Notes on Plücker Coordinates

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1 Introduction

Plücker Coordinates were first introduced by the 19th-century mathematician Julius Plücker. **This representation uses a point in the six-dimensional $\mathbb{P}^5$ space to represent a line in the four-dimensional $\mathbb{P}^3$ space. This method features a one-to-one correspondence between a point on the $\mathbb{P}^5$ quadric and a line in $\mathbb{P}^3$ space, due to its inherent constraints.** For more detailed information about projective space, see this post.

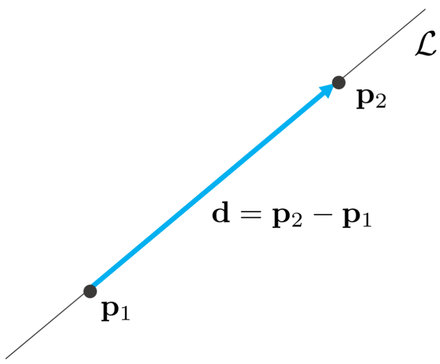
In the MVG book, a point in three-dimensional space is typically denoted as $\mathbf{X} = [X, Y, Z, W] \in \mathbb{P}^3$, but in Plücker Coordinates, to facilitate l_{ij} indexing, the order $[W, X, Y, Z]$ is sometimes used. If denoted in the order $[X, Y, Z, W]$, it appears as (28). **Therefore, attention must be paid to the order when actually using it.**

Plücker Coordinate representation is frequently used in the field of computer graphics, and also in robotics and kinematics for representing Screws and Wrenches using this line representation method. In the SLAM field, this representation method is also used to track or optimize line features. See [4][5].

NOMENCLATURE

- A Plücker line in three-dimensional space is denoted as $\mathcal{L} \in \mathbb{P}^5$.
- A line on the image plane is denoted as $l \in \mathbb{P}^2$. Note that l is not a scalar in this context.
- The Plücker matrix is denoted as $\mathbf{L} \in \mathbb{R}^{6\times 6}$. Note that this is a 6x6 matrix, not a Plücker line.
- The Plücker matrix of a line on the image plane is denoted as $\mathbf{l} \in \mathbb{R}^{3\times 3}$. Note that this refers to a 3x3 matrix, not a line.
- $(*)^\wedge$ denotes the skew-symmetric matrix of vector $*$ e.g., for $\mathbf{x} = [x, y, z]^\intercal$, $\mathbf{x}^\wedge = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$. Other documents may denote this as $[*]_\times$.

2 How to represent a line in 3D space

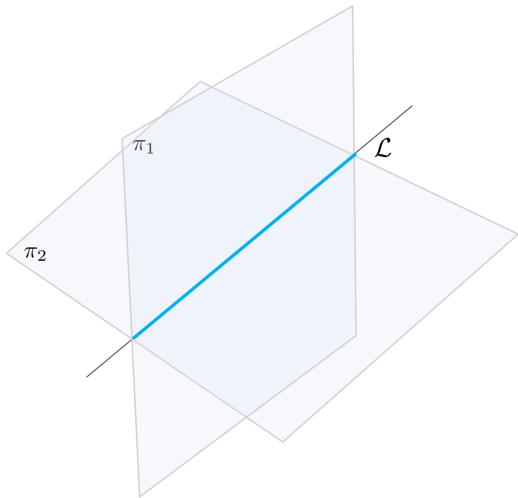


There are various methods to represent a line in three-dimensional space. For example, a line $\mathcal{L}$ can be represented using two points $\mathbf{p}_1, \mathbf{p}_2$.

$$\mathcal{L}(\mathbf{p}_1, \mathbf{p}_2) \tag{1}$$

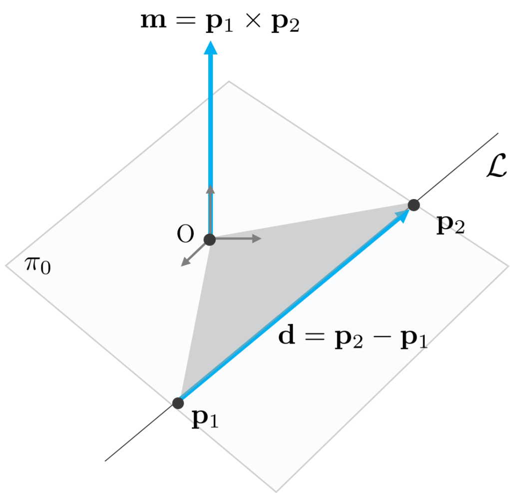
Alternatively, a line can be represented using a point $\mathbf{p}_1$ on the line and a direction vector $\mathbf{d}$.

$$\begin{aligned} &\mathcal{L}(\mathbf{p}_1, \mathbf{d}) \\ &\text{where, } \mathbf{d} = \mathbf{p}_2 - \mathbf{p}_1 \end{aligned} \tag{2}$$



Alternatively, a line can be represented as the intersection of two non-parallel planes π_1, π_2.

$$\mathcal{L}(\pi_1, \pi_2) \tag{3}$$



Alternatively, a line can be represented using the direction vector $\mathbf{d}$ of the line and the direction vector $\mathbf{m}$ of a plane π_0 that includes the origin and contains the line.

$$\begin{aligned} &\mathcal{L}(\mathbf{d}, \mathbf{m}) \in \mathbb{P}^5 \\ &\text{where, } \mathbf{d} = \mathbf{p}_2 - \mathbf{p}_1 \\ &\mathbf{m} = \mathbf{p}_1 \times \mathbf{p}_2 \end{aligned} \tag{4}$$

Here, $\mathbf{m}$ can be considered as the moment vector generated when a point $\mathbf{p}_1$ with unit mass moves to $\mathbf{p}_2$, and the magnitude of $\mathbf{m}$ is twice the area of the triangle formed by $\mathbf{p}_1, \mathbf{p}_2$, and the origin. Since the direction vector $\mathbf{d}$ and the moment vector $\mathbf{m}$ are orthogonal, the following equation holds.

$$\mathbf{m}^\intercal \cdot \mathbf{d} = 0 \tag{5}$$

While $\mathbf{d}$ or $\mathbf{m}$ alone cannot uniquely represent the line $\mathcal{L}$, using the pair $(\mathbf{d}, \mathbf{m})$ allows for the unique representation of the line up to scale.

$$(\mathbf{d} : \mathbf{m}) = (d_x : d_y : d_z : m_x : m_y : m_z) \tag{6}$$

Among the various methods of line representation, using (d, m) is known as Plücker Coordinate representation. This representation method is homogeneous, so for any non-zero scalar λ, $(\mathrm{d} : \mathrm{m}) = (\lambda\mathrm{d} : \lambda\mathrm{m})$ holds.

3 Plücker coordinate representation

The method of representing a line using Plücker Coordinates is as follows. When two points $\mathbf{p}_1, \mathbf{p}_2$ exist in the $\mathbb{P}^3$ space, they can be represented as

$$\begin{aligned}\mathbf{p}_1 &= [W_1, X_1, Y_1, Z_1] \\ \mathbf{p}_2 &= [W_2, X_2, Y_2, Z_2]\end{aligned} \tag{7}$$

For ease of description, they are denoted in the order $[W, X, Y, Z]$. When the direction vector of the line $\mathbf{d} = \mathbf{p}_2 - \mathbf{p}_1$ and the direction vector of the plane containing the origin and the line $\mathbf{m} = \mathbf{p}_2 \times \mathbf{p}_1$ are available, the line $\mathcal{L}$ can be represented as follows.

$$\mathcal{L} = (\mathbf{d} : \mathbf{m}) = (d_x : d_y : d_z : m_x : m_y : m_z) \tag{8}$$

Here, $\mathbf{m}$ is the moment vector between the origin and the line, and if the moment vector is zero, it implies that the line includes the origin. To derive the values in the above equation, consider the matrix $\mathbf{M} \in \mathbb{R}^{2\times 4}$ consisting of the two points as rows and define the matrix determinant operation l_{ij} for the 2x2 submatrix as follows.

$$\begin{aligned}\mathbf{M} &= \begin{bmatrix} W_1 & X_1 & Y_1 & Z_1 \\ W_2 & X_2 & Y_2 & Z_2 \end{bmatrix} \\ l_{ij} &= \begin{vmatrix} X_i & Y_i \\ X_j & Y_j \end{vmatrix} = X_iY_j - X_jY_i\end{aligned} \tag{9}$$

$l_{ii} = 0, l_{ij} = -l_{ji}$ holds true. Consequently, the line $\mathcal{L}$ can be defined as follows.

$$\begin{aligned}\mathcal{L} = (\mathbf{d} : \mathbf{m}) &= (d_x : d_y : d_z : m_x : m_y : m_z) \\ &= (l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12})\end{aligned} \tag{10}$$

This method of line representation allows for a unique representation of the line up to scale, with at least one component non-zero. Typically, Plücker Coordinates are represented in the homogeneous $\mathbb{P}^5$ space using a colon(:) to emphasize the ratios between the elements rather than the values themselves, due to the up to scale nature.

When detailing the components of l_{ij},

$$\begin{aligned} l_{01} &= W_1X_2 - W_2X_1 \quad & l_{23} &= Y_1Z_2 - Z_1Y_2 \\ l_{02} &= W_1Y_2 - W_2Y_1 \quad & l_{31} &= Z_1X_2 - X_1Z_2 \\ l_{03} &= W_1Z_2 - W_2Z_1 \quad & l_{12} &= X_1Y_2 - Y_1X_2 \end{aligned} \tag{11}$$

$$\mathbf{d} = \begin{bmatrix} W_1X_2 - W_2X_1 \\ W_1Y_2 - W_2Y_1 \\ W_1Z_2 - W_2Z_1 \end{bmatrix} \qquad \mathbf{m} = \begin{bmatrix} Y_1Z_2 - Z_1Y_2 \\ Z_1X_2 - X_1Z_2 \\ X_1Y_2 - Y_1X_2 \end{bmatrix} \tag{12}$$

Here, it is evident that $\mathbf{m} = \mathbf{p}_1 \times \mathbf{p}_2$. If $W_1 = W_2 = 1$, then l_{01}, l_{02}, l_{03} can be succinctly represented as follows.

$$l_{01} = X_2 - X_1 \quad l_{02} = Y_2 - Y_1 \quad l_{03} = Z_2 - Z_1 \tag{13}$$

This definition aligns with that of $\mathbf{d} = \mathbf{p}_2 - \mathbf{p}_1$.

3.1 Graßmann–Plücker relations

Plücker Coordinates are characterized by a constraint known as the Graßmann–Plücker relations, which establish a one-to-one correspondence between a point on the $\mathbb{P}^5$ quadric and a line in $\mathbb{P}^3$ space. The corresponding $\mathbb{P}^5$ quadric is specifically referred to as the Klein quadric and satisfies the following constraint.

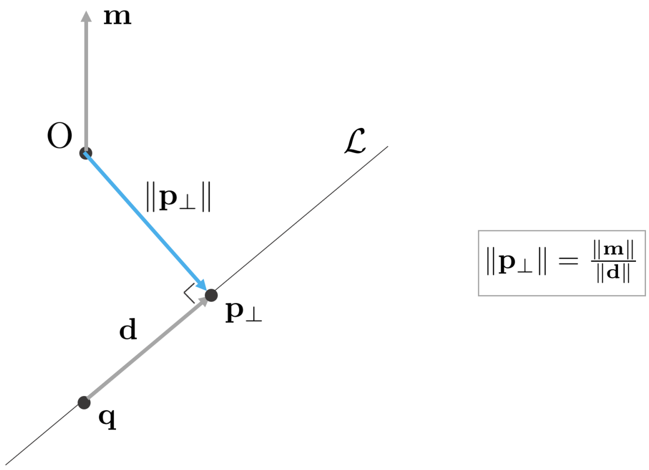
$$l_{01}l_{23} + l_{02}l_{31} + l_{03}l_{12} = 0 \tag{14}$$

Alternatively, since $l_{31} = -l_{13}$, it can also be written as follows.

$$l_{01}l_{23} - l_{02}l_{13} + l_{03}l_{12} = 0 \tag{15}$$

This equation is derived from (5) $\mathbf{m}^\intercal\mathbf{d} = 0$ and can also be determined through $\det \mathbf{L} = 0$.

3.2 Distance to the origin



The distance $\|\mathbf{p}_\perp\|$ from the origin to the line can be calculated as follows. Given any point $\mathbf{q}$ on the line $\mathcal{L}$, and the foot of the perpendicular dropped from the origin to the line being $\mathbf{p}_\perp$, $\mathbf{p}_\perp$ can be expressed through $\mathbf{q}$.

$$\mathbf{q} = \mathbf{p}_\perp + k\mathbf{d} \quad (\text{any } \mathbf{q} \text{ on } \mathcal{L}) \tag{16}$$

Accordingly, the moment vector $\mathbf{m}$ can be expanded as follows.

$$\begin{aligned} \mathbf{m} &= \mathbf{d} \times \mathbf{q} \\ &= \mathbf{d} \times (\mathbf{p}_\perp + k\mathbf{d}) \\ &= \mathbf{d} \times \mathbf{p}_\perp \end{aligned} \tag{17}$$

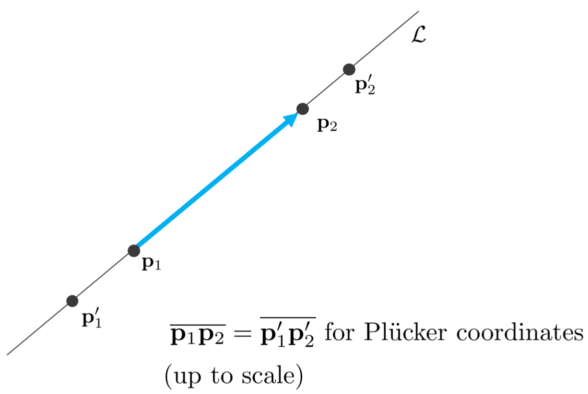
By the definition of the magnitude of a cross product, it can be represented as follows:

$$\|\mathbf{m}\| = \|\mathbf{d}\| \, \|\mathbf{p}_\perp\| \sin \frac{\pi}{2} = \|\mathbf{d}\| \, \|\mathbf{p}_\perp\| \tag{18}$$

Thus, the distance from the origin to the line $\mathcal{L}$ is as follows.

$$\|\mathbf{p}_\perp\| = \frac{\|\mathbf{m}\|}{\|\mathbf{d}\|} \tag{19}$$

3.3 Up to scale uniqueness



When arbitrary points $\mathbf{p}_1, \mathbf{p}_2$ on the line are replaced with $\mathbf{p}'_1, \mathbf{p}'_2$, these can be expressed through the existing points.

$$\begin{aligned} \mathbf{p}'_1 &= \lambda\mathbf{p}_1 + (1-\lambda)\mathbf{p}_2 \\ \mathbf{p}'_2 &= \mu\mathbf{p}_1 + (1-\mu)\mathbf{p}_2 \end{aligned} \tag{20}$$

The direction vector $\mathbf{d}'$ and moment vector $\mathbf{m}$ can also be expressed in scale through the existing points.

$$\begin{aligned} \mathbf{d}' &= \mathbf{p}_2' - \mathbf{p}_1' = (\lambda - \mu)\mathbf{d} \\ \mathbf{m}' &= \mathbf{p}_1' \times \mathbf{p}_2' = (\lambda - \mu)\mathbf{m} \end{aligned} \tag{21}$$

Thus, Plücker Coordinates can uniquely represent a line up to scale.

3.4 Plücker matrix

Let us assume that two points $\mathbf{A}, \mathbf{B}$ exist in $\mathbb{P}^3$ space.

$$\begin{aligned} \mathbf{A} &= (W_1, X_1, Y_1, Z_1) \\ \mathbf{B} &= (W_2, X_2, Y_2, Z_2) \end{aligned} \tag{22}$$

Using these two points, the following Plücker matrix can be defined.

$$\mathbf{L} = \mathbf{A}\mathbf{B}^\intercal - \mathbf{B}\mathbf{A}^\intercal = \begin{bmatrix} 0 & -l_{01} & -l_{02} & -l_{03} \\ l_{01} & 0 & -l_{12} & -l_{13} \\ l_{02} & l_{12} & 0 & -l_{23} \\ l_{03} & l_{13} & l_{23} & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \tag{23}$$

Plücker matrix is a 4x4 skew-symmetric matrix, and the six elements of the lower-triangle matrix become the Plücker coordinates.

$$\mathbf{L} = \mathcal{L}^\wedge \tag{24}$$

$$\begin{aligned} &\mathcal{L} = (l_{01} : l_{02} : l_{03} : l_{12} : l_{13} : l_{23}) \\ &l_{ij} = A_i B_j - B_i A_j \end{aligned} \tag{25}$$

Tip

Note that the order of Plücker coordinates previously described and the sequence used to represent m in the equation above slightly differ.

$$\begin{aligned} &(1)\ \mathcal{L} = (l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12}) \\ &(2)\ \mathcal{L} = (l_{01} : l_{02} : l_{03} : l_{12} : l_{13} : l_{23}) \end{aligned} \tag{26}$$

While [3] lists the sequence as (1), [1] and [2] present it as (2). It appears that method (1) is correct as it satisfies $(l_{23} : l_{31} : l_{12}) = (m_x : m_y : m_z)$, but method (2) is also widely used, so care must be taken with the order.

3.4.1 Properties of the plücker matrix

The Plücker matrix has the following properties.

- The $\mathbf{L}$ matrix has a rank of 2, and the two-dimensional null space of the matrix is spanned by a pencil of planes that pass through the line.
- The $\mathbf{L}$ matrix has four degrees of freedom. Due to the homogeneous nature of the six elements l_{ij}, one degree of freedom is lost, and an additional degree of freedom is lost due to the constraint $\det \mathbf{L} = 0$, resulting in a total of four degrees of freedom.
- $\mathbf{L} = \mathbf{A}\mathbf{B}^\intercal - \mathbf{B}\mathbf{A}^\intercal$ can be seen as a $\mathbb{P}^3$ version of the method of representing a line connecting two points in $\mathbb{P}^2$ space as $\mathbf{l} = \mathbf{a} \times \mathbf{b}$.
  - $\mathbf{a}, \mathbf{b}$: Points $\mathbf{A}, \mathbf{B}$ projected onto the image plane

- The $\mathbf{L}$ matrix is independent of the two points $\mathbf{A}, \mathbf{B}$ that defined it. If a point $\mathbf{C}$ on the line is given, it can be represented as $\mathbf{C} = \mathbf{A} + \mu\mathbf{B}$, and the matrix transforms as follows:

$$\begin{aligned} \hat{\mathbf{L}} &= \mathbf{A}\mathbf{C}^\intercal - \mathbf{C}\mathbf{A}^\intercal = \mathbf{A}(\mathbf{A}^\intercal + \mu\mathbf{B}^\intercal) - (\mathbf{A} + \mu\mathbf{B})\mathbf{A}^\intercal \\ &= \mathbf{A}\mathbf{B}^\intercal - \mathbf{B}\mathbf{A}^\intercal = \mathbf{L} \end{aligned} \tag{27}$$

- Homography matrix $\mathbf{H} \in \mathbb{R}^{4\times 4}$ in $\mathbb{P}^3$ space transforms a 3D point as $\mathbf{X}' = \mathbf{H}\mathbf{X}$. Correspondingly, the Homography transformation for the line is $\mathbf{L}' = \mathbf{H}\mathbf{L}\mathbf{H}^\intercal$.

Tip

- If the coordinates are not in the order $[W, X, Y, Z]$ but in the order $[X, Y, Z, W]$, the original $\mathcal{L}$ transforms as follows.

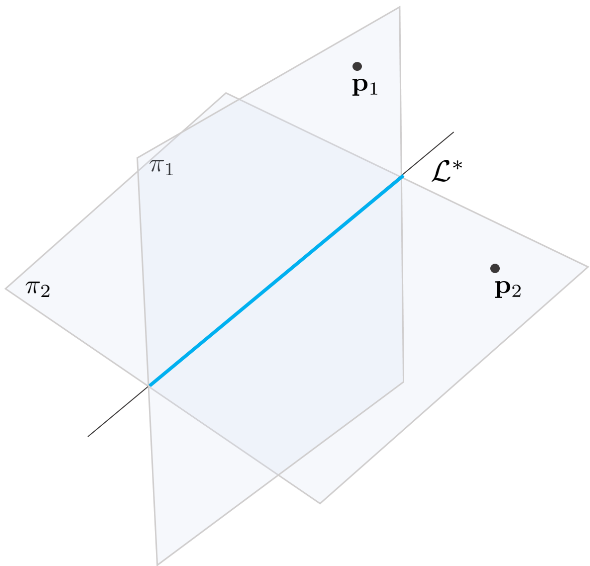
$$\begin{aligned} \mathcal{L} &= (\mathbf{d} : \mathbf{m}) \\ &= (l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12}) \rightarrow (l_{30} : l_{31} : l_{32} : l_{12} : l_{20} : l_{01}) \end{aligned} \tag{28}$$

  - In this case, the $\mathbf{L}$ matrix can be simply constructed as follows:

$$\mathbf{L} = \begin{bmatrix} 0 & -l_{01} & -l_{02} & -l_{03} \\ l_{01} & 0 & -l_{12} & -l_{13} \\ l_{02} & l_{12} & 0 & -l_{23} \\ l_{03} & l_{13} & l_{23} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{m}^\wedge & \mathbf{d} \\ -\mathbf{d}^\intercal & 0 \end{bmatrix} \tag{29}$$

  - Note that it is only possible in the coordinate order $[X, Y, Z, W]$, not $[W, X, Y, Z]$.

4 Dual plücker coordinate representation



Just as points in 3D space are used to represent a line, the intersection of two non-parallel planes can be used to represent a line using Plücker Coordinates. **This method of representing a line by the intersection of two planes is called Dual Plücker Coordinate.**

Using two non-parallel planes, a 2x4 matrix $\mathbf{M}^*$ can be created, and a 2x2 submatrix operation $\mathbf{l}^*_{ij}$

Tip

When points $\mathbf{p}_1, \mathbf{p}_2$ in 3D space are each on planes π_1, π_2, the following formula holds.

$$\begin{aligned} \pi_1^\intercal \mathbf{p}_1 = 0, \quad \pi_2^\intercal \mathbf{p}_2 = 0 \\ \begin{bmatrix} a_w & a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} W_1 \\ X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = 0 \\ \begin{bmatrix} b_w & b_x & b_y & b_z \end{bmatrix} \begin{bmatrix} W_2 \\ X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = 0 \end{aligned} \tag{30}$$

Planes can be parameterized as $\pi_1 = \begin{bmatrix} a_w & a_x & a_y & a_z \end{bmatrix}, \pi_2 = \begin{bmatrix} b_w & b_x & b_y & b_z \end{bmatrix}$.

can be defined.

$$\begin{aligned} \mathbf{M}^* = \begin{bmatrix} a_w & a_x & a_y & a_z \\ b_w & b_x & b_y & b_z \end{bmatrix} \\ l_{ij}^* = \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} = a_x b_y - a_y b_x \end{aligned} \tag{31}$$

The dual representation expressed line $\mathcal{L}^*$ is as follows.

$$\mathcal{L}^* = (l_{23}^* : l_{31}^* : l_{12}^* : l_{01}^* : l_{02}^* : l_{03}^*) \tag{32}$$

This is the same as changing the indices $\{1, 2, 3, 4\}$ in Plücker Coordinate $\mathcal{L}$'s l_{ij} to other values. For example, $l_{01} \leftrightarrow l_{23}^*, l_{12} \leftrightarrow l_{03}^*$ happens.

$$(l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12}) = (l_{23}^* : l_{31}^* : l_{12}^* : l_{01}^* : l_{02}^* : l_{03}^*) \tag{33}$$

4.1 Dual plücker matrix

Let us assume that two planes $\mathbf{P}, \mathbf{Q}$ exist in $\mathbb{P}^3$ space.

$$\begin{aligned} \mathbf{P} = [W_1, X_1, Y_1, Z_1] \\ \mathbf{Q} = [W_2, X_2, Y_2, Z_2] \end{aligned} \tag{34}$$

The dual Plücker matrix $\mathbf{L}^*$ can be defined using the above two planes similarly to the Plücker matrix $\mathbf{L}$.

$$\mathbf{L}^* = \mathbf{P}\mathbf{Q}^\intercal - \mathbf{Q}\mathbf{P}^\intercal = \begin{bmatrix} 0 & l_{23}^* & -l_{13}^* & l_{12}^* \\ -l_{23}^* & 0 & l_{03}^* & -l_{02}^* \\ l_{13}^* & -l_{03}^* & 0 & l_{01}^* \\ -l_{12}^* & l_{02}^* & -l_{01}^* & 0 \end{bmatrix} \in \mathbb{R}^{4\times4} \tag{35}$$

4.1.1 Properties of the dual plücker matrix

The properties of the dual Plücker matrix are as follows.

- A Homography matrix $\mathbf{H} \in \mathbb{R}^{4\times4}$ in $\mathbb{P}^3$ space applies to a 3D point as $\mathbf{X}' = \mathbf{H}\mathbf{X}$. Correspondingly, the dual line's Homography transformation is $\mathbf{L}^{*'} = \mathbf{H}^{-\intercal}\mathbf{L}\mathbf{H}^{-1}$.
- When a line $\mathcal{L}$ and a plane π are given, the intersection point can be found through $\mathbf{X} = \mathbf{L}\pi$. Also, when a line $\mathcal{L}$ and a point $\mathbf{X}$ are given, the plane containing both can be found through $\pi = \mathbf{L}^*\mathbf{X}$. For more details, refer to the content of this section.

– If the line $\mathcal{L}$ exists on the plane π or the point $\mathbf{X}$ exists on the line $\mathcal{L}$, then each yields $\mathbf{L}\pi = 0$, $\mathbf{L}^*\mathbf{X} = 0$. Extending this expression yields the following.

$$\begin{aligned} \mathbf{X} &= \mathbf{L}\pi = 0 \\ \mathbf{L}^*\mathbf{X} &= \mathbf{L}^*\mathbf{L}\pi = 0 \\ \therefore \mathbf{L}^*\mathbf{L} &= 0 \in \mathbb{R}^{4\times 4} \end{aligned} \tag{36}$$

– Explaining the above formula in detail yields the following.

$$\begin{aligned} \begin{bmatrix} 0 & l^*_{23} & -l^*_{13} & l^*_{12} \\ -l^*_{23} & 0 & l^*_{03} & -l^*_{02} \\ l^*_{13} & -l^*_{03} & 0 & l^*_{01} \\ -l^*_{12} & l^*_{02} & -l^*_{01} & 0 \end{bmatrix} \begin{bmatrix} 0 & -l_{01} & -l_{02} & -l_{03} \\ l_{01} & 0 & -l_{12} & -l_{13} \\ l_{02} & l_{12} & 0 & -l_{23} \\ l_{03} & l_{13} & l_{23} & 0 \end{bmatrix} &= 0 \\ [l_{01}l_{23} - l_{02}l_{13} + l_{03}l_{12}] \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} &= 0 \end{aligned} \tag{37}$$

– This equation satisfies the zero condition due to the Graßmann–Plücker relation constraint.

Tip

- If the order $[W, X, Y, Z]$ is changed to $[X, Y, Z, W]$, the original $\mathcal{L}^*$ transforms as follows.

$$\begin{aligned} \mathcal{L} &= (\mathbf{d} : \mathbf{m}) \\ &= (l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12}) \rightarrow (l_{30} : l_{31} : l_{32} : l_{12} : l_{20} : l_{01}) \\ \mathcal{L}^* &= (-\mathbf{m} : -\mathbf{d}) \\ &= (l^*_{23} : l^*_{31} : l^*_{12} : l^*_{01} : l^*_{02} : l^*_{03}) \rightarrow (l^*_{21} : l^*_{02} : l^*_{10} : l^*_{03} : l^*_{13} : l^*_{23}) \end{aligned} \tag{38}$$

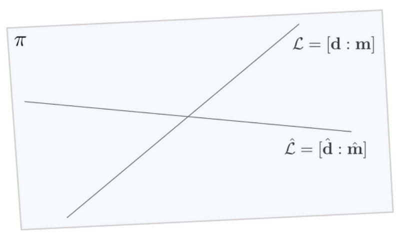
  – At this time, the $\mathbf{L}^*$ matrix can be simply obtained as follows.

$$\mathbf{L}^* = \begin{bmatrix} 0 & l^*_{23} & -l^*_{13} & l^*_{12} \\ -l^*_{23} & 0 & l^*_{03} & -l^*_{02} \\ l^*_{13} & -l^*_{03} & 0 & l^*_{01} \\ -l^*_{12} & l^*_{02} & -l^*_{01} & 0 \end{bmatrix} = \begin{bmatrix} -\mathbf{d}^\wedge & \mathbf{m} \\ -\mathbf{m}^\intercal & 0 \end{bmatrix} \tag{39}$$

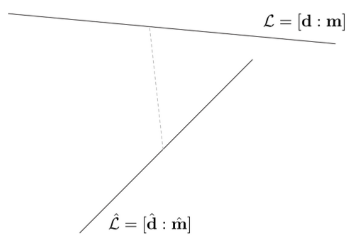
  – Note that this simple construction is only possible when in the order $[X, Y, Z, W]$ and not in $[W, X, Y, Z]$.

5 Uses

5.1 Line-line crossing



$(\mathcal{L}|\hat{\mathcal{L}}) = 0$ if and only if two lines are coplanar



$(\mathcal{L}|\hat{\mathcal{L}}) \neq 0$ if two lines are skew

Assume two lines $\mathcal{L}, \hat{\mathcal{L}}$ exist on the same plane (coplanar) in $\mathbb{P}^3$ space. Line $\mathcal{L}$ is determined by two points $\mathbf{A}, \mathbf{B}$ and line $\hat{\mathcal{L}}$ is determined by two points $\hat{\mathbf{A}}, \hat{\mathbf{B}}$.

$$\begin{aligned} \det[\mathbf{A}, \mathbf{B}, \hat{\mathbf{A}}, \hat{\mathbf{B}}] &= l_{12}\hat{l}_{34} + \hat{l}_{12}l_{34} + l_{13}\hat{l}_{42} + \hat{l}_{13}l_{42} + l_{14}\hat{l}_{23} + \hat{l}_{14}l_{23} \\ &= (\mathcal{L}|\hat{\mathcal{L}}) \end{aligned} \tag{40}$$

As explained in this section, since $\mathcal{L}$ and $\hat{\mathcal{L}}$ are independent of the points that determine them, $(\mathcal{L}|\hat{\mathcal{L}})$ holds true for all points on these lines, not just $\mathbf{A}, \mathbf{B}, \hat{\mathbf{A}}, \hat{\mathbf{B}}$. Two lines must satisfy $\det[\mathbf{A}, \mathbf{B}, \hat{\mathbf{A}}, \hat{\mathbf{B}}] = 0$ to exist on the same plane.

$$(\mathcal{L}|\hat{\mathcal{L}}) = 0 \qquad \text{if and only if two lines are coplanar} \tag{41}$$

This results in the following useful properties:

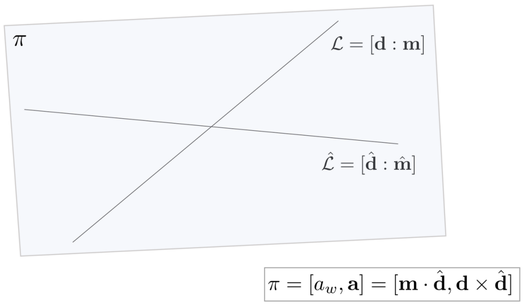
- If any 6-dimensional vector $\mathcal{L}$ satisfies $(\mathcal{L}|\mathcal{L}) = 0$, it represents a line in $\mathbb{P}^3$ space. This is identical to the previously mentioned Klein quadric constraint.
- Two lines $\mathcal{L}, \hat{\mathcal{L}}$ can be determined from two planes. If $\mathcal{L}$ is determined from two planes $\mathbf{P}, \mathbf{Q}$ and $\hat{\mathcal{L}}$ from two planes $\hat{\mathbf{P}}, \hat{\mathbf{Q}}$, then they satisfy the following constraint.

  $$(\mathcal{L}|\hat{\mathcal{L}}) = \det[\mathbf{P}, \mathbf{Q}, \hat{\mathbf{P}}, \hat{\mathbf{Q}}] \tag{42}$$

  As explained previously, two lines intersect when $(\mathcal{L}|\mathcal{L}) = 0$.
- If line $\mathcal{L}$ is determined from two planes $\mathbf{P}, \mathbf{Q}$ and line $\hat{\mathcal{L}}$ is determined from two points $\mathbf{A}, \mathbf{B}$, then they satisfy the following constraint.

  $$(\mathcal{L}|\hat{\mathcal{L}}) = (\mathbf{P}^\intercal\mathbf{A})(\mathbf{Q}^\intercal\mathbf{B}) - (\mathbf{Q}^\intercal\mathbf{A})(\mathbf{P}^\intercal\mathbf{B}) \tag{43}$$

5.2 Line-line join (Plane)



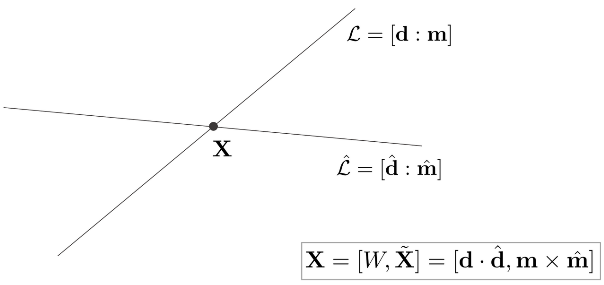
In $\mathbb{P}^3$ space, if two lines $\mathcal{L}, \hat{\mathcal{L}}$ are coplanar and not parallel, the plane π can be determined through these two lines.

$$\begin{aligned} \mathcal{L} &= [\mathbf{d} : \mathbf{m}] \\ \hat{\mathcal{L}} &= [\hat{\mathbf{d}} : \hat{\mathbf{m}}] \\ \pi &= [a_w, a_x, a_y, a_z] = [a_w, \mathbf{a}] \end{aligned} \tag{44}$$

The plane π can be found using the following formula:

$$\pi = [a_w, \mathbf{a}] = [\mathbf{m} \cdot \hat{\mathbf{d}}, \mathbf{d} \times \hat{\mathbf{d}}] \tag{45}$$

5.3 Line-line meet (Point)



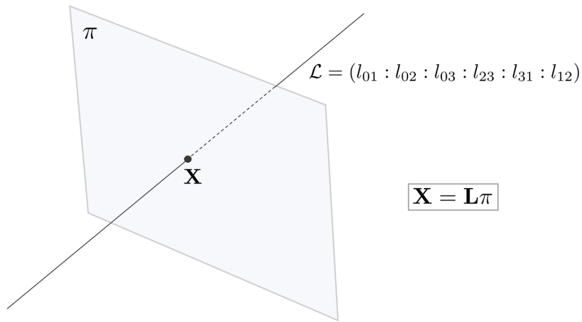
In $\mathbb{P}^3$ space, if two lines $\mathcal{L}, \hat{\mathcal{L}}$ are coplanar and not parallel, they meet at a point $\mathbf{X}$.

$$\begin{aligned}
&\mathcal{L} = [\mathbf{d} : \mathbf{m}], \\
&\hat{\mathcal{L}} = [\hat{\mathbf{d}} : \hat{\mathbf{m}}] \\
&\mathbf{X} = [W, X, Y, Z] = [W, \tilde{\mathbf{X}}]
\end{aligned} \tag{46}$$

The 3D point $\mathbf{X}$ can be obtained through the following formula:

$$\mathbf{X} = [W, \tilde{\mathbf{X}}] = [\mathbf{d} \cdot \hat{\mathbf{d}}, \mathbf{m} \times \hat{\mathbf{m}}] \tag{47}$$

5.4 Plane-line meet (Point)



The intersection point $\mathbf{X}$ created by a line $\mathcal{L}$ and a plane π in 3D space is defined as follows.

$$\mathbf{X} = \mathbf{L}\pi \tag{48}$$

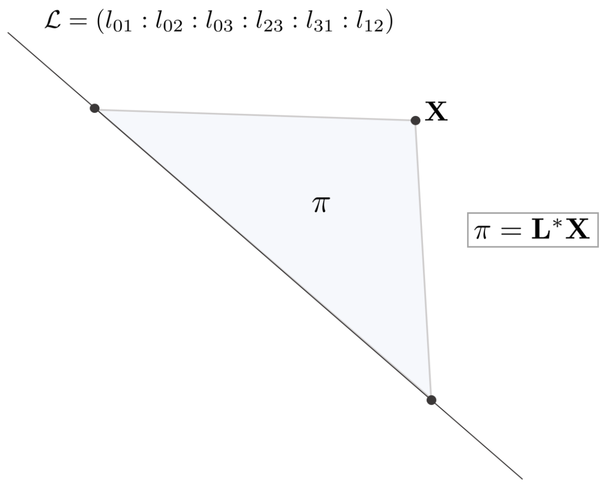
If the line $\mathcal{L}$ exists on the plane π, then $\mathbf{L}\pi = 0$.
For a more detailed explanation using vectors:

$$\begin{aligned}
&\mathcal{L} = [\mathbf{d} : \mathbf{m}], \\
&\pi = [a_w, a_x, a_y, a_z] = [a_w, \mathbf{a}] \\
&\mathbf{X} = [W, X, Y, Z] = [W, \tilde{\mathbf{X}}]
\end{aligned} \tag{49}$$

Therefore, the point $\mathbf{X}$ generated by the intersection of a line and a plane is as follows.

$$\mathbf{X} = [W, \tilde{\mathbf{X}}] = [\mathbf{a} \cdot \mathbf{d}, \mathbf{a} \times \mathbf{m} - a_w \mathbf{d}] \tag{50}$$

5.5 Point-line join (Plane)



The plane π generated by a point $\mathbf{X}$ and a line $\mathcal{L}$ in 3D space can be represented as follows.

$$\pi = \mathbf{L}^*\mathbf{X} \tag{51}$$

If $\mathbf{X}$ exists on the line $\mathcal{L}$, then $\mathbf{L}^*\mathbf{X} = 0$. A detailed explanation of the Dual plücker matrix $\mathbf{L}^*$ is provided in this section.

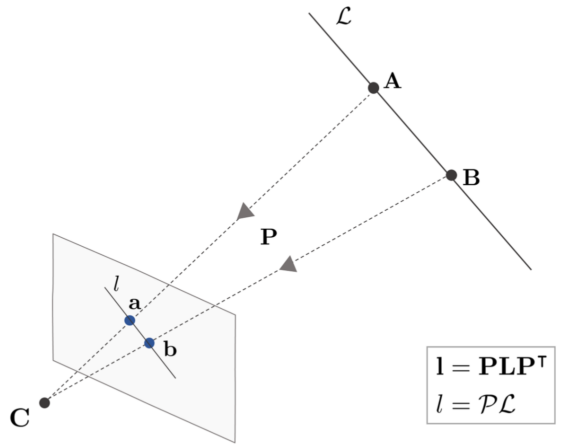
For a more detailed explanation using vectors:

$$\begin{aligned} &\mathcal{L} = [\mathbf{d} : \mathbf{m}], \\ &\mathbf{X} = [W, X, Y, Z] = [W, \tilde{\mathbf{X}}] \\ &\pi = [a_w, a_x, a_y, a_z] = [a_w, \mathbf{a}] \end{aligned} \tag{52}$$

Therefore, the plane π generated by a line and a point is as follows.

$$\pi = [a_w, \mathbf{a}] = [\tilde{\mathbf{X}} \cdot \mathbf{m}, \tilde{\mathbf{X}} \times \mathbf{d} - W\mathbf{m}] \tag{53}$$

5.6 Line projection to the image plane



Consider points $\mathbf{A}, \mathbf{B}$ in 3D space and the line $\mathcal{L}$ connecting these points. Given a camera projection matrix $\mathbf{P}$, the image of the line l projected onto the image plane, and its plücker matrix $\mathbf{l}$ can be represented as follows.

$$l^\wedge = \mathbf{l} = \mathbf{P}\mathbf{L}\mathbf{P}^\intercal \in \mathbb{R}^{3\times 3} \tag{54}$$

- $\mathcal{L}$: Line in 3D space
- $\mathbf{L} = \mathcal{L}^\wedge$: plücker matrix of the line in 3D space (antisymmetric matrix)
- l: Line projected onto the image plane
- $\mathbf{l} = l^\wedge$: plücker matrix of the projected line (antisymmetric matrix)

5.6.1 Line projection matrix

Using the line projection matrix $\mathcal{P}$, it is possible to directly compute l without converting the 3D space line $\mathcal{L}$ into its plücker matrix form $\mathbf{L}$ and then calculating $\mathbf{l}$. Given any camera projection matrix $\mathbf{P} = [\mathbf{N}|\mathbf{n}] \in \mathbb{R}^{3\times 4}$, $\mathcal{P}$ can be defined as follows.

$$\mathcal{P} = [\det(\mathbf{N})\mathbf{N}^{-\intercal}|\mathbf{n}^\wedge\mathbf{N}] \in \mathbb{R}^{3\times 6} \tag{55}$$

If $\mathbf{P} = [\mathbf{R}|\mathbf{t}]$, the formula can be expressed as follows.

$$\mathcal{P} = [\mathbf{R} \mid \mathbf{t}^\wedge\mathbf{R}] \in \mathbb{R}^{3\times 6} \tag{56}$$

This derivation can be followed through the following formula, given two points $\mathbf{A} = [\tilde{\mathbf{A}}|a], \mathbf{B} = [\tilde{\mathbf{B}}|b] \in \mathbb{P}^3$ and their projected image plane points $\mathbf{a}, \mathbf{b} \in \mathbb{P}^2$. The line l can be determined as follows.

$$
\begin{aligned}
l &= \mathbf{a} \times \mathbf{b} \\
&= \mathbf{PA} \times \mathbf{PB} \\
&= (\mathbf{N}\tilde{\mathbf{A}} + a\mathbf{n}) \times (\mathbf{N}\tilde{\mathbf{B}} + b\mathbf{n}) \\
&= (\mathbf{N}\tilde{\mathbf{A}}) \times (\mathbf{N}\tilde{\mathbf{B}}) + a\mathbf{n} \times (\mathbf{N}\tilde{\mathbf{B}}) - b\mathbf{n} \times (\mathbf{N}\tilde{\mathbf{A}}) \\
&= \det(\mathbf{N})\mathbf{N}^{-\intercal}(\tilde{\mathbf{A}} \times \tilde{\mathbf{B}}) + \mathbf{n}^{\wedge}\mathbf{N}(a\tilde{\mathbf{B}} - b\tilde{\mathbf{A}}) \\
&= [\det(\mathbf{N})\mathbf{N}^{-\intercal}|\mathbf{n}^{\wedge}\mathbf{N}] \cdot [\mathbf{m}^\intercal|\mathbf{d}^\intercal]^\intercal \\
&= \mathcal{P}\mathcal{L}
\end{aligned} \tag{57}
$$

- $\mathbf{m} = \tilde{\mathbf{A}} \times \tilde{\mathbf{B}}$
- $\mathbf{d} = a\tilde{\mathbf{B}} - b\tilde{\mathbf{A}}$

The fifth line derivation can be checked at this link.
Alternatively, $\mathcal{P}$ can be defined in another way as follows.

$$
\mathcal{P} = \begin{bmatrix} \mathbf{p}_{2,row} \wedge \mathbf{p}_{3,row} \\ \mathbf{p}_{3,row} \wedge \mathbf{p}_{1,row} \\ \mathbf{p}_{1,row} \wedge \mathbf{p}_{2,row} \end{bmatrix} \in \mathbb{R}^{3\times 6} \tag{58}
$$

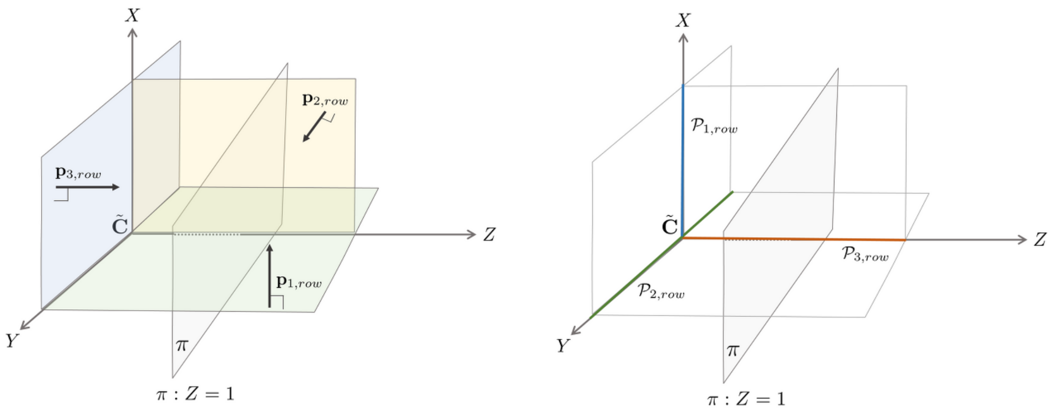
- $\mathbf{P} = \begin{bmatrix} \mathbf{p}_{1,row}^\intercal \\ \mathbf{p}_{2,row}^\intercal \\ \mathbf{p}_{3,row}^\intercal \end{bmatrix} \in \mathbb{R}^{3\times 4}$: Camera projection matrix
- $\mathbf{p}_{i,row} \in \mathbb{R}^{1\times 4}$: The i-th row vector of $\mathbf{P}$. $i = 1, 2, 3$ represents planes parallel to the X, Y, Z axes, respectively.
- $\mathbf{p}_{i,row} \wedge \mathbf{p}_{j,row} \in \mathbb{R}^{1\times 6}$: Plücker coordinates of the line generated by the intersection of planes $\mathbf{p}_{i,row}$ and $\mathbf{p}_{j,row}$

Using this, l can be determined as follows.

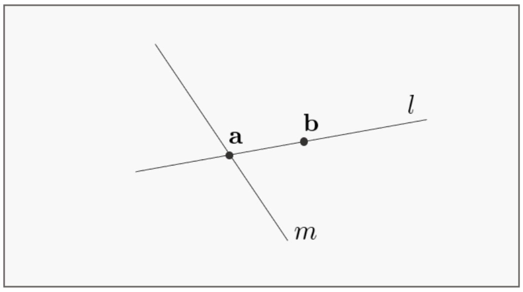
$$
l = \mathcal{P}\mathcal{L} = \begin{bmatrix} (\mathbf{p}_{2,row} \wedge \mathbf{p}_{3,row}|\mathcal{L}) \\ (\mathbf{p}_{3,row} \wedge \mathbf{p}_{1,row}|\mathcal{L}) \\ (\mathbf{p}_{1,row} \wedge \mathbf{p}_{2,row}|\mathcal{L}) \end{bmatrix} \tag{59}
$$

Projecting two points $\mathbf{A}, \mathbf{B}$ in 3D space to $\mathbf{a}, \mathbf{b}$ on the image plane can be represented as $\mathbf{a} = \mathbf{PA}, \mathbf{b} = \mathbf{PB}$, and thus the image of the line $l = \mathbf{a} \times \mathbf{b} = (\mathbf{PA} \times \mathbf{PB})$ can be expanded according to (43).

$$
\begin{aligned}
l &= \mathbf{PA} \times \mathbf{PB} \\
&= \left(\begin{bmatrix} \mathbf{p}_{1,row} \\ \mathbf{p}_{2,row} \\ \mathbf{p}_{3,row} \end{bmatrix} \mathbf{A} \right) \times \left(\begin{bmatrix} \mathbf{p}_{1,row} \\ \mathbf{p}_{2,row} \\ \mathbf{p}_{3,row} \end{bmatrix} \mathbf{B} \right) \\
&= \begin{bmatrix} \mathbf{p}_{1,row}\mathbf{A} \\ \mathbf{p}_{2,row}\mathbf{A} \\ \mathbf{p}_{3,row}\mathbf{A} \end{bmatrix} \times \begin{bmatrix} \mathbf{p}_{1,row}\mathbf{B} \\ \mathbf{p}_{2,row}\mathbf{B} \\ \mathbf{p}_{3,row}\mathbf{B} \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\mathbf{p}_{3,row}\mathbf{A} & \mathbf{p}_{2,row}\mathbf{A} \\ \mathbf{p}_{3,row}\mathbf{A} & 0 & -\mathbf{p}_{1,row}\mathbf{A} \\ -\mathbf{p}_{2,row}\mathbf{A} & \mathbf{p}_{1,row}\mathbf{A} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{1,row}\mathbf{B} \\ \mathbf{p}_{2,row}\mathbf{B} \\ \mathbf{p}_{3,row}\mathbf{B} \end{bmatrix} \\
&= \begin{bmatrix} (\mathbf{p}_{2,row}^\intercal\mathbf{A})(\mathbf{p}_{3,row}^\intercal\mathbf{B}) - (\mathbf{p}_{2,row}^\intercal\mathbf{B})(\mathbf{p}_{3,row}^\intercal\mathbf{A}) \\ (\mathbf{p}_{3,row}^\intercal\mathbf{A})(\mathbf{p}_{1,row}^\intercal\mathbf{B}) - (\mathbf{p}_{3,row}^\intercal\mathbf{B})(\mathbf{p}_{1,row}^\intercal\mathbf{A}) \\ (\mathbf{p}_{1,row}^\intercal\mathbf{A})(\mathbf{p}_{2,row}^\intercal\mathbf{B}) - (\mathbf{p}_{1,row}^\intercal\mathbf{B})(\mathbf{p}_{2,row}^\intercal\mathbf{A}) \end{bmatrix} \\
&= \begin{bmatrix} (\mathbf{p}_{2,row} \wedge \mathbf{p}_{3,row}|\mathcal{L}) \\ (\mathbf{p}_{3,row} \wedge \mathbf{p}_{1,row}|\mathcal{L}) \\ (\mathbf{p}_{1,row} \wedge \mathbf{p}_{2,row}|\mathcal{L}) \end{bmatrix} \qquad \cdots \text{ by (43)}
\end{aligned} \tag{60}
$$



5.6.2 Point and line duality



duality on homogeneous coordinates

$l = \mathbf{a} \times \mathbf{b} \in \mathbb{R}^3$
$\mathbf{a} = l \times m \in \mathbb{R}^3$

duality on plücker coordinates

$l^\wedge = \mathbf{l} = \mathbf{a}\mathbf{b}^\intercal - \mathbf{b}\mathbf{a}^\intercal \in \mathbb{R}^{3\times3}$
$\mathbf{a}^\wedge = lm^\intercal - ml^\intercal \in \mathbb{R}^{3\times3}$

As mentioned earlier, the skew-symmetric matrix of the projected line l can be calculated through $\mathbf{l} = \mathbf{P}\mathbf{L}\mathbf{P}^\intercal$. This can also be expressed as follows:

$$\begin{aligned} l^\wedge &= (\mathbf{a} \times \mathbf{b})^\wedge \\ &= \mathbf{a}\mathbf{b}^\intercal - \mathbf{b}\mathbf{a}^\intercal = \begin{bmatrix} 0 & l_2 & -l_1 \\ -l_2 & 0 & l_0 \\ l_1 & -l_0 & 0 \end{bmatrix} \\ &= \mathbf{l} \end{aligned} \tag{61}$$

- $l \in \mathbb{R}^3$: line in image space
- $l^\wedge \in \mathbb{R}^{3\times3}$: skew-symmetric matrix of line l
- $\mathbf{a}, \mathbf{b}$: points in 3D space $\mathbf{A}, \mathbf{B}$ projected onto the image plane

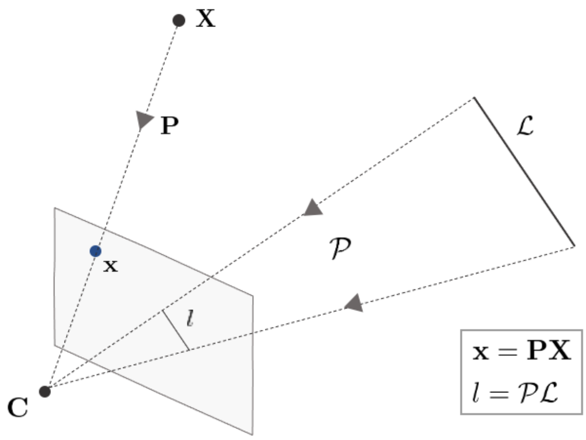
According to the properties of the cross product, when $\mathbf{a} = \mathbf{b} \times \mathbf{c}$, then $\mathbf{a}^\wedge = \mathbf{c}\mathbf{b}^\intercal - \mathbf{b}\mathbf{c}^\intercal$.

Duality: The intersection point $\mathbf{a}$ of two projected lines l, m from lines $\mathcal{L}, \mathcal{M}$ in 3D space can be represented as follows:

$$\mathbf{a}^\wedge = lm^\intercal - ml^\intercal \in \mathbb{R}^{3\times3} \tag{62}$$

- $\mathbf{a}^\wedge$: skew-symmetric matrix of $\mathbf{a}$

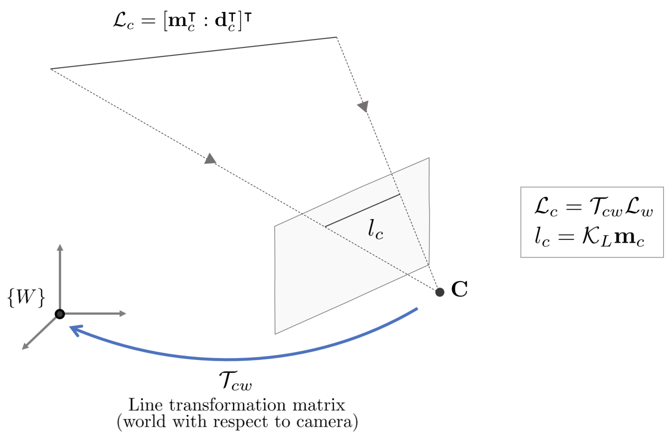
5.6.3 Projection matrix duality



The projection matrix $\mathcal{P}$ for lines performs the same function as the projection matrix $\mathbf{P}$ for 3D points.

$$\begin{aligned} \mathbf{x} &= \mathbf{P}\mathbf{X} \\ l &= \mathcal{P}\mathcal{L} \end{aligned} \tag{63}$$

6 Plücker line-based optimization



3D lines can be represented as a 6-dimensional column vector using Plücker Coordinates.

$$\mathcal{L} = [\mathbf{m}^\intercal : \mathbf{d}^\intercal]^\intercal = [m_x : m_y : m_z : d_x : d_y : d_z]^\intercal \tag{64}$$

Unlike the earlier mentioned order $[\mathbf{d} : \mathbf{m}]$, most papers using Plücker Coordinates utilize the order $[\mathbf{m} : \mathbf{d}]$, so this section will represent lines in that order as well. This line representation has a scale ambiguity (up to scale) and hence has five degrees of freedom as the ratios of the vector values of $\mathbf{m}, \mathbf{d}$ uniquely determine the line even if not unit vectors.

6.1 Line Transformation and projection

If a line in world coordinates is $\mathcal{L}_w$, it can be transformed into camera coordinates as follows:

$$\mathcal{L}_c = \begin{bmatrix} \mathbf{m}_c \\ \mathbf{d}_c \end{bmatrix} = \mathcal{T}_{cw}\mathcal{L}_w = \begin{bmatrix} \mathbf{R}_{cw} & \mathbf{t}^\wedge \mathbf{R}_{cw} \\ 0 & \mathbf{R}_{cw} \end{bmatrix} \begin{bmatrix} \mathbf{m}_w \\ \mathbf{d}_w \end{bmatrix} \tag{65}$$

- $\mathcal{T}_{cw} \in \mathbb{R}^{6\times 6}$: transformation matrix for Plücker lines

The projection of this line onto the image plane is as follows:

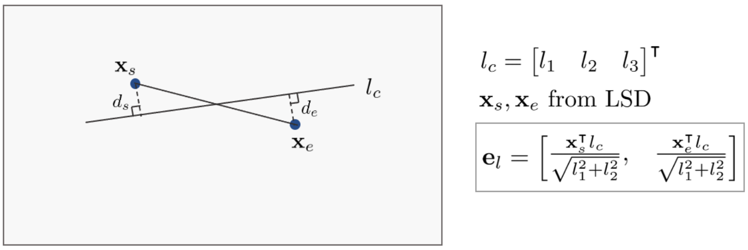
$$l_c = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \mathcal{K}_L \mathbf{m}_c = \begin{bmatrix} f_y & & \\ & f_x & \\ -f_y c_x & -f_x c_y & f_x f_y \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \tag{66}$$

- $\mathcal{K}_L$: internal parameter matrix of the line (line intrinsic matrix)

$\mathcal{K}_L$ refers to the case where $\mathbf{P} = K[\mathbf{I}|\mathbf{0}]$ in (55), hence $\mathcal{P} = [\det(\mathbf{K})\mathbf{K}^{-\intercal}|\mathbf{0}]$, and in (57) the term $\mathbf{d}$ of $\mathcal{L}$ is eliminated. Therefore, when $\mathbf{K} = \begin{bmatrix} f_x & & c_x \\ & f_y & c_y \\ & & 1 \end{bmatrix}$, the following equation is derived:

$$\mathcal{K}_L = \det(\mathbf{K})\mathbf{K}^{-\intercal} = \begin{bmatrix} f_y & & \\ & f_x & \\ -f_y c_x & -f_x c_y & f_x f_y \end{bmatrix} \in \mathbb{R}^{3\times 3} \tag{67}$$

6.2 Line reprojection error



The reprojection error $\mathbf{e}_l$ of the line can be represented as follows:

$$\mathbf{e}_l = \begin{bmatrix} d_s, & d_e \end{bmatrix} = \left[\frac{\mathbf{x}_s^\intercal l_c}{\sqrt{l_1^2+l_2^2}}, \quad \frac{\mathbf{x}_e^\intercal l_c}{\sqrt{l_1^2+l_2^2}} \right] \in \mathbb{R}^2 \tag{68}$$

This is according to the formula for distance from a point to a line, where $\{\mathbf{x}_s, \mathbf{x}_e\}$ represent respectively the start and end points of the line extracted using a line feature extractor (e.g., LSD).

6.3 Orthonormal representation

When performing BA optimization using the calculated $\mathbf{e}_l$, using Plücker Coordinate representation as is can pose problems. Plücker Coordinates always satisfy the Klein quadric constraint, $\mathbf{m}^\intercal\mathbf{d} = 0$, and therefore have five degrees of freedom, making them over-parameterized compared to the minimal number of parameters needed, which is four.

The disadvantages of using an over-parameterized representation are as follows:

- Increased computational load due to the need to compute redundant parameters.
- Additional degrees of freedom can lead to numerical instability issues.
- Parameters need to be checked continuously to ensure they satisfy the constraint.

Therefore, when optimizing lines, it is common to change to a minimal parameter representation of four degrees of freedom using the orthonormal representation method. In other words, although lines are represented using Plücker Coordinates, during optimization, they are transformed into orthonormal representation, updated for optimal values, and then converted back to Plücker Coordinates.

The orthonormal representation is as follows: a 3D line can always be represented in the following manner:

$$(\mathbf{U}, \mathbf{W}) \in SO(3) \times SO(2) \tag{69}$$

- $\mathbf{U} \in SO(3)$: rotation matrix for the 3D line
- $\mathbf{W} \in SO(2)$: matrix containing distance information from the origin for the 3D line

Any Plücker line $\mathcal{L} = [\mathbf{m}^\intercal : \mathbf{d}^\intercal]^\intercal$ always corresponds to a $(\mathbf{U}, \mathbf{W})$, and this representation method is known as orthonormal representation. Given a world line $\mathcal{L}_w = [\mathbf{m}_w^\intercal : \mathbf{d}_w^\intercal]^\intercal$, $(\mathbf{U}, \mathbf{W})$ can be obtained through QR decomposition of $\mathcal{L}_w$.

$$\begin{bmatrix}\mathbf{m}_w \mid \mathbf{d}_w\end{bmatrix} = \mathbf{U}\begin{bmatrix} w_1 & 0 \\ 0 & w_2 \\ 0 & 0 \end{bmatrix}, \quad \text{with set: } \mathbf{W} = \begin{bmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{bmatrix} \tag{70}$$

In this case, the $(1, 2)$ element of the upper triangle matrix $\mathbf{R}$ is always 0 due to the Plücker constraint (Klein quadric). $\mathbf{U}, \mathbf{W}$ represent respectively 3D and 2D rotation matrices, so $\mathbf{U} = \mathbf{R}(\boldsymbol{\theta}), \mathbf{W} = \mathbf{R}(\theta)$ can be represented as follows:

$$\begin{aligned} \mathbf{R}(\boldsymbol{\theta}) = \mathbf{U} &= \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{bmatrix} = \begin{bmatrix}\frac{\mathbf{m}_w}{\|\mathbf{m}_w\|} & \frac{\mathbf{d}_w}{\|\mathbf{d}_w\|} & \frac{\mathbf{m}_w \times \mathbf{d}_w}{\|\mathbf{m}_w \times \mathbf{d}_w\|}\end{bmatrix} \\ \mathbf{R}(\theta) = \mathbf{W} &= \begin{bmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \\ &= \frac{1}{\sqrt{\|\mathbf{m}_w\|^2 + \|\mathbf{d}_w\|^2}}\begin{bmatrix} \|\mathbf{m}_w\| & \|\mathbf{d}_w\| \\ -\|\mathbf{d}_w\| & \|\mathbf{m}_w\| \end{bmatrix} \end{aligned} \tag{71}$$

- $\boldsymbol{\theta} \in \mathbb{R}^3$: corresponding parameters for the SO(3) rotation matrix
- $\theta \in \mathbb{R}$: corresponding parameter for the SO(2) rotation matrix
- $\mathbf{u}_i$: ith column vector

When performing actual optimization, $\mathbf{U} \leftarrow \mathbf{U}\mathbf{R}(\boldsymbol{\theta}), \mathbf{W} \leftarrow \mathbf{W}\mathbf{R}(\theta)$ is updated as follows. Therefore, the orthonormal representation allows a line in 3D space to be represented with 4 degrees of freedom through $\delta_{\boldsymbol{\theta}} = [\boldsymbol{\theta}^\intercal, \theta] \in \mathbb{R}^4$. The updated $[\boldsymbol{\theta}^\intercal, \theta]$ is then transformed into $\mathcal{L}_w$ as follows:

$$\mathcal{L}_w = \begin{bmatrix} w_1\mathbf{u}_1^\intercal & w_2\mathbf{u}_2^\intercal \end{bmatrix} \tag{72}$$

6.4 Error function formulation

To optimize the reprojection error $\mathbf{e}_l$ of a line, iterative non-linear least squares methods such as Gauss-Newton (GN) and Levenberg-Marquardt (LM) must be used to iteratively update the optimal variables. The error function using reprojection error is expressed as follows:

$$\begin{aligned} \mathbf{E}_l(\mathcal{X}) &= \arg\min_{\mathcal{X}^*} \sum_i \sum_j \|\mathbf{e}_{l,ij}\|^2 \\ &= \arg\min_{\mathcal{X}^*} \sum_i \sum_j \mathbf{e}_{l,ij}^\intercal \mathbf{e}_{l,ij} \end{aligned} \tag{73}$$

- $\mathcal{X} = [\delta_{\boldsymbol{\theta}}, \delta_\xi]$: State variable
- $\delta_{\boldsymbol{\theta}} = [\boldsymbol{\theta}^\intercal, \theta] \in \mathbb{R}^4$: State variable of orthonormal representation
- $\delta_\xi = [\delta\xi] \in se(3)$: Update method through Lie theory can be found in this link

The optimal state of $\mathbf{E}_l(\mathcal{X}^*)$ satisfying $\|\mathbf{e}_l(\mathcal{X}^*)\|^2$ can be computed iteratively through non-linear least squares. Small increments $\Delta\mathcal{X}$ are repeatedly updated to $\mathcal{X}$ to find the optimal state.

$$\mathbf{E}_l(\mathcal{X} + \Delta\mathcal{X}) = \arg\min_{\mathcal{X}^*} \sum_i \sum_j \|\mathbf{e}_l(\mathcal{X} + \Delta\mathcal{X})\|^2 \tag{74}$$

Strictly speaking, the state increment $\Delta\mathcal{X}$ includes the SE(3) transformation matrix, so it should be added to the existing state $\mathcal{X}$ using the $\oplus$ operator, but the $+$ operator is used for simplicity of expression.

$$\mathbf{e}_l(\mathcal{X} \oplus \Delta\mathcal{X}) \quad \rightarrow \quad \mathbf{e}_l(\mathcal{X} + \Delta\mathcal{X}) \tag{75}$$

- $\oplus$: Operator that updates the state variables $\delta_{\boldsymbol{\theta}}, \delta_\xi$ all at once

This equation can be expressed through first-order Taylor approximation as follows:

$$\begin{aligned} \mathbf{e}_l(\mathcal{X} + \Delta\mathcal{X}) &\approx \mathbf{e}_l(\mathcal{X}) + \mathbf{J}\Delta\mathcal{X} \\ &= \mathbf{e}_l(\mathcal{X}) + \mathbf{J}_{\boldsymbol{\theta}}\Delta\delta_{\boldsymbol{\theta}} + \mathbf{J}_\xi\Delta\delta_\xi \\ &= \mathbf{e}_l(\mathcal{X}) + \frac{\partial \mathbf{e}_l}{\partial \delta_{\boldsymbol{\theta}}}\Delta\delta_{\boldsymbol{\theta}} + \frac{\partial \mathbf{e}_l}{\partial \delta_\xi}\Delta\delta_\xi \end{aligned} \tag{76}$$

- $\mathbf{J} = \frac{\partial \mathbf{e}_l}{\partial \mathcal{X}} = \frac{\partial \mathbf{e}_l}{\partial [\delta_{\boldsymbol{\theta}}, \delta_\xi]}$

$$\mathbf{E}_l(\mathcal{X} + \Delta\mathcal{X}) \approx \arg\min_{\mathcal{X}^*} \sum_i \sum_j \|\mathbf{e}_l(\mathcal{X}) + \mathbf{J}\Delta\mathcal{X}\|^2 \tag{77}$$

Differentiating this yields the optimal increment $\Delta\mathcal{X}^*$ as follows. The detailed derivation process is omitted here. If you want to know more about the derivation process, please refer to this link.

$$\begin{aligned} \mathbf{J}^\intercal \mathbf{J} \Delta\mathcal{X}^* &= -\mathbf{J}^\intercal \mathbf{e} \\ \mathbf{H}\Delta\mathcal{X}^* &= -\mathbf{b} \end{aligned} \tag{78}$$

6.4.1 The analytical jacobian of 3d line

As explained in the previous section, to perform non-linear optimization, $\mathbf{J}$ must be calculated. $\mathbf{J}$ is composed as follows:

$$\mathbf{J} = [\mathbf{J}_{\boldsymbol{\theta}}, \mathbf{J}_\xi] \tag{79}$$

$[\mathbf{J}_{\boldsymbol{\theta}}, \mathbf{J}_\xi]$ can be expanded as follows:

$$\begin{aligned} \mathbf{J}_{\boldsymbol{\theta}} &= \frac{\partial \mathbf{e}_l}{\partial \delta_{\boldsymbol{\theta}}} = \frac{\partial \mathbf{e}_l}{\partial l} \frac{\partial l}{\partial \mathcal{L}_c} \frac{\partial \mathcal{L}_c}{\partial \mathcal{L}_w} \frac{\partial \mathcal{L}_w}{\partial \delta_{\boldsymbol{\theta}}} \\ \mathbf{J}_\xi &= \frac{\partial \mathbf{e}_l}{\partial \delta_\xi} = \frac{\partial \mathbf{e}_l}{\partial l} \frac{\partial l}{\partial \mathcal{L}_c} \frac{\partial \mathcal{L}_c}{\partial \delta_\xi} \end{aligned} \tag{80}$$

$\frac{\partial \mathbf{e}_l}{\partial l}$ can be determined as follows. Note that l is a vector and l_i is a scalar.

$$\frac{\partial \mathbf{e}_l}{\partial l} = \frac{1}{\sqrt{l_1^2 + l_2^2}} \begin{bmatrix} x_s - \frac{l_1 \mathbf{x}_s l}{\sqrt{l_1^2+l_2^2}} & y_s - \frac{l_2 \mathbf{x}_s l}{\sqrt{l_1^2+l_2^2}} & 1 \\ x_e - \frac{l_1 \mathbf{x}_e l}{\sqrt{l_1^2+l_2^2}} & y_e - \frac{l_2 \mathbf{x}_e l}{\sqrt{l_1^2+l_2^2}} & 1 \end{bmatrix} \in \mathbb{R}^{2\times3} \tag{81}$$

$\frac{\partial l}{\partial \mathcal{L}_c}$ can be determined as follows:

$$\frac{\partial l}{\partial \mathcal{L}_c} = \frac{\partial \mathcal{K}_L \mathbf{m}_c}{\partial \mathcal{L}_c} = \begin{bmatrix} \mathcal{K}_L & \mathbf{0}_{3\times3} \end{bmatrix} = \begin{bmatrix} f_y & & & 0 & 0 & 0 \\ & f_x & & 0 & 0 & 0 \\ -f_y c_x & -f_x c_y & f_x f_y & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times6} \tag{82}$$

$\frac{\partial \mathcal{L}_c}{\partial \mathcal{L}_w}$ can be determined as follows:

$$\frac{\partial \mathcal{L}_c}{\partial \mathcal{L}_w} = \frac{\partial \mathcal{T}_{cw} \mathcal{L}_w}{\partial \mathcal{L}_w} = \mathcal{T}_{cw} = \begin{bmatrix} \mathbf{R}_{cw} & \mathbf{t}^\wedge \mathbf{R}_{cw} \\ 0 & \mathbf{R}_{cw} \end{bmatrix} \in \mathbb{R}^{6\times6} \tag{83}$$

The Jacobian of orthonormal representation $\frac{\partial \mathcal{L}_w}{\partial \delta_{\boldsymbol{\theta}}}$ can be determined as follows[8].

$$\frac{\partial \mathcal{L}_w}{\partial \delta_{\boldsymbol{\theta}}} = \begin{bmatrix} \mathbf{0}_{3\times1} & -w_1 \mathbf{u}_3 & w_1 \mathbf{u}_2 & -w_2 \mathbf{u}_1 \\ w_2 \mathbf{u}_3 & \mathbf{0}_{3\times1} & -w_2 \mathbf{u}_1 & w_1 \mathbf{u}_2 \end{bmatrix} \in \mathbb{R}^{6\times4} \tag{84}$$

The Jacobian for camera pose $\frac{\partial \mathcal{L}_c}{\partial \delta_\xi}$ can be determined as follows[9].

$$\frac{\partial \mathcal{L}_c}{\partial \delta_\xi} = \begin{bmatrix} -(\mathbf{Rm})^\wedge - (\mathbf{t}^\wedge \mathbf{Rd})^\wedge & -(\mathbf{Rd})^\wedge \\ -(\mathbf{Rd})^\wedge & \mathbf{0}_{3\times3} \end{bmatrix} \in \mathbb{R}^{6\times6} \tag{85}$$

7 References

[1] Hartley, Richard, and Andrew Zisserman. Multiple view geometry in computer vision. Cambridge university press, 2003

[2] Plücker Matrix

[3] Plücker Coordinates

[4] Robust Visual SLAM with Point and Line Features

[5] Elaborate Monocular Point and Line SLAM With Robust Initialization

[6] Bartoli, Adrien, and Peter Sturm. "Structure-from-motion using lines: Representation, triangulation, and bundle adjustment." Computer vision and image understanding 100.3 (2005): 416-441.

[7] Shu, Fangwen, et al. "Structure PLP-SLAM: Efficient Sparse Mapping and Localization using Point, Line and Plane for Monocular, RGB-D and Stereo Cameras." arXiv preprint arXiv:2207.06058 (2022).

[8] Bartoli, Adrien, and Peter Sturm. "The 3D line motion matrix and alignment of line reconstructions." Proceedings of the 2001 IEEE Computer Society Conference on Computer Vision and Pattern Recognition. CVPR 2001. Vol. 1. IEEE, 2001.

[9] Huang, Shi-Sheng. "The Derivation of Jacobian Matrix for the Point-Line-Plane Reprojection Factor." matrix 4: 4.

8 Revision log

- 1st: 2022-01-05
- 2nd: 2022-12-28
- 3rd: 2022-12-31
- 4th: 2023-01-01
- 5th: 2023-01-05
- 6th: 2023-01-06
- 7th: 2023-01-22
- 8th: 2024-02-08
- 9th: 2024-05-04