Notes on
Various Errors and Jacobian Derivations for SLAM

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1 Introduction

In this post, we discuss the definitions of various errors used in SLAM and the Jacobians utilized for their optimization. The errors covered in this post are as follows...

- Reprojection error

$$\mathbf{e} = \mathbf{p} - \hat{\mathbf{p}} \in \mathbb{R}^2 \tag{1}$$

- Photometric error

$$\mathbf{e} = \mathbf{I}_1(\mathbf{p}_1) - \mathbf{I}_2(\mathbf{p}_2) \in \mathbb{R}^1 \tag{2}$$

- Relative pose error (PGO)

$$\mathbf{e}_{ij} = \mathrm{Log}(\mathbf{z}_{ij}^{-1}\hat{\mathbf{z}}_{ij}) \in \mathbb{R}^6 \tag{3}$$

- Line reprojection error

$$\mathbf{e}_l = \begin{bmatrix} \frac{\mathbf{x}_s^\intercal l_c}{\sqrt{l_1^2+l_2^2}}, & \frac{\mathbf{x}_e^\intercal l_c}{\sqrt{l_1^2+l_2^2}} \end{bmatrix} \in \mathbb{R}^2 \tag{4}$$

- IMU measurement error :

$$\mathbf{e}_{\mathcal{B}} = \begin{bmatrix} \delta\alpha_{b_{k+1}}^{b_k} \\ \delta\boldsymbol{\theta}_{b_{k+1}}^{b_k} \\ \delta\beta_{b_{k+1}}^{b_k} \\ \delta\mathbf{b}_a \\ \delta\mathbf{b}_g \end{bmatrix} = \begin{bmatrix} \mathbf{R}_w^{b_k}(\mathbf{p}_{b_{k+1}}^w - \mathbf{p}_{b_k}^w - \mathbf{v}_{b_k}^w \Delta t_k + \frac{1}{2}\mathbf{g}^w \Delta t_k^2) - \hat{\alpha}_{b_{k+1}}^{b_k} \\ 2\Big[\Big(\hat{\gamma}_{b_{k+1}}^{b_k}\Big)^{-1} \otimes (\mathbf{q}_{b_k}^w)^{-1} \otimes \mathbf{q}_{b_{k+1}}^w\Big]_{xyz} \\ \mathbf{R}_w^{b_k}(\mathbf{v}_{b_{k+1}}^w - \mathbf{v}_{b_k}^w + \mathbf{g}^w \Delta t_k) - \hat{\beta}_{b_{k+1}}^{b_k} \\ \mathbf{b}_{a_{k+1}} - \mathbf{b}_{a_k} \\ \mathbf{b}_{g_{k+1}} - \mathbf{b}_{g_k} \end{bmatrix} \tag{5}$$

Depending on whether the camera pose is expressed as a rotation matrix $\mathbf{R} \in SO(3)$ or a transformation matrix $\mathbf{T} \in SE(3)$, different Jacobians are derived. Jacobians for reprojection errors are derived for SO(3), and Jacobians for photometric errors are derived for SE(3). The representation of a point in 3D space as $\mathbf{X} = [X, Y, Z, W]^\intercal$ or using inverse depth ρ also affects the Jacobian derivation. The derivation processes for both cases are explained.

The Jacobians discussed in this post are as follows.

- Camera pose (SO(3)-based)

$$\frac{\partial \mathbf{e}}{\partial \mathbf{R}} \rightarrow \frac{\partial \mathbf{e}}{\partial \Delta\mathbf{w}} \tag{6}$$

- Camera pose (SE(3)-based)

$$\frac{\partial \mathbf{e}}{\partial \mathbf{T}} \rightarrow \frac{\partial \mathbf{e}}{\partial \Delta\xi} \tag{7}$$

- Map point

$$\frac{\partial \mathbf{e}}{\partial \mathbf{X}} \tag{8}$$

- Relative pose (SE(3)-based)

$$\frac{\partial \mathbf{e}_{ij}}{\partial \Delta\xi_i}, \frac{\partial \mathbf{e}_{ij}}{\partial \Delta\xi_j} \tag{9}$$

- 3D plücker line

$$\frac{\partial \mathbf{e}_l}{\partial l}, \frac{\partial l}{\partial \mathcal{L}_c}, \frac{\partial \mathcal{L}_c}{\partial \mathcal{L}_w}, \frac{\partial \mathcal{L}_w}{\partial \delta_{\boldsymbol{\theta}}} \tag{10}$$

- Quaternion representation

$$\frac{\partial \mathbf{X}'}{\partial \mathbf{q}} \tag{11}$$

- Camera intrinsics

$$\frac{\partial \mathbf{e}}{\partial f_x}, \frac{\partial \mathbf{e}}{\partial f_y}, \frac{\partial \mathbf{e}}{\partial c_x}, \frac{\partial \mathbf{e}}{\partial c_y} \tag{12}$$

- Inverse depth

$$\frac{\partial \mathbf{e}}{\partial \rho} \tag{13}$$

- IMU error-state system kinematics :

$$\mathbf{J}_{b_{k+1}}^{b_k} \tag{14}$$

- IMU measurement :

$$\frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial [\mathbf{p}_{b_k}^w, \mathbf{q}_{b_k}^w]}, \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial [\mathbf{v}_{b_k}^w, \mathbf{b}_{a_k}, \mathbf{b}_{g_k}]}, \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial [\mathbf{p}_{b_{k+1}}^w, \mathbf{q}_{b_{k+1}}^w]}, \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial [\mathbf{v}_{b_{k+1}}^w, \mathbf{b}_{a_{k+1}}, \mathbf{b}_{g_{k+1}}]} \tag{15}$$

2 Optimization formulation

2.1 Error derivation

In SLAM, the error is defined as the difference between the observed value (measurement) $\mathbf{z}$ and the predicted value (estimate) $\hat{\mathbf{z}}$ based on sensor data.

$$\boxed{\mathbf{e}(\mathbf{x}) = \mathbf{z} - \hat{\mathbf{z}}(\mathbf{x})} \tag{16}$$

- $\mathbf{x}$: model state variables

As such, the difference between the observed and predicted values is defined as the error, and the optimal state variables $\mathbf{x}$ that minimize this error become the optimization problem in SLAM. In general, since the state variables in SLAM include non-linear terms related to rotation, the non-linear least squares method is mainly used.

2.2 Error function derivation

Typically, when a large amount of sensor data comes in, dozens to hundreds of errors are calculated in vector form. At this time, it is assumed that the error follows a normal distribution, and the work of converting it into an error function is performed.

$$\mathbf{e}(\mathbf{x}) = \mathbf{z} - \hat{\mathbf{z}} \sim \mathcal{N}(0, \mathbf{\Sigma}) \tag{17}$$

Tip

The multivariate normal distribution of the probability variable $\mathbf{x}$ for modeling the error function is as follows.

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n |\mathbf{\Sigma}|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^\intercal \mathbf{\Omega}(\mathbf{x}-\mu)\right) \sim \mathcal{N}(\mu, \mathbf{\Sigma}) \tag{18}$$

- $\mathbf{\Omega} = \mathbf{\Sigma}^{-1}$: information matrix (inverse of covariance matrix)

The error can be modeled as a multivariate normal distribution with mean 0 and variance $\mathbf{\Sigma}$. Applying the log-likelihood to this equation, $\ln p(\mathbf{e})$ is as follows.

$$\begin{aligned} \ln p(\mathbf{e}) &\propto -\frac{1}{2}(\mathbf{z}-\hat{\mathbf{z}})^T \mathbf{\Omega}(\mathbf{z}-\hat{\mathbf{z}}) \\ &\propto -\frac{1}{2}\mathbf{e}^\intercal \mathbf{\Omega} \mathbf{e} \end{aligned} \tag{19}$$

Finding $\mathbf{x}^*$ where log-likelihood $\ln p(\mathbf{e})$ is maximized results in the highest probability of the multivariate normal distribution. This is called Maximum Likelihood Estimation (MLE). Since $\ln p(\mathbf{e})$ has a negative (-) sign in front, finding the minimum of the negative log-likelihood $\ln p(\mathbf{e})$ is as follows.

$$\mathbf{x}^* = \arg\max p(\mathbf{e}) = \arg\min \mathbf{e}^T \mathbf{\Omega} \mathbf{e} \tag{20}$$

If all errors are added instead of a single error, it is expressed as follows, and this is called the error function E. In actual optimization problems, not the single error $\mathbf{e}_i$ but the error function E that minimizes $\mathbf{x}^*$ is found.

$$\boxed{\begin{aligned}\mathbf{E}(\mathbf{x}) &= \sum_i \mathbf{e}_i^T \mathbf{\Omega}_i \mathbf{e}_i \\ \mathbf{x}^* &= \arg\min \mathbf{E}(\mathbf{x})\end{aligned}} \tag{21}$$

2.3 Non-linear least squares

The final optimization equation to be solved is as follows.

$$\mathbf{x}^* = \arg\min \mathbf{E}(\mathbf{x}) = \arg\min \sum_i \mathbf{e}_i^T \mathbf{\Omega}_i \mathbf{e}_i \tag{22}$$

In the above formula, the optimal parameter $\mathbf{x}^*$ that minimizes the error must be found. **However, the above formula typically includes non-linear terms related to rotation in SLAM, so no closed-form solution exists. Therefore, non-linear optimization methods (Gauss-Newton (GN), Levenberg-Marquardt (LM)) must be used to solve the problem.** Among the actual implemented SLAM codes, the information matrix $\mathbf{\Omega}_i$ is often set to $\mathbf{I}_3$ to find the optimal value for $\mathbf{e}_i^\intercal \mathbf{e}_i$.

For example, let's assume that the problem is solved using the GN method. The order of solving the problem is as follows.

- Define the error function
- Approximate linearization using Taylor expansion
- Set the first derivative to zero.
- Calculate the value and substitute it into the error function
- Repeat until convergence.

If the error function $\mathbf{e}$ is detailed, it appears as $\mathbf{e}(\mathbf{x})$, meaning that the value of the error function changes according to the robot's pose vector $\mathbf{x}$. The GN method updates the increment $\Delta\mathbf{x}$ iteratively in a direction that reduces the error for $\mathbf{e}(\mathbf{x})$.

$$\mathbf{e}(\mathbf{x}+\Delta\mathbf{x})^\intercal \Omega \mathbf{e}(\mathbf{x}+\Delta\mathbf{x}) \tag{23}$$

When $\mathbf{e}(\mathbf{x}+\Delta\mathbf{x})$ is used near $\mathbf{x}$ with a first-order Taylor expansion, the above equation is approximated as follows.

$$\begin{aligned}\mathbf{e}(\mathbf{x}+\Delta\mathbf{x})|_{\mathbf{x}} &\approx \mathbf{e}(\mathbf{x}) + \mathbf{J}(\mathbf{x}+\Delta\mathbf{x}-\mathbf{x}) \\ &= \mathbf{e}(\mathbf{x}) + \mathbf{J}\Delta\mathbf{x}\end{aligned} \tag{24}$$

At this time, $\mathbf{J} = \frac{\partial \mathbf{e}(\mathbf{x}+\Delta\mathbf{x})}{\partial \mathbf{x}}$. When this is applied to the entire error function, it is as follows.

$$\mathbf{e}(\mathbf{x}+\Delta\mathbf{x})^\intercal \Omega \mathbf{e}(\mathbf{x}+\Delta\mathbf{x}) \approx (\mathbf{e}+\mathbf{J}\Delta\mathbf{x})^\intercal \Omega (\mathbf{e}+\mathbf{J}\Delta\mathbf{x}) \tag{25}$$

After expanding the above equation and substituting, it is as follows.

$$\begin{aligned}&= \underbrace{\mathbf{e}^\intercal \Omega \mathbf{e}}_{\mathbf{c}} + 2\underbrace{\mathbf{e}^\intercal \Omega \mathbf{J}}_{\mathbf{b}}\Delta\mathbf{x} + \Delta\mathbf{x}^\intercal \underbrace{\mathbf{J}^\intercal \Omega \mathbf{J}}_{\mathbf{H}}\Delta\mathbf{x} \\ &= \mathbf{c} + 2\mathbf{b}\Delta\mathbf{x} + \Delta\mathbf{x}^\intercal \mathbf{H}\Delta\mathbf{x}\end{aligned} \tag{26}$$

The overall error applied is as follows.

$$\mathbf{E}(\mathbf{x}+\Delta\mathbf{x}) = \sum_i \mathbf{e}_i^\intercal \Omega_i \mathbf{e}_i = \mathbf{c} + 2\mathbf{b}\Delta\mathbf{x} + \Delta\mathbf{x}^T \mathbf{H}\Delta\mathbf{x} \tag{27}$$

$\mathbf{E}(\mathbf{x}+\Delta\mathbf{x})$ is in a quadratic form about $\Delta\mathbf{x}$ and since $\mathbf{H} = \mathbf{J}^\intercal \Omega \mathbf{J}$ is a positive definite matrix, the first derivative of $\mathbf{E}(\mathbf{x}+\Delta\mathbf{x})$ set to zero determines the minimum of $\Delta\mathbf{x}$.

$$\frac{\partial \mathbf{E}(\mathbf{x}+\Delta\mathbf{x})}{\partial \Delta\mathbf{x}} \approx 2\mathbf{b} + 2\mathbf{H}\Delta\mathbf{x} = 0 \tag{28}$$

This leads to the following formula being derived.

$$\mathbf{H}\Delta\mathbf{x} = -\mathbf{b} \tag{29}$$

Thus obtained $\Delta\mathbf{x} = -\mathbf{H}^{-1}\mathbf{b}$ is updated to $\mathbf{x}$.

$$\mathbf{x} \leftarrow \mathbf{x} + \Delta\mathbf{x} \tag{30}$$

The algorithm that iteratively performs the process so far is called the Gauss-Newton method. The LM method, compared to the GN method, has the same overall process, however, in the formula for calculating the increment, a damping factor λ term is added.

$$\boxed{\begin{array}{l}\text{(GN) } \mathbf{H}\Delta\mathbf{x} = -\mathbf{b} \\ \text{(LM) } (\mathbf{H} + \lambda\mathbf{I})\Delta\mathbf{x} = -\mathbf{b}\end{array}} \tag{31}$$

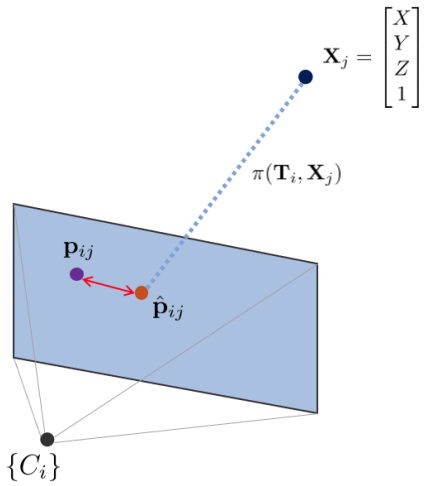
3 Reprojection Error

Reprojection error is primarily used in feature-based Visual SLAM. It is commonly used when performing feature-based method visual odometry (VO) or bundle adjustment (BA). For more details on BA, refer to the post [SLAM] Bundle Adjustment Concept Review.

NOMENCLATURE of reprojection error

- $\tilde{\mathbf{p}} = \pi_h(\cdot) : \begin{bmatrix} X' \\ Y' \\ Z' \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} X'/Z' \\ Y'/Z' \\ 1 \end{bmatrix} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ 1 \end{bmatrix}$
  - Point $\mathbf{X}'$ in 3D space non-homogeneously transformed to be projected onto the image plane
- $\hat{\mathbf{p}} = \pi_k(\cdot) = \tilde{\mathbf{K}}\tilde{\mathbf{p}} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ 1 \end{bmatrix} = \begin{bmatrix} f\tilde{u} + c_x \\ f\tilde{v} + c_y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$
  - Point projected onto the image plane after lens distortion correction. If distortion correction has already been performed at the input stage, $\pi_k(\cdot) = \tilde{\mathbf{K}}(\cdot)$.
- $\mathbf{K} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}$: Camera's intrinsic parameters
- $\tilde{\mathbf{K}} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \end{bmatrix}$: Omitting the last row of the intrinsic parameters for projection from $\mathbb{P}^2 \rightarrow \mathbb{R}^2$.
- $\mathcal{X} = [\mathcal{T}_1, \cdots, \mathcal{T}_m, \mathbf{X}_1, \cdots, \mathbf{X}_n]^\intercal$: Model's state variables
- m: Number of camera poses
- n: Number of 3D points
- $\mathcal{T}_i = [\mathbf{R}_i, \mathbf{t}_i]$
- $\mathbf{e}_{ij} = \mathbf{e}_{ij}(\mathcal{X})$: Notation simplified by omitting $\mathcal{X}$
- $\mathbf{p}_{ij}$: Observed pixel coordinates of a feature point
- $\hat{\mathbf{p}}_{ij}$: Estimated pixel coordinates of a feature point
- $\mathbf{T}_i\mathbf{X}_j$: Transformation, 3D point $\mathbf{X}_j$ transformed to camera coordinate system $\{i\}$, $\left(\mathbf{T}_i\mathbf{X}_j = \begin{bmatrix} \mathbf{R}_i\mathbf{X}_j + \mathbf{t}_i \\ 1 \end{bmatrix} \in \mathbb{R}^{4\times 1}\right)$
  - $\mathbf{X}' = \mathbf{T}\mathbf{X} = [X', Y', Z', 1]^\intercal = [\tilde{\mathbf{X}}', 1]^\intercal$
- $\oplus$: Operator that updates rotation matrix $\mathbf{R}$ and 3D vector $\mathbf{t}, \mathbf{X}$ simultaneously.

- $\mathbf{J} = \frac{\partial \mathbf{e}}{\partial \mathcal{X}} = \frac{\partial \mathbf{e}}{\partial [\mathcal{T}, \mathbf{X}]}$
- $\mathbf{w} = \begin{bmatrix} w_x & w_y & w_z \end{bmatrix}^\intercal$: Angular velocity
- $[\mathbf{w}]_\times = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}$: Skew-symmetric matrix of angular velocity $\mathbf{w}$



When there is a pinhole camera pose $\{C_i\}$ and a world point $\mathbf{X}_j$, $\mathbf{X}_j$ is projected onto the image plane through the following transformation.

$$\text{projection model:} \quad \hat{\mathbf{p}}_{ij} = \pi(\mathbf{T}_i, \mathbf{X}_j) \tag{32}$$

3차원 점	$\mathbf{T}_i\mathbf{X}_j$ Transformation	$\tilde{\mathbf{p}} = \pi_h(\cdot)$ Non-homogeneous	$\hat{\mathbf{p}} = \pi_k(\cdot)$ Undistort	Project to Image Plane	
$\mathbf{X}_j$ $[X_j, Y_j, Z_j, 1]^\intercal$	$\mathbf{X}'_j = \begin{bmatrix} \mathbf{R}_i\mathbf{X}_j + \mathbf{t}_i \\ 1 \end{bmatrix} = [X'_j, Y'_j, Z'_j, 1]^\intercal$	$\tilde{\mathbf{p}}_{ij} = [X'_j/Z'_j,\ Y'_j/Z'_j,\ 1]^\intercal = [\tilde{u}_{ij}, \tilde{v}_{ij}, 1]^\intercal$	$\begin{cases} \bar{u}_{ij} = \tilde{u}_{ij}(1 + k_1 r^2 + k_2 r^4) \\ \bar{v}_{ij} = \tilde{v}_{ij}(1 + k_1 r^2 + k_2 r^4) \end{cases}$	$\begin{cases} u_{ij} = f_x \bar{u}_{ij} + c_x \\ v_{ij} = f_y \bar{v}_{ij} + c_y \end{cases}$	$\hat{\mathbf{p}}_{ij}$ $[u_{ij}, v_{ij}]^\intercal$

The model utilizing the camera's intrinsic/extrinsic parameters is called the projection model. Reprojection error is defined as follows:

$$\boxed{\begin{aligned} \mathbf{e}_{ij} &= \mathbf{p}_{ij} - \hat{\mathbf{p}}_{ij} \\ &= \mathbf{p}_{ij} - \pi(\mathbf{T}_i, \mathbf{X}_j) \\ &= \mathbf{p}_{ij} - \pi_k(\pi_h(\mathbf{T}_i\mathbf{X}_j)) \end{aligned}} \tag{33}$$

The error function for all camera poses and 3D points is defined as follows.

$$\mathbf{E}(\mathcal{X}) = \sum_i \sum_j \|\mathbf{e}_{ij}\|^2 \tag{34}$$

$$\begin{aligned} \mathcal{X}^* &= \arg\min_{\mathcal{X}^*} \mathbf{E}(\mathcal{X}) \\ &= \arg\min_{\mathcal{X}^*} \sum_i \sum_j \|\mathbf{e}_{ij}\|^2 \\ &= \arg\min_{\mathcal{X}^*} \sum_i \sum_j \mathbf{e}_{ij}^\intercal \mathbf{e}_{ij} \\ &= \arg\min_{\mathcal{X}^*} \sum_i \sum_j (\mathbf{p}_{ij} - \hat{\mathbf{p}}_{ij})^\intercal (\mathbf{p}_{ij} - \hat{\mathbf{p}}_{ij}) \end{aligned} \tag{35}$$

The error $\|\mathbf{e}(\mathcal{X}^*)\|^2$ satisfying $\mathbf{E}(\mathcal{X}^*)$ can be calculated iteratively through non-linear least squares. By repeatedly updating a small increment $\Delta\mathcal{X}$ to $\mathcal{X}$, the optimal state is found.

$$\arg\min_{\mathcal{X}^*} \mathbf{E}(\mathcal{X} + \Delta\mathcal{X}) = \arg\min_{\mathcal{X}^*} \sum_i \sum_j \|\mathbf{e}(\mathcal{X} + \Delta\mathcal{X})\|^2 \tag{36}$$

Strictly speaking, since the state increment $\Delta\mathcal{X}$ includes an SO(3) rotation matrix, it is correct to add it using the $\oplus$ operator to the existing state $\mathcal{X}$, but the $+$ operator is used for convenience of expression.

$$\mathbf{e}(\mathcal{X} \oplus \Delta\mathcal{X}) \quad \rightarrow \quad \mathbf{e}(\mathcal{X} + \Delta\mathcal{X}) \tag{37}$$

This equation can be expressed through the first-order Taylor approximation as follows.

$$\begin{aligned} \mathbf{e}(\mathcal{X} + \Delta\mathcal{X}) &\approx \mathbf{e}(\mathcal{X}) + \mathbf{J}\Delta\mathcal{X} \\ &= \mathbf{e}(\mathcal{X}) + \mathbf{J}_c\Delta\mathcal{T} + \mathbf{J}_p\Delta\mathbf{X} \\ &= \mathbf{e}(\mathcal{X}) + \frac{\partial \mathbf{e}}{\partial \mathcal{T}}\Delta\mathcal{T} + \frac{\partial \mathbf{e}}{\partial \mathbf{X}}\Delta\mathbf{X} \end{aligned} \tag{38}$$

$$\arg\min_{\mathcal{X}^*} \mathbf{E}(\mathcal{X} + \Delta\mathcal{X}) \approx \arg\min_{\mathcal{X}^*} \sum_i \sum_j \|\mathbf{e}(\mathcal{X}) + \mathbf{J}\Delta\mathcal{X}\|^2 \tag{39}$$

The optimal increment $\Delta\mathcal{X}^*$ is found by differentiating the above expression. The derivation process is omitted in this section. For detailed information on the derivation process, refer to the previous section.

$$\begin{aligned} \mathbf{J}^\intercal\mathbf{J}\Delta\mathcal{X}^* &= -\mathbf{J}^\intercal\mathbf{e} \\ \mathbf{H}\Delta\mathcal{X}^* &= -\mathbf{b} \end{aligned} \tag{40}$$

This equation is in the form of a linear system $\mathbf{Ax} = \mathbf{b}$, thus $\Delta\mathcal{X}^*$ can be found using various linear algebra techniques like schur complement and cholesky decomposition. In this case, $\mathbf{t}, \mathbf{X}$ among the existing states $\mathcal{X}$ exist in a linear vector space, so there is no difference depending on whether they are added from the right or from the left, but **since the rotation matrix R belongs to the non-linear SO(3) group, it depends on whether it is multiplied from the right or from the left whether to update the pose seen in the local coordinate system (right) or the pose seen in the global coordinate system (left). Reprojection error updates the transformation matrix of the global coordinate system, so it generally uses the left multiplication method.**

$$\mathcal{X} \leftarrow \mathcal{X} \oplus \Delta\mathcal{X}^* \tag{41}$$

$\mathcal{X}$ consists of $[\mathcal{T}, \mathbf{X}]$, so it can be described as follows.

$$\begin{aligned} \mathcal{T} &\leftarrow \mathcal{T} \oplus \Delta\mathcal{T}^* \\ \mathbf{X} &\leftarrow \mathbf{X} \oplus \Delta\mathbf{X}^* \end{aligned} \tag{42}$$

The definition of the left multiplication $\oplus$ operation is as follows.

$$\begin{aligned} \mathbf{R} \oplus \Delta\mathbf{R}^* &= \Delta\mathbf{R}^*\mathbf{R} \\ &= \exp([\Delta\mathbf{w}^*]_\times)\mathbf{R} \quad \cdots \text{ globally updated (left mult)} \\ \mathbf{t} \oplus \Delta\mathbf{t}^* &= \mathbf{t} + \Delta\mathbf{t}^* \\ \mathbf{X} \oplus \Delta\mathbf{X}^* &= \mathbf{X} + \Delta\mathbf{X}^* \end{aligned} \tag{43}$$

3.1 Jacobian of the Reprojection Error

3.1.1 Jacobian of Camera Pose

The Jacobian of the pose $\mathbf{J}_c$ can be decomposed as follows.

$$\begin{aligned} \mathbf{J}_c = \frac{\partial \mathbf{e}}{\partial \mathcal{T}} &= \frac{\partial}{\partial \mathcal{T}}(\mathbf{p} - \hat{\mathbf{p}}) \\ &= \frac{\partial}{\partial \mathcal{T}}\Big(\mathbf{p} - \pi_k(\pi_h(\mathbf{T}_i\mathbf{X}_j))\Big) \\ &= \frac{\partial}{\partial \mathcal{T}}\Big(\ - \pi_k(\pi_h(\mathbf{T}_i\mathbf{X}_j))\Big) \end{aligned} \tag{44}$$

Using the chain rule, the above formula is organized as follows. For convenience, $\mathbf{T}_i\mathbf{X}_j \rightarrow \mathbf{X}'$ is denoted.

$$\begin{aligned}\mathbf{J}_c &= \frac{\partial \hat{\mathbf{p}}}{\partial \tilde{\mathbf{p}}} \frac{\partial \tilde{\mathbf{p}}}{\partial \mathbf{X}'} \frac{\partial \mathbf{X}'}{\partial [\mathbf{w}, \mathbf{t}]} \\ &= \mathbb{R}^{2\times 3} \cdot \mathbb{R}^{3\times 4} \cdot \mathbb{R}^{4\times 6} = \mathbb{R}^{2\times 6}\end{aligned} \tag{45}$$

The reason for calculating the Jacobian $\frac{\partial \mathbf{X}'}{\partial \mathbf{w}}$ for the angular velocity w instead of the Jacobian $\frac{\partial \mathbf{X}'}{\partial \mathbf{R}}$ for the rotation matrix R is explained in the next section. Also, depending on whether the error is defined as $\mathbf{p} - \hat{\mathbf{p}}$ or $\hat{\mathbf{p}} - \mathbf{p}$, the sign of $\mathbf{J}_c$ also changes, so this should be carefully applied when implementing the actual code. The sign is considered as + and marked in this material.

If it is assumed that undistortion has already been performed during the image input process, $\frac{\partial \hat{\mathbf{p}}}{\partial \tilde{\mathbf{p}}}$ is as follows.

$$\boxed{\begin{aligned}\frac{\partial \hat{\mathbf{p}}}{\partial \tilde{\mathbf{p}}} &= \frac{\partial}{\partial \tilde{\mathbf{p}}} \tilde{\mathbf{K}} \tilde{\mathbf{p}} \\ &= \tilde{\mathbf{K}} \\ &= \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \end{bmatrix} \in \mathbb{R}^{2\times 3}\end{aligned}} \tag{46}$$

Next, $\frac{\partial \tilde{\mathbf{p}}}{\partial \mathbf{X}'}$ is as follows.

$$\boxed{\begin{aligned}\frac{\partial \tilde{\mathbf{p}}}{\partial \mathbf{X}'} &= \frac{\partial [\tilde{u}, \tilde{v}, 1]}{\partial [X', Y', Z', 1]} \\ &= \begin{bmatrix} \frac{1}{Z'} & 0 & \frac{-X'}{Z'^2} & 0 \\ 0 & \frac{1}{Z'} & \frac{-Y'}{Z'^2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 4}\end{aligned}} \tag{47}$$

Next, $\frac{\partial \mathbf{X}'}{\partial \mathbf{t}}$ needs to be calculated. This can be relatively simply obtained as follows.

$$\boxed{\begin{aligned}\frac{\partial \mathbf{X}'}{\partial \mathbf{t}} &= \frac{\partial}{\partial [t_x, t_y, t_z]} \begin{bmatrix} \mathbf{R}\mathbf{X} + \mathbf{t} \\ 1 \end{bmatrix} \\ &= \frac{\partial}{\partial [t_x, t_y, t_z]} \begin{bmatrix} \mathbf{t} \\ 1 \end{bmatrix} \\ &= \frac{\partial}{\partial [t_x, t_y, t_z]} \left(\begin{bmatrix} t_x \\ t_y \\ t_z \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 3}\end{aligned}} \tag{48}$$

3.1.2 Lie Theory-Based SO(3) Optimization

Finally, $\frac{\partial \mathbf{X}'}{\partial \mathbf{w}}$ needs to be calculated. In this case, the rotation-related parameter was represented as angular velocity $\mathbf{w}$ instead of the rotation matrix $\mathbf{R}$. The rotation matrix $\mathbf{R}$ has 9 parameters, whereas the actual rotation is limited to 3 degrees of freedom, therefore it is over-parameterized. The disadvantages of an over-parameterized representation are as follows:

- Due to the calculation of redundant parameters, the amount of computation required for optimization increases.
- Additional degrees of freedom can cause problems with numerical instability.
- Each time parameters are updated, it must be checked whether they always satisfy the constraints.

Lie theory allows optimization to be performed free from constraints. Therefore, instead of the lie group SO(3) $\mathbf{R}$, the lie algebra so(3) $[\mathbf{w}]_\times$ is used to freely update parameters from constraints. Here, $\mathbf{w} \in \mathbb{R}^3$ denotes the angular velocity vector.

$$\mathbf{J}_c = \frac{\partial \mathbf{e}}{\partial [\mathbf{R}, \mathbf{t}]} \rightarrow \frac{\partial \mathbf{e}}{\partial [\mathbf{w}, \mathbf{t}]} \tag{49}$$

However, since $\mathbf{w}$ is not directly visible from $\mathbf{X}'$, $\mathbf{X}'$ must be represented in lie algebra. At this time, since the Jacobian for the $\mathbf{w}$ term related to rotation needs to be calculated, let's assume that the 3D point $\mathbf{X}_t$ is the point $\mathbf{X}$ translated by $\mathbf{t}$ and then $\mathbf{X}'$ is the point $\mathbf{X}_t$ rotated by $\mathbf{R}$.

$$\begin{aligned}\mathbf{X}_t &= \mathbf{X} + \mathbf{t} \\ \mathbf{X}' &= \mathbf{R}\mathbf{X}_t \\ &= \exp([\mathbf{w}]\times)\mathbf{X}_t\end{aligned} \tag{50}$$

Tip

$\exp([\mathbf{w}]_\times) \in SO(3)$ denotes the operation of converting angular velocity $\mathbf{w}$ into a 3D rotation matrix $\mathbf{R}$ using exponential mapping. For detailed information on exponential mapping, see this link.

$$\exp([\mathbf{w}]_\times) = \mathbf{R} \tag{51}$$

In this case, depending on how the small lie algebra increment $\Delta\mathbf{w}$ is updated to the existing $\exp([\mathbf{w}]_\times)$, there are two ways to update. First, there is [1] the basic lie algebra update method. Next, there is [2] the update method using the perturbation model.

$$\begin{aligned}&\exp([\mathbf{w}]_\times) \leftarrow \exp([\mathbf{w} + \Delta\mathbf{w}]_\times) \quad \cdots [1] \\ &\exp([\mathbf{w}]_\times) \leftarrow \exp([\Delta\mathbf{w}]_\times)\exp([\mathbf{w}]_\times) \quad \cdots [2]\end{aligned} \tag{52}$$

Tip

There is the following relationship between the above two methods. For detailed information, see this link chapter 4.3.3.

$$\begin{aligned}\exp([\Delta\mathbf{w}]_\times)\exp([\mathbf{w}]_\times) &= \exp([\mathbf{w} + \mathbf{J}_l^{-1}\Delta\mathbf{w}]_\times) \\ \exp([\mathbf{w} + \Delta\mathbf{w}]_\times) &= \exp([\mathbf{J}_l\Delta\mathbf{w}]_\times)\exp([\mathbf{w}]_\times)\end{aligned} \tag{53}$$

[1] **Lie Algebra-Based Update:** First, using method [1] to directly calculate the Jacobian $\frac{\partial \mathbf{R}\mathbf{X}_t}{\partial \mathbf{w}}$ results in the following complex formula.

$$\begin{aligned}\frac{\partial \mathbf{R}\mathbf{X}_t}{\partial \mathbf{w}} &= \lim_{\Delta\mathbf{w}\to 0} \frac{\exp([\mathbf{w} + \Delta\mathbf{w}]_\times)\mathbf{X}_t - \exp([\mathbf{w}]_\times)\mathbf{X}_t}{\Delta\mathbf{w}} \\ &\approx \lim_{\Delta\mathbf{w}\to 0} \frac{\exp([\mathbf{J}_l\Delta\mathbf{w}]_\times)(\exp([\mathbf{w}]_\times)\mathbf{X}_t - \exp([\mathbf{w}]_\times)\mathbf{X}_t}{\Delta\mathbf{w}} \\ &\approx \lim_{\Delta\mathbf{w}\to 0} \frac{(\mathbf{I} + [\mathbf{J}_l\Delta\mathbf{w}]_\times)(\exp([\mathbf{w}]_\times)\mathbf{X}_t - \exp([\mathbf{w}]_\times)\mathbf{X}_t}{\Delta\mathbf{w}} \\ &= \lim_{\Delta\mathbf{w}\to 0} \frac{[\mathbf{J}_l\Delta\mathbf{w}]_\times\mathbf{R}\mathbf{X}_t}{\Delta\mathbf{w}} \qquad (\because \exp([\mathbf{w}]_\times)\mathbf{X}_t = \mathbf{R}\mathbf{X}_t) \\ &= \lim_{\Delta\mathbf{w}\to 0} \frac{-[\mathbf{R}\mathbf{X}_t]_\times\mathbf{J}_l\Delta\mathbf{w}}{\Delta\mathbf{w}} \\ &= -[\mathbf{R}\mathbf{X}_t]_\times\mathbf{J}_l \\ &= -[\mathbf{X}']_\times\mathbf{J}_l\end{aligned} \tag{54}$$

[2] **Perturbation Model-Based Update:** To calculate a simpler Jacobian without using $\mathbf{J}_l$, the perturbation model of lie algebra so(3) is generally used. Calculating the Jacobian $\frac{\partial \mathbf{R}\mathbf{X}_t}{\partial \Delta\mathbf{w}}$ using the perturbation model

Tip

In the above formula, the second row uses the BCH approximation to derive the left Jacobian (left jacobian) $\mathbf{J}_l$, and the third row applies the first-order Taylor approximation for small rotation $\exp([\mathbf{J}_l\Delta\mathbf{w}]_\times)$. For more information on $\mathbf{J}_l$, see Visual SLAM Introduction Chapter 4.
To understand the third row's approximation, given an arbitrary rotation vector $\mathbf{w} = [w_x, w_y, w_z]^\intercal$, the rotation matrix can be expanded in exponential mapping form as follows.

$$\mathbf{R} = \exp([\mathbf{w}]_\times) = \mathbf{I} + [\mathbf{w}]_\times + \frac{1}{2}[\mathbf{w}]_\times^2 + \frac{1}{3!}[\mathbf{w}]_\times^3 + \frac{1}{4!}[\mathbf{w}]_\times^4 + \cdots \tag{55}$$

For a small rotation matrix $\Delta\mathbf{R}$, higher-order terms beyond the second can be ignored, and it can be approximated as follows.

$$\Delta\mathbf{R} \approx \mathbf{I} + [\Delta\mathbf{w}]_\times \tag{56}$$

results in the following.

$$\begin{aligned}
\frac{\partial \mathbf{R}\mathbf{X}_t}{\partial \Delta\mathbf{w}} &= \lim_{\Delta\mathbf{w}\to 0} \frac{\exp([\Delta\mathbf{w}]_\times)\exp([\mathbf{w}]_\times)\mathbf{X}_t - \exp([\mathbf{w}]_\times)\mathbf{X}_t}{\Delta\mathbf{w}} \\
&\approx \lim_{\Delta\mathbf{w}\to 0} \frac{(\mathbf{I} + [\Delta\mathbf{w}]_\times)\exp([\mathbf{w}]_\times)\mathbf{X}_t - \exp([\mathbf{w}]_\times)\mathbf{X}_t}{\Delta\mathbf{w}} \\
&= \lim_{\Delta\mathbf{w}\to 0} \frac{[\Delta\mathbf{w}]_\times \mathbf{R}\mathbf{X}_t}{\Delta\mathbf{w}} \qquad (\because \exp([\mathbf{w}]_\times)\mathbf{X}_t = \mathbf{R}\mathbf{X}_t) \\
&= \lim_{\Delta\mathbf{w}\to 0} \frac{-[\mathbf{R}\mathbf{X}_t]_\times \Delta\mathbf{w}}{\Delta\mathbf{w}} \\
&= -[\mathbf{R}\mathbf{X}_t]_\times \\
&= -[\mathbf{X}']_\times
\end{aligned} \tag{57}$$

The second row in the above formula uses the approximation $\exp([\Delta\mathbf{w}]_\times) \approx \mathbf{I} + [\Delta\mathbf{w}]_\times$ for a small rotation matrix. **Therefore, using method [2], there is an advantage that the Jacobian can be simply calculated using the skew-symmetric matrix of the 3D point $\mathbf{X}'$. In the case of reprojection error optimization, since the error of feature points in sequentially incoming images is optimized, the camera pose changes are not large, and thus $\Delta\mathbf{w}$ is also not large, so the above Jacobian is commonly used.** Using method [2], the existing rotation matrix $\mathbf{R}$ is updated with a small increment $\Delta\mathbf{w}$ as in (43).

$$\mathbf{R} \leftarrow \Delta\mathbf{R}^*\mathbf{R} \quad \text{where, } \Delta\mathbf{R}^* = \exp([\Delta\mathbf{w}^*]_\times) \tag{58}$$

Therefore, the existing Jacobian changes from $\frac{\partial \mathbf{X}'}{\partial[\mathbf{w},\mathbf{t}]}$ to $\frac{\partial \mathbf{X}'}{\partial[\Delta\mathbf{w},\mathbf{t}]}$ and this is as follows.

$$\boxed{\frac{\partial}{\partial[\Delta\mathbf{w},\mathbf{t}]}\begin{bmatrix}\mathbf{R}\mathbf{X}+\mathbf{t}\\1\end{bmatrix} = \begin{bmatrix}0 & Z' & -Y' & 1 & 0 & 0\\ -Z' & 0 & X' & 0 & 1 & 0\\ Y' & -X' & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \in \mathbb{R}^{4\times 6}} \tag{59}$$

The final Jacobian of the pose $\mathbf{J}_c$ is as follows.

$$\boxed{\begin{aligned}
\mathbf{J}_c &= \frac{\partial \hat{\mathbf{p}}}{\partial \tilde{\mathbf{p}}}\frac{\partial \tilde{\mathbf{p}}}{\partial \mathbf{X}'}\frac{\partial \mathbf{X}'}{\partial[\Delta\mathbf{w},\mathbf{t}]} \\
&= \begin{bmatrix}f & 0 & c_x\\ 0 & f & c_y\end{bmatrix}\begin{bmatrix}\frac{1}{Z'} & 0 & \frac{-X'}{Z'^2} & 0\\ 0 & \frac{1}{Z'} & \frac{-Y'}{Z'^2} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & Z' & -Y' & 1 & 0 & 0\\ -Z' & 0 & X' & 0 & 1 & 0\\ Y' & -X' & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \\
&= \begin{bmatrix}-\frac{fX'Y'}{Z'^2} & \frac{f(1+X'^2)}{Z'^2} & -\frac{fY'}{Z'} & \frac{f}{Z'} & 0 & -\frac{fX'}{Z'^2}\\ -\frac{f(1+y^2)}{Z'^2} & \frac{fX'Y'}{Z'^2} & \frac{fX'}{Z'} & 0 & \frac{f}{Z'} & -\frac{fY'}{Z'^2}\end{bmatrix} \in \mathbb{R}^{2\times 6}
\end{aligned}} \tag{60}$$

3.2 Jacobian of Map Point

The Jacobian $\mathbf{J}_p$ of the 3D point $\mathbf{X}$ can be calculated as follows.

$$\begin{aligned}\mathbf{J}_p = \frac{\partial \mathbf{e}}{\partial \mathbf{X}} &= \frac{\partial}{\partial \mathbf{X}}(\mathbf{p} - \hat{\mathbf{p}}) \\ &= \frac{\partial}{\partial \mathbf{X}}\Big(\mathbf{p} - \pi_k(\pi_h(\mathbf{T}_i\mathbf{X}_j))\Big) \\ &= \frac{\partial}{\partial \mathbf{X}}\Big(- \pi_k(\pi_h(\mathbf{T}_i\mathbf{X}_j))\Big)\end{aligned} \tag{61}$$

Using the chain rule, the above formula is organized as follows.

$$\begin{aligned}\mathbf{J}_p &= \frac{\partial \hat{\mathbf{p}}}{\partial \tilde{\mathbf{p}}}\frac{\partial \tilde{\mathbf{p}}}{\partial \mathbf{X}'}\frac{\partial \mathbf{X}'}{\partial \mathbf{X}} \\ &= \mathbb{R}^{2\times 3}\cdot\mathbb{R}^{3\times 4}\cdot\mathbb{R}^{4\times 4} = \mathbb{R}^{2\times 4}\end{aligned} \tag{62}$$

Among these, $\frac{\partial \hat{\mathbf{p}}}{\partial \tilde{\mathbf{p}}}\frac{\partial \tilde{\mathbf{p}}}{\partial \mathbf{X}'}$ is the same as the Jacobian calculated earlier. Therefore, only $\frac{\partial \mathbf{X}'}{\partial \mathbf{X}}$ needs to be calculated.

$$\boxed{\begin{aligned}\frac{\partial \mathbf{X}'}{\partial \mathbf{X}} &= \frac{\partial}{\partial \mathbf{X}}\begin{bmatrix}\mathbf{R}\mathbf{X}+\mathbf{t}\\ 1\end{bmatrix} \\ &= \begin{bmatrix}\mathbf{R}\\ 0\end{bmatrix}\end{aligned}} \tag{63}$$

Therefore, $\mathbf{J}_p$ is as follows.

$$\boxed{\begin{aligned}\mathbf{J}_p &= \begin{bmatrix}f & 0 & c_x\\ 0 & f & c_y\end{bmatrix}\begin{bmatrix}\frac{1}{Z'} & 0 & \frac{-X'}{Z'^2} & 0\\ 0 & \frac{1}{Z'} & \frac{-Y'}{Z'^2} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{R}\\ 0\end{bmatrix} \\ &= \begin{bmatrix}\frac{f}{Z'} & 0 & -\frac{fX'}{Z'^2} & 0\\ 0 & \frac{f}{Z'} & -\frac{fY'}{Z'^2} & 0\end{bmatrix}\begin{bmatrix}\mathbf{R}\\ 0\end{bmatrix} \in \mathbb{R}^{2\times 4}\end{aligned}} \tag{64}$$

Typically, the last column of $\mathbf{J}_p$ is always 0, so it is often omitted and represented in non-homogeneous form.

$$\boxed{\mathbf{J}_p = \begin{bmatrix}\frac{f}{Z'} & 0 & -\frac{fX'}{Z'^2}\\ 0 & \frac{f}{Z'} & -\frac{fY'}{Z'^2}\end{bmatrix}\mathbf{R} \in \mathbb{R}^{2\times 3}} \tag{65}$$

3.3 Code Implementations

- g2o code: edge_project_xyz.cpp#L80
- g2o code: edge_project$_x yz.cpp\#L82$

4 Photometric Error

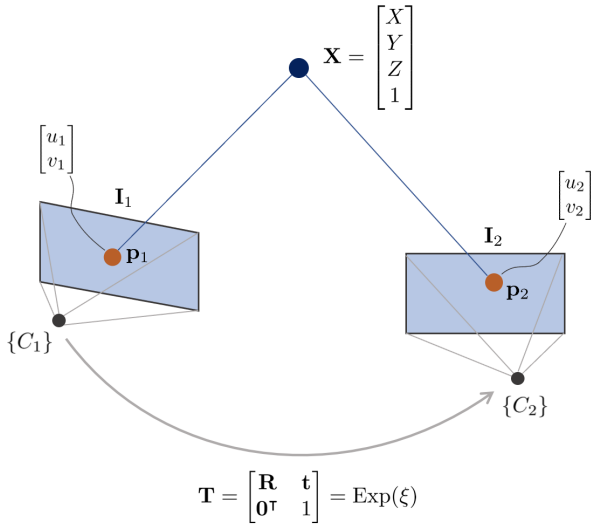
Photometric error is primarily used in direct Visual SLAM. It is commonly utilized in direct method-based visual odometry (VO) or bundle adjustment (BA). For more detailed information on the direct method, refer to the post at [SLAM] Optical Flow and Direct Method Concept and Code Review.

NOMENCLATURE of Photometric Error

- $\tilde{\mathbf{p}}_2 = \pi_h(\cdot) : \begin{bmatrix}X'\\ Y'\\ Z'\\ 1\end{bmatrix} \rightarrow \begin{bmatrix}X'/Z'\\ Y'/Z'\\ 1\end{bmatrix} = \begin{bmatrix}\tilde{u}_2\\ \tilde{v}_2\\ 1\end{bmatrix}$
  - The point $\mathbf{X}'$ in 3D space transformed to a non-homogeneous point on the image plane.
- $\mathbf{p}_2 = \pi_k(\cdot) = \tilde{\mathbf{K}}\tilde{\mathbf{p}}_2 = \begin{bmatrix}f & 0 & c_x\\ 0 & f & c_y\end{bmatrix}\begin{bmatrix}\tilde{u}_2\\ \tilde{v}_2\\ 1\end{bmatrix} = \begin{bmatrix}f\tilde{u}+c_x\\ f\tilde{v}+c_y\end{bmatrix} = \begin{bmatrix}u_2\\ v_2\end{bmatrix}$

– The point projected onto the image plane after correcting for lens distortion. If distortion correction has already been performed at the input stage, $\pi_k(\cdot) = \tilde{\mathbf{K}}(\cdot)$.

- $\mathbf{K} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}$: Camera's intrinsic parameters.
- $\tilde{\mathbf{K}} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \end{bmatrix}$: Omitting the last row of the intrinsic parameters for projection from $\mathbb{P}^2 \to \mathbb{R}^2$.
- $\mathcal{P}$: Set of all feature points in the image.
- $\mathbf{e}(\mathbf{T}) \to \mathbf{e}$: Generally abbreviated for simplicity.
- $\mathbf{p}_1^i, \mathbf{p}_2^i$: Pixel coordinates of the ith feature point in the first and second images.
- $\oplus$: Operator for combining two SE(3) groups (composition).
- $\mathbf{J} = \frac{\partial \mathbf{e}}{\partial \mathbf{T}} = \frac{\partial \mathbf{e}}{\partial [\mathbf{R}, \mathbf{t}]}$
- $\mathbf{X}' = [X, Y, Z, 1]^\intercal = [\tilde{\mathbf{X}}', 1]^\intercal = \mathbf{T}\mathbf{X}$
- $\mathbf{TX}$: Transformation, transforming the 3D point $\mathbf{X}$ into camera coordinates, $\left(\mathbf{TX} = \begin{bmatrix} \mathbf{R}\mathbf{X} + \mathbf{t} \\ 1 \end{bmatrix} \in \mathbb{R}^{4\times 1} \right)$
- $\mathbf{X}' = [X', Y', Z', 1]^\intercal = [\tilde{\mathbf{X}}', 1]^\intercal$
- $\xi = [\mathbf{w}, \mathbf{v}]^\intercal = [w_x, w_y, w_z, v_x, v_y, v_z]^\intercal$: Vector consisting of 3D angular velocity and velocity, called a twist.
- $[\xi]_\times = \begin{bmatrix} [\mathbf{w}]_\times & \mathbf{v} \\ \mathbf{0}^\intercal & 0 \end{bmatrix} \in \mathrm{se}(3)$: Lie algebra of the twist applied with the hat operator (4x4 matrix)
- $\mathcal{J}_l$: Jacobian for left multiplication. It is not used in actual calculations and hence not detailed here.



In the above figure, the world coordinates of the 3D point $\mathbf{X}$ are $[X, Y, Z, 1]^\intercal \in \mathbb{P}^3$, and the corresponding pixel coordinates on the two camera image planes are $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{P}^2$. Assuming the internal parameters $\mathbf{K}$ of the two cameras $\{C_1\}, \{C_2\}$ are the same. When camera $\{C_1\}$ is considered the origin ($\mathbf{R} = \mathbf{I}, \mathbf{t} = \mathbf{0}$), the pixel coordinates $\mathbf{p}_1, \mathbf{p}_2$ are projected through the 3D point $\mathbf{X}$ as follows:

$$\mathbf{p} = \pi(\mathbf{T}, \mathbf{X}) \tag{66}$$

3차원 점	$\mathbf{TX}$ Transformation	$\tilde{\mathbf{p}} = \pi_h(\cdot)$ Non-homogeneous	$\mathbf{p} = \pi_k(\cdot)$ Undistort	Project to Image Plane	2차원 이미지 픽셀
$\mathbf{X}$ $[X, Y, Z, 1]^\intercal$	$\mathbf{X}' = \begin{bmatrix} \mathbf{R}\mathbf{X} + \mathbf{t} \\ 1 \end{bmatrix} = [X', Y', Z', 1]^\intercal$	$\tilde{\mathbf{p}} = [X'/Z',\ Y'/Z',\ 1]^\intercal = [\tilde{u}, \tilde{v}, 1]^\intercal$	$\begin{cases} \bar{u} = \tilde{u}(1 + k_1 r^2 + k_2 r^4) \\ \bar{v} = \tilde{v}(1 + k_1 r^2 + k_2 r^4) \end{cases}$	$\begin{cases} u = f_x \bar{u} + c_x \\ v = f_y \bar{v} + c_y \end{cases}$	$\hat{\mathbf{p}}$ $[u, v]^\intercal$

$$\begin{aligned}\mathbf{p}_1 &= \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \pi(\mathbf{I}, \mathbf{X}) = \pi_k(\pi_h(\mathbf{X})) \\ \mathbf{p}_2 &= \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \pi(\mathbf{T}, \mathbf{X}) = \pi_k(\pi_h(\mathbf{T}\mathbf{X}))\end{aligned} \tag{67}$$

One characteristic of the direct method, unlike feature-based methods, is the absence of a way to determine which $\mathbf{p}_2$ matches $\mathbf{p}_1$. Therefore, the position of $\mathbf{p}_2$ is found based on the current pose estimate. Thus, the camera's pose is optimized to make $\mathbf{p}_2$ and $\mathbf{p}_1$ similar, and this problem is solved by minimizing the photometric error. The photometric error is as follows:

$$\boxed{\begin{aligned}\mathbf{e}(\mathbf{T}) &= \mathbf{I}_1(\mathbf{p}_1) - \mathbf{I}_2(\mathbf{p}_2) \\ &= \mathbf{I}_1\Big(\pi_k(\pi_h(\mathbf{X}))\Big) - \mathbf{I}_2\Big(\pi_k(\pi_h(\mathbf{T}\mathbf{X}))\Big)\end{aligned}} \tag{68}$$

Photometric error is based on the assumption of grayscale invariance and holds scalar values. The following error function $\mathbf{E}(\mathbf{T})$ can be defined to solve non-linear least squares:

$$\mathbf{E}(\mathbf{T}) = \sum_{i \in \mathcal{P}} \|\mathbf{e}_i\|^2 \tag{69}$$

$$\begin{aligned}\mathbf{T}^* &= \arg\min_{\mathbf{T}^*} \mathbf{E}(\mathbf{T}) \\ &= \arg\min_{\mathbf{T}^*} \sum_{i \in \mathcal{P}} \|\mathbf{e}_i\|^2 \\ &= \arg\min_{\mathbf{T}^*} \sum_{i \in \mathcal{P}} \mathbf{e}_i^\intercal \mathbf{e}_i \\ &= \arg\min_{\mathbf{T}^*} \sum_{iin\mathcal{P}} \Big(\mathbf{I}_1(\mathbf{p}_1^i) - \mathbf{I}_2(\mathbf{p}_2^i)\Big)^\intercal \Big(\mathbf{I}_1(\mathbf{p}_1^i) - \mathbf{I}_2(\mathbf{p}_2^i)\Big)\end{aligned} \tag{70}$$

$\mathbf{E}(\mathbf{T}^*)$ that satisfies $\|\mathbf{e}(\mathbf{T}^*)\|^2$ can be calculated iteratively through non-linear least squares. Small increments $\Delta\mathbf{T}$ are iteratively updated to $\mathbf{T}$ to find the optimal state.

$$\arg\min_{\mathbf{T}^*} \mathbf{E}(\mathbf{T} + \Delta\mathbf{T}) = \arg\min_{\mathbf{T}^*} \sum_{i \in \mathcal{P}} \|\mathbf{e}_i(\mathbf{T} + \Delta\mathbf{T})\|^2 \tag{71}$$

Technically, since the state increment $\Delta\mathbf{T}$ is a SE(3) transformation matrix, it should be added to the existing state $\mathbf{T}$ using the $\oplus$ operator, but the $+$ operator is used here for convenience of expression.

$$\mathbf{T} \oplus \Delta\mathbf{T} \quad \rightarrow \quad \mathbf{T} + \Delta\mathbf{T} \tag{72}$$

It is expressed through the first-order Taylor approximation as follows.

$$\begin{aligned}\mathbf{e}(\mathbf{T} + \Delta\mathbf{T}) &\approx \mathbf{e}_i(\mathbf{T}) + \mathbf{J}\Delta\mathbf{T} \\ &= \mathbf{e}_i(\mathbf{T}) + \frac{\partial \mathbf{e}}{\partial \mathbf{T}}\Delta\mathbf{T}\end{aligned} \tag{73}$$

$$\arg\min_{\mathbf{T}^*} \mathbf{E}(\mathbf{T} + \Delta\mathbf{T}) = \arg\min_{\mathbf{T}^*} \sum_{i \in \mathcal{P}} \|\mathbf{e}_i(\mathbf{T}) + \mathbf{J}\Delta\mathbf{T}\|^2 \tag{74}$$

When differentiating to find the optimal increment $\Delta\mathbf{T}^*$, the following results. The detailed derivation process is omitted in this section. If you want to know more about the derivation process, refer to the previous section here.

$$\begin{aligned}\mathbf{J}^\intercal\mathbf{J}\Delta\mathbf{T}^* &= -\mathbf{J}^\intercal\mathbf{e} \\ \mathbf{H}\Delta\mathbf{T}^* &= -\mathbf{b}\end{aligned} \tag{75}$$

Since the above formula forms a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, various linear algebra techniques such as schur complement, cholesky decomposition can be used to find $\Delta\mathbf{T}^*$. The optimal increment found in this way is then added to the current state. **Depending on whether it is multiplied to the right or left of the existing state T, it changes whether to update the pose in the local coordinate system (right) or the pose in the global coordinate system (left). Since photometric error updates the transformation matrix of the global coordinate system, the left multiplication method is generally used.**

$$\mathbf{T} \leftarrow \mathbf{T} \oplus \Delta\mathbf{T}^* \tag{76}$$

The definition of the left multiplication $\oplus$ operation is as follows.

$$\begin{aligned}\mathbf{T} \oplus \Delta\mathbf{T}^* &= \Delta\mathbf{T}^*\mathbf{T} \\ &= \exp([\Delta\xi^*]_\times)\mathbf{T} \quad \cdots \text{ globally updated (left mult)}\end{aligned} \tag{77}$$

4.1 Jacobian of the Photometric Error

To perform (75), the Jacobian $\mathbf{J}$ of the photometric error must be determined. It can be represented as follows.

$$\begin{aligned}\mathbf{J} &= \frac{\partial \mathbf{e}}{\partial \mathbf{T}} \\ &= \frac{\partial \mathbf{e}}{\partial [\mathbf{R}, \mathbf{t}]}\end{aligned} \tag{78}$$

Expanding this in detail results in the following.

$$\begin{aligned}\mathbf{J} = \frac{\partial \mathbf{e}}{\partial \mathbf{T}} &= \frac{\partial}{\partial \mathbf{T}} \Big(\mathbf{I}_1(\mathbf{p}_1) - \mathbf{I}_2(\mathbf{p}_2) \Big) \\ &= \frac{\partial}{\partial \mathbf{T}} \Big(\mathbf{I}_1 \Big(\pi_k(\pi_h(\mathbf{X})) \Big) - \mathbf{I}_2 \Big(\pi_k(\pi_h(\mathbf{TX})) \Big) \Big) \\ &= \frac{\partial}{\partial \mathbf{T}} \Big(- \mathbf{I}_2 \Big(\pi_k(\pi_h(\mathbf{TX})) \Big) \Big) \\ &= \frac{\partial}{\partial \mathbf{T}} \Big(- \mathbf{I}_2 \Big(\pi_k(\pi_h(\mathbf{X}')) \Big) \Big)\end{aligned} \tag{79}$$

Applying the chain rule re-expresses the above equation as follows.

$$\begin{aligned}\frac{\partial \mathbf{e}}{\partial \xi} &= \frac{\partial \mathbf{I}}{\partial \mathbf{p}_2} \frac{\partial \mathbf{p}_2}{\partial \tilde{\mathbf{p}}_2} \frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'} \frac{\partial \mathbf{X}'}{\partial \xi} \\ &= \mathbb{R}^{1\times 2} \cdot \mathbb{R}^{2\times 3} \cdot \mathbb{R}^{3\times 4} \cdot \mathbb{R}^{4\times 6} = \mathbb{R}^{1\times 6}\end{aligned} \tag{80}$$

The reason for computing the Jacobian $\frac{\partial \mathbf{X}'}{\partial \xi}$ instead of $\frac{\partial \mathbf{X}'}{\partial \mathbf{T}}$ will be explained in the next section.
First, $\frac{\partial \mathbf{I}}{\partial \mathbf{p}_2}$ refers to the gradient of the image.

$$\boxed{\begin{aligned}\frac{\partial \mathbf{I}}{\partial \mathbf{p}_2} &= \begin{bmatrix} \frac{\partial \mathbf{I}}{\partial u} & \frac{\partial \mathbf{I}}{\partial v} \end{bmatrix} \\ &= \begin{bmatrix} \nabla \mathbf{I}_u & \nabla \mathbf{I}_v \end{bmatrix}\end{aligned}} \tag{81}$$

If it is assumed that undistortion was already performed during image input, $\frac{\partial \mathbf{p}_2}{\partial \tilde{\mathbf{p}}_2}$ is as follows.

$$\boxed{\begin{aligned}\frac{\partial \mathbf{p}_2}{\partial \tilde{\mathbf{p}}_2} &= \frac{\partial}{\partial \tilde{\mathbf{p}}_2} \tilde{\mathbf{K}} \tilde{\mathbf{p}}_2 \\ &= \tilde{\mathbf{K}} \\ &= \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \end{bmatrix} \in \mathbb{R}^{2\times 3}\end{aligned}} \tag{82}$$

Next, $\frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'}$ is as follows.

$$\boxed{\begin{aligned}\frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'} &= \frac{\partial [\tilde{u}_2, \tilde{v}_2, 1]}{\partial [X', Y', Z', 1]} \\ &= \begin{bmatrix} \frac{1}{Z'} & 0 & \frac{-X'}{Z'^2} & 0 \\ 0 & \frac{1}{Z'} & \frac{-Y'}{Z'^2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 4}\end{aligned}} \tag{83}$$

4.1.1 Lie Theory-based SE(3) Optimization

Finally, $\frac{\partial \mathbf{X}'}{\partial \mathbf{T}} = \frac{\partial \mathbf{X}'}{\partial [\mathbf{R}, \mathbf{t}]}$ must be computed. At this time, the term related to position $\mathbf{t}$ is a 3D vector, and the size of this vector is the minimum degree of freedom, 3 degrees of freedom, for representing 3D position, so there is no separate constraint when performing optimization updates. **On the other hand, the rotation matrix R has 9 parameters, which is more than the minimum degrees of freedom, 3 degrees of freedom, for representing 3D rotation, so various constraints exist. This is called being over-parameterized. The disadvantages of over-parameterized representation are as follows.**

- It is necessary to calculate redundant parameters, which increases the computation during optimization.

- Additional degrees of freedom can cause numerical instability.
- It is necessary to check whether the constraints are satisfied each time the parameters are updated.

Therefore, the optimization method based on lie theory, which is free from constraints, is generally used. **The lie group SE(3) based optimization method refers to the method of updating SE(3) by finding the optimal twist $\Delta\xi^*$ after changing the term related to rotation from $\mathbf{R} \rightarrow \mathbf{w}$ and the term related to position from $\mathbf{t} \rightarrow \mathbf{v}$, and then updating SE(3) through exponential mapping of lie algebra se(3)** $[\Delta\xi]_\times$.

$$\Delta\mathbf{T}^* \rightarrow \Delta\xi^* \tag{84}$$

The Jacobian ξ is as follows.

$$\begin{aligned} \mathbf{J} = \frac{\partial \mathbf{e}}{\partial [\mathbf{R}, \mathbf{t}]} \quad &\rightarrow \frac{\partial \mathbf{e}}{\partial [\mathbf{w}, \mathbf{v}]} \\ &\rightarrow \frac{\partial \mathbf{e}}{\partial \xi} \end{aligned} \tag{85}$$

The existing equation is changed as follows through this.

$$\begin{aligned} \mathbf{e}(\mathbf{T}) &\rightarrow \mathbf{e}(\xi) \\ \mathbf{E}(\mathbf{T}) &\rightarrow \mathbf{E}(\xi) \\ \mathbf{e}(\mathbf{T}) + \mathbf{J}'\Delta\mathbf{T} &\rightarrow \mathbf{e}(\xi) + \mathbf{J}\Delta\xi \\ \mathbf{H}\Delta\mathbf{T}^* = -\mathbf{b} &\rightarrow \mathbf{H}\Delta\xi^* = -\mathbf{b} \\ \mathbf{T} \leftarrow \Delta\mathbf{T}^*\mathbf{T} &\rightarrow \mathbf{T} \leftarrow \exp([\Delta\xi^*]_\times)\mathbf{T} \end{aligned} \tag{86}$$

- $\mathbf{J}' = \frac{\partial \mathbf{e}}{\partial \mathbf{T}}$
- $\mathbf{J} = \frac{\partial \mathbf{e}}{\partial \xi}$

> Tip
>
> $\exp([\xi]_\times) \in \mathrm{SE}(3)$ refers to the operation of transforming the twist ξ through exponential mapping into a 3D pose. For more details on exponential mapping, refer to the related link.
>
> $$\exp([\Delta\xi]_\times) = \Delta\mathbf{T} \tag{87}$$

Until now, the Jacobians were easy to calculate, whereas $\frac{\partial \mathbf{X}'}{\partial \xi}$ requires changing $\mathbf{X}'$ into a term related to lie algebra as it is not immediately apparent from $\mathbf{X}'$ parameters ξ.

$$\mathbf{X}' \rightarrow \mathbf{T}\mathbf{X} \rightarrow \exp([\xi]_\times)\mathbf{X} \tag{88}$$

At this time, depending on the update method of the small lie algebra increment $\Delta\xi$ to the existing $\exp([\xi]_\times)$, it is divided into two methods. First, there is [1] the basic update method using lie algebra. Next, there is [2] the update method using the perturbation model.

$$\begin{aligned} &\exp([\xi]_\times) \leftarrow \exp([\xi + \Delta\xi]_\times) \quad \cdots [1] \\ &\exp([\xi]_\times) \leftarrow \exp([\Delta\xi]_\times)\exp([\xi]_\times) \quad \cdots [2] \end{aligned} \tag{89}$$

Among the two methods, method [1] is a method of adding a fine increment $\Delta\xi$ to the existing ξ and performing exponential mapping to obtain the Jacobian, while method [2] is a method of updating the existing state by multiplying the perturbation model $\exp([\Delta\xi]_\times)$ to the left of the existing ξ.

> Tip
>
> The following transformation exists between the two methods, known as the BCH approximation. For more details, refer to Introduction to Visual SLAM Chapter 4.
>
> $$\begin{aligned} \exp([\Delta\xi]_\times)\exp([\xi]_\times) &= \exp([\xi + \mathcal{J}_l^{-1}\Delta\xi]_\times) \\ \exp([\xi + \Delta\xi]_\times) &= \exp([\mathcal{J}_l\Delta\xi]_\times)\exp([\xi]_\times) \end{aligned} \tag{90}$$

Since a very complex equation is derived when using method [1], this method is not commonly used and method [2] of the perturbation model is mainly used. Therefore, $\frac{\partial \mathbf{X}'}{\partial \xi}$ is transformed as follows.

$$\frac{\partial \mathbf{X}'}{\partial \xi} \rightarrow \frac{\partial \mathbf{X}'}{\partial \Delta\xi} \tag{91}$$

The Jacobian for $\frac{\partial \mathbf{X}'}{\partial \Delta\xi}$ can be calculated as follows.

$$\begin{aligned}
\frac{\partial \mathbf{X}'}{\partial \Delta\xi} &= \lim_{\Delta\xi \to 0} \frac{\exp([\Delta\xi]_\times)\mathbf{X}' - \mathbf{X}'}{\Delta\xi} \\
&\approx \lim_{\Delta\xi \to 0} \frac{(\mathbf{I} + [\Delta\xi]_\times)\mathbf{X}' - \mathbf{X}'}{\Delta\xi} \\
&= \lim_{\Delta\xi \to 0} \frac{[\Delta\xi]_\times \mathbf{X}'}{\Delta\xi} \\
&= \lim_{\Delta\xi \to 0} \frac{\begin{bmatrix} [\Delta\mathbf{w}]_\times & \Delta\mathbf{v} \\ \mathbf{0}^\intercal & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{X}}' \\ 1 \end{bmatrix}}{\Delta\xi} \\
&= \lim_{\Delta\xi \to 0} \frac{\begin{bmatrix} [\Delta\mathbf{w}]_\times \tilde{\mathbf{X}}' + \Delta\mathbf{v} \\ \mathbf{0}^\intercal \end{bmatrix}}{[\Delta\mathbf{w}, \Delta\mathbf{v}]^\intercal} = \begin{bmatrix} -[\tilde{\mathbf{X}}']_\times & \mathbf{I} \\ \mathbf{0}^\intercal & \mathbf{0}^\intercal \end{bmatrix} \in \mathbb{R}^{4\times 6}
\end{aligned} \tag{92}$$

Therefore, using method [2] of the perturbation model has the advantage of simplifying the Jacobian calculation using the skew-symmetric matrix of the 3D point $\mathbf{X}'$. Since photometric error optimization generally involves optimizing the error in brightness changes in sequentially incoming images, the camera pose changes are not large, and thus $\Delta\xi$ is also not large, so the above Jacobian is commonly used. Method [2] of the perturbation model is used, so the small increment $\Delta\xi^*$ is updated as (77).

$$\mathbf{T} \leftarrow \Delta\mathbf{T}^*\mathbf{T} = \exp([\Delta\xi^*]_\times)\mathbf{T} \tag{93}$$

Tip

The second row of the above equation is a form where the first-order Taylor approximation is applied to a small twist increment $\exp([\Delta\xi]_\times)$. To understand the approximation in the second row, when an arbitrary twist $\xi = [\mathbf{w}, \mathbf{v}]^\intercal$ is given, the transformation matrix $\mathbf{T}$ can be expanded into an exponential mapping form as follows.

$$\begin{aligned}
\mathbf{T} = \exp([\xi]_\times) &= \mathbf{I} + \begin{bmatrix} [\mathbf{w}]_\times & \mathbf{v} \\ \mathbf{0}^\intercal & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} [\mathbf{w}]_\times^2 & [\mathbf{w}]_\times \mathbf{v} \\ \mathbf{0}^\intercal & 0 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} [\mathbf{w}]_\times^3 & [\mathbf{w}]_\times^2 \mathbf{v} \\ \mathbf{0}^\intercal & 0 \end{bmatrix} + \cdots \\
&= \mathbf{I} + [\xi]_\times + \frac{1}{2!}[\xi]_\times^2 + \frac{1}{3!}[\xi]_\times^3 + \cdots
\end{aligned} \tag{94}$$

For a small magnitude of twist increment $\Delta\xi$, higher-order terms can be ignored to approximately express it as follows.

$$\exp([\Delta\xi]_\times) \approx \mathbf{I} + [\Delta\xi]_\times \tag{95}$$

The final Jacobian $\mathbf{J}$ for the pose is as follows.

$$\begin{aligned}
\mathbf{J} = \frac{\partial \mathbf{e}}{\partial \Delta\xi} &= \frac{\partial \mathbf{I}}{\partial \mathbf{p}_2} \frac{\partial \mathbf{p}_2}{\partial \tilde{\mathbf{p}}_2} \frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'} \frac{\partial \mathbf{X}'}{\partial \Delta\xi} \\
&= \begin{bmatrix} \nabla\mathbf{I}_u & \nabla\mathbf{I}_v \end{bmatrix} \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \end{bmatrix} \begin{bmatrix} \frac{1}{Z'} & 0 & \frac{-X'}{Z'^2} & 0 \\ 0 & \frac{1}{Z'} & \frac{-Y'}{Z'^2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -[\tilde{\mathbf{X}}']_\times & \mathbf{I} \\ \mathbf{0}^\intercal & \mathbf{0}^\intercal \end{bmatrix} \\
&= \begin{bmatrix} \nabla\mathbf{I}_u & \nabla\mathbf{I}_v \end{bmatrix} \begin{bmatrix} -\frac{fX'Y'}{Z'^2} & \frac{f(1+X'^2)}{Z'^2} & -\frac{fY'}{Z'} & \frac{f}{Z'} & 0 & -\frac{fX'}{Z'^2} \\ -\frac{f(1+Y'^2)}{Z'^2} & \frac{fX'Y'}{Z'^2} & \frac{fX'}{Z'} & 0 & \frac{f}{Z'} & -\frac{fY'}{Z'^2} \end{bmatrix} \in \mathbb{R}^{1\times 6}
\end{aligned} \tag{96}$$

Since the last row of $\frac{\partial \mathbf{X}'}{\partial \Delta\xi}$ is always 0, it is often omitted and calculated.

4.2 Code Implementations

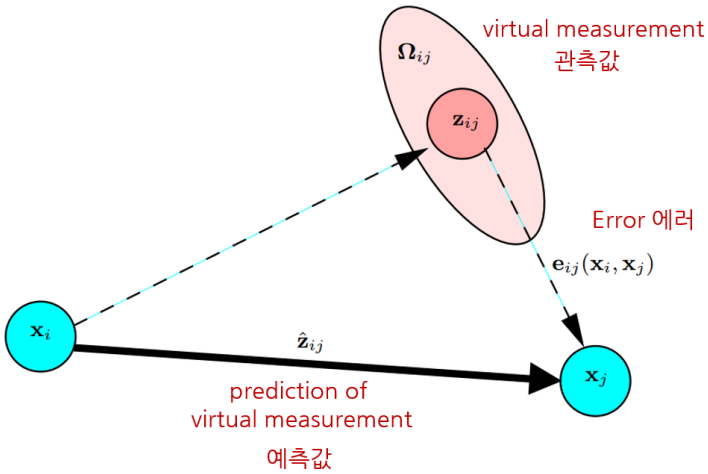
- Introduction to Visual SLAM Chapter 8 code: direct_sparse.cpp#L111
- DSO code: CoarseInitializer.cpp#L430
- DSO code2: CoarseTracker.cpp#L320

5 Relative pose error

Relative pose error is commonly used in pose graph optimization (PGO). For more information about PGO, refer to the post [SLAM] Conceptual explanation and example code analysis of Pose Graph Optimization.

NOMENCLATURE of relative pose error

- (Node) $\mathbf{x}_i = \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0}^\intercal & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$
- (Edge) $\mathbf{z}_{ij} = \begin{bmatrix} \mathbf{R}_{ij} & \mathbf{t}_{ij} \\ \mathbf{0}^\intercal & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$
- $\hat{\mathbf{z}}_{ij} = \mathbf{x}_i^{-1}\mathbf{x}_j$: Predicted value
- $\mathbf{z}_{ij}$: Observed value (virtual measurement)
- $\mathbf{x} = [\mathbf{x}_1, \cdots, \mathbf{x}_n]$: All pose nodes in the pose graph
- $\mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) \leftrightarrow \mathbf{e}_{ij}$: Notation is simplified for convenience.
- $\mathbf{J} = \frac{\partial \mathbf{e}}{\partial \mathbf{x}}$
- $\oplus$: Operator that combines two SE(3) groups (composition)
- Log$(\cdot)$: Operator that transforms SE(3) into a twist $\xi \in \mathbb{R}^6$. For detailed information about Logarithm mapping, refer to this post.



When two nodes $\mathbf{x}_i, \mathbf{x}_j$ are given on the pose graph, the difference between the newly calculated relative pose (observed value) $\mathbf{z}_{ij}$ and the known relative pose (predicted value) $\hat{\mathbf{z}}_{ij}$ is defined as the relative pose error. (Refer to the figure from Freiburg Univ. Robot Mapping Course).

$$\mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{z}_{ij}^{-1}\hat{\mathbf{z}}_{ij} = \mathbf{z}_{ij}^{-1}\mathbf{x}_i^{-1}\mathbf{x}_j \tag{97}$$

The process of optimizing the relative pose error is called pose graph optimization (PGO), and it is also known as the back-end algorithm of graph-based SLAM. The nodes $\mathbf{x}_i, \mathbf{x}_{i+1}, \cdots$, sequentially calculated by the front-end visual odometry (VO) or lidar odometry (LO), do not undergo PGO because the observed and predicted values are the same. However, when loop closing occurs and a non-sequential edge connects two nodes $\mathbf{x}_i, \mathbf{x}_j$, a difference between the observed and predicted values arises, leading to the execution of PGO.

In other words, PGO is typically performed when special situations like loop closing occur. When the robot revisits the same location while moving, a loop detection algorithm operates to determine the loop. At this time, if a loop is detected, the existing node $\mathbf{x}_i$ and the node $\mathbf{x}_j$ created by revisiting are connected by a loop edge, and an observed value is produced by various matching algorithms (GICP, NDT, etc...). **Such observed values, not actually observed but created by matching algorithms, are called virtual measurements.**

The relative pose error for all nodes on the pose graph can be defined as follows.

$$\mathbf{E}(\mathbf{x}) = \sum_i \sum_j \|\mathbf{e}_{ij}\|^2 \tag{98}$$

$$\begin{aligned} \mathbf{x}^* &= \arg\min_{\mathbf{x}^*} \mathbf{E}(\mathbf{x}) \\ &= \arg\min_{\mathbf{x}^*} \sum_i \sum_j \|\mathbf{e}_{ij}\|^2 \\ &= \arg\min_{\mathbf{x}^*} \sum_i \sum_j \mathbf{e}_{ij}^\intercal \mathbf{e}_{ij} \end{aligned} \tag{99}$$

$\mathbf{E}(\mathbf{x}^*)$ can be calculated iteratively through non-linear least squares by updating a small increment $\Delta\mathbf{x}$ to $\mathbf{x}$ repeatedly to find the optimal state.

$$\arg\min_{\mathbf{x}^*} \mathbf{E}(\mathbf{x} + \Delta\mathbf{x}) = \arg\min_{\mathbf{x}^*} \sum_i \sum_j \|\mathbf{e}_{ij}(\mathbf{x}_i + \Delta\mathbf{x}_i, \mathbf{x}_j + \Delta\mathbf{x}_j)\|^2 \tag{100}$$

Technically speaking, since the state increment $\Delta\mathbf{x}$ is an SE(3) transformation matrix, it should be added to the existing state $\mathbf{x}$ through the $\oplus$ operator, but for convenience of expression, the $+$ operator is used.

$$\mathbf{e}_{ij}(\mathbf{x}_i \oplus \Delta\mathbf{x}_i, \mathbf{x}_j \oplus \Delta\mathbf{x}_j) \quad \rightarrow \quad \mathbf{e}_{ij}(\mathbf{x}_i + \Delta\mathbf{x}_i, \mathbf{x}_j + \Delta\mathbf{x}_j) \tag{101}$$

This equation can be expressed through a first-order Taylor approximation as follows.

$$\begin{aligned} \mathbf{e}_{ij}(\mathbf{x}_i + \Delta\mathbf{x}_i, \mathbf{x}_j + \Delta\mathbf{x}_j) &\approx \mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) + \mathbf{J}_{ij} \begin{bmatrix} \Delta\mathbf{x}_i \\ \Delta\mathbf{x}_j \end{bmatrix} \\ &= \mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) + \mathbf{J}_i\Delta\mathbf{x}_i + \mathbf{J}_j\Delta\mathbf{x}_j \\ &= \mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) + \frac{\partial \mathbf{e}_{ij}}{\partial \mathbf{x}_i}\Delta\mathbf{x}_i + + \frac{\partial \mathbf{e}_{ij}}{\partial \mathbf{x}_j}\Delta\mathbf{x}_j \end{aligned} \tag{102}$$

$$\arg\min_{\mathbf{x}^*} \mathbf{E}(\mathbf{x} + \Delta\mathbf{x}) \approx \arg\min_{\mathbf{x}^*} \sum_i \sum_j \left\| \mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) + \mathbf{J}_{ij} \begin{bmatrix} \Delta\mathbf{x}_i \\ \Delta\mathbf{x}_j \end{bmatrix} \right\|^2 \tag{103}$$

Differentiating this to find the optimal increment $\Delta\mathbf{x}^*$ for all nodes results in the following. The derivation process is omitted in this section. If you want to know the detailed derivation process, refer to the previous section.

$$\begin{aligned} \mathbf{J}^\intercal\mathbf{J}\Delta\mathbf{x}^* &= -\mathbf{J}^\intercal\mathbf{e} \\ \mathbf{H}\Delta\mathbf{x}^* &= -\mathbf{b} \end{aligned} \tag{104}$$

This equation forms a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, and the optimal increment $\Delta\mathbf{x}^*$ can be found using various linear algebra techniques such as the schur complement and Cholesky decomposition. The obtained optimal increment is then added to the current state. Depending on whether it is multiplied on the right or the left of the existing state $\mathbf{x}$, it updates the pose viewed from the local coordinate system (right) or the global coordinate system (left). Since the relative pose error is related to the relative pose of the two nodes, right multiplication, which updates in the local coordinate system, is applied.

$$\mathbf{x} \leftarrow \mathbf{x} \oplus \Delta\mathbf{x}^* \tag{105}$$

The definition of the right multiplication $\oplus$ operation is as follows.

$$\begin{aligned} \mathbf{x} \oplus \Delta\mathbf{x}^* &= \mathbf{x}\Delta\mathbf{x}^* \\ &= \mathbf{x}\exp([\Delta\xi^*]_\times) \quad \cdots \text{ locally updated (right mult)} \end{aligned} \tag{106}$$

5.1 Jacobian of relative pose error

To perform (104), it is necessary to compute the Jacobian $\mathbf{J}$ of the relative pose error. For the given non-sequential nodes $\mathbf{x}_i, \mathbf{x}_j$, their Jacobian $\mathbf{J}_{ij}$ can be expressed as follows.

$$\begin{aligned}\mathbf{J}_{ij} &= \frac{\partial \mathbf{e}_{ij}}{\partial \mathbf{x}_{ij}} \\ &= \frac{\partial \mathbf{e}_{ij}}{\partial [\mathbf{x}_i, \mathbf{x}_j]} \\ &= [\mathbf{J}_i, \mathbf{J}_j]\end{aligned} \tag{107}$$

If we elaborate on this, it looks like the following.

$$\begin{aligned}\mathbf{J}_{ij} = \frac{\partial \mathbf{e}_{ij}}{\partial [\mathbf{x}_i, \mathbf{x}_j]} &= \frac{\partial}{\partial [\mathbf{x}_i, \mathbf{x}_j]} \left(\mathbf{z}_{ij}^{-1} \hat{\mathbf{z}}_{ij} \right) \\ &= \frac{\partial}{\partial [\mathbf{R}_i, \mathbf{t}_i, \mathbf{R}_j, \mathbf{t}_j]} \left(\mathbf{z}_{ij}^{-1} \hat{\mathbf{z}}_{ij} \right)\end{aligned} \tag{108}$$

5.1.1 Lie theory-based SE(3) optimization

When calculating the above Jacobian, since the term $\mathbf{t}$ related to the position is a 3-dimensional vector and the size of this vector is the minimum degrees of freedom to represent 3-dimensional position, which is 3 degrees of freedom, there is no separate constraint when performing optimization updates. **However, the rotation matrix R has 9 parameters, which is more than the minimum degrees of freedom to represent 3-dimensional rotation, which is 3 degrees of freedom, thus various constraints exist. This is referred to as being over-parameterized. The disadvantages of an over-parameterized representation are as follows.**

- Because redundant parameters must be calculated, the computational load increases during optimization.
- Additional degrees of freedom can cause numerical instability issues.
- Parameters must always be checked to satisfy constraints whenever they are updated.

Therefore, a minimal parameter representation free from constraints, a Lie theory-based optimization method, is generally used. **The Lie group SE(3) based optimization method refers to the method of updating SE(3) by calculating the optimal twist $\Delta\xi^*$ by changing the term related to rotation from $\mathbf{R} \rightarrow \mathbf{w}$ and the term related to position from $\mathbf{t} \rightarrow \mathbf{v}$, and then using exponential mapping of the lie algebra se(3) $[\Delta\xi]_\times$.**

$$\left[\Delta \mathbf{x}_i^*, \Delta \mathbf{x}_j^*\right] \rightarrow \left[\Delta \xi_i^*, \Delta \xi_j^*\right] \tag{109}$$

The Jacobian for ξ is as follows.

$$\mathbf{J}_{ij} = \frac{\partial \mathbf{e}_{ij}}{\partial [\mathbf{x}_i, \mathbf{x}_j]} \quad \rightarrow \frac{\partial \mathbf{e}_{ij}}{\partial [\xi_i, \xi_j]} \tag{110}$$

This changes the existing formula as follows.

$$\begin{aligned}
\mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) &\rightarrow \mathbf{e}_{ij}(\xi_i, \xi_j) \\
\mathbf{E}(\mathbf{x}) &\rightarrow \mathbf{E}(\xi) \\
\mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) + \mathbf{J}_i' \Delta \mathbf{x}_i + \mathbf{J}_j' \Delta \mathbf{x}_j &\rightarrow \mathbf{e}_{ij}(\xi_i, \xi_j) + \mathbf{J}_i \Delta \xi_i + \mathbf{J}_j \Delta \xi_j \\
\mathbf{H} \Delta \mathbf{x}^* = -\mathbf{b} &\rightarrow \mathbf{H} \Delta \xi^* = -\mathbf{b} \\
\mathbf{x} \leftarrow \Delta \mathbf{x}^* \mathbf{x} &\rightarrow \mathbf{x} \leftarrow \exp([\Delta \xi^*]_\times) \mathbf{x}
\end{aligned} \tag{111}$$

- $\mathbf{J}_{ij}' = \frac{\partial \mathbf{e}}{\partial [\mathbf{x}_i, \mathbf{x}_j]}$
- $\mathbf{J}_{ij} = \frac{\partial \mathbf{e}}{\partial [\xi_i, \xi_j]}$

$\frac{\partial}{\partial \xi}(\mathbf{z}_{ij}^{-1} \hat{\mathbf{z}}_{ij})$ does not directly appear in the parameters ξ from $\mathbf{z}_{ij}^{-1} \hat{\mathbf{z}}_{ij}$, so it needs to be changed into a term related to lie algebra.

$$\mathbf{z}_{ij}^{-1} \hat{\mathbf{z}}_{ij} \rightarrow \mathrm{Log}(\mathbf{z}_{ij}^{-1} \hat{\mathbf{z}}_{ij}) \tag{113}$$

At this time, $\mathrm{Log}(\cdot)$ means logarithm mapping that changes SE(3) into twist $\xi \in \mathbb{R}^6$. For detailed information about Logarithm mapping, refer to this post. Therefore, the SE(3) version of the relative pose error $\mathbf{e}_{ij}$ is changed as follows.

$$\boxed{\mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{z}_{ij}^{-1} \hat{\mathbf{z}}_{ij} \quad \rightarrow \quad \mathbf{e}_{ij}(\xi_i, \xi_j) = \mathrm{Log}(\mathbf{z}_{ij}^{-1} \hat{\mathbf{z}}_{ij})} \tag{114}$$

Tip

$\exp([\xi]_\times) \in \mathrm{SE}(3)$ refers to the operation of transforming the twist ξ through exponential mapping into a 3-dimensional pose. For detailed information about exponential mapping, refer to this link.

$$\exp([\Delta\xi]_\times) = \Delta\mathbf{x} \tag{112}$$

This is elaborated as follows.

$$\begin{aligned}\mathbf{e}_{ij}(\xi_i, \xi_j) &= \mathrm{Log}(\mathbf{z}_{ij}^{-1}\hat{\mathbf{z}}_{ij}) \\ &= \mathrm{Log}(\mathbf{z}_{ij}^{-1}\mathbf{x}_i^{-1}\mathbf{x}_j) \\ &= \mathrm{Log}(\exp([-\xi_{ij}]_\times)\exp([-\xi_i]_\times)\exp([\xi_j]_\times))\end{aligned} \tag{115}$$

From this equation, we can see that the parameters ξ_i, ξ_j in $\mathbf{z}_{ij}$ are connected through exponential mapping. If we apply the left perturbation model to the second line of the formula and express the increment, it looks like this.

$$\mathbf{e}_{ij}(\xi_i + \Delta\xi_i, \xi_j + \Delta\xi_j) = \mathrm{Log}(\hat{\mathbf{z}}_{ij}^{-1}\mathbf{x}_i^{-1}\exp(-[\Delta\xi_i]_\times)\exp([\Delta\xi_j]_\times)\mathbf{x}_j) \tag{116}$$

Tip

To arrange the term in the form $\mathbf{e} + \mathbf{J}\Delta\xi$ by moving the incremental term to the left or right, the following property of the adjoint matrix of SE(3) must be used. For more information about the adjoint matrix, refer to this post.

$$\exp([\mathrm{Ad}_\mathbf{T} \cdot \xi]_\times) = \mathbf{T} \cdot \exp([\xi]_\times) \cdot \mathbf{T}^{-1} \tag{117}$$

Transforming the above formula for $\mathbf{T} \to \mathbf{T}^{-1}$, we get the following.

$$\exp([\mathrm{Ad}_{\mathbf{T}^{-1}} \cdot \xi]_\times) = \mathbf{T}^{-1} \cdot \exp([\xi]_\times) \cdot \mathbf{T} \tag{118}$$

And simplifying gives the following formula.

$$\exp([\xi]_\times) \cdot \mathbf{T} = \mathbf{T}\exp([\mathrm{Ad}_{\mathbf{T}^{-1}} \cdot \xi]_\times) \tag{119}$$

Using (119), it is possible to move the $\exp(\cdot)\exp(\cdot)$ term in the middle of (116) to the right or left. This post describes the process of moving it to the right. This is expanded for each $\Delta\xi_i, \Delta\xi_j$ as follows.

$$\begin{aligned}\mathbf{e}_{ij}(\xi_i + \Delta\xi_i, \xi_j) &= \mathrm{Log}(\hat{\mathbf{z}}_{ij}^{-1}\mathbf{x}_i^{-1}\exp(-[\Delta\xi_i]_\times)\mathbf{x}_j) \\ &= \mathrm{Log}(\mathbf{z}_{ij}^{-1}\mathbf{x}_i^{-1}\mathbf{x}_j\exp([-\mathrm{Ad}_{\mathbf{x}_j^{-1}}\Delta\xi_i]_\times)) \quad \cdots [1] \\ \mathbf{e}_{ij}(\xi_i, \xi_j + \Delta\xi_j) &= \mathrm{Log}(\hat{\mathbf{z}}_{ij}^{-1}\mathbf{x}_i^{-1}\exp([\Delta\xi_j]_\times)\mathbf{x}_j) \\ &= \mathrm{Log}(\mathbf{z}_{ij}^{-1}\mathbf{x}_i^{-1}\mathbf{x}_j\exp([\mathrm{Ad}_{\mathbf{x}_j^{-1}}\Delta\xi_j]_\times)) \quad \cdots [2]\end{aligned} \tag{120}$$

To express this simply using substitution, [1], [2] are as follows.

$$\begin{aligned}&\mathrm{Log}(\exp([\mathbf{a}]_\times)\exp([\mathbf{b}]_\times)) \quad \cdots [1] \\ &\mathrm{Log}(\exp([\mathbf{a}]_\times)\exp([\mathbf{c}]_\times)) \quad \cdots [2]\end{aligned} \tag{121}$$

- $\exp([\mathbf{a}]_\times) = \mathbf{z}_{ij}^{-1}\mathbf{x}_i^{-1}\mathbf{x}_j$: Transformation matrix expressed as an exponential term. According to (114), $\mathbf{a} = \mathbf{e}_{ij}(\xi_i, \xi_j)$.
- $\mathbf{b} = -\mathrm{Ad}_{\mathbf{x}_j^{-1}}\Delta\xi_i$
- $\mathbf{c} = \mathrm{Ad}_{\mathbf{x}_j^{-1}}\Delta\xi_j$

This formula can be organized using the right BCH approximation.
Using the BCH approximation, (121) is organized as follows.

$$\begin{aligned}\mathrm{Log}(\exp([\mathbf{a}]_\times)\exp([\mathbf{b}]_\times)) &= \mathrm{Log}(\exp([\mathbf{a} + \mathcal{J}_r^{-1}\mathbf{b}]_\times)) \\ &= \mathbf{a} + \mathcal{J}_r^{-1}\mathbf{b} \quad \cdots [1] \\ \mathrm{Log}(\exp([\mathbf{a}]_\times)\exp([\mathbf{c}]_\times)) &= \mathrm{Log}(\exp([\mathbf{a} + \mathcal{J}_r^{-1}\mathbf{c}]_\times)) \\ &= \mathbf{a} + \mathcal{J}_r^{-1}\mathbf{c} \quad \cdots [2]\end{aligned} \tag{123}$$

Tip

The right BCH approximation is as follows.

$$\begin{aligned}\exp([\xi]_\times)\exp([\Delta\xi]_\times) &= \exp([\xi + \mathcal{J}_r^{-1}\Delta\xi]_\times) \\ \exp([\xi + Delta\xi]_\times) &= \exp([\xi]_\times)\exp([\mathcal{J}_r\Delta\xi]_\times)\end{aligned} \tag{122}$$

For detailed information, refer to Introduction to Visual SLAM Chapter 4.

Finally, undoing the substitution and combining the $\Delta\xi_i, \Delta\xi_j$ formulas gives the SE(3) version of the formula in (102).

$$\boxed{\begin{aligned}\mathbf{e}_{ij}(\xi_i + \Delta\xi_i, \xi_j + \Delta\xi_j) &= \mathbf{a} + \mathcal{J}_r^{-1}\mathbf{b} + \mathcal{J}_r^{-1}\mathbf{c} \\ &= \mathbf{e}_{ij}(\xi_i, \xi_j) - \mathcal{J}_r^{-1}\mathrm{Ad}_{\mathbf{x}_j^{-1}}\Delta\xi_i + \mathcal{J}_r^{-1}\mathrm{Ad}_{\mathbf{x}_j^{-1}}\Delta\xi_j \\ &= \mathbf{e}_{ij}(\xi_i, \xi_j) + \frac{\partial \mathbf{e}_{ij}}{\partial\Delta\xi_i}\Delta\xi_i + \frac{\partial \mathbf{e}_{ij}}{\partial\Delta\xi_j}\Delta\xi_j\end{aligned}} \tag{124}$$

Therefore, the final relative pose error Jacobian for SE(3) is as follows.

$$\boxed{\begin{aligned}\frac{\partial \mathbf{e}_{ij}}{\partial\Delta\xi_i} &= -\mathcal{J}_r^{-1}\mathrm{Ad}_{\mathbf{x}_j^{-1}} \in \mathbb{R}^{6\times 6} \\ \frac{\partial \mathbf{e}_{ij}}{\partial\Delta\xi_j} &= \mathcal{J}_r^{-1}\mathrm{Ad}_{\mathbf{x}_j^{-1}} \in \mathbb{R}^{6\times 6}\end{aligned}} \tag{125}$$

At this time, $\mathcal{J}_r^{-1}$ is generally approximated as follows or used by setting it as $\mathbf{I}_6$.

$$\mathcal{J}_r^{-1} \approx \mathbf{I}_6 + \frac{1}{2}\begin{bmatrix}[\mathbf{w}]_\times & [\mathbf{v}]_\times \\ \mathbf{0} & [\mathbf{w}]_\times\end{bmatrix} \in \mathbb{R}^{6\times 6} \tag{126}$$

If $\mathcal{J}_r^{-1} = \mathbf{I}_6$ is assumed and optimization is performed, there is a reduction in computational load, but the optimization performance is slightly superior when using the approximated Jacobian as above. For detailed information, refer to Introduction to Visual SLAM Chapter 11.

5.2 Code implementations

- g2o code: edge_se3_expmap.cpp#L55
  - In the above g2o code, the error is defined as $\mathbf{e}_{ij} = \mathbf{x}_j^{-1}\mathbf{z}_{ij}\mathbf{x}_i$, so the Jacobian is slightly different from the explanation above.
  - $\frac{\partial \mathbf{e}_{ij}}{\partial\Delta\xi_i} = \mathcal{J}_l^{-1}\mathrm{Ad}_{\mathbf{x}_j^{-1}\mathbf{z}_{ij}}$
  - $\frac{\partial \mathbf{e}_{ij}}{\partial\Delta\xi_j} = -\mathcal{J}_r^{-1}\mathrm{Ad}_{\mathbf{x}_i^{-1}\mathbf{z}_{ij}^{-1}}$
  - This follows the same form as combining after arranging the $\Delta\xi_i$ to the left and $\Delta\xi_j$ to the right in (120).
  - It also appears that $\mathcal{J}_l^{-1} \approx \mathbf{I}_6, \mathcal{J}_r^{-1} \approx \mathbf{I}_6$ is approximated. Thus, the actual implemented code is as follows.
    - $\frac{\partial \mathbf{e}_{ij}}{\partial\Delta\xi_i} \approx \mathrm{Ad}_{\mathbf{x}_j^{-1}\mathbf{z}_{ij}}$
    - $\frac{\partial \mathbf{e}_{ij}}{\partial\Delta\xi_j} \approx -\mathrm{Ad}_{\mathbf{x}_i^{-1}\mathbf{z}_{ij}^{-1}}$

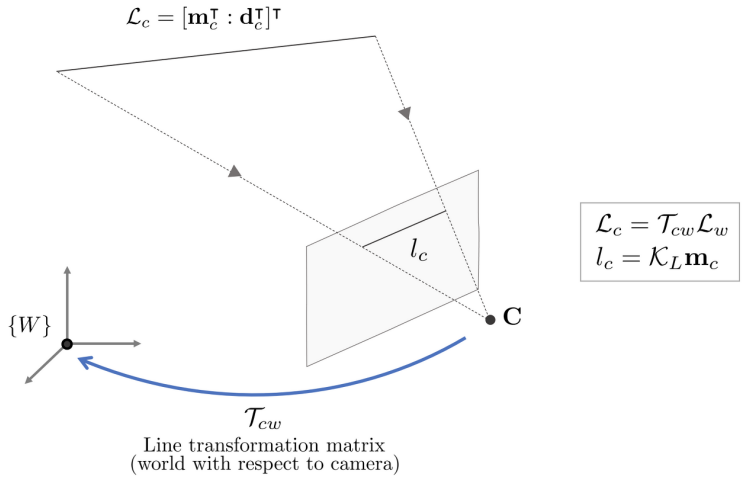
6 Line Reprojection Error

Line reprojection error is used to optimize a 3D line expressed in Plücker coordinates. For more details on Plücker coordinates, refer to the post Plücker Coordinate Concept Summary.

NOMENCLATURE of line reprojection error

- $\mathcal{T}_{cw} \in \mathbb{R}^{6\times 6}$: Transformation matrix for the Plücker line

- $\mathcal{K}_L$: Internal parameter matrix for the line (line intrinsic matrix)
- $\mathbf{U} \in SO(3)$: Rotation matrix for the 3D line
- $\mathbf{W} \in SO(2)$: Matrix containing distance information of the 3D line from the origin
- $\boldsymbol{\theta} \in \mathbb{R}^3$: Parameters corresponding to the SO(3) rotation matrix
- $\theta \in \mathbb{R}$: Parameter corresponding to the SO(2) rotation matrix
- $\mathbf{u}_i$: ith column vector
- $\mathcal{X} = [\delta_{\boldsymbol{\theta}}, \delta_\xi]$: State variable
- $\delta_{\boldsymbol{\theta}} = [\boldsymbol{\theta}^\intercal, \theta] \in \mathbb{R}^4$: State variable in orthonormal representation
- $\delta_\xi = [\delta\xi] \in se(3)$: Update method through Lie theory, refer to this link
- $\oplus$: Operator to update the state variables $\delta_{\boldsymbol{\theta}}, \delta_\xi$ at once.
- $\mathbf{J} = \frac{\partial \mathbf{e}_l}{\partial \mathcal{X}} = \frac{\partial \mathbf{e}_l}{\partial [\delta_{\boldsymbol{\theta}}, \delta_\xi]}$



A line in 3D space can be expressed as a 6-dimensional column vector using Plücker Coordinates.

$$\mathcal{L} = [\mathbf{m}^\intercal : \mathbf{d}^\intercal]^\intercal = [m_x : m_y : m_z : d_x : d_y : d_z]^\intercal \tag{127}$$

The order in papers using Plücker Coordinates is mostly $[\mathbf{m} : \mathbf{d}]$, hence this section uses this order to represent the line. This line representation has scale ambiguity (up to scale), so it has 5 degrees of freedom; $\mathbf{m}, \mathbf{d}$ do not need to be unit vectors, and the line can be uniquely represented by the ratio of the two vector values.

6.1 Line Transformation and Projection

If we refer to a line in the world coordinate system as $\mathcal{L}_w$, then its transformation to the camera coordinate system can be expressed as follows:

$$\mathcal{L}_c = \begin{bmatrix} \mathbf{m}_c \\ \mathbf{d}_c \end{bmatrix} = \mathcal{T}_{cw}\mathcal{L}_w = \begin{bmatrix} \mathbf{R}_{cw} & \mathbf{t}^\wedge\mathbf{R}_{cw} \\ 0 & \mathbf{R}_{cw} \end{bmatrix} \begin{bmatrix} \mathbf{m}_w \\ \mathbf{d}_w \end{bmatrix} \tag{128}$$

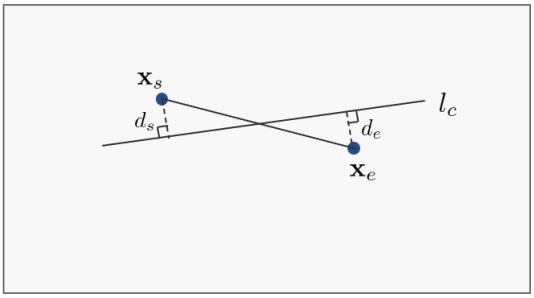
The projection of this line onto the image plane is as follows:

$$l_c = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \mathcal{K}_L\mathbf{m}_c = \begin{bmatrix} f_y & & \\ & f_x & \\ -f_y c_x & -f_x c_y & f_x f_y \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \tag{129}$$

$\mathcal{K}_L$ means $\mathcal{P} = [\det(\mathbf{N})\mathbf{N}^{-\intercal}|\mathbf{n}^\wedge\mathbf{N}]$ where $\mathbf{P} = K[\mathbf{I}|\mathbf{0}]$. Thus, $\mathcal{P} = [\det(\mathbf{K})\mathbf{K}^{-\intercal}|\mathbf{0}]$, so the $\mathbf{d}$ term of $\mathcal{L}$ is eliminated. Therefore, when $\mathbf{K} = \begin{bmatrix} f_x & & c_x \\ & f_y & c_y \\ & & 1 \end{bmatrix}$, the following equation is derived:

$$\mathcal{K}_L = \det(\mathbf{K})\mathbf{K}^{-\intercal} = \begin{bmatrix} f_y & & \\ & f_x & \\ -f_y c_x & -f_x c_y & f_x f_y \end{bmatrix} \in \mathbb{R}^{3\times 3} \tag{130}$$

6.2 Line Reprojection Error



$l_c = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^\intercal$

$\mathbf{x}_s, \mathbf{x}_e$ from LSD

$$\mathbf{e}_l = \left[\frac{\mathbf{x}_s^\intercal l_c}{\sqrt{l_1^2+l_2^2}}, \quad \frac{\mathbf{x}_e^\intercal l_c}{\sqrt{l_1^2+l_2^2}} \right]$$

The reprojection error $\mathbf{e}_l$ of the line can be expressed as follows:

$$\mathbf{e}_l = \begin{bmatrix} d_s, & d_e \end{bmatrix} = \left[\frac{\mathbf{x}_s^\intercal l_c}{\sqrt{l_1^2+l_2^2}}, \quad \frac{\mathbf{x}_e^\intercal l_c}{\sqrt{l_1^2+l_2^2}} \right] \in \mathbb{R}^2 \tag{131}$$

This can be expressed using the distance from a point to a line formula. Here, $\{\mathbf{x}_s, \mathbf{x}_e\}$ represent the starting and ending points of the line extracted using a line feature extractor (e.g., LSD). **In other words, l_c is the predicted value obtained through modeling, and the line connecting $\mathbf{x}_s, \mathbf{x}_e$ becomes the observed value measured through sensor data.**

6.3 Orthonormal Representation

Using the previously calculated $\mathbf{e}_l$ for BA optimization poses problems when using the Plücker Coordinate representation directly because Plücker Coordinates must always satisfy the Klein quadric constraint $\mathbf{m}^\intercal \mathbf{d} = 0$, which implies 5 degrees of freedom, making it over-parameterized compared to the minimum 4 parameters needed to represent a line. The disadvantages of an over-parameterized representation are as follows:

- Redundant parameters must be calculated, increasing computational load during optimization.
- Additional degrees of freedom can lead to numerical instability.
- The parameters must always be checked to satisfy the constraint after each update.

Therefore, when optimizing a line, it is common to change to a 4-degree of freedom using the orthonormal representation method. **That is, while lines are represented using Plücker Coordinates, optimization is performed using the orthonormal representation, and then the optimized values are converted back to Plücker Coordinates.**

Orthonormal representation is as follows. A line in 3D space can always be represented as follows:

$$(\mathbf{U}, \mathbf{W}) \in SO(3) \times SO(2) \tag{132}$$

Any given Plücker line $\mathcal{L} = [\mathbf{m}^\intercal : \mathbf{d}^\intercal]^\intercal$ always has a corresponding $(\mathbf{U}, \mathbf{W})$, and this representation method is called the orthonormal representation. When a line $\mathcal{L}_w = [\mathbf{m}_w^\intercal : \mathbf{d}_w^\intercal]^\intercal$ in the world is given, $\mathcal{L}_w$ can be obtained through QR decomposition as follows:

$$\begin{bmatrix} \mathbf{m}_w \mid \mathbf{d}_w \end{bmatrix} = \mathbf{U} \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \\ 0 & 0 \end{bmatrix}, \quad \text{with set: } \mathbf{W} = \begin{bmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{bmatrix} \tag{133}$$

In this case, the upper triangle matrix $\mathbf{R}$'s $(1,2)$ element is always 0 due to the Plücker constraint (Klein quadric). $\mathbf{U}, \mathbf{W}$ represent 3D and 2D rotation matrices, respectively, so $\mathbf{U} = \mathbf{R}(\boldsymbol{\theta}), \mathbf{W} = \mathbf{R}(\theta)$ can be represented as follows:

$$\begin{aligned} \mathbf{R}(\boldsymbol{\theta}) = \mathbf{U} &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{m}_w}{\|\mathbf{m}_w\|} & \frac{\mathbf{d}_w}{\|\mathbf{d}_w\|} & \frac{\mathbf{m}_w \times \mathbf{d}_w}{\|\mathbf{m}_w \times \mathbf{d}_w\|} \end{bmatrix} \\ \mathbf{R}(\theta) = \mathbf{W} &= \begin{bmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \frac{1}{\sqrt{\|\mathbf{m}_w\|^2 + \|\mathbf{d}_w\|^2}} \begin{bmatrix} \|\mathbf{m}_w\| & \|\mathbf{d}_w\| \\ -\|\mathbf{d}_w\| & \|\mathbf{m}_w\| \end{bmatrix} \end{aligned} \tag{134}$$

When actually performing optimization, $\mathbf{U} \leftarrow \mathbf{U}\mathbf{R}(\boldsymbol{\theta}), \mathbf{W} \leftarrow \mathbf{W}\mathbf{R}(\theta)$ are updated as follows. **Therefore, the orthonormal representation can represent a 3D line through $\delta_{\boldsymbol{\theta}} = [\boldsymbol{\theta}^\intercal, \theta] \in \mathbb{R}^4$.** The updated $[\boldsymbol{\theta}^\intercal, \theta]$ is converted back to $\mathcal{L}_w$ as follows:

$$\mathcal{L}_w = \begin{bmatrix} w_1 \mathbf{u}_1^\intercal & w_2 \mathbf{u}_2^\intercal \end{bmatrix} \tag{135}$$

6.4 Error Function Formulation

To optimize the line reprojection error $\mathbf{e}_l$, nonlinear least squares methods such as Gauss-Newton (GN), Levenberg-Marquardt (LM), etc., are used to iteratively update the optimal variables. The error function using reprojection error is expressed as follows:

$$\mathbf{E}_l(\mathcal{X}) = \sum_i \sum_j \|\mathbf{e}_{l,ij}\|^2 \tag{136}$$

$$\begin{aligned} \mathcal{X}^* &= \arg\min_{\mathcal{X}^*} \mathbf{E}_l(\mathcal{X}) \\ &= \arg\min_{\mathcal{X}^*} \sum_i \sum_j \|\mathbf{e}_{l,ij}\|^2 \\ &= \arg\min_{\mathcal{X}^*} \sum_i \sum_j \mathbf{e}_{l,ij}^\intercal \mathbf{e}_{l,ij} \end{aligned} \tag{137}$$

The $\mathbf{E}_l(\mathcal{X}^*)$ that satisfies $\|\mathbf{e}_l(\mathcal{X}^*)\|^2$ can be computed iteratively through non-linear least squares. A small increment $\Delta\mathcal{X}$ is iteratively updated to $\mathcal{X}$ to find the optimal state.

$$\arg\min_{\mathcal{X}^*} \mathbf{E}_l(\mathcal{X} + \Delta\mathcal{X}) = \arg\min_{\mathcal{X}^*} \sum_i \sum_j \|\mathbf{e}_l(\mathcal{X} + \Delta\mathcal{X})\|^2 \tag{138}$$

Strictly speaking, the state increment $\Delta\mathcal{X}$ includes an SE(3) transformation matrix, so it is correct to add it to the existing state $\mathcal{X}$ through the $\oplus$ operator, but the $+$ operator is used for simplicity of expression.

$$\mathbf{e}_l(\mathcal{X} \oplus \Delta\mathcal{X}) \quad \rightarrow \quad \mathbf{e}_l(\mathcal{X} + \Delta\mathcal{X}) \tag{139}$$

The above equation can be expressed through a Taylor first-order approximation as follows:

$$\begin{aligned} \mathbf{e}_l(\mathcal{X} + \Delta\mathcal{X}) &\approx \mathbf{e}_l(\mathcal{X}) + \mathbf{J}\Delta\mathcal{X} \\ &= \mathbf{e}_l(\mathcal{X}) + \mathbf{J}_{\boldsymbol{\theta}}\Delta\delta_{\boldsymbol{\theta}} + \mathbf{J}_\xi\Delta\delta_\xi \\ &= \mathbf{e}_l(\mathcal{X}) + \frac{\partial \mathbf{e}_l}{\partial \delta_{\boldsymbol{\theta}}}\Delta\delta_{\boldsymbol{\theta}} + \frac{\partial \mathbf{e}_l}{\partial \delta_\xi}\Delta\delta_\xi \end{aligned} \tag{140}$$

$$\arg\min_{\mathcal{X}^*} \mathbf{E}_l(\mathcal{X} + \Delta\mathcal{X}) \approx \arg\min_{\mathcal{X}^*} \sum_i \sum_j \|\mathbf{e}_l(\mathcal{X}) + \mathbf{J}\Delta\mathcal{X}\|^2 \tag{141}$$

The optimal increment $\Delta\mathcal{X}^*$ is obtained by differentiating the above. The detailed derivation process is omitted in this section. For detailed information on the derivation process, refer to the previous section.

$$\begin{aligned} \mathbf{J}^\intercal\mathbf{J}\Delta\mathcal{X}^* &= -\mathbf{J}^\intercal\mathbf{e} \\ \mathbf{H}\Delta\mathcal{X}^* &= -\mathbf{b} \end{aligned} \tag{142}$$

6.4.1 The Analytical Jacobian of 3D Line

As explained in the previous section, to perform nonlinear optimization, $\mathbf{J}$ must be calculated. $\mathbf{J}$ is composed as follows:

$$\mathbf{J} = [\mathbf{J}_{\boldsymbol{\theta}}, \mathbf{J}_\xi] \tag{143}$$

$[\mathbf{J}_{\boldsymbol{\theta}}, \mathbf{J}_\xi]$ can be expanded as follows:

$$\begin{aligned} \mathbf{J}_{\boldsymbol{\theta}} &= \frac{\partial \mathbf{e}_l}{\partial \delta_{\boldsymbol{\theta}}} = \frac{\partial \mathbf{e}_l}{\partial l}\frac{\partial l}{\partial \mathcal{L}_c}\frac{\partial \mathcal{L}_c}{\partial \mathcal{L}_w}\frac{\partial \mathcal{L}_w}{\partial \delta_{\boldsymbol{\theta}}} \\ \mathbf{J}_\xi &= \frac{\partial \mathbf{e}_l}{\partial \delta_\xi} = \frac{\partial \mathbf{e}_l}{\partial l}\frac{\partial l}{\partial \mathcal{L}_c}\frac{\partial \mathcal{L}_c}{\partial \delta_\xi} \end{aligned} \tag{144}$$

$\frac{\partial \mathbf{e}_l}{\partial l}$ can be obtained as follows. Note that l is a vector and l_i is a scalar.

$$\boxed{\frac{\partial \mathbf{e}_l}{\partial l} = \frac{1}{\sqrt{l_1^2 + l_2^2}} \begin{bmatrix} x_s - \frac{l_1 \mathbf{x}_s l}{\sqrt{l_1^2+l_2^2}} & y_s - \frac{l_2 \mathbf{x}_s l}{\sqrt{l_1^2+l_2^2}} & 1 \\ x_e - \frac{l_1 \mathbf{x}_e l}{\sqrt{l_1^2+l_2^2}} & y_e - \frac{l_2 \mathbf{x}_e l}{\sqrt{l_1^2+l_2^2}} & 1 \end{bmatrix} \in \mathbb{R}^{2\times 3}} \tag{145}$$

$\frac{\partial l}{\partial \mathcal{L}_c}$ can be obtained as follows:

$$\boxed{\frac{\partial l}{\partial \mathcal{L}_c} = \frac{\partial \mathcal{K}_L \mathbf{m}_c}{\partial \mathcal{L}_c} = \begin{bmatrix} \mathcal{K}_L & \mathbf{0}_{3\times 3} \end{bmatrix} = \begin{bmatrix} f_y & & & 0 & 0 & 0 \\ & f_x & & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 6}} \tag{146}$$

$\frac{\partial \mathcal{L}_c}{\partial \mathcal{L}_w}$ can be obtained as follows:

$$\boxed{\frac{\partial \mathcal{L}_c}{\partial \mathcal{L}_w} = \mathbf{J}_{SE(3)}(\mathcal{L}_c) = \begin{bmatrix} \mathbf{I}_{3\times 3} & -[\mathbf{t}]_\times \end{bmatrix} = \begin{bmatrix} 1 & 0 & -y_c \\ 0 & 1 & x_c \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}} \tag{147}$$

$\frac{\partial \mathcal{L}_w}{\partial \delta_{\boldsymbol{\theta}}}$ can be obtained as follows:

$$\boxed{\frac{\partial \mathcal{L}_w}{\partial \delta_{\boldsymbol{\theta}}} = \frac{\partial \mathcal{L}_w}{\partial \mathbf{R}} \frac{\partial \mathbf{R}}{\partial \delta_{\boldsymbol{\theta}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathbf{W} \\ \mathbf{W} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{2\times 4}} \tag{148}$$

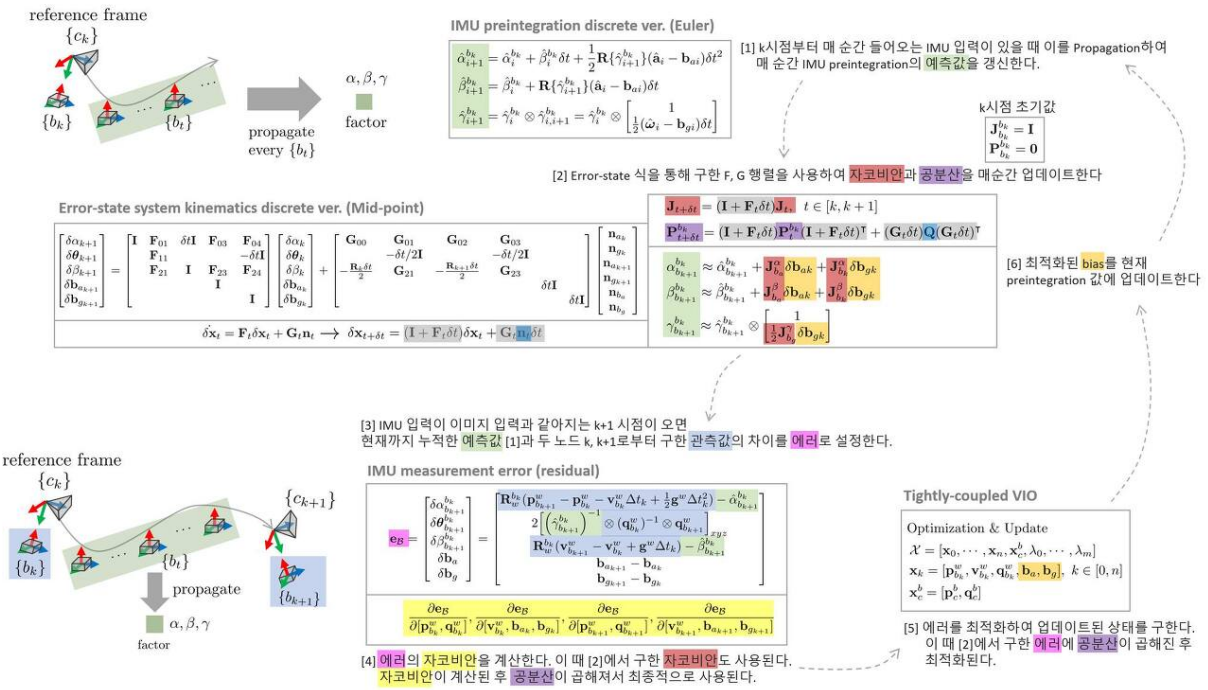
$\frac{\partial \mathcal{L}_c}{\partial \delta_\xi}$ can be obtained as follows:

$$\boxed{\frac{\partial \mathcal{L}_c}{\partial \delta_\xi} = \mathbf{J}_{SO(3)}(\mathcal{L}_c) = \frac{\partial \mathcal{L}_c}{\partial \mathbf{R}} \frac{\partial \mathbf{R}}{\partial \delta_\xi} = \begin{bmatrix} -\mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \end{bmatrix} \in \mathbb{R}^{6\times 1}} \tag{149}$$

6.5 Code implementations

- Structure PLP SLAM 코드: g2o/se3/pose_opt_edge_line3d_orthonormal.h#L62
- Structure PLP SLAM 코드2: g2o/se3/pose_opt_edge_line3d_orthonormal.h#L81

7 IMU measurement error

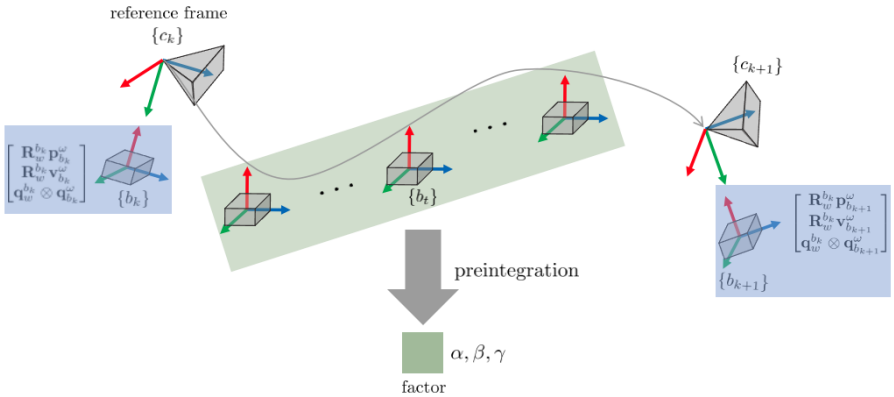


IMU measurement 에러를 구하기 위해서는 우선 IMU preintegration 기법과 error-state 모델링에 대해 알아야 한다. 전반적인 IMU measurement 에러 기반 최적화 과정을 표현한 그림은 위와 같다. [1]-[6] 순서대로 보면 된다.

보다 자세한 내용은 [SLAM] Formula Derivation and Analysis of the VINS-mono 내용 정리를 참조하면 된다.

NOMENCLATURE of IMU measurement error

- $\alpha_{b_{k+1}}^{b_k} \in \mathbb{R}^{3\times1}$: $t \in [b_k, b_{k+1}]$ 시간 동안 누적된 위치의 관측값
- $\hat{\alpha}_{b_{k+1}}^{b_k} \in \mathbb{R}^{3\times1}$: $t \in [b_k, b_{k+1}]$ 시간 동안 누적된 위치의 예측값
- $\beta_{b_{k+1}}^{b_k} \in \mathbb{R}^{3\times1}$: $t \in [b_k, b_{k+1}]$ 시간 동안 누적된 속도의 관측값
- $\hat{\beta}_{b_{k+1}}^{b_k} \in \mathbb{R}^{3\times1}$: $t \in [b_k, b_{k+1}]$ 시간 동안 누적된 속도의 예측값
- $\gamma_{b_{k+1}}^{b_k} \in \mathbb{R}^{3\times1}$: $t \in [b_k, b_{k+1}]$ 시간 동안 누적된 방향(orientation)의 관측값
- $\hat{\gamma}_{b_{k+1}}^{b_k} \in \mathbb{R}^{3\times1}$: $t \in [b_k, b_{k+1}]$ 시간 동안 누적된 방향(orientation)의 예측값
- $\mathcal{X} = [\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{x}_c^b, \lambda_0, \lambda_1, \cdots, \lambda_m]$: 모든 상태 변수
- $\mathbf{x}_k = [\mathbf{p}_{b_k}^w, \mathbf{v}_{b_k}^w, \mathbf{q}_{b_k}^w, \mathbf{b}_a, \mathbf{b}_g]$: 특정 k 시점에서 IMU 모델의 상태 변수
- $\mathbf{x}_c^b = [\mathbf{p}_c^b, \mathbf{q}_c^b]$: 카메라와 IMU의 외부 파라미터(extrinsic parameter)
- $\mathcal{X}_k$: 특정 두 시점 $[b_k, b_{k+1}]$의 상태변수. 이는 즉 $\mathcal{X}_k = (\mathbf{x}_k, \mathbf{x}_{k+1})$과 같다.
- λ: 특징점의 inverse depth
- $\otimes$: 쿼터니언 곱셈 연산자. (e.g., $\mathbf{q} = \mathbf{q}_1 \otimes \mathbf{q}_2$)
- $\mathcal{B}$: 모든 IMU b_k 값들의 집합
- $\ominus$: 벡터와 쿼터니언을 한 번에 뺄셈 연산하는 연산자
- $\mathbf{P}_\mathcal{B}$: 모든 IMU b_k 값들의 공분산
- $\mathbf{\Omega}_\mathcal{B}$: 공분산 $\mathbf{P}_\mathcal{B}$의 역행렬. Information 행렬을 의미한다.
- $\mathbf{e}_{\mathcal{B},k} = \mathbf{e}_\mathcal{B}(\mathcal{X}_k)$



$$\mathbf{e}_\mathcal{B} = \begin{bmatrix} \delta\alpha_{b_{k+1}}^{b_k} \\ \delta\boldsymbol{\theta}_{b_{k+1}}^{b_k} \\ \delta\beta_{b_{k+1}}^{b_k} \\ \delta\mathbf{b}_a \\ \delta\mathbf{b}_g \end{bmatrix} = \begin{bmatrix} \mathbf{R}_w^{b_k}\left(\mathbf{p}_{b_{k+1}}^w - \mathbf{p}_{b_k}^w - \mathbf{v}_{b_k}^w \Delta t_k + \frac{1}{2}\mathbf{g}^w \Delta t_k^2\right) - \hat{\alpha}_{b_{k+1}}^{b_k} \\ 2\left[\left(\hat{\gamma}_{b_{k+1}}^{b_k}\right)^{-1} \otimes \left(\mathbf{q}_{b_k}^w\right)^{-1} \otimes \mathbf{q}_{b_{k+1}}^w\right]_{xyz} \\ \mathbf{R}_w^{b_k}\left(\mathbf{v}_{b_{k+1}}^w - \mathbf{v}_{b_k}^w + \mathbf{g}^w \Delta t_k\right) - \hat{\beta}_{b_{k+1}}^{b_k} \\ \mathbf{b}_{a_{k+1}} - \mathbf{b}_{a_k} \\ \mathbf{b}_{g_{k+1}} - \mathbf{b}_{g_k} \end{bmatrix}$$

IMU 또한 이전 섹션에서 설명한 에러들과 동일하게 관측값 - 예측값을 에러로 정의하며 이를 IMU measurement 에러 $\mathbf{e}_\mathcal{B}$라고 한다. **자세히 설명하면, IMU measurement 에러 $\mathbf{e}_\mathcal{B}$는 $t \in [b_k, b_{k+1}]$ 시간 동안 들어오는 IMU**

데이터를 누적한 preintegration과 bias $[\alpha, \beta, \gamma, \mathbf{b}_a, \mathbf{b}_g]$의 관측값($\mathbf{z}_{b_{k+1}}^{b_k}$)과 예측값($\hat{\mathbf{z}}_{b_{k+1}}^{b_k}$)의 차이를 의미한다.

$$\mathbf{e}_{\mathcal{B}}(\mathcal{X}_k) = \mathbf{z}_{b_{k+1}}^{b_k} \ominus \hat{\mathbf{z}}_{b_{k+1}}^{b_k} = \begin{bmatrix} \alpha_{b_{k+1}}^{b_k} - \hat{\alpha}_{b_{k+1}}^{b_k} \\ \beta_{b_{k+1}}^{b_k} - \hat{\beta}_{b_{k+1}}^{b_k} \\ \gamma_{b_{k+1}}^{b_k} \otimes \hat{\gamma}_{b_{k+1}}^{b_k} \\ \mathbf{b}_{a_k} - \hat{\mathbf{b}}_a \\ \mathbf{b}_g - \hat{\mathbf{b}}_g \end{bmatrix} \tag{150}$$

관측값과 예측값에 대해 자세히 알아보자. **우선, 관측값은 두 시점 b_k, b_{k+1}의 위치 p와 속도 v 그리고 방향 q 값을 사용하여 구할 수 있다.** 관측값을 구하기 위해 $[b_k, b_{k+1}]$ 구간의 IMU kinematics 공식을 보면 다음과 같다.

$$\begin{aligned} \mathbf{R}_w^{b_k} \mathbf{p}_{b_{k+1}}^w &= \mathbf{R}_w^{b_k} (\mathbf{p}_{b_k}^w + \mathbf{v}_{b_k}^w \Delta t - \frac{1}{2} \mathbf{g}^w \Delta t_k^2) + \alpha_{b_{k+1}}^{b_k} \\ \mathbf{R}_w^{b_k} \mathbf{v}_{b_{k+1}}^w &= \mathbf{R}_w^{b_k} (\mathbf{v}_{b_k}^w - \mathbf{g}^w \Delta t_k) + \beta_{b_{k+1}}^{b_k} \\ \mathbf{q}_w^{b_k} \otimes \mathbf{q}_{b_{k+1}}^w &= \gamma_{b_{k+1}}^{b_k} \end{aligned} \tag{151}$$

따라서 관측값은 다음과 같이 구할 수 있다.

$$\mathbf{z}_{b_{k+1}}^{b_k} = \begin{bmatrix} \alpha_{b_{k+1}}^{b_k} \\ \beta_{b_{k+1}}^{b_k} \\ \gamma_{b_{k+1}}^{b_k} \\ \mathbf{b}_{a_{k+1}} - \mathbf{b}_{a_k} \\ \mathbf{b}_{g_{k+1}} - \mathbf{b}_{g_k} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_w^{b_k} (\mathbf{p}_{b_{k+1}}^w - \mathbf{p}_{b_k}^w - \mathbf{v}_{b_k}^w \Delta t_k + \frac{1}{2} \mathbf{g}^w \Delta t_k^2) \\ \mathbf{R}_w^{b_k} (\mathbf{v}_{b_{k+1}}^w - \mathbf{v}_{b_k}^w + \mathbf{g}^w \Delta t_k) \\ (\mathbf{q}_{b_k}^w)^{-1} \otimes \mathbf{q}_{b_{k+1}}^w \\ \mathbf{b}_{a_{k+1}} - \mathbf{b}_{a_k} \\ \mathbf{b}_{g_{k+1}} - \mathbf{b}_{g_k} \end{bmatrix} \tag{152}$$

다음으로 예측값은 $t \in [b_k, b_{k+1}]$ 시간동안 누적한 preintegration 값을 통해 구할 수 있다. 예측값을 구하기 위해 preintegration 식을 자세히 보면 다음과 같다.

$$\begin{aligned} \hat{\alpha}_{b_{k+1}}^{b_k} &= \iint_{t \in [k, k+1]} \mathbf{R}_t^{b_k} (\hat{\mathbf{a}}_t - \mathbf{b}_{at} - \mathbf{n}_a) dt^2 \\ \hat{\beta}_{b_{k+1}}^{b_k} &= \int_{t \in [k, k+1]} \mathbf{R}_t^{b_k} (\hat{\mathbf{a}}_t - \mathbf{b}_{at} - \mathbf{n}_a) dt \\ \hat{\gamma}_{b_{k+1}}^{b_k} &= \int_{t \in [k, k+1]} \frac{1}{2} \Omega_R (\hat{\boldsymbol{\omega}}_t - \mathbf{b}_{gt} - \mathbf{n}_g) \gamma_t^{b_k} dt \end{aligned} \tag{153}$$

위 식은 연속 신호에서 사용 가능한 공식이다. 하지만 실제 IMU 신호는 이산 신호(discrete signal)로 들어오므로 미분 방정식(differential equation)을 차분 방정식(difference equation)으로 표현해야 한다. 해당 과정에서 다양한 수치적분 알고리즘이 사용되는데, 수치적분에는 zero-order hold(euler), first-order hold(mid-point), higher order (RK4) 등이 존재한다. **이 중 VINS-mono에서 사용한 mid-point method를 사용해 차분 방정식을 표현하면 다음과 같다.**

$$\begin{aligned} \hat{\alpha}_{t+1}^{b_k} &= \hat{\alpha}_t^{b_k} + \frac{1}{2} (\hat{\beta}_t^{b_k} + \hat{\beta}_{t+1}^{b_k}) \delta t \\ &= \hat{\alpha}_t^{b_k} + \hat{\beta}_t^{b_k} \delta t + \frac{1}{4} [\mathbf{R}\{\hat{\gamma}_t^{b_k}\} (\hat{\mathbf{a}}_t - \mathbf{b}_{at}) + \mathbf{R}\{\hat{\gamma}_{t+1}^{b_k}\} (\hat{\mathbf{a}}_{t+1} - \mathbf{b}_{at})] \delta t^2 \\ \hat{\beta}_{t+1}^{b_k} &= \hat{\beta}_t^{b_k} + \frac{1}{2} [\mathbf{R}\{\hat{\gamma}_t^{b_k}\} (\hat{\mathbf{a}}_t - \mathbf{b}_{at}) + \mathbf{R}\{\hat{\gamma}_{t+1}^{b_k}\} (\hat{\mathbf{a}}_{t+1} - \mathbf{b}_{at})] \delta t \\ \hat{\gamma}_{t+1}^{b_k} &= \hat{\gamma}_t^{b_k} \otimes \hat{\gamma}_{t,t+1}^{b_k} = \hat{\gamma}_t^{b_k} \otimes \begin{bmatrix} 1 \\ 1/4 (\hat{\boldsymbol{\omega}}_t + \hat{\boldsymbol{\omega}}_{t+1} - 2\mathbf{b}_{gt}) \delta t \end{bmatrix} \end{aligned} \tag{154}$$

따라서 예측값은 (154) 식을 $t \in [b_k, b_{k+1}]$ 시간 동안 누적한 값으로 구할 수 있다. Bias 값은 예측값을 구할 수 없기 때문에 0으로 설정한다.

$$\hat{\mathbf{z}}_{b_{k+1}}^{b_k} = \begin{bmatrix} \hat{\alpha}_{b_{k+1}}^{b_k} \\ \hat{\beta}_{b_{k+1}}^{b_k} \\ \hat{\gamma}_{b_{k+1}}^{b_k} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{155}$$

지금까지 구한 값을 토대로 (150)를 풀어쓰면 IMU measurement 에러는 다음과 같이 나타낼 수 있다.

$$\mathbf{e}_{\mathcal{B}}(\mathcal{X}_k) = \mathbf{z}_{b_{k+1}}^{b_k} \ominus \hat{\mathbf{z}}_{b_{k+1}}^{b_k} = \begin{bmatrix} \mathbf{R}_w^{b_k}(\mathbf{p}_{b_{k+1}}^w - \mathbf{p}_{b_k}^w - \mathbf{v}_{b_k}^w \Delta t_k + \frac{1}{2}\mathbf{g}^w \Delta t_k^2) - \hat{\alpha}_{b_{k+1}}^{b_k} \\ \mathbf{R}_w^{b_k}(\mathbf{v}_{b_{k+1}}^w - \mathbf{v}_{b_k}^w + \mathbf{g}^w \Delta t_k) - \hat{\beta}_{b_{k+1}}^{b_k} \\ \left(\hat{\gamma}_{b_{k+1}}^{b_k}\right)^{-1} \otimes (\mathbf{q}_{b_k}^w)^{-1} \otimes \mathbf{q}_{b_{k+1}}^w \\ \mathbf{b}_{a_{k+1}} - \mathbf{b}_{a_k} \\ \mathbf{b}_{g_{k+1}} - \mathbf{b}_{g_k} \end{bmatrix} \tag{156}$$

7.1 Error function formulation

모든 preintegration, bias들에 대한 에러 함수는 다음과 같이 정의된다.

$$\mathbf{E}_{\mathcal{B}}(\mathcal{X}) = \sum_{k \in \mathcal{B}} \|\mathbf{e}_{\mathcal{B},k}\|_{\mathbf{P}_{\mathcal{B}}}^2 \tag{157}$$

$$\begin{aligned} \mathcal{X}^* &= \arg\min_{\mathcal{X}^*} \mathbf{E}_{\mathcal{B}}(\mathcal{X}) \\ &= \arg\min_{\mathcal{X}^*} \sum_{k \in \mathcal{B}} \|\mathbf{e}_{\mathcal{B},k}\|_{\mathbf{P}_{\mathcal{B}}}^2 \\ &= \arg\min_{\mathcal{X}^*} \sum_{k \in \mathcal{B}} \mathbf{e}_{\mathcal{B},k}^{\intercal} \mathbf{\Omega}_{\mathcal{B}} \mathbf{e}_{\mathcal{B},k} \\ &= \arg\min_{\mathcal{X}^*} \sum_{k \in \mathcal{B}} (\mathbf{z}_{b_{k+1}}^{b_k} \ominus \hat{\mathbf{z}}_{b_{k+1}}^{b_k})^{\intercal} \mathbf{\Omega}_{\mathcal{B}} (\mathbf{z}_{b_{k+1}}^{b_k} \ominus \hat{\mathbf{z}}_{b_{k+1}}^{b_k}) \end{aligned} \tag{158}$$

위 식에서 $\mathbf{e}_{\mathcal{B},k} = \mathbf{e}_{\mathcal{B}}(\mathcal{X}_k)$를 의미한다.

Tip

실제 VINS-mono에서는 IMU measurement 에러 뿐만 아니라 visual residual $\mathbf{r}_{\mathcal{C}}$, marginalization prior residual $\mathbf{r}_p$ 값도 동시에 최적화하여 tightly-coupled VIO를 수행한다. VINS-mono에서는 IMU measurement 에러를 residual $\mathbf{r}_{\mathcal{B}}(\hat{\mathbf{z}}_{b_{k+1}}^{b_k}, \mathcal{X})$로 표현하였다.

$$\min_{\mathcal{X}} \left\{ \|\mathbf{r}_p - \mathbf{J}_p \mathcal{X}\|_{\mathbf{P}_M} + \sum_{k \in \mathcal{B}} \left\| \mathbf{r}_{\mathcal{B}}\left(\hat{\mathbf{z}}_{b_{k+1}}^{b_k}, \mathcal{X}\right) \right\|_{\mathbf{P}_B} + \sum_{(l,j) \in \mathcal{C}} \left\| \mathbf{r}_{\mathcal{C}}\left(\hat{\mathbf{z}}_l^{c_j}, \mathcal{X}\right) \right\|_{\mathbf{P}_l^{c_j}} \right\} \tag{159}$$

본 섹션에서는 이 중 IMU measurement 에러 $\mathbf{r}_{\mathcal{B}}(\hat{\mathbf{z}}_{b_{k+1}}^{b_k}, \mathcal{X})$에 대한 내용만 설명한다.

$\mathbf{E}_{\mathcal{B}}(\mathcal{X}^*)$를 만족하는 $\|\mathbf{e}_{\mathcal{B}}(\mathcal{X}_k^*)\|_{\mathbf{P}_{\mathcal{B}}}^2$를 non-linear least squares를 통해 반복적으로 계산할 수 있다. 작은 증분량 $\Delta\mathcal{X}$를 반복적으로 $\mathcal{X}$에 업데이트함으로써 최적의 상태를 찾는다.

$$\arg\min_{\mathcal{X}^*} \mathbf{E}_{\mathcal{B}}(\mathcal{X} + \Delta\mathcal{X}) = \arg\min_{\mathcal{X}^*} \sum_{k \in \mathcal{B}} \|\mathbf{e}_{\mathcal{B}}(\mathcal{X}_k + \Delta\mathcal{X}_k)\|^2 \tag{160}$$

엄밀하게 말하면 상태 증분량 $\Delta\mathcal{X}$은 쿼터니언을 포함하므로 $\oplus$ 연산자를 통해 기존 상태 $\mathcal{X}$에 더해지는게 맞지만 표현의 편의를 위해 $+$ 연산자를 사용하였다.

$$\mathbf{e}_{\mathcal{B}}(\mathcal{X}_k \oplus \Delta\mathcal{X}_k) \quad \rightarrow \quad \mathbf{e}_{\mathcal{B}}(\mathcal{X}_k + \Delta\mathcal{X}_k) \tag{161}$$

위 식은 테일러 1차 근사를 통해 다음과 같이 표현이 가능하다.

$$\mathbf{e}_{\mathcal{B}}(\mathcal{X}_k + \Delta\mathcal{X}_k) \approx \mathbf{e}_{\mathcal{B}}(\mathcal{X}) + \mathbf{J}\Delta\mathcal{X}_k$$

$$\begin{aligned}
&= \mathbf{e}_{\mathcal{B}}(\mathcal{X}_k) + \begin{bmatrix} \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{p}^w_{b_k}, \mathbf{q}^w_{b_k}]} & \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{v}^w_{b_k}, \mathbf{b}_{ak}, \mathbf{b}_{gk}]} & \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{p}^w_{b_{k+1}}, \mathbf{q}^w_{b_{k+1}}]} & \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{v}^w_{b_{k+1}}, \mathbf{b}_{ak+1}, \mathbf{b}_{ak+1}]} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{p}^w_k \\ \Delta\mathbf{q}^w_k \\ \Delta\mathbf{v}^w_k \\ \Delta\mathbf{b}_{ak} \\ \Delta\mathbf{b}_{gk} \\ \Delta\mathbf{p}^w_{k+1} \\ \Delta\mathbf{q}^w_{k+1} \\ \Delta\mathbf{v}^w_{k+1} \\ \Delta\mathbf{b}_{ak+1} \\ \Delta\mathbf{b}_{gk+1} \end{bmatrix} \\
&= \mathbf{e}_{\mathcal{B}}(\mathcal{X}_k) + \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{p}^w_{b_k}, \mathbf{q}^w_{b_k}]}(\Delta\mathbf{p}^w_k, \Delta\mathbf{q}^w_k) + \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{v}^w_{b_k}, \mathbf{b}_{ak}, \mathbf{b}_{gk}]}(\Delta\mathbf{v}^w_k, \Delta\mathbf{b}_{ak}, \Delta\mathbf{b}_{gk}) \\
&+ \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{p}^w_{b_{k+1}}, \mathbf{q}^w_{b_{k+1}}]}(\Delta\mathbf{p}^w_{k+1}, \Delta\mathbf{q}^w_{k+1}) + \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{v}^w_{b_{k+1}}, \mathbf{b}_{ak+1}, \mathbf{b}_{ak+1}]}(\Delta\mathbf{v}^w_{k+1}, \Delta\mathbf{b}_{ak+1}, \Delta\mathbf{b}_{gk+1})
\end{aligned} \tag{162}$$

b_k 시점의 $[\mathbf{p}^w_{b_k}, \mathbf{v}^w_{b_k}, \mathbf{q}^w_{b_k}, \mathbf{b}_{a_k}, \mathbf{b}_{g_k}]$와 b_{k+1} 시점의 $[\mathbf{p}^w_{b_{k+1}}, \mathbf{v}^w_{b_{k+1}}, \mathbf{q}^w_{b_{k+1}}, \mathbf{b}_{a_{k+1}}, \mathbf{b}_{g_{k+1}}]$ 모두 에러 값에 관여하기 때문에 총 10개 변수에 대한 자코비안을 모두 계산해야 한다. VINS-mono에서는 다음과 같이 4개의 그룹으로 묶어서 상태 변수를 표현하였다.

$$\begin{aligned}
&[\mathbf{p}^w_{b_k}, \mathbf{q}^w_{b_k}] && \cdots \text{ for } \mathbf{J}[0] \\
&[\mathbf{v}^w_{b_k}, \mathbf{b}_{ak}, \mathbf{b}_{gk}] && \cdots \text{ for } \mathbf{J}[1] \\
&[\mathbf{p}^w_{b_{k+1}}, \mathbf{q}^w_{b_{k+1}}] && \cdots \text{ for } \mathbf{J}[2] \\
&[\mathbf{v}^w_{b_{k+1}}, \mathbf{b}_{ak+1}, \mathbf{b}_{ak+1}] && \cdots \text{ for } \mathbf{J}[3]
\end{aligned} \tag{163}$$

Tightly-coupled VIO에서 최적화하는 상태 변수 $\mathcal{X}$은 inverse depth λ와 외부 파라미터(extrinsic parameter) $\mathbf{x}^b_c$, time difference td의 상태를 포함하지만 IMU measurement 에러에서는 위와 같이 두 시점 $[b_k, b_{k+1}]$에서 포즈와 속도, bias 값들만 업데이트함에 유의한다.

에러 함수는 다음과 같이 근사할 수 있다.

$$\arg\min_{\mathcal{X}^*} \mathbf{E}_{\mathcal{B}}(\mathcal{X} + \Delta\mathcal{X}) \approx \arg\min_{\mathcal{X}^*} \sum_{k\in\mathcal{B}} \|\mathbf{e}_{\mathcal{B}}(\mathcal{X}_k) + \mathbf{J}\Delta\mathcal{X}_k\|^2_{\mathbf{P}_{\mathcal{B}}} \tag{164}$$

이를 미분하여 최적의 증분량 $\Delta\mathcal{X}^*$ 값을 구하면 다음과 같다. 자세한 유도 과정은 본 섹션에서는 생략한다. 유도 과정에 대해 자세히 알고 싶으면 이전 섹션을 참조하면 된다.

$$\begin{aligned}
\mathbf{J}^\intercal\mathbf{J}\Delta\mathcal{X}^* &= -\mathbf{J}^\intercal\mathbf{e} \\
\mathbf{H}\Delta\mathcal{X}^* &= -\mathbf{b}
\end{aligned} \tag{165}$$

위 식은 선형시스템 $\mathbf{Ax} = \mathbf{b}$ 형태이므로 schur complement, cholesky decomposition과 같은 다양한 선형대수학 테크닉을 사용하여 $\Delta\mathcal{X}^*$를 구할 수 있다. 이렇게 구한 최적의 증분량을 현재 상태에 더한다. **이 때, 기존 상태 x의 오른쪽에 곱하느냐 왼쪽에 곱하느냐에 따라서 각각 로컬 좌표계에서 본 포즈를 업데이트할 것 인지(오른쪽) 전역 좌표계에서 본 포즈를 업데이트할 것 인지(왼쪽) 달라지게 된다. IMU measurement 에러는 두 노드 b_k, b_{k+1}과 관련되어 있으므로 로컬 좌표계에서 업데이트하는 오른쪽 곱셈이 적용된다.**

$$\mathcal{X} \leftarrow \mathcal{X} \oplus \Delta\mathcal{X}^* \tag{166}$$

$\mathcal{X}$ 중에서 IMU measurement 에러에 의해 업데이트되는 $\mathcal{X}_k$는 $[\mathbf{p}^w_{b_k}, \mathbf{v}^w_{b_k}, \mathbf{q}^w_{b_k}, \mathbf{b}_{a_k}, \mathbf{b}_{g_k}, \mathbf{p}^w_{b_{k+1}}, \mathbf{v}^w_{b_{k+1}}, \mathbf{q}^w_{b_{k+1}}, \mathbf{b}_{a_{k+1}}, \mathbf{b}_{g_{k+1}}]$로 구성되어 있으므로 다음과 같이 풀어 쓸 수 있다.

$$\begin{aligned}
\mathbf{p}^w_{b_k} &\leftarrow \mathbf{p}^w_{b_k} \oplus \Delta\mathbf{p}^{w*}_{b_k} \\
\mathbf{q}^w_{b_k} &\leftarrow \mathbf{q}^w_{b_k} \oplus \Delta\mathbf{q}^{w*}_{b_k} \\
\mathbf{v}^w_{b_k} &\leftarrow \mathbf{v}^w_{b_k} \oplus \Delta\mathbf{v}^{w*}_{b_k} \\
\mathbf{b}_{ak} &\leftarrow \mathbf{b}_{ak} \oplus \Delta\mathbf{b}^*_{ak} \\
\mathbf{b}_{gk} &\leftarrow \mathbf{b}_{gk} \oplus \Delta\mathbf{b}^*_{gk} \\
\mathbf{p}^w_{b_{k+1}} &\leftarrow \mathbf{p}^w_{b_{k+1}} \oplus \Delta\mathbf{p}^{w*}_{b_{k+1}} \\
\mathbf{q}^w_{b_{k+1}} &\leftarrow \mathbf{q}^w_{b_{k+1}} \oplus \Delta\mathbf{q}^{w*}_{b_{k+1}} \\
\mathbf{v}^w_{b_{k+1}} &\leftarrow \mathbf{v}^w_{b_{k+1}} \oplus \Delta\mathbf{v}^{w*}_{b_{k+1}} \\
\mathbf{b}_{ak+1} &\leftarrow \mathbf{b}_{ak+1} \oplus \Delta\mathbf{b}^*_{ak+1} \\
\mathbf{b}_{gk+1} &\leftarrow \mathbf{b}_{gk+1} \oplus \Delta\mathbf{b}^*_{gk+1}
\end{aligned} \tag{167}$$

오른쪽 곱셈 $\oplus$ 연산의 정의는 다음과 같다.

$$
\begin{aligned}
\mathbf{p}_{b_k}^{w} &\leftarrow \mathbf{p}_{b_k}^{w} + \Delta\mathbf{p}_{b_k}^{w*} \\
\mathbf{q}_{b_k}^{w} &\leftarrow \mathbf{q}_{b_k}^{w} \otimes \Delta\mathbf{q}_{b_k}^{w*} \quad \cdots \text{ locally updated (right mult)} \\
\mathbf{v}_{b_k}^{w} &\leftarrow \mathbf{v}_{b_k}^{w} + \Delta\mathbf{v}_{b_k}^{w*} \\
\mathbf{b}_{ak} &\leftarrow \mathbf{b}_{ak} + \Delta\mathbf{b}_{ak}^{*} \\
\mathbf{b}_{gk} &\leftarrow \mathbf{b}_{gk} + \Delta\mathbf{b}_{gk}^{*} \\
\mathbf{p}_{b_{k+1}}^{w} &\leftarrow \mathbf{p}_{b_{k+1}}^{w} + \Delta\mathbf{p}_{b_{k+1}}^{w*} \\
\mathbf{q}_{b_{k+1}}^{w} &\leftarrow \mathbf{q}_{b_{k+1}}^{w} \otimes \Delta\mathbf{q}_{b_{k+1}}^{w*} \quad \cdots \text{ locally updated (right mult)} \\
\mathbf{v}_{b_{k+1}}^{w} &\leftarrow \mathbf{v}_{b_{k+1}}^{w} + \Delta\mathbf{v}_{b_{k+1}}^{w*} \\
\mathbf{b}_{ak+1} &\leftarrow \mathbf{b}_{ak+1} + \Delta\mathbf{b}_{ak+1}^{*} \\
\mathbf{b}_{gk+1} &\leftarrow \mathbf{b}_{gk+1} + \Delta\mathbf{b}_{gk+1}^{*}
\end{aligned}
\tag{168}
$$

7.2 Jacobian of IMU measurement error

(165)를 수행하기 위해서는 IMU measurement 에러에 대한 자코비안 $\mathbf{J}$를 구해야 한다. 이를 풀어쓰면 다음과 같이 나타낼 수 있다.

$$
\begin{aligned}
\mathbf{J} &= \begin{bmatrix} \mathbf{J}[0] & \mathbf{J}[1] & \mathbf{J}[2] & \mathbf{J}[3] \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{p}_{b_k}^{w}, \mathbf{q}_{b_k}^{w}]} & \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{v}_{b_k}^{w}, \mathbf{b}_{ak}, \mathbf{b}_{gk}]} & \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{p}_{b_{k+1}}^{w}, \mathbf{q}_{b_{k+1}}^{w}]} & \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{v}_{b_{k+1}}^{w}, \mathbf{b}_{ak+1}, \mathbf{b}_{ak+1}]} \end{bmatrix} \\
&= \frac{\partial}{\partial[\mathbf{p}_{b_k}^{w}, \mathbf{q}_{b_k}^{w}], [\mathbf{v}_{b_k}^{w}, \mathbf{b}_{ak}, \mathbf{b}_{gk}], [\mathbf{p}_{b_{k+1}}^{w}, \mathbf{q}_{b_{k+1}}^{w}], [\mathbf{v}_{b_{k+1}}^{w}, \mathbf{b}_{ak+1}, \mathbf{b}_{ak+1}]} \begin{bmatrix} \mathbf{R}_{w}^{b_k}(\mathbf{p}_{b_{k+1}}^{w} - \mathbf{p}_{b_k}^{w} - \mathbf{v}_{b_k}^{w}\Delta t_k + \frac{1}{2}\mathbf{g}^{w}\Delta t_k^2) - \hat{\alpha}_{b_{k+1}}^{b_k} \\ \mathbf{R}_{w}^{b_k}(\mathbf{v}_{b_{k+1}}^{w} - \mathbf{v}_{b_k}^{w} + \mathbf{g}^{w}\Delta t_k) - \hat{\beta}_{b_{k+1}}^{b_k} \\ \left(\hat{\gamma}_{b_{k+1}}^{b_k}\right)^{-1} \otimes (\mathbf{q}_{b_k}^{w})^{-1} \otimes \mathbf{q}_{b_{k+1}}^{w} \\ \mathbf{b}_{a_{k+1}} - \mathbf{b}_{a_k} \\ \mathbf{b}_{g_{k+1}} - \mathbf{b}_{g_k} \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{R}^{15\times 7} & \mathbb{R}^{15\times 9} & \mathbb{R}^{15\times 7} & \mathbb{R}^{15\times 9} \end{bmatrix} = \mathbb{R}^{15\times 32}
\end{aligned}
\tag{169}
$$

7.2.1 Lie theory-based SO(3) optimization

위 자코비안을 구할 때, 위치 p, 속도 v, bias $\mathrm{b}_a, \mathrm{b}_g$에 관련된 항은 각각 3차원 벡터이므로 최적화 업데이트를 수행할 때 별도의 제약조건이 존재하지 않는다. 반면에, 쿼터니언 q는 파라미터의 개수가 4개이고 이는 3차원 회전을 표현하는 최소 자유도인 3 자유도보다 많으므로 다양한 제약조건이 존재한다. 이를 over-parameterized 되었다고 한다. over-parameterized 표현법의 단점은 다음과 같다.

- 중복되는 파라미터를 계산해야 하기 때문에 최적화 수행 시 연산량이 증가한다.
- 추가적인 자유도로 인해 수치적인 불안정성(numerical instability) 문제가 야기될 수 있다.
- 파라미터가 업데이트될 때마다 항상 제약조건을 만족하는 지 체크해줘야 한다.

lie theory를 사용하면 제약조건으로부터 자유롭게 최적화를 수행할 수 있다. 따라서 쿼터니언 $\mathbf{q}$을 사용하는 대신 lie algebra so(3) $[\boldsymbol{\theta}]_{\times}$을 사용하여 제약조건으로부터 자유롭게 파라미터를 업데이트할 수 있게 된다. 이 때, $\boldsymbol{\theta} \in \mathbb{R}^3$는 각속도 벡터를 의미한다. SO(3)-based 최적화에 대한 디테일한 내용은 reproejction 에러 섹션과 동일하므로 이에 대한 자세한 설명은 생략한다.

각속도 벡터 $\boldsymbol{\theta}$를 사용하면 기존의 쿼터니언 $\mathbf{q}$의 자코비안은 다음과 같이 변경된다.

$$
\begin{aligned}
\frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial \mathbf{q}_{b_k}^{w}} &\rightarrow \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial \begin{bmatrix} 1 & \frac{1}{2}\boldsymbol{\theta}_{b_k}^{w} \end{bmatrix}} \\
\frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial \mathbf{q}_{b_{k+1}}^{w}} &\rightarrow \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial \begin{bmatrix} 1 & \frac{1}{2}\boldsymbol{\theta}_{b_{k+1}}^{w} \end{bmatrix}}
\end{aligned}
\tag{170}
$$

일반적으로 최적화에 사용하는 에러는 크기가 작으므로 γ에 대한 에러 $\left(\hat{\gamma}_{b_{k+1}}^{b_k}\right)^{-1} \otimes (\mathbf{q}_{b_k}^{w})^{-1} \otimes \mathbf{q}_{b_{k+1}}^{w}$ 또한 크기가 작다고 가정한다. 따라서 실제 쿼터니언의 $\mathbf{q} = [w, x, y, z]$ 중에서 허수 부분 $[x, y, z] = \frac{1}{2}\boldsymbol{\theta}$만 최적화에 사용한다. 이를 통해 γ 부분은 다음과 같이 변형된다.

$$
\begin{aligned}
\gamma &\rightarrow 2[\gamma]_{xyz} = 2[x, y, z] = \boldsymbol{\theta} \\
\left(\hat{\gamma}_{b_{k+1}}^{b_k}\right)^{-1} \otimes (\mathbf{q}_{b_k}^{w})^{-1} \otimes \mathbf{q}_{b_{k+1}}^{w} &\rightarrow 2\left[\left(\hat{\gamma}_{b_{k+1}}^{b_k}\right)^{-1} \otimes (\mathbf{q}_{b_k}^{w})^{-1} \otimes \mathbf{q}_{b_{k+1}}^{w}\right]_{xyz}
\end{aligned}
\tag{173}
$$

Tip

임의의 angle-axis 벡터 $\boldsymbol{\theta} = \theta\mathbf{u}$가 주어졌을 때, 이에 대한 exponential map은 오일러 공식의 확장 버전으로 표현할 수 있다.

$$\mathbf{q} \triangleq \mathrm{Exp}(\boldsymbol{\theta}) = \mathrm{Exp}(\theta\mathbf{u}) = e^{\theta\mathbf{u}/2} = \cos\frac{\theta}{2} + \mathbf{u}\sin\frac{\theta}{2} = \begin{bmatrix} \cos(\theta/2) \\ \mathbf{u}\sin(\theta/2) \end{bmatrix} \tag{171}$$

충분히 작은 $\boldsymbol{\theta}$값에 대해 $\cos\frac{\theta}{2} \approx 1$과 $\sin\frac{\theta}{2} \approx \frac{\theta}{2}$이 만족하므로 충분히 작은 쿼터니언에 대한 다음 식이 성립한다.

$$\mathbf{q} \approx \begin{bmatrix} 1 \\ \frac{1}{2}\boldsymbol{\theta} \end{bmatrix} \tag{172}$$

이에 대한 자세한 내용은 Quaternion kinematics for the error-state Kalman filter 내용 정리 포스트의 4.4 챕터를 참고하면 된다.

최종적인 SO(3) 버전 IMU measurement 에러 $\mathbf{e}_{\mathcal{B}}$는 다음과 같다.

$$\mathbf{e}_{\mathcal{B}}(\mathcal{X}_k) = \begin{bmatrix} \mathbf{R}_w^{b_k}(\mathbf{p}_{b_{k+1}}^w - \mathbf{p}_{b_k}^w - \mathbf{v}_{b_k}^w \Delta t_k + \frac{1}{2}\mathbf{g}^w \Delta t_k^2) - \hat{\alpha}_{b_{k+1}}^{b_k} \\ 2\left[\left(\hat{\gamma}_{b_{k+1}}^{b_k}\right)^{-1} \otimes (\mathbf{q}_{b_k}^w)^{-1} \otimes \mathbf{q}_{b_{k+1}}^w\right]_{xyz} \\ \mathbf{R}_w^{b_k}(\mathbf{v}_{b_{k+1}}^w - \mathbf{v}_{b_k}^w + \mathbf{g}^w \Delta t_k) - \hat{\beta}_{b_{k+1}}^{b_k} \\ \mathbf{b}_{a_{k+1}} - \mathbf{b}_{a_k} \\ \mathbf{b}_{g_{k+1}} - \mathbf{b}_{g_k} \end{bmatrix} \tag{174}$$

$[\mathbf{p}, \mathbf{q}], [\mathbf{v}, \mathbf{b}_a, \mathbf{b}_g]$에 대한 자코비안을 편하게 계산하기 위해 기존 상태 변수의 두번째 줄 β와 세번째 줄 γ의 순서를 서로 변경하였다.

최종적으로 SO(3) 버전 IMU measurement 에러의 자코비안은 다음과 같이 구할 수 있다. 자세한 유도 과정은 Formula Derivation and Analysis of the VINS-Mono 논문의 Appendix 섹션을 참고하면 된다.

$$\mathbf{J}[0]_{15\times 6} = \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{p}_{b_k}^w, \mathbf{q}_{b_k}^w]} = \begin{bmatrix} -\mathbf{R}_w^{b_k} & [\mathbf{R}_w^{b_k}(\mathbf{p}_{b_{k+1}}^w - \mathbf{p}_{b_k}^w - \mathbf{v}_{b_k}^w \Delta t_k + \frac{1}{2}\mathbf{g}^w \Delta t_k^2)]_\times \\ 0 & [\gamma_{b_{k+1}}^{b_k}]_R[(\mathbf{q}_{b_{k+1}}^w)^{-1} \otimes \mathbf{q}_w^{b_k}]_{L,3\times 3} \\ 0 & [\mathbf{R}_w^{b_k}(\mathbf{p}_{b_{k+1}}^w - \mathbf{p}_{b_k}^w + \mathbf{g}^w \Delta t_k)]_\times \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{175}$$

$$\mathbf{J}[1]_{15\times 9} = \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{v}_{b_k}^w, \mathbf{b}_{a_k}, \mathbf{b}_{g_k}]} = \begin{bmatrix} -\mathbf{R}_w^{b_k}\Delta t_k & -\mathbf{J}_{b_a}^{\alpha} & -\mathbf{J}_{b_g}^{\alpha} \\ 0 & 0 & -[(\hat{\gamma}_{b_{k+1}}^{b_k})^{-1} \otimes (\mathbf{q}_{b_k}^w)^{-1} \otimes \mathbf{q}_{b_{k+1}}^w]_{R,3\times 3}\mathbf{J}_{b_g}^{\gamma} \\ -\mathbf{R}_w^{b_k} & -\mathbf{J}_{b_a}^{\beta} & -\mathbf{J}_{b_g}^{\beta} \\ 0 & -\mathbf{I} & 0 \\ 0 & 0 & -\mathbf{I} \end{bmatrix} \tag{176}$$

$$\mathbf{J}[2]_{15\times 6} = \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{p}_{b_{k+1}}^w, \mathbf{q}_{b_{k+1}}^w]} = \begin{bmatrix} \mathbf{R}_w^{b_k} & 0 \\ 0 & [(\hat{\gamma}_{b_{k+1}}^{b_k})^{-1} \otimes (\mathbf{q}_{b_k}^w)^{-1} \otimes \mathbf{q}_{b_{k+1}}^w]_L \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{177}$$

$$\mathbf{J}[3]_{15\times 9} = \frac{\partial \mathbf{e}_{\mathcal{B}}}{\partial[\mathbf{v}_{b_{k+1}}^w, \mathbf{b}_{a_{k+1}}, \mathbf{b}_{g_{k+1}}]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbf{R}_w^{b_k} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \tag{178}$$

NOTICE: 기존의 $\mathbf{J}[0], \mathbf{J}[2] \in \mathbb{R}^{15\times 7}$이었으나 쿼터니언을 SO(3) 기반으로 업데이트하면서 $[xyz]$ 부분만 사용하므로 w 부분은 항상 0이된다. w 부분을 생략하여 표기하면 $\mathbf{J}[0], \mathbf{J}[2] \in \mathbb{R}^{15\times 6}$이 된다.

NOTICE: 위 식을 보면 자코비안 안에 또다른 자코비안 $\mathbf{J}_{b_a}^{\alpha}, \mathbf{J}_{b_g}^{\alpha}, \mathbf{J}_{b_a}^{\beta}, \mathbf{J}_{b_g}^{\beta} \mathbf{J}_{b_g}^{\gamma}$이 사용된 것을 알 수 있다. 이는 IMU의 에러 상태 방정식(error-state equation)에서 파생된 자코비안 $\mathbf{J}_{b_{k+1}}^{b_k}$의 부분 자코비안을 의미한다.

Tip

IMU의 이산(discrete) 신호에 대한 에러 상태 방정식은 다음과 같다. (Mid-point 근사 방법 사용)

$$\begin{bmatrix}\delta\alpha_{k+1}\\ \delta\boldsymbol{\theta}_{k+1}\\ \delta\beta_{k+1}\\ \delta\mathbf{b}_{a_{k+1}}\\ \delta\mathbf{b}_{g_{k+1}}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{F}_{01} & \delta t\mathbf{I} & \mathbf{F}_{03} & \mathbf{F}_{04}\\ & \mathbf{F}_{11} & & & -\delta t\mathbf{I}\\ & \mathbf{F}_{21} & \mathbf{I} & \mathbf{F}_{23} & \mathbf{F}_{24}\\ & & & \mathbf{I} & \\ & & & & \mathbf{I}\end{bmatrix}\begin{bmatrix}\delta\alpha_k\\ \delta\boldsymbol{\theta}_k\\ \delta\beta_k\\ \delta\mathbf{b}_{a_k}\\ \delta\mathbf{b}_{g_k}\end{bmatrix} + \begin{bmatrix}\mathbf{G}_{00} & \mathbf{G}_{01} & \mathbf{G}_{02} & \mathbf{G}_{03} & & \\ & -\delta t/2\mathbf{I} & & -\delta t/2\mathbf{I} & & \\ -\frac{\mathbf{R}_k\delta t}{2} & \mathbf{G}_{21} & -\frac{\mathbf{R}_{k+1}\delta t}{2} & \mathbf{G}_{23} & & \\ & & & & \delta t\mathbf{I} & \\ & & & & & \delta t\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{n}_{a_k}\\ \mathbf{n}_{g_k}\\ \mathbf{n}_{a_{k+1}}\\ \mathbf{n}_{g_{k+1}}\\ \mathbf{n}_{b_a}\\ \mathbf{n}_{b_g}\end{bmatrix} \tag{179}$$

이 때 상태 변수에 대한 자코비안 $\mathbf{J}_t^{b_k}$은 다음과 같이 업데이트 된다.

$$\mathbf{J}_{t+\delta t}^{b_k} = (\mathbf{I} + \mathbf{F}_t\delta t)\mathbf{J}_t^{b_k}, \quad t \in [k, k+1] \tag{180}$$

보다 자세한 내용은 [SLAM] Formula Derivation and Analysis of the VINS-mono 내용 정리 포스트의 2.3, 2.4 섹션을 참고하면 된다.

7.3 Code implementations

- VINS-mono 코드: integration_base.h#L180
  - SO(3) 버전 IMU measurement 에러 $\mathbf{e}_{\mathcal{B}}$가 구현되어 있다.
- VINS-mono 코드: imu_factor.h#L86
  - $\mathbf{J}[0], \mathbf{J}[1], \mathbf{J}[2], \mathbf{J}[3]$가 구현되어 있다.
  - 자코비안과 에러 함수에 공분산의 제곱근 역함수 $\sqrt{(\mathbf{P}_{b_{k+1}}^{b_k})^{-1}} = \sqrt{\boldsymbol{\Omega}_{\mathcal{B}}}$이 information 행렬의 형태로 곱해진다.
    - $\mathbf{e}_{\mathcal{B},k} \quad \rightarrow \quad \sqrt{\boldsymbol{\Omega}_{\mathcal{B}}}^{\intercal}\mathbf{e}_{\mathcal{B},k}$: 실제 코드에서는 오른쪽 에러항이 최적화된다.
    - 이는 실제 코드 구현에서 에러 함수 $\mathbf{E}_{\mathcal{B}}(\mathcal{X}) = \mathbf{e}_{\mathcal{B},k}^{\intercal}\boldsymbol{\Omega}_{\mathcal{B}}\mathbf{e}_{\mathcal{B},k}$의 제곱근 $\sqrt{\boldsymbol{\Omega}_{\mathcal{B}}}^{\intercal}\mathbf{e}_{\mathcal{B},k}$을 에러로 설정했기 때문이다.
- VINS-mono 코드: integration_base.h#L90
  - 에러 상태 방정식을 Mid-point 방법으로 근사한 $\mathbf{F}, \mathbf{G}$ 상태천이행렬이 구현되어 있다.
  - IMU 상태 변수의 자코비안 업데이트 공식 $\mathbf{J}_{t+\delta t}^{b_k} = (\mathbf{I} + \mathbf{F}_t\delta t)\mathbf{J}_t^{b_k}$이 구현되어 있다.
  - IMU 상태 변수의 공분산 업데이트 공식 $\mathbf{P}_{t+\delta t}^{b_k} = (\mathbf{I}+\mathbf{F}_t\delta t)\mathbf{P}_t^{b_k}(\mathbf{I}+\mathbf{F}_t\delta t)^{\intercal}+(\mathbf{G}_t\delta t)\mathbf{Q}(\mathbf{G}_t\delta t)^{\intercal}$이 구현되어 있다.

8 Other jacobians

8.1 Jacobian of unit quaternion

NOMENCLATURE of jacobian of unit quaternion

- $\mathbf{X} = [X, Y, Z, 1]^{\intercal} = [\tilde{\mathbf{X}}, 1]^{\intercal} \in \mathbb{P}^3$
- $\tilde{\mathbf{X}} = [X, Y, Z]^{\intercal} \in \mathbb{P}^2$
- $\mathbf{q} = [w, x, y, z]^{\intercal} = [w, \mathbf{v}]^{\intercal}$
  - hamilton 표기법으로 표현한 쿼터니언. 이에 대한 자세한 내용은 해당 포스트를 참조하면 된다.

앞서 reprojection 에러 섹션에서 설명했던 자코비안은 다음과 같다.

$$\mathbf{J}_c = \frac{\partial\hat{\mathbf{p}}}{\partial\tilde{\mathbf{p}}}\frac{\partial\tilde{\mathbf{p}}}{\partial\mathbf{X}'}\frac{\partial\mathbf{X}'}{\partial[\mathbf{R}, \mathbf{t}]} \tag{181}$$

이 중, $\frac{\partial\mathbf{X}'}{\partial\mathbf{R}}$은 회전을 회전행렬 $\mathbf{R}$로 표현하였을 때 사용할 수 있는 자코비안이다. 해당 섹션에서는 회전을 단위 쿼터니언 $\mathbf{q}$로 표현하였을 때 사용할 수 있는 자코비안 $\frac{\partial\mathbf{X}'}{\partial\mathbf{q}}$에 대해 설명한다.

3차원 공간 상의 점 $\mathbf{X}$이 주어졌을 때 임의의 단위 쿼터니언 $\mathbf{q}$을 통해 회전한 점 $\mathbf{X}'$는 다음과 같이 나타낼 수 있다.

$$\tilde{\mathbf{X}}' = \mathbf{q} \otimes \tilde{\mathbf{X}} \otimes \mathbf{q}^* \tag{182}$$

이를 다시 풀어서 전개하면 다음과 같다.

$$\begin{aligned}
\tilde{\mathbf{X}}' &= \mathbf{q} \otimes \tilde{\mathbf{X}} \otimes \mathbf{q}^* \\
&= (w + \mathbf{v}) \otimes \tilde{\mathbf{X}} \otimes (w - \mathbf{v}) \\
&= w^2\tilde{\mathbf{X}} + w(\mathbf{v} \otimes \tilde{\mathbf{X}} - \tilde{\mathbf{X}} \otimes \mathbf{v}) - \mathbf{v} \otimes \tilde{\mathbf{X}} \otimes \mathbf{v} \\
&= w^2\tilde{\mathbf{X}} + 2w(\mathbf{v} \times \tilde{\mathbf{X}}) - [(-\mathbf{v}^\intercal\tilde{\mathbf{X}} + \mathbf{v} \times \tilde{\mathbf{X}}) \otimes \mathbf{v}] \\
&= w^2\tilde{\mathbf{X}} + 2w(\mathbf{v} \times \tilde{\mathbf{X}}) - [(-\mathbf{v}^\intercal\tilde{\mathbf{X}})\mathbf{v} + (\mathbf{v} \times \tilde{\mathbf{X}}) \otimes \mathbf{v}] \\
&= w^2\tilde{\mathbf{X}} + 2w(\mathbf{v} \times \tilde{\mathbf{X}}) - [(-\mathbf{v}^\intercal\tilde{\mathbf{X}})\mathbf{v} - \cancel{(\mathbf{v} \times \tilde{\mathbf{X}})^\intercal\mathbf{v}} + (\mathbf{v} \times \tilde{\mathbf{X}}) \times \mathbf{v}] \\
&= w^2\tilde{\mathbf{X}} + 2w(\mathbf{v} \times \tilde{\mathbf{X}}) - [(-\mathbf{v}^\intercal\tilde{\mathbf{X}})\mathbf{v} + (\mathbf{v}^\intercal\mathbf{v})\tilde{\mathbf{X}} - (\mathbf{v}^\intercal\tilde{\mathbf{X}})\mathbf{v}] \\
&= w^2\tilde{\mathbf{X}} + 2w(\mathbf{v} \times \tilde{\mathbf{X}}) + 2(\mathbf{v}^\intercal\tilde{\mathbf{X}})\mathbf{v} - (\mathbf{v}^\intercal\mathbf{v})\tilde{\mathbf{X}}
\end{aligned} \tag{183}$$

이를 사용하여 쿼터니언에 대한 자코비안 $\frac{\partial\tilde{\mathbf{X}}'}{\partial\mathbf{q}}$을 구할 수 있다. 스칼라 파트 $\frac{\partial\tilde{\mathbf{X}}'}{\partial w}$와 벡터 파트 $\frac{\partial\tilde{\mathbf{X}}'}{\partial\mathbf{v}}$로 나누어 구하면 다음과 같다.

$$\begin{aligned}
\frac{\partial\tilde{\mathbf{X}}'}{\partial w} &= 2(w\tilde{\mathbf{X}} + \mathbf{v} \times \tilde{\mathbf{X}}) \\
\frac{\partial\tilde{\mathbf{X}}'}{\partial\mathbf{v}} &= -2w[\tilde{\mathbf{X}}]_\times + 2(\mathbf{v}^\intercal\tilde{\mathbf{X}}\mathbf{I} + \mathbf{v}\tilde{\mathbf{X}}^\intercal) - 2\tilde{\mathbf{X}}\mathbf{v}^\intercal \\
&= 2(\mathbf{v}^\intercal\tilde{\mathbf{X}}\mathbf{I} + \mathbf{v}\tilde{\mathbf{X}}^\intercal - \tilde{\mathbf{X}}\mathbf{v}^\intercal - w[\tilde{\mathbf{X}}]_\times)
\end{aligned} \tag{184}$$

이 때, 쿼터니언 곱셈 중앙에 들어가는 $\tilde{\mathbf{X}}$는 실제로는 3차원 벡터가 들어가지 않고 스칼라 값이 0인 순수 쿼터니언(pure quaternion) $[0, X, Y, Z]^\intercal$ 형태로 변형되어 들어간다. **따라서 위 식에서 스칼라에 대한 자코비안 $\frac{\partial\tilde{\mathbf{X}}'}{\partial w}$는 실제 최적화 수행 시 사용되지 않기 때문에 별도로 구하지 않고 벡터에 대한 자코비안 $\frac{\partial\tilde{\mathbf{X}}'}{\partial\mathbf{v}}$만 구한다.**

$$\begin{gathered}
\tilde{\mathbf{X}}' = \mathbf{q} \otimes \tilde{\mathbf{X}} \otimes \mathbf{q}^* \quad \rightarrow \quad \begin{bmatrix} 0 \\ \tilde{\mathbf{X}}' \end{bmatrix} = \mathbf{q} \otimes \begin{bmatrix} 0 \\ \tilde{\mathbf{X}} \end{bmatrix} \otimes \mathbf{q}^* \quad \cdots \text{ strict notation} \\
\text{Then, } \frac{\partial\tilde{\mathbf{X}}'}{\partial w} \text{ is going to be useless}
\end{gathered} \tag{185}$$

또한, 충분히 작은 회전행렬을 $\mathbf{R} \approx \mathbf{I} + [\mathbf{w}]_\times$로 근사했던 방법과 동일하게 쿼터니언 $\mathbf{q}$이 충분히 작다고 가정하면 이는 identity로 근사할 수 있다($\mathbf{q} \approx \mathbf{q}_1 = [1, 0, 0, 0]^\intercal$).

$$\begin{aligned}
\left.\frac{\partial\tilde{\mathbf{X}}'}{\partial\mathbf{v}}\right|_{\mathbf{q}\approx\mathbf{q}_1} &= 2(\mathbf{v}^\intercal\tilde{\mathbf{X}}\mathbf{I} + \mathbf{v}\tilde{\mathbf{X}}^\intercal - \tilde{\mathbf{X}}\mathbf{v}^\intercal - w[\tilde{\mathbf{X}}]_\times) \\
&= -2[\tilde{\mathbf{X}}]_\times
\end{aligned} \tag{186}$$

따라서 최종적인 쿼터니언에 대한 자코비안 $\frac{\partial\tilde{\mathbf{X}}'}{\partial\mathbf{q}}$은 다음과 같다.

$$\boxed{\frac{\partial\tilde{\mathbf{X}}'}{\partial\mathbf{q}} = -2[\tilde{\mathbf{X}}]_\times = -2\begin{bmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{bmatrix} \in \mathbb{R}^{3\times3}} \tag{187}$$

8.1.1 Code Implementations

- ProSLAM 코드: trajectory_analyzer.cpp#L253
  - jinyongjeong님의 블로그 포스트를 참고하였다.

8.2 Jacobian of camera intrinsics

NOMENCLATURE of jacobian of camera intrinsics

- $\pi^{-1}(\cdot) = Z\mathbf{K}^{-1}(\cdot)$: 이미지 상의 점을 3차원 공간 상에 back projection하는 함수
- $\pi(\cdot) = \pi_k(\pi_h(\cdot)) = \mathbf{K}(\frac{1}{Z}\cdot)$: 3차원 공간 상의 점을 이미지 평면 상에 프로젝션하는 함수

- $\mathbf{K} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$: 카메라 내부(intrinsic) 파라미터
- $\mathbf{K}^{-1} = \begin{bmatrix} f_x^{-1} & 0 & -f_x^{-1}c_x \\ 0 & f_y^{-1} & -f_y^{-1}c_y \\ 0 & 0 & 1 \end{bmatrix}$
- $\tilde{\mathbf{K}} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \end{bmatrix}$: $\mathbb{P}^2 \to \mathbb{R}^2$로 프로젝션하기 위해 내부 파라미터의 마지막 행을 생략했다.
- $\mathbf{X} = [\tilde{\mathbf{X}}, 1]^\intercal$

SLAM을 수행하기 위해 카메라 캘리브레이션을 수행하면 내부 파라미터(intrinsic matrix) $\mathbf{c} = [f_x, f_y, c_x, c_y]$와 렌즈 왜곡 파라미터 $\mathbf{d} = [k_1, k_2, p_1, p_2]$를 구할 수 있다. 하지만 캘리브레이션 값이 정확히 실제 센서의 파라미터와 일치하지는 않으므로 이를 최적화를 통해 fine tuning할 수 있다. 본 섹션에서는 이 중 $\mathbf{c}$에 대한 자코비안 $\mathbf{J_c}$을 유도하는 과정에 대해 설명한다. 이 때, 초점 거리(focal length)는 $f_x \neq f_y$라고 가정한다.

예를 들어, (68) photometric 에러 대한 $\mathbf{J_c}$를 구한다고 가정해보자. 이는 다음과 같이 나타낼 수 있다.

$$\begin{aligned} \mathbf{J_c} &= \frac{\partial \mathbf{e}}{\partial \mathbf{c}} \\ &= \frac{\partial \mathbf{I}}{\partial \mathbf{p}_2} \frac{\partial \mathbf{p}_2}{\partial \tilde{\mathbf{p}}_2} \frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'} \frac{\partial \mathbf{X}'}{\partial \mathbf{c}} \\ &= \mathbb{R}^{1\times 2} \cdot \mathbb{R}^{2\times 3} \cdot \mathbb{R}^{3\times 4} \cdot \mathbb{R}^{4\times 4} = \mathbb{R}^{1\times 4} \end{aligned} \tag{188}$$

이 때, 맨 앞의 $\frac{\partial \mathbf{I}}{\partial \mathbf{p}_2}$ 항은 photometric 에러를 구하기 위해 구해야 하는 자코비안이고 나머지 세 자코비안은 reprojection, photometric 에러 항과 관계없이 항상 구해야하는 항이다. **따라서 $\frac{\partial \mathbf{p}_2}{\partial \tilde{\mathbf{p}}_2} \frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'} \frac{\partial \mathbf{X}'}{\partial \mathbf{c}}$ 를 구하면 일반적으로 SLAM에서 사용하는 에러 항에 대해 모두 적용이 가능하다.**

두 카메라 $\{C_1\}, \{C_2\}$의 이미지 평면 상의 점 $\mathbf{p}_1, \mathbf{p}_2$의 관계는 다음과 같이 풀어 쓸 수 있다.

$$\begin{aligned} \mathbf{p}_1 &= \begin{bmatrix} u_1 & v_1 \end{bmatrix}^\intercal \\ \mathbf{p}_2 &= \begin{bmatrix} u_2 & v_2 \end{bmatrix}^\intercal \end{aligned} \tag{189}$$

$$\begin{aligned} \mathbf{p}_2 &= \pi(\mathbf{X}') \\ &= \pi(\mathbf{R}\mathbf{X} + \mathbf{t}) \\ &= \pi(\mathbf{R}\pi^{-1}(\mathbf{p}_1) + \mathbf{t}) \quad \cdots \text{ apply back-proj} \\ &= \pi(\mathbf{R}(Z\mathbf{K}^{-1}\mathbf{p}_1) + \mathbf{t}) \\ &= \pi_k(\pi_h(\mathbf{R}(Z\mathbf{K}^{-1}\mathbf{p}_1) + \mathbf{t})) \\ &= \pi_k(\frac{Z}{Z'}\mathbf{R}\mathbf{K}^{-1}\mathbf{p}_1 + \frac{1}{Z'}\mathbf{t}) \quad \cdots \text{ apply } \pi_h(\cdot) \\ &= \frac{Z}{Z'}\tilde{\mathbf{K}}\mathbf{R}\mathbf{K}^{-1}\mathbf{p}_1 + \frac{1}{Z'}\tilde{\mathbf{K}}\mathbf{t} \quad \cdots \text{ apply } \pi_k(\cdot) \end{aligned} \tag{190}$$

$\mathbf{p}_1$에 back projection $\to$ 변환행렬 적용 $\to$ 프로젝션이 연쇄적으로 발생하여 $\mathbf{p}_2$가 되기 때문에 위와 같은 복잡한 형태의 공식이 얻어진다. 위 식에서 보다시피 $\frac{\partial \mathbf{p}_2}{\partial \tilde{\mathbf{p}}_2} \frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'} \frac{\partial \mathbf{X}'}{\partial \mathbf{c}}$는 $\mathbf{p}_2$부터 $\mathbf{c}$ 파라미터를 포함한다. 따라서 세 자코비안을 묶어서 $\frac{\partial \mathbf{p}_2}{\partial \mathbf{c}}$를 한 번에 계산해야 한다.

$$\begin{aligned} \frac{\partial \mathbf{p}_2}{\partial \mathbf{c}} &= \frac{\partial}{\partial \mathbf{c}} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \\ &= \frac{\partial}{\partial [f_x, f_y, c_x, c_y]} \begin{bmatrix} f_x\tilde{u}_2 + c_x \\ f_y\tilde{v}_2 + c_y \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial u_2}{\partial f_x} & \frac{\partial u_2}{\partial f_y} & \frac{\partial u_2}{\partial c_x} & \frac{\partial u_2}{\partial c_y} \\ \frac{\partial v_2}{\partial f_x} & \frac{\partial v_2}{\partial f_y} & \frac{\partial v_2}{\partial c_x} & \frac{\partial v_2}{\partial c_y} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{u}_2 + f_x\frac{\partial \tilde{u}_2}{\partial f_x} & f_x\frac{\partial \tilde{u}_2}{\partial f_y} & f_x\frac{\partial \tilde{u}_2}{\partial c_x} + 1 & f_x\frac{\partial \tilde{u}_2}{\partial c_y} \\ f_y\frac{\partial \tilde{v}_2}{\partial f_x} & \tilde{v}_2 + f_y\frac{\partial \tilde{v}_2}{\partial f_y} & f_y\frac{\partial \tilde{v}_2}{\partial c_x} & f_y\frac{\partial \tilde{v}_2}{\partial c_y} + 1 \end{bmatrix} \in \mathbb{R}^{2\times 4} \end{aligned} \tag{191}$$

다음으로 위 식의 원소를 계산해야 한다.

$$\begin{pmatrix} \frac{\partial \tilde{u}_2}{\partial f_x} & \frac{\partial \tilde{u}_2}{\partial f_y} & \frac{\partial \tilde{u}_2}{\partial c_x} & \frac{\partial \tilde{u}_2}{\partial c_y} \\ \frac{\partial \tilde{v}_2}{\partial f_x} & \frac{\partial \tilde{v}_2}{\partial f_y} & \frac{\partial \tilde{v}_2}{\partial c_x} & \frac{\partial \tilde{v}_2}{\partial c_y} \end{pmatrix} \tag{192}$$

이를 구하기 위해 우선 $\tilde{\mathbf{p}}_2 = [\tilde{u}_2, \tilde{v}_2, 1]^\intercal$을 구하면 다음과 같다.

$$\begin{aligned}
\tilde{\mathbf{p}}_2 &= [\tilde{u}_2, \tilde{v}_2, 1]^\intercal \\
&= \frac{1}{Z'}\tilde{\mathbf{X}}' \\
&= \frac{1}{Z'}(\mathbf{R}\tilde{\mathbf{X}} + \mathbf{t}) \\
&= \frac{Z}{Z'}\mathbf{R}\mathbf{K}^{-1}\mathbf{p}_1 + \frac{1}{Z'}\mathbf{t} \\
&= \frac{Z}{Z'}\begin{bmatrix} & \mathbf{R} & \end{bmatrix}\begin{bmatrix} f_x^{-1} & & -f_x^{-1}c_x \\ & f_y^{-1} & -f_y^{-1}c_y \\ & & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix} + \frac{1}{Z'}\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \\
&= \frac{Z}{Z'}\begin{bmatrix} & \mathbf{R} & \end{bmatrix}\begin{bmatrix} f_x^{-1}(u_1 - c_x) \\ f_y^{-1}(v_1 - c_y) \\ 1 \end{bmatrix} + \frac{1}{Z'}\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \\
&= \frac{Z}{Z'}\begin{bmatrix} r_{11}f_x^{-1}(u_1 - c_x) + r_{12}f_y^{-1}(v_1 - c_y) + r_{13} \\ r_{21}f_x^{-1}(u_1 - c_x) + r_{22}f_y^{-1}(v_1 - c_y) + r_{23} \\ r_{31}f_x^{-1}(u_1 - c_x) + r_{32}f_y^{-1}(v_1 - c_y) + r_{33} \end{bmatrix} + \frac{1}{Z'}\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}
\end{aligned} \tag{193}$$

위 식을 정리하면 다음과 같다.

$$\begin{bmatrix} \tilde{u}_2 \\ \tilde{v}_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{r_{11}f_x^{-1}(u_1 - c_x) + r_{12}f_y^{-1}(v_1 - c_y) + r_{13} + \frac{1}{Z}t_x}{r_{31}f_x^{-1}(u_1 - c_x) + r_{32}f_y^{-1}(v_1 - c_y) + r_{33} + \frac{1}{Z}t_z} \\ \frac{r_{21}f_x^{-1}(u_1 - c_x) + r_{22}f_y^{-1}(v_1 - c_y) + r_{23} + \frac{1}{Z}t_y}{r_{31}f_x^{-1}(u_1 - c_x) + r_{32}f_y^{-1}(v_1 - c_y) + r_{33} + \frac{1}{Z}t_z} \\ 1 \end{bmatrix} \tag{194}$$

이를 바탕으로 (192)을 구해보면 다음과 같다.

$$\begin{aligned}
\frac{\partial \tilde{u}_2}{\partial f_x} &= \frac{Z}{Z'}(r_{31}\tilde{u}_2 - r_{11})f_x^{-2}(u_1 - c_x) \\
\frac{\partial \tilde{u}_2}{\partial f_y} &= \frac{Z}{Z'}(r_{32}\tilde{u}_2 - r_{12})f_y^{-2}(v_1 - c_y) \\
\frac{\partial \tilde{u}_2}{\partial c_x} &= \frac{Z}{Z'}(r_{31}\tilde{u}_2 - r_{11})f_x^{-1} \\
\frac{\partial \tilde{u}_2}{\partial c_y} &= \frac{Z}{Z'}(r_{32}\tilde{u}_2 - r_{12})f_y^{-1} \\
\frac{\partial \tilde{v}_2}{\partial f_x} &= \frac{Z}{Z'}(r_{31}\tilde{v}_2 - r_{21})f_x^{-2}(u_1 - c_x) \\
\frac{\partial \tilde{v}_2}{\partial f_y} &= \frac{Z}{Z'}(r_{32}\tilde{v}_2 - r_{22})f_y^{-2}(v_1 - c_y) \\
\frac{\partial \tilde{v}_2}{\partial c_x} &= \frac{Z}{Z'}(r_{31}\tilde{v}_2 - r_{21})f_x^{-1} \\
\frac{\partial \tilde{u}_2}{\partial c_y} &= \frac{Z}{Z'}(r_{32}\tilde{v}_2 - r_{22})f_y^{-1}
\end{aligned} \tag{195}$$

최종적으로 (191)는 다음과 같다.

$$\begin{aligned}
\frac{\partial \mathbf{p}_2}{\partial \mathbf{c}} &= \begin{bmatrix} \frac{\partial u_2}{\partial f_x} & \frac{\partial u_2}{\partial f_y} & \frac{\partial u_2}{\partial c_x} & \frac{\partial u_2}{\partial c_y} \\ \frac{\partial v_2}{\partial f_x} & \frac{\partial v_2}{\partial f_y} & \frac{\partial v_2}{\partial c_x} & \frac{\partial v_2}{\partial c_y} \end{bmatrix} \\
&= \begin{bmatrix} \tilde{u}_2 + f_x\frac{\partial \tilde{u}_2}{\partial f_x} & f_x\frac{\partial \tilde{u}_2}{\partial f_y} & f_x\frac{\partial \tilde{u}_2}{\partial c_x} + 1 & f_x\frac{\partial \tilde{u}_2}{\partial c_y} \\ f_y\frac{\partial \tilde{v}_2}{\partial f_x} & \tilde{v}_2 + f_y\frac{\partial \tilde{v}_2}{\partial f_y} & f_y\frac{\partial \tilde{v}_2}{\partial c_x} & f_y\frac{\partial \tilde{v}_2}{\partial c_y} + 1 \end{bmatrix} \\
&= \begin{bmatrix} \tilde{u}_2 + \frac{Z}{Z'}f_x^{-1}(r_{31}\tilde{u}_2 - r_{11})(u_1 - c_x) & \frac{Z}{Z'}f_xf_y^{-2}(r_{32}\tilde{u}_2 - r_{12})(v_1 - c_y) & \frac{Z}{Z'}(r_{31}\tilde{u}_2 - r_{11}) + 1 & \frac{Z}{Z'}f_xf_y^{-1}(r_{32}\tilde{u}_2 - r_{12} \\ \frac{Z}{Z'}f_x^{-2}f_y(r_{31}\tilde{v}_2 - r_{21})(u_1 - c_x) & \tilde{v}_2 + \frac{Z}{Z'}f_y^{-1}(r_{32}\tilde{v}_2 - r_{22})(v_1 - c_y) & \frac{Z}{Z'}f_x^{-1}f_y(r_{31}\tilde{v}_2 - r_{21}) & \frac{Z}{Z'}(r_{32}\tilde{u}_2 - r_{12}) + 1 \end{bmatrix}
\end{aligned} \tag{196}$$

8.2.1 Code Implementations

- DSO 코드: Residuals.cpp#L123

– 코드에 대한 자세한 설명은 [SLAM] Direct Sparse Odometry (DSO) 논문 및 코드 리뷰 (2)를 참조하면 된다.

8.3 Jacobian of inverse depth

NOMENCLATURE of jacobian of inverse depth

- $\mathbf{X} = [X, Y, Z, 1]^\intercal = [\tilde{\mathbf{X}}, 1]^\intercal \in \mathbb{P}^3$
- $\tilde{\mathbf{X}} = [X, Y, Z]^\intercal \in \mathbb{P}^2$
- $\rho = \frac{1}{Z}, \rho^{-1} = Z$

8.3.1 Inverse depth parameterization

SLAM에서 inverse depth parameterization란 3차원 점 $\mathbf{X}$를 표현할 때 3개의 파라미터 $[X, Y, Z, 1]$을 사용하는 것이 아닌 1개의 파라미터 (Z의 역수 ρ)만 사용하여 표현하는 방법을 말한다. 이를 통해 이미지 평면 상의 픽셀 $\mathbf{p} = [u, v]$의 위치만 알고 있으면 오직 inverse depth ρ를 사용하여 3차원 점 $\mathbf{X}$를 완벽하게 표현할 수 있다. 이는 최적화를 수행할 때 1개의 파라미터만 추정하면 되므로 계산 상의 이점을 지닌다.

8.3.2 Jacobian of inverse depth

inverse depth의 자코비안을 $\mathbf{J}_\rho$라고 했을 때 photometric 에러에 대한 $\mathbf{J}_\rho$를 구한다고 가정해보자. 이는 다음과 같이 나타낼 수 있다.

$$\begin{aligned}\mathbf{J}_\rho &= \frac{\partial \mathbf{e}}{\partial \rho} \\ &= \frac{\partial \mathbf{I}}{\partial \mathbf{p}_2}\frac{\partial \mathbf{p}_2}{\partial \tilde{\mathbf{p}}_2}\frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'}\frac{\partial \mathbf{X}'}{\partial \rho} \\ &= \mathbb{R}^{1\times 2}\cdot\mathbb{R}^{2\times 3}\cdot\mathbb{R}^{3\times 4}\cdot\mathbb{R}^{4\times 1} = \mathbb{R}^{1\times 1}\end{aligned} \tag{197}$$

이 때, 맨 앞의 $\frac{\partial \mathbf{I}}{\partial \mathbf{p}_2}$ 항은 photometric 에러를 구하기 위해 구해야 하는 자코비안이고 나머지 세 자코비안은 reprojection, photometric 에러 항과 관계없이 항상 구해야하는 항이다. **따라서 $\frac{\partial \mathbf{p}_2}{\partial \tilde{\mathbf{p}}_2}\frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'}\frac{\partial \mathbf{X}'}{\partial \rho}$ 를 구하면 일반적으로 SLAM에서 사용하는 에러 항에 대해 모두 적용이 가능하다.**

우선 $\frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'}$를 inverse depth로 표현하면 아래와 같다. 이는 $\rho' = \frac{1}{Z'}$로 치환하여 표현한 것과 같다.

$$\boxed{\begin{aligned}\frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'} &= \frac{\partial [\tilde{u}_2, \tilde{v}_2, 1]}{\partial \mathbf{X}'} \\ &= \begin{bmatrix}\rho' & 0 & -\rho'^2 X' & 0 \\ 0 & \rho' & -\rho'^2 Y' & 0 \\ 0 & 0 & 0 & 0\end{bmatrix} \in \mathbb{R}^{3\times 4}\end{aligned}} \tag{198}$$

다음으로 $\frac{\partial \mathbf{X}'}{\partial \rho}$를 구해야 한다. 우선 $\mathbf{X}'$는 다음과 같이 풀어서 쓸 수 있다.

$$\begin{aligned}\mathbf{X}' = \begin{bmatrix}\tilde{\mathbf{X}}' \\ 1\end{bmatrix} &= \begin{bmatrix}\mathbf{R}\tilde{\mathbf{X}} + \mathbf{t} \\ 1\end{bmatrix} \\ &= \begin{bmatrix}\mathbf{R}\left(Z\mathbf{K}^{-1}\tilde{\mathbf{X}}\right) + \mathbf{t} \\ 1\end{bmatrix} \\ &= \begin{bmatrix}\mathbf{R}\left(\frac{\mathbf{K}^{-1}\tilde{\mathbf{X}}}{\rho}\right) + \mathbf{t} \\ 1\end{bmatrix}\end{aligned} \tag{199}$$

위 식을 참고하여 $\frac{\partial \mathbf{X}'}{\partial \rho}$를 구하면 다음과 같다.

$$\boxed{\begin{aligned}\frac{\partial \mathbf{X}'}{\partial \rho} &= \begin{bmatrix}-\mathbf{R}\left(\frac{\mathbf{K}^{-1}\tilde{\mathbf{X}}}{\rho^2}\right) \\ 0\end{bmatrix} \\ &= \begin{bmatrix}-\frac{\tilde{\mathbf{X}}' - \mathbf{t}}{\rho} \\ 0\end{bmatrix} \\ &= -\rho^{-1}\begin{bmatrix}X' - t_x \\ Y' - t_y \\ Z' - t_z \\ 0\end{bmatrix} \in \mathbb{R}^{4\times 1}\end{aligned}} \tag{200}$$

위 식에서 두 번째 줄은 (199)을 변형하여 구할 수 있다. 위 두 자코비안을 사용하여 최종적으로 $\frac{\partial \mathbf{p}_2}{\partial \rho}$를 구하면 다음과 같다.

$$\begin{aligned}\frac{\partial \mathbf{p}_2}{\partial \rho} &= \frac{\partial \mathbf{p}_2}{\partial \tilde{\mathbf{p}}_2}\frac{\partial \tilde{\mathbf{p}}_2}{\partial \mathbf{X}'}\frac{\partial \mathbf{X}'}{\partial \rho} \\ &= \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \end{bmatrix} \begin{bmatrix} \rho' & 0 & -\rho'^2 X' & 0 \\ 0 & \rho' & -\rho'^2 Y' & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot -\rho^{-1} \begin{bmatrix} X'-t_x \\ Y'-t_y \\ Z'-t_z \\ 0 \end{bmatrix} \\ &= -\rho^{-1}\rho' \begin{bmatrix} f_x(\tilde{u}_2 t_z - t_x) \\ f_y(\tilde{v}_2 t_z - t_y) \end{bmatrix} \in \mathbb{R}^{2\times 1}\end{aligned} \tag{201}$$

- $\tilde{u}_2 = \frac{X'}{Z'} = \rho' X'$
- $\tilde{v}_2 = \frac{Y'}{Z'} = \rho' Y'$

8.3.3 Code Implementations

- DSO 코드: CoarseInitializer.cpp#L424
  - 코드에 대한 자세한 설명은 [SLAM] Direct Sparse Odometry (DSO) 논문 및 코드 리뷰 (2)를 참조하면 된다.

9 References

[1] [Blog] [SLAM] Bundle Adjustment 개념 리뷰: Reprojection error

[2] [Blog] [SLAM] Optical Flow와 Direct Method 개념 및 코드 리뷰: Photometric error

[3] [Blog] [SLAM] Pose Graph Optimization 개념 설명 및 예제 코드 분석: Relative pose error

[4] [Blog] Plücker Coordinate 개념 정리: Line projection error

[5] [Blog] [SLAM] Formula Derivation and Analysis of the VINS-mono 내용 정리: IMU measurement error

10 Revision log

- 1st: 2023-01-21
- 2nd: 2023-01-22
- 3rd: 2023-01-25
- 4th: 2023-01-28
- 5th: 2023-09-26
- 6th: 2023-11-14
- 7th: 2024-02-06
- 8th: 2024-04-02