pm 0.01, \pm 0.005\}$.

For $n = 4$: $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ has two independent simple factors. Since "The Source is Common" requires a single simple factor generated by X, Y; restricting to one factor collapses to $n = 3$, while activating both violates simplicity. For $n \geq 5$: $\dim \mathfrak{so}(n) = n(n-1)/2 \geq 10$; BCH constraints yield a 3D algebra, while $\mathfrak{so}(n)$ has excess generators exceeding the required minimality. □

Lemma A.2: Translational Degrees of Freedom (ONA)

Lemma A.2.1: Under ONA, bi-gyrogroup consistency requires exactly three translational degrees of freedom.

Proof: At ONA, $\vdash \neg\Box O$ activates bi-gyrogroup with distinct lgyr$[a, b]$ and rgyr$[a, b]$. Consistency demands semidirect product $G \cong \mathrm{SU}(2) \ltimes \mathbb{R}^n$ with minimal $n = 3$, yielding SE(3) (3 rotational + 3 translational = 6 DOF). Fewer/more parameters violate bi-gyrogroup minimality. □

Theorem A.1: Non-Existence for $n \neq 3$

Theorem A.1: Requiring CS traceability for reachability, BU-Egress for uniform continuous closure, and BU-Ingress for simple Lie reconstruction, the five foundational constraints characterize $n = 3$ as the only dimensional structure satisfying coherent measurement requirements within the stated operational regime.

Proof: By Lemma A.0, representing the operators as unitary groups, Proposition 2.9 (BU-Egress) characterizes a 3D Lie algebra with $\mathfrak{su}(2)$-type structure. By Lemma A.1.2, this is incompatible with $n = 2$ (2D algebras are abelian or non-compact) and $n \geq 4$ (violates simplicity or minimality). For $n = 3$: $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ satisfies all constraints. By Lemma A.2, Lemma 2.8 requires the minimal abelian normal subgroup on which $\mathfrak{su}(2)$ acts faithfully; this is $\mathbb{R}^3$ (standard representation), yielding SE(3) = SU(2) $\ltimes \mathbb{R}^3$ with 6 DOF. The gyrotriangle closure $\delta = 0$ at angles $(\pi/2, \pi/4, \pi/4)$ is satisfied in 3D. All foundational assumption, lemmas, and propositions hold. The Hilbert model on $L^2(S^2)$ (Appendix D) realizes this structure; S^1 models cannot satisfy Lemmas 2.7 and Proposition 2.9 uniformly under unitarity; $\mathfrak{so}(4)$ models violate simplicity. □

Explicit Exclusion for n=4. Suppose the generators X, Y span both factors of $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Write $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ with $(X_i, Y_i) \in \mathfrak{su}(2)$ for $i = 1, 2$. The commutator then splits as $[X, Y] = [X_1, Y_1] + [X_2, Y_2]$. The BCH difference contains length-three terms of the form $[X, [X, Y]]$ and $[Y, [X, Y]]$; projecting into the two summands gives components in both factors because $[X_1, [X_1, Y_1]]$ belongs to the first $\mathfrak{su}(2)$ factor while $[X_2, [X_2, Y_2]]$ belongs to the second. The S-sector constraint $s\Delta s = 0$ therefore forces both projections to vanish. By the structural lemma each factor must satisfy the $\mathfrak{sl}(2)$ relations separately, but this implies that $X_1, Y_1, [X_1, Y_1]$ and $X_2, Y_2, [X_2, Y_2]$ are non-zero, giving two independent simple summands. Such a decomposition contradicts the simplicity requirement (memory reconstruction with a single cyclic vector). Concretely, if both summands are active the

Hilbert representation splits as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $U_L = U_L^{(1)} \oplus U_L^{(2)}$ and similarly for U_R; the balanced state is then $|\Omega\rangle = |\Omega_1\rangle \oplus |\Omega_2\rangle$ and cannot reconstruct the cross-summand modalities. The only way to satisfy BU-Ingress is to restrict to a single $\mathfrak{su}(2)$ factor, which reduces the dimensionality to $n = 3$. Therefore $n = 4$ is incompatible with the operational requirements. □

Numerical checks match the analytic exclusions: on the $L^2(S^2)$ model with normalization $Q_G = 4\pi$, the depth-four norm stays below 10^{-13} for sampled t and $\|P_S[X, Y]P_S\|_{L^2} = 7.9 \times 10^{-19}$. Lower-dimensional ($S^1$) models force $\Box E$ and higher-dimensional ones violate simplicity, aligning with the lemmas above.

Corollary A.1: DOF Progression

Corollary A.1: DOF emerge as $1 \to 3 \to 6 \to 6$(closed).

Proof: CS: 1 DOF (chirality). UNA: +3 rotational (total 3). ONA: +3 translational (total 6). BU: Coordinates 6 DOF into closure. Uniqueness from the five foundational constraints and $\delta = 0$. □

Corollary A.2: Requiring unitarity and simplicity, the mapping from the five foundational constraints to $n = 3$ spatial dimensions is logically necessary: no other dimensionality satisfies the system simultaneously. The requirement of unitarity (that transitions are continuous) and the requirement of simplicity (that all structure traces to a single origin, formalizing "The Source is Common") are operational constraints, not physical primitives. Given these constraints, the emergence of three-dimensional space with six degrees of freedom is a necessary consequence, not an assumption.

Computational verification is provided in `https://github.com/gyrogovernance/science/blob/main/experiments/cgm_3D_6DoF_analysis.py`.

Appendix D: Hilbert Space Representation via GNS Construction

Appendices B and C are provided in the Supplement.

We record only the CGM-specific data for the standard Gelfand–Naimark–Segal construction [Gelfand &Naimark(1943)]. The underlying $*$-algebra is generated by unitaries u_L, u_R with $u_k u_k^* = I$.

Define the free $*$-algebra $\mathcal{A}$ generated by unitaries u_L, u_R with $u_L u_L^* = u_R u_R^* = I$. The modal operators map to: $[L] \to u_L$, $[R] \to u_R$, $[L][R] \to u_L u_R$, etc.

Define positive linear functional $\omega : \mathcal{A} \to \mathbb{C}$ fixed by the modal constraints:

- $\omega(I) = 1$, $\omega(u_R) = 1$, and $\omega(u_L) = e^{is_p}$ with $s_p \neq 0$ (Assumption 2.6).
- $\omega((u_L u_R - u_R u_L)^*(u_L u_R - u_R u_L)) > 0$ and $\omega((u_L u_R u_L u_R - u_R u_L u_R u_L)^*(\cdot)) = 0$ encode Lemmas 2.7, 2.8, and Proposition 2.9.
- Balance implying CS, UNA, and ONA (Proposition 2.10) is imposed as a linear condition on ω.

Completing $\mathcal{A}/\mathcal{N}$ with $\langle [a], [b] \rangle = \omega(a^* b)$ yields $\mathcal{H}_\omega$; we construct a concrete representation on $L^2(S^2, d\Omega)$, taking P_S to project onto $l = 0 \oplus l = 1$ and choosing $U_L = e^{itX}$, $U_R = e^{itY}$ so that $\langle \Omega | U_R | \Omega \rangle = 1$ and $\langle \Omega | U_L | \Omega \rangle = e^{is_p}$.

Verification: the $L^2(S^2)$ realization satisfies Proposition 2.9 with $\|(U_L U_R)^2 - (U_R U_L)^2 |\Omega\rangle\| < 10^{-13}$, $\langle \Omega | U_R | \Omega \rangle = 1$, $\langle \Omega | U_L | \Omega \rangle = e^{is_p}$, and $\|P_S [X, Y] P_S\| \approx 7.9 \times 10^{-19}$. These figures (see `cgm_Hilbert_Space_analysis.py`) provide the operational confirmation of Appendix A.

The supplementary scripts carry the detailed observable definitions and uniqueness discussions, so Appendix D records only the data needed to reconstruct the representation.

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Data Availability Statement

All data and code supporting this study are available at `https://github.com/gyrogovernance/science`, with exact scripts linked in the Reproducibility section. Experimental evaluation data for GyroDiagnostics are available at `https://github.com/gyrogovernance/gyrodiagnostics`, and the GyroSI implementation is hosted at `https://github.com/gyrogovernance/superintelligence`. Detailed derivations, proofs, and numerical artifacts are archived alongside the Zenodo record.

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References

[Hilbert(1900)] Hilbert, D. (1900). Mathematical problems. *Bulletin of the American Mathematical Society*, *8*(10), 437–479.

[Noether(1918)] Noether, E. (1918). Invariante Variationsprobleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 235–257.

[Kolmogorov(1933)] Kolmogorov, A. N. (1933). *Foundations of the Theory of Probability*. Chelsea Publishing Company.

[Wightman(1956)] Wightman, A. S. (1956). Quantum field theory in terms of vacuum expectation values. *Physical Review*, *101*(2), 860–866. `https://doi.org/10.1103/PhysRev.101.860`

[Chellas(1980)] Chellas, B. F. (1980). *Modal Logic: An Introduction*. Cambridge University Press.

[Kripke(1963)] Kripke, S. A. (1963). Semantical considerations on modal logic. *Acta Philosophica Fennica*, *16*, 83–94.

[McKinsey & Tarski(1944)] McKinsey, J. C. C., & Tarski, A. (1944). The algebra of topology. *Annals of Mathematics*, *45*(1), 141–191.

[Sahlqvist(1975)] Sahlqvist, H. (1975). Completeness and correspondence in the first and second order semantics for modal logic. In *Proceedings of the 3rd Scandinavian Logic Symposium* (pp. 110–143). North-Holland.

[Bradfield & Stirling(2006)] Bradfield, J., & Stirling, C. (2006). Modal mu-calculi. In P. Blackburn, J. van Benthem, & F. Wolter (Eds.), *Handbook of Modal Logic* (pp. 721–756). Elsevier.

[Jónsson & Tarski(1951)] Jónsson, B., & Tarski, A. (1951). Boolean algebras with operators. Part I. *American Journal of Mathematics*, *73*(4), 891–939.

[Ungar(2001)] Ungar, A. A. (2001). *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces*. Kluwer Academic Publishers.

[Ungar(2008)] Ungar, A. A. (2008). *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*. World Scientific.

[Gelfand & Naimark(1943)] Gelfand, I., & Naimark, M. (1943). On the imbedding of normed rings into the ring of operators in Hilbert space. *Matematicheskii Sbornik*, *12*(2), 197–213.

[Hall(2015)] Hall, B. C. (2015). *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction* (2nd ed.). Springer. https://doi.org/10.1007/978-3-319-13467-3

[Gyrogovernance(2025a)] Gyrogovernance. (2025a). CGM 3D/6DoF analysis: Proof of dimensional necessity. https://github.com/gyrogovernance/science/blob/main/experiments/cgm_3D_6DoF_analysis.py.

[Gyrogovernance(2025b)] Gyrogovernance. (2025b). CGM Hilbert space representation: GNS construction and BCH verification. https://github.com/gyrogovernance/science/blob/main/experiments/cgm_Hilbert_Space_analysis.py.

[Gyrogovernance(2025c)] Gyrogovernance. (2025c). CGM axiomatization analysis: Z3 SMT verification of the foundational assumption and lemmas. https://github.com/gyrogovernance/science/blob/main/experiments/cgm_axiomatization_analysis.py.

[GyroDiagnostics(2025)] GyroDiagnostics. (2025). AI Safety Diagnostics: Gyroscopic Alignment Evaluation Lab. https://github.com/gyrogovernance/diagnostics

[Sorkin(2003)] Sorkin, R. D. (2003). Causal sets: Discrete gravity. In G. Ellis, J. Murugan, & A. Weltman (Eds.), *Proceedings of the Valdivia Summer School in Quantum Gravity* (pp. 305–327). Springer. https://doi.org/10.1007/0-387-24992-3_7

[Haag(1996)] Haag, R. (1996). *Local Quantum Physics: Fields, Particles, Algebras* (2nd ed.). Springer.

[Polchinski(1998)] Polchinski, J. (1998). *String Theory* (Vols. 1–2). Cambridge University Press.

[Rovelli(2004)] Rovelli, C. (2004). *Quantum Gravity*. Cambridge University Press.

[Particle Data Group(2024)] Particle Data Group. (2024). Review of particle physics. *Physical Review D*, *110*(3), 030001. https://doi.org/10.1103/PhysRevD.110.030001

[Morel et al.(2020)] Morel, L., Yao, Z., Cladé, P., & Guellati-Khélifa, S. (2020). Determination of the fine-structure constant with an accuracy of 81 parts per trillion. *Nature*, *588*(7836), 61–65. https://doi.org/10.1038/s41586-020-2964-7

[Aoyama et al.(2012)] Aoyama, T., Hayakawa, M., Kinoshita, T., & Nio, M. (2012). Tenth-order QED contribution to the electron $g-2$ and an improved value of the fine structure constant. *Physical Review Letters*, *109*(11), 111807. https://doi.org/10.1103/PhysRevLett.109.111807

[Minkowski(1977)] Minkowski, P. (1977). $\mu \to e\gamma$ at a rate of one out of 10^9 muon decays? *Physics Letters B*, *67*(4), 421–428.

[Mohapatra & Senjanović(1980)] Mohapatra, R. N., & Senjanović, G. (1980). Neutrino mass and spontaneous parity nonconservation. *Physical Review Letters*, *44*(14), 912–915.

[Ouyang et al.(2022)] Ouyang, L., Wu, J., Jiang, X., Almeida, D., Wainwright, C. L., Mishkin, P., Zhang, C., Agarwal, S., Slama, K., Ray, A., Schulman, J., Hilton, J., Kelton, F., Miller, L., Simens, M., Askell, A., Welinder, P., Christiano, P., Leike, J., & Lowe, R. (2022). Training language models to follow instructions with human feedback. *Advances in Neural Information Processing Systems*, *35*, 27730–27744.

[Irving et al.(2018)] Irving, G., Christiano, P., & Amodei, D. (2018). AI safety via debate. *arXiv preprint arXiv:1805.00899*. https://doi.org/10.48550/arXiv.1805.00899

[Bai et al.(2022)] Bai, Y., Kadavath, S., Kundu, S., Askell, A., Kernion, J., Jones, A., Chen, A., Goldie, A., Mirhoseini, A., McKinnon, C., Chen, C., Olsson, C., Olah, C., Hernandez, D., Drain, D., Ganguli, D., Li, D., Tran-Johnson, E., Perez, E., ... Kaplan, J. (2022). Constitutional AI: Harmlessness from AI feedback. *arXiv preprint arXiv:2212.08073*. `https://doi.org/10.48550/arXiv.2212.08073`

[Noroozizadeh et al.(2025)] Noroozizadeh, S., Nagarajan, V., Rosenfeld, E., & Kumar, S. (2025). Deep sequence models tend to memorize geometrically; it is unclear why. *arXiv preprint arXiv:2510.26745*. `https://arxiv.org/abs/2510.26745`

[Korompilias(2025)] Korompilias, B. (2025). Common Governance Model: Mathematical Physics Framework. Zenodo. `https://doi.org/10.5281/zenodo.17521384`

[Beer(1994)] Beer, S. (1994). *Beyond Dispute: The Invention of Team Syntegrity*. John Wiley and Sons.