ODT Eddy Events

This document provides a derivation of the equations dictating eddy events in the one dimensional turbulence (ODT) model along with implementation details in a Lagrangian, adaptive ODT code. The eddy events are prescribed by the triplet map, and the frequency, location, and size of eddies. The eddy sampling procedure is also described.

Triplet Maps

Eddy events occur through the imposition of triplet maps. These maps are the one dimensional model analog of turbulent advention. These triplet maps are implimented by a physical mapping of fluid at given locations to other locations [1]:

$$\phi(y) \rightarrow \phi(f(y)). \tag{1}$$

That is, fluid at location $f(y)$ is mapped to location y. The map is defined as

$$f(y) = \begin{cases} 3(y - y_0) & \text{if } y_0 \leq y \leq y_0 + \frac{1}{3}l, \\ 2l - 3(y - y_0) & \text{if } y_0 + \frac{1}{3}l \leq y \leq y_0 + \frac{2}{3}l, \\ 3(y - y_0) - 2l & \text{if } y_0 + \frac{2}{3}l \leq y \leq y_0 + l, \\ y - y_0 & \text{otherwise.} \end{cases} \tag{2}$$

This mapping is plotted in Fig. 1. Fluid at a given position is spread to three new positions along the eddy region l. For example, for $l = 3$, $y_0 = 0$, fluid at location 1.5 is moved to locations 0.5, 1.5, and 2.5; or, fluid at $f(0.5) = 1.5$, $f(1.5) = 1.5$, $f(2.5) = 1.5$ is moved to positions 0.5, 1.5, and 2.5, respectively. The triplet map essentially takes a scalar profile on an eddy region and replaces it with three copies of the original, each compressed by a factor of three, with the middle copy inverted. For example, a linear increasing profile would be replaced by a profile of the shape of $f(y)$ in Fig. 1. In an adaptive, Lagrangian implementation with piecewise constant profiles in cells, the triplet map is applied as indicated:

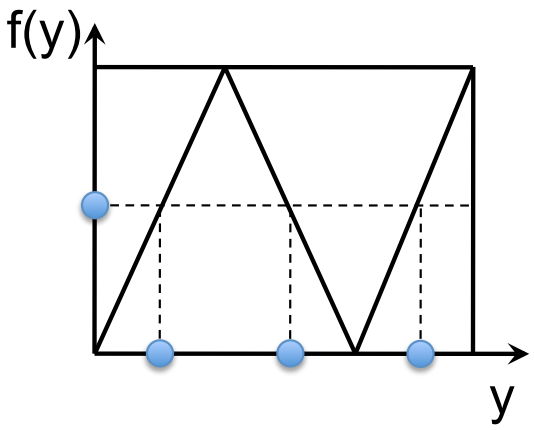

Figure 1: Illustration of triplet map. Fluid at $f(y)$ is mapped to y.

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                 :                                                   :
               | | :           |           |           |           : | |
Before TM:     | | :     a     |     b     |     c     |     d     : | |
               | | :           |           |           |           : | |
                 :                                                   :
                 :                                                   :
               | | :   |   |   |   |   |   |   |   |   |   |   |   : | |
After TM:      | | : a | b | c | d | d | c | b | a | a | b | c | d : | |
               | | :   |   |   |   |   |   |   |   |   |   |   |   : | |
                 :                                                   :
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Here, vertical bars indicate cell boundaries and the dashed bars are the eddy boundaries (not necessarily coincident with cell boundaries). The triplet mapping of a region results in three times as many cells and cell properties (e.g., velocity, density, etc.) are the same in given marked cells before and after the map. The triplet map is continuous and measure preserving (i.e., momentum, kinetic energy, etc. are conserved). Further details on the triplet map are provided in [1, **?**, 2].

Vector Formulation

In addition to the triplet map itself, the vector formulation of ODT modifies the velocity components to allow for energy transfer between the components in order to incoroporate return to isotropy processes associated with pressure-velocity coupling. The following derivations follow [3].

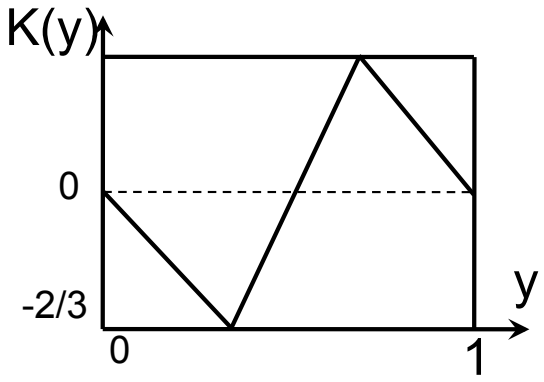

Figure 2: Illustration of the K(y) kernel.

Constant Density

For the triplet map operation for velocity components, Eq. (1) is replaced with

$$v_i(y) \rightarrow v_i(f(y)) + c_i K(y), \tag{3}$$

where c_i is a constant coefficient and $K(y)$ is a kernel function [1]. $K(y)$ is defined as

$$K(y) = y - f(y), \tag{4}$$

This function is illustrated in Fig. 2. The integral of $K(y)$ over the eddy region is zero, enforcing momentum conservation.

The constant c_i is different for each velocity component, and is computed by redistributing energy among the three velocity components. We consider the energy in the whole eddy by integrating over the eddy. Each (eddy) velocity component has an energy before the triplet map, and an energy after the triplet map plus kernel addition. For a given component, c_i can be adjusted to give an arbitrarily large increase in the energy of that component. However, energy added to one component must be taken from the other two. Each component has a finite amount of energy that can be removed (through the specification of c_i). This maximum Q_i is computed and the redistribution is written as a fraction α of this value. Consider velocity component 1. If $\alpha = 1$, then we remove Q_1 from the energy in component 1 and add $Q_1/2$ to components 2 and 3. The same procedure is done for the other two components: remove Q_2 from component 2 and give $Q_2/2$ to components 1 and 3; remove Q_3 from component 3 and give $Q_3/2$ to components 1 and 2. Therefore, the change in the energy of component 1 is its loss Q_1 and its gain $Q_2/2 + Q_3/2$. This is summarized as

$$\Delta E_i = -\alpha Q_i + \frac{\alpha}{2} Q_j + \frac{\alpha}{2} Q_k, \tag{5}$$

where i, j, k are an permutation of component indicies 1, 2, 3.

The change in the eddy energy in a given component is given by

$$\Delta E_i = \frac{1}{2}\rho \int (v_i(f(y)) + c_i K(y))^2 - v_i^2(y) dy. \tag{6}$$

Noting that the triplet map conserves energy, we obtain

$$\Delta E_i = \frac{1}{2}\rho K K l^3 c_i^2 + \rho v_{iK} l^2 c_i, \tag{7}$$

where,

$$KK \equiv \frac{1}{l^3}\int K^2(y)dy, \tag{8}$$

and

$$v_{i,K} \equiv \frac{1}{l^2}\int v_i(f(y))K(y)dy. \tag{9}$$

Here, the $1/l^2$ preserves the units on v. Also, $KK = 4/27$, but is written here as KK to preserve generality. That is, a computer program with peicewise constant properties in cells implements a peicewise constant version of $K(y)$, so that $KK = \frac{1}{l^3}\sum_i K_i^2 * dy_i$ is implemented, and $KK \to 4/27$ as $dy_i \to 0$.

Now, the maximum energy removed from a component Q_i is the minimum change in the energy of that component. Equation (7) is quadratic in c_i, that is, a parabola with a minimum. By differenting Eq. (7) with respect to c_i, equating to zero, solving for $c_{i,min}$, and substituing into Eq. (7) gives

$$Q_i = \frac{1}{2KK}\rho l v_{i,K}^2. \tag{10}$$

Inserting Q_i here into Eq. (5), and the result for ΔE_i into Eq. (7), and solving for c_i gives

$$c_i = \frac{1}{KKl}\left(-v_{i,K} \pm \sqrt{(1-\alpha)v_{i,K}^2 + \frac{\alpha}{2}v_{j,K}^2 + \frac{\alpha}{2}v_{k,K}^2}\right). \tag{11}$$

The sign, $\pm$ is chosen so that if $\alpha = 0$, then $c_i = 0$ and no energy is transferred among components. Hence replace $\pm$ with $+\text{sgn}(v_{i,K})$, which gives $-$ for negative $v_{i,K}$ and $+$ for positive $v_{i,K}$.

Variable Density–Approach 1

When the density varies with position along the ODT line, the shape of $K(y)$ is no longer sufficient to ensure conservation of momentum because while $\int K(y)dy = 0$, $\int \rho(y)K(y)dy \neq 0$. To ensure conservation of momentum when applying kernels to the triplet mapped velocities, an additional $J(y)$ kernel is added with coefficient b_i:

$$v_i(y) \to v_i(f(y)) + c_i K(y) + b_i J(y), \tag{12}$$

The addition of b_i gives an additional degree of freedom to enforce momentum conservation. $J(y)$ is defined (for convenience) as

$$J(y) = |K(y)|, \tag{13}$$

so that $J^2 = K^2$. The constants c_i and b_i are determined from momentum and energy conservation. Following Ashurst and Kerstein [3], denote primed quantities (e.g., ρ') as those following a triplet map, and double primed quantities as those following triplet map and kernel operations (e.g., v_i'').

For momentum conservation, we have $\int \rho v dy = \int \rho v'' dy$, or $\int \rho v dy = \int \rho v' + c_i \rho K + b_i \rho J dy$. The first two terms are equal, giving

$$b_i = A c_i, \tag{14}$$

where

$$A \equiv -\frac{\rho_K}{\rho_J}, \tag{15}$$

$$\rho_J \equiv \int \rho' J dy, \tag{16}$$

$$\rho_K \equiv \int \rho' K dy. \tag{17}$$

So, knowing c_i provides b_i. The constant c_i is computed from conservation of kinetic energy in the same manner as for the constant density case. We have

$$\Delta E_i = \frac{1}{2} \int \rho' (v_i''^2 - v_i'^2) dy \tag{18}$$

$$= \frac{1}{2} \int \rho' (b_i J + c_i K)^2 + 2\rho' v_i' (b_i J + c_i K) dy. \tag{19}$$

Define the following quantities:

$$\rho_{JJ} \equiv \frac{1}{l^3} \int \rho' J^2 dy, \tag{20}$$

$$\rho_{KK} \equiv \frac{1}{l^3} \int \rho' K^2 dy, \tag{21}$$

$$\rho_{JK} \equiv \frac{1}{l^3} \int \rho' JK dy, \tag{22}$$

$$v_{i,\rho J} \equiv \frac{1}{l^2} \int \rho' v_i J dy, \tag{23}$$

$$v_{i,\rho K} \equiv \frac{1}{l^2} \int \rho' v_i K dy, \tag{24}$$

Here, $\rho_{JJ} = \rho_{KK}$. Using These quantities, Eq. (19) becomes

$$\Delta E_i = \frac{1}{2} b_i^2 \rho_{JJ} l^3 + \frac{1}{2} c_i^2 \rho_{KK} l^3 + b_i c_i \rho_{JK} l^3 + b_i v_{i,\rho J} l^2 + c_i v_{i\rho K} l^2, \tag{25}$$

$$= \frac{1}{2} A^2 c_i^2 \rho_{JJ} l^3 + \frac{1}{2} c_i^2 \rho_{KK} l^3 - A c_i^2 \rho_{JK} l^3 - A c_i v_{i,\rho J} l^2 + c_i v_{i\rho K} l^2, \tag{26}$$

$$= c_i l^2 (v_{i\rho K} - A v_{i\rho J}) + c_i^2 l^3 (-A \rho_{JK} + \frac{1}{2}(A^2 + 1)\rho_{KK}) \tag{27}$$

$$= c_i P_i + c_i^2 S, \tag{28}$$

where

$$P_i \equiv l^2(v_{i\rho K} - Av_{i\rho J}), \tag{29}$$

$$S \equiv l^3(\frac{1}{2}(A^2+1)\rho_{KK} - A\rho_{JK}). \tag{30}$$

Now, the available energy is found by minimizing ΔE_i with respect to c_i to get $c_{i,min}$, which is substituted into Eq. (28) to give

$$Q_i = \frac{P_i^2}{4S}. \tag{31}$$

Substitution of this result into Eq. (5), and the result for ΔE_i into Eq. (28), and solving for c_i gives

$$c_i = \frac{1}{2S}\left(-P_i + \mathrm{sgn}(P_i)\sqrt{(1-\alpha)P_i^2 + \frac{\alpha}{2}P_j^2 + \frac{\alpha}{2}P_k^2}\right). \tag{32}$$

All of the above integrals are implemented numerically as sums over peicewise constant cell values making up the eddy interval.

Variable Density–Approach 2

A somewhat simpler approach to the kernel operation with variable density flows is to consider the kernel applied to momentum components rather than velocity components. This approach requires only a single kernel, and conserves momentum and kinetic energy of the eddy event. Equation (12) is replaced with

$$\rho v_i \rightarrow \rho' v_i' + c_i K(y), \tag{33}$$

where, again, primed quantities have been triplet mapped. We can rewrite this equation as

$$m_i \rightarrow m_i' + c_i K(y), \tag{34}$$

where m_i is momentum of component i. The analog of Eq. (**??**) is

$$\Delta E_i = \frac{1}{2}\int \frac{1}{\rho'}(m_i' + c_i K)^2 - \frac{1}{\rho}m_i^2 dy. \tag{35}$$

The triplet map conserves energy resulting in

$$\Delta E_i = \frac{1}{2}\int \frac{1}{\rho'}c_i^2 K^2 dy + \int \frac{1}{\rho'}c_i m_i' K dy \tag{36}$$

Now define

$$I_1 = \int \frac{K^2}{\rho'} dy, \tag{37}$$

and

$$I_{2,i} = \int \frac{m_i' K}{\rho'} dy. \tag{38}$$

We proceed as we did in deriving Eqs. (10), (11), to obtain

$$Q_i = \frac{1}{2}\frac{I_{2,i}^2}{I_1}, \tag{39}$$

and

$$c_i = \frac{1}{I_1}\left(-I_{2,i} \pm \sqrt{(1-\alpha)I_{2,i}^2 - \frac{\alpha}{2}I_{2,j}^2 - \frac{\alpha}{2}I_{2,k}^2}\right). \tag{40}$$

This equation is applied with

$$\rho \to \rho', \tag{41}$$

and

$$v \to v' + \frac{c_i K}{\rho'}. \tag{42}$$

This latter equation is Eq. (33) divided through by ρ, where we recognize that ρ and ρ' have the same numerical value. If this is not obvious, note that we could redo the derivation beginning with this equation rather than Eq. (33), which was applied only to emphasize that the Kernel is defined in terms of momentum.

Eddy Sampling

ODT proceeds as a one dimensional diffusion problem interspersed with eddy events in the form of triplet maps. The primary concerns in specifying the eddy events are to determine

1. the frequency of events,
2. the location x_0 of events, and
3. the size l of the eddies.

These three quantities are determined dynamically as a function of the current line state, specifically, the local momentum and kinetic energy fields of the line. The frequency of eddy events is related to a given eddy (x_0, l) and is discussed below. However, we can compute this frequency and it is denoted by

$$\lambda(x_0, l), \qquad \textit{rate of eddies per unit size and location}$$

where $\lambda(x_0, l)dx_0 dl$ is the rate of eddies between locations x_0 and $x_0 + dx_0$ and between sizes l and $l + dl$. The rate of all eddies is given by

$$\Lambda = \int\int \lambda(x_0, l)dx_0 dl. \qquad \textit{rate of all eddies}$$

The ratio of λ to Λ is the instantaneous joint PDF of eddy sizes and locations:

$$P(x_0, l) = \frac{\lambda(x_0, l)}{\Lambda}. \qquad \textit{joint PDF of eddies}$$

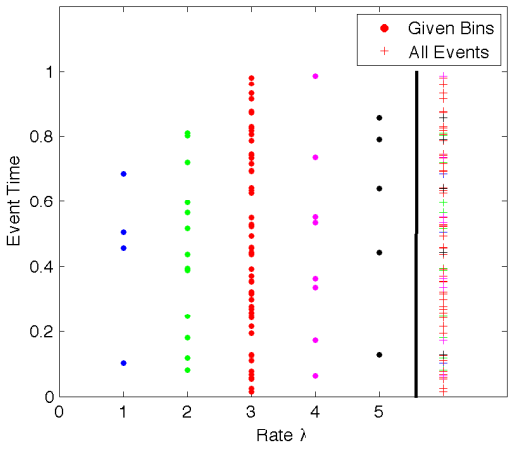

Figure 3: Eddy occurance times based on Poisson statistics for five rates (corresponding to five eddy states S_i), with the combination of all at the far right.

Eddies are assumed to occur independently, with mean rate Λ. A random process that generates time sequences of independent random events with a given overall rate is termed a Poisson process. On property of Poisson processes is that the time intervals Δt between randomly occurring, independent events, with a mean rate of occurence Λ are exponentially distributed, i.e., $P(\Delta t) = \Lambda \exp(-\Lambda \Delta t)$. The most straightforward approach to eddy sampling is to sample the next eddy occurence time from this distribution, and sample the eddy location x_0 and size l from $P(x_0, l)$. This is computationally prohibitive, however, since it would require the full computation of $P(x_0, l)$, and sampling eddies from it would require its numerical integration and inversion. This process would have to be repeated whenever the line state changes through eddy events or diffusive advancement.

A more economical sampling method that achieves the same results is introduced. Consider the eddy rate $\lambda(x_0, l)$, with the state (x_0, l) denoted (S) for simplicity. Each eddy S has an associated rate λ. Technically, λ is a rate density and λdS is the rate of eddies between S and $S + dS$. When we consider discrete S, we simply refer to λ as the rate of that eddy though it is better to write $\lambda \Delta S$. Consider a discrete $\lambda_i = \lambda(S = S_i)$, say five S values (e.g., five eddy sizes). Each S_i has its own rate λ_i. Figure 3 illustrates this for $\lambda_i = 5$, 10, 50, 10, 5, where exponentially distributed eddy occurance times are shown for each rate. At the far right are occurance times for all these eddies, which are also exponentially distributed with rate $\Lambda = \sum_i \lambda_i = 80$.

Figure 4 shows the eddy rate distribution λ, along with a simpler assumed rate distribution $\hat{\lambda}$.

Our goal is to recover (statistically) the rate distribution (times of eddy occurances and their size distribution) shown in Figs. 3, 4, (which is defined by λ_i), using only $\hat{\lambda}_i$ (and sampled values of λ_i), where $\hat{\lambda}_i > \lambda_i$. In specifying $\hat{\lambda}_i$, we also have $\hat{\Lambda}$.

Consider just a single bin S_i. If we sample occurance times from the bin with mean rate $\hat{\lambda}_i$, then we will get too many eddies. Instead, we sample candidate occurence times from bin S_i with rate

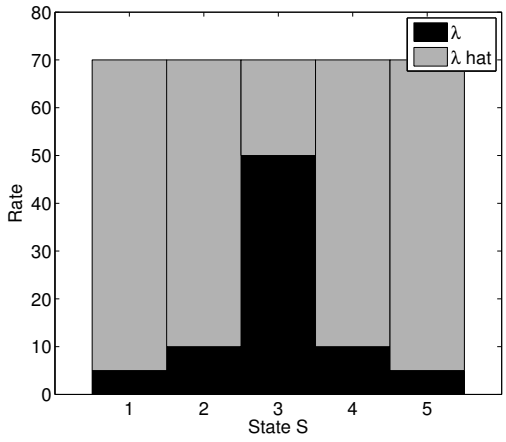

Figure 4: Example eddy rate distributions λ, and $\hat{\lambda}$.

$\hat{\lambda}_i$, but only accept these samples with probability $\lambda_i/\hat{\lambda}_i$. This is called *thinning* [4] and results in sample times governed by Poisson statistics with rate λ_i, as desired.

When we consider all bins together, we do the following:

1. At current time t, sample a time increment $t+\Delta t$ (which is the next candidate eddy occurrence time) from the exponential distribution (Poisson process) with mean rate $\hat{\Lambda}$ (the rate of all eddies).
2. Randomly sample a particular eddy S_i from the probability distribution $\hat{P}(S_i) = \hat{\lambda}_i/\hat{\Lambda}$. If $\hat{\lambda}_i$ is uniform as shown in Figure 4, the randomly sampled S_i would be uniform.
3. Accept this eddy with probability $P_a = \lambda_i/\hat{\lambda}_i$.
4. Advance the time to $t + \Delta t$, whether or not the eddy was accepted.
5. Repeat from step 1.

In the example problem given here, λ_i is known for all S_i, so that thinning is not required and a direct sampling could be performed. The benefit of thinning is for cases in which λ_i is not known in advance, as in ODT eddy sampling. For eddy sampling, λ_i is evaluated based on the current flow state using Eq. (45) below. The key point is that λ is evaluated only for the sampled S rather than for all S, which is a huge cost reduction due to the many possible eddy states (a two parameter continuum of states) in ODT. The tradeoff is that we must over-sample the eddies, rejecting many of them. As a note to the proceedure given above, when an eddy is accepted, diffusive advancement of the system is performed for a time equal to the time between the last two accepted eddies. If no eddy events have occured over a suffienctly long time interval (related to the characteristic diffusion time on the grid Δx), then diffusive advancement is performed for that time interval.

There is nothing specific about this process using a discrete distribution. For a continuous state S, or rather, for continuous x_0, l, we specify $\hat{\Lambda}$, and specify $\hat{P}(x_0, l)$ (which is $\hat{\lambda}/\hat{\Lambda}$). An eddy (x_0, l)

is sampled from $\hat{P}(x_0, l)$ and a time increment for the next candidate eddy occurrence is sampled from the exponential distribution with mean rate $\hat{\Lambda}$. Finally, that eddy is accepted with probability $P_a = \lambda(x_0, l)/\hat{\Lambda}\hat{P}(x_0, l)$. This means we draw a random number $r \in [0, 1]$ and accept the eddy if $r \leq Pa$. Note that $1/\hat{\Lambda}$ is a mean time $\overline{\Delta t}_s$ between samplings of candidate eddies, and we use

$$P_a = \frac{\overline{\Delta t}_s \lambda(x_0, l)}{\hat{P}(x_0, l)}. \tag{43}$$

$\overline{\Delta t}_s$ is set so that P_a never exceeds unity. It is inially set low, then increased (or decreased) to achieve some desired average acceptance probability.

A successful validation of the method was performed on a distribution $\lambda(l) = P(l) = 1.5(1 - (2l - 1)^2)$, with $l \in [0, 1]$, where $\Lambda = 1$. $\hat{\Lambda}$ may be taken as 1.5, with $\hat{\lambda}(l)$ uniform (that is, uniform $\hat{P}(l)$. The thinning method recovered the analytic eddy distribution and eddy rate as expected.

Eddy PDF

The PDF selected for eddy sampling is modeled as $\hat{P}(x_0, l) = f(l) \cdot g(x_0)$. The PDF $f(l)$ was given by Wunsch (as described in [5]):

$$f(l) = \frac{-2l_p/l^2}{\exp(-2l_p/l_{max}) - \exp(-2l_p/l_{min})} \exp(-2l_p/l), \tag{44}$$

where l_p, l_{max}, and l_{min} are user-specified most probable, maximum, and minimum eddy sizes, respectively. The PDF $g(x_0)$ is taken to be uniform on the domain $x \in [0, L_d - l]$, where L_d is the domain size. That is, for a given eddy size l, the eddy location is sampled uniformly on the domain such that the eddy fits in the domain.

Eddy Rate

The eddy rate $\lambda(x_0, l)$ may be modeled using dimensional arguments. This follows [3]. The units on λ are $1/(s \cdot m^2)$, so that λ is taken as

$$\lambda = \frac{1}{\tau l^2}. \tag{45}$$

Consider the constant density case. Eddies occur based on the local kinetic energy $E_{kin} = \frac{1}{2}\rho l v^2$, which scales to $E_{kin} = \frac{1}{2}\rho l^3/\tau^2$. Solving for $1/\tau$ (an eddy rate) gives

$$\frac{1}{\tau} = C\sqrt{\frac{2}{\rho l^3} E_{kin}}. \tag{46}$$

Here, an adjustable eddy rate parameter C, has been introduced. Now, take $E_{kin} = \frac{1}{2}\rho l \sum_i v_{k,i}^2 = KK \sum_i Q_i$.

For the variable density case, we proceed by analogy. In the above expression for τ, we take ρ to be an average density $\rho \Rightarrow \rho_{KK}/KK$, and $E_{kin} = KK \sum_i Q_i$, where Q_i is the variable density version.

An alternative approach is to scale the kinetic energy as $E_{kin} = \frac{1}{2\tau^2} \int \rho' \Delta^2 dy$. Here, Δ is the displacement of a fluid element at a given position under a triplet map, and equals $K(y)$, so that $E_{kin} = \frac{1}{2\tau^2} \rho KK l^3$ for constant density, or $E_{kin} = \frac{1}{2\tau^2} \rho_{KK} l^3$ for variable density. Solving for $1/\tau$ and using $E_{kin} = KK \sum_i Q_i$ gives

$$\frac{1}{\tau} = C \sqrt{\frac{2}{\rho KK l^3} E_{kin}}, \tag{47}$$

for constant density, and

$$\frac{1}{\tau} = C \sqrt{\frac{2}{\rho_{KK} l^3} E_{kin}}, \tag{48}$$

for variable density. The difference with this approach is the factor of KK in the denominator. Also, the ρ_{KK} appears naturally for the variable density case. In [3], the first approach is implied for the constant density case, and the second approach (with a slightly different E_{kin}) is used for the variable density case, so that the variable density case in the second approach does not reduce to the constant density case of the first approach. This is not an error, but it is not consistent. This was noted in an erratam [6]. The first approach is what is implemented in the ODT code, but the second approach is somewhat cleaner.

In order to avoid the allowance of artifically small eddies, that are not energetically possible, the kinetic energy is reduced by a viscous penalty term E_{vp}. This term is also defined by scaling as for the kinetic energy, $E_{vp} = \frac{1}{2} \bar{\rho} l^3 / \tau^2$, but using $\tau = l^2/\nu$, where ν is an average kinematic viscosity and $\bar{\rho}$ the average density. This gives $E_{vp} = \frac{1}{2} \bar{\rho} \nu^2 / l$. E_{vp} is multiplied by an adjustable viscous penalty parameter Z, and subtracted from E_{kin} to give, for constant density,

$$\frac{1}{\tau} = C \sqrt{\frac{2}{\rho l^3} (E_{kin} - Z E_{vp})}, \tag{49}$$

(or $\rho \Rightarrow \rho_{KK}/KK$ for variable density). Eddies with negative arguments of the radical are disallowed.

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