Decentered FDM for non-uniform grid

Take a function f. We want to approximate the value of $f'(x_1)$ knowing the value of $f(x_1)$, $f(x_2)$ and $f(x_3)$ for $x_3 > x_2 > x_1$. Denote $\alpha := x_2 - x_1$ and $\beta := x_3 - x_1$. We call $Df(x_1)$ the approximate value of $f'(x_1)$. We want $|f'(x_1) - Df(x_1)| = o\left(\max(\alpha, \beta)^2\right)$. To this end, write

$$Df(x_1) = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3), \tag{1}$$

and expand f to the second order.

$$\begin{aligned} Df(x_1) = c_1 f(x_1) + c_2 &\left(f(x_1) + \alpha f'(x_1) + \frac{1}{2}\alpha^2 f''(x_1) + o(\alpha^2) \right) \\ + c_3 &\left(f(x_1) + (\alpha+\beta) f'(x_1) + \frac{1}{2}(\alpha+\beta)^2 f''(x_1) + o((\alpha+\beta)^2) \right). \end{aligned}$$

A term-by-term comparison gives us the following system:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ \alpha c_2 + (\alpha+\beta) c_3 &= 1 \\ \alpha^2 c_2 + (\alpha+\beta)^2 c_3 &= 0. \end{aligned}$$

The solution is therefore given by

$$c_1 = -\frac{2\alpha+\beta}{\alpha^2+\alpha\beta} \qquad c_2 = \frac{1}{\alpha} + \frac{1}{\beta} \qquad c_3 = -\frac{\alpha}{\beta(\alpha+\beta)}. \tag{2}$$

The same computation can be made for the backward and the centered FDM scheme, and lead to these three formulas:

$$\begin{aligned} Df(x_1) &= -\frac{2\alpha+\beta}{\alpha(\alpha+\beta)} f(x_1) + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) f(x_2) - \frac{\alpha}{\beta(\alpha+\beta)} f(x_3) \\ Df(x_2) &= -\frac{\beta}{\alpha(\alpha+\beta)} f(x_1) + \left(\frac{1}{\alpha} - \frac{1}{\beta}\right) f(x_2) + \frac{\alpha}{\alpha(\alpha+\beta)} f(x_3) \\ Df(x_3) &= \frac{2\alpha+\beta}{\alpha(\alpha+\beta)} f(x_1) - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) f(x_2) + \frac{\alpha}{\beta(\alpha+\beta)} f(x_3). \end{aligned}$$

for $\alpha = |x_1 - x_2|$ and $\beta = |x_2 - x_3|$.