Notes on Multiple View Geometry in Computer Vision

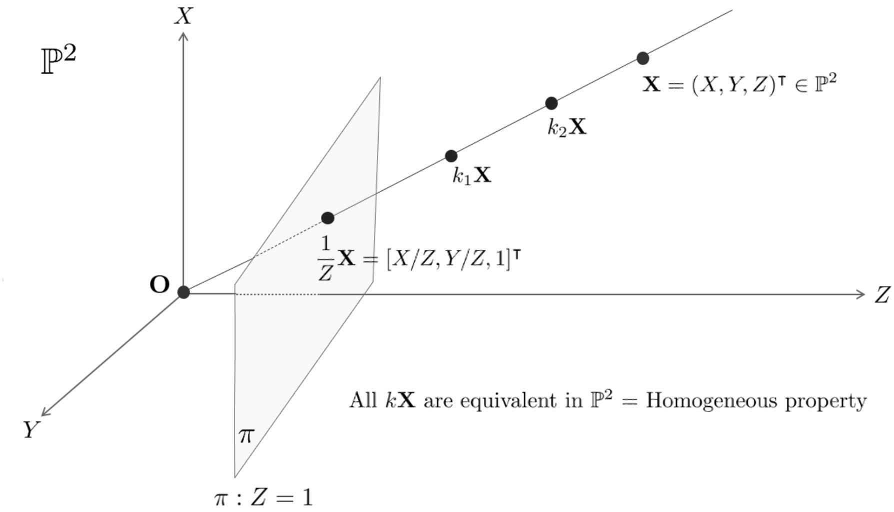
Gyubeom Edward Im*
(Orig. by Richard Hartley and Andrew Zisserman)

March 23, 2024

Contents

*blog: alida.tistory.com, email: criterion.im@gmail.com

1 Projective Space



The projective space $\mathbb{P}^n$ is defined as the set of lines passing through the origin in $\mathbb{R}^{n+1}$ space. Thus, it includes every element of $\mathbb{R}^{n+1}$ except the origin. Strictly speaking, since it only deals with real numbers excluding imaginary numbers, it should be written as $\mathbb{RP}^n$, but for convenience in this posting, we will use $\mathbb{P}^n$.

$$\mathbb{P}^n = \mathbb{R}^{n+1} - \{0\} \tag{1}$$

Let's say a point $\mathbf{X}$ in 3-dimensional space is given as follows.

$$\mathbf{X} = [X, Y, Z] \in \mathbb{P}^2 \tag{2}$$

Even if every element of X is multiplied by an arbitrary value k, it still exists on the line connecting the origin and X, and this property is called the homogeneous property. If $k = 1/Z$ is multiplied, it geometrically means the same as projecting the 3-dimensional point onto the plane where $Z = 1$.

$$[X, Y, Z] \rightarrow [X/Z, Y/Z, 1] \tag{3}$$

Therefore, by using $\mathbb{P}^2$, points in 3-dimensional space can be projected onto a specific plane and expressed in 2-dimensional space like $\mathbb{R}^2$, and it also has the expressive advantage of representing additional elements such as the point at infinity $\mathbf{x}_\infty$ and the line at infinity $\mathbf{l}_\infty$. Furthermore, it has the operational advantage of computing points and lines with the same 3-dimensional vector. Details will be explained in the following section.

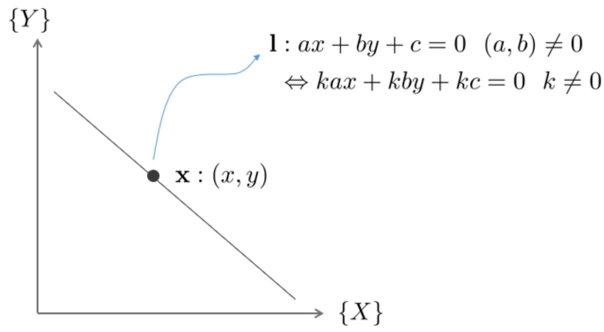
2 Projective Geometry and Transformations in 2D

The 2D projective plane

Typically, a point $\mathbf{x}$ on the plane is expressed as $(x, y) \in \mathbb{R}^2$. If $\mathbb{R}^2$ is a vector space, $\mathbf{x}$ can be represented as a single vector. Moreover, a line $\mathbf{l}$ that includes two points $\mathbf{x}_1, \mathbf{x}_2$ can be represented by subtracting the two vectors. This section explains Homogeneous Notation, which allows points and lines on the plane to be expressed by the same vector.

Points and lines

Homogeneous representation of line



A line $\mathbf{l}$ can be represented as follows:

$$\mathbf{l} : ax + by + c = 0 \quad (a, b) \neq 0 \tag{4}$$

If a point $\mathbf{x} = (x, y, 1)$ exists on the line $\mathbf{l}$, according to the formula $ax + by + c = 0$, the line $\mathbf{l}$ can be expressed as follows:

$$\mathbf{l} : (a, b, c) \tag{5}$$

In this case, (a, b, c) does not uniquely represent line $\mathbf{l}$. Even if a non-zero arbitrary constant k is multiplied, such as (ka, kb, kc), it can still represent the same line $\mathbf{l}$.

$$\mathbf{l} : (ka, kb, kc) \tag{6}$$

Therefore, **the line l on the plane represents the same line regardless of scale value**. All vectors in this equivalent relationship are called Homogeneous vectors. **The set of all vectors in the $\mathbb{R}^3$ space that are in this equivalence relationship is called the projective space $\mathbb{P}^2$.**

Homogeneous representation of points

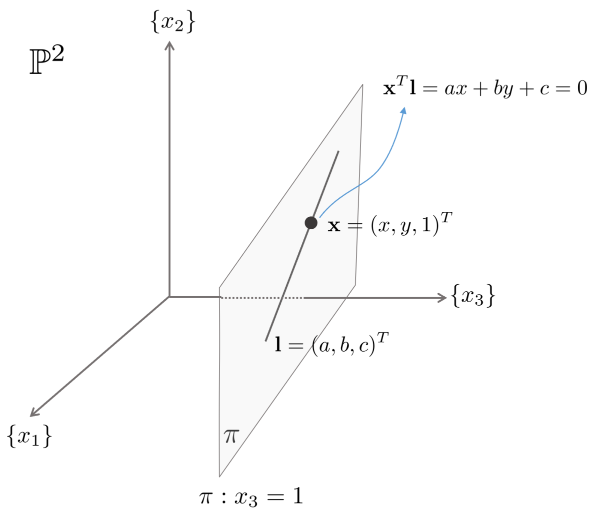
There is the following formula between the line $\mathbf{l} = (a, b, c)^\intercal$ and a point $\mathbf{x} = (x, y)^\intercal$ on the line:

$$ax + by + c = 0 \tag{7}$$

This can be expressed as the dot product of two vectors $\mathbf{l}$ and $\mathbf{x}$.

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} \mathbf{l} = 0 \tag{8}$$

This can be seen as adding 1 at the end of the coordinates of a point $\mathbf{x} = (x, y)$ and dot producting it with the line. Since line $\mathbf{l}$ can represent a single line regardless of scale, under the premise that $(x, y, 1)\mathbf{l} = 0$ holds, for all values of k, $(kx, ky, k)\mathbf{l} = 0$ also holds. Therefore, **for any arbitrary constant k, (kx, ky, k) represents one point $\mathbf{x} = (x, y)$ in $\mathbb{R}^2$ space, hence a point can also be represented as a Homogeneous vector like a line**. Generalizing this representation, an arbitrary point $\mathbf{x} = (x, y, z)^\intercal$ represents the point $(x/z, y/z)$ in $\mathbb{R}^2$ space.



Therefore, if an arbitrary point $\mathbf{x}$ exists on the line $\mathbf{l}$ in the space $\mathbb{P}^2$,

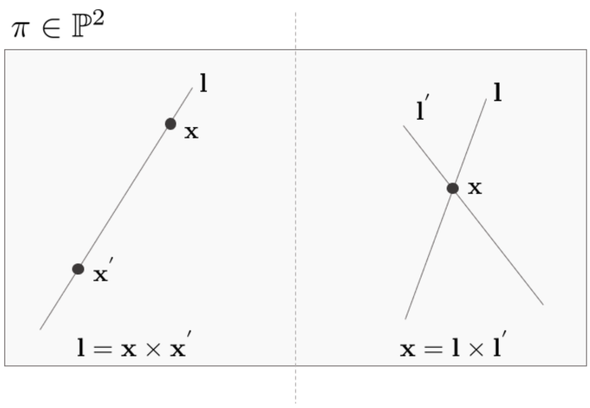
$$\begin{aligned} \mathbf{x}^\intercal \mathbf{l} &= \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= ax + by + c \\ &= 0 \\ &\therefore \mathbf{x}^\intercal \mathbf{l} = 0 \end{aligned} \tag{9}$$

The above formula is established.

Degrees of freedom (dof)

In the space $\mathbb{P}^2$, a single point is uniquely determined by two values (x, y). Similarly, to uniquely determine a line, two independent ratios $\{a : b : c\}$ must be provided. Thus, in the $\mathbb{P}^2$ space, both points and lines have two degrees of freedom.

Intersection of lines



Given two lines $\mathbf{l}, \mathbf{l}'$ in the space $\mathbb{P}^2$, their equations can be written as follows:

$$\begin{aligned} \mathbf{x}^\intercal \mathbf{l} &= 0 \\ \mathbf{x}^\intercal \mathbf{l}' &= 0 \end{aligned} \tag{10}$$

At this point, the intersection $\mathbf{x}$, regardless of the scale value, signifies a single point and can thus be represented as a multiple of the Cross Product of the two lines $\mathbf{l}, \mathbf{l}'$.

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \tag{11}$$

For instance, in $\mathbb{P}^2$ space, the line $x = 1$ and the line $y = 1$ intersect at $(1, 1)$. Using the above formula, the line $x = 1$ can be expressed as $-x+1 = 0 \Rightarrow (-1, 0, 1)^\intercal$ and the line $y = 1$ as $-y+1 = 0 \Rightarrow (0, -1, 1)^\intercal$, resulting in

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{12}$$

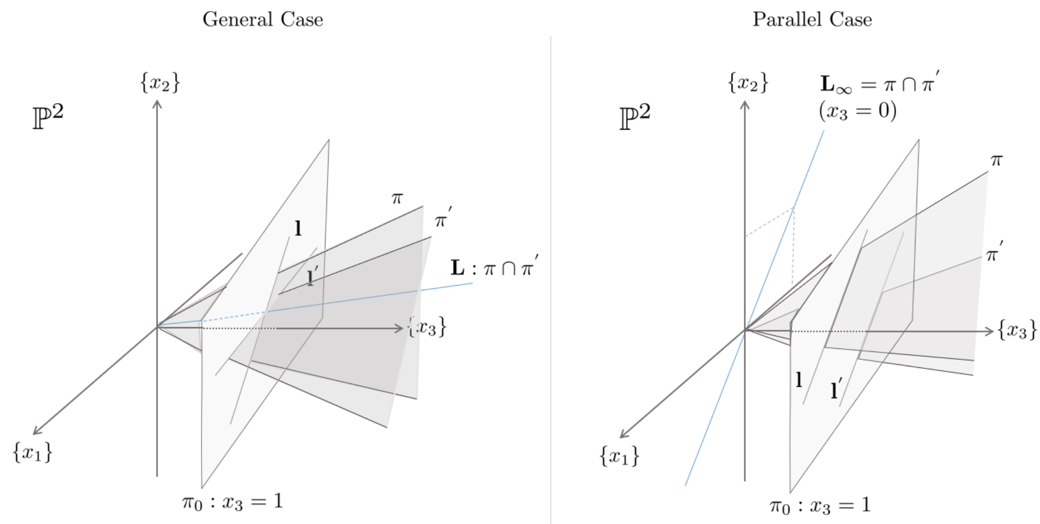
This equation holds true. $(1, 1, 1)^\intercal$ in $\mathbb{R}^2$ space corresponds to $(1, 1)$.

Line joining points

Similarly to the formula for finding the intersection of two lines, in $\mathbb{P}^2$ space, given two points $\mathbf{x}, \mathbf{x}'$, the line $\mathbf{l}$ passing through these two points can be determined as follows:

$$\mathbf{l} = \mathbf{x} \times \mathbf{x}' \tag{13}$$

Ideal points and the line at infinity



Intersection of parallel lines

If two lines $\mathbf{l}, \mathbf{l}'$ are parallel, their intersection point does not meet in $\mathbb{R}^2$ space but does meet in $\mathbb{P}^2$ space.

$$\mathbb{P}^2 = \mathbb{R}^2 \cup \mathbf{l}_\infty \tag{14}$$

The parallel lines $\mathbf{l}, \mathbf{l}'$ can be expressed as follows:

$$\begin{aligned} \mathbf{l} &: (a, b, c)^\intercal \\ \mathbf{l}' &: (a, b, c')^\intercal \end{aligned} \tag{15}$$

The parallel lines intersect at the point $\mathbf{x}_\infty$ located at infinity, therefore

$$\begin{aligned} \mathbf{x}_\infty &= \mathbf{l} \times \mathbf{l}' \\ &= (c' - c) \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} \sim \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} \end{aligned} \tag{16}$$

The formula holds. The infinite point $\mathbf{x}_\infty$ transformed into $\mathbb{R}^2$ space becomes $(b/0, -a/0)$, rendering it invalid. Therefore, the point at infinity $\mathbf{x}_\infty = (x, y, 0)^\intercal$ in $\mathbb{P}^2$ space does not transform into $\mathbb{R}^2$ space. This indicates that while parallel lines do not meet in Euclidean space, they intersect at infinity in projective space.

For example, in $\mathbb{P}^2$ space, the parallel lines $x = 1$ and $x = 2$ intersect at infinity. In Homogeneous Notation, this is represented as $-x + 1 = 0 \Rightarrow (-1, 0, 1)^\intercal$ and $-x + 2 = 0 \Rightarrow (-1, 0, 2)^\intercal$, resulting in

$$\mathbf{x}_\infty = \mathbf{l} \times \mathbf{l}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{17}$$

Here, $\mathbf{x}_\infty$ represents a point at infinity in the direction of the y-axis.

Ideal points and the line at infinity

A homogeneous vector $\mathbf{x} = (x_1, x_2, x_3)^\intercal$ corresponds to a point in $\mathbb{R}^2$ space when $x_3 \neq 0$. However, if $x_3 = 0$, the point does not correspond to $\mathbb{R}^2$ space and exists only in $\mathbb{P}^2$ space, where it is known as an Ideal Point or point at infinity. Points at infinity are

$$\mathbf{x}_\infty = \begin{pmatrix} x_1 & x_2 & 0 \end{pmatrix}^\intercal \tag{18}$$

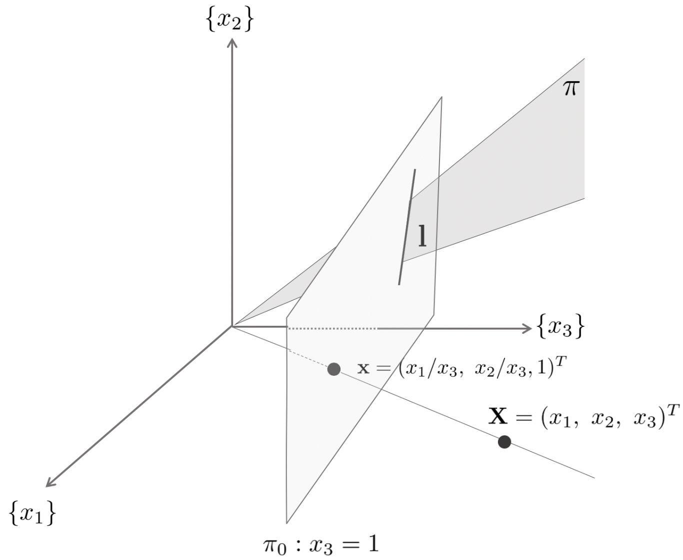
and exist on a specific line known as the line at infinity.

$$\mathbf{l}_\infty = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^\intercal \tag{19}$$

Thus, $\mathbf{x}_\infty^\intercal \mathbf{l}_\infty = \begin{pmatrix} x_1 & x_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^\intercal = 0$ holds true.

From the previous section, we can see that two parallel lines $\mathbf{l} = (a, b, c)^\intercal$ and $\mathbf{l}'_\infty = (a, b, c')^\intercal$ intersect at the infinite point $\mathbf{x}_\infty = (b, -a, 0)^\intercal$. This reveals that while parallel lines do not intersect in $\mathbb{R}^2$ space, any two distinct lines in $\mathbb{P}^2$ space necessarily intersect at a point.

A model for the projective plane



Geometrically, $\mathbb{P}^2$ represents the set of all lines passing through the origin in the three-dimensional space $\mathbb{R}^3$. Every vector in $\mathbb{P}^2$ can be described as $k(x_1, x_2, x_3)^\intercal$, where the position of a point $(x_1, x_2, x_3)^\intercal$ is determined by the value of k. Since k is a real number, $k = 0$ represents the origin, and $k \neq 0$ represents a set of points forming a line. Conversely, a line passing through the origin in $\mathbb{R}^3$ space corresponds to a point in $\mathbb{P}^2$ space. Extending this, a line $\mathbf{l}$ in $\mathbb{P}^2$ space corresponds to a plane π in $\mathbb{R}^3$ space that includes the origin.

In $\mathbb{P}^2$ space, a point is uniquely defined regardless of the scale value, so the normalized value $(x_1/x_3, x_2/x_3, 1)$ is typically considered the representative value for a point. Therefore, a line in $\mathbb{R}^3$ space passing through the origin and intersecting the plane $x_3 = 1$ corresponds to a point in $\mathbb{P}^2$ space.

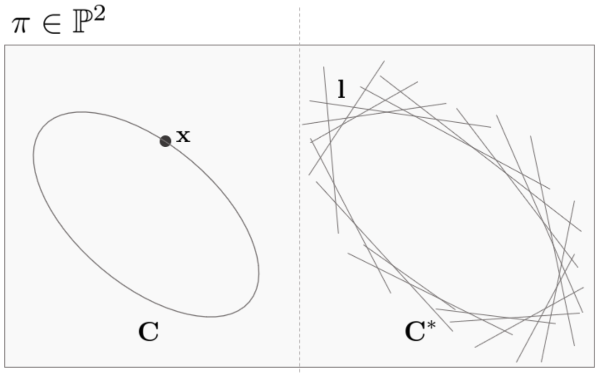
Duality

In $\mathbb{P}^2$ space, points and lines exhibit symmetry (duality). For example, a point $\mathbf{x}$ on a line $\mathbf{l}$ can be expressed in two ways: $\mathbf{x}^\intercal \mathbf{l} = 0$ or $\mathbf{l}^\intercal \mathbf{x} = 0$. Additionally, the point where two lines $\mathbf{l}, \mathbf{l}'$ intersect is

$\mathbf{x} = \mathbf{l} \times \mathbf{l}'$, and the line $\mathbf{l}$ passing through two points $\mathbf{x}, \mathbf{x}'$ can be described as $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$, using the same formula but with the positions of points and lines switched.

Thus, in $\mathbb{P}^2$ space, points and lines have symmetry, meaning that the formula for a line passing through two points is symmetrical to the formula for the intersection point of two lines.

Conics and dual conics



A conic is a curve defined by a quadratic equation in a plane. The general equation is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{20}$$

Depending on the coefficients, it can represent various curves such as circles, ellipses, hyperbolas, and parabolas. When represented in Homogeneous Form, it is expressed as

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0 \tag{21}$$

When organized in matrix form, it looks like

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \tag{22}$$

In this form, the symmetric matrix $\begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$ is referred to as the Conic $\mathbf{C}$.

Five points define a conic

A Conic $\mathbf{C}$ is uniquely determined by five points. The equation for a point on a conic can be rewritten as:

$$\begin{aligned} &ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0 \\ &\Rightarrow \begin{pmatrix} x_i^2 & x_i y_i & y_i^2 & x_i & y_i & 1 \end{pmatrix} \mathbf{c} = 0 \end{aligned} \tag{23}$$

Here, $\mathbf{c} = \begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \in \mathbb{R}^6$ represents a vector. Since $\mathbf{c}$ has 5 degrees of freedom,

$$\underbrace{\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}}_{\mathbf{A}} \mathbf{c} = 0 \tag{24}$$

As stated above, using a total of 5 points, the Null Space vector of matrix $\mathbf{A} \in \mathbb{R}^{5\times 6}$ becomes the unique solution for $\mathbf{c}$, thus uniquely determining the conic.

Tangent lines to conics

The tangent line $\mathbf{l}$ at a point $\mathbf{x}$ on a Conic $\mathbf{C}$ is expressed as

$$\mathbf{l} = \mathbf{C}\mathbf{x} \tag{25}$$

A conic $\mathbf{C}$ that includes any two lines $\mathbf{l}, \mathbf{m}$ can be written as

$$\mathbf{C} = \mathbf{l}\mathbf{m}^\intercal + \mathbf{m}\mathbf{l}^\intercal \tag{26}$$

Dual conics

The **projective space** $\mathbb{P}^n$ refers to a collection of lines passing through the origin in $\mathbb{R}^{n+1}$ space. In contrast, the **Dual Projective Space** $(\mathbb{P}^n)^\vee$ represents a collection of n-dimensional sub-linear spaces in $\mathbb{R}^n$ space.

An n-dimensional sub-linear space $\mathbf{H}$ is defined as

$$\mathbf{H} = \{\sum_{i=0}^{n} a_i x_i = 0 \mid a_i \neq 0 \ \text{ for some i}\}. \tag{27}$$

Here $a_0, \cdots, a_n \in \mathbb{P}^n$ can be considered as a single projective space, and a Dual projective space has a symmetric relationship with a projective space.

Given a Conic $\mathbf{C}$ on $\mathbb{P}^2$, the **Dual Conic $\mathbf{C}^*$** on the space $(\mathbb{P}^2)^\vee$ represents a conic and contains information about the tangents to the Conic $\mathbf{C}$. $(\mathbb{P}^2)^\vee$ can parameterize lines on $\mathbb{P}^2$. $\hat{\mathbf{C}}^*$ can be represented as follows.

$$\mathbf{C}^*_{ij} = (-1)^{i+j} \det(\hat{\mathbf{C}}_{ij}) \tag{28}$$

Where $\hat{\mathbf{C}}_{ij}$ denotes the matrix obtained by removing the i-th row and j-th column from $\mathbf{C}_{ij}$.

The necessary and sufficient condition for a line $\mathbf{l}$ to be tangent to a Conic $\mathbf{C}$ is as follows.

$$\mathbf{l}^\intercal \mathbf{C}^* \mathbf{l} = 0 \tag{29}$$

Proof

($\Rightarrow$) Assuming that the Conic $\mathbf{C} \in \mathbb{R}^{3\times 3}$ has rank 3 and is non-singular, it can be expressed as $\mathbf{C}^* = \det(\mathbf{C}^{-1})$. When a point $\mathbf{x} \in \mathbf{C}$ is given on $\mathbf{C}$, the tangent line $\mathbf{l}$ at $\mathbf{x}$ can be represented as $\mathbf{l} = \mathbf{C}\mathbf{x}$. Substituting into the above formula gives

$$\begin{aligned} \mathbf{l}^\intercal \mathbf{C}^* \mathbf{l} &= (\mathbf{C}\mathbf{x})^\intercal \mathbf{C}^* \mathbf{C}\mathbf{x} \\ &= \mathbf{x}^\intercal \mathbf{C}^\intercal \mathbf{C}^* \mathbf{C}\mathbf{x} \\ &= \det(\mathbf{x}^\intercal \mathbf{C}^\intercal \mathbf{x}) \qquad \because \mathbf{C}^* = \det(\mathbf{C}^{-1}) \\ &= 0 \qquad \because \mathbf{x} \in \mathbf{C}, (\mathbf{x}^\intercal \mathbf{C}\mathbf{x})^\intercal = 0 \end{aligned} \tag{30}$$

($\Leftarrow$) When the line $\mathbf{l}$ and Dual Conic $\mathbf{C}^*$ satisfy $\mathbf{l}^\intercal \mathbf{C}^* \mathbf{l} = 0$, it must be proven that $\mathbf{l}$ and $\mathbf{C}^*$ meet at a point $\mathbf{x}$. In this case, the formula $\mathbf{l} = \mathbf{C}\mathbf{x}$ holds.

Since $\mathbf{C}$ is non-singular, it has an inverse, hence $\mathbf{x} = \mathbf{C}^{-1}\mathbf{l}$ can be represented as follows. Therefore, $\mathbf{x}^\intercal \mathbf{l}$ is

$$\begin{aligned} \mathbf{x}^\intercal \mathbf{l} &= (\mathbf{C}^{-1}\mathbf{l})^\intercal \mathbf{l} \\ &= \mathbf{l}^\intercal \mathbf{C}^{-\intercal} \mathbf{l} = 0 \\ &(\mathbf{C}^* \sim \mathbf{C}^{-1} \ \text{ by assumption.}) \end{aligned} \tag{31}$$

And $\mathbf{x}^\intercal \mathbf{C}\mathbf{x}$ is

$$\begin{aligned} \mathbf{x}^\intercal \mathbf{C}\mathbf{x} &= (\mathbf{C}^{-1}\mathbf{l})^\intercal \mathbf{C}\mathbf{C}^{-1}\mathbf{l} \\ &= \mathbf{l}^\intercal \mathbf{C}^{-\intercal} \mathbf{C}\mathbf{C}^{-1}\mathbf{l} \\ &= \mathbf{l}^\intercal \mathbf{C}^{-1}\mathbf{l} = 0 \\ &(\mathbf{C}^{-\intercal} = \mathbf{C}^{-1} \ \ \mathbf{C} is symmetric.) \end{aligned} \tag{32}$$

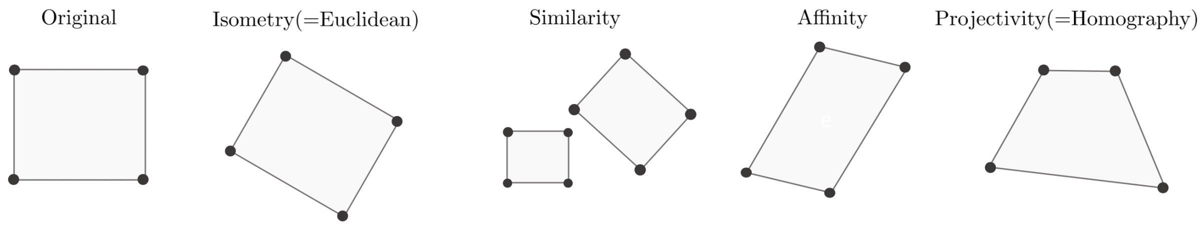
This confirms that $\mathbf{x}$ is the intersection of $\mathbf{l}$ and $\mathbf{C}$.

$$\{\mathbf{x}\} = \mathbf{l} \cap \mathbf{C} \tag{33}$$

Projective transformations

Projective Transformation in $\mathbb{P}^2$ refers to a $\mathbb{P}^2 \Rightarrow \mathbb{P}^2$ mapping defined by a non-singular 3×3 matrix, and it has the property of mapping lines to lines. **Projective Transformation is also known as Collineation, Projectivity, or Homography**.

A hierarchy of transformations



There are various types of transformation matrices depending on what properties are preserved between before and after the transformation.

Class 1: Isometries

If the size of an object remains the same before and after transformation, such transformation is called an Isometry.

$$\mathbf{H}_{iso} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \tag{34}$$

Here, $\mathbf{A} \in \mathbb{R}^{2\times2}$ represents a matrix including 2-dimensional rotation and reflection, and $\mathbf{t} \in \mathbb{R}^2$ is the translation vector for the 2-dimensional object.

Class 2: Similarity transformations

Transformations that include a scale factor s in addition to the Isometry are called Similarity transformations, and **additionally transform the scale along with the translation and rotation of the object**. Here, the $\mathbf{R}$ matrix, where the Reflection property is removed from the existing $\mathbf{A}$ matrix, is used.

$$\mathbf{H}_S = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \tag{35}$$

Similarity transformation preserves the angle and the ratio of lengths of an object, but not the scale. When two objects are the same up to a Similarity transformation, it means their shapes are identical but differ in scale.

Class 3: Affine transformations

An Affine transformation refers to a transformation matrix without any restrictions on the matrix $\mathbf{A}$ in Isometry transformations. Generally, the object has a different shape after the transformation.

$$\mathbf{H}_A = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \tag{36}$$

$\mathbf{H}_A$ has six degrees of freedom, so it can uniquely determine $\mathbf{H}_A$ from three pairs of corresponding points.

Affine transformations preserve the ratio of lengths of an object and also the property of parallel lines being parallel. Thus, even if the line at infinity $\mathbf{l}_\infty$ is transformed by an Affine transformation, it still remains $\mathbf{l}_\infty$.

$$\mathbf{H}_A(\mathbf{l}_\infty) = \mathbf{l}_\infty \tag{37}$$

Class 4: Projective transformations

Finally, a Projective transformation refers to a transformation matrix where the last row is not necessarily $(0, 0, 1)$. Projective transformations have the property of mapping lines to lines, but do not preserve any of the properties mentioned earlier. Even parallel lines become non-parallel under a Projective transformation, and the ratio of lengths of an object also changes.

$$\mathbf{H}_p = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\intercal & v \end{bmatrix} \tag{38}$$

Here, $\mathbf{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^\intercal$ represents an arbitrary 2-dimensional vector, and v also represents an arbitrary scalar value. The Projective transformation matrix $\mathbf{H}_p$ has 8 degrees of freedom, so it can typically be uniquely determined from four pairs of corresponding points.

Decomposition of a projective transformation

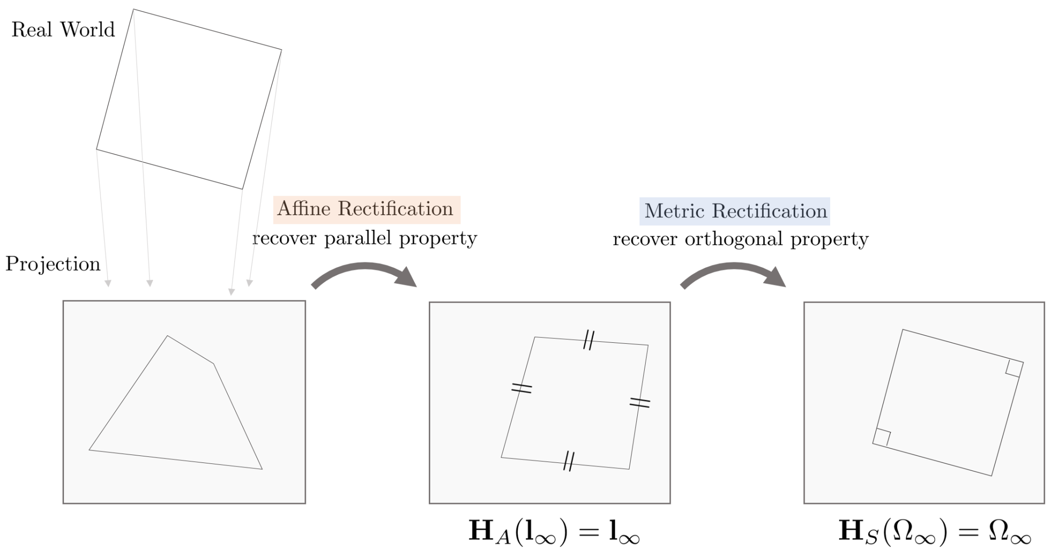
According to the hierarchical structure of the transformation matrices described earlier, a Projective transformation matrix can be expressed as a product of other transformation matrices. Conversely, **a Projective transformation can be decomposed into other transformation matrices.** When an arbitrary Projective transformation $\mathbf{H}_p$ is given,

$$\begin{aligned} \mathbf{H}_p &= \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P \\ &= \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\intercal & v \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\intercal & v \end{bmatrix} \end{aligned} \tag{39}$$

The Projective transformation $\mathbf{H}_p$ can be decomposed into the Similarity transformation $\mathbf{H}_S$, the Affine transformation $\mathbf{H}_A$, and the remaining transformation $\mathbf{H}_P$. Here, $\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathbf{t}\mathbf{v}^\intercal$ and $\mathbf{K}$ represents a normalized upper-triangle matrix with $\det(\mathbf{K}) = 1$. However, such decomposition is only possible when $v \neq 0$ and is uniquely determined when $s > 0$.

$\mathbf{H}^{-1} = \mathbf{H}_P^{-1}\mathbf{H}_A^{-1}\mathbf{H}_S^{-1} = \mathbf{H}'_P\mathbf{H}'_A\mathbf{H}'_S$ also represents the inverse operation of homography in the opposite direction of $\mathbf{H}$. Here, the detailed values of $\mathbf{R}, \mathbf{t}, \mathbf{K}, \mathbf{v}, s, v$ differ between $\mathbf{H}$ and $\mathbf{H}^{-1}$.

Recovery of affine and metric properties from images



When an arbitrary image is given, it is possible to restore the Affine and Metric properties of the image using lines that are parallel and orthogonal in the real world.

The line at infinity

Affine transformation means preserving the Affine property where parallel lines are preserved, and even if the line at infinity $\mathbf{l}_\infty = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\intercal$ is transformed by an Affine transformation, it still retains the property of being at infinity.

$$\mathbf{H}_A(\mathbf{l}_\infty) = \mathbf{H}_A^{-\intercal}\mathbf{l}_\infty = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ 0 & 1 \end{bmatrix}^{-\intercal} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{-\intercal} & 0 \\ -\mathbf{t}^\intercal\mathbf{A}^{-\intercal} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{l}'_\infty \tag{40}$$

Thus, $\mathbf{l}_\infty$ remains at infinity even after an Affine transformation.

Recovery of affine properties from images

Recovering affine properties means restoring two lines that are parallel in the real world but appear non-parallel on the image plane due to a projective transformation. If an arbitrary homography $\mathbf{H}$ preserves affine properties, it means that even if the line at infinity $\mathbf{l}_\infty$ is transformed by $\mathbf{H}$, it still remains a line at infinity. That is, if there is a point $\mathbf{x}_\infty$ on the line at infinity, the following holds:

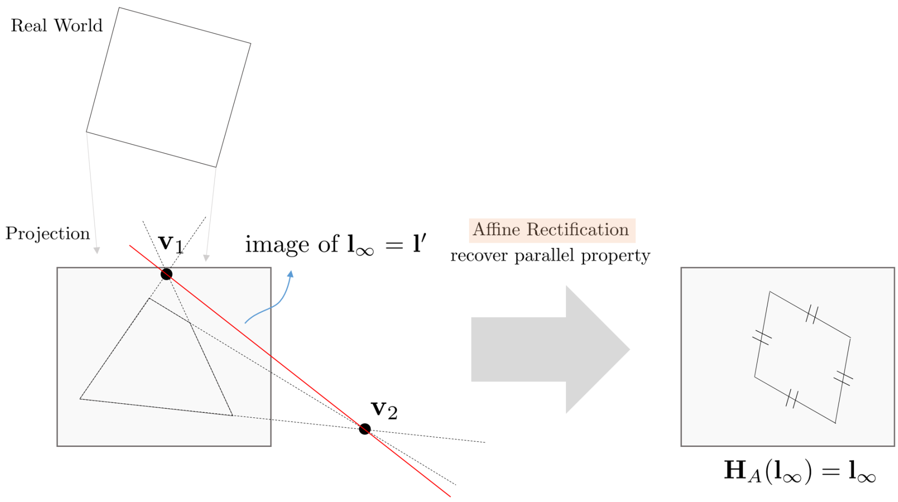
$$\mathbf{H}(\mathbf{x}_\infty) = \mathbf{H}\mathbf{x}_\infty = \mathbf{x}'_\infty \tag{41}$$

A point $\mathbf{x}_\infty$ on the line at infinity is characterized by the last term being 0, such as $\mathbf{x}_\infty = (x, y, 0)^\intercal$, so any homography $\mathbf{H}$ satisfies

$$\mathbf{H}\mathbf{x}_\infty = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v} & v \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix} \tag{42}$$

Therefore, $\mathbf{v} = (0, 0)$ and v becomes a scale constant, allowing transformation to 1.

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A}/v & \mathbf{t}/v \\ \mathbf{0} & 1 \end{bmatrix} \tag{43}$$



However, in images captured through real-world cameras, projective transformations are applied, so the properties of $\mathbf{l}_\infty$ are not preserved and are projected onto the image. Therefore, the process of finding a homography $\mathbf{H}$ that transforms an arbitrary projected line $\mathbf{l}'$ into $\mathbf{l}_\infty$ becomes affine rectification.

$$\mathbf{H}(\mathbf{l}') = \mathbf{H}^{-\intercal}\mathbf{l}' = \mathbf{l}_\infty \tag{44}$$

An arbitrary line can be represented as $\mathbf{l}' = \begin{bmatrix} a & b & c \end{bmatrix}^\intercal$ and $\mathbf{l}_\infty = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\intercal$, so rephrasing it gives the following.

$$\mathbf{H}(\mathbf{l}') = \mathbf{H}^{-\intercal} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{45}$$

Next, the components of $\mathbf{H}$ must be found. The projective transformation can be divided into three parts as follows:

$$\begin{aligned}\mathbf{H}_p &= \mathbf{H}_S\mathbf{H}_A\mathbf{H}_P \\ &= \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\intercal & v \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\intercal & v \end{bmatrix}\end{aligned} \tag{46}$$

Among these, $\mathbf{H}_P$ does not preserve the property of the line at infinity characteristic of projective transformations. Therefore, the form of $\mathbf{H}$ is as follows.

$$\mathbf{H} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\intercal & v \end{bmatrix} \tag{47}$$

The $\mathbf{H}$ that transforms $\mathbf{l}^{'}$ to $\mathbf{l}_\infty$ while satisfying the above form is as follows.

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix} \tag{48}$$

$$\mathbf{H}^{-\intercal}\mathbf{l}^{'} = \mathbf{l}_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}^{-\intercal} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{49}$$

The steps for affine rectification are as follows.

1. Obtain the coordinates of two pairs of parallel lines from the real world.
2. Calculate the vanishing point $\mathbf{v}$ for each pair of parallel lines. Since there are two pairs, two $\mathbf{v}_1, \mathbf{v}_2$ are obtained.
3. Obtain the image of the line at infinity $\mathbf{l}^{'} = [a, b, c]^\intercal$ that connects $\mathbf{v}_1$ and $\mathbf{v}_2$.
4. Based on $\mathbf{l}^{'}$, calculate the recovery homography $\mathbf{H}$.

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix} \tag{50}$$

5. Apply $\mathbf{H}$ to the entire image to complete the affine rectification. The resulting image of affine rectification preserves parallel lines.

Recovery of metric properties from images

The recovery of metric properties means restoring two lines in the real world that are perpendicular but not orthogonal on the image plane due to a projective transformation. This restored image cannot determine the exact scale value (up to similarity, up to scale). Thus, metric rectification means restoring the image to only differ in scale value from the original image. To perform this, it is necessary to use the features of the absolute dual conic $\mathbf{C}^*_\infty$.

- Circular Point

Circular points (or absolute points) $\mathbf{x}_c, \mathbf{x}_{-c}$ are defined as follows.

$$\mathbf{x}_{\pm c} = \begin{bmatrix} 1 \\ \pm i \\ 0 \end{bmatrix} \in \mathbb{CP}^2 \tag{51}$$

- $i = \sqrt{-1}$
- $\mathbb{CP}^2$: complex projective space

If a homography $\mathbf{H}$ preserves the set of circular points, then $\mathbf{H}$ has the property of preserving similarity.

$$\mathbf{H}(\mathbf{x}_{\pm c}) = \mathbf{x}_{\pm c} \quad \text{then, } \mathbf{H} \in \mathbf{H}_s \tag{52}$$

Thus, the form of $\mathbf{H}$ is as follows.

$$\mathbf{H} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s\mathbf{R} & t \\ 0 & 1 \end{bmatrix} \tag{53}$$

- s : scale factor
- $\mathbf{R}$: rotation matrix

- Dual Conic Properties

When two points $\mathbf{P}$ and $\mathbf{Q}$ exist in $\mathbb{P}^2$ space, the dual conic $\mathbf{C}^*$ tangent to the line connecting the two points can be expressed as follows.

$$\mathbf{C}^* = \mathbf{P}\mathbf{Q}^\intercal + \mathbf{Q}\mathbf{P}^\intercal \tag{54}$$

- $\mathbf{P} = [p_1, p_2, p_3]^\intercal$

This means that $\mathbf{C}^*$ is a dual conic parameterizing the line $\mathbf{l}$ passing through two points $\mathbf{P}$ and $\mathbf{Q}$. The relationship between the dual conic $\mathbf{C}^*$ and the tangent line $\mathbf{l}$ is as follows.

$$\mathbf{l}^\intercal \mathbf{C}^* \mathbf{l} = 0 \tag{55}$$

$$\mathbf{l}^\intercal (\mathbf{P}\mathbf{Q}^\intercal + \mathbf{Q}\mathbf{P}^\intercal)\mathbf{l} = 0 \tag{56}$$

Since line $\mathbf{l}$ includes two points $\mathbf{P}$ and $\mathbf{Q}$, either $\mathbf{P}^\intercal\mathbf{l} = 0$ or $\mathbf{Q}^\intercal\mathbf{l} = 0$ holds, satisfying the above equation.

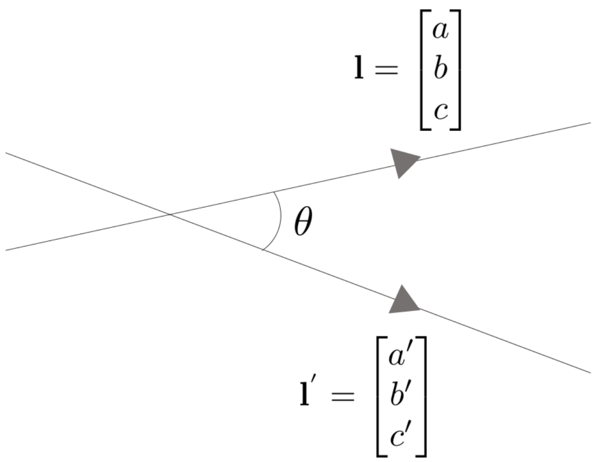
- Absolute Dual Conic

The absolute dual conic $\mathbf{C}^*_\infty$ refers to a dual conic that parameterizes a line passing through two circular points.

$$\mathbf{C}^*_\infty = \mathbf{x}_c \mathbf{x}_{-c}^\intercal + \mathbf{x}_{-c}\mathbf{x}_c^\intercal \tag{57}$$

$$\begin{aligned} \mathbf{C}^*_\infty &= \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -i & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \begin{bmatrix} 1 & i & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{58}$$

When two lines $\mathbf{l}, \mathbf{l}'$ exist in $\mathbb{P}^2$ space, the angle between the two lines can be expressed as follows.



$$\cos\theta = \frac{aa' + bb'}{\sqrt{a^2 + b^2}\sqrt{a'^2 + b'^2}} \tag{59}$$

When using $\mathbf{C}^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the equation becomes as follows.

$$\cos\theta = \frac{\mathbf{l}^\intercal \mathbf{C}^*_\infty \mathbf{l}'}{\sqrt{\mathbf{l}^\intercal \mathbf{C}^*_\infty \mathbf{l}}\sqrt{\mathbf{l}'^\intercal \mathbf{C}^*_\infty \mathbf{l}'}} \tag{60}$$

- $aa' + bb' = \mathbf{l}^\intercal \mathbf{C}^*_\infty \mathbf{l}'$
- $\sqrt{a^2+b^2} = \sqrt{\mathbf{l}^\intercal \mathbf{C}^*_\infty \mathbf{l}}$
- $\sqrt{a'^2+b'^2} = \sqrt{\mathbf{l}'^\intercal \mathbf{C}^*_\infty \mathbf{l}'}$

- Homography of Dual Conic

The dual conic and $\mathbf{C}^*$ and the tangent line $\mathbf{l}$ have the following relationship.

$$\mathbf{l}^\intercal \mathbf{C}^* \mathbf{l} = 0 \tag{61}$$

When performing Homography $\mathbf{H} : \mathbb{P}^2 \mapsto \mathbb{P}^2$, the following occurs. Since $\mathbf{H}(\mathbf{l}) = \mathbf{H}^{-\intercal}\mathbf{l}$,

$$(\mathbf{H}^{-\intercal}\mathbf{l})^\intercal \mathbf{H}(\mathbf{C}^*)(\mathbf{H}^{-\intercal}\mathbf{l}) = 0 \tag{62}$$

$$\therefore \mathbf{H}(\mathbf{C}^*) = \mathbf{H}\mathbf{C}^*\mathbf{H}^\intercal \tag{63}$$

The image of the absolute dual conic $\mathbf{w}$ is referred to as $\mathbf{H}(\mathbf{C}^*)$.

- Image of Absolute Dual Conic

If two lines $\mathbf{l}, \mathbf{m}$ in $\mathbb{P}^2$ space are orthogonal, the following equation holds.

$$\mathbf{l}^\intercal \mathbf{w} \mathbf{m} = 0 \tag{64}$$

- $\mathbf{w}$: image of absolute conic $\mathbf{C}^*_\infty$

Since $\mathbf{w} = \mathbf{H}\mathbf{C}^*\mathbf{H}^\intercal$, decomposing an arbitrary projective homography $\mathbf{H}$ results in the following.

$$\begin{aligned} \mathbf{H} &= \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P \\ &= \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\intercal & v \end{bmatrix} \end{aligned} \tag{65}$$

$\mathbf{H}^{-1} = \mathbf{H}_P^{-1}\mathbf{H}_A^{-1}\mathbf{H}_S^{-1} = \mathbf{H}'_P\mathbf{H}'_A\mathbf{H}'_S$ also means the same homography operations. For convenience, $\mathbf{H}'_P\mathbf{H}'_A\mathbf{H}'_S$ is denoted as $\mathbf{H}_P\mathbf{H}_A\mathbf{H}_S$. In this case, the detailed values of $\mathbf{R}, \mathbf{t}, \mathbf{K}, \mathbf{v}, s, v$ of each matrix are different for $\mathbf{H}$ and $\mathbf{H}^{-1}$. Therefore, reversing the decomposition order of $\mathbf{H}$ and multiplying gives the following expansion.

$$\mathbf{H}\mathbf{C}^*\mathbf{H}^\intercal = \mathbf{H}_P\mathbf{H}_A\mathbf{H}_S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{H}_S^\intercal \mathbf{H}_A^\intercal \mathbf{H}_P^\intercal \tag{66}$$

$$\mathbf{H}\mathbf{C}^*\mathbf{H}^\intercal = \mathbf{H}_P\mathbf{H}_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{H}_A^\intercal \mathbf{H}_P^\intercal \tag{67}$$

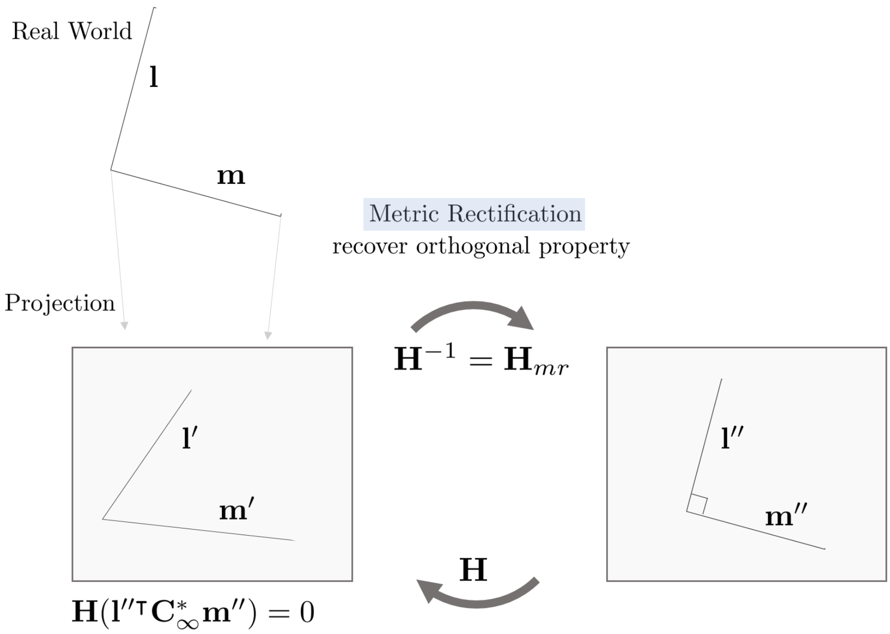
- $\because \mathbf{H}_S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{H}_S^\intercal = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\mathbf{w} = \begin{bmatrix} \mathbf{K}\mathbf{K}^\intercal & \mathbf{K}\mathbf{K}^\intercal\mathbf{v} \\ \mathbf{v}^\intercal\mathbf{K}^\intercal\mathbf{K} & \mathbf{v}^\intercal\mathbf{K}\mathbf{K}^\intercal\mathbf{v} \end{bmatrix} \tag{68}$$

Assuming there is no projective transformation and only a similarity transformation, $\mathbf{v} = 0$ and $\mathbf{w}$ is as follows.

$$\mathbf{w} = \begin{bmatrix} \mathbf{K}\mathbf{K}^\intercal & 0 \\ 0 & 0 \end{bmatrix} \tag{69}$$

- Metric Rectification



$$\mathbf{H}(\mathbf{l}''^{\intercal}\mathbf{C}^{*}_{\infty}\mathbf{m}'') = 0$$
$$\mathbf{l}'^{\intercal}\mathbf{H}(\mathbf{C}^{*}_{\infty})\mathbf{m}' = 0$$
$$\therefore \mathbf{l}'^{\intercal}\mathbf{w}\mathbf{m}' = 0$$

As previously mentioned, the image of absolute dual conic by $\mathbf{H}$ can be expressed as $\mathbf{w} = \begin{bmatrix} \mathbf{KK}^{\intercal} & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, the result of applying the homography transformation $\mathbf{H}$ to $\mathbf{l}''$, $\mathbf{m}''$ in the figure above can be expressed as follows.

$$\mathbf{H}(\mathbf{l}''\mathbf{C}^{*}_{\infty}\mathbf{m}'') = \mathbf{l}'\mathbf{w}\mathbf{m}' = 0 \tag{70}$$

- $\mathbf{H}(\mathbf{l}'') = \mathbf{l}'$
- $\mathbf{H}(\mathbf{C}^{*}_{\infty}) = \mathbf{w}$
- $\mathbf{H}(\mathbf{m}'') = \mathbf{m}'$

Expanding this further

$$\mathbf{l}' \begin{bmatrix} \mathbf{KK}^{\intercal} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{m}' = 0 \tag{71}$$

$$\begin{bmatrix} l'_1 & l'_2 \end{bmatrix} \mathbf{KK}^{\intercal} \begin{bmatrix} m'_1 \\ m'_2 \end{bmatrix} = 0 \tag{72}$$

- $\mathbf{KK}^{\intercal} \in \mathbb{R}^{2\times 2}$: symmetric matrix & $\det \mathbf{KK}^{\intercal} = 1$

Therefore, $\mathbf{KK}^{\intercal}$ can be computed from a pair of perpendicular lines and $\mathbf{w}$ can be determined. When $\mathbf{KK}^{\intercal} = \mathbf{S}$ is substituted, a symmetric and positive definite matrix can be decomposed as follows.

$$\begin{bmatrix} l'_1 & l'_2 \end{bmatrix} \mathbf{S} \begin{bmatrix} m'_1 \\ m'_2 \end{bmatrix} = 0 \tag{73}$$

$$\mathbf{S} = \mathbf{UDU}^{\intercal} \tag{74}$$

- $\mathbf{U}$: orthogonal matrix
- $\mathbf{D}$: diagonal matrix

Furthermore, the diagonal matrix $\mathbf{D}$ can be expressed as the product of two matrices $\mathbf{D} = \mathbf{EE}^{\intercal}$, so it can be rearranged as follows.

$$\mathbf{S} = \mathbf{UE}(\mathbf{UE})^{\intercal} \tag{75}$$

Next, performing QR decomposition on $\mathbf{UE}$ yields an upper triangle matrix $\mathbf{R}(= \mathbf{K})$ and an orthogonal matrix $\mathbf{Q}$. Expanding this further results in the following.

$$\mathbf{S} = \mathbf{KQQ}^{\intercal}\mathbf{K}^{\intercal} = \mathbf{KK}^{\intercal} \tag{76}$$

- $\mathbf{QQ}^{\intercal} = \mathbf{I}$

Next, $\mathbf{S}$ is extracted through Cholesky or SVD to find $\mathbf{K}$ and thus determine the final metric rectify homography $\mathbf{H}^{-1} = \mathbf{H}_{mr}$.

$$\mathbf{H} = \begin{bmatrix} \mathbf{K} & 0 \\ 0 & 1 \end{bmatrix} \tag{77}$$

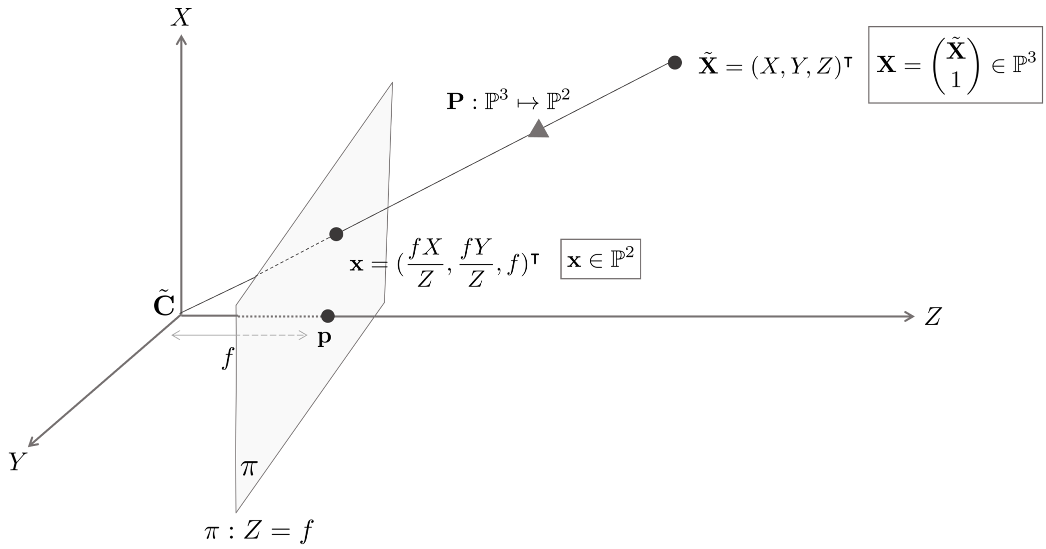
$$\mathbf{H}_{mr} = \mathbf{H}^{-1} = \begin{bmatrix} \mathbf{K} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \tag{78}$$

The steps for metric rectification are summarized as follows:
1. Select a pair of perpendicular lines $\mathbf{l}', \mathbf{m}'$ and determine their coordinates.
2. Use the formula $\begin{bmatrix} l_1' & l_2' \end{bmatrix} \mathbf{S} \begin{bmatrix} m_1' \\ m_2' \end{bmatrix} = 0$ to compute $\mathbf{S} = \mathbf{K}\mathbf{K}^\intercal$.
3. Find $\mathbf{K}$ through Cholesky or SVD, and from this, compute $\mathbf{H}_{mr} = \begin{bmatrix} \mathbf{K} & 0 \\ 0 & 1 \end{bmatrix}^{-1}$.
4. Apply $\mathbf{H}_{mr}$ to the image to perform metric rectification. The restored image will have the same form as the original image except for scale values (up to scale).

3 Camera Models

Finite cameras

The basic pinhole model



A pinhole camera is a mathematical camera modeling method that represents an image by projecting a point $\tilde{\mathbf{X}}$ in $\mathbb{R}^3$ space towards a specific center $\tilde{\mathbf{C}}$, and forming an image at a point $\mathbf{x}$ on the image plane $\pi \in \mathbb{R}^2$ that intersects in the middle. $\tilde{\mathbf{X}}, \tilde{\mathbf{C}}$ are inhomogeneous coordinates representing $\mathbf{X}$.

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} X & Y & Z & 1 \end{bmatrix}^\intercal \\ \tilde{\mathbf{X}} &= \begin{bmatrix} X & Y & Z \end{bmatrix}^\intercal \\ \mathbf{C} &= \begin{bmatrix} c_x & c_y & c_z & 1 \end{bmatrix}^\intercal \\ \tilde{\mathbf{C}} &= \begin{bmatrix} c_x & c_y & c_z \end{bmatrix}^\intercal \end{aligned} \tag{79}$$

If we consider any $\mathbb{R}^3$ space as a camera coordinate system, the origin of the coordinate system becomes the camera center $\tilde{\mathbf{C}}$. Typically, the image plane π is positioned perpendicular to the Z axis, and the point where the Principal Axis meets the image plane is called the Principal Point $\mathbf{p}$.

Given a point $\tilde{\mathbf{X}} = \begin{bmatrix} X & Y & Z \end{bmatrix}^\intercal$ in 3D space, when only considering the XZ plane, we can calculate the focal length f_x for the X axis, the distance between the image plane π and the camera center $\tilde{\mathbf{C}}$. f_x, f_y imply the pixel aspect ratio, but most sensors produced recently have a 1:1 ratio, so we assume $f = f_x = f_y$.

$$f\frac{Y}{Z} = y \tag{80}$$

Similarly, f can be calculated when looking at the image plane from the YZ plane.

$$f\frac{X}{Z} = x \tag{81}$$

Thus, **the pinhole camera matrix P** can be seen as a linear mapping that projects a point $\tilde{\mathbf{X}} = \begin{pmatrix} X & Y & Z \end{pmatrix}^\intercal \in \mathbb{R}^3$ to a 2D image plane $\pi \in \mathbb{R}^2$.

$$\mathbf{P} : (X,\ Y,\ Z)^\intercal \ \mapsto \ (f\frac{X}{Z},\ f\frac{Y}{Z})^\intercal \tag{82}$$

Central projection using homogeneous coordinates

The pinhole camera matrix $\mathbf{P}$ can also be considered as moving a Homogeneous Point. In other words, the pinhole camera matrix $\mathbf{P}$ projects the point $\mathbf{X} = \begin{pmatrix} X & Y & Z & 1 \end{pmatrix}^\intercal$ in $\mathbb{P}^3$ space to the point $\mathbf{x} = \begin{pmatrix} fX & fY & Z \end{pmatrix}^\intercal$ in $\mathbb{P}^2$ space.

$$\mathbf{P} : \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \tag{83}$$

This is expressed in matrix form as

$$\mathbf{x} = \mathbf{P}\mathbf{X} \tag{84}$$

At this time, $\mathbf{P} = \text{diag}(f, f, 1)[\mathbf{I} \mid 0]_{3\times 4}$ can be expressed as follows.

Principal point offset

Typically, the Principal Point $\mathbf{p}$ is not the origin of the image plane π. Therefore, to properly correspond the linear mapping through the pinhole camera matrix to the image plane π, it must be corrected as

$$\begin{pmatrix} X & Y & Z \end{pmatrix}^\intercal \mapsto \begin{pmatrix} fX/Z + p_x & fY/Z + p_y \end{pmatrix}^\intercal \tag{85}$$

Expressing this correction in the camera matrix $\mathbf{P}$ at once,

$$\mathbf{P} : \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ 1 \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \tag{86}$$

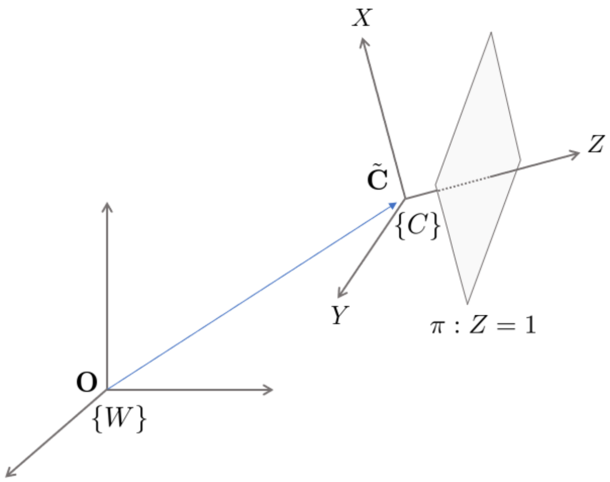
This is represented as $\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$, which is succinctly denoted as $\mathbf{K}$, and this is called the intrinsic parameter matrix or camera calibration matrix.

$$\mathbf{K} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \tag{87}$$

Consequently, the linear mapping $\mathbf{X} \in \mathbb{P}^3 \mapsto \mathbf{x} \in \mathbb{P}^2$ is possible through the camera matrix $\mathbf{P}$ including the Principal Point Offset.

$$\mathbf{x} = \mathbf{K}[\mathbf{I} \mid 0]\mathbf{X} \tag{88}$$

Camera rotation and translation



- $\tilde{\mathbf{X}}_{\mathbf{C}} = \mathbf{R}(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})$

Typically, the camera coordinate system is not the same as the world coordinate system. Given a world coordinate system $\{W\}$ in $\mathbb{R}^3$ space, the camera coordinate system $\{C\}$, which is rotated by $\mathbf{R}$ and positioned $\mathbf{C} = \begin{pmatrix} c_x & c_y & c_z & 1 \end{pmatrix}^\intercal$ from it, the formula to transform a point $\tilde{\mathbf{X}}$ in the world coordinate system $\{W\}$ to a point $\tilde{\mathbf{X}}_{\mathbf{C}}$ in the camera coordinate system $\{C\}$ is

$$\tilde{\mathbf{X}}_{\mathbf{C}} = \mathbf{R}(\tilde{\mathbf{X}} - \tilde{\mathbf{C}}) \tag{89}$$

The point $\mathbf{X}_{\mathbf{C}}$, represented in Homogeneous Coordinates, when projected onto the image plane π, is expressed as

$$\mathbf{x} = \mathbf{P}\mathbf{X}_{\mathbf{C}} = \mathbf{K}[\mathbf{I} \mid 0]\mathbf{X}_{\mathbf{C}} \tag{90}$$

Detailed expression of $\mathbf{X}_{\mathbf{C}}$ is

$$\begin{aligned} \mathbf{X}_{\mathbf{C}} &= \mathbf{R}\begin{bmatrix} 1 & & & -c_x \\ & 1 & & -c_y \\ & & 1 & -c_z \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \end{aligned} \tag{91}$$

This can be reformulated and inserted into the formula $\mathbf{x} = \mathbf{K}[\mathbf{I} \mid 0]\mathbf{X}_{\mathbf{C}}$ resulting in

$$\begin{aligned} \mathbf{X}_{\mathbf{C}} &= \mathbf{K}[\mathbf{I} \mid 0]\mathbf{X}_{\mathbf{C}} \\ &= \mathbf{K}[\mathbf{I} \mid 0]\begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \\ &= \mathbf{K}[\mathbf{R} \mid -\mathbf{R}\tilde{\mathbf{C}}]\mathbf{X} \\ &= \mathbf{K}\mathbf{R}[\mathbf{I} \mid -\tilde{\mathbf{C}}]\mathbf{X} \end{aligned} \tag{92}$$

Typically, $\tilde{\mathbf{X}}_{\mathbf{C}}$ can also often be expressed based on the world coordinate system as $\tilde{\mathbf{X}}_{\mathbf{C}} = \mathbf{R}\tilde{\mathbf{X}} + \mathbf{t}$. In this case, the camera matrix $\mathbf{P}$ can be expressed as

$$\mathbf{P} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}] \tag{93}$$

where the relation $\mathbf{t} = -\mathbf{R}\tilde{\mathbf{C}}$ holds.

CCD cameras

Among the cameras commonly used today, CCD cameras record image coordinates in terms of the number of pixels. Therefore, when the image coordinates are given in mm as $(x, y)[mm]$, in CCD cameras they are represented as $(m_x x,\ m_y y)$. Here m_x, m_y represent the number of pixels in the x-axis or y-axis direction per 1 mm^2. Therefore, to convert a given general camera calibration matrix $\mathbf{K}$ in mm to the coordinate system of a CCD camera, it needs to be transformed as

$$\begin{pmatrix} m_x & & \\ & m_y & \\ & & 1 \end{pmatrix} \mathbf{K} = \begin{pmatrix} m_x & & \\ & m_y & \\ & & 1 \end{pmatrix} \begin{pmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{pmatrix} = \begin{pmatrix} fm_x & & p_x m_x \\ & fm_y & p_y m_y \\ 0 & 0 & 1 \end{pmatrix} \tag{94}$$

Finite projective camera

When a camera matrix is given as $\mathbf{P} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}]$, it can be represented as

$$\mathbf{P} = \mathbf{KR}[\mathbf{I} \mid -\tilde{\mathbf{C}}] \quad \text{where, } \mathbf{t} = -\mathbf{R}\tilde{\mathbf{C}} \tag{95}$$

This type of camera matrix is referred to as a Finite Projective Camera, and it requires that $\mathbf{KR}$ be a non-singular matrix. Given any non-singular matrix $\mathbf{M} \in \mathbb{R}^{3\times3}$, it can be decomposed into an upper triangular matrix $\mathbf{K}$ and an orthogonal matrix $\mathbf{R}$ through QR factorization, thus the set of camera matrices are

$$\{\text{set of camera matrix}\} = \{\mathbf{P} = [\mathbf{M} \mid \mathbf{t}] \mid \mathbf{M} \text{ is non-singular } 3\times3 \text{ matrix.}, \mathbf{t} \in \mathbb{R}^3\} \tag{96}$$

General projective cameras

Unlike Finite Projective Cameras, General Projective Cameras do not require the matrix $\mathbf{M} \in \mathbb{R}^{3\times3}$ in $\mathbf{P} = [\mathbf{M} \mid \mathbf{t}]$ to be non-singular, and are defined by camera matrices $\mathbf{P}$ with a rank of 3.

The projective camera

Camera anatomy

Camera center

Given any Finite Projective camera matrix $\mathbf{P} = \mathbf{KR}[\mathbf{I} \mid -\tilde{\mathbf{C}}]$

$$\mathbf{PC} = \mathbf{KR}(\mathbf{C} - \mathbf{C}) = \mathbf{0} \tag{97}$$

This means $\mathbf{C} \in \mathbb{R}^4$ represents the camera's center or location in the world coordinate system, and can be determined by finding the Null Space vector of the rank 3 camera matrix $\mathbf{P} \in \mathbb{R}^{3\times4}$.

Given $\mathbf{PC} = \mathbf{0}$, let's consider a point in the world $\mathbf{X}(\lambda)$ as

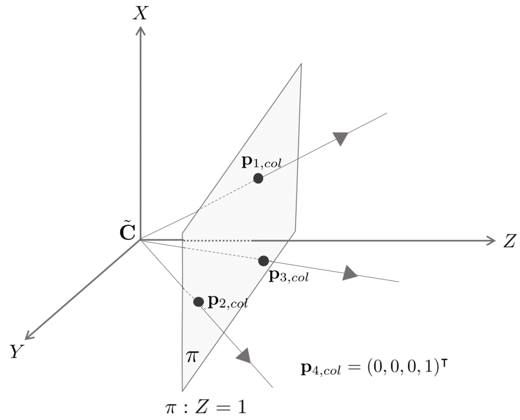
$$\mathbf{X}(\lambda) = \lambda\mathbf{A} + (1-\lambda)\mathbf{C} \tag{98}$$

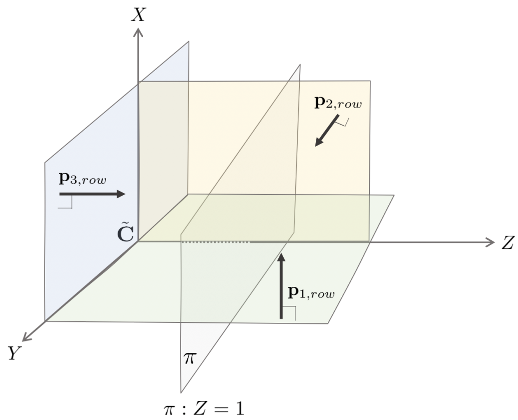
This represents the line connecting $\mathbf{A}$ and $\mathbf{C}$, and when $\mathbf{X}(\lambda)$ is projected onto the camera, it becomes

$$\mathbf{x} = \mathbf{PX}(\lambda) = \lambda\mathbf{PA} + (1-\lambda)\mathbf{PC} = \lambda\mathbf{PA} \tag{99}$$

This means the line connecting points $\mathbf{A}$ and $\mathbf{C}$ projects onto a single point $\mathbf{x} = \lambda\mathbf{PA}$ in the image plane, regardless of $\mathbf{C}$'s value, illustrating the properties of the camera's center. This property is also true for a General Projective Camera where the Null Space vector of $\mathbf{P}$ represents the camera's center $\mathbf{C}$.

Column vectors





The camera matrix $\mathbf{P}$ can be expressed in terms of column vectors as follows.

$$\begin{aligned} \mathbf{P} &= \begin{bmatrix} \mathbf{p}_{1,col} & \mathbf{p}_{2,col} & \mathbf{p}_{3_col} & \mathbf{p}_{4_col} \end{bmatrix} \\ &\text{where, } \mathbf{p}_{i,col} \in \mathbb{R}^{3\times 1}, \ i = 1, \cdots, 4 \end{aligned} \tag{100}$$

Here, $\mathbf{p}_{i,col}, \ i = 1,2,3$ represent the vanishing points located at the infinite plane π_∞ for axes X, Y, Z, respectively. And $\mathbf{p}_{4_col} = \mathbf{P}\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^\intercal$ represents the origin of the world coordinate system.

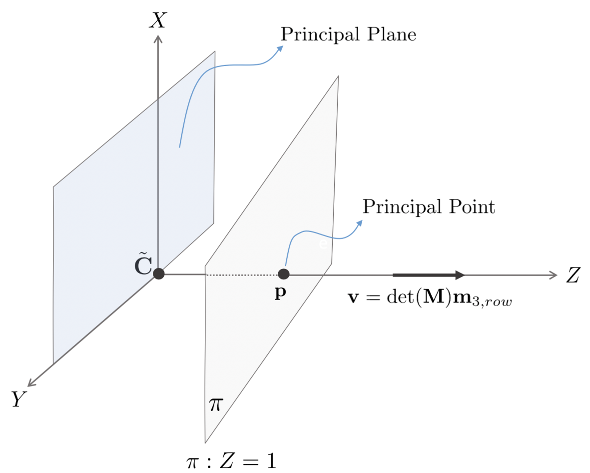
Row vectors

The camera matrix $\mathbf{P}$ can also be expressed using row vectors.

$$\begin{aligned} \mathbf{P} &= \begin{bmatrix} \mathbf{p}_{1,row}^\intercal \\ \mathbf{p}_{2,row}^\intercal \\ \mathbf{p}_{3,row}^\intercal \end{bmatrix} \\ &\text{where, } \mathbf{p}_{i,row} \in \mathbb{R}^{4\times 1}, \ i = 1,2,3 \end{aligned} \tag{101}$$

Row vector $\mathbf{p}_{i,row}, \ i = 1,2,3$ correspond to planes parallel to axes X, Y, Z in the camera coordinate system, respectively.

The principal plane



The principal plane π_{pp} includes the camera's center and is parallel to the image plane. In the camera coordinate system $\{C\}$, it is equivalent to the XY plane with characteristic $Z = 0$. A point $\mathbf{X} \in \pi_{pp}$ on the principal plane meets the image plane π at the line at infinity, thus

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \begin{pmatrix} x & y & 0 \end{pmatrix}^\intercal \tag{102}$$

Therefore, the necessary and sufficient condition for a point $\mathbf{X}$ to be located on the principal plane is $\mathbf{p}_{3,row}^\intercal \mathbf{X} = 0$, meaning that the third row vector of the camera matrix $\mathbf{p}_{3,row}$ represents the camera's principal plane.

The Principal Point

The principal point $\mathbf{p}$ refers to the intersection of the principal axis with the image plane π. The principal point $\mathbf{p}$ is located on the image plane π, and the line connecting the camera center $\mathbf{C}$ and the principal point is perpendicular to the image plane.

$$\mathbf{p} - \mathbf{C} \perp \pi \tag{103}$$

The principal point can also be defined as follows. Since the principal plane is the third row vector of the camera matrix $\mathbf{p}_{3,row}$, for a point $\mathbf{X}$ located on the principal plane:

$$\mathbf{p}_{3,row}^\intercal \mathbf{X} = 0 \tag{104}$$

Here, $\mathbf{p}_{3,row} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix}^\intercal$ represents the normal vector of the plane $\mathbf{p}_{3,row}$ in Dual Projective Space $(\mathbb{P}^3)^\vee$. The intersection line between the principal plane $\mathbf{p}_{3,row}$ and the plane at infinity π_∞ has the normal vector existing on the plane at infinity as $\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & 0 \end{pmatrix}^\intercal$. Consequently, the point projected onto the image plane is the principal point $\mathbf{p}$.

$$\mathbf{p} = \mathbf{P} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ 0 \end{pmatrix} \tag{105}$$

The normal vector existing on the plane at infinity $\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & 0 \end{bmatrix}^\intercal$, passing through the camera center $\mathbf{C}$, is identical to the principal axis.

$$\mathbf{P}\left(\lambda \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ 0 \end{pmatrix} + (1-\lambda)\mathbf{C} \right) = \lambda \mathbf{P} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ 0 \end{pmatrix} = \mathbf{p} \tag{106}$$

Thus, projecting the principal axis onto the image plane results in the principal point $\mathbf{p}$. In conclusion, the principal point $\mathbf{p}$ is defined by the first three terms of the third row vector of the camera matrix $\mathbf{P}$, $\mathbf{p}_{3,row} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & p_4 \end{bmatrix}^\intercal$ as $\mathbf{p} = (\pi_1, \pi_2, \pi_3)^\intercal$.

The Principal Axis Vector

When a camera matrix $\mathbf{P} = [\mathbf{M} \mid \mathbf{p}_{4,col}]$ is given, the third row vector of the matrix $\mathbf{M} \in \mathbb{R}^{3\times3}$, $\mathbf{m}_{3,row}$, represents the principal point. This section considers the direction of the principal axis in the camera coordinate system as equivalent to the $+Z$ direction indicated by $\mathbf{m}_{3,row}$. However, since the camera matrix $\mathbf{P}$ is uniquely determined up to sign, it is impossible to know whether $\mathbf{m}_{3,row}$ or $-\mathbf{m}_{3,row}$ signifies $+Z$.

By multiplying the determinant of $\mathbf{M}$ to $\mathbf{m}_{3,row} = \det(\mathbf{M})\mathbf{m}_{3,row} = (0,0,1)^\intercal$, it always indicates a positive direction. Even if the scale changes from $\mathbf{P} \to k\mathbf{P}$, $\mathbf{v} \to k^4\mathbf{v}$ remains in the same direction. When a typical camera matrix $k\mathbf{P} = k\mathbf{K}\mathbf{R}[\mathbf{I} \mid -\tilde{\mathbf{C}}]$ is provided, $\mathbf{M} = k\mathbf{K}\mathbf{R}$ and since $\det(\mathbf{R}) > 0$, the direction vector of the axis $\mathbf{v} = \det(\mathbf{M})\mathbf{m}_{3,row}$ remains consistent. Accordingly,

$$\mathbf{v} = \det(\mathbf{M})\mathbf{m}_{3,row} \tag{107}$$

This vector represents the direction vector of the principal axis.

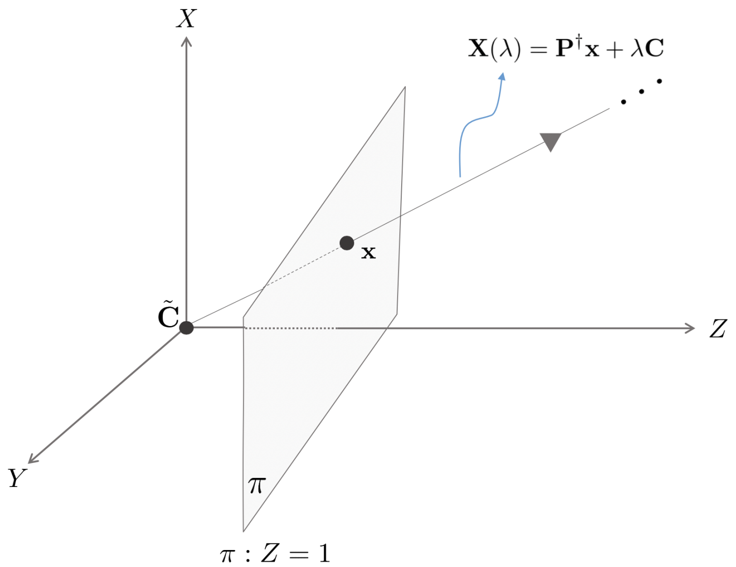
Action of a Projective Camera on Points

Forward Projection

Forward projection, commonly referred to as projection, signifies the operation of transforming a given point $\mathbf{X}$ in the world into a point $\mathbf{x}$ on the image plane. The following formula holds for any camera matrix $\mathbf{P}$:

$$\mathbf{x} = \mathbf{P}\mathbf{X} \tag{108}$$

Back-projection of Points to Rays



Back-projection is the opposite of forward projection and means the operation of transforming a given point $\mathbf{x}$ on the image plane into a line in the world. Typically, the depth value of $\mathbf{x}$ is unknown, hence it cannot be directly transformed into a point $\mathbf{X}$ in the world. Since the rank of any camera matrix $\mathbf{P}$ is 3, a Right Pseudo Inverse $\mathbf{P}^{\dagger}$ exists.

$$\mathbf{P}^{\dagger} = \mathbf{P}^{\intercal}(\mathbf{P}\mathbf{P}^{\intercal})^{-1} \tag{109}$$

Here, $\mathbf{P}\mathbf{P}^{\dagger} = \mathbf{P}\mathbf{P}^{\intercal}(\mathbf{P}\mathbf{P}^{\intercal})^{-1} = \mathbf{I}$ holds true. The line back-projected by $\mathbf{P}^{\dagger}\mathbf{x}$ passes through the camera's center $\mathbf{C}$, thus

$$\mathbf{X}(\lambda) = \mathbf{P}^{\dagger}\mathbf{x} + \lambda\mathbf{C} \tag{110}$$

is represented. Re-projecting the back-projection line results in $\mathbf{P}\mathbf{X}(\lambda) = \mathbf{P}\mathbf{P}^{\dagger}\mathbf{x} + \lambda\mathbf{P}\mathbf{C} = \mathbf{x}$.

In the case of a Finite Projective camera, back-projection can be expressed differently. When an arbitrary Finite Projective camera matrix $\mathbf{P} = [\mathbf{M} \mid \mathbf{p}_{4,col}]$ is given, the camera's center point is represented as $\tilde{\mathbf{C}} = -\mathbf{M}^{-1}\mathbf{p}_{4,col}$. Here, the line back-projected from a point $\mathbf{x}$ on the image plane is tangent to the plane at infinity π_{∞} and $\mathbf{D} = ((\mathbf{M}^{-1}\mathbf{x})^{\intercal}, 0)$, so the back-projection line is

$$\mathbf{X}(\mu) = \mu \begin{pmatrix} \mathbf{M}^{-1}\mathbf{x} \\ 0 \end{pmatrix} + \begin{pmatrix} -\mathbf{M}^{-1}\mathbf{p}_{4_col} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{M}^{-1}(\mu\mathbf{x} - \mathbf{p}_{4_col}) \\ 1 \end{pmatrix} \tag{111}$$

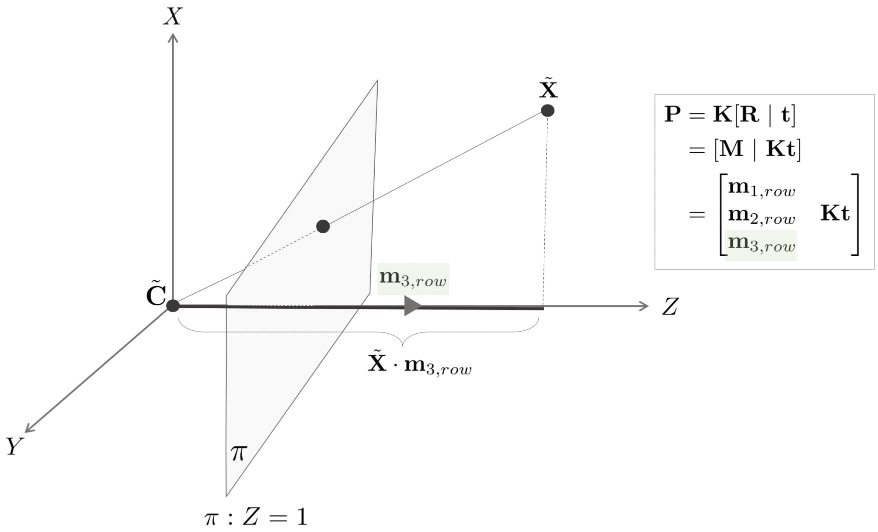
Depth of points

General Projective 카메라 $\mathbf{P}$와 월드 상의 한 점 $\mathbf{X} = \begin{pmatrix} X & Y & Z & 1 \end{pmatrix}^{\intercal}$가 주어졌을 때 이를 이미지 평면으로 프로젝션시키면

$$\mathbf{x} = \mathbf{P}\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ w \end{pmatrix} \tag{112}$$

와 같이 하나의 점 $\mathbf{x}$를 얻을 수 있다.

Result 6.1



The depth of a point $\mathbf{X}$ in the world with respect to the camera matrix $\mathbf{P}$ is expressed as

$$\text{depth}(\mathbf{X}; \mathbf{P}) = \frac{\text{sign}(\det(M))w}{\|\mathbf{m}_{3,row}\|} \tag{113}$$

where $\mathbf{m}_{3,row} \in \mathbb{R}^{3\times 3}$ is the third row vector of the matrix $\mathbf{M}$.

Proof

The row vector $\mathbf{m}_{3,row}$ represents the direction of the principal axis, so projecting the world point $\tilde{\mathbf{X}}$ onto $\mathbf{m}_{3,row}$ gives the depth along the Z-axis. The projection onto the principal axis is

$$\text{depth} = \frac{(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})\mathbf{m}_{3.row}}{\|\mathbf{m}_{3,row}\|} \tag{114}$$

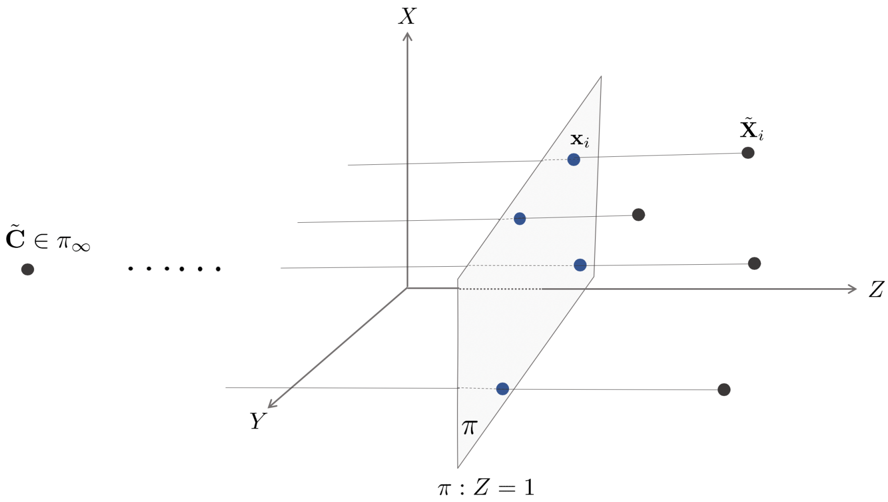
In the case of a Finite Projective camera, $\mathbf{m}_{3,row} = \mathbf{r}_{3,row} = 1$.
The depth value is the third row w of $\mathbf{x} = \mathbf{PX}$, hence

$$\begin{aligned} w &= (\mathbf{PX})_{3,row} \\ &= (\mathbf{P}(\mathbf{X} - \mathbf{C}))_{3,row} \\ &= (\tilde{\mathbf{X}} - \tilde{\mathbf{C}})\mathbf{m}_{3,row} \end{aligned} \tag{115}$$

Thus, considering that the depth value can be located behind the camera depending on the sign of $\det(\mathbf{M})$, it is expressed as follows.

$$\text{depth}(\mathbf{X}; \mathbf{P}) = \frac{\text{sign}(\det(M))w}{\|\mathbf{m}_{3,row}\|} \tag{116}$$

Cameras at infinity

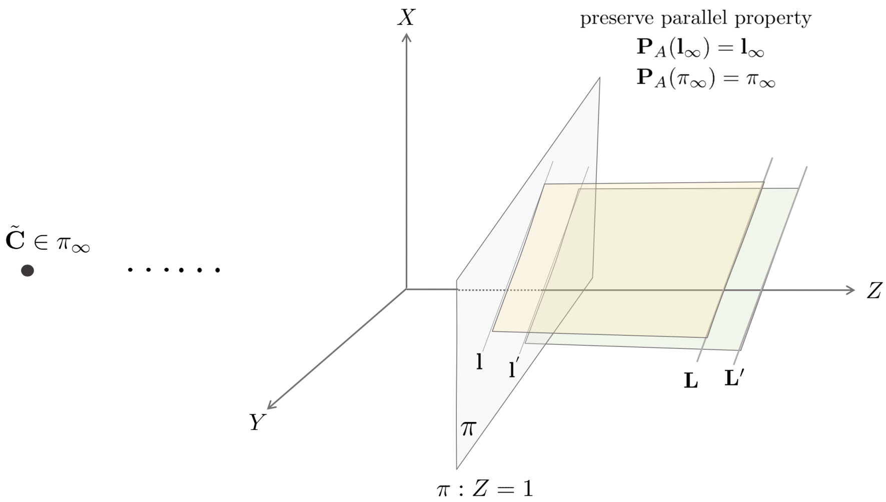


A General Projective camera whose center point C exists on the infinite plane π_∞ is referred to as a camera at infinity.

$$\mathbf{C} = (*,\ *,\ *, 0)^\intercal \in \pi_\infty \tag{117}$$

An equivalent case occurs when the camera matrix $\mathbf{P} = [\mathbf{M} \mid \mathbf{p}_{4,col}]$ is given and matrix $\mathbf{M}$ is singular. Cameras at infinity are broadly classified into Affine cameras and Non-affine cameras.

Definition 6.3

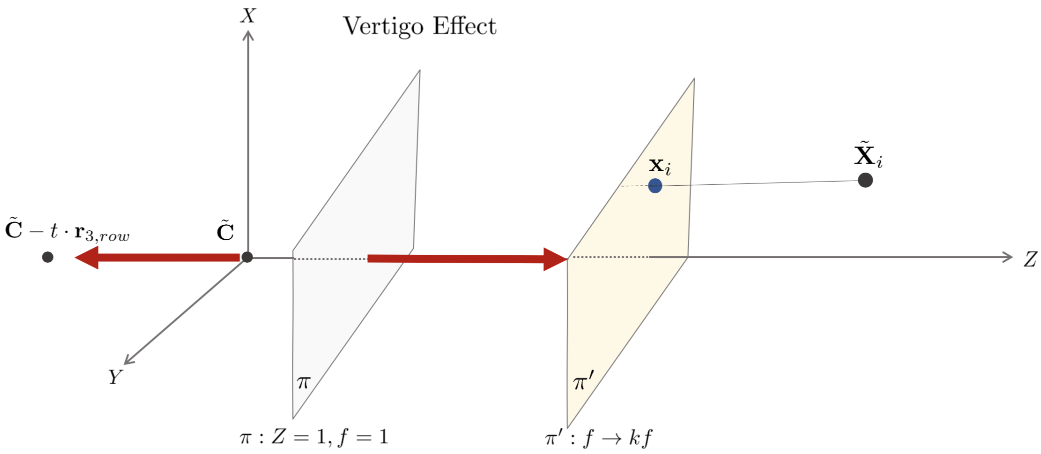


An Affine camera $\mathbf{P}_A$ is defined as a camera that projects the infinite plane to again be the infinite plane.

$$\mathbf{P}_A(\pi_\infty) = \pi_\infty \tag{118}$$

In this case, $\mathbf{P}_A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$ form.

Affine cameras



Consider a Finite Projective camera matrix $\mathbf{P} = \mathbf{KR}[\mathbf{I} \mid 0]$ with objects existing in the world. If we perform a Zoom In on the object while simultaneously moving the camera in the opposite direction of the principal axis, a Vertigo Effect occurs. The Vertigo Effect is named after the technique first used in the movie Vertigo by director Hitchcock.

To understand this mathematically, reconsider the depth value d of an object in the world given a camera center point $\tilde{\mathbf{C}}$ and a world point $\tilde{\mathbf{X}}$:

$$d = -(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})\mathbf{r}_{3,row} \tag{119}$$

Here, $\mathbf{r}_{3,row}$ is the third row vector of the rotation matrix $\mathbf{R}$ and represents the principal axis. If we denote the distance between the camera center and the world origin as d_0, then for $\tilde{\mathbf{X}} = 0$,

$$d_0 = -\tilde{\mathbf{C}}\mathbf{r}_{3,row} \tag{120}$$

As the camera moves in the opposite direction of the principal axis, the camera center $\tilde{\mathbf{C}}$ becomes

$$\tilde{\mathbf{C}} - t \cdot \mathbf{r}_{3,row} \tag{121}$$

and here, t represents time. As the camera moves backward over time, the camera matrix becomes

$$\begin{aligned} \mathbf{P}_t &= \mathbf{KR}[\mathbf{I} \mid -(\tilde{\mathbf{C}} - t \cdot \mathbf{r}_{3,row})] \\ &= \mathbf{K}\begin{bmatrix} & & -\tilde{\mathbf{C}}\mathbf{r}_{1,row} \\ & \mathbf{R} & -\tilde{\mathbf{C}}\mathbf{r}_{2,row} \\ & & t - \tilde{\mathbf{C}}\mathbf{r}_{3,row} \end{bmatrix} \\ &= \mathbf{K}\begin{bmatrix} & & -\tilde{\mathbf{C}}\mathbf{r}_{1,row} \\ & \mathbf{R} & -\tilde{\mathbf{C}}\mathbf{r}_{2,row} \\ & & d_0 + t \end{bmatrix} \end{aligned} \tag{122}$$

Thus, moving the camera in the opposite direction of the principal axis results in $\mathbf{P}_t$ with only the $(3, 4)$ term being modified to $d_0 + t$. If we define $d_0 + t = d_t$,

$$\mathbf{P}_t = \mathbf{K}\begin{bmatrix} - & - & - & - \\ - & nochange & - & - \\ - & - & - & d_t \end{bmatrix} \tag{123}$$

Next, consider performing a Zoom In. Mathematically, Zoom In involves increasing the focal length f,

$$ZoomIn : f \to kf \quad \forall k > 0 \tag{124}$$

When expressed as a matrix, Zoom In becomes

$$\mathbf{P} \to \begin{bmatrix} k & & \\ & k & \\ & & 1 \end{bmatrix} \mathbf{P} \tag{125}$$

When moving the camera in the direction of the principal axis and simultaneously Zooming In by a factor of k, it is possible to create the Vertigo Effect without changing the depth of the object. The appropriate Zoom In factor k is

$$k = d_t/d_0 \tag{126}$$

Ultimately, the camera matrix over time $\mathbf{P}_t$ becomes

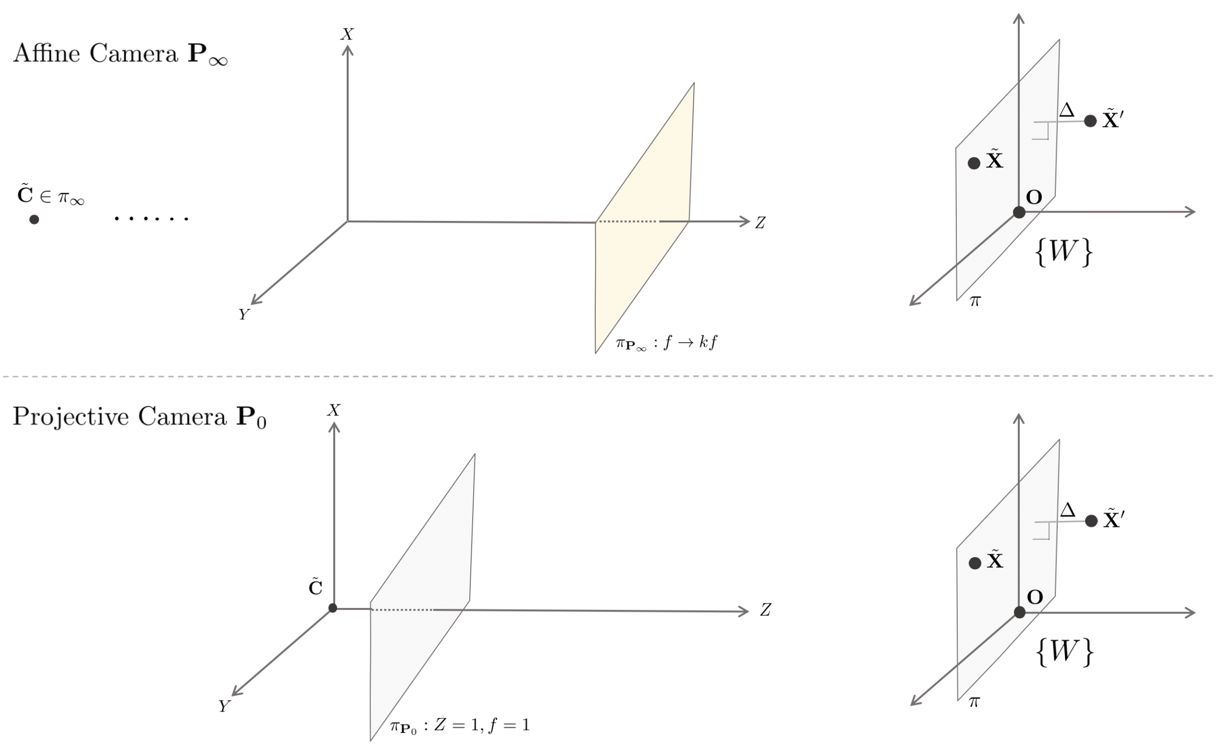
$$\begin{aligned} \begin{bmatrix} d_t/d_0 & & \\ & d_t/d_0 & \\ & & 1 \end{bmatrix} \mathbf{P}_t &= \mathbf{K} \begin{bmatrix} d_t/d_0 & & \\ & d_t/d_0 & \\ & & 1 \end{bmatrix} \left[\begin{array}{c|c} \mathbf{R} & \begin{matrix} * \\ * \\ d_t \end{matrix} \end{array} \right] \\ &= \frac{1}{k}\mathbf{K} \begin{bmatrix} 1 & & \\ & 1 & \\ & & d_0/d_t \end{bmatrix} \left[\begin{array}{c|c} \mathbf{R} & \begin{matrix} * \\ * \\ d_t \end{matrix} \end{array} \right] \\ &= \frac{1}{k}\mathbf{K} \begin{bmatrix} - & - & - & - \\ - & \textit{nochange} & - & - \\ & d_0/d_t \cdot \mathbf{r}_{3,row} & & d_0 \end{bmatrix} \end{aligned} \tag{127}$$

$\frac{1}{k}$ is a scaling factor and can be omitted. Assuming time progresses infinitely,

$$\mathbf{P}_\infty = \lim_{t\to\infty} \mathbf{P}_t = \mathbf{K} \begin{bmatrix} \mathbf{r}^\intercal_{1,row} & -\mathbf{r}^\intercal_{1,row}\tilde{\mathbf{C}} \\ \mathbf{r}^\intercal_{2,row} & -\mathbf{r}^\intercal_{2,row}\tilde{\mathbf{C}} \\ \mathbf{0}^\intercal & d_0 \end{bmatrix} \tag{128}$$

As the third row of $\mathbf{P}$ consists of $\mathbf{0}^\intercal$, this corresponds to an Affine camera.

Error in employing an affine camera



In this section, the significant differences between capturing the same object with a General Projective camera and an Affine camera are discussed. The General Projective camera is denoted by $\mathbf{P}_0$, the Affine camera by $\mathbf{P}_\infty$, and changes in the camera matrix over time t are denoted by $\mathbf{P}_t$.

When a plane π that includes the origin of the world coordinate system and is perpendicular to the image plane of camera $\mathbf{P}_t$ is given, the points on plane π remain constant in the image obtained by $\mathbf{P}_t$ over time, when performing the Vertigo Effect (zoom in + backward moving).

To prove this, consider a point $\mathbf{X} \in \pi$ on plane π. Since π includes the origin of the world coordinate system, it can be expressed as:

$$\mathbf{X} = \begin{pmatrix} \alpha \mathbf{r}_{1,row} + \beta \mathbf{r}_{2,row} \\ 1 \end{pmatrix} \in \mathbb{R}^4 \tag{129}$$

Projecting this through $\mathbf{P}_t$ yields:

$$\begin{aligned} \mathbf{P}_t \mathbf{X} &= \mathbf{K} \begin{bmatrix} \mathbf{r}_{1,row}^\intercal & -\mathbf{r}_{1,row}^\intercal \tilde{\mathbf{C}} \\ \mathbf{r}_{2,row}^\intercal & -\mathbf{r}_{2,row}^\intercal \tilde{\mathbf{C}} \\ d_0/d_t \cdot \mathbf{r}_{3,row} & d_0 \end{bmatrix} \begin{bmatrix} \alpha \mathbf{r}_{1,row} + \beta \mathbf{r}_{2,row} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} * \\ * \\ d_0 \end{bmatrix} \\ & \because \mathbf{r}_{1,row} \cdot \mathbf{r}_{3,row} = \mathbf{r}_{2,row} \cdot \mathbf{r}_{3,row} = 0 \end{aligned} \tag{130}$$

Therefore, the depth (depth) of the point $\mathbf{X}$ on plane π, which includes the origin of the world coordinate system, is always constant at d_0, thus appearing constant in size over time t. **In other words, $\mathbf{X}$ is transformed into the same point in the image for both the General Projective and Affine cameras.**

$$\mathbf{P}_0 \mathbf{X} = \mathbf{P}_t \mathbf{X} = \mathbf{P}_\infty \mathbf{X} \tag{131}$$

However, **if a world point $\mathbf{X}'$, which is Δ away from plane π passing through the origin of the world coordinate system and perpendicular to the image plane, is captured by the two cameras, $\mathbf{P}_0 \mathbf{X}' \neq \mathbf{P}_\infty \mathbf{X}'$.** $\mathbf{X}'$ can be expressed as follows.

$$\mathbf{X}' = \begin{pmatrix} \alpha \mathbf{r}_{1,row} + \beta \mathbf{r}_{2,row} + \Delta \mathbf{r}_{3,row} \\ 1 \end{pmatrix} \tag{132}$$

Here, $\mathbf{r}_{3,row}$ represents the camera's principal axis. Projecting $\mathbf{X}'$ with both cameras results in:

$$\mathbf{x}_{\text{proj}} = \mathbf{P}_0 \mathbf{X}' = \mathbf{K} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z}_{\text{proj}} \end{pmatrix} = \mathbf{K} \begin{pmatrix} \alpha - \mathbf{r}_{1,row}^\intercal \tilde{\mathbf{C}} \\ \beta - \mathbf{r}_{2,row}^\intercal \tilde{\mathbf{C}} \\ d_0 + \Delta \end{pmatrix} \tag{133}$$

$$\mathbf{x}_{\text{affine}} = \mathbf{P}_\infty \mathbf{X}' = \mathbf{K} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z}_{\text{affine}} \end{pmatrix} = \mathbf{K} \begin{pmatrix} \alpha - \mathbf{r}_{1,row}^\intercal \tilde{\mathbf{C}} \\ \beta - \mathbf{r}_{2,row}^\intercal \tilde{\mathbf{C}} \\ d_0 \end{pmatrix} \tag{134}$$

$\tilde{z}_{\text{proj}}$ can be calculated as follows.

$$\begin{aligned} \tilde{z}_{\text{proj}} &= [\mathbf{r}_{3,row} | -\mathbf{r}_{3,row} \tilde{\mathbf{C}}] \mathbf{X}' \\ &= [\mathbf{r}_{3,row} | -\mathbf{r}_{3,row} \tilde{\mathbf{C}}] \begin{pmatrix} \alpha \mathbf{r}_{1,row} + \beta \mathbf{r}_{2,row} + \Delta \mathbf{r}_{3,row} \\ 1 \end{pmatrix} \\ &= -\mathbf{r}_{3,row} \tilde{\mathbf{C}} + \Delta \\ &= d_0 + \Delta \end{aligned} \tag{135}$$

The camera calibration matrix $\mathbf{K}$ can be represented as follows.

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{2\times2} & \tilde{\mathbf{x}}_0 \\ \tilde{0}^\intercal & 1 \end{bmatrix} \tag{136}$$

Here, $\mathbf{K}_{2\times2}$ represents an upper-triangular matrix of size 2×2 and $\tilde{\mathbf{x}}_0 = \begin{pmatrix} x_0 & y_0 \end{pmatrix}^\intercal$ represents the origin of the image plane. Considering these, the formulas can be reorganized as follows.

$$\begin{aligned} \mathbf{x}_{\text{proj}} &= \begin{pmatrix} \mathbf{K}_{2\times 2}\tilde{\mathbf{x}} + (d_0+\Delta)\tilde{\mathbf{x}}_0 \\ d_0+\Delta \end{pmatrix} \\ \mathbf{x}_{\text{affine}} &= \begin{pmatrix} \mathbf{K}_{2\times 2}\tilde{\mathbf{x}} + d_0\tilde{\mathbf{x}}_0 \\ d_0 \end{pmatrix} \end{aligned} \tag{137}$$

Here $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} & \tilde{y} \end{pmatrix}^\intercal$. Calculating the Inhomogeneous coordinates of the points $\mathbf{x}_{\text{proj}}$ and $\mathbf{x}_{\text{affine}}$ results in values obtained by dividing by the last term.

$$\begin{aligned} \tilde{\mathbf{x}}_{\text{proj}} &= \tilde{\mathbf{x}}_0 + \mathbf{K}_{2\times 2}\tilde{\mathbf{x}}/(d_0+\Delta) \\ \tilde{\mathbf{x}}_{\text{affine}} &= \tilde{\mathbf{x}}_0 + \mathbf{K}_{2\times 2}\tilde{\mathbf{x}}/d_0 \end{aligned} \tag{138}$$

In conclusion, the difference between the two points projected through the General Projective and Affine cameras is as follows.

$$\tilde{\mathbf{x}}_{\text{affine}} - \tilde{\mathbf{x}}_0 = \frac{d_0+\Delta}{d_0}(\tilde{\mathbf{x}}_{\text{proj}} - \tilde{\mathbf{x}}_0) \tag{139}$$

This equation is called the Discrepancy Equation, and if $\Delta = 0$, i.e., the point is on the plane π which includes the world coordinate system origin and is perpendicular to the image plane, the object captured by the two cameras projects to the same point in the image. This phenomenon can be observed in movies like Vertigo or Jaws, where the protagonist's face remains unchanged while the surroundings zoom out.

4 Computation of the Camera Matrix P

This section describes the method to numerically determine the camera matrix $\mathbf{P}$ using several pairs of corresponding points $(\mathbf{x}_i, \mathbf{X}_i)$ in $\mathbb{P}^3$ space and $\mathbb{P}^2$ space. This method is commonly referred to as Resectioning or Calibration.

Basic equations

When a pair of corresponding points $(\mathbf{x}_i, \mathbf{X}_i)$ is given, the correspondence between the two points is as follows.

$$\mathbf{x}_i = \mathbf{P}\mathbf{X}_i \tag{140}$$

Here, $\mathbf{P}\mathbf{X}_i$ is:

$$\mathbf{P}\mathbf{X}_i = \begin{bmatrix} \mathbf{p}^\intercal_{1,row} \\ \mathbf{p}^\intercal_{2,row} \\ \mathbf{p}^\intercal_{3,row} \end{bmatrix} \mathbf{X}_i \quad = \begin{bmatrix} \mathbf{p}^\intercal_{1,row}\mathbf{X}_i \\ \mathbf{p}^\intercal_{2,row}\mathbf{X}_i \\ \mathbf{p}^\intercal_{3,row}\mathbf{X}_i \end{bmatrix} \tag{141}$$

Using row vectors (row vector), $\mathbf{p}_{i,row} \in \mathbb{R}^{4\times 1}$ is meant. If $\mathbf{x} = \begin{pmatrix} x & y & w \end{pmatrix}^\intercal$, then $\mathbf{x}_i \times \mathbf{P}\mathbf{X}_i = 0$ thus,

$$\mathbf{x}_i \times \mathbf{P}\mathbf{X}_i = \begin{pmatrix} y_i \cdot \mathbf{p}^\intercal_{2,row}\mathbf{X}_i - w_i \cdot \mathbf{p}_{3,row}\mathbf{X}_i \\ w_i \cdot \mathbf{p}^\intercal_{1,row}\mathbf{X}_i - x_i \cdot \mathbf{p}_{3,row}\mathbf{X}_i \\ x_i \cdot \mathbf{p}^\intercal_{2,row}\mathbf{X}_i - y_i \cdot \mathbf{p}_{1,row}\mathbf{X}_i \end{pmatrix} = 0 \tag{142}$$

This can be arranged in the form $\mathbf{A}\mathbf{p} = 0$,

$$\begin{bmatrix} \mathbf{0}^\intercal & -w_i\mathbf{X}_i^\intercal & y_i\mathbf{X}_i^\intercal \\ w_i\mathbf{X}_i^\intercal & \mathbf{0}^\intercal & -x_i\mathbf{X}_i^\intercal \\ -y_i\mathbf{X}_i^\intercal & x_i\mathbf{X}_i^\intercal & \mathbf{0}^\intercal \end{bmatrix} \begin{pmatrix} \mathbf{p}_{1,row} \\ \mathbf{p}_{2,row} \\ \mathbf{p}_{3,row} \end{pmatrix} = 0 \tag{143}$$

Since the last row (row) of the left matrix is linearly dependent, only the first and second rows are represented as

$$\underbrace{\begin{bmatrix} \mathbf{0}^\intercal & -w_i\mathbf{X}_i^\intercal & y_i\mathbf{X}_i^\intercal \\ w_i\mathbf{X}_i^\intercal & \mathbf{0}^\intercal & -x_i\mathbf{X}_i^\intercal \end{bmatrix}}_{\mathbf{A}} \begin{pmatrix} \mathbf{p}_{1,row} \\ \mathbf{p}_{2,row} \\ \mathbf{p}_{3,row} \end{pmatrix} = 0 \tag{144}$$

Here, the matrix $\mathbf{A}$ is of size $\mathbb{R}^{2n\times 12}$, and the vector $\begin{pmatrix} \mathbf{p}_{1,row} \\ \mathbf{p}_{2,row} \\ \mathbf{p}_{3,row} \end{pmatrix}$ is 12×1 in size. Since this equation is in the form $\mathbf{Ap} = 0$, vector $\mathbf{p}$ can be determined using methods such as Singular Value Decomposition (SVD).

Minimal solution

To determine the vector $\mathbf{p} \in \mathbb{R}^{12}$ up to scale, a total of 11 equations are necessary. Using one pair of corresponding points $(\mathbf{x}_i, \mathbf{X}_i)$ generates 2 equations, thus a minimum of 5.5 pairs of corresponding points are required to solve for $\mathbf{p}$. In the case of 5.5 noise-free corresponding point pairs, the rank of matrix $\mathbf{A}$ becomes 11, making the Null Space vector the unique solution vector $\mathbf{p}$.

Over-determined solution

Typically, more than six pairs of corresponding points $(\mathbf{x}_i, \mathbf{X}_i)$ can be obtained, and since the data contains noise, the rank of matrix $\mathbf{A}$ becomes 12. Hence, there is no Null Space, making it impossible to solve for the solution vector $\mathbf{p}$. **In such an Over-determined system of linear equations $\mathbf{Ap} = 0$, the approximate solution vector $\hat{\mathbf{p}}$, which minimizes $\|\mathbf{Ap}\|$ under the condition $\|\mathbf{p}\| = 1$, must be found.**

Degenerate configurations

If more than 5.5 pairs of corresponding points $(\mathbf{x}_i, \mathbf{X}_i)$ are not linearly independent, it becomes impossible to determine the unique solution vector $\mathbf{p}$, and such sets of corresponding points are called Degenerate Configurations. The inability to determine a unique solution vector $\mathbf{p}$ implies that for a point $\mathbf{X}$ in the world,

$$
\begin{aligned}
&{}^{\exists}\mathbf{P}', \quad \mathbf{P} \neq \mathbf{P}' \\
&\mathbf{PX}_i = \mathbf{P}'\mathbf{X}_i \quad {}^{\forall}i
\end{aligned}
\tag{145}
$$

suggests the existence of another camera matrix $\mathbf{P}'$ that satisfies the conditions, equivalent to saying that $\mathbf{PX} = -\theta\mathbf{P}'\mathbf{X}$ for some constant θ, leading to

$$
\underbrace{(\mathbf{P} + \theta\mathbf{P}')}_{\mathbf{P}_\theta}\mathbf{X} = 0 \quad \text{for some } \theta \tag{146}
$$

and the set of points $\mathbf{X}$ in the world that satisfy $\mathbf{P}_\theta\mathbf{X} = 0$ cannot differentiate between $\mathbf{P}$ and $\mathbf{P}'$. Such sets are denoted as $\mathcal{S}_\theta$:

$$
\mathcal{S}_\theta = \{\mathbf{X} \mid \mathbf{P}_\theta\mathbf{X} = 0\} \tag{147}
$$

Points $\mathbf{X}$ that satisfy $\mathcal{S}_\theta$ include:

- All $\mathbf{X}_i$s positioned on a Twisted Cubic
- All $\mathbf{X}_i$s existing on the same plane, including a line through the camera's center point

Twisted Cubic refers to a curve in $\mathbb{P}^3$ space. The set $\mathcal{C}_\theta$ consisting of Twisted Cubic $\mathbf{C}_\theta$:

$$
\mathcal{C}_\theta = \{\mathbf{C}_\theta \mid \mathbf{P}_\theta\mathbf{C}_\theta = 0 \text{ and } \mathbf{P}_\theta\text{'s rank is 3}\} \tag{148}
$$

signifies that $\mathbf{C}_\theta$, typically in the form of a cubic polynomial, lies in the row space of $\mathbf{P}_\theta$. Therefore,

$$
\mathbf{C}_\theta = \text{Row } \mathbf{P}_\theta \tag{149}
$$

implies that $\mathbf{C}_\theta = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}$, and

$$
\det \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ - & - & - & - \\ - & \mathbf{P}_\theta & - & - \\ - & - & - & - \end{pmatrix} = 0 \tag{150}
$$

results in $\det(2\ 3\ 4)c_1 - \det(1\ 3\ 4)c_2 + \det(1\ 2\ 4)c_3 - \det(1\ 2\ 3)c_4 = 0$. **From this expansion, $\mathbf{C}_\theta$ is expressed as**

$$\mathbf{C}_\theta = (\det(2\ 3\ 4), -\det(1\ 3\ 4), \det(1\ 2\ 4), -\det(1\ 2\ 3)) \tag{151}$$

becoming a Twisted Cubic with each term being a cubic. $\mathbf{P}, \mathbf{P}'$ may share common roots among the terms of $\mathbf{C}_\theta$, and the degrees of each term may fall below cubic. Such cases are referred to as the Degenerate Configuration of $\mathbf{C}_\theta$, and such $\mathbf{C}_\theta$ are not Twisted Cubics.

Line correspondences

When a line $\mathbf{L}$ in the world is projected by the camera matrix $\mathbf{P}$ onto the image plane as line $\mathbf{l}$, unlike points, $\mathbf{l} \neq \mathbf{PL}$.

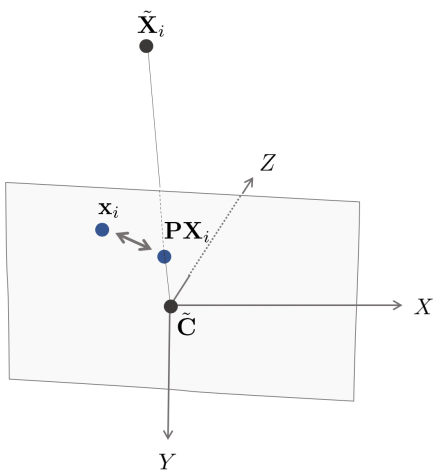
$$\mathbf{x} = \mathbf{PX} \quad \text{but,} \quad \mathbf{l} \neq \mathbf{PL} \tag{152}$$

Since a point $\mathbf{X}$ on the line $\mathbf{L}$ projected by the camera results in point $\mathbf{x}$ existing on line $\mathbf{l}$,

$$\begin{aligned} \mathbf{l}^\intercal \mathbf{x} = \mathbf{l}^\intercal \mathbf{PX} = 0 \\ \Rightarrow \mathbf{Ap} = 0 \end{aligned} \tag{153}$$

A linear equation for vector $\mathbf{p}$ holds as shown above. **Therefore, by using multiple points $\mathbf{X}_i$ on the world line L, a linear equation for vector p holds, and through this, the camera matrix P can be determined.**

Geometric error



As previously explained, an over-determined linear system of the form $\mathbf{Ap} = 0$ can be constructed using pairs of corresponding points $(\mathbf{x}_i, \mathbf{X}_i)$, and an approximate solution vector $\hat{\mathbf{p}}$ can be obtained that minimizes the magnitude of $\|\mathbf{Ap}\|$ while $\|\mathbf{p}\| = 1$. This section explains a method to minimize geometric error in order to obtain a more accurate camera matrix $\mathbf{P}$. **Geometric error refers to the pixel distance between the given $\mathbf{x}_i$ in the image plane and the projected point $\mathbf{PX}_i$ of the world point $\mathbf{X}_i$. In real data, since $\mathbf{x}_i \neq \mathbf{PX}_i$ due to noise, the camera matrix P that minimizes the distance $d(\mathbf{x}_i, \mathbf{PX}_i)$ between the two points must be found.**

$$\min_{\mathbf{P}} d(\mathbf{x}_i, \mathbf{PX}_i)^2 \tag{154}$$

Since $d(\mathbf{x}_i, \mathbf{PX}_i)^2$ is generally non-linear, the optimal camera matrix $\mathbf{P}$ can be determined using non-linear least squares methods such as Gauss-Newton (GN) or Levenberg-Marquardt (LM).

Algorithm 7.1

- **Objective:** Find the MLE (maximum likelihood estimation) value for $\mathbf{P}$ for given pairs of corresponding points $(\mathbf{x}_i, \mathbf{X}_i),\ i = 1, \cdots, 6, \cdots$ to minimize $\sum_i d(\mathbf{x}_i, \mathbf{P}\mathbf{X}_i)^2$.

- **Normalization:** Normalize the image points $\mathbf{x}_i$ through a matrix $\mathbf{T}$ such that $\mathbf{x}_i \to \bar{\mathbf{x}}_i$ and normalize the world points $\mathbf{X}_i$ through a matrix $\mathbf{U}$ such that $\mathbf{X}_i \to \bar{\mathbf{X}}_i$. Without performing normalization, direct linear transformation (DLT) results in the last term being very small, equivalent to 1, while other terms are very large, thus not yielding a proper solution.

- **DLT:** Form the normalized pairs of corresponding points into an over-determined system $\bar{\mathbf{A}}\bar{\mathbf{p}} = 0$. Next, find the approximate solution $\bar{\mathbf{p}}$ that minimizes $\|\bar{\mathbf{A}}\bar{\mathbf{p}}\|$ while $\|\bar{\mathbf{p}}\| = 1$ through DLT, and set it as the initial value $\bar{\mathbf{P}}_0$.

- **Minimize geometric error:** Calculate the optimal normalized camera matrix $\bar{\mathbf{P}}$ by minimizing the following geometric error using GN or LM methods.

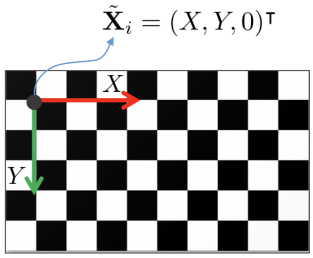
$$\min_{\bar{\mathbf{P}}} \sum_i d(\bar{\mathbf{x}}_i, \bar{\mathbf{P}}\bar{\mathbf{X}}_i)^2 \quad \text{start at } \bar{\mathbf{P}}_0 \tag{155}$$

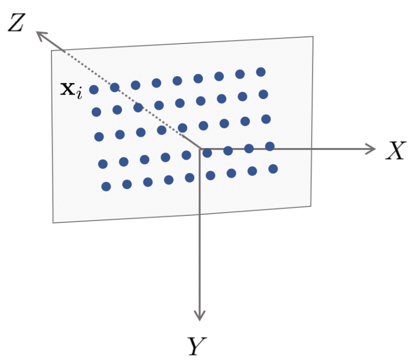
- **Denormalization:** Convert the normalized camera matrix back to the original camera matrix.

$$\mathbf{P} = \mathbf{T}^{-1}\bar{\mathbf{P}}\mathbf{U} \tag{156}$$

This algorithm is commonly referred to as **The Gold Standard algorithm for estimating P.**

Zhang's method





When using the Gold Standard algorithm, instead of using arbitrary pairs of corresponding points, pairs of corresponding points on a checkerboard are used. This algorithm to estimate the camera matrix $\mathbf{P}$ using a checkerboard is called Zhang's Method. When the checkerboard plane π_0 in the world is given, the origin of the world is set to the top left of the checkerboard, and the checkerboard plane is set as the plane $Z = 0$.

$$\pi_0 = \{\mathbf{X} = (X, Y, Z)^\intercal \mid Z = 0\} \tag{157}$$

Accordingly, any point $\mathbf{X}_i$ on the checkerboard plane π_0 becomes a point with $Z = 0$.

$$\mathbf{X}_i = (*,\ *,\ 0)^\intercal \tag{158}$$

Projecting a point $\mathbf{X} = (X,\ Y,\ 0,\ 1)^\intercal$ on the checkerboard results in

$$\begin{aligned}\mathbf{PX} = \mathbf{P}\begin{pmatrix}X\\Y\\0\\1\end{pmatrix} &= \mathbf{K}[\mathbf{R} \mid \mathbf{t}]\begin{pmatrix}X\\Y\\0\\1\end{pmatrix} \\ &= \mathbf{K}[\mathbf{r}_{1,col}\ \mathbf{r}_{2,col}\ \mathbf{t}]\mathbf{X}\end{aligned} \tag{159}$$

Since $z = 0$, the third column vector of the matrix $\mathbf{R}$ becomes 0. The matrix $\mathbf{K}[\mathbf{r}_{1,col}\ \mathbf{r}_{2_col}\ \mathbf{t}] \in \mathbb{R}^{3\times3}$ can be seen as a Homography $\mathbf{H}$ transforming from the checkerboard plane π_0 to the image plane π.

$$\mathbf{H} = \mathbf{K}[\mathbf{r}_{1_col}\ \mathbf{r}_{2_col}\ \mathbf{t}] \tag{160}$$

Since the length and number of patterns on the checkerboard are known, the points $\mathbf{x}_i, i = 1, \cdots$ on the checkerboard plane π_0 can be automatically determined. Next, using a Feature Extraction algorithm, the points $\mathbf{x}'_i, i = 1, \cdots$ seen on the image plane π from π_0 can be determined. Thus, pairs of corresponding points $\mathbf{x}_i \in \pi_0 \leftrightarrow \mathbf{x}'_i \in \pi$ can be obtained. From this, the Homography $\mathbf{H}$ mapping from $\pi_0 \mapsto \pi$ can be calculated.

$$\mathbf{H} = \begin{bmatrix}\mathbf{h}_{1_col} & \mathbf{h}_{2_col} & \mathbf{h}_{3_col}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{r}_{1_col} & \mathbf{r}_{2_col} & \mathbf{t}\end{bmatrix} \tag{161}$$

This can be summarized as

$$\mathbf{K}^{-1}\begin{bmatrix}\mathbf{h}_{1_col} & \mathbf{h}_{2_col} & \mathbf{h}_{3_col}\end{bmatrix} = \begin{bmatrix}\mathbf{r}_{1_col} & \mathbf{r}_{2_col} & \mathbf{t}\end{bmatrix} \tag{162}$$

At this time, since the column vectors of the orthogonal matrix $\mathbf{R}$, $\mathbf{r}_{1_col}$ and $\mathbf{r}_{2_col}$, are orthogonal to each other, $\mathbf{r}^\intercal_{1_col}\mathbf{r}_{2_col} = 0$ holds. Using this constraint, $\mathbf{K}^{-1}\mathbf{h}_{1_col} = \mathbf{r}_{1_col}$ and $\mathbf{K}^{-1}\mathbf{h}_{2_col} = \mathbf{r}_{2_col}$, so by the orthogonality condition,

$$\mathbf{h}^\intercal_{1_col}\mathbf{K}^{-\intercal}\mathbf{K}^{-1}\mathbf{h}_{2_col} = 0 \tag{163}$$

It holds. Also, due to the orthogonal matrix condition (up to scale), $\mathbf{r}^\intercal_{1_col}\mathbf{r}_{1_col} = \mathbf{r}^\intercal_{2_col}\mathbf{r}_{2_col}$, so

$$\mathbf{h}^\intercal_{1_col}\mathbf{K}^{-\intercal}\mathbf{K}^{-1}\mathbf{h}_{1_col} = \mathbf{h}^\intercal_{2_col}\mathbf{K}^{-\intercal}\mathbf{K}^{-1}\mathbf{h}_{2_col} \tag{164}$$

The formula holds. Two equations like the above can be obtained from a single checkerboard photograph. The general camera calibration matrix $\mathbf{K}$ is

$$\mathbf{K} = \begin{bmatrix}f_x & s & x_0\\ & f_y & y_0\\ & & 1\end{bmatrix} \tag{165}$$

Therefore, $\mathbf{K}^{-\intercal}\mathbf{K}^{-1}$ also becomes a matrix with 5 variables. Therefore, when at least three Homography $\mathbf{H}_j,\ j = 1, 2, 3$ are given, $\mathbf{K}^{-\intercal}\mathbf{K}^{-1}$ can be determined.

- To determine the matrix $\mathbf{K}$ which has 5 parameters, at least three checkerboard images are acquired. Two equations per image can be obtained, so more than three must be acquired. Homography $\mathbf{H}_j,\ j = 1, 2, 3$ can be obtained for each image, and equations (163), (164) are formulated.
- Set $\mathbf{S} = \mathbf{K}^{-\intercal}\mathbf{K}^{-1}$. Perform Cholesky Decomposition or Singular Value Decomposition (SVD) on matrix $\mathbf{S}$ to find $\mathbf{K}^{-1}$. Matrix $\mathbf{S}$ is a symmetric and Positive Definite matrix, so it can be decomposed as $\mathbf{U}^\intercal\mathbf{DU}$, and a square root matrix of the diagonal matrix $\mathbf{D} exists$.

$$\text{SVD}(\mathbf{S}) = \mathbf{U}^\intercal\mathbf{DU} = (\mathbf{U}\sqrt{\mathbf{D}})(\mathbf{U}\sqrt{\mathbf{D}})^\intercal \tag{166}$$

This allows $\mathbf{K}$ to be determined.

- Using the formula $\mathbf{K}^{-1}\mathbf{H} = [\mathbf{r}_{1_col}\ \mathbf{r}_{2_col}\ \mathbf{t}]$, $\mathbf{r}_{1_col}, \mathbf{r}_{2_col}, \mathbf{t}$ are determined, and then

$$\mathbf{r}_{3_col} = \mathbf{r}_1 \times \mathbf{r}_2 \tag{167}$$

is determined. Consequently, the camera's rotation $\mathbf{R}$, translation $\mathbf{t}$, and internal parameter matrix $\mathbf{K}$ can be determined through each Homography $\mathbf{H}_j,\ j = 1, 2, 3$.

Radial Distortion

Real camera images contain radial distortion, which differs from the ideal pinhole camera model, making it impossible to obtain accurate $\mathbf{R}, \mathbf{t}, \mathbf{K}$ without proper calibration. By using actual calibration tools, you can obtain not only $\mathbf{R}, \mathbf{t}, \mathbf{K}$ but also Distortion parameters that correct the radial distortion in the image.

With a small focal length f, the Field of View (FOV) widens, and significant radial distortion occurs near the edges of the image. Conversely, a small f results in a narrow FOV and relatively less distortion.

Let a point on the distorted image plane be $(\breve{u}, \breve{v})$ and a point on the undistorted image plane be (u, v), both in $[pixel]$. Also, let a point on the distorted normalized image plane be $(\breve{x}, \breve{y})$ and a point on the undistorted normalized image plane be (x, y), both in $[mm]$. Being normalized implies that the origin u_0, v_0 of the image plane is zero. The relationship between these normalized points is

$$\begin{aligned} \breve{x} &= x + x(k_1 r^2 + k_2 r^4) \\ \breve{y} &= y + y(k_1 r^2 + k_2 r^4) \quad \text{where, } r^2 = x^2 + y^2 \end{aligned} \tag{168}$$

This indicates that distortion increases with distance from the origin. These normalized points can be represented back on the pixel unit image plane as

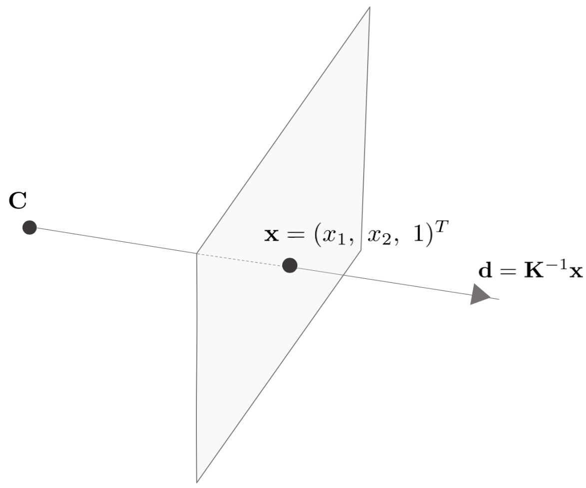
$$\begin{aligned} \breve{u} &= u_0 + \alpha \breve{x} \\ \breve{v} &= v_0 + \beta \breve{v} \end{aligned} \tag{169}$$

Here, u_0, v_0 denote the origin of the image plane and α, β are coefficients that convert points from mm to pixel units. Thus, by determining the radial distortion parameters k_1, k_2, the relationship between the distorted and actual points can be understood.

5 More Single View Geometry

Camera calibration and the image of the absolute conic

Result 8.15

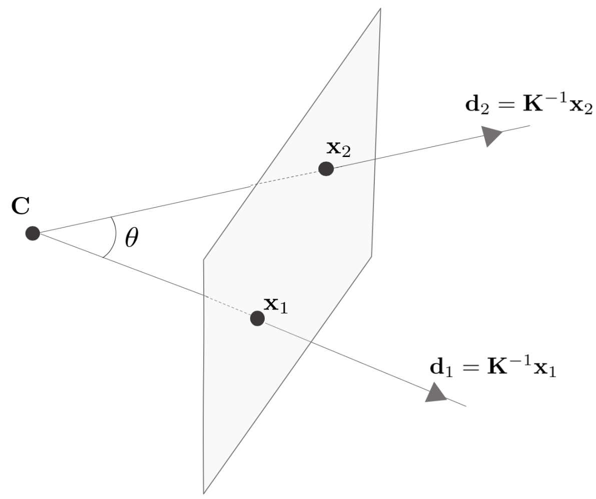


When a camera $\mathbf{C}$ located at the origin back-projects a point $\mathbf{x}$, it creates a line $\mathbf{d}$ passing through the camera center, where $\mathbf{d} = \mathbf{K}^{-1}\mathbf{x}$.

$$\begin{aligned} \mathbf{x} &= \mathbf{P} \begin{bmatrix} \lambda \mathbf{d} \\ 1 \end{bmatrix} \\ &= \mathbf{K}[\mathbf{I}|0] \begin{bmatrix} \lambda \mathbf{d} \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{d} \end{aligned} \tag{170}$$

Therefore, the following formula holds:

$$
\begin{aligned}
\mathbf{x} &= \mathbf{K}\mathbf{d} \\
\mathbf{d} &= \mathbf{K}^{-1}\mathbf{x}
\end{aligned} \tag{171}
$$



The angle between two lines $\mathbf{d}_1, \mathbf{d}_2$ created by back-projecting two points $\mathbf{x}_1, \mathbf{x}_2$ on the image plane can be calculated as follows:

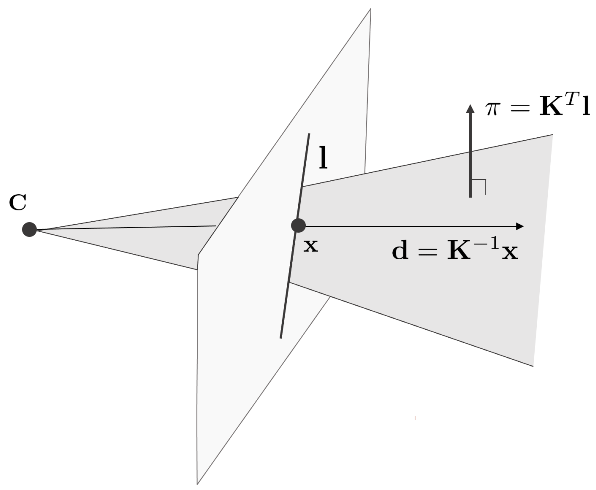
$$
\begin{aligned}
\cos\theta &= \frac{\mathbf{d}_1^\intercal \mathbf{d}_2}{\sqrt{\mathbf{d}_1^\intercal \mathbf{d}_1}\sqrt{\mathbf{d}_2^\intercal \mathbf{d}_2}} = \frac{(\mathbf{K}^{-1}\mathbf{x}_1)^\intercal(\mathbf{K}^{-1}\mathbf{x}_2)}{\sqrt{(\mathbf{K}^{-1}\mathbf{x}_1)^\intercal(\mathbf{K}^{-1}\mathbf{x}_1)}\sqrt{(\mathbf{K}^{-1}\mathbf{x}_2)^\intercal(\mathbf{K}^{-1}\mathbf{x}_2)}} \\
&= \frac{\mathbf{x}_1^\intercal(\mathbf{K}^{-\intercal}\mathbf{K}^{-1})\mathbf{x}_2}{\sqrt{\mathbf{x}_1^\intercal \mathbf{K}^{-\intercal}\mathbf{K}^{-1}\mathbf{x}_1}\sqrt{\mathbf{x}_2^\intercal \mathbf{K}^{-\intercal}\mathbf{K}^{-1}\mathbf{x}_2}}
\end{aligned} \tag{172}
$$

Here, **the image of the Absolute Conic is $\mathbf{K}^{-\intercal}\mathbf{K}^{-1}$.**

Result 8.16

When a point $\mathbf{x}$ lies on a line $\mathbf{l}$ on the image plane, the following formula holds:

$$
\mathbf{x}^\intercal \mathbf{l} = 0 \tag{173}
$$



When $\mathbf{l}$ is back-projected, it creates a plane π and when $\mathbf{x}$ is back-projected, it creates a line $\mathbf{d} = \mathbf{K}^{-1}\mathbf{x}$, and the relationship $(\mathbf{K}^{-1}\mathbf{x})^\intercal\pi = 0$ holds. Rearranging this gives $\mathbf{x}^\intercal(\mathbf{K}^{-\intercal}\pi) = 0$. According to the formula $\mathbf{x}^\intercal\mathbf{l} = 0$, the following formula holds conclusively:

$$
\begin{aligned}
\mathbf{K}^{-\intercal}\pi &= \mathbf{l} \\
\pi &= \mathbf{K}^{\intercal}\mathbf{l}
\end{aligned} \tag{174}
$$

The image of the absolute conic

When there is an infinite plane π_∞, and an infinite point $\mathbf{X}_\infty = (\mathbf{d}^\intercal, 0)^\intercal$ exists on π_∞, projecting it with camera $\mathbf{P} = \mathbf{KR}[\mathbf{I}| - \tilde{\mathbf{C}}]$ results in the following:

$$
\mathbf{x} = \mathbf{PX}_\infty = \mathbf{KR}[\mathbf{I}| - \tilde{\mathbf{C}}] \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix} = \mathbf{KRd} \tag{175}
$$

Thus, a Homography $\mathbf{H}$ exists such that $\mathbf{x} = \mathbf{Hd}$, where Homography $\mathbf{H}$ is $\mathbf{H} = \mathbf{KR}$. The Absolute Conic Ω_∞ at infinity is $\mathbf{I}_3 \in \pi_\infty$. Transforming Ω_∞ by Homography results in

$$
\begin{aligned}
\mathbf{H}(\Omega_\infty) &= \mathbf{H}^{-\intercal}\mathbf{I}_3\mathbf{H}^{-1} \\
&= (\mathbf{KR})^{-\intercal}\mathbf{I}_3(\mathbf{KR})^{-1} \\
&= \mathbf{K}^{-\intercal}\mathbf{R}^{-\intercal}\mathbf{R}^{-1}\mathbf{K}^{-1} \\
&= \mathbf{K}^{-\intercal}\mathbf{K}^{-1}
\end{aligned} \tag{176}
$$

Result 8.17

Accordingly, **the Image of Absolute Conic $\mathbf{w} = \mathbf{K}^{-\intercal}\mathbf{K}^{-1}$** is established.

When two points $\mathbf{x}_1, \mathbf{x}_2$ on the image plane are back-projected, the angle between the two lines is

$$
\cos\theta = \frac{\mathbf{x}_1^\intercal \mathbf{w} \mathbf{x}_2}{\sqrt{\mathbf{x}_1^\intercal \mathbf{w} \mathbf{x}_1}\sqrt{\mathbf{x}_2^\intercal \mathbf{w} \mathbf{x}_2}} \tag{177}
$$

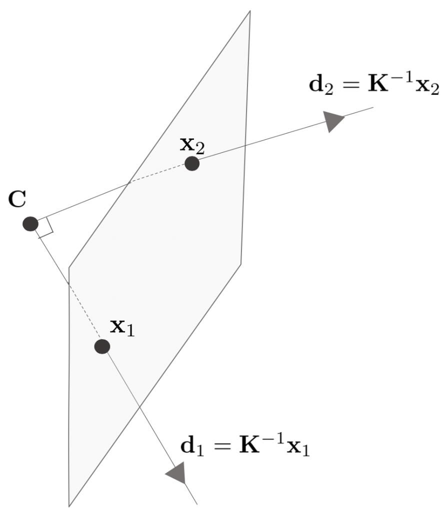
and transforming it by Homography results in

$$
\cos\theta = \frac{(\mathbf{Hx}_1)^\intercal \mathbf{H}^{-\intercal}\mathbf{w}\mathbf{H}^{-1}(\mathbf{Hx}_2)}{\sqrt{*}\sqrt{*}} \tag{178}
$$

Thus, the angles are preserved even after the Homography transformation. If the two lines $\mathbf{K}^{-1}\mathbf{x}_1$ and $\mathbf{K}^{-1}\mathbf{x}_2$ are orthogonal, the following formula holds:

$$
\mathbf{x}_1^\intercal \mathbf{w} \mathbf{x}_2 = 0 \tag{179}
$$

Orthogonality and w



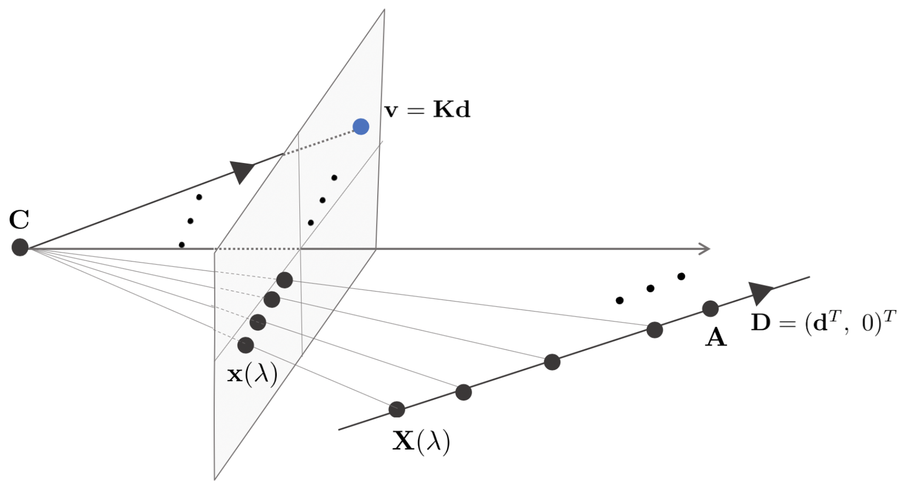
When two points $\mathbf{x}_1, \mathbf{x}_2$ on the image plane are back-projected, the two lines $\mathbf{K}^{-1}\mathbf{x}_1, \mathbf{K}^{-1}\mathbf{x}_2$ are created. If these two lines are orthogonal, the formula $\mathbf{x}_1^\intercal \mathbf{w}\mathbf{x}_2 = 0$ holds. Also, if $\mathbf{x}_1$ is included in line $\mathbf{l}$, then $\mathbf{x}_1^\intercal \mathbf{l} = 0$ holds.

Result 8.19

Combining the two formulas results in

$$\begin{aligned} &\mathbf{x}_1^\intercal \mathbf{w}\mathbf{x}_2 = 0 \\ &\mathbf{x}_1^\intercal \mathbf{l} = 0 \\ &\therefore \mathbf{l} = \mathbf{w}\mathbf{x}_2 \end{aligned} \tag{180}$$

Vanishing points and vanishing lines



When a point $\mathbf{A}$ and the direction of a line $\mathbf{D} = (\mathbf{d}^\intercal, 0)^\intercal$ exist in the world, a point $\mathbf{X}(\lambda)$ on the line is defined as follows:

$$\mathbf{X}(\lambda) = \mathbf{A} + \lambda\mathbf{D} = \begin{bmatrix} \tilde{\mathbf{A}} + \lambda\mathbf{d} \\ 1 \end{bmatrix} \tag{181}$$

The point $\mathbf{x}(\lambda) = \mathbf{P}\mathbf{X}(\lambda)$, where $\mathbf{P} = \mathbf{K}[\mathbf{I}|0]$ projected on the image plane is defined as follows:

$$\mathbf{x}(\lambda) = \mathbf{P}\mathbf{X}(\lambda) = \mathbf{P}\mathbf{A} + \lambda\mathbf{P}\mathbf{D} = \mathbf{a} + \lambda\mathbf{K}\mathbf{d} \tag{182}$$

Here, $\mathbf{a}$ represents the Image of $\mathbf{A}$.

Result 8.20

Conclusively, the vanishing point $\mathbf{v}$ is defined as follows:

$$\begin{aligned} &\mathbf{v} = \lim_{\lambda\to\infty} \mathbf{x}(\lambda) = \lim_{\lambda\to\infty} (\mathbf{a} + \lambda\mathbf{K}\mathbf{d}) = \mathbf{K}\mathbf{d} \\ &\mathbf{v} = \mathbf{K}\mathbf{d} \end{aligned} \tag{183}$$

Camera rotation from vanishing points

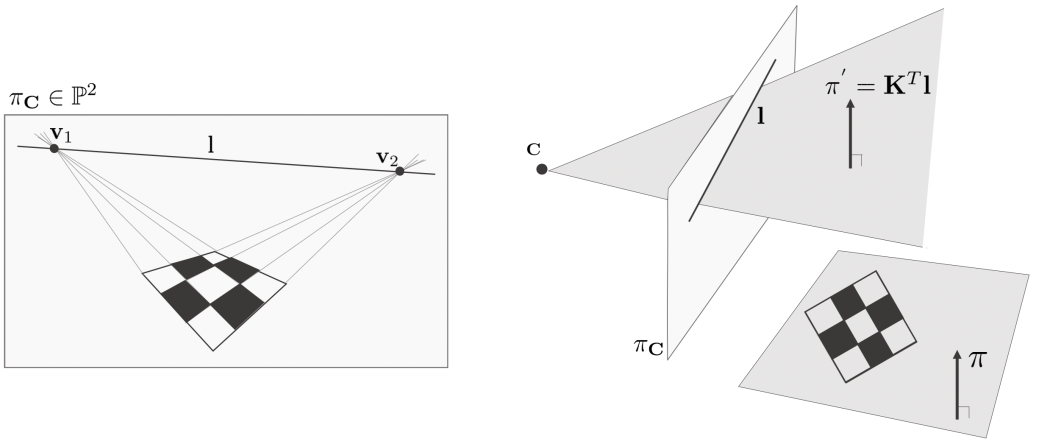
Using vanishing points, the rotation of the camera can be calculated. If there are vanishing points $\mathbf{v}_1$ from Image 1 and $\mathbf{v}_2$ from Image 2, the direction of $\mathbf{v}_1$ is $\mathbf{d}_1 = \mathbf{K}^{-1}\mathbf{v}_1$ and the direction of $\mathbf{v}_2$ is $\mathbf{d}_2 = \mathbf{K}^{-1}\mathbf{v}_2$.

Assuming the values of $\mathbf{K}, \mathbf{v}_1, \mathbf{v}_2$ are all known, the values of $\mathbf{d}_1, \mathbf{d}_2$ can be calculated and the two direction vectors have the following relationship:

$$\mathbf{d}_2 = \mathbf{R}\mathbf{d}_1 \tag{184}$$

In this case, **since the degree of freedom of the rotation matrix R is 3, the rotation matrix can be recovered by using more than two pairs of vanishing points**.

Vanishing Lines



Lines connecting two or more vanishing points $\mathbf{v}_i, i = 1, 2, \cdots$ are called vanishing lines $\mathbf{l}$. For instance, consider an image with a checkerboard leading to two vanishing points $\mathbf{v}_1, \mathbf{v}_2$. In this case, **the line connecting the two vanishing points $\mathbf{v}_1, \mathbf{v}_2$ is referred to as the vanishing line $\mathbf{l}$ of the checkerboard plane π.**

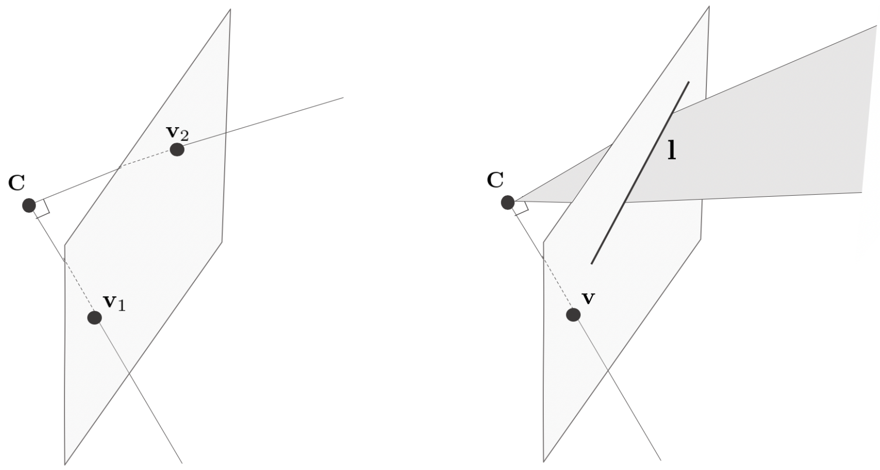
$$\mathbf{l} = \text{image of } \pi \cap \pi_\infty \tag{185}$$

Ultimately, **the vanishing line $\mathbf{l}$ represents the intersection of a plane π', parallel to the checkerboard plane π and containing the camera center, with the image plane.** π' can be computed by back-projecting the vanishing line $\mathbf{l}$. Depending on the relationship between the plane and line,

$$\pi' = \mathbf{K}^\intercal \mathbf{l} \tag{186}$$

can be calculated as such.

Orthogonality relationships amongst vanishing points and lines



The condition for lines back-projected from two vanishing points $\mathbf{v}_1, \mathbf{v}_2$ on the image plane to be orthogonal is as follows.

$$\mathbf{v}_1^\intercal \mathbf{w} \mathbf{v}_2 = 0 \tag{187}$$

In this case, $\mathbf{w} = \mathbf{K}^{-\intercal}\mathbf{K}^{-1}$ is the Image of Absolute Conic.

The condition for the line back-projected from the vanishing point $\mathbf{v}$ and the plane back-projected from the vanishing line $\mathbf{l}$ to be orthogonal is as follows.

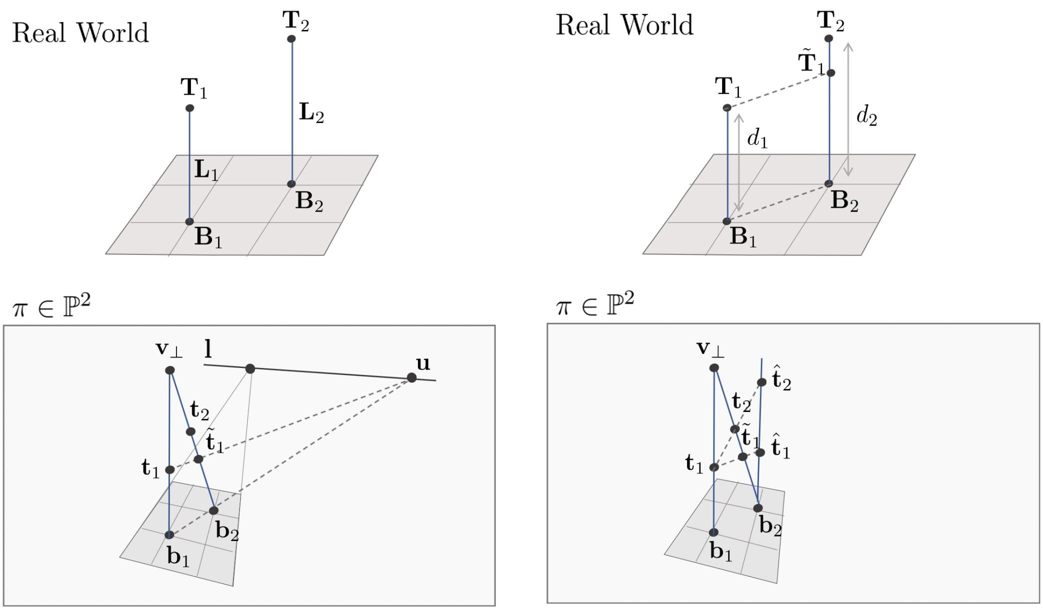
$$\mathbf{l} = \mathbf{w}\mathbf{v} \tag{188}$$

The condition for the planes back-projected from two lines $\mathbf{l}_1, \mathbf{l}_2$ on the image plane to be orthogonal to each other is as follows.

$$\mathbf{l}_1^\intercal \mathbf{w}^* \mathbf{l}_2 = 0 \tag{189}$$

In this case, $\mathbf{w}^*$ is the Image of Dual Absolute Conic.

Affine 3D measurements and reconstruction



The vanishing point perpendicular to the 3D space plane π is called the vertical vanishing point $\mathbf{v}_\perp$.

Result 8.24

Using the vanishing line $\mathbf{l}$ and the vertical vanishing point, the size of objects can be calculated up to a scale parameter. To be precise, **if you know the vanishing line $\mathbf{l}$ and the vertical vanishing point $\mathbf{v}_\perp$, you can determine the relative lengths of line segments perpendicular to the plane π.**

For example, suppose there are two points $\mathbf{B}_1, \mathbf{B}_2$ located on the 3D space plane π and lines $\mathbf{L}_1, \mathbf{L}_2$ passing through them and perpendicular to π. If there are endpoints $\mathbf{T}_1, \mathbf{T}_2$ of lines $\mathbf{L}_1, \mathbf{L}_2$, then **the relative lengths of $\mathbf{T}_1, \mathbf{T}_2$ can be measured using the vanishing line $\mathbf{l}$ and the vertical vanishing point $\mathbf{v}_\perp$.**

First, all elements are projected onto the image plane. At this time, define the point where the line connecting $\mathbf{b}_1, \mathbf{b}_2$ meets the vanishing line $\mathbf{l}$ as $\mathbf{u}$. Also, if you draw a line parallel to $\overline{\mathbf{b}_1\mathbf{b}_2}$ from $\mathbf{t}_1$, this line will touch $\mathbf{u}$. At this time, the intersection of $\overline{\mathbf{t}_1\mathbf{u}}$ and $\overline{\mathbf{v}_\perp\mathbf{b}_2}$ is defined as $\tilde{\mathbf{t}}_1$.

Next, project the line $\overline{\mathbf{v}_\perp\mathbf{b}_2}$ onto the $\mathbb{P}^1$ space where the vertical vanishing point $\mathbf{v}_\perp$ becomes the infinite point $(1, 0)$ and $\mathbf{b}_1$ becomes the origin $(0, 1)$. The Homography $\mathbf{H}_{2\times 2}$ used at this time is as follows.

$$\mathbf{H}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 1 & -\mathbf{v}_\perp \end{bmatrix} \tag{190}$$

$\mathbf{H}_{2\times 2}$ preserves the Cross-Ratio. Next, using the ratio of lengths, calculate the ratio $d_1 : d_2 = \mathbf{b}_1\tilde{\mathbf{t}}_1 : \mathbf{b}_1\mathbf{t}_2$.

$$\frac{d_1}{d_2} = \frac{\tilde{\mathbf{t}}_1(\mathbf{v}_\perp - \mathbf{t}_2)}{\mathbf{t}_2(\mathbf{v}_\perp - \tilde{\mathbf{t}}_1)} \tag{191}$$

If the vertical vanishing point $\mathbf{v}_\perp$ and the camera's principal axis are perpendicular in the image plane, the vertical vanishing point does not intersect, and the ratio can be calculated simply as follows.

$$\frac{d_1}{d_2} = \frac{\tilde{\mathbf{t}}_1 - \mathbf{b}_2}{\mathbf{t}_2 - \mathbf{b}_2} \tag{192}$$

Determining camera calibration K from a single view

In a single-view image, two constraints are needed to determine the internal parameters $\mathbf{K}$. **These are the image constraint and the internal parameter constraint**.

The image constraint includes two orthogonal vanishing points $\mathbf{v}_1, \mathbf{v}_2$ on the image plane, such that $\mathbf{v}_1^\intercal \mathbf{w} \mathbf{v}_2 = 0$, and the case where the vanishing line $\mathbf{l}$ and the vanishing point $\mathbf{v}$ are orthogonal, for which their back-projections are $\mathbf{l} = \mathbf{wx}$.

$$\begin{aligned} &\mathbf{v}_1^\intercal \mathbf{w} \mathbf{v}_2 = 0 \\ &\mathbf{l} \times (\mathbf{wv}) = 0 \end{aligned} \tag{193}$$

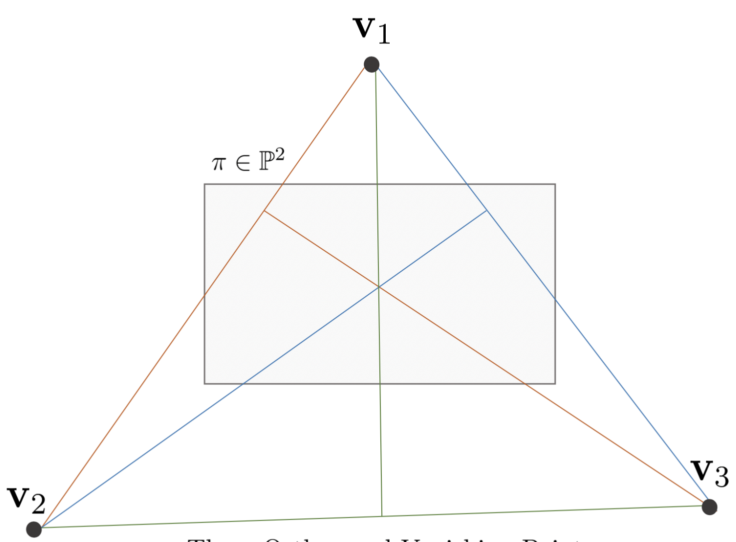
Result 8.26

The internal parameter constraint includes the case where $\mathbf{K}$ has Zero-Skew, where $w_{12} = w_{21} = 0$, and also, where it has Square Pixels, where $w_{12} = w_{21} = 0, w_{11} = w_{22}$.

$$\begin{aligned} &w_{12} = w_{21} \quad \text{for zero skew} \\ &w_{12} = w_{21} = 0, w_{11} = w_{22} \quad \text{for square pixel} \end{aligned} \tag{194}$$

As such, $\mathbf{w} = \begin{bmatrix} w_1 & w_2 & w_4 \\ w_2 & w_3 & w_5 \\ w_4 & w_5 & w_6 \end{bmatrix}$ parameters can be found after securing sufficient constraints and $\mathbf{w}$ is aligned as a 6-dimensional vector to form a $\mathbf{Ax} = 0$ linear system. Next, **use Singular Value Decomposition (SVD) to calculate the value of $\mathbf{w} = \mathbf{K}^{-\intercal}\mathbf{K}^{-1}$, then use Cholesky Decomposition to decompose $\mathbf{K}^{-\intercal}\mathbf{K}^{-1}$**. This allows the calculation of the internal parameters $\mathbf{K}$.

Calibration from three orthogonal vanishing points



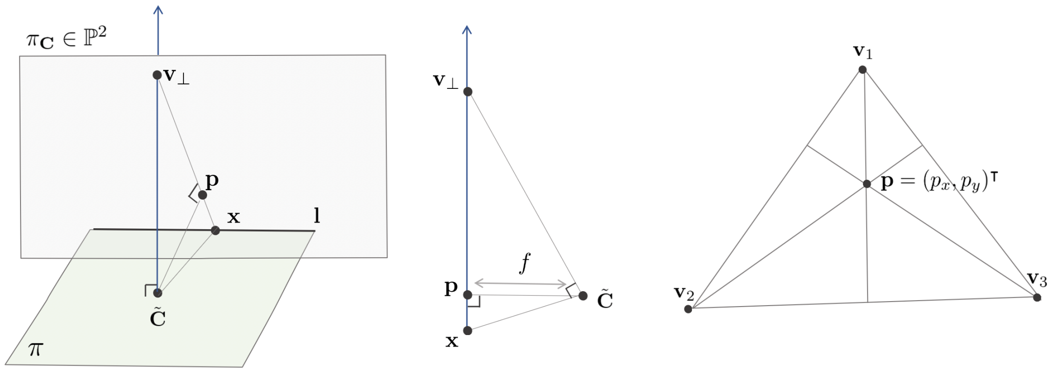
Three Orthogonal Vanishing Points

Using three mutually orthogonal vanishing points $\mathbf{v}_i$, $i = 1, 2, 3$, the internal parameters $\mathbf{K}$ can be calculated when $\mathbf{K}$ has Zero-Skew and Square Pixel. First, since $\mathbf{K}$ has Zero-Skew and Square Pixel, the following holds.

$$\mathbf{w} = \begin{bmatrix} w_1 & 0 & w_2 \\ 0 & w_1 & w_3 \\ w_2 & w_3 & w_4 \end{bmatrix} \tag{195}$$

Having a total of 4 degrees of freedom, $\mathbf{K}$ can be determined using three orthogonal vanishing points. First, use the orthogonal characteristics of each vanishing point to calculate $\mathbf{v}_i^\intercal \mathbf{w} \mathbf{v}_j$, $^\forall i \neq j$. Then, convert $\mathbf{w}$ into a vectorized form to form a $\mathbf{Ax} = 0$ linear system, and by calculating Nul $(\mathbf{A})$, find $\mathbf{w}$. Next, use Cholesky Decomposition to decompose $\mathbf{w} = \mathbf{K}^{-\intercal}\mathbf{K}^{-1}$. This way, the internal parameters $\mathbf{K}$ can be calculated.

Computation of focal length and principal point using vanishing point and vanishing line

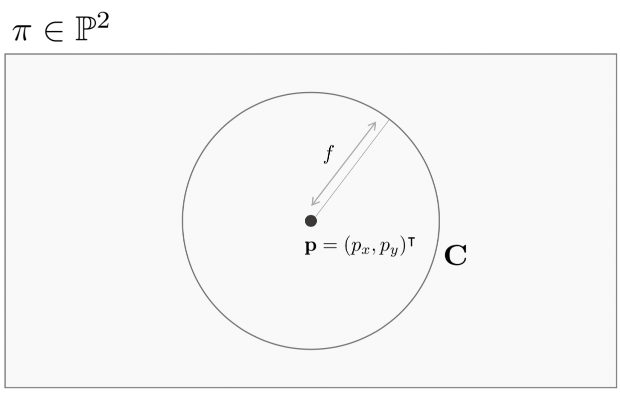


If there is a vanishing line $\mathbf{l}$ that can be obtained from the 3D space plane π and a vertical vanishing point $\mathbf{v}_\perp$ perpendicular to π, these can be used to determine the focal length f and the principal point.

The method for determining the focal length f is as follows. If the vertical vanishing point is $\mathbf{v}_\perp$ and the intersection line of the image plane $\pi_\mathbf{C}$ and the plane π is $\mathbf{l}$, then $\mathbf{v}_\perp\tilde{\mathbf{C}}$ and $\mathbf{x}\tilde{\mathbf{C}}$ are orthogonal to each other. If there is a point P where a perpendicular is dropped from the camera center to the image plane, then the length of $\tilde{\mathbf{C}}\mathbf{p}$ is the focal length f. **Draw a circle with diameter $\mathbf{v}_\perp x$, and if the line drawn horizontally from point p intersects the circle at points $\mathbf{a}, \mathbf{b}$, one of these points is the camera center $\tilde{\mathbf{C}}$, and this equals ap $=$ bp, which is the focal length f.**

The method for determining the principal point P is as follows. If the vertical vanishing point is $\mathbf{v}_1$ and the intersection line of the image plane and the plane π is $\mathbf{l}_1$, then dropping a perpendicular from $\mathbf{v}_1$ to $\mathbf{l}_1$ positions the principal point $\mathbf{p}$ on this perpendicular. Because of this feature, **if there are three different vertical vanishing points, their orthocenter becomes the principal point p**.

The calibrating conic



$$\mathbf{C} = \mathbf{K}^{-\intercal}\mathbf{D}\mathbf{K}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -f^2 \end{bmatrix}$$

where, $\mathbf{K} = \begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix}$

The IAC (image of absolute conic) is a useful tool that can measure the angles between lines back-projected from points on the image plane and perform Metric Rectification, but it has the disadvantage of being unvisualizable due to the nature of Circular Points not having real roots. To compensate for this, the Calibration Conic was devised. The Calibration Conic means the Image Conic projected from the Cone $X^2 + Y^2 = Z^2$ and has the advantage of being visualizable.

When there is a camera projection $\mathbf{P} = \mathbf{K}[\mathbf{I}|0]$, points on the Calibration Conic are projected as follows.

$$\mathbf{C} = \mathbf{K}^{-\intercal} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \mathbf{K}^{-1} \tag{196}$$

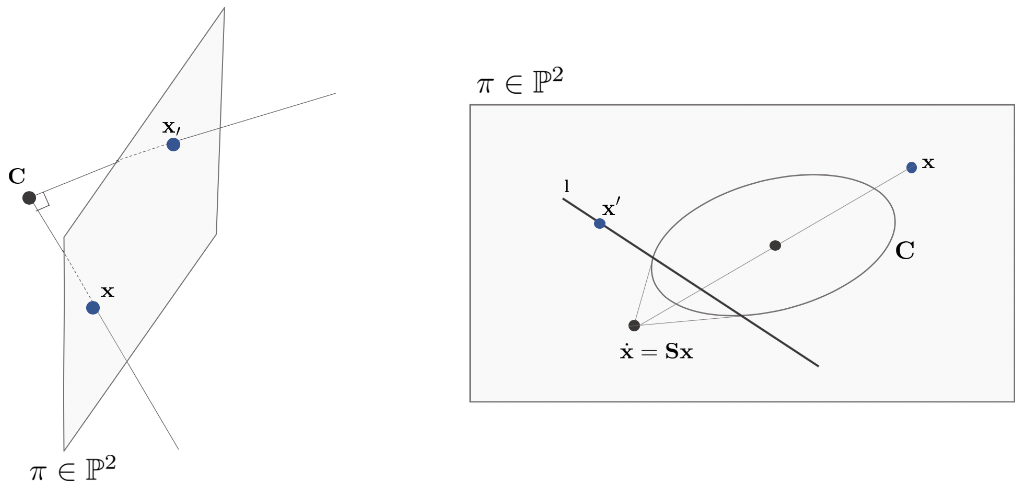
If $\mathbf{D} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$, then $\mathbf{C} = \mathbf{K}^{-\intercal}\mathbf{D}\mathbf{K}^{-1}$ can be represented as such. If $\mathbf{K} = \begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix}$, then the Calibration Conic can be represented as

$$\mathbf{C} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -f^2 \end{bmatrix} \tag{197}$$

and in this case, **the Calibration Conic represents a circle on the image with the principal point as the origin and the radius as the focal length f**.

$\mathbf{C}$ can be redefined as follows.

$$\begin{aligned} \mathbf{C} &= \mathbf{K}^{-\intercal}\mathbf{D}\mathbf{K}^{-1} = \mathbf{K}^{-\intercal}\mathbf{K}^{-1}\mathbf{K}\mathbf{D}\mathbf{K}^{-1} \\ \mathbf{C} &= \mathbf{w}\mathbf{S} \quad \text{where, } \mathbf{S} = \mathbf{K}\mathbf{D}\mathbf{K}^{-1} \end{aligned} \tag{198}$$



For any point $\mathbf{x} = \mathbf{K}\tilde{\mathbf{X}}$ on the image, $\mathbf{S}\mathbf{x}$ represents the reflected point by the Calibration Conic.

$$\begin{aligned} \mathbf{S}\mathbf{x} &= \mathbf{K}\mathbf{D}\mathbf{K}^{-1}\mathbf{K}\tilde{\mathbf{X}} \\ &= \mathbf{K}\mathbf{D} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{K} \begin{bmatrix} X \\ Y \\ -Z \end{bmatrix} \end{aligned} \tag{199}$$

For two points $\mathbf{x}, \mathbf{x}'$ on the image plane, if the lines back-projected from these two points are orthogonal to each other, $\mathbf{x}'^\intercal \mathbf{w}\mathbf{x} = 0$ holds. Rewriting this equation,

$$\begin{aligned} &\mathbf{x}'^\intercal \mathbf{w}\mathbf{x} = \mathbf{x}'^\intercal \mathbf{C}\mathbf{S}^{-1}\mathbf{x} = \mathbf{x}'^\intercal \mathbf{C}\mathbf{S}\mathbf{x} = \mathbf{x}'^\intercal \mathbf{C}\dot{\mathbf{x}} \\ &\because \mathbf{S}^{-1} = \mathbf{S} \end{aligned} \tag{200}$$

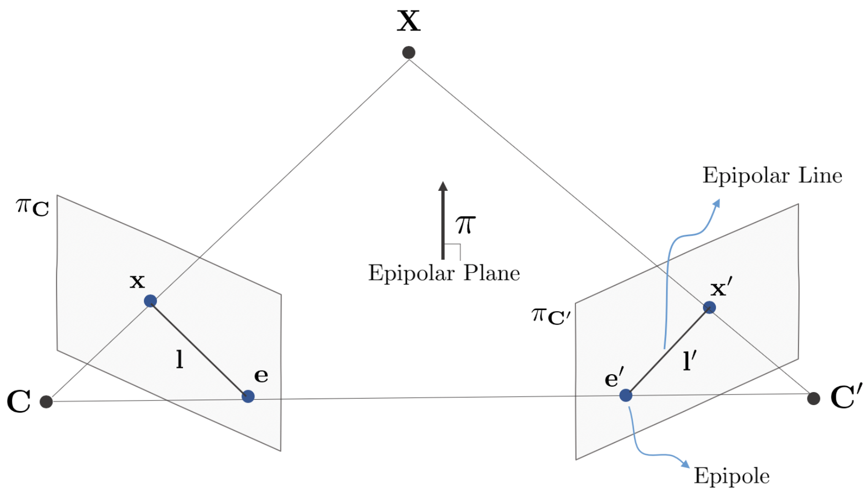
Here, $\dot{\mathbf{x}} = \mathbf{S}\mathbf{x}$ represents the point reflected by the Calibration Conic.

Result 8.30

Conclusively, **the line $\mathbf{C}\dot{\mathbf{x}}$ becomes the line connecting the reflected point $\dot{\mathbf{x}}$ and the tangents of the Calibration Conic, and the point $\mathbf{x}'$ exists on the line $\mathbf{C}\dot{\mathbf{x}}$**.

6 Epipolar Geometry and the Fundamental Matrix

Epipolar geometry



Epipolar geometry refers to the **geometric relationship defined between two camera images**. It is independent of the structure of the 3D object and **depends solely on the internal parameters of the cameras and the relative pose between the two cameras**. As shown in the figure above, given the centers of two cameras $\mathbf{C}, \mathbf{C}'$ and a point $\mathbf{X}$ in three-dimensional space, the plane uniquely determined through these three points is called the **Epipolar Plane** π. Additionally, the point $\mathbf{P}'\mathbf{C} = \mathbf{e}'$, which is the projection of the camera center $\mathbf{C}$ through $\mathbf{P}'$, is referred to as the **Epipole** $\mathbf{e}'$ on the image plane $\pi_{\mathbf{C}'}$. The line connecting $\mathbf{e}'$ and $\mathbf{x}'$ is called the **Epipolar Line** $\mathbf{l}'$. Similarly, $\mathbf{e}, \mathbf{l}$ refer to the Epipole and Epipolar Line on the image plane $\pi_{\mathbf{C}}$ respectively.

The fundamental matrix F

The Fundamental Matrix is a **matrix $\mathbf{F} \in \mathbb{R}^{3\times 3}$ with rank 2 given the centers of two cameras $\mathbf{C}, \mathbf{C}'$**. The matrix $\mathbf{F}$ satisfies the following for corresponding point pairs $\mathbf{x}, \mathbf{x}'$ on the image planes of the two cameras:

$$\mathbf{x}'^{\intercal}\mathbf{F}\mathbf{x} = 0 \tag{201}$$

The meaning varies depending on the order of multiplication. If $\mathbf{x}'^{\intercal}\mathbf{F}\mathbf{x} = 0$, then it is referred to as the Fundamental Matrix between the two cameras $\mathbf{C}, \mathbf{C}'$, and if $\mathbf{x}^{\intercal}\mathbf{F}'\mathbf{x}' = 0$, it is the Fundamental Matrix between $\mathbf{C}', \mathbf{C}$. Here, $\mathbf{F}^{\intercal} = \mathbf{F}'$ holds true.

From a geometric perspective, in $\mathbf{x}'^{\intercal}\mathbf{F}\mathbf{x} = 0$, **$\mathbf{F}\mathbf{x}$ signifies the Epipolar Line $\mathbf{l}' \in \pi_{\mathbf{C}'}$ corresponding to the point $\mathbf{x}$ on the image plane $\pi_{\mathbf{C}}$**. Therefore, **$\mathbf{F}$ can be considered as a function mapping the point $\mathbf{x}$ on the image plane $\pi_{\mathbf{C}}$ to the Epipolar Line $\mathbf{l}'$**.

$$\begin{aligned} &\mathbf{F}: \ \mathbf{x} \mapsto \mathbf{l}' \\ &\text{where,} \quad \mathbf{x} \in \pi_{\mathbf{C}} \in \mathbb{P}^2 \\ &\mathbf{l}' \in (\mathbb{P}^2)^{\vee} \end{aligned} \tag{202}$$

Proof

($\Rightarrow$) If $\mathbf{l}' = \mathbf{F}\mathbf{x}$, then for a point $\mathbf{x}'$ on the Epipolar Line $\mathbf{l}'$, the following holds:

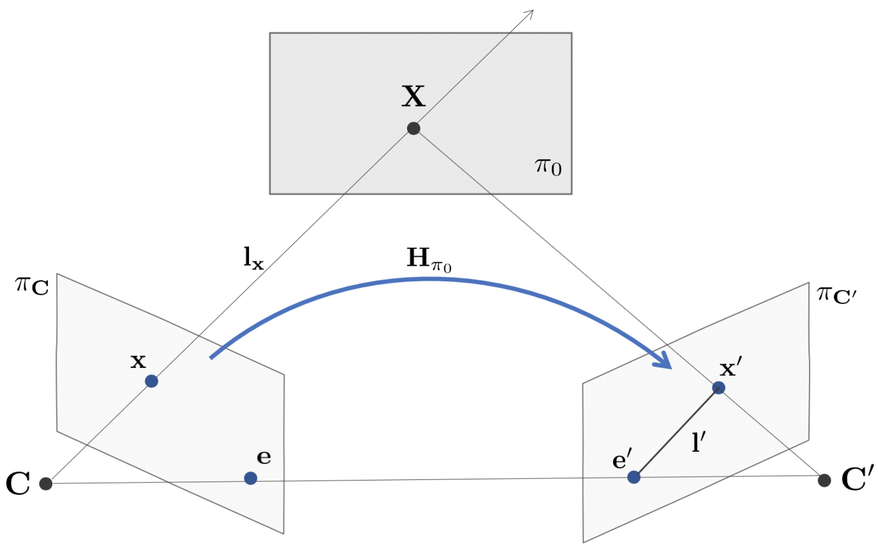
$$
\mathbf{x}'^\intercal \mathbf{l}' = \mathbf{x}'^\intercal \mathbf{F}\mathbf{x} = 0 \quad {}^\forall \mathbf{x} \leftrightarrow \mathbf{x}' \tag{203}
$$

($\Leftarrow$) Assuming that points $\mathbf{X}_1, \mathbf{X}_2$ are created through Back-projection of the point $\mathbf{x}$ on the image plane $\pi_{\mathbf{C}}$, and projecting these onto $\pi_{\mathbf{C}'}$ results in the points $\mathbf{x}'_1 = \mathbf{P}'\mathbf{X}_1$, $\mathbf{x}'_2 = \mathbf{P}'\mathbf{X}_2$. Then, due to the Fundamental Matrix $\mathbf{F}$, the following relations hold:

$$
\begin{aligned}
\mathbf{x}_1'^\intercal \mathbf{F}\mathbf{x} = 0 \\
\mathbf{x}_2'^\intercal \mathbf{F}\mathbf{x} = 0
\end{aligned} \tag{204}
$$

Thus, $\mathbf{F}\mathbf{x}$ signifies the Epipolar Line $\mathbf{l}'$ orthogonal to $\mathbf{x}_i'^\intercal$, $i = 1, 2$.

Geometric derivation



When considering two cameras $\mathbf{C}, \mathbf{C}'$, let us assume that the line $\mathbf{l}_\mathbf{x}$, Back-projected from the point $\mathbf{x}$ on the image plane $\pi_{\mathbf{C}}$, intersects a random plane π_0 at a point in three-dimensional space. If we consider the Homography $\mathbf{H}_{\pi_0}$ that connects the image points of $\mathbf{C}, \mathbf{C}'$,

$$
\begin{aligned}
\mathbf{H}_{\pi_0} : & \pi_{\mathbf{C}} \mapsto \pi_{\mathbf{C}'} \\
& \mathbf{x} \mapsto \mathbf{x}' \\
& \mathbf{x} \mapsto \mathbf{P}'(\mathbf{l}_\mathbf{x} \cap \pi_0)
\end{aligned} \tag{205}
$$

If the point projected onto $\pi_{\mathbf{C}'}$ is $\mathbf{x}' = \mathbf{P}'(\mathbf{l}_\mathbf{x} \cap \pi_0)$, and we define the Epipolar Line $\mathbf{l}'$ as the line connecting $\mathbf{x}'$ and the Epipole $\mathbf{e}'$,

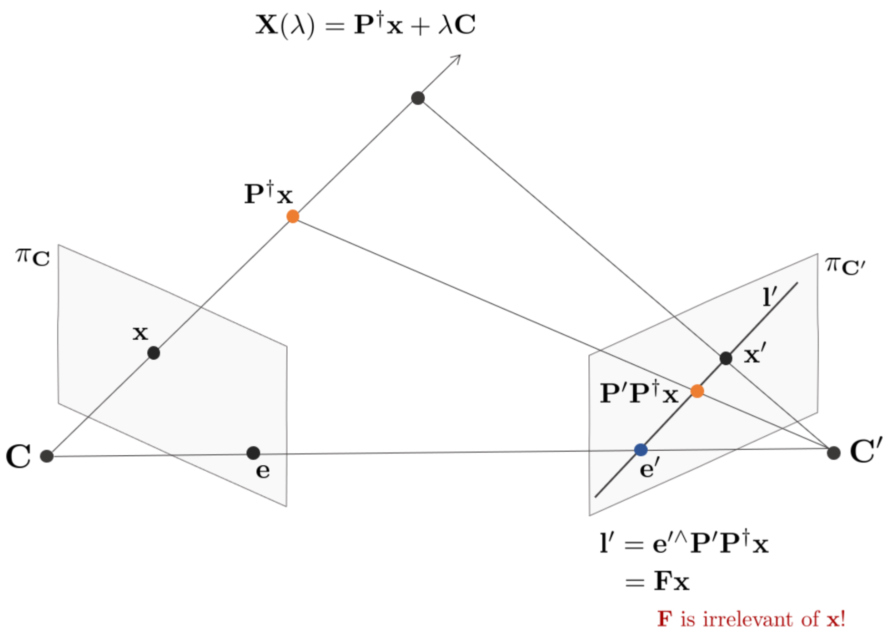
$$
\begin{aligned}
\mathbf{e}' \times \mathbf{x}' &= \mathbf{e}'^\wedge \mathbf{x}' \\
&= \mathbf{e}'^\wedge \mathbf{H}_{\pi_0}(\mathbf{x}) \quad \text{is Epipolar Line.}
\end{aligned} \tag{206}
$$

In this case, using the formula $\mathbf{l}' = \mathbf{F}\mathbf{x}$,

$$
\begin{aligned}
\mathbf{l}' &= \mathbf{F}\mathbf{x} \\
&= \mathbf{e}'^\wedge \mathbf{H}_{\pi_0} \mathbf{x}
\end{aligned} \tag{207}
$$

Thus, $\mathbf{F} = \mathbf{e}'^\wedge \mathbf{H}_{\pi_0}$ is established. Here, $\mathbf{H}_{\pi_0}$ is a rank 3 matrix and $\mathbf{e}'^\wedge$ is a rank 2 matrix, so $\mathbf{F}$ is a rank 2 matrix.

Algebraic derivation



The Epipolar Line $\mathbf{l}' = \mathbf{P}'(\mathbf{X}(\lambda))$. Here, $\mathbf{X}(\lambda)$ represents the Back-projection line of $\mathbf{x}$ passing through the center of camera $\mathbf{C}$. $\mathbf{X}(\lambda)$ can be rewritten as follows.

$$\mathbf{X}(\lambda) = \mathbf{P}^{\dagger}\mathbf{x} + \lambda\mathbf{C} \quad \lambda \in \mathbb{R} \tag{208}$$

Here, $\mathbf{P}^{\dagger}$ represents the Pseudo Inverse of $\mathbf{P}$ and

$$\mathbf{P}^{\dagger} = \mathbf{P}^{\intercal}(\mathbf{P}\mathbf{P}^{\intercal})^{-1} \tag{209}$$

Then, $\mathbf{l}' = \mathbf{P}'(\mathbf{X}(\lambda))$ is as follows.

$$\begin{aligned} \mathbf{P}'(\mathbf{X}(\lambda)) &= \mathbf{P}'\mathbf{P}^{\dagger}\mathbf{x} + \lambda\mathbf{P}'\mathbf{C} \\ &\mathbf{P}'\mathbf{P}^{\dagger}\mathbf{x} + \lambda\mathbf{e}' \end{aligned} \tag{210}$$

$\mathbf{P}'\mathbf{C}$ signifies the point projected onto $\pi_{\mathbf{C}'}$ from the center of camera $\mathbf{C}$, thus becoming the Epipole $\mathbf{e}'$. When $\lambda = 0$, $\mathbf{P}'(\mathbf{X}(0)) = \mathbf{P}'\mathbf{P}^{\dagger}\mathbf{x}$ and when $\lambda = \infty$, $\mathbf{P}'(\mathbf{X}(\infty)) = \mathbf{P}'\mathbf{C} = \mathbf{e}'$. Therefore, the Epipolar Line $\mathbf{l}'$ connects these two points, so

$$\begin{aligned} \mathbf{l}' &= \mathbf{e}'^{\wedge}\mathbf{P}'\mathbf{P}^{\dagger}\mathbf{x} \\ &= \mathbf{F}\mathbf{x} \\ \therefore \mathbf{F} &= \mathbf{e}'^{\wedge}\mathbf{P}'\mathbf{P}^{\dagger} \end{aligned} \tag{211}$$

Additionally, since Epipole $\mathbf{e}'$ is included in the Epipolar Line $\mathbf{l}'$ and holds for all $\mathbf{x}_i$,

$$\mathbf{e}'^{\intercal}\mathbf{F}\mathbf{x}_i = (\mathbf{e}'^{\intercal}\mathbf{F})\mathbf{x}_i = 0 \quad {}^{\forall}\mathbf{x}_i \tag{212}$$

$\mathbf{e}'^{\intercal}\mathbf{F} = 0$ holds true. **In conclusion, $\mathbf{e}'$ is the Left null vector of $\mathbf{F}$.** Similarly, $\mathbf{e}$ becomes the (right) null vector of $\mathbf{F}$.

Properties of the fundamental matrix

The properties of the Fundamental Matrix $\mathbf{F}$ are as follows:

- **Transpose:** If $\mathbf{F}$ is the Fundamental Matrix for two cameras $(\mathbf{P}, \mathbf{P}')$, then $\mathbf{F}^{\intercal}$ becomes the Fundamental Matrix for $(\mathbf{P}', \mathbf{P})$.

- **Epipolar Lines:** For a corresponding point $\mathbf{x}$ in the first image, the corresponding Epipolar Line in the second image can be expressed as $\mathbf{l}' = \mathbf{F}\mathbf{x}$. Similarly, for a corresponding point $\mathbf{x}'$ in the second image, the corresponding Epipolar Line in the first image can be expressed as $\mathbf{l} = \mathbf{F}^\intercal\mathbf{x}'$.
- **The Epipole:** Epipolar Line $\mathbf{l}' = \mathbf{F}\mathbf{x}$ always passes through $\mathbf{e}'$ for any point $\mathbf{x}$ other than $\mathbf{e}$. Therefore, $\mathbf{e}'$ satisfies $\mathbf{e}'^\intercal(\mathbf{F}\mathbf{x}) = (\mathbf{e}'^\intercal\mathbf{F})\mathbf{x} = 0$ for all $\mathbf{x}$, indicating that $\mathbf{e}'^\intercal\mathbf{F} = 0$. Hence, $\mathbf{e}'$ is the Left null-vector of $\mathbf{F}$. Similarly, $\mathbf{F}\mathbf{e} = 0$ means that $\mathbf{e}$ is the Right null-vector of $\mathbf{F}$.
- **Fundamental Matrix is a rank 2 Homogeneous matrix with 7 degrees of freedom (DOF) and lacks an inverse matrix.** As a 3x3 matrix, it loses one degree of freedom due to the Scale Ambiguity in Homogeneous coordinates, and an additional degree of freedom is lost due to the constraint $\det \mathbf{F} = 0$, resulting in 7 DOF.

The Epipolar Line Homography

Let there be two cameras $\mathbf{C}, \mathbf{C}'$ and let's denote a point on their image planes as $\mathbf{x}, \mathbf{x}'$ respectively. Corresponding to these, there exist Epipolar Lines $\mathbf{l}, \mathbf{l}'$. There is a specific relationship between $\mathbf{l}$ and $\mathbf{l}'$ which is represented by

$$\mathbb{P}(\pi)^\vee \mapsto \mathbb{P}(\pi')^\vee \tag{213}$$

This relationship is provided by the **Homography matrix**.

Result 9.5

When there exists a line $\mathbf{k}$ that passes through the camera center $\mathbf{C}$ but does not pass through the Epipole $\mathbf{e}$, $\mathbf{k}$ will necessarily meet the Epipolar Line $\mathbf{l}$ at a point $\mathbf{p}$. $\mathbf{p}$ can be calculated as

$$\mathbf{p} = \mathbf{k}^\wedge \mathbf{l} \tag{214}$$

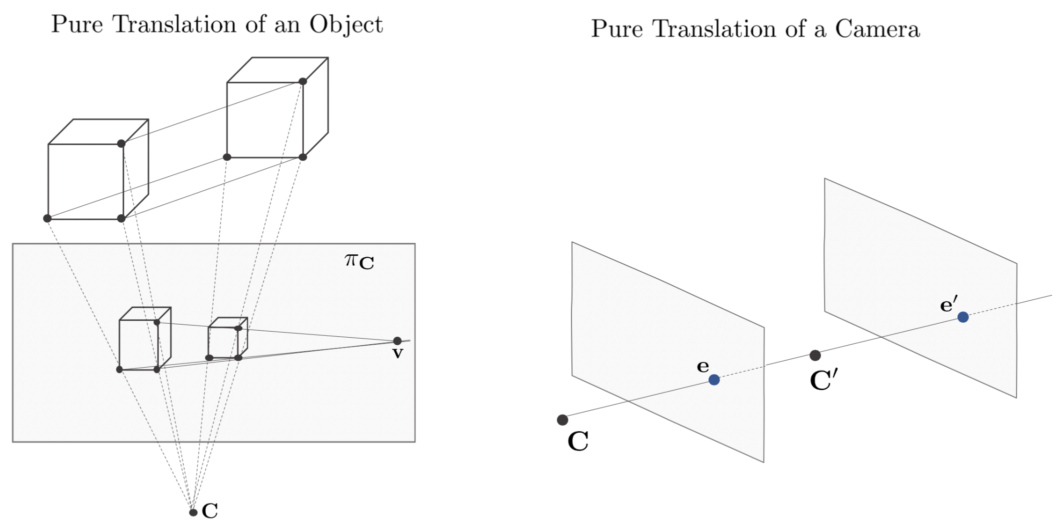
and since $\mathbf{p}$ is projected onto Epipolar Line $\mathbf{l}'$ by the Fundamental Matrix, the following relation holds:

$$\begin{aligned} \mathbf{l}' &= \mathbf{F}\mathbf{k}^\wedge \mathbf{l} \\ &= \mathbf{H}\mathbf{l} \end{aligned} \tag{215}$$

Thus, **the relationship between $\mathbf{l}, \mathbf{l}'$ is established by the Homography matrix $\mathbf{H} = \mathbf{F}\mathbf{k}^\wedge$**.

Fundamental matrices arising from special motions

Pure translation



Pure translation implies moving the camera center without any rotation. In this scenario, moving the camera while fixing all objects in the world is equivalent to fixing the camera and moving all objects in the world. If we denote the initial camera matrix and camera center as $\mathbf{P}, \mathbf{C}$ respectively, and the camera matrix and camera center after pure translation as $\mathbf{P}', \mathbf{C}'$ respectively, **the length of the baseline between the two cameras is equal to the amount of pure translation of the camera center**.

Also, **if the Epipoles of both cameras are denoted as e, e' respectively, the position of both Epipoles is the same, and this is referred to as the vanishing point**.

$$\mathbf{e} = \mathbf{e}' = \mathbf{v} \tag{216}$$

In such a pure translation situation, $\mathbf{e} = \mathbf{e}'$ is referred to as Auto Epipolar.

Algebraically deriving the Fundamental Matrix, if the initial camera matrix is $\mathbf{P} = \mathbf{K}[\mathbf{I}|0]$ and after pure translation the camera matrix is $\mathbf{P}' = \mathbf{K}[\mathbf{I}|\mathbf{t}]$, then the Fundamental Matrix $\mathbf{F}$ from the previous section is $\mathbf{F} = \mathbf{e}'^{\wedge}\mathbf{P}'\mathbf{P}^{\dagger}$ therefore,

$$\begin{aligned}
\mathbf{F} &= \mathbf{e}'^{\wedge}\mathbf{P}'\mathbf{P}^{\dagger} \\
&= \mathbf{e}'^{\wedge}\mathbf{K}\mathbf{K}^{-1} \\
&= \mathbf{e}'^{\wedge}
\end{aligned} \tag{217}$$

Therefore, **in a situation of pure translation, the Fundamental Matrix is $\mathbf{F} = \mathbf{e}'^{\wedge}$**.

Retrieving the camera matrices

Projective invariance and canonical cameras

Result 9.8

When there exist points $\mathbf{x}, \mathbf{x}'$ on the image planes corresponding to two cameras $\mathbf{C}, \mathbf{C}'$ and a Fundamental Matrix $\mathbf{F}$, $\mathbf{F}$ remains the same regardless of the Homography transformation of the $\mathbf{x} \leftrightarrow \mathbf{x}'$ pair. In other words, when there exists a Homography transformation $\mathbf{H} \in \mathbb{R}^{4\times 4}$ that satisfies $\mathbb{P}^3 \mapsto \mathbb{P}^3$, $\mathbf{H}$ satisfies

$$(\mathbf{P}, \mathbf{P}') \mapsto (\mathbf{P}\mathbf{H}, \mathbf{P}'\mathbf{H}) \tag{218}$$

and **irrespective of such $\mathbf{H}$, the Fundamental Matrix $\mathbf{F}$ remains the same**.

Proof

For the corresponding point pairs $(\mathbf{P}\mathbf{H}, \mathbf{P}'\mathbf{H})$,

$$\mathbf{x}'^{\intercal}\tilde{\mathbf{F}}\mathbf{x} = 0 \tag{219}$$

holds. Since $\mathbf{x} = \mathbf{P}\mathbf{H}\mathbf{X}, \mathbf{x}' = \mathbf{P}'\mathbf{H}\mathbf{X}$, substituting this into the above equation,

$$\mathbf{X}^{\intercal}\mathbf{H}^{\intercal}\mathbf{P}'^{\intercal}\tilde{\mathbf{F}}\mathbf{P}\mathbf{H}\mathbf{X} = 0 \tag{220}$$

holds for all ${}^{\forall}\mathbf{X} \in \mathbb{P}^3$. Replacing $\mathbf{X} = \mathbf{H}^{-1}(\mathbf{H}\mathbf{X}), \tilde{\mathbf{X}} = \mathbf{H}\mathbf{X}$ in the above equation, we get

$$\begin{aligned}
&\tilde{\mathbf{X}}^{\intercal}\mathbf{H}^{-\intercal}\mathbf{H}^{\intercal}\mathbf{P}'^{\intercal}\tilde{\mathbf{F}}\mathbf{P}\mathbf{H}\mathbf{H}^{-1}\tilde{\mathbf{X}} = 0 \\
&\tilde{\mathbf{X}}^{\intercal}\mathbf{P}'^{\intercal}\tilde{\mathbf{F}}\mathbf{P}\tilde{\mathbf{X}} = 0 \\
&\tilde{\mathbf{x}}'^{\intercal}\tilde{\mathbf{F}}\tilde{\mathbf{x}} = 0
\end{aligned} \tag{221}$$

Thus, **in conclusion, $\mathbf{F} = \tilde{\mathbf{F}}$ is valid**.

Canonical form of camera matrices

Following this property, multiple $(\mathbf{P}, \mathbf{P}')$ pairs correspond to the same Fundamental Matrix $\mathbf{F}$, hence $\mathbf{F}$ and $(\mathbf{P}, \mathbf{P}')$ have a One-to-Many Correspondence relationship. Therefore, despite this ambiguity, to accurately represent the transformation of $\mathbf{F}$, the initial camera matrix $\mathbf{P}$ is simply represented as Canonical Form $\mathbf{P} = [\mathbf{I}|0]$, $\mathbf{P}' = [\mathbf{M}|\mathbf{m}]$. However, stating $\mathbf{P} = [\mathbf{I}|0]$ does not uniquely determine $\mathbf{P}'$.

When any camera matrix $\mathbf{P} \in \mathbb{R}^{3\times 4}$ is given, it can be changed to a matrix $\mathbf{P}^{*} = \begin{bmatrix} \mathbf{P} \\ \mathbf{r}_4 \end{bmatrix} \in \mathbb{R}^{4\times 4}$ that has an inverse, then

$$\mathbf{P}^*\mathbf{H} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \tag{222}$$

satisfies $\mathbf{PH} = [\mathbf{I}|\mathbf{0}]$. Therefore, **assuming the existence of H, any arbitrary camera matrix P can be written in Canonical Form.**

What then can the Fundamental Matrix $\mathbf{F}$ be written in terms of $\mathbf{M}, \mathbf{m}$?

Firstly, when there exist arbitrary camera matrices $\mathbf{P} = \mathbf{K}[\mathbf{I}|0], \mathbf{P}' = \mathbf{K}'[\mathbf{R}|\mathbf{t}]$, the following formulas hold.

$$\begin{aligned}
&\mathbf{P}\mathbf{P}^\intercal = \mathbf{K}^2 \\
&\mathbf{P}^\dagger = \mathbf{P}^\intercal(\mathbf{P}\mathbf{P}^\intercal)^{-1} = \begin{bmatrix} \mathbf{K}^{-1} \\ 0 \end{bmatrix} \\
&\mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}^\intercal \\
&\mathbf{e}' = \mathbf{P}'\mathbf{C} = \mathbf{K}'\mathbf{t}
\end{aligned} \tag{223}$$

Therefore, the corresponding Fundamental Matrix $\mathbf{F}$ is as follows.

$$\begin{aligned}
\mathbf{F} &= \mathbf{e}'^\wedge \mathbf{P}'\mathbf{P}^\dagger \\
&= (\mathbf{K}'\mathbf{t})^\wedge \mathbf{K}'[\mathbf{R}|\mathbf{t}]\mathbf{K}^{-1} \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix} \\
&= (\mathbf{K}'\mathbf{t})^\wedge \mathbf{K}'\mathbf{R}\mathbf{K}^{-1}
\end{aligned} \tag{224}$$

Result 9.9

Considering the Canonical Form, the Fundamental Matrix $\mathbf{F}$ for the two camera matrices $\mathbf{P} = [\mathbf{I}|0], \mathbf{P}' = [\mathbf{M}|\mathbf{m}]$ satisfies the following formula.

$$\mathbf{F} = \mathbf{m}^\wedge \mathbf{M} \tag{225}$$

The F of the Canonical Form is $\mathbf{F} = \mathbf{m}^\wedge \mathbf{M}$.

Projective Ambiguity of Cameras given F

Theorem 9.10

The Fundamental Matrix $\mathbf{F}$ is invariant under Homography transformation, leading to inherent ambiguity. If the $\mathbf{F}$ for camera matrix pairs $(\mathbf{P}, \mathbf{P}')$ and $(\tilde{\mathbf{P}}, \tilde{\mathbf{P}}')$ are the same, there exists a Homography $\mathbf{H}$ that connects $(\mathbf{P}, \mathbf{P}')$ and $(\tilde{\mathbf{P}}, \tilde{\mathbf{P}}')$.

$$\begin{aligned}
&{}^\exists\mathbf{H} \in \mathrm{PGL}_4 \\
&(\tilde{\mathbf{P}}, \tilde{\mathbf{P}}') = (\mathbf{P}, \mathbf{P}')\mathbf{H}
\end{aligned} \tag{226}$$

Proof

Given two pairs of camera matrices $(\mathbf{P}, \mathbf{P}')$ and $(\tilde{\mathbf{P}}, \tilde{\mathbf{P}}')$, when written in Canonical Form, they are as follows:

$$\begin{aligned}
&\mathbf{P} = \tilde{\mathbf{P}} = [\mathbf{I}|0] \\
&\mathbf{P}' = [\mathbf{A}|\mathbf{a}] \\
&\tilde{\mathbf{P}}' = [\tilde{\mathbf{A}}|\tilde{\mathbf{a}}]
\end{aligned} \tag{227}$$

Here, the Fundamental Matrix $\mathbf{F}$ is defined as:

$$\mathbf{F} = \mathbf{a}^\wedge \mathbf{A} = \tilde{\mathbf{a}}^\wedge \tilde{\mathbf{A}} \tag{228}$$

Due to the properties of $\mathbf{F}$, $\mathbf{a}^\intercal\mathbf{F} = 0$ and $\tilde{\mathbf{a}}^\intercal\mathbf{F} = 0$ hold true:

$$\begin{aligned}
&\mathbf{a}^\intercal\mathbf{F} = \mathbf{a}^\intercal\mathbf{a}^\wedge\mathbf{A} = 0 \\
&\tilde{\mathbf{a}}^\intercal\mathbf{F} = \tilde{\mathbf{a}}^\intercal\tilde{\mathbf{a}}^\wedge\tilde{\mathbf{A}} = 0
\end{aligned} \tag{229}$$

Here, $\mathbf{a}$ and $\tilde{\mathbf{a}}$ form the rank 1 Left Null Space of $\mathbf{F}. Thus, a, arerelatedby:$

$$\tilde{\mathbf{a}} = k\mathbf{a}, \quad k \neq 0 \in \mathbb{R} \tag{230}$$

Substituting $\tilde{\mathbf{a}}$ and re-expressing it leads to:

$$\begin{aligned} \mathbf{F} &= \mathbf{a}^{\wedge}\mathbf{A} = \tilde{\mathbf{a}}^{\wedge}\tilde{\mathbf{A}} = k\mathbf{a}^{\wedge}\tilde{\mathbf{A}} = 0 \\ &= \mathbf{a}^{\wedge}(k\tilde{\mathbf{A}} - \mathbf{A}) = 0 \end{aligned} \tag{231}$$

Therefore, $(k\tilde{\mathbf{A}} - \mathbf{A})$ is parallel to $\mathbf{a}$. Hence, each column of the matrices are in a scalar relationship:

$$k\tilde{\mathbf{A}} - \mathbf{A} = \mathbf{a}\mathbf{v}^{\intercal} \quad \text{for some } \mathbf{v} \tag{232}$$

Thus, $\tilde{\mathbf{A}} = k^{-1}(\mathbf{A} + \mathbf{a}\mathbf{v}^{\intercal})$. The camera matrix pairs can be represented as:

$$\begin{aligned} \mathbf{P} &= \tilde{\mathbf{P}} = [\mathbf{I}|0] \\ \mathbf{P}' &= [\mathbf{A}|\mathbf{a}] \\ \tilde{\mathbf{P}}' &= [k^{-1}(\mathbf{A} + \mathbf{a}\mathbf{v}^{\intercal}) \mid k\mathbf{a}] \end{aligned} \tag{233}$$

In conclusion, if the matrix transforming camera matrix pair $(\mathbf{P}, \mathbf{P}')$ to Canonical Form is $\mathbf{H}_1$, and the matrix transforming $(\tilde{\mathbf{P}}, \tilde{\mathbf{P}}')$ to Canonical Form is $\mathbf{H}_2$, then

$$(\mathbf{P}, \mathbf{P}')\mathbf{H}_1\mathbf{H} = (\tilde{\mathbf{P}}, \tilde{\mathbf{P}}')\mathbf{H}_2 \tag{234}$$

a Homography H exists satisfying this, thereby making $(\mathbf{P}, \mathbf{P}')$ and $(\tilde{\mathbf{P}}, \tilde{\mathbf{P}}')$ projectively equivalent.

$$(\mathbf{P}, \mathbf{P}') \sim (\tilde{\mathbf{P}}, \tilde{\mathbf{P}}') \tag{235}$$

Canonical Cameras given F

Result 9.12

The necessary and sufficient condition for an arbitrary square matrix $\mathbf{F} \in \mathbb{R}^{3\times3}$ to be the Fundamental Matrix corresponding to camera matrix pair $(\mathbf{P}, \mathbf{P}')$ is that $\mathbf{P}'^{\intercal}\mathbf{F}\mathbf{P}$ is a skew symmetric matrix.

$$\begin{aligned} &\mathbf{x}'\mathbf{F}\mathbf{x} = 0 \\ &\Leftrightarrow \mathbf{X}^{\intercal}\mathbf{P}'^{\intercal}\mathbf{F}\mathbf{P}\mathbf{X} = 0 \ \ {}^{\forall}\mathbf{X} \\ &\Leftrightarrow \mathbf{P}'^{\intercal}\mathbf{F}\mathbf{P} \text{ is a skew symmetric matrix.} \end{aligned} \tag{236}$$

Result 9.13

Given an arbitrary square matrix $\mathbf{F} \in \mathbb{R}^{3\times3}$ and an arbitrary skew symmetric matrix $\mathbf{S} \in \mathbb{R}^{3\times3}$, if the camera matrix pair $(\mathbf{P}, \mathbf{P}')$ is as follows:

$$\begin{aligned} \mathbf{P} &= [\mathbf{I}|0] \\ \mathbf{P}' &= [\mathbf{S}\mathbf{F}|\mathbf{e}'] \ \text{ where, } \mathbf{e}' \ \text{ is epipole.} \end{aligned} \tag{237}$$

Then **$(\mathbf{P}, \mathbf{P}')$ possesses F as its Fundamental Matrix.**
To prove this, expand $\mathbf{P}'\mathbf{F}\mathbf{P}$:

$$\begin{aligned} \mathbf{P}'\mathbf{F}\mathbf{P} &= [\mathbf{S}\mathbf{F}|\mathbf{e}']^{\intercal}\mathbf{F}[\mathbf{I}|0] \\ &= \begin{bmatrix} \mathbf{F}^{\intercal}\mathbf{S}^{\intercal}\mathbf{F} & \mathbf{0} \\ \mathbf{e}'\mathbf{F} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{F}^{\intercal}\mathbf{S}^{\intercal}\mathbf{F} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \end{aligned} \tag{238}$$

Since it forms a skew symmetric matrix, this confirms $\mathbf{F}$ as the Fundamental Matrix.
The skew symmetric matrix $\mathbf{S}$ can be expressed in the form of a 3-dimensional vector $\mathbf{s}^{\wedge}$, and if $\mathbf{s}^{\intercal}\mathbf{e}' \neq 0$, then the rank of $\mathbf{P}' = [\mathbf{s}^{\wedge}\mathbf{F}|\mathbf{e}']$ is 3. For $\mathbf{P}'$ to have a rank of 3, the ranks of $\mathbf{s}^{\wedge}\mathbf{F}$ and $\mathbf{e}'$ must

be 2 and 1, respectively. First, to prove $\mathbf{s}^\wedge\mathbf{F}$ has rank 2, since $\mathbf{e}'\mathbf{F} = 0$ satisfies the Fundamental Matrix property, the column space Col $\mathbf{F}$ and $\mathbf{e}'$ are orthogonal:

$$\text{Col } \mathbf{F} \perp \mathbf{e}' \tag{239}$$

Furthermore, since $\mathbf{s}^\wedge\mathbf{e}' \neq 0$, $\mathbf{s}$ does not reside within Col $\mathbf{F}$. The column space of $\mathbf{s}^\wedge\mathbf{F}$ is:

$$\begin{aligned} \text{Col } \mathbf{s}^\wedge\mathbf{F} &= \mathbf{s}^\wedge \text{Col } \mathbf{F} \\ &= \mathbf{s} \times \text{Col } \mathbf{F} \end{aligned} \tag{240}$$

And since Col F has rank 2, $\mathbf{s} \times$ Col F also has rank 2. Next, to prove $\mathbf{e}'$ is linearly independent from other column vectors, $\mathbf{s}^\intercal \text{Col } \mathbf{s}^\wedge\mathbf{F} = 0$ and $\mathbf{s}^\wedge\mathbf{e}' \neq 0$ therefore,

$$\mathbf{e}' \notin \text{Col } \mathbf{s}^\wedge\mathbf{F} \tag{241}$$

Conclusively, the rank of $\mathbf{P}' = [\mathbf{s}^\wedge\mathbf{F}|\mathbf{e}']$ is 3.

Result 9.14

When selecting $\mathbf{s}$ such that the rank of $\mathbf{P}'$ is 3, $\mathbf{s}$ that is not orthogonal to $\mathbf{e}'$ should be used, hence setting $\mathbf{s} = \mathbf{e}'$ leads to:

$$\begin{aligned} \mathbf{P} &= [\mathbf{I}|0] \\ \mathbf{P}' &= [\mathbf{e}'^\wedge\mathbf{F} \mid \mathbf{e}'] \end{aligned} \tag{242}$$

Thus, given the Fundamental Matrix F, computing the Left Null Space $\mathbf{e}'$ allows for the calculation of the camera pair $(\mathbf{P}, \mathbf{P}')$.

Using the proportionality defined earlier for the camera matrix pairs,

$$\begin{aligned} (\mathbf{P}, \mathbf{P}') &= ([\mathbf{I}|0], [\mathbf{A}|\mathbf{a}]) \\ (\tilde{\mathbf{P}}, \tilde{\mathbf{P}}') &= ([\mathbf{I}|0], [k^{-1}(\mathbf{A} + \mathbf{a}\mathbf{v}^\intercal) \mid k\mathbf{a}]) \end{aligned} \tag{243}$$

can be generalized as follows.

Result 9.15

$$\begin{aligned} (\mathbf{P}, \mathbf{P}') &= ([\mathbf{I}|0], [\mathbf{e}'\mathbf{F} \mid \mathbf{e}']) \\ (\tilde{\mathbf{P}}, \tilde{\mathbf{P}}') &= ([\mathbf{I}|0], [\mathbf{e}'\mathbf{F} + \mathbf{e}'\mathbf{v}^\intercal \mid \lambda\mathbf{e}']) \end{aligned} \tag{244}$$

The Null Space value $\mathbf{e}'$ being scale-invariant, using $\frac{1}{k}\mathbf{e}'$ results in the same outcome. Thus, it can be represented in the above form. The implication of this formula is that regardless of the arbitrary vector $\mathbf{e}'\mathbf{v}^\intercal$ and the arbitrary scale value λ multiplied with the existing $\mathbf{F}$, there is no change in the Fundamental Matrix. This is because substituting into $\mathbf{F} = \mathbf{e}'^\wedge\mathbf{P}'\mathbf{P}^\dagger = \mathbf{e}'^\wedge\tilde{\mathbf{P}}'\tilde{\mathbf{P}}^\dagger$ results in a consistent $\mathbf{F}$. **Thus, the above formula embodies the Projective Ambiguity of the Fundamental Matrix.**

The essential matrix

The Essential Matrix E refers to the Fundamental Matrix when the corresponding point pairs $\mathbf{x} \leftrightarrow \mathbf{x}'$ are in the normalized image coordinate system. Historically, the Essential Matrix was introduced by Longuet-Higgins before the Fundamental Matrix, and later, the Fundamental Matrix for uncalibrated corresponding point pairs was introduced as a generalized version of the Essential Matrix.

Normalized coordinates

Given any camera matrix $\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$, the point $\mathbf{x}$ on the image plane satisfies $\mathbf{x} = \mathbf{P}\mathbf{X} = \mathbf{K}[\mathbf{R}|\mathbf{t}]\mathbf{X}$. In this case, **the point on the normalized image plane is $\bar{\mathbf{x}}$**,

$$\bar{\mathbf{x}} = \mathbf{K}^{-1}\mathbf{x} \tag{245}$$

It holds that,

$$\mathbf{K}^{-1}\mathbf{P} = [\mathbf{R}|\mathbf{t}] \tag{246}$$

is referred to as the **normalized camera**. In a normalized camera, the camera matrix can be considered where $\mathbf{K} = \mathbf{I}$. The corresponding pair of normalized camera matrices $(\mathbf{P}, \mathbf{P}')$ is

$$(\mathbf{P}, \mathbf{P}') = ([\mathbf{I}|0],\ [\mathbf{R}|\mathbf{t}]) \tag{247}$$

thus in this case **the Fundamental Matrix $\mathbf{F} = \mathbf{t}^{\wedge}\mathbf{R}$ is specifically called the Essential Matrix $\mathbf{E}$**,

$$\mathbf{E} = \mathbf{t}^{\wedge}\mathbf{R} \tag{248}$$

Definition 9.16.

The Essential Matrix is a 3×3 square matrix, and $\mathbf{E} \in \mathbb{R}^{3\times3}$ is used to express the correlation between points in the normalized image coordinate system $\bar{\mathbf{x}} \leftrightarrow \bar{\mathbf{x}}'$.

$$\bar{\mathbf{x}}'^{\intercal}\mathbf{E}\bar{\mathbf{x}} = 0 \tag{249}$$

At this time, since $\bar{\mathbf{x}} = \mathbf{K}^{-1}\mathbf{x}, \bar{\mathbf{x}}' = \mathbf{K}^{-1}\mathbf{x}'$,

$$\mathbf{x}'^{\intercal}\mathbf{K}'^{-\intercal}\mathbf{E}\mathbf{K}^{-1}\mathbf{x} = 0 \tag{250}$$

this relationship holds between the Fundamental Matrix $\mathbf{F}$ and the Essential Matrix $\mathbf{E}$.

$$\begin{aligned} \mathbf{F} &= \mathbf{K}'^{-\intercal}\mathbf{E}\mathbf{K}^{-1} \\ \mathbf{E} &= \mathbf{K}'^{\intercal}\mathbf{F}\mathbf{K} \end{aligned} \tag{251}$$

Properties of the essential matrix

- **The Essential Matrix $\mathbf{E} = \mathbf{t}^{\wedge}\mathbf{R}$ has 5 degrees of freedom (DOF).** Both rotation $\mathbf{R}$ and translation $\mathbf{t}$ have 3 degrees of freedom each, but like the Fundamental Matrix, they lose 1 degree of freedom due to the homogeneous property, which introduces a scale ambiguity. This reduced degree of freedom forms additional constraints in the Essential Matrix compared to the Fundamental Matrix.
- **When the Essential Matrix is decomposed using Singular Value Decomposition (SVD), the diagonal matrix $\mathbf{D} = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$ should have the two largest singular values σ_1, σ_2 equal, and the third singular value $\sigma_3 = 0$ should be satisfied.** More details are explained in the following section.

Result 9.17

The necessary and sufficient condition for any square matrix $\mathbf{E} \in \mathbb{R}^{3\times3}$ to be an Essential Matrix is that the two largest singular values σ_1, σ_2 are equal and the third singular value $\sigma_3 = 0$.

Proof

Forward Proof: Suppose there is a Fundamental Matrix $\mathbf{F}$ corresponding to the camera matrix pair $(\mathbf{P}, \mathbf{P}') = ([\mathbf{I}|0],\ [\mathbf{R}|\mathbf{t}])$

$$\mathbf{F} = \mathbf{t}^{\wedge}\mathbf{R} = \mathbf{S}\mathbf{R} \tag{252}$$

이 된다. 반대칭행렬 $\mathbf{t}^{\wedge} = \mathbf{S}$가 rank 2를 가지는 경우 $\mathbf{S}$는 항상 기저(basis)를 변경하여 다음과 같은 형태로 변경할 수 있다.

$$\mathbf{S} = k\mathbf{U}\mathbf{Z}\mathbf{U}^{\intercal} \tag{253}$$

이 때, k는 임의의 스케일 값을 의미하고 일반적으로 고려하지 않는다. $\mathbf{U}$는 임의의 직교 행렬이며 $\mathbf{Z}$는 반대칭 행렬이다. $\mathbf{U}\mathbf{Z}\mathbf{U}^{\intercal}$를 SVD 형태에 맞게 변경하기 위해 약간의 대수적 트릭을 이용한다. 반대칭행렬 $\mathbf{Z}$와 직교행렬 $\mathbf{W}$를 아래와 같이 정의한다.

$$
\begin{aligned}
\mathbf{Z} &= \begin{bmatrix} & 1 & \\ -1 & & \\ & & \end{bmatrix} \\
\mathbf{W} &= \begin{bmatrix} & -1 & \\ 1 & & \\ & & 1 \end{bmatrix}
\end{aligned} \tag{254}
$$

두 행렬 사이에는 다음과 같은 유용한 성질이 존재한다. 해당 성질들은 본 섹션의 증명 과정에서 자주 등장하니 익혀두는 것을 권장한다.

$$
\begin{aligned}
\mathbf{Z} &= \begin{bmatrix} 1 & & \\ & 1 & \\ & & \end{bmatrix} \mathbf{W} \quad \text{up to sign} \\
&= \text{diag}(1,1,0)\mathbf{W} \quad \text{up to sign}
\end{aligned} \tag{255}
$$

$$
\begin{aligned}
\mathbf{ZW} &= -\mathbf{ZW}^\intercal \\
&= \begin{bmatrix} 1 & & \\ & 1 & \\ & & \end{bmatrix} \\
&= \text{diag}(1,1,0)
\end{aligned} \tag{256}
$$

$$
\begin{aligned}
\mathbf{W}^\intercal &= \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \mathbf{W} \\
\mathbf{WW}^\intercal &= \mathbf{I}
\end{aligned} \tag{257}
$$

위 식에 따라 $\mathbf{UZU}^\intercal = \mathbf{U}\text{diag}(1,1,0)\mathbf{WU}^\intercal$과 같이 나타낼 수 있다. 이를 사용하여 Fundamental Matrix $\mathbf{F}$를 다시 나타내면 다음과 같다.

$$
\begin{aligned}
\mathbf{F} &= \mathbf{SR} \\
&= \mathbf{UZU}^\intercal\mathbf{R} \\
&= \mathbf{U}\text{diag}(1,1,0)\mathbf{WU}^\intercal\mathbf{R} \\
&\sim \mathbf{U}\text{diag}(1,1,0)\mathbf{V}^\intercal \quad \text{up to similarity}
\end{aligned} \tag{258}
$$

위 식에 따라 **$\mathbf{F}$의 가장 큰 두 특이값은 같으며 마지막 특이값은 0이 된다. 이에 따라 Fundamental Matrix $\mathbf{F}$는 Essential Matrix $\mathbf{E}$의 성질을 만족하므로 Essential Matrix**가 된다.

Reverse Proof: 반대로 임의의 정방행렬 $\mathbf{E} \in \mathbb{R}^{3\times3}$의 특이값 분해(SVD)가

$$
\mathbf{E} \sim \mathbf{U} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \mathbf{V}^\intercal \tag{259}
$$

을 만족하는 경우 이는

$$
\begin{aligned}
\mathbf{E} &\sim \mathbf{U} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \mathbf{V}^\intercal \\
&= \mathbf{U}\text{diag}(1,1,0)\mathbf{V}^\intercal \\
&= \mathbf{UZWV}^\intercal \qquad \because \mathbf{ZW} = \text{diag}(1,1,0) \\
&= \mathbf{UZU}^\intercal(\mathbf{UWV}^\intercal) \quad \because \mathbf{U}^\intercal\mathbf{U} = \mathbf{I} \\
&= \mathbf{SR} \\
&= \mathbf{t}^\wedge\mathbf{R}
\end{aligned} \tag{260}
$$

이 된다. 네번째 행에서 3개의 직교행렬의 곱 $\mathbf{UWV}^\intercal$은 회전행렬의 성질을 만족하므로 $\mathbf{R}$로 표시할 수 있다. 따라서 **이 때 $\mathbf{E}$는 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}') = ([\mathbf{I}|0], [\mathbf{R}|\mathbf{t}])$에 대응하는 Fundamental Matrix가 된다.**

Extraction of cameras from the essential matrix

Result 9.18

Essential Matrix $\mathbf{E}$가 다음과 같이 주어졌을 때

$$\mathbf{E} = \mathbf{U}\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}\mathbf{V}^{\intercal} \tag{261}$$

행렬 $\mathbf{E}$는 SR Factorization을 통해 $\mathbf{R}, \mathbf{t}$를 분해할 수 있다. 이 때, **$\mathbf{R}$는 서로 부호가 다른 두 개의 SR Factorization 해를 가진다(up to sign).**

$$\mathbf{R} = \mathbf{U}\mathbf{W}\mathbf{V}^{\intercal} \quad \text{or} \quad \mathbf{U}\mathbf{W}^{\intercal}\mathbf{V}^{\intercal} \tag{262}$$

Proof

(260)에서 Essential Matrix $\mathbf{E}$는 다음과 같이 2개의 행렬로 분해가 가능하다.

$$\begin{aligned}\mathbf{E} &= (\mathbf{U}\mathbf{Z}\mathbf{U}^{\intercal})(\mathbf{U}\mathbf{W}\mathbf{V}^{\intercal}) = \mathbf{S}_0\mathbf{R}_0 \\ &= (\mathbf{U}\mathbf{Z}\mathbf{U}^{\intercal})(\mathbf{U}\mathbf{W}^{\intercal}\mathbf{V}^{\intercal}) = \mathbf{S}_0\mathbf{R}_0'\end{aligned} \tag{263}$$

위 식에서 두번째 행의 $(\mathbf{U}\mathbf{W}^{\intercal}\mathbf{V}^{\intercal})$는 $-\mathbf{Z}\mathbf{W}^{\intercal}$을 사용하여 $\text{diag}(1,1,0)$을 표현했을 때 얻는 행렬이다(up to sign). Essential Matrix $\mathbf{E}$가 위 두 가지 케이스로만 분해된다는 것을 증명하려면

$$\mathbf{E} = \mathbf{S}_0\mathbf{R}_0 = \mathbf{S}\mathbf{R} \tag{264}$$

에 대하여 $\mathbf{S} = \mathbf{S}_0, \mathbf{R} = \mathbf{R}_0 = \mathbf{R}_0'$임을 증명해야 한다.
반대칭행렬 $\mathbf{S}, \mathbf{S}_0$는 rank가 2인 행렬이므로 $\mathbf{S}_0^{\intercal} = \mathbf{s}_0^{\wedge}, \mathbf{S}^{\intercal} = \mathbf{s}^{\wedge}$과 같이 벡터 형태로 나타낼 수 있고

$$\mathbf{s}, \mathbf{s}_0 \in \text{Nul } \mathbf{E}^{\intercal} \tag{265}$$

과 같이 $\mathbf{E}$의 Left Null Space에 포함된다. 따라서 $\mathbf{s}, \mathbf{s}_0$는 비례관계가 성립한다.

$$\mathbf{s} = \alpha\mathbf{s}_0 \quad \alpha \neq 0 \in \mathbb{R} \tag{266}$$

위 식에 따라 $\mathbf{S}\mathbf{R} = \alpha\mathbf{S}_0\mathbf{R}_0$이 성립하고 **이 때 $\|\mathbf{R}\| = \|\mathbf{R}_0\| = 1$이므로 Frobenius Norm을 비교해보면 $\alpha = \pm 1$인 것**을 알 수 있다.

$$\mathbf{S} = \pm\mathbf{S}_0 \tag{267}$$

다음으로 $\mathbf{R} = \mathbf{R}_0 = \mathbf{R}_0'$라는 것을 증명한다. 지금까지 구한 결과에 따라 Essential Matrix $\mathbf{E}$는 다음과 같이 쓸 수 있다.

$$\begin{aligned}\mathbf{E} &= \mathbf{U}\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}\mathbf{V}^{\intercal} \\ &= \mathbf{S}_0\mathbf{R} \\ &= \mathbf{S}_0(\mathbf{U}\mathbf{X}\mathbf{V}^{\intercal}) \quad \because \mathbf{R} = \mathbf{U}\mathbf{X}\mathbf{V}^{\intercal} \\ &= \mathbf{U}\mathbf{Z}\mathbf{U}^{\intercal}\mathbf{U}\mathbf{X}\mathbf{V}^{\intercal} \quad \because \mathbf{U}^{\intercal}\mathbf{U} = \mathbf{I} \\ &= \mathbf{U}(\mathbf{Z}\mathbf{X})\mathbf{V}^{\intercal}\end{aligned} \tag{268}$$

이 때, 회전행렬 $\mathbf{R}$은 $\mathbf{R} = \mathbf{U}\mathbf{X}\mathbf{V}^{\intercal}$와 같이 서로 다른 3개의 직교행렬로 분해가 가능하다.
$\mathbf{Z}\mathbf{X} = \begin{bmatrix} & 1 & \\ -1 & & \\ & & \end{bmatrix}\mathbf{X} = \text{diag}(1,1,0)$이므로 이를 전개하면

$$\begin{aligned}\mathbf{X} &= \begin{bmatrix} & -1 & \\ 1 & & \\ & & \pm 1\end{bmatrix} \\ &= \mathbf{W} = -\mathbf{W}^{\intercal}\end{aligned} \tag{269}$$

이므로 **따라서 부호의 변화를 포함한(up to sign) $\mathbf{R} = \mathbf{R}_0 = \mathbf{R}_0'$가 성립한다.**

$$\mathbf{R} = \mathbf{U}\mathbf{W}\mathbf{V}^\intercal \quad \text{or} \quad \mathbf{U}\mathbf{W}^\intercal\mathbf{V}^\intercal \tag{270}$$

다음으로 $\mathbf{t}$를 구해보자. 반대칭행렬 $\mathbf{S} = \mathbf{U}\text{diag}(1,1,0)\mathbf{U}^\intercal = \mathbf{t}^\wedge$이므로 다음이 성립한다.

$$\mathbf{S}\mathbf{t} = \mathbf{t}^\wedge\mathbf{t} = 0 \tag{271}$$

위 식에서 **t의 해는 S의 Null Space가 되므로 행렬 U의 세 번째 열(third column)인 u_3가 된다.** 하지만 $\mathbf{S} = \pm\mathbf{S}_0$이므로

$$\mathbf{t} = \pm\mathbf{u}_3 \tag{272}$$

이 되어 정확한 $\mathbf{t}$ 값을 결정할 수 없다.

Result 9.19

따라서 $t = \pm u_3$와 앞서 구한 $R = UWV^\intercal$ 또는 $R = UW^\intercal V^\intercal$에 의해 Essential Matrix $E = t^\wedge R$에 대한 총 네 가지 경우의 수가 존재한다.

두 개의 카메라 행렬 $\mathbf{P}, \mathbf{P}'$와 Essential Matrix $\mathbf{E}$가 주어졌을 때, $\mathbf{P} = [\mathbf{I} \mid 0]$이라고 하면 $\mathbf{P}'$에 대한 다음과 같은 네 가지 해가 존재한다.

$$\mathbf{P}' = [\mathbf{U}\mathbf{W}\mathbf{V}^\intercal | +\mathbf{u}_3] \text{ or } [\mathbf{U}\mathbf{W}\mathbf{V}^\intercal | -\mathbf{u}_3] \text{ or } [\mathbf{U}\mathbf{W}^\intercal\mathbf{V}^\intercal | +\mathbf{u}_3] \text{ or } [\mathbf{U}\mathbf{W}^\intercal\mathbf{V}^\intercal | -\mathbf{u}_3] \tag{273}$$

Geometrical interpretation of the four solutions

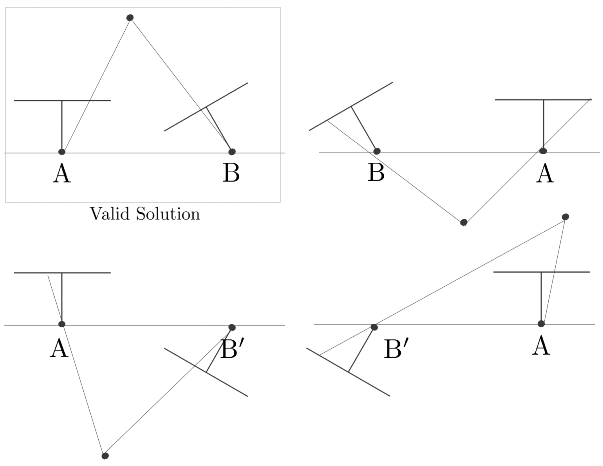
위 네가지 해에서 처음 두 개의 솔루션을 보면 $\mathbf{u}_3$의 부호만 다른 것을 알 수 있는데 이는 첫 번째와 두 번째 카메라가 반대 방향으로 뒤집혀 있는 상태를 의미한다.

첫 번째 해와 세 번째 해는 다음과 같은 관계를 가진다.

$$\begin{aligned}
\mathbf{E} &= [\mathbf{U}\mathbf{W}^\intercal\mathbf{V}^\intercal \mid \mathbf{u}_3] & & \cdots \text{3rd solution} \\
&= [\mathbf{U}(\mathbf{W}\mathbf{V}^\intercal\mathbf{V}\mathbf{W}^\intercal)\mathbf{W}^\intercal\mathbf{V}^\intercal \mid \mathbf{u}_3] & & \because \mathbf{W}\mathbf{V}^\intercal\mathbf{V}\mathbf{W}^\intercal = \mathbf{I} \\
&= [\mathbf{U}\mathbf{W}\mathbf{V}^\intercal \mid \mathbf{u}_3] \begin{bmatrix} \mathbf{V}\mathbf{W}^\intercal\mathbf{W}^\intercal\mathbf{V}^\intercal & \\ & 1 \end{bmatrix} & & \\
&= [\mathbf{U}\mathbf{W}\mathbf{V}^\intercal \mid \mathbf{u}_3] \begin{bmatrix} \mathbf{V}\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\mathbf{V}^\intercal & \\ & 1 \end{bmatrix} & & \cdots \text{1st solution} \cdot [*]
\end{aligned} \tag{274}$$

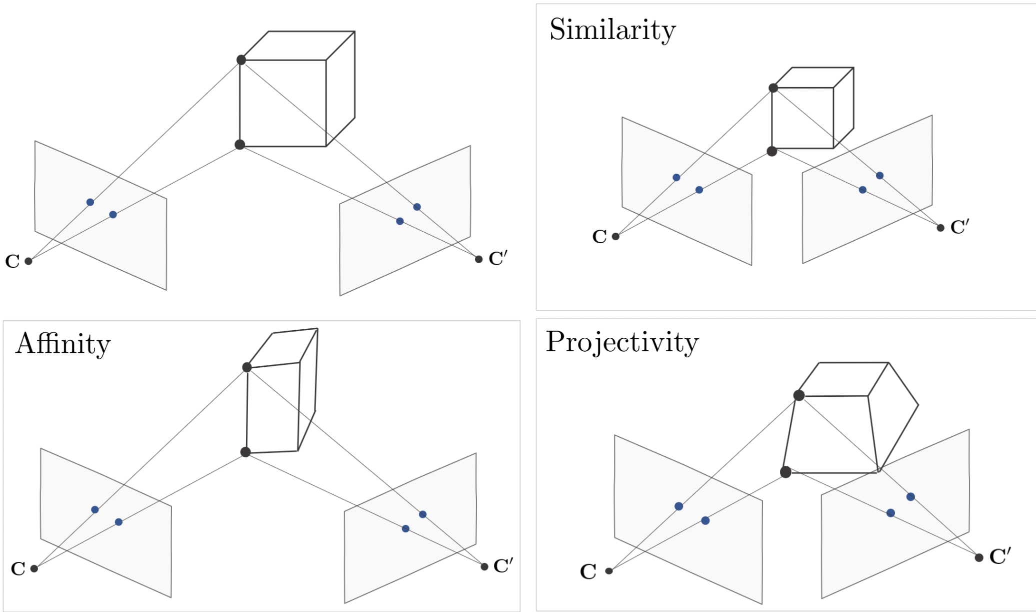
이 때 $\mathbf{V}\mathbf{W}^\intercal\mathbf{W}^\intercal\mathbf{V}^\intercal$는 Baseline의 수직한 방향 2개를 상하대칭하는 행렬이 된다. 즉, **Baseline을 축으로 180 degree 회전하는 형태**가 된다.

Four Solutions of Essential Matrix



네 개의 해를 기하학적으로 표현하면 위 그림과 같다. **수학적으로는 총 네 개의 해가 도출되지만 실제 유효한 값은 하나만 존재한다. 따라서 3차원 공간 상의 점 X가 두 카메라 앞에 존재하는 유일한 해를 선택하면 Essential Matrix E를 분해한 R, t를 성공적으로 얻을 수 있다.**

7 3D Reconstruction of Cameras and Structure



이전 섹션에서 언급했듯이 **Fundamental Matrix F는 여러 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}')$에 대하여 사영 모호성(projective ambiguity)이 존재하기 때문에 결과물인 $\mathbf{P}, \mathbf{P}'$를 통해 계산한 3차원 공간 상의 점 또한 모호성을 가지게 된다.** 해당 섹션에서는 Scene Constraint와 Internal Constraint를 사용하여 모호성을 제거하는 방법에 대해서 설명한다.

The projective reconstruction theorem

Theorem 10.1 (Projective reconstruction theorem)

두 카메라의 이미지 평면 상 대응점 쌍들인 $\mathbf{x}, \mathbf{x}'$이 충분히 주어져서 이를 통해 Fundamental Matrix $\mathbf{F}$를 구했다고 가정해보자. 이 때, 사영모호성으로 인해 다음과 같은 두 카메라 행렬 대응쌍

$$\begin{aligned} &(\mathbf{P}_1, \mathbf{P}'_1, \{\mathbf{X}_{1,i}\}) \\ &(\mathbf{P}_2, \mathbf{P}'_2, \{\mathbf{X}_{2,i}\}) \end{aligned} \tag{275}$$

이 동일한 Fundamental Matrix $\mathbf{F}$를 갖게 되는 경우

$$(\mathbf{P}_2, \mathbf{P}'_2, \{\mathbf{X}_{2,i}\}) = \mathbf{H} \cdot (\mathbf{P}_1, \mathbf{P}'_1, \{\mathbf{X}_{1,i}\}) \tag{276}$$

를 만족하는 Homography 행렬 $\mathbf{H} \in \mathrm{PGL}_4$가 반드시 존재한다. 이 때 $\cdot$ 연산은 다음과 같이 성립한다.

$$\begin{aligned} \mathbf{P}_2 &= \mathbf{H} \cdot \mathbf{P}_1 = \mathbf{P}_1\mathbf{H}^{-1} \\ \mathbf{P}'_2 &= \mathbf{H} \cdot \mathbf{P}'_1 = \mathbf{P}'_1\mathbf{H}^{-1} \\ \mathbf{X}_{2i} &= \mathbf{H} \cdot \mathbf{X}_{1i} = \mathbf{H}\mathbf{X}_{1i} \end{aligned} \tag{277}$$

Proof

이전 섹션에서 동일한 Fundamental Matrix $\mathbf{F}$를 공유하는 카메라 행렬 대응쌍 사이에 $\mathbf{P}_2 = \mathbf{P}_1\mathbf{H}^{-1}, \mathbf{P}'_2 = \mathbf{P}'_1\mathbf{H}^{-1}$를 만족하는 Homography 행렬 $\mathbf{H} \in \mathbb{R}^{4\times4}$가 존재한다는 것을 증명하였다. 이를 적용해보면

$$\begin{aligned} \mathbf{P}_2\mathbf{X}_{2i} &= \mathbf{P}_1\mathbf{H}^{-1}\mathbf{X}_{2i} = \mathbf{x}_i = \mathbf{P}_1\mathbf{X}_{1i} \\ \mathbf{P}'_2\mathbf{X}_{2i} &= \mathbf{P}'_1\mathbf{H}^{-1}\mathbf{X}_{2i} = \mathbf{x}'_i = \mathbf{P}'_1\mathbf{X}_{1i} \end{aligned} \tag{278}$$

가 성립한다. $\mathbf{P}_1, \mathbf{P}_2$에 의해 Back Projection된 직선을 $\mathcal{R}$이라고 하고 $\mathbf{P}'_1, \mathbf{P}'_2$에 의해 Back Projection 된 직선을 $\mathcal{R}'$이라고 하면

$$\{\mathbf{H}^{-1}\mathbf{X}_{2i}, \mathbf{X}_{1i}\} \in \mathcal{R} \cap \mathcal{R}' \tag{279}$$

와 같이 3차원 공간 상의 점 $\{\mathbf{H}^{-1}\mathbf{X}_{2i}, \mathbf{X}_{1i}\}$는 두 직선의 교차점이 된다. 따라서 **두 직선 $\mathcal{R}, \mathcal{R}'$이 Baseline과 같이 동일한 직선인 경우를 제외하면 $\{\mathbf{H}^{-1}\mathbf{X}_{2i}, \mathbf{X}_{1i}\}$는 교차점인 한 점을 의미**하므로

$$\begin{aligned} \mathbf{H}^{-1}\mathbf{X}_{2i} &= \mathbf{X}_{1i} \\ \mathbf{X}_{2i} &= \mathbf{H}\mathbf{X}_{1i} \end{aligned} \tag{280}$$

가 성립한다. 따라서 추가적인 기하학적 원리를 사용하지 않으면 위와 같은 상황에서는 사영모호성을 포함하여(up to projectivity) 복원할 수 있다.

Stratified reconstruction

Fundamental Matrix $\mathbf{F}$를 사용하여 계산한 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}')$의 사영모호성 문제를 해결하기 위해 Ground Truth 카메라 행렬 대응쌍인

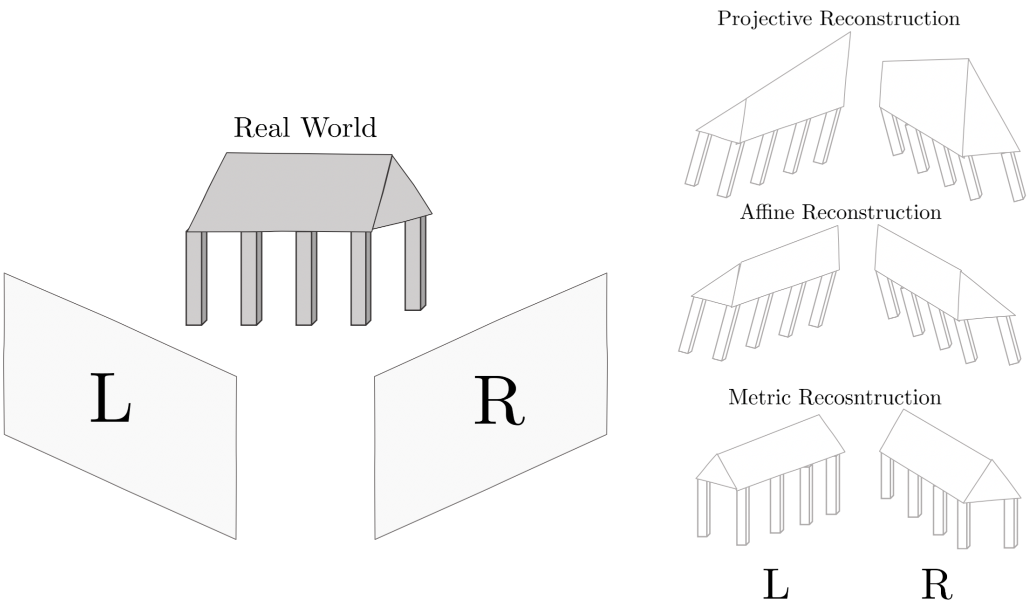
$$(\mathbf{P}_0, \mathbf{P}'_0, \{\hat{\mathbf{X}}_i\}) \tag{281}$$

이 존재한다고 가정하면

$$\begin{aligned} (\mathbf{P}_1, \mathbf{P}'_1, \{\mathbf{X}_i\}) &= \mathbf{H} \cdot (\mathbf{P}_0, \mathbf{P}'_0, \{\hat{\mathbf{X}}_i\}) \\ \mathcal{T} &= \mathbf{H} \cdot \mathcal{T}_0 \end{aligned} \tag{282}$$

을 만족하는 Homography 행렬 $\mathbf{H}$가 존재한다. 이 때, 이미지 평면 상에서 평행한 직선의 특성을 활용하여 Affine Reconstruction을 수행한 다음, 이미지 평면 상에서 직교한 선들의 특성을 사용하여 Similarity Reconstruction을 순차적으로 수행한다.

The step to affine reconstruction



Affine Reconstruction 단계에서는 평행한 직선들의 특성을 보존하는 Homography 행렬 $\mathbf{H}_A$를 찾아야 한다.

$$\mathbf{H}_A(\pi_\infty) = \pi_\infty \tag{283}$$

3차원 공간 상의 점들의 집합 $\{\mathbf{X}_i\}$를 사용하여 소실점을 3개 이상 구한 경우 이미지 평면에 프로젝션된 무한대 평면 π'_∞를 계산할 수 있다. 이미지 평면 상에 프로젝션된 무한대 평면 π'_∞를

$$\pi'_\infty = \begin{bmatrix} a & b & c & 1 \end{bmatrix}^\intercal \tag{284}$$

라고 하면 실제 무한대 평면 $\pi_\infty = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^\intercal$을 π'_∞로 보내는 Homography 행렬 $\mathbf{H}_A$는 다음과 같이 구할 수 있다.

$$\mathbf{H}_A = \begin{bmatrix} \mathbf{I}_3 \mid 0 \\ \pi_\infty'^{\intercal} \end{bmatrix} = \begin{bmatrix} & \mathbf{I}_3 & & 0 \\ a & b & c & 1 \end{bmatrix} \tag{285}$$

이 때 $\mathbf{H}_A^{\intercal}\pi_\infty = \pi_\infty'$ 관계가 성립한다.

$$\pi_\infty = \mathbf{H}_A^{-\intercal}\pi_\infty' \tag{286}$$

이를 $\mathcal{T}$에 적용하면

$$\begin{aligned} &\mathcal{T}_a = \mathbf{H} \cdot \mathcal{T} \\ &(\mathbf{P}\mathbf{H}_A^{-1}, \mathbf{P}'\mathbf{H}_A^{-1}, \{\mathbf{H}_A\mathbf{X}_i\}) = \mathbf{H} \cdot (\mathbf{P}, \mathbf{P}', \{\mathbf{X}_i\}) \end{aligned} \tag{287}$$

연산을 통해 이미지를 Affine Reconstruction할 수 있다. 여기까지 진행했다면 Ground Truth $\mathcal{T}_0$와 $\mathcal{T}_a$는 Affine만큼만 차이가 나게 된다. 즉, 평행한 직선들은 모두 복원되었으나 아직 직교한 직선들은 복원되지 않은 상태이다.

The step to metric reconstruction

Metric Reconstruction 단계에서는 직교하는 선들을 보존하는 Homography 행렬 $\mathbf{H}_S$를 찾아야 한다.

$$\mathbf{H}_S(\Omega_\infty) = \Omega_\infty \tag{288}$$

이 때, Ω_∞는 무한대 평면상에 존재하는 Absolute Conic을 의미한다. Affine Reconstruction 단계에서 얻은 $\mathcal{T}_a$는 무한대 평면 π_∞가 실제 무한대에 위치하게 된다. 이 때, $\{\mathbf{X}_i\}$들 중 직교하는 점들을 사용하여 Ω_∞를 구할 수 있다. 직교하는 점들의 대응쌍을 $\mathbf{d}_1, \mathbf{d}_2$라고 하면

$$\mathbf{d}_1^{\intercal}\omega\mathbf{d}_2 = 0 \tag{289}$$

을 만족하는 ω를 찾을 수 있고 ω는 이미지 평면 상에 프로젝션된 Absolute Conic Ω_∞ (Image of Absolute Conic, IAC)가 된다.

카메라 행렬 $\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}] = [\mathbf{M} \mid \mathbf{m}]$가 주어졌을 때 무한대 평면 π_∞는 $\mathbf{M}$에 의해 이미지 평면 π에 프로젝션된다.

$$\mathbf{M} : \pi_\infty \mapsto \pi \tag{290}$$

반대로 $\mathbf{M}^{-1}$는 π 평면 상의 점들을 무한대 평면 π_∞으로 보내는 매핑이라고 할 수 있다.

$$\mathbf{M}^{-1} : \pi \mapsto \pi_\infty \tag{291}$$

이 때, 이미지 평면에 프로젝션된 Absolute Conic에 $\mathbf{M}^{-1}$을 적용하면 $\mathbf{M}^{-1}(\omega) = \tilde{\Omega}_\infty$가 된다. $\tilde{\Omega}_\infty$는 아직 Metric Reconstruction이 되기 전이므로 $\Omega_\infty \neq \tilde{\Omega}_\infty$이다. 둘 사이에는 다음 공식이 성립한다.

$$\mathbf{H}_S(\tilde{\Omega}_\infty) = \Omega_\infty = \mathbf{I}_3 \tag{292}$$

Result 10.5

이미지 평면에 프로젝션된 Absolute Conic을 ω라고 하고 Affine Reconstruction으로 구한 카메라 행렬을 $\mathbf{P} = [\mathbf{M} \mid \mathbf{m}]$이라고 할 때 Metric Reconstruction을 수행하는 Homography $\mathbf{H}_S$는 다음과 같이 구할 수 있다.

$$\mathbf{H}_S = \begin{bmatrix} \mathbf{A}^{-1} & \\ & 1 \end{bmatrix} \tag{293}$$

이 때, $\mathbf{A}\mathbf{A}^{\intercal} = (\mathbf{M}^{\intercal}\omega\mathbf{M})^{-1}$이고 $\mathbf{A}$는 이를 Cholesky Decomposition을 통해 얻을 수 있다.

Proof

Metric Reconstruction을 수행하는 Homography $\mathbf{H}_S = \begin{bmatrix} \mathbf{A}^{-1} & \\ & 1 \end{bmatrix}$이라고 하고 이 때 이미지 평면을 π_s, 카메라 행렬을 $\mathbf{P}_s$라고 하면 $\mathbf{H}_S(\pi) = \pi_s$ 관계가 성립하고 무한대 평면을 Homography 변환한 공식은

$$\begin{aligned} \pi_\infty \mathbf{A}^{-1} &= \pi_{\infty,s} \\ \mathbf{H}_S|_{\pi_\infty} &= \mathbf{A}^{-1} \end{aligned} \tag{294}$$

가 된다. 각각 무한대 평면 $\pi_\infty, \pi_{\infty,s}$을 이미지 평면 π, π_s에 투영하는 공식은

$$\begin{aligned} \mathbf{P}|_{\pi_\infty} &= \mathbf{M} \\ \mathbf{P}_s|_{\pi_\infty} &= \mathbf{MA} \end{aligned} \tag{295}$$

과 같이 변형된다. 따라서 $\mathbf{H}_S(\tilde{\Omega}_\infty)$는 다음과 같다.

$$\begin{aligned} \mathbf{H}_S(\tilde{\Omega}_\infty) &= (\mathbf{A}^{-1})^{-\intercal} \tilde{\Omega}_\infty \mathbf{A} \\ &= \mathbf{A}^\intercal (\mathbf{M}^\intercal \omega \mathbf{M}) \mathbf{A} \end{aligned} \tag{296}$$

따라서 Absolute Conic 특성에 의해 $\mathbf{A}^\intercal(\mathbf{M}^\intercal\omega\mathbf{M})\mathbf{A}$는 $\mathbf{I}$이 되어야 하므로

$$\begin{aligned} \mathbf{A}^\intercal (\mathbf{M}^\intercal \omega \mathbf{M}) \mathbf{A} &= \mathbf{I} \\ (\mathbf{M}^\intercal \omega \mathbf{M})^{-1} &= \mathbf{A}\mathbf{A}^\intercal \end{aligned} \tag{297}$$

가 성립한다. 결론적으로 $\mathbf{H}_S = \begin{bmatrix} \mathbf{A}^{-1} & \\ & 1 \end{bmatrix}$를 구하기 위해서는 **이미지 평면 상에서 직교하는 직선들의 특성을 사용하여 ω를 구한 후, $(\mathbf{M}^\intercal\omega\mathbf{M})^{-1}$를 Cholesky Decomposition하여 $\mathbf{A}\mathbf{A}^\intercal$를 구한다. 이를 통해 $\mathbf{A}^{-1}$ 값을 구할 수 있고 최종적으로 Metric Reconstruction을 수행하는 Homography $\mathbf{H}_S$를 구할 수 있다.**

8 Computation of the Fundamental Matrix F

Basic equations

두 카메라의 대응점쌍 $\mathbf{x} \leftrightarrow \mathbf{x}'$이 존재할 때

$$\mathbf{x}'^\intercal \mathbf{F} \mathbf{x} = 0 \tag{298}$$

을 만족하는 3x3 크기의 행렬 $\mathbf{F}$를 Fundamental Matrix라고 한다. 만약 세 개의 대응점쌍 $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{x}' = (x'_1, x'_2, x'_3)$이 주어졌을 때 $\mathbf{x}'^\intercal\mathbf{F}\mathbf{x} = 0$이를 다시 정리하면 $\sum \mathbf{x}'_i \mathbf{x}_j \mathbf{f}_{ij} = 0$과 같이 정리할 수 있다. 해당 식은 선형시스템

$$\mathbf{A}\mathbf{f} = 0 \tag{299}$$

꼴로 나타낼 수 있고 이 때 $\mathbf{A}$와 $\mathbf{f}$를 풀어쓰면 다음과 같다.

$$\begin{aligned} &\mathbf{A} = (\mathbf{a}_{ik}) \in \mathbb{R}^{N\times 9}, \quad \text{i-th row is } (x'_{i1}x_{i1}, x'_{i1}x_{i2}, \cdots, x'_{i3}x_{i3}) \\ &\mathbf{f} = (\mathbf{f}_{11}, \mathbf{f}_{12}, \mathbf{f}_{13}, \cdots, \mathbf{f}_{33})^\intercal \in \mathbb{R}^{9\times 1} \end{aligned} \tag{300}$$

만약 주어진 대응점쌍의 개수 N이 8보다 크고 노이즈가 없는 완벽한 데이터로 생성이 되었으면 $\mathbf{A}$는 rank 8 행렬이 되고 유일한 해 $\mathbf{f}$가 존재한다. 이 때 $\mathbf{f}$는 $\mathbf{A}$의 Null Space의 원소가 된다.

하지만 대부분의 경우 주어진 데이터에는 항상 노이즈가 존재하므로 $\mathbf{A}$ 행렬은 rank 9인 full rank 행렬이 된다. 이 때 해를 구하면 $\mathbf{f}$는 항상 영벡터가 계산되므로 $\|\mathbf{f}\| = 1$이라는 조건하에 $\|\mathbf{A}\mathbf{f}\|$의 크기를 최소화하는 근사해 $\mathbf{f}$를 찾아야 한다. 해를 찾는 방법은 $\mathbf{A}$를 특이값 분해(SVD)한 다음

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\intercal \tag{301}$$

대각행렬 $\mathbf{D}$의 eigenvalue 중 가장 절대값이 작은 것에 대응되는 $\mathbf{V}$의 열벡터를 $\mathbf{f}$의 일반해로 간주한다. 해당 열벡터 $\mathbf{f}$가 $\|\mathbf{A}\mathbf{f}\|$의 크기를 최소화하는 벡터가 되며 이를 Linear Solution이라고 부른다.

하지만 **이 때, Linear Solution $\mathbf{f}$로 복원한 $\mathbf{F}_0$가 rank 2 행렬임을 보장할 수 없다.** 수치적으로 해를 구하면 일반적으로 rank 3인 행렬이 나온다. 따라서 한 번 더 SVD를 사용하여 $\mathbf{F}_0$와 가장 가까운 rank 2 행렬을 찾아야 한다.

$$\mathbf{F}_0 = \mathbf{U}_0\mathbf{D}_0\mathbf{V}_0^\intercal \tag{302}$$

이 때 대각행렬 $\mathbf{D}_0 = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$는 일반적으로 마지막 특이값 σ_3가 0이 아니므로 rank 3인 행렬이 된다. 따라서 **임의로 $\sigma_3 = 0$으로 변경한 행렬이 $\mathbf{F}_0$과 가장 가까운 rank 2 행렬이 된다.**

The minimum case - seven point correspondences

7개의 대응점쌍이 주어진 경우 $\mathbf{A} \in \mathbb{R}^{7\times 9}$인 rank 7 행렬이 된다. 이에 따라 Null Space의 차원이 2가 되므로 무수한 해가 존재한다. $\mathbf{Af} = 0$의 풀어서 두 개의 선형독립인 Fundamental Matrix $\mathbf{F}_1, \mathbf{F}_2$를 구하고

$$\mathbf{F} = \alpha\mathbf{F}_1 + (1-\alpha)\mathbf{F}_2 \tag{303}$$

를 통해 $\mathbf{F}$를 구한다. $\mathbf{F}$ 행렬의 rank는 2이므로 $\det(\mathbf{F}) = 0$을 만족하게 되고 이에 따라 α에 대한 3차 방정식을 얻는다. 3차 방정식이 세 개의 실수해를 가지는 경우는 Degenerate Configuration이 된다.

The normalized 8-point algorithm

행렬 $\mathbf{A}$는 주어진 대응점쌍 $\mathbf{x} \leftrightarrow \mathbf{x}'$으로 구성된 행렬이므로 **대응점쌍들을 정규화(normalization)하지 않고 바로 해를 구하는 경우 $\mathbf{x}_i = (1000000, 20000000, 1)$과 같이 마지막 값에 비해서 처음 두 값의 크기가 매우 크게 되므로 수치적으로 불안정한 문제**를 가진다. 따라서 $\mathbf{x}_i, \mathbf{x}'_i$를 Centroid가 0이고 Centroid로부터 평균거리가 $\sqrt{2}$가 되도록하는 Homography 변환 $\mathbf{H}, \mathbf{H}'$를 적용한다.

$$\begin{aligned} \mathbf{x}_i &\rightarrow \mathbf{H}\mathbf{x}_i \\ \mathbf{x}'_i &\rightarrow \mathbf{H}'\mathbf{x}'_i \end{aligned} \tag{304}$$

이렇게 변환된 대응점 쌍들을 사용하여 수치적으로 안정적인 Fundamental Matrix $\mathbf{F}'$ 계산할 수 있다. 앞서 설명한 내용을 따라 $\mathbf{F}'$를 계산한 다음 다시 원래의 대응점쌍 $\mathbf{x} \leftrightarrow \mathbf{x}'$으로 복원하기 위해

$$\begin{aligned} 0 &= (\mathbf{H}'\mathbf{x}')^\intercal\mathbf{F}'(\mathbf{H}\mathbf{x}) \\ &= \mathbf{x}^\intercal(\mathbf{H}'^\intercal\mathbf{F}'\mathbf{H})\mathbf{x} \end{aligned} \tag{305}$$

처럼 $\mathbf{F} = \mathbf{H}'^\intercal\mathbf{F}'\mathbf{H}$를 통해 원래 대응점쌍의 Fundamental Matrix $\mathbf{F}$를 계산할 수 있다.

Degenrate configurations

두 개의 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}', \{\mathbf{X}_i\})$, $(\mathbf{Q}, \mathbf{Q}', \{\mathbf{Y}_i\})$가 다음의 조건을 만족하면 서로 **Conjugate Configuration**이라고 한다.

$$\begin{aligned} &\mathbf{P}\mathbf{X}_i = \mathbf{Q}\mathbf{Y}_i, \quad \mathbf{P}'\mathbf{x}_i = \mathbf{Q}'\mathbf{Y}' \quad {}^{\forall}s \\ &(\mathbf{P}, \mathbf{P}), (\mathbf{Q}, \mathbf{Q}') \quad \text{share the same} \quad \mathbf{F} \end{aligned} \tag{306}$$

이 때, **Conjuate Configuration이 있는 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}', \{\mathbf{X}_i\})$을 Critical Triple이라고 부른다.**

임의의 카메라 대응쌍 $\mathbf{P}, \mathbf{P}', \{\mathbf{X}_i\}$가 Critical Triple일 필요충분조건은 각각의 카메라 중심점 $\mathbf{C}, \mathbf{C}'$과 3차원 공간 상의 점 $\mathbf{X}_i$가 어떤 Ruled Quadric Surface, 즉 직선을 포함하는 Quadric Surface 상에 포함되는 것이다.

Proof

임의의 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}', \{\mathbf{X}_i\})$이 Critical Triple이라는 것을 보이기 위해서는 이와 대응하는 카메라 행렬 대응쌍 $(\mathbf{Q}, \mathbf{Q}', \{\mathbf{Y}_i\})$이 같은 Fundamental Matrix $\mathbf{F}$를 가진다는 사실을 사용해야 한다. $\mathbb{P}^3$ 공간에서 이차곡면은 4x4 크기의 행렬로 정의된다.

$$\mathbf{S}_p := \mathbf{P}'\mathbf{F}_{QQ'}\mathbf{P} \tag{307}$$

여기서 이차곡면 $\mathbf{S}_p \in \mathbb{R}^{4\times 4}$는 카메라 중심점 $\mathbf{C}, \mathbf{C}'$과 3차원 공간 상의 점 $\{\mathbf{X}_i\}$를 포함해야 한다.

$\mathbf{S}_p$는 Ruled Quadric Surface이므로 직선을 포함하고 있음을 보이면 되므로 카메라 중심선을 연결한 직선인 baseline을 $\mathbf{S}_p$가 포함하고 있음을 보이면 된다. baseline 위에 있는 임의의 점 $\mathbf{X}$가 존재할 때

$$\mathbf{S}_p\mathbf{X} = \mathbf{P}'\mathbf{F}_{QQ'}\mathbf{P}\mathbf{X} = 0 \quad \because \mathbf{F}_{QQ'}\mathbf{P}\mathbf{X} = 0 \tag{308}$$

을 만족하므로 $\mathbf{S}_p$는 Ruled Quadric Surface임을 알 수 있다.
반대로, 카메라 중심점 $\mathbf{C}, \mathbf{C}'$와 3차원 공간 상의 점 $\{\mathbf{X}_i\}$가 $\mathbf{S}_p$에 속하면 다음 공식이 성립한다.

$$(\mathbf{P}'\mathbf{X}_i)^\intercal \mathbf{F}_{QQ'}(\mathbf{P}\mathbf{X}_i) = 0 \tag{309}$$

여기서 $\mathbf{x}' = \mathbf{P}'\mathbf{X}_i, \mathbf{x} = \mathbf{P}\mathbf{X}_i$이고 $\mathbf{F}_{QQ'}$이므로 $\mathbf{F}_{QQ'}$에 대한 대응점 쌍이 $\mathbf{x}, \mathbf{x}'$ 존재한다는 의미이다. 따라서

$$\begin{aligned} \mathbf{x}' &= \mathbf{P}'\mathbf{X}_i = \mathbf{Q}'\mathbf{Y}_i \\ \mathbf{x} &= \mathbf{P}\mathbf{X}_i = \mathbf{Q}\mathbf{Y}_i \end{aligned} \tag{310}$$

를 만족하므로 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}', \{\mathbf{X}_i\})$과 $(\mathbf{Q}, \mathbf{Q}', \{\mathbf{Y}_i\})$는 같은 Fundamental Matrix $\mathbf{F} = \mathbf{F}_{PP'} = \mathbf{F}_{QQ'}$를 공유하는 Conjugate Triple이라는 것을 알 수 있다.

The Gold Standard method

이미지 평면 상의 대응점 쌍 $\mathbf{x} \leftrightarrow \mathbf{x}'$는 일반적으로 노이즈를 포함하고 있으므로 이를 통해 Fundamental Matrix $\mathbf{F}$를 정확하게 계산할 수 없고 $\|\mathbf{Af}\|$의 크기를 최소화하는 근사해 $\mathbf{F}'$를 계산할 수 밖에 없다. 따라서 **$\mathbf{F}'$를 더 정확하게 계산하기 위해 2D Homography 문제와 마찬가지로 주어진 데이터 $\mathbf{x}, \mathbf{x}'$를 Ground Truth $\hat{\mathbf{x}}, \hat{\mathbf{x}}'$과 가까워지게 보정하는 방법을 The Gold Standard method**라고 한다.

- 8개 이상의 대응점 쌍 $\mathbf{x} \leftrightarrow \mathbf{x}'$이 주어졌을 때 우선 앞서 설명한 것과 같이 총 두 번 SVD를 사용하여 rank가 2인 Fundamental Matrix $\mathbf{F}_0$를 계산한다. 이 때, $\mathbf{F}_0$를 Linear Solution이라고 부른다.
- 다음으로 Epipole $\mathbf{e}, \mathbf{e}'$에 대하여 다음 공식이 성립한다.

$$\mathbf{e}'\mathbf{F}_0 = 0, \quad \mathbf{F}_0\mathbf{e} = 0 \tag{311}$$

- 위 공식으로부터 $\mathbf{e}, \mathbf{e}'$를 각각 계산한다.
- Epipole을 계산하면 Caninocal Form의 카메라 행렬 $\mathbf{P}, \mathbf{P}'$를 계산할 수 있다.

$$\begin{aligned} &\mathbf{P} = [\mathbf{I}|0], \quad \mathbf{P}' = [\mathbf{M}|\mathbf{m}] \\ &\text{where, } \ \mathbf{M} = \mathbf{e}'^{\wedge}\mathbf{F}, \ \mathbf{m} = \mathbf{e}' \end{aligned} \tag{312}$$

- 이를 통해 $\mathbf{x}, \mathbf{x}'$의 Back-projection된 직선들을 각각 구할 수 있다.
- 두 Back-projection 직선과 가장 근접한 3차원 공간 상의 점 $\mathbf{X}_i$를 계산할 수 있다.
- 이미지 평면에 3차원 공간 상의 점 $\mathbf{X}_i$를 재투영(reprojection) 한다.

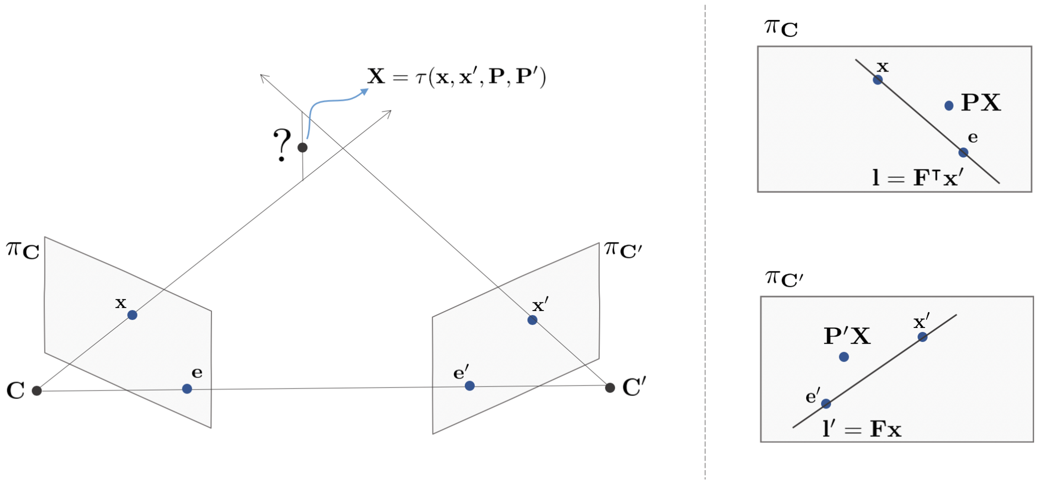
$$\begin{aligned} \hat{\mathbf{x}} &= \mathbf{P}\mathbf{X}_i \\ \hat{\mathbf{x}}' &= \mathbf{P}'\mathbf{X}_i \end{aligned} \tag{313}$$

다음과 같이 재투영 오차(reprojection error)를 Gauss-Newton, Levenerg-Marquardt 등과 같은 비선형 최적화 방법을 사용하여 최소화한다.

$$\sum_i d(\mathbf{x}, \hat{\mathbf{x}})^2 + d(\mathbf{x}', \hat{\mathbf{x}}')^2 \tag{314}$$

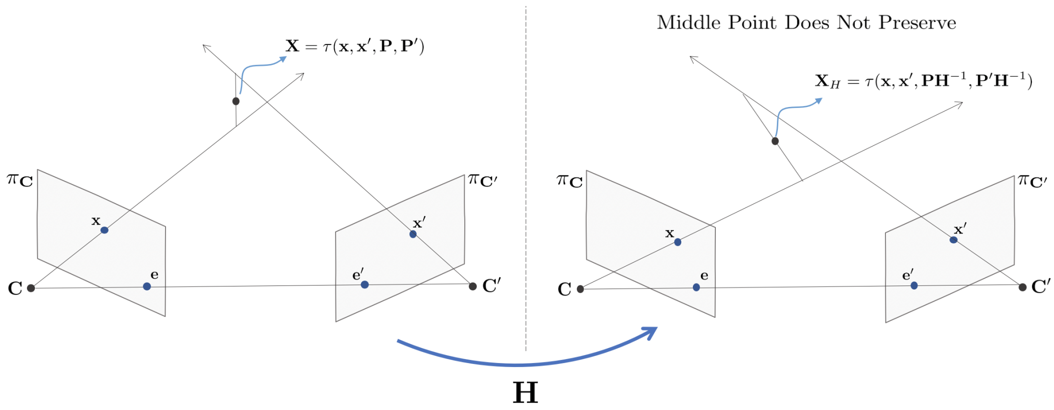
이 때 최적화되는 파라미터는 $\mathbf{X}_i$와 카메라 행렬 $\mathbf{P}' = [\mathbf{M}|\mathbf{m}]$이다.

9 Structure Computation



Problem statement

해당 섹션에서는 두 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}')$와 이미지 평면의 대응점 쌍 $(\mathbf{x}, \mathbf{x}')$가 주어진 경우 $\mathbb{P}^3$ 공간 상의 점 $\mathbf{X}$를 구하는 방법에 대해 설명한다. **이론적으로는 대응점 쌍 $\mathbf{x}, \mathbf{x}'$를 Back-projection하여 얻은 두 직선의 교차점을 통해 $\mathbf{X}$를 계산할 수 있으나 실제 데이터는 노이즈가 존재하기 때문에 두 Back-projection 직선이 교차하지 않는다.** 또한 노이즈로 인해 두 이미지 평면 상에서 Epipolar Line $\mathbf{l}, \mathbf{l}'$ 또한 $\mathbf{x}, \mathbf{x}'$와 만나지 않는다.



이를 해결하기 위해서 두 Back-projection 직선들에 동시에 수직인 최단거리 직선을 구한 다음 최단거리 직선의 중점(mid-point)를 계산하여 $\mathbf{X}$를 계산하는 방법이 존재한다. 해당 방법은 체크보드를 이용한 캘리브레이션 방법과 같이 $\mathbf{P}, \mathbf{P}'$가 고정된 경우에는 사용하면 좋은 방법이지만, 대부분의 경우 $\mathbf{P}, \mathbf{P}'$는 사영모호성(projective ambiguity)으로 인해 유일하게 결정되지 않는다.

현실 세계에서는 카메라 행렬을 Affinity 또는 Projectivity까지 알고 있는 경우(up to affinity, projectivity)가 대부분이다. 정확한 $\mathbf{X}$를 구하기 위해서는 triangulation이 사영모호성에 대해서도 불변이어야 한다. Triangulation τ은 주어진 대응점 쌍과 카메라 행렬 대응쌍으로부터 $\mathbf{X} \in \mathbb{P}^3$를 계산하는 방법을 말한다.

$$\mathbf{X} = \tau(\mathbf{x}, \mathbf{x}', \mathbf{P}, \mathbf{P}') \tag{315}$$

τ가 사영모호성에 불변이기 위해서는 임의의 Homography $\mathbf{H} \in \text{PGL}_4$에 대하여

$$\begin{aligned} \mathbf{X}_H &= \tau(\mathbf{x}, \mathbf{x}', \mathbf{P}\mathbf{H}^{-1}, \mathbf{P}'\mathbf{H}^{-1}) \\ &= \mathbf{H} \cdot \tau(\mathbf{x}, \mathbf{x}', \mathbf{P}, \mathbf{P}') \\ &= \mathbf{H}\mathbf{X} \end{aligned} \tag{316}$$

공식이 성립해야 한다. 위와 같이 불변성이 성립하는 경우

$$\begin{aligned}\mathbf{P}\mathbf{H}^{-1}\mathbf{X}_H &= \mathbf{P}\mathbf{H}^{-1}\mathbf{H}\mathbf{X} \\ &= \mathbf{P}\mathbf{X} \\ &= \mathbf{x}\end{aligned} \tag{317}$$

이 된다. 마찬가지로 $\mathbf{P}'\mathbf{H}^{-1}\mathbf{X}_H = \mathbf{x}'$이 된다. 예를 들어, Fundemantal Matrix $\mathbf{F}$와 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}')$, 대응점 쌍 $(\mathbf{x}, \mathbf{x}')$이 주어졌을 때 $(\mathbf{P}, \mathbf{P}')$를 Homography 변환한 카메라 행렬 대응쌍 $(\mathbf{P}\mathbf{H}^{-1}, \mathbf{P}'\mathbf{H}^{-1})$ 또한 동일한 $\mathbf{F}$를 공유한다. 이를 통해

$$\begin{aligned} \{\mathbf{X}\}: &\quad \text{3D Points from } (\mathbf{x}, \mathbf{x}') \text{ and } (\mathbf{P}, \mathbf{P}'). \\ \{\mathbf{X}_H\}: &\quad \text{3D Points from } (\mathbf{x}, \mathbf{x}') \text{ and } (\mathbf{P}\mathbf{H}^{-1}, \mathbf{P}'\mathbf{H}^{-1}). \end{aligned} \tag{318}$$

과 같이 두 PointCloud $\{\mathbf{X}\}, \{\mathbf{X}_H\}$를 구할 수 있고 **PointCloud가 Homography에 불변이기 위해서는 두 PointCloud 사이에**

$$\mathbf{H}\{\mathbf{X}\} = \{\mathbf{X}_H\} \tag{319}$$

인 변환 성질을 만족해야 한다. 앞서 설명한 중점(mid-point)을 사용한 방법은 두 Back-projection 직선들에 서로 수직인 최단거리 직선이 Homography 변환을 수행하면 더 이상 직교하지 않게 되고 중점 또한 더 이상 중심이 위치하지 않게 된다. 따라서 **중점을 사용한 방법은 Homography에 불변하지 않다.**

Linear triangulation methods

해당 섹션에서는 $\mathbf{X} \in \mathbb{P}^3$를 구하기 위해 선형방정식을 사용하는 방법을 설명한다. $\mathbf{x} = \mathbf{P}\mathbf{X}, \mathbf{x}' = \mathbf{P}'\mathbf{X}$로부터

$$\begin{aligned} \mathbf{x}^\wedge(\mathbf{P}\mathbf{X}) = 0 \\ \mathbf{x}'^\wedge(\mathbf{P}'\mathbf{X}) = 0 \end{aligned} \tag{320}$$

공식이 성립한다.

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} x & y & 1 \end{bmatrix}^\intercal \\ \mathbf{P} &= \begin{bmatrix} \mathbf{p}_{1,row}^\intercal & \mathbf{p}_{2,row}^\intercal & \mathbf{p}_{3,row}^\intercal \end{bmatrix}^\intercal \\ \mathbf{X} &= \begin{bmatrix} X & Y & Z & W \end{bmatrix}^\intercal \end{aligned} \tag{321}$$

$\mathbf{p}_{i,row}^\intercal$를 $\mathbf{P}$의 i번째 행(row)라고 했을 때 $\mathbf{x}^\wedge(\mathbf{P}\mathbf{X})$를 전개한 후 정리하면

$$\begin{aligned} x(\mathbf{p}_{3,row}^\intercal\mathbf{X}) - (\mathbf{p}_{1,row}^\intercal\mathbf{X}) = 0 \\ y(\mathbf{p}_{3,row}^\intercal\mathbf{X}) - (\mathbf{p}_{2,row}^\intercal\mathbf{X}) = 0 \\ x(\mathbf{p}_{2,row}^\intercal\mathbf{X}) - y(\mathbf{p}_{1,row}^\intercal\mathbf{X}) = 0 \end{aligned} \tag{322}$$

과 같고 이를 $\mathbf{x}'^\wedge(\mathbf{P}'\mathbf{X})$도 같이 구한 후 $\mathbf{X}$에 대한 선형시스템으로 다시 작성하면

$$\underbrace{\begin{bmatrix} x\mathbf{p}_{3,row}^\intercal - \mathbf{p}_{1,row}^\intercal \\ y\mathbf{p}_{3,row}^\intercal - \mathbf{p}_{2,row}^\intercal \\ x'\mathbf{p}_{3,row}'^\intercal - \mathbf{p}_{1,row}'^\intercal \\ y'\mathbf{p}_{3,row}'^\intercal - \mathbf{p}_{2,row}'^\intercal \end{bmatrix}}_{\mathbf{A}} \mathbf{X} = 0 \tag{323}$$

같이 $\mathbf{A}\mathbf{X} = 0$ 형태로 정리할 수 있다. 데이터에 노이즈가 없는 경우 $\mathbf{A} \in \mathbb{R}^{4\times4}$의 rank가 3이 되어 Null Space 벡터를 통해 유일한 $\mathbf{X}$를 계산할 수 있지만 노이즈가 존재하는 경우 $\mathbf{A}$의 rank는 4가 되어 무수히 많은 해가 존재한다. 따라서 $\mathbf{A}$를 특이값 분해(SVD)하여 $\|\mathbf{X}\| = 1$일 때 $\|\mathbf{A}\mathbf{X}\|$의 크기를 최소화하는 근사해 $\hat{\mathbf{X}}$를 계산해야 한다.

선형 방법에 Homography를 적용해보면

$$\begin{aligned} y(\mathbf{p}_{3,row}) - \mathbf{p}_{2,row} \rightarrow y(\mathbf{p}\mathbf{H}^{-1})_{3,row} - (\mathbf{p}\mathbf{H}^{-1})_{2,row} \\ = y(\mathbf{p}_{3,row})\mathbf{H}^{-1} - \mathbf{p}_{2,row}\mathbf{H}^{-1} \\ = (y(\mathbf{p}_{3,row}) - \mathbf{p}_{2,row})\mathbf{H}^{-1} \end{aligned} \tag{324}$$

과 같이 $\mathbf{A} \Rightarrow \mathbf{A}\mathbf{H}^{-1}$로 변환되어 $\mathbf{X} \Rightarrow \mathbf{X}_H$로 변환이 된다. 이를 통해

$$\|\mathbf{AX}\| = \|\mathbf{AH}^{-1}\mathbf{X}_H\| \tag{325}$$

이 성립하지만 **Homography 변환을 할 때 $\|\mathbf{X}_H\| \neq 1$이 되어 $\|\mathbf{X}\| = 1$의 성질을 보존하지 않는다. 따라서 선형방정식을 통한 방법은 Homography에 대해 불변이 아니므로 최적의 솔루션이 아니다.**

Inhomogeneous method

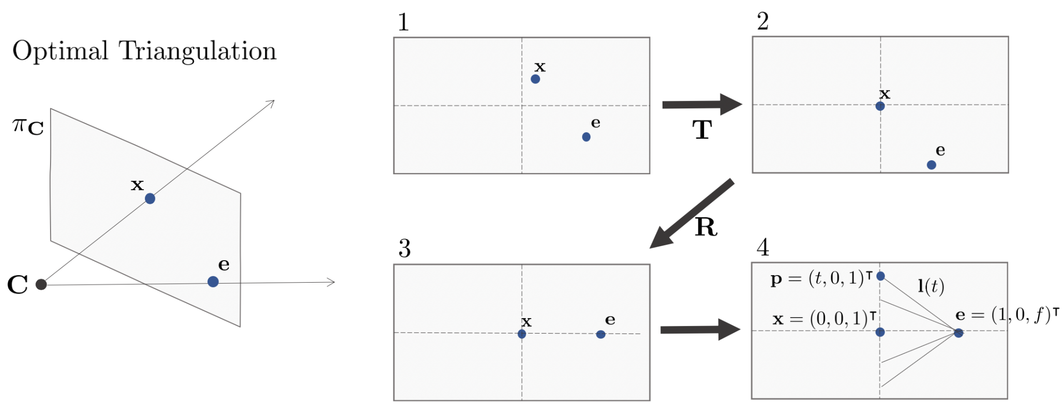
카메라 행렬 대응쌍 $\mathbf{P}, \mathbf{P}'$이 Affine 변환까지(up to affinity) 결정되는 경우 $\mathbf{X} = (X,\ Y,\ Z,\ 1)^\intercal$로 놓고 $\|\mathbf{AX}\|$를 최소화하는 근사해 $\hat{\mathbf{X}}$를 구하는 방법을 Inhomogeneous method라고 한다. $\mathbf{X}$는 Affine Point이므로 $\|\mathbf{X}\| = 1$ 제약조건이 사라지게 된다. 따라서, **Inhomogeneous method는 임의의 Affine 변환 $\mathbf{H}_A$에 대해 불변인 특성을 지닌다.**

An optimial solution

해당 섹션에서는 최적의 $\mathbf{X} \in \mathbb{P}^3$을 구하기 위한 Optimal Triangulation 방법에 대해 설명한다. 우선 비선형 최소제곱법(non-linear least square)을 통해 $\mathbf{X}$를 구하는 방법이 있다.

- 주어진 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}')$와 대응점 쌍 $(\mathbf{x}, \mathbf{x}')$에 대해여 이전 섹션에서 설명한 선형방정식 $\mathbf{AX} = 0$을 통해 구한 근사해 $\hat{\mathbf{X}}$를 초기값 $\mathbf{X}_0$로 설정한다.
- $\mathbf{X}_0$를 각각 카메라에 프로젝션하면 $\hat{\mathbf{x}}, \hat{\mathbf{x}}'$가 생성되는데 이 때 $d(\mathbf{x}, \hat{\mathbf{x}})^2 + d(\mathbf{x}', \hat{\mathbf{x}}')^2$를 최소화한다. $d(\mathbf{x}_1, \mathbf{x}_2)$은 $\mathbf{x}_1$과 $\mathbf{x}_2$ 사이의 거리를 의미한다.
- Gauss-Newton 또는 Levenberg-Marquardt와 같은 비선형 최소제곱법을 통해 반복적으로(iterative) 최적해 $\mathbf{X}^*$를 구한다.

Reformulation of the minimization problem



반복법을 사용하지 않고도(non-iterative) Optimal Triangulation을 수행할 수 있다. 해당 방법은 **$d(\mathbf{x}, \hat{\mathbf{x}})$를 최적화하는 대신 $\mathbf{x}$와 파라미터화된 Epipolar Line $\mathbf{l}(t)$ 사이의 거리 $d(\mathbf{x}, \mathbf{l}(t))$를 최적화한다. 즉, $\mathbf{x}$와 $\mathbf{l}(t)$ 사이의 거리를 최소화하는 파라미터 t를 구함으로써 최적해를 구한다.**

$$\min_t d(\mathbf{x}, \mathbf{l}(t))^2 + d(\mathbf{x}', \mathbf{l}'(t))^2 \tag{326}$$

Details of the minimization

우선 $\mathbf{x} = (x, y, 1)^\intercal$와 $\mathbf{x}' = (x', y', 1)^\intercal$를 각각 원점 $(0, 0, 1)^\intercal$로 변환시킨다.

$$\begin{aligned} \mathbf{x} &= (x, y, 1)^\intercal \rightarrow (0, 0, 1)^\intercal \\ \mathbf{x}' &= (x', y', 1)^\intercal \rightarrow (0, 0, 1)^\intercal \end{aligned} \tag{327}$$

이를 변환해주는 행렬을

$$\mathbf{T} = \begin{bmatrix} 1 & & -x \\ & 1 & -y \\ & & 1 \end{bmatrix} \quad \mathbf{T}' = \begin{bmatrix} 1 & & -x' \\ & 1 & -y' \\ & & 1 \end{bmatrix} \tag{328}$$

로 설정한다. 다음으로 Epipole $\mathbf{e} = (e_1, e_2, e_3)^\intercal$, $\mathbf{e}' = (e'_1, e'_2, e'_3)^\intercal$을 각각 x축 상의 점인 $(1, 0, f)^\intercal, (1, 0, f')^\intercal$로 변환한다.

$$\begin{aligned}\mathbf{e} = (e_1, e_2, e_3)^\intercal \rightarrow (1, 0, f)^\intercal \\ \mathbf{e}' = (e'_1, e'_2, e'_3)^\intercal \rightarrow (1, 0, f')^\intercal\end{aligned} \tag{329}$$

이는 이미지 평면에서 각각 $(1/f, 0)^\intercal, (1/f', 0)^\intercal$을 의미한다. 우선 $e_1^2 + e_2^2 = e_1'^2 + e_2'^2 = 1$이 되도록 Epipole을 정규화한 다음, 이를 x축 상의 점으로 회전하는 행렬을 $\mathbf{R}, \mathbf{R}'$ 다음과 같이 정의한다.

$$\mathbf{R} = \begin{bmatrix} e_1 & e_2 & \\ -e_2 & e_1 & \\ & & 1 \end{bmatrix} \quad \mathbf{R}' = \begin{bmatrix} e'_1 & e'_2 & \\ -e'_2 & e'_1 & \\ & & 1 \end{bmatrix} \tag{330}$$

이를 통해 $\mathbf{Re} = (1, 0, e_3)^\intercal, \mathbf{R}'\mathbf{e}' = (1, 0, e'_3)^\intercal$로 변환한다. 이 때, $e_3 = f, e'_3 = f'$이다. 다음으로 Epipolar Line $\mathbf{l}$을 $\mathbf{l}(t)$로 파라미터화한다. Epipole $\mathbf{e} = (1, 0, f)^\intercal$는 x축 선상에 위치한 점이므로 이를 지나는 Epipolar Line은 y축을 기준으로 $(0, t, 1)^\intercal$과 같이 파라미터화할 수 있다. 따라서 $\mathbf{l}(t)$는

$$\begin{aligned}\mathbf{l}(t) &= \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ f \end{pmatrix} \\ &= \begin{pmatrix} tf \\ 1 \\ -t \end{pmatrix}\end{aligned} \tag{331}$$

가 된다. 이전 단계에서 $\mathbf{x}$를 원점 $(0, 0, 1)^\intercal$로 옮겼으므로 $d(\mathbf{x}, \mathbf{l}(t))$는

$$d(\mathbf{x}, \mathbf{l}(t)) = \frac{t^2}{1^2 + (tf)^2} \tag{332}$$

이 된다. 다음으로 $\mathbf{l}'(t)$를 계산해야 한다. Fundmamental Matrix $\mathbf{F}$를 사용하여 $\mathbf{l}'(t)$를 계산하면

$$\begin{aligned}\mathbf{l}'(t) = \mathbf{F} \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} \\ \because \mathbf{l}(t) \text{ is the epipolar line of } \begin{pmatrix} 0 & t & 1 \end{pmatrix}^\intercal.\end{aligned} \tag{333}$$

과 같다. $\mathbf{F}$는 주어진 카메라 행렬 대응쌍에서 $\mathbf{F}_0$를 계산한 후 $\mathbf{F}_0$로부터 $\mathbf{T}, \mathbf{R}, \mathbf{T}', \mathbf{R}'$을 이용하여 계산할 수 있다. Epipole과 $\mathbf{F}$ 사이에는

$$\begin{aligned}\mathbf{F} \begin{pmatrix} 1 \\ 0 \\ f \end{pmatrix} = 0 \\ \begin{pmatrix} 1 & 0 & f' \end{pmatrix} \mathbf{F} = 0\end{aligned} \tag{334}$$

이 성립하므로 $\mathbf{F}_{1,col} = -f\mathbf{F}_{3,col}$ 그리고 $\mathbf{F}_{1,row} = -f'\mathbf{F}_{3,row}$가 되고 이를 정리하면

$$\mathbf{F} = \begin{bmatrix} f'fd & -f'c & -f'd \\ -fb & a & b \\ -fd & c & d \end{bmatrix} \tag{335}$$

가 된다. 이를 통해 $\mathbf{l}'(t)$를 다시 표현하면

$$\begin{aligned}\mathbf{l}'(t) = \mathbf{F} \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} &= t\mathbf{F}_{2,col} + \mathbf{F}_{3,col} \\ &= \begin{pmatrix} -f'(ct + d) \\ at + b \\ ct + d \end{pmatrix}^\intercal\end{aligned} \tag{336}$$

이 된다. 다음으로 원점 $\mathbf{x}'$와 $\mathbf{l}'(t)$ 사이의 거리 $d(\mathbf{x}', \mathbf{l}'(t))$를 구하면

$$d(\mathbf{x}', \mathbf{l}'(t))^2 = \frac{(ct + d)^2}{(at + b)^2 + f'^2(ct + d)^2} \tag{337}$$

이 된다. 이를 통해 최적화하고자 하는 목적함수 $d(\mathbf{x}, \mathbf{l}(t))^2 + d(\mathbf{x}', \mathbf{l}'(t))^2$는

$$s(t) = \frac{t^2}{1 + f^2 t^2} + \frac{(ct + d)^2}{(at + b)^2 + f'^2 (ct + d)^2} \tag{338}$$

가 되고 이를 미분하여 $0(s'(t) = 0)$이 되는 t를 찾으면 총 6개의 $t_i,\ i = 1, \cdots, 6$이 나오고 이 때 $s(t_i)$ 값을 비교하여 최소가 되는 t_i를 찾는다. 이렇게 찾은 t_{min} 값을 사용한 $\mathbf{l}(t_{\text{min}}), \mathbf{l}'(t_{\text{min}})$가 최적의 Epipolar Line이 된다. 다음으로

$$\begin{aligned} \hat{\mathbf{x}} &= \mathbf{l}(t_{\text{min}}) \\ \hat{\mathbf{x}}' &= \mathbf{l}'(t_{\text{min}}) \\ \hat{\mathbf{x}} &\leftarrow \mathbf{T}^{-1}\mathbf{R}^{-1}\hat{\mathbf{x}} \\ \hat{\mathbf{x}}' &\leftarrow \mathbf{T}'^{-1}\mathbf{R}'^{-1}\hat{\mathbf{x}}' \end{aligned} \tag{339}$$

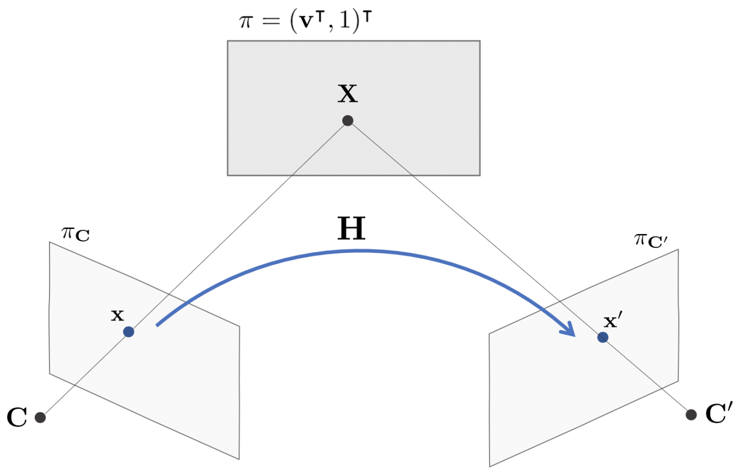
순서대로 변환하기 전 원래의 $\hat{\mathbf{x}}, \hat{\mathbf{x}}'$을 복원한 다음

$$\begin{aligned} \hat{\mathbf{x}} \times \mathbf{P}\hat{\mathbf{X}} &= 0 \\ \hat{\mathbf{x}}' \times \mathbf{P}\hat{\mathbf{X}}' &= 0 \end{aligned} \tag{340}$$

식을 **$\mathbf{A}\hat{\mathbf{X}} = 0$ 꼴로 정리하여 최종적으로 특이값 분해(SVD)를 통해 최적해 $\hat{\mathbf{X}}$를 구한다. 이와 같이 $\mathbf{X}$의 근사해를 구하는 방법을 Optimal Triangulation 방법이라고 한다.**

10 Scene planes and homographies

Homographies given the plane and vice versa



해당 섹션에서는 월드 상의 평면 π이 주어졌을 때 이를 사용하여 하나의 이미지 평면에서 다른 이미지 평면으로 가는 Homography 변환에 대해 설명한다.

Result 13.1

Canonical Form으로 변환한 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}')$과 월드 상의 평면 π 다음과 같이 주어졌을 때

$$\begin{aligned} \mathbf{P} &= [\mathbf{I} \mid 0] \\ \mathbf{P}' &= [\mathbf{A} \mid \mathbf{a}] \\ \pi &= (\mathbf{v}^\intercal, 1)^\intercal \quad \mathbf{v} \in \mathbb{R}^3 \end{aligned} \tag{341}$$

이 때, Homography $\mathbf{H}$는

$$\mathbf{H} = \mathbf{A} - \mathbf{a}\mathbf{v}^\intercal \tag{342}$$

로 주어진다.

Proof

두 카메라의 이미지 평면 $\pi_{\mathbf{P}}, \pi_{\mathbf{P}'}$가 주어졌을 때, 월드 평면 상의 점 $\mathbf{X} \in \pi$를 $\pi_{\mathbf{P}}$ 평면으로 프로젝션한 점 $\mathbf{x}$는 다음과 같다.

$$\begin{aligned} \mathbf{PX} &= [\mathbf{I} \mid 0] \begin{bmatrix} \tilde{\mathbf{X}} \\ 1 \end{bmatrix} \\ &= \tilde{\mathbf{X}} = \mathbf{x} \end{aligned} \tag{343}$$

따라서 임의의 스칼라 ρ에 대해 $\tilde{\mathbf{X}} = \rho^{-1}\mathbf{x}$가 성립한다. $\mathbf{X}$를 다시 쓰면 $\mathbf{X} = \begin{bmatrix} x & \rho \end{bmatrix}^\intercal$이 되고 $\mathbf{X}$는 π 평면 위의 점이므로

$$\begin{bmatrix} \mathbf{v}^\intercal & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \rho \end{bmatrix} = 0 \tag{344}$$

이 성립한다. 위 식을 풀면 $\rho = -\mathbf{v}^\intercal\mathbf{x}$이 된다. $\mathbf{X}$를 $\pi_{\mathbf{P}'}$에 프로젝션하면

$$\begin{aligned} \mathbf{x}' = \mathbf{P}'\mathbf{X} &= [\mathbf{A} \mid \mathbf{a}] \begin{bmatrix} \mathbf{x} \\ -\mathbf{v}^\intercal\mathbf{x} \end{bmatrix} \\ &= \mathbf{Ax} - \mathbf{av}^\intercal\mathbf{x} \\ &= (\mathbf{A} - \mathbf{av}^\intercal)\mathbf{x} \end{aligned} \tag{345}$$

이 되므로 따라서 $\mathbf{H} = \mathbf{A} - \mathbf{av}^\intercal$이 된다.

A calibrated stereo rig

캘리브레이션 된 스테레오 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}')$가 다음과 같이 주어지고

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{I} \mid 0] \\ \mathbf{P}' &= \mathbf{K}'[\mathbf{R} \mid \mathbf{t}] \end{aligned} \tag{346}$$

월드 상의 평면 $\pi = (\mathbf{n}^\intercal, d)^\intercal$이 주어진 경우, π 평면 상의 점 $\mathbf{X}$는

$$\mathbf{n}^\intercal\tilde{\mathbf{X}} + d = 0 \tag{347}$$

이 성립한다. 이를 다시 정리하면 $-\frac{\mathbf{n}^\intercal\tilde{\mathbf{X}}}{d} = 1$이 된다. $\mathbf{X}$를 이미지 평면 $\pi_{\mathbf{P}}$ 상에 프로젝션한 점 $\mathbf{x}$는 다음과 같이 나타낼 수 있다.

$$\begin{aligned} \mathbf{x} = \mathbf{PX} &= \mathbf{K}[\mathbf{I} \mid 0] \begin{bmatrix} \tilde{\mathbf{X}} \\ 1 \end{bmatrix} \\ &= \mathbf{K}\tilde{\mathbf{X}} \end{aligned} \tag{348}$$

따라서 $\tilde{\mathbf{X}} = \mathbf{K}^{-1}\mathbf{x}$가 되고 다음으로 $\mathbf{X}$를 이미지 평면 $\pi_{\mathbf{P}'}$에 프로젝션한 점 $\mathbf{x}'$는

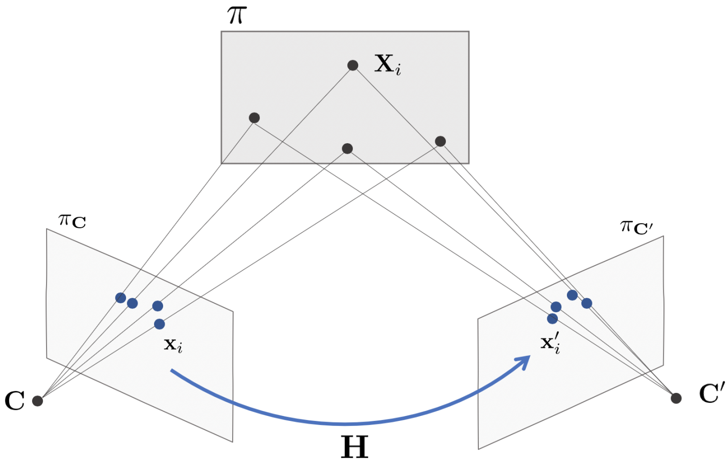
$$\begin{aligned} \mathbf{x}' = \mathbf{P}'\mathbf{X} &= \mathbf{K}'[\mathbf{R} \mid \mathbf{t}] \begin{bmatrix} \tilde{\mathbf{X}} \\ 1 \end{bmatrix} \\ &= \mathbf{K}'\mathbf{R}\tilde{\mathbf{X}} + \mathbf{K}'\mathbf{t}(-\frac{\mathbf{n}^\intercal\tilde{\mathbf{X}}}{d}) \\ &= \mathbf{K}'(\mathbf{R} - \frac{\mathbf{t}\mathbf{n}^\intercal}{d})\tilde{\mathbf{X}} \\ &= \mathbf{K}'(\mathbf{R} - \frac{\mathbf{t}\mathbf{n}^\intercal}{d})\mathbf{K}^{-1}\mathbf{x} \end{aligned} \tag{349}$$

가 되어 결론적으로 캘리브레이션된 스테레오 카메라에서 Homography $\mathbf{H}$는

$$\mathbf{H} = \mathbf{K}'(\mathbf{R} - \frac{\mathbf{t}\mathbf{n}^\intercal}{d})\mathbf{K}^{-1} \tag{350}$$

가 된다.

Homographies compatible with epipolar geometry



월드 상의 평면 π 위에 존재하는 4개의 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i), \ i = 1, \cdots, 4$가 주어졌다고 가정해보자. 4개의 대응점 쌍으로부터 8개의 제약조건을 얻을 수 있으며 이를 통해 Homograpy $\mathbf{H}$를 유일하게 결정할 수 있다.

다음으로 월드 상에 서로 일직선으로 있지 않은(no three are collinear) 임의의 4개의 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i), i = 1, \cdots, 4$가 주어졌다고 가정해보자. **이 때, 한 이미지 평면 $\pi_{\mathbf{P}}$에서 다른 이미지 평면 $\pi_{\mathbf{P}'}$으로 점들을 변환하는 Homography $\mathbf{H}$가 존재하게 되는데 이러한 $\mathbf{H}$가 두 카메라 사이의 Fundamental Matrix $\mathbf{F}$와 서로 호환(compatible)이 되기 위해서는 $\mathbf{H}$가 월드 평면 상의 평면 π에 대한 Homography 변환이어야 한다. 다시 말하면, $\mathbf{H}$가 Epipolar Geometry를 따르기 위해서는 $\mathbf{H}$가 π에 대한 Homography이어야 한다.** Epipolar Geometry에 의해 $\mathbf{x}$가 $\mathbf{Hx}$와 대응이 되기 위한 조건은

$$(\mathbf{Hx})^\intercal \mathbf{F} \mathbf{x} = 0 \tag{351}$$

이다. 즉, $\mathbf{x}^\intercal \mathbf{H}^\intercal \mathbf{F} \mathbf{x} = 0$을 만족해야 하므로 $\mathbf{H}^\intercal \mathbf{F}$가 반대칭행렬(skew-symmetric)이어야 한다.

$$\mathbf{H}^\intercal \mathbf{F} + \mathbf{F}^\intercal \mathbf{H} = 0 \tag{352}$$

위 조건은 $\mathbf{H}$가 Epipolar Geometry를 따르기 위한 필요충분조건이다.

Result 13.3

일반적으로 $\mathbf{F} = \mathbf{e}'^\wedge \mathbf{A}$와 같이 주어졌을 때 $\mathbf{H}$는

$$\mathbf{H} = \mathbf{A} - \mathbf{e}' \mathbf{v}^\intercal \tag{353}$$

이 된다. 이 때, $\mathbf{H}$는 $\mathbf{v} \in \mathbb{R}^3$로부터 3자유도를 가진다.

Proof

$\mathbf{F} = \mathbf{e}'^\wedge \mathbf{A}$와 같이 주어진 경우 Projective Reconstruction을 수행하면 두 카메라 행렬 대응쌍을

$$\begin{aligned} \mathbf{P} &= [\mathbf{I} \mid 0] \\ \mathbf{P}' &= [\mathbf{A} \mid \mathbf{e}'] \end{aligned} \tag{354}$$

과 같이 구할 수 있다. 월드 상의 평면 $\pi = (\mathbf{v}^\intercal, 1)^\intercal$이 주어졌을 때 앞의 정리에 의해

$$\mathbf{H} = \mathbf{A} - \mathbf{e}' \mathbf{v} \tag{355}$$

와 같이 구할 수 있다. 이를 $\mathbf{F}^\intercal \mathbf{H}$에 대입하면

$$\begin{aligned} \mathbf{F}^\intercal \mathbf{H} &= -\mathbf{A}^\intercal \mathbf{e}'^\wedge (\mathbf{A} - \mathbf{e}'^\intercal \mathbf{v}) \\ &= -\mathbf{A}^\intercal \mathbf{e}'^\wedge \mathbf{A} \quad \because \mathbf{e}'^\wedge \mathbf{e}' = 0 \end{aligned} \tag{356}$$

와 같이 반대칭행렬(skew-symmetric)이 된다.

Corollary

두 카메라 사이의 임의의 Homography $\mathbf{H}$가 Epipolar Geometry를 따르기 위해서는 $\mathbf{H}$가 월드 상의 평면 π에 대한 Homography이어야 한다. $\mathbf{H}$가 월드 상의 평면 π에 대한 Homography이기 위한 필요충분조건은 다음과 같다.

$$\mathbf{F} = \mathbf{e}'^{\wedge}\mathbf{H} \tag{357}$$

Proof

$(\Rightarrow)$ $\mathbf{H}$가 월드 상의 평면 $\pi = (\mathbf{v}^\intercal, 1)^\intercal$에 대한 Homography이면서 $\mathbf{F} = \mathbf{e}'^{\wedge}\mathbf{A}$일 때 앞서 정리에 의해

$$\mathbf{H} = \mathbf{H}_{\begin{pmatrix}\mathbf{v}\\1\end{pmatrix}} = \mathbf{A} - \mathbf{e}'^\intercal\mathbf{v} \tag{358}$$

가 성립한다. 따라서

$$\begin{aligned}\mathbf{e}'^{\wedge}\mathbf{H} &= \mathbf{e}'^{\wedge}(\mathbf{A} - \mathbf{e}'^\intercal\mathbf{v}) \\ &= \mathbf{e}'^{\wedge}\mathbf{A} = \mathbf{F} \quad \because \mathbf{e}'^{\wedge}\mathbf{e}' = 0\end{aligned} \tag{359}$$

가 된다.

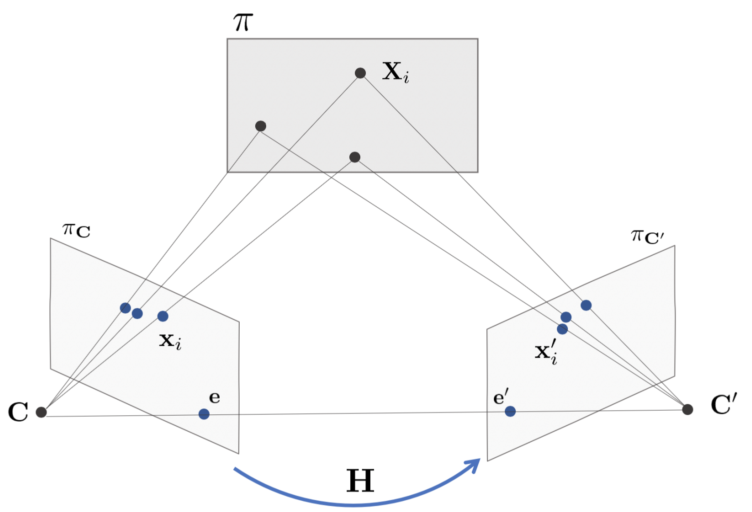
$(\Leftarrow)$ $\mathbf{F} = \mathbf{e}'^{\wedge}\mathbf{H}$인 경우 $\mathbf{H}$는 월드 상의 임의의 평면 π에 의한 Homography이다. 예를 들어, $\mathbf{H}$가 무한대 평면 $\pi_\infty = (\mathbf{0}^\intercal, 1)^\intercal$에 의한 Homography라고 하면

$$\mathbf{H} = \mathbf{H}_{\begin{pmatrix}\mathbf{0}\\1\end{pmatrix}} = \mathbf{H} - \mathbf{e}' \cdot 0 = \mathbf{H} \tag{360}$$

가 된다.

Plane induced homographies given F and image correspondences

Three points



두 개의 이미지 평면 $\pi_\mathbf{P}, \pi_{\mathbf{P}'}$에 세 개의 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i),\ i = 1,2,3$이 주어졌을 때 이들을 Back-projection한 월드 상의 점 $\mathbf{X}_i,\ i = 1,2,3$을 통해 월드 상의 평면 π를 유일하게 결정할 수 있고 이를 통한 Homography $\mathbf{H}$를 계산할 수 있다. 해당 섹션에서는 이러한 Homography $\mathbf{H}$를 구하는 방법에 대해 설명한다.

$\mathbf{H}$를 구하는 방법에는 크게 두 가지 방법이 존재한다. **첫 번째 방법은 월드 상의 평면 π를 직접 구하는 방법이 있다.** 세 개의 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i),\ i = 1,2,3$과 Fundamental Matrix $\mathbf{F} = \mathbf{e}'^{\wedge}\mathbf{A}$가 주어진 경우 이를 통해 카메라 행렬을 구할 수 있다(projective reconstrcution). 주어진 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i),\ i = 1,2,3$과 $\mathbf{F}$를 통해

$$
\begin{aligned}
\mathbf{P} &= [\mathbf{I} \mid 0] \\
\mathbf{P}' &= [\mathbf{A} \mid \mathbf{e}']
\end{aligned} \tag{361}
$$

을 사영모호성을 포함하여(up to projectivity) 결정할 수 있다. 그리고 $\mathbf{P}, \mathbf{P}'$를 사용하여 Back-projection 한 월드 상의 점 $\mathbf{X}_i,\ i = 1, 2, 3$ 또한 계산할 수 있다. 이 때, $\mathbf{X}_i,\ i = 1, 2, 3$은 동일한 직선 상에 존재하면 안 된다(not colinear). 다음으로 월드 상의 세 개의 점 $\mathbf{X}_i,\ i = 1, 2, 3$을 포함하는 월드 상의 평면 π를 유일하게 결정할 수 있고 $\pi = (\mathbf{v}^\intercal, d)^\intercal$일 때

$$
\mathbf{H} = \mathbf{A} - \mathbf{e}'\mathbf{v}^\intercal \tag{362}
$$

과 같이 π를 기반으로 $\pi_{\mathbf{P}} \leftrightarrow \pi_{\mathbf{P}'}$ 사이의 점들을 변환하는 Homography $\mathbf{H}$를 계산할 수 있다.

두 번째 방법은 $\mathbf{Hx}_i = \mathbf{x}'_i,\ \ i = 1, 2, 3$을 대수적으로 푸는 방법이다. 이 때 $\mathbf{x}'_i \times (\mathbf{Hx}_i) = 0$의 공식을 선형방정식의 $\mathbf{Ah} = 0$ 형태로 변형함으로써 Homography $\mathbf{H}$를 구할 수 있다. 하지만 Homography $\mathbf{H}$를 구하려면 총 네 개의 대응점 쌍이 필요한데 주어진 대응점 쌍은 세 개이므로 추가적으로 한 개의 대응점 쌍이 필요하다. 미리 알고 있는 Fundamental Matrix $\mathbf{F}$를 사용하여 두 이미지 평면의 Epipole $\mathbf{e}, \mathbf{e}'$을 구하고 이를 Back-projection함으로써 한 개의 대응점쌍을 추가하여 Homography $\mathbf{H}$를 계산할 수 있다. 단, 이 때 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i), i = 1, 2, 3$들이 Epipolar Line 위에 존재하면 안된다는 가정이 필요하다.

Result 13.6

Fundamental Matrix $\mathbf{F}$와 세 개의 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i), i = 1, 2, 3$이 주어졌을 때 이들을 통해 계산한 월드 상의 평면 π로 인한 Homography $\mathbf{H}$는

$$
\mathbf{H} = \mathbf{A} - \mathbf{e}'(\mathbf{M}^{-1}\mathbf{b})^\intercal \tag{363}
$$

와 같이 계산할 수 있다. 이 때,

$$
\begin{aligned}
\mathbf{M} &= \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \\
\mathbf{b} &= (\mathbf{x}'_i \times (\mathbf{Ax}_i))^\intercal (\mathbf{x}'_i \times \mathbf{e}') / \|\mathbf{x}'_i \times \mathbf{e}'\|
\end{aligned} \tag{364}
$$

Proof

Canonical 카메라 행렬 $\mathbf{P} = [\mathbf{I} \mid 0], \mathbf{P}' = [\mathbf{A} \mid \mathbf{e}']$일 때 Fundmaental Matrix $\mathbf{F} = \mathbf{e}'^\wedge \mathbf{A}$와 같이 구할 수 있고 이 때 $\mathbf{e}'^\wedge \mathbf{F} = \mathbf{e}'^\wedge \mathbf{e}'^\wedge \mathbf{A}$가 되어서

$$
\mathbf{e}'^\wedge \mathbf{F} = \mathbf{e}'^\wedge \mathbf{e}'^\wedge \mathbf{A} \sim \mathbf{A} \tag{365}
$$

와 같이 $\mathbf{A}$와 비례 관계가 된다. 따라서 $\mathbf{A} = \mathbf{e}'^\wedge \mathbf{F}$가 성립한다.

월드 상의 평면 $\pi = (\mathbf{v}^\intercal, d)^\intercal$이 주어졌을 때 임의의 Homography $\mathbf{H}$는 $\mathbf{H} = \mathbf{A} - \mathbf{e}'\mathbf{v}$와 같이 나타낼 수 있고 이 때 $\mathbf{v}$를 구하기 위해

$$
\mathbf{x}'_i \times \mathbf{Hx}_i = 0 \tag{366}
$$

식을 세운 후 정리하면

$$
\begin{aligned}
\mathbf{x}'_i \times \mathbf{Hx}_i &= 0 \\
\mathbf{x}'_i \times (\mathbf{Ax}_i - \mathbf{e}'\mathbf{v}^\intercal \mathbf{x}_i) &= 0 \\
\mathbf{x}'_i \times \mathbf{Ax}_i &= (\mathbf{x}_i \times \mathbf{e}')\mathbf{v}^\intercal \mathbf{x}_i
\end{aligned} \tag{367}
$$

꼴이 된다. 양변에 $(\mathbf{x}_i \times \mathbf{e}')^\intercal$을 곱한 후 정리하면

$$
\mathbf{x}_i^\intercal \mathbf{v} = \mathbf{b}_i \tag{368}
$$

형태가 나온다. 이를 세 개의 점에 대해 모두 적용하면

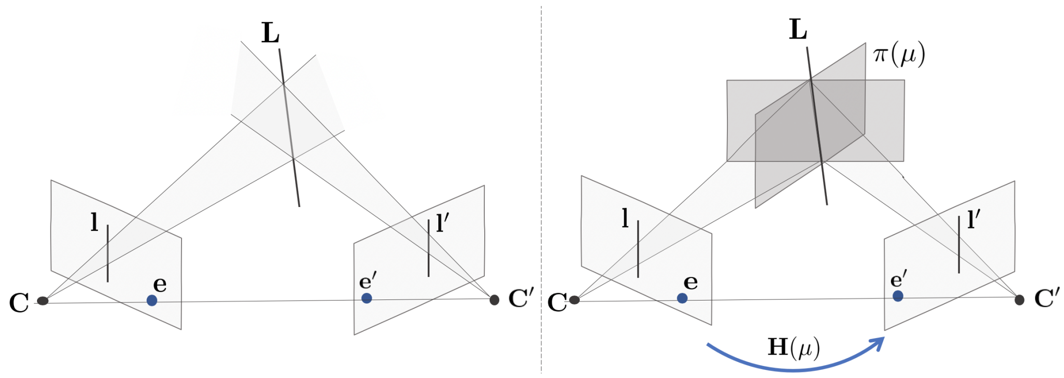
$$
\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \mathbf{v} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} \tag{369}
$$

가 된다 이 때, $\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}$를 행렬 $\mathbf{M}$으로 치환하면

$$\mathbf{M}\mathbf{v} = \mathbf{b} \tag{370}$$

꼴이 되고 이를 정리하면 **결론적으로 $\mathbf{v} = \mathbf{M}^{-1}\mathbf{b}$가 된다. 이에 따라 Homography $\mathbf{H}$는 $\mathbf{H} = \mathbf{A} - \mathbf{e}'(\mathbf{M}^{-1}\mathbf{b})^{\intercal}$과 같이 나타낼 수 있다.**

A point and line



해당 섹션에서는 월드 상에 하나의 선과 하나의 점이 주어졌을 때 Homography $\mathbf{H}$를 구하는 방법에 대해 설명한다.

Result 13.7

월드 상에 직선 $\mathbf{L}$에 대응하는 이미지 평면 상에 대응선 쌍 $\mathbf{l} \leftrightarrow \mathbf{l}'$이 주어졌을 때 월드 상의 직선 $\mathbf{L}$을 사용하여 $\mathbf{l}$을 $\mathbf{l}'$로 변환하는 Homography $\mathbf{H}$는

$$\mathbf{H}(\mu) = \mathbf{l}'^{\wedge}\mathbf{F} + \mu\mathbf{e}'\mathbf{l}^{\intercal} \tag{371}$$

과 같이 나타낼 수 있다. 이 때, $\mathbf{l}'^{\intercal}\mathbf{e}' \neq 0$이어야 하며 μ는 $\mu \in \mathbb{P}^1$인 파라미터이다.

Proof

카메라 행렬 대응쌍 $\mathbf{P} = [\mathbf{I} \mid 0], \mathbf{P}' = [\mathbf{A} \mid \mathbf{e}']$이 주어졌을 때 이미지 평면 상의 직선 $\mathbf{l}, \mathbf{l}'$을 Back-projection 한 평면 $\pi_{\mathbf{l}} = \mathbf{P}^{\intercal}\mathbf{l}, \pi_{\mathbf{l}'} = \mathbf{P}'^{\intercal}\mathbf{l}'$과 같이 나타낼 수 있다. 두 개의 Back-projection 평면들은 $\pi_{\mathbf{l}}$, $\pi_{\mathbf{l}'}$이 교차하는 선이 월드 상의 직선 $\mathbf{L}$이 되고 이 때 $\mathbf{L}$을 포함하는 평면 π는

$$\begin{aligned}\pi(\mu) &= \mu\mathbf{P}^{\intercal}\mathbf{l} + \mathbf{P}'^{\intercal}\mathbf{l}' \\ &= \mu\begin{pmatrix}\mathbf{l}\\ 0\end{pmatrix} + \begin{pmatrix}\mathbf{A}^{\intercal}\mathbf{l}'\\ \mathbf{e}'^{\intercal}\mathbf{l}'\end{pmatrix} \\ &= \begin{pmatrix}\mu\mathbf{l} + \mathbf{A}^{\intercal}\mathbf{l}'\\ \mathbf{e}'^{\intercal}\mathbf{l}'\end{pmatrix}\end{aligned} \tag{372}$$

과 같이 μ로 파라미터화하여 $\mathbf{L}$을 포함하는 무수한 평면을 나타낼 수 있다. Result 13.1로부터 평면 $\pi(\mu)$를 사용한 Homography $\mathbf{H}$는

$$\mathbf{H}(\mu) = \mathbf{A} - \mathbf{e}'\mathbf{v}(\mu)^{\intercal} \tag{373}$$

과 같이 나타낼 수 있고 이 때, $\mathbf{v}$는 위에서 설명한 $\pi(\mu)$ 공식에 따라 $\mathbf{v}(\mu) = (\mu\mathbf{l} + \mathbf{A}^{\intercal}\mathbf{l}')/(\mathbf{e}'^{\intercal}\mathbf{l}')$이 된다. 위 공식에 행렬 $\mathbf{A}$에 $\mathbf{A} = \mathbf{e}'^{\wedge}\mathbf{F}$를 대입하여 다시 정리하면

$$\mathbf{H}(\mu) = -(\mathbf{l}'^{\wedge}\mathbf{F} + \mu\mathbf{e}'\mathbf{l}^{\intercal})/(\mathbf{e}'^{\intercal}\mathbf{l}') \sim \mathbf{l}'^{\wedge}\mathbf{F} + \mu\mathbf{e}'\mathbf{l}^{\intercal} \tag{374}$$

이 된다. 따라서 **$\mathbf{H}$는 따라서 $\mathbf{H}(\mu) = \mathbf{l}'^{\wedge}\mathbf{F} + \mu\mathbf{e}'\mathbf{l}^{\intercal}$에 비례하므로 해당 식과 같이 쓸 수 있다.**

The homography for a corresponding point and line

앞서 설명한 대응선 쌍 $\mathbf{l} \leftrightarrow \mathbf{l}'$을 이용한 Homography $\mathbf{H}(\mu)$는 μ 값에 따라 Homography가 변하는 특징이 있다.

Result 13.8

하나의 대응선 쌍 $\mathbf{l} \leftrightarrow \mathbf{l}'$과 하나의 대응점 쌍 $\mathbf{x} \leftrightarrow \mathbf{x}'$이 주어지면 이를 통해 μ 값을 특정되어 다음과 같은 유일한 Homography $\mathbf{H}$가 도출된다.

$$\mathbf{H} = \mathbf{l}'^{\wedge}\mathbf{F} + \frac{(\mathbf{x}' \times \mathbf{e}')^{\intercal}(\mathbf{x}' \times ((\mathbf{F}\mathbf{x}) \times \mathbf{l}'))}{\|\mathbf{x}' \times \mathbf{e}'\|^2(\mathbf{l}^{\intercal}\mathbf{x})}\mathbf{e}'\mathbf{l}^{\intercal} \tag{375}$$

대응선 쌍 $\mathbf{l} \leftrightarrow \mathbf{l}'$를 사용하면 월드 상의 평면 $\pi(\mu)$를 통해 $\mathbf{H}(\mu) = \mathbf{l}'^{\wedge}\mathbf{F} + \mu\mathbf{e}'\mathbf{l}^{\intercal}$와 같이 Homography를 구할 수 있다. 이 때, $\mathbf{H}\mathbf{x} = \mathbf{x}'$ 식을 통해 $\mathbf{x}' \times (\mathbf{H}\mathbf{x}) = 0$ 식을 전개해보면

$$\mathbf{x}' \times (\mathbf{l}'^{\wedge}\mathbf{F}\mathbf{x} + \mu\mathbf{e}'\mathbf{l}^{\intercal}\mathbf{x}) = 0 \tag{376}$$

이 된다. 이를 전개 후 정리해보면

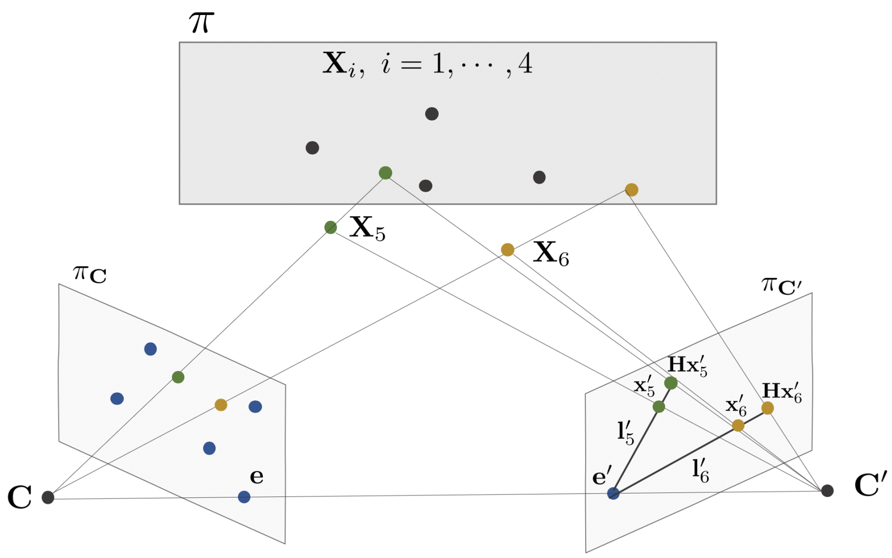
$$\begin{aligned}(\mathbf{x}' \times \mathbf{e}')\mathbf{l}^{\intercal}\mathbf{x} \cdot \mu &= -\mathbf{x}' \times \mathbf{l}' \times \mathbf{F}\mathbf{x} \\ &= \mathbf{x}' \times \mathbf{F}\mathbf{x} \times \mathbf{l}'\end{aligned} \tag{377}$$

이 되고 양변에 $(\mathbf{x}' \times \mathbf{e}')^{\intercal}$을 곱하여 μ에 대하여 정리 후 $\mathbf{H}(\mu)$식 에 넣어주면

$$\mathbf{H} = \mathbf{l}'^{\wedge}\mathbf{F} + \frac{(\mathbf{x}' \times \mathbf{e}')^{\intercal}(\mathbf{x}' \times ((\mathbf{F}\mathbf{x}) \times \mathbf{l}'))}{\|\mathbf{x}' \times \mathbf{e}'\|^2(\mathbf{l}^{\intercal}\mathbf{x})}\mathbf{e}'\mathbf{l}^{\intercal} \tag{378}$$

공식을 얻을 수 있다.

Computing F given the homography induced by a plane



일반적으로 두 개의 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}')$에 대한 Fundamental Matrix $\mathbf{F}$를 계산하기 위해서는 최소 8개의 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i),\ i = 1, \cdots, 8$이 필요하다. 하지만 **Scene Plane Homography를 이용하면 6개의 대응점 쌍만 사용해도 F를 계산할 수 있다.** 단, 6개의 대응점 쌍들 중 4개 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i),\ i = 1, \cdots, 4$에 해당하는 월드 상의 점들의 동일한 평면 π 위에 존재해야 한다는 제약조건이 있다.

6개의 대응점 쌍으로 Fundmaental Matrix $\mathbf{F}$를 계산하는 알고리즘의 순서는 다음과 같다.

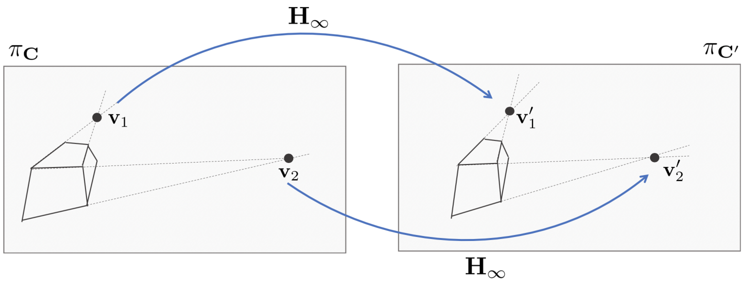
- 월드 상의 평면 π 위에 존재하는 4개의 점들 $\mathbf{X}_i,\ i = 1, \cdots, 4$를 프로젝션한 대응점 쌍들 $(\mathbf{x}_i, \mathbf{x}'_i),\ i = 1, \cdots, 4$를 사용하여 $\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$를 만족하는 Homorgaphy $\mathbf{H}$를 계산한다. 이 때 4쌍의 대응점은 Homography $\mathbf{H}$를 유일하게 결정한다.

- 남은 2개의 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i),\ i = 5, 6$을 사용하여 Epipole $\mathbf{e}'$를 구한다. $\mathbf{x}_5$를 Homography 변환한 $\mathbf{H}\mathbf{x}_5$와 $\mathbf{x}'_5$를 사용하여 두 점을 잇는 직선을 $(\mathbf{H}\mathbf{x}_5) \times \mathbf{x}'_5$와 같이 구한다. $\mathbf{x}_6$에 대해서도 마찬가지로 적용하여 직선 $(\mathbf{H}\mathbf{x}_6) \times \mathbf{x}'_6$를 구한 다음 두 직선이 교차하는 점을 구하면 그 점이 곧 Epipole $\mathbf{e}'$가 된다.

$$\mathbf{e}' = (\mathbf{H}\mathbf{x}_5) \times \mathbf{x}'_5 \cap (\mathbf{H}\mathbf{x}_6) \times \mathbf{x}'_6 \tag{379}$$

- Fundamental Matrix $\mathbf{F}$를 $\mathbf{F} = \mathbf{e}'^{\wedge}\mathbf{H}$를 통해 구한다.

The infinite homography $\mathbf{H}_\infty$



Definition 13.10

Scene Plane이 무한대 평면(plane at infinity) π_∞인 경우, 이 때 계산한 Homography를 Infinity Homography $\mathbf{H}_\infty$라고 한다.

카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}')$가 다음과 같이 주어졌을 때

$$\mathbf{P} = \mathbf{K}[\mathbf{I} \mid 0] \quad \mathbf{P}' = \mathbf{K}'[\mathbf{R} \mid \mathbf{t}] \tag{380}$$

월드 상의 평면 $\pi(d) = \begin{pmatrix} \mathbf{v} \\ d \end{pmatrix}$에 대해 Homography는 다음과 같이 구할 수 있다.

$$\mathbf{H}_{\pi(d)} = \mathbf{K}'(\mathbf{R} - \frac{\mathbf{t}\mathbf{n}^\intercal}{d})\mathbf{K}^{-1} \tag{381}$$

Inifinite Homography $\mathbf{H}_\infty$는 d가 무한대인 경우에 해당하므로

$$\mathbf{H}_\infty = \lim_{d \to \infty} \mathbf{H}_{\pi(d)} = \mathbf{K}'\mathbf{R}\mathbf{K}^{-1} \tag{382}$$

가 된다. 따라서 **$\mathbf{H}_\infty$는 $\mathbf{t}$에 의존하지 않고 오직 카메라의 회전에만 의존한다는 것을 알 수 있다.**

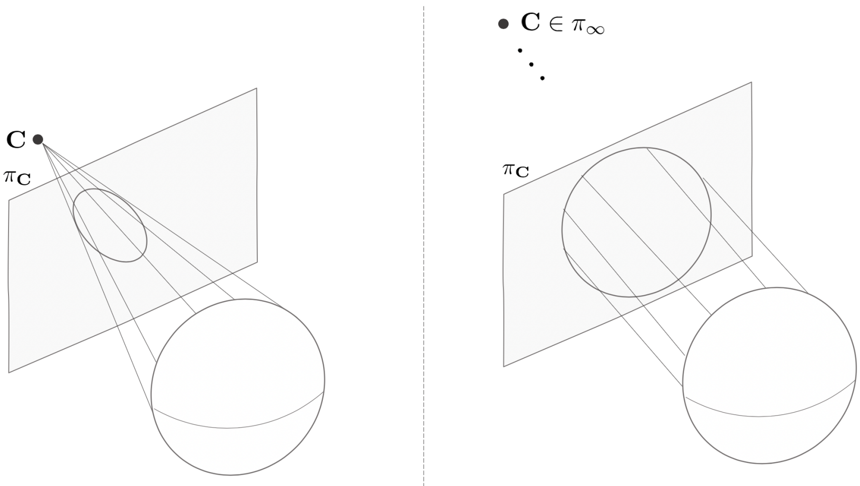
Affine and metric reconstruction

만약 월드좌표계가 Affine인 경우 π_∞는 $\pi_\infty = (0, 0, 0, 1)^\intercal$과 같이 나타낼 수 있고 두 카메라 행렬이 $\mathbf{P} = [\mathbf{M} \mid \mathbf{m}],\ \mathbf{P}' = [\mathbf{M}' \mid \mathbf{m}']$으로 주어진 경우 무한대 평면 상의 점 $\mathbf{X} \in \pi_\infty$를 각각 카메라에 프로젝션하면

$$\begin{aligned} \mathbf{x} &= \mathbf{P}\mathbf{X} = [\mathbf{M} \mid \mathbf{m}] \begin{bmatrix} \tilde{\mathbf{X}} \\ 0 \end{bmatrix} = \mathbf{M}\tilde{\mathbf{X}} \\ \mathbf{x}' &= \mathbf{P}'\mathbf{X} = [\mathbf{M}' \mid \mathbf{m}'] \begin{bmatrix} \tilde{\mathbf{X}} \\ 0 \end{bmatrix} = \mathbf{M}'\tilde{\mathbf{X}} = (\mathbf{M}'\mathbf{M}^{-1})\mathbf{x} \end{aligned} \tag{383}$$

따라서 $\mathbf{x}' = (\mathbf{M}'\mathbf{M}^{-1})\mathbf{x}$가 되어 월드좌표계가 Affine일 때 두 카메라 사이의 Infinite Homography는 $\mathbf{H}_\infty = \mathbf{M}'\mathbf{M}^{-1}$가 된다.

11 Affine Epipolar Geometry



Pinhole Camera　　　Affine Camera

핀홀 카메라(pinhole camera)는 월드 상의 물체가 이미지 평면 상에 프로젝션될 때 초점이라 부르는 하나의 점을 통과하여 이미지 평면상에 투과되는 카메라 모델을 의미한다. 따라서 핀홀 카메라 모델은 사영공간(projective space) $\mathbb{P}^3 \mapsto \mathbb{P}^2$로 매핑하는 함수로 모델링 할 수 있고 원근법에 의한 왜곡이 발생한다. 이와 반대로 **Affine 카메라는 월드 상의 물체가 이미지 평면 상에 프로젝션 될 때 마치 무한대 광원에 의한 그림자 상이 맺히는 것과 같이 이미지 평면에 프로젝션되는 카메라를 의미한다. 따라서 Affine 카메라는 원근법에 의한 왜곡이 발생하지 않는다.**

Affine 카메라도 Canonical Form으로 나타낼 수 있다. 두 개의 Affine 카메라 $\mathbf{P}_A, \mathbf{P}'_A$가 주어졌을 때 월드 좌표계와 카메라 좌표계가 동일한 경우 $\mathbf{P}_A$의 주축(principal axis)는 Z축이 되고 이 때의 프로젝션은 XY 평면으로 프로젝션이 된다. 그리고 $\mathbf{P}'_A$는 Affine 변환 $\mathbf{M}$과 카메라 이동 $\mathbf{t}$를 통해 표현할 수 있다. **이 때, 두 개의 Affine 카메라는 모두 마지막 행(row)이 $(0,0,0,1)$인 성질을 지닌다.**

$$\mathbf{P}_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{P}'_A = \begin{bmatrix} \mathbf{M} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \quad \text{where, } \mathbf{M} \in \mathbb{R}^{2\times 3},\ \mathbf{t} \in \mathbb{R}^2 \tag{384}$$

Affine 카메라 행렬 $\mathbf{P}_A$는 Affine 변환의 특성 상 **무한대 평면(plane at infinity) $\pi_\infty = (X,Y,Z,0)^\intercal$ 상의 한 점을 무한대 직선(line at infinity) $\mathbf{l}_\infty$ 상의 한 점으로 변환한다. 즉, 월드 상 물체의 평행한 성질이 보존된다.**

$$\mathbf{P}_A \begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix} \in \mathbf{l}_\infty \tag{385}$$

일반적인 Affine 카메라 행렬 $\begin{bmatrix} \mathbf{M} & \mathbf{m} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{R}^{3\times 4}$에서 행렬 $\mathbf{M} \in \mathbb{R}^{2\times 3}$의 rank는 2이므로

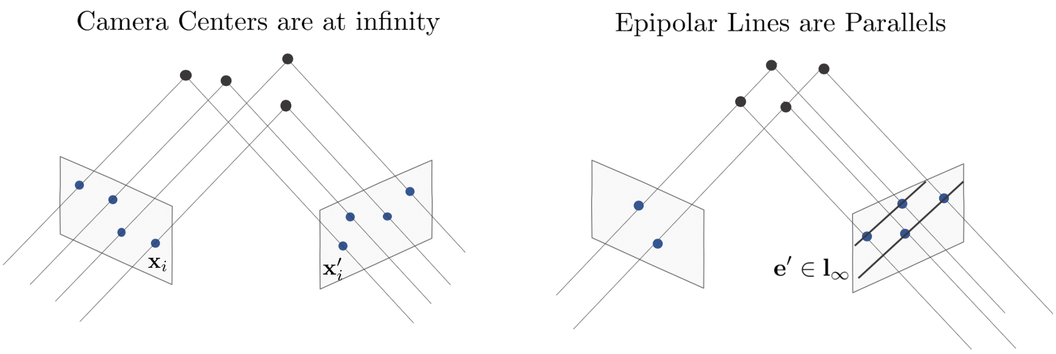
$$\mathbf{M}\tilde{\mathbf{C}} = 0 \tag{386}$$

을 만족하는 Affine 카메라의 중심점 $\mathbf{C}$이 존재한다. **이 때, $\mathbf{C}$는 무한대 평면 상에 존재한다.**

$$\mathbf{C} = \begin{pmatrix} \tilde{\mathbf{C}} \\ 0 \end{pmatrix} \in \pi_\infty \tag{387}$$

Affine 카메라 행렬 $\mathbf{P}_A$의 중심점 $\mathbf{C}$는 $\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\intercal$이다. **즉, Affine 카메라의 중심점은 주축(principal axis)의 방향과 동일하다.**

Affine epipolar geometry



Epipolar lines

두 개의 동일한 Affine 카메라가 주어졌을 때, 이들의 Back-projection 직선을 생각해보자. 첫 번째 Affine 카메라의 이미지 평면 상의 한 점 $\mathbf{x}$를 Back-projection하면

$$\mathbf{X}(\lambda) = \mathbf{P}_A^{\dagger}\mathbf{x} + \lambda \begin{bmatrix} \tilde{\mathbf{C}} \\ 0 \end{bmatrix} \tag{388}$$

이 되므로 Back-projection 직선들의 방향이 곧 Affine 카메라의 중심점 방향 $\begin{bmatrix} \tilde{\mathbf{C}} \\ 0 \end{bmatrix}$이 된다. **즉, Affine 카메라의 이미지 평면 상 모든 점들의 Back-projection 직선들은 평행하므로 이를 두 번째 Affine 카메라에 프로젝션한 Epipolar Line들 또한 모두 평행하다.**

The epipoles

Epipolar Line들이 모두 평행하므로 **Epipole들은 모두 무한대 직선(line at infinity) 상에 존재한다.**

The affine fundamental matrix

Result 14.1

두 개의 동일한 Affine 카메라가 주어졌을 때 이를 통해 Affine Fundamental Matrix $\mathbf{F}_A$를 정의할 수 있다. 이 때, $\mathbf{F}_A$는

$$\mathbf{F}_A = \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix} \tag{389}$$

꼴로 나타난다. $*$는 0이 아닌 값을 의미한다. 일반적으로 $\mathbf{F}_A$는

$$\mathbf{F}_A = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & d & e \end{bmatrix} \tag{390}$$

와 같이 작성한다. $\mathbf{F}_A$는 일반적인 Fundamental Matrix와 동일하게 rank 2를 가진다.

Derivation

Geometric derivation

두 개의 Affine 카메라 행렬 $\mathbf{P}_A, \mathbf{P}'_A$가 주어졌을 때 둘 사이에는 Affine 변환의 성질에 의해 평행한 선들이 보존된다. 따라서 두 이미지 평면의 Homography $\mathbf{H}_A$는 Affine Homography가 되어

$$\mathbf{x}' = \mathbf{H}_A\mathbf{x} \quad \text{where, } \mathbf{H}_A = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{bmatrix} \tag{391}$$

을 만족한다. Affine 카메라의 Epipole $\mathbf{e}'$은 무한대 직선(line at infinity) 상에 존재하므로 $\mathbf{e}'$의 Cross Product는

$$\mathbf{e}'^{\wedge} = \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{bmatrix} \tag{392}$$

꼴이다. Affine Fundamental Matrix $\mathbf{F}_A$는 $\mathbf{F}_A = \mathbf{e}'^{\wedge}\mathbf{H}_A$를 통해 계산할 수 있으므로

$$\begin{aligned} \mathbf{F}_A &= \mathbf{e}'^{\wedge}\mathbf{H}_A \\ &= \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix} \end{aligned} \tag{393}$$

와 같은 형태가 된다.

Properties

The epipoles

Affine Fundamental Matrix $\mathbf{F}_A$가 주어졌을 때 이를 통해 Epipole $\mathbf{e}, \mathbf{e}'$를 계산할 수 있다. $\mathbf{F}_A$가

$$\mathbf{F}_A = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & d & e \end{bmatrix} \tag{394}$$

일 때 **Fundamental Matrix의 성질로 인해 $\mathbf{F}_A\mathbf{e} = 0$와 $\mathbf{e}'^{\intercal}\mathbf{F}_A = 0$가 성립하므로 다음과 같이 계산할 수 있다.**

$$\begin{aligned} \mathbf{e} &= \begin{bmatrix} -d & c & 0 \end{bmatrix}^{\intercal} \\ \mathbf{e}' &= \begin{bmatrix} -b & a & 0 \end{bmatrix}^{\intercal} \end{aligned} \tag{395}$$

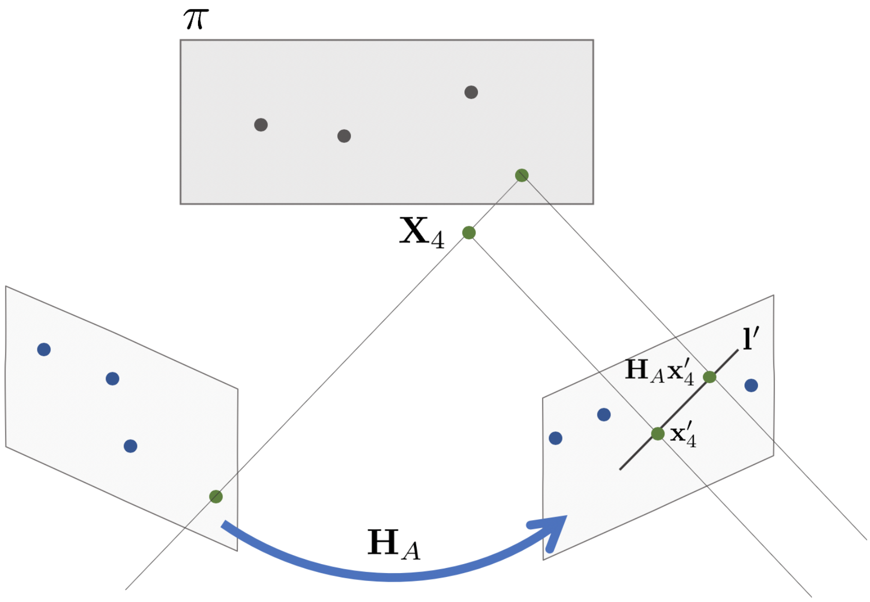
Epipolar lines

첫 번째 Affine 이미지 평면 상의 점 $\mathbf{x}$와 Affine Fundamental Matrix $\mathbf{F}_A$가 주어졌을 때 두 번째 Affine 이미지 평면 상의 Epipolar Line $\mathbf{l}'$는 다음과 같이 계산할 수 있다.

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x & y & 1 \end{pmatrix}^{\intercal} \\ \mathbf{F}_A &= \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & d & e \end{bmatrix} \\ \mathbf{l}' &= \mathbf{F}_A\mathbf{x} = \begin{pmatrix} a & b & cx + dy + e \end{pmatrix}^{\intercal} \end{aligned} \tag{396}$$

Epipolar Line $\mathbf{l}'$의 처음 두 개의 항 (a, b)가 이미지 평면 상의 점 $\mathbf{x} = (x, y, 1)^{\intercal}$에 독립적이므로 이는 곧 $\mathbf{x}$에 관계없이 Epipolar Line들이 평행하다는 것을 의미한다.

Estimating $\mathbf{F}_A$ from image point correspondences



Projective 카메라의 Fundamental Matrix $\mathbf{F}$를 계산하기 위해서는 두 이미지 평면 사이에 대응점 쌍 $(\mathbf{x}, \mathbf{x}')$가 최소 8개 이상이 필요하다. 하지만 Affine 카메라의 Affine Fundamental Matrix $\mathbf{F}_A$는 앞서 설명한 것과 같이 좌상단 2×2 항이 0이므로 최소 4개 이상의 대응점 쌍 $(\mathbf{x}, \mathbf{x}')$만으로도 $\mathbf{F}_A$를 구할 수 있다.

Algorithm 14.2

- **Objective:** Affine 이미지 평면에서 4개의 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i),\ i = 1, \cdots, 4$가 주어졌을 때 이를 통해 Affine Fundamental Matrix를 구한다.

- 우선 처음 세 개의 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i),\ i = 1, 2, 3$을 Back-projection함으로써 월드 상의 점 $\mathbf{X}_i,\ i = 1, 2, 3$을 구할 수 있고 이 점들을 Span함으로써 유일한 평면 π가 결정할 수 있다.

$$\pi = \text{Span}\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\} \tag{397}$$

- **월드 상의 평면 π에 대해 $\mathbf{x}' = \mathbf{H}_A\mathbf{x}$를 만족하는 Affine Homography $\mathbf{H}_A$를 구한다.** 일반적으로 Homography를 구하기 위해서는 네 개 이상의 대응점 쌍이 필요하지만 Affine Homography는 마지막 행이 항상 $(0, 0, 1)$이므로 세 개의 대응점 쌍을 통해서도 $\mathbf{H}_A$를 계산할 수 있다.

- 나머지 한 점 $\mathbf{x}_4$를 사용하여 $\mathbf{H}_A\mathbf{x}_4$와 $\mathbf{x}_4$를 잇는 **Epipolar Line $\mathbf{l}'$을 계산한다.**

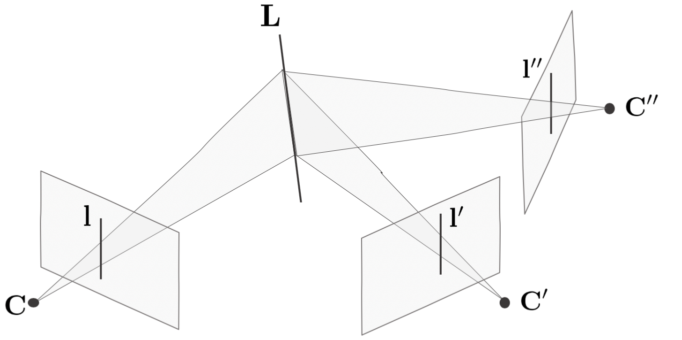
$$\mathbf{l}' = \mathbf{H}_A\mathbf{x}_4 \times \mathbf{x}'_4 \tag{398}$$

  Epipolar Line $\mathbf{l}'$를 계산했으면 이를 통해 Epipole $\mathbf{e}' = (-l'_2, l'_1, 0)^\intercal$를 계산할 수 있다. 만약 $\mathbf{x}_4$를 Back-projection한 월드 상의 점 $\mathbf{X}_4$가 월드 상의 평면 π 위에 존재하는 경우 Epipolar Line을 구할 수 없다.

- $\mathbf{F}_A = \mathbf{e}'^{\wedge}\mathbf{H}_A$를 통해 Affine Fundamental Matrix를 계산한다.

$$\mathbf{F}_A = \begin{bmatrix} -l'_2 & l'_1 & 0 \end{bmatrix}^{\intercal\wedge} \mathbf{H}_A \tag{399}$$

12 The Trifocal Tensor



Trifocal Tensor $\mathcal{T}$

$$\begin{aligned}\mathbf{l}^\intercal &= \mathbf{l}'^\intercal[\mathbf{T}_1,\ \mathbf{T}_2,\ \mathbf{T}_3]\mathbf{l}'' \\ &= \mathbf{l}'^\intercal\mathcal{T}\mathbf{l}''\end{aligned}$$

The geometric basis for the trifocal tensor

Three-View Geometry에서 Trifocal Tensor $\mathcal{T}$는 Two-View Geometry에서 Fundamental Matrix **F**와 유사한 역할을 한다. **F**와 유사하게 $\mathcal{T}$은 서로 다른 세 개의 카메라 이미지 평면들을 특별한 제약조건으로 구속시킨다.

Incidence relations for lines

월드 평면 상의 한 직선 **L**과 이를 바라보고 있는 서로 다른 세 개의 카메라가 주어졌다고 가정하자. 세 개의 카메라의 이미지 평면을 각각 $\pi_{\mathbf{P}}, \pi_{\mathbf{P}'}, \pi_{\mathbf{P}''}$이라고 하고 **L**을 이미지 평면 상에 프로젝션한 직선들은 $\mathbf{l}, \mathbf{l}', \mathbf{l}''$이라고 했을 때 $\mathbf{l} \leftrightarrow \mathbf{l}' \leftrightarrow \mathbf{l}''$ 사이의 관계를 알아보자.

카메라 행렬을 각각 $(\mathbf{P}, \mathbf{P}', \mathbf{P}'')$이라고 했을 때 이를 Canonical Form으로 나타내면

$$\begin{aligned}\mathbf{P} &= [\mathbf{I} \mid 0] \\ \mathbf{P}' &= [\mathbf{A} \mid \mathbf{a}_4] \\ \mathbf{P}'' &= [\mathbf{B} \mid \mathbf{b}_4]\end{aligned} \tag{400}$$

와 같이 항상 사영모호성을 포함하여(up to projectivity) 나타낼 수 있다. 월드 상의 직선 **L**은 $\mathbf{l}, \mathbf{l}', \mathbf{l}''$을 Back-projection한 평면들$\pi, \pi', \pi''$의 교차선이므로 $\mathbf{l}, \mathbf{l}', \mathbf{l}''$를 Back-projection 해보면

$$\begin{aligned}\pi &= \mathbf{P}^\intercal\mathbf{l} = \begin{pmatrix}\mathbf{l} \\ 0\end{pmatrix} \\ \pi' &= \mathbf{P}'^\intercal\mathbf{l}' = \begin{pmatrix}\mathbf{A}^\intercal\mathbf{l}' \\ \mathbf{a}_4^\intercal\mathbf{l}'\end{pmatrix} \\ \pi'' &= \mathbf{P}''^\intercal\mathbf{l}'' = \begin{pmatrix}\mathbf{B}^\intercal\mathbf{l}'' \\ \mathbf{b}_4^\intercal\mathbf{l}''\end{pmatrix}\end{aligned} \tag{401}$$

같은 평면들의 법선벡터(normal vector)를 얻을 수 있다. 이러한 세 평면의 법선벡터를 열벡터로 가지는 행렬 $\mathbf{M} \in \mathbb{R}^{4\times 3}$이 다음과 같이 주어졌을 때

$$\mathbf{M} = \begin{bmatrix}\pi & \pi' & \pi''\end{bmatrix} \tag{402}$$

M의 열공간(column space)는 월드 상의 직선 **L**과 수직인 2차원 평면을 의미하므로 **M**의 rank는 2이다. 따라서

$$\mathbf{M} = \begin{bmatrix}\mathbf{m}_1 & \mathbf{m}_2 & \mathbf{m}_3\end{bmatrix} = \begin{bmatrix}\mathbf{l} & \mathbf{A}^\intercal\mathbf{l}' & \mathbf{B}^\intercal\mathbf{l}'' \\ 0 & \mathbf{a}_4^\intercal\mathbf{l}' & \mathbf{b}_4^\intercal\mathbf{l}''\end{bmatrix} \tag{403}$$

에서 $\mathbf{m}_1 = \alpha\mathbf{m}_2 + \beta\mathbf{m}_3$가 성립한다. 이 때, 행렬 **M**의 두 번째 행에서 첫번째 열 $(2,1)$ 값이 0이므로 α, β를 구할 수 있다.

$$0 = k(\mathbf{b}_4^\intercal\mathbf{l}'')\mathbf{m}_2 - (k\mathbf{a}_4^\intercal\mathbf{l}')\mathbf{m}_3 \tag{404}$$

따라서 임의의 상수 k에 대하여 $\alpha = k(\mathbf{b}_4^\intercal\mathbf{l}'), \beta = k(\mathbf{a}_4^\intercal\mathbf{l}')$이 된다. 다음으로 행렬 **M**의 첫번째 행을 전개해보면

$$
\begin{aligned}
\mathbf{l} &= (\mathbf{b}_4^\intercal \mathbf{l}'')\mathbf{A}^\intercal \mathbf{l}' - (\mathbf{a}_4^\intercal \mathbf{l}')\mathbf{B}^\intercal \mathbf{l}'' \\
&= (\mathbf{l}''^\intercal \mathbf{b}_4)\mathbf{A}^\intercal \mathbf{l}' - (\mathbf{l}'^\intercal \mathbf{a}_4)\mathbf{B}^\intercal \mathbf{l}''
\end{aligned} \tag{405}
$$

이 된다. $\mathbf{a}_4^\intercal \mathbf{l}', \mathbf{b}_4^\intercal \mathbf{l}''$은 스칼라 값이므로 전치(transpose)를 취해도 같은 값을 나타낸다. 직선 $\mathbf{l}$의 i번째 좌표값을 l_i라고 하면

$$
\begin{aligned}
l_i &= \mathbf{l}''^\intercal (\mathbf{b}_4 \mathbf{a}_i^\intercal)\mathbf{l}' - \mathbf{l}'^\intercal (\mathbf{a}_4 \mathbf{b}_i^\intercal)\mathbf{l}'' \\
&= \mathbf{l}'^\intercal (\mathbf{a}_i \mathbf{b}_4^\intercal)\mathbf{l}'' - \mathbf{l}'^\intercal (\mathbf{a}_4 \mathbf{b}_i^\intercal)\mathbf{l}'' \\
&= \mathbf{l}'^\intercal (\mathbf{a}_i \mathbf{b}_4^\intercal - \mathbf{a}_4 \mathbf{b}_i^\intercal)\mathbf{l}''
\end{aligned} \tag{406}
$$

과 같이 나타낼 수 있다. 이 때 $\mathbf{a}_i \mathbf{b}_4^\intercal - \mathbf{a}_4 \mathbf{b}_i^\intercal$를 $\mathbf{T}_i$로 치환하면

$$
l_i = \mathbf{l}'^\intercal \mathbf{T}_i \mathbf{l}'' \tag{407}
$$

으로 간결하게 표현할 수 있다.

Definition 15.1

이 때, **행렬의 집합 $\{\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3\}$은 Trifocal Tensor $\mathcal{T}$을 행렬로 표현한 것을 의미한다. 이를 사용하여 직선 $\mathbf{l}$을 다시 표현해보면 다음과 같다.**

$$
\begin{aligned}
\mathbf{l}^\intercal &= \mathbf{l}'^\intercal \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3 \end{bmatrix} \mathbf{l}'' \\
&= (\mathbf{l}'^\intercal \mathbf{T}_1 \mathbf{l}'', \mathbf{l}'^\intercal \mathbf{T}_2 \mathbf{l}'', \mathbf{l}'^\intercal \mathbf{T}_3 \mathbf{l}'')
\end{aligned} \tag{408}
$$

Homographies induced by a plane

Result 15.2

서로 다른 세 카메라의 이미지 평면을 각각 $\pi_{\mathbf{P}}, \pi_{\mathbf{P}'}, \pi_{\mathbf{P}''}$ 하자. 두 번째 카메라 이미지 평면 상의 직선 $\mathbf{l}'$을 Back-projection하여 얻은 월드 상의 평면을 π'이라고 하면 π'을 통해 $\pi_{\mathbf{P}}$에서 $\pi_{\mathbf{P}''}$으로 변환하는 Homography $\mathbf{H}_{13}$가 존재한다. 해당 섹션에서는 $\mathbf{H}_{13}$를 $\mathbf{l}'$을 통해 기술하는 방법에 대해 설명한다.

첫 번째 이미지 평면 상의 직선 $\mathbf{l}$은 Trifocal Tensor에 의해

$$
\mathbf{l}^\intercal = \mathbf{l}'^\intercal \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3 \end{bmatrix} \mathbf{l}'' \tag{409}
$$

과 같이 나타낼 수 있고 이를 정리하여 $\mathbf{l} = \mathbf{H}_{13}^\intercal \mathbf{l}''$ 꼴로 나타내면

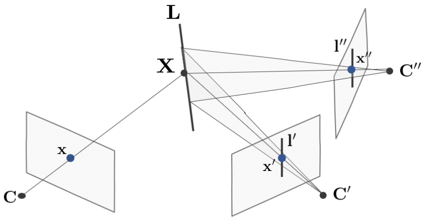
$$
\begin{aligned}
\mathbf{l}^\intercal &= \mathbf{l}'^\intercal \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3 \end{bmatrix} \mathbf{l}'' \\
&= (\begin{bmatrix} \mathbf{T}_1^\intercal & \mathbf{T}_2^\intercal & \mathbf{T}_3^\intercal \end{bmatrix} \mathbf{l}')^\intercal \mathbf{l}'' \\
&= \mathbf{H}_{13}^\intercal \mathbf{l}''
\end{aligned} \tag{410}
$$

이 된다. 따라서 $\mathbf{H}_{13} = \begin{bmatrix} \mathbf{T}_1^\intercal, & \mathbf{T}_2^\intercal & \mathbf{T}_3^\intercal \end{bmatrix} \mathbf{l}'$과 같다. $\mathbf{H}_{13}$을 사용하면 $\mathbf{x}'' = \mathbf{H}_{13}\mathbf{x}$와 같이 $\pi_{\mathbf{P}} \rightarrow \pi_{\mathbf{P}''}$으로 Homography 변환을 할 수 있다.

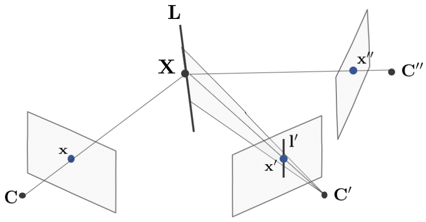
이와 유사하게 $\mathbf{H}_{12}$는 $\mathbf{x}' = \mathbf{H}_{12}\mathbf{x}$ 공식을 만족하며 다음과 같다.

$$
\mathbf{H}_{12} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3 \end{bmatrix} \mathbf{l}'' \quad \forall \mathbf{l}'' \tag{411}
$$

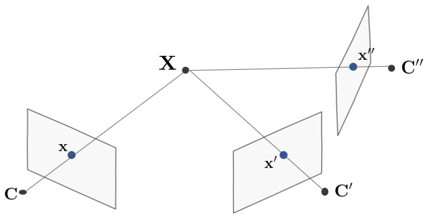
Point and line incidence relations



Point-Line-Line



Point-Line-Point



Point-Point-Point

이전 섹션에서는 $\mathbf{l} = \mathbf{l}'^{\intercal}\mathcal{T}\mathbf{l}''$ 공식을 통해 세 이미지 평면에 존재하는 직선들이 Trifocal Tensor $\mathcal{T}$에 의해 제약된다는 것에 대해 설명했다. **해당 섹션에서는 세 개의 직선들 뿐만 아니라 점과 직선들의 관계들이 Trifocal Tensor $\mathcal{T}$에 의해 어떻게 제약되는 지 설명한다.**

직선 $\mathbf{l}$ 위에 존재하는 임의의 점 $\mathbf{x}$에 대해 $\mathbf{x}^{\intercal}\mathbf{l} = 0$ 이 성립하고 이를 Tensor 표현법으로 다시 나타내면

$$\begin{aligned} \mathbf{x}^{\intercal}\mathbf{l} &= \sum_i x^i l_i \\ &= \sum_i x^i \mathbf{l}'^{\intercal}\mathbf{T}_i\mathbf{l}'' \quad \because l_i = \mathbf{l}'^{\intercal}\mathbf{T}_i\mathbf{l}'' \\ &= \mathbf{l}'^{\intercal}(\sum_i x^i\mathbf{T}_i)\mathbf{l}'' = 0 \end{aligned} \tag{412}$$

같이 나타낼 수 있고 이는 **한 점과 두 직선(point-line-line) 사이의 관계를 의미한다.**

이전 섹션에서 설명한 $\mathbf{H}_{13}$은 첫 번째 이미지 평면에서 세 번째 이미지 평면으로 변환하는 Homography를 의미한다. 이를 통해 $\mathbf{x}'' = \mathbf{H}_{13}\mathbf{x}$와 같이 세 번째 이미지 평면 상의 점 $\mathbf{x}''$을 구할 수 있고

$$\mathbf{x}'' = \mathbf{H}_{13}\mathbf{x} = \begin{bmatrix}\mathbf{T}_1{}^{\intercal}\mathbf{l}' & \mathbf{T}_2^{\intercal}\mathbf{l}' & \mathbf{T}_3^{\intercal}\mathbf{l}'\end{bmatrix}\mathbf{x} = (\sum_i x^i\mathbf{T}_i^{\intercal})\mathbf{l}' \tag{413}$$

같이 전개할 수 있다. 이 때, $\mathbf{x}''$은 Scale Factor를 포함하고 있으므로 유일하게 $\mathbf{x}''$를 결정하기 위해 $\mathbf{x}''^{\wedge}$를 곱하면

$$\mathbf{x}''^{\intercal}\mathbf{x}''^{\wedge} = \mathbf{l}'^{\intercal}(\sum_i x^i\mathbf{T}_i)\mathbf{x}''^{\wedge} = \mathbf{0}^{\intercal} \tag{414}$$

이 된다. 이는 **두 개의 점과 하나의 직선 사이(point-line-point)의 관계를 의미한다.**

이와 유사하게 세 개의 점이 주어졌을 때

$$\mathbf{x}'^{\wedge}(\sum_i x^i\mathbf{T}_i)\mathbf{x}''^{\wedge} = \mathbf{0} \tag{415}$$

을 통해 Scale Factor를 제거하여 유일한 $\mathbf{x}'$을 구할 수 있으며 **해당 공식은 세 점 사이(point-point-point)의 관계를 의미한다.**

Epipolar lines

Result 15.3

Trifocal Tensor $\mathcal{T}$를 통해 두 카메라의 Epipoar Line을 구할 수 있다. 세 이미지 평면 $\pi_{\mathbf{P}}, \pi_{\mathbf{P}'}, \pi_{\mathbf{P}''}$이 주어지고 $\pi_{\mathbf{P}}$ 위의 한 점 $\mathbf{x}$가 존재하며 $\pi_{\mathbf{P}'}, \pi_{\mathbf{P}''}$ 평면에 Epipolar Line $\mathbf{l}', \mathbf{l}''$이 존재한다고 가정하면 이들 사이에는

$$\begin{aligned} \mathbf{l}'^{\intercal}(\sum_i x^i \mathbf{T}_i) = 0 \\ (\sum_i x^i \mathbf{T}_i)\mathbf{l}'' = 0 \end{aligned} \tag{416}$$

관계가 성립한다. 즉, **$\mathbf{l}'^{\intercal}$은 $(\sum_i x^i \mathbf{T}_i)$의 Left Null Vector이며 $\mathbf{l}''$은 $(\sum_i x^i \mathbf{T}_i)$의 Right Null Vector가 된다.**

Proof

두 번째 이미지 평면 $\pi_{\mathbf{P}'}$ 상의 Epipolar Line $\mathbf{l}'$을 Back-projection하여 생성된 평면을 π'이라고 하면 π'과 $\pi_{\mathbf{P}}$ 사이에는 교차선이 생성되고 해당 교차선은 $\pi_{\mathbf{P}}$ 평면의 Epipolar Line $\mathbf{l}$이 된다. 이 때 세 번째 이미지 평면 $\pi_{\mathbf{P}''}$ 상의 임의의 직선 $\mathbf{l}''$을 Back-projection한 평면 π''과 π'은 월드 평면 상의 직선 $\mathbf{L}$에서 교차선을 생성하며 이를 다시 $\pi_{\mathbf{P}}$로 프로젝션하면 $\mathbf{L}$은 항상 Epipolar Line $\mathbf{l}$ 위에 한 점 $\mathbf{x} \in \mathbf{l}$로 프로젝션된다.

이러한 $\mathbf{l} \leftrightarrow \mathbf{l}' \leftrightarrow \mathbf{l}''$ 관계를 통해 Trifocal Tensor $\mathcal{T}$을 구할 수 있으며 이전 섹션에서 설명한 **한 점 x와 두 직선 사이(point-line-line)의 관계 공식이 성립한다.**

$$\begin{aligned} &\mathbf{x} \in \mathbf{l} = \mathbf{l}'^{\intercal}\mathcal{T}\mathbf{l}'' \\ &\mathbf{l}'^{\intercal}(\sum_i x^i \mathbf{T}_i)\mathbf{l}'' = 0 \quad {}^{\forall}\mathbf{l}'' \\ &\therefore \mathbf{l}'^{\intercal}(\sum_i x^i \mathbf{T}_i) = 0 \end{aligned} \tag{417}$$

위와 같이 **모든 $\mathbf{l}''$에 대해 위 공식을 만족해야 하므로 $\mathbf{l}'^{\intercal}(\sum_i x^i \mathbf{T}_i) = 0$ 공식이 성립하게 된다. 이를 모든 $\mathbf{l}'$에 대해서도 마찬가지로 성립하므로 $(\sum_i x^i \mathbf{T}_i)\mathbf{l}'' = 0$ 또한 성립한다.**

Result 15.4

추가적으로 Epipole $\mathbf{e}', \mathbf{e}''$은 모든 ${}^{\forall}\mathbf{x}$에 대해 구할 수 있는 $\mathbf{l}', \mathbf{l}''$들의 교차점을 계산함으로써 구할 수 있다.

Extracting the fundamental matrices

이전 섹션에서 설명한 것과 같이 여러 점과 직선들과 관계를 통해 세 개의 이미지 평면에 대한 Trifocal Tensor $\mathcal{T}$를 구할 수 있다. 해당 섹션에서는 $\mathcal{T}$를 통해 세 카메라에 대한 Fundamental Matrix $\mathbf{F}$를 구하는 방법에 대해 설명한다.

Fundamental Matrix $\mathbf{F}_{21}$는 $\mathbf{F}_{21} = \mathbf{e}'^{\wedge}\mathbf{H}_{21}$를 통해 구할 수 있다. $\mathbf{F}_{ij}$는 i번째 이미지 평면과 j번째 이미지 평면 사이의 Fundamental Matrix를 의미한다. $\mathbf{e}'$은 이전 섹션에서 설명한 방법대로 $\mathbf{T}_i$의 Left Null Vector를 계산하여 $\mathbf{l}', \mathbf{e}'$을 통해 구하고 $\mathbf{H}_{21}$은 $\begin{bmatrix}\mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3\end{bmatrix}\mathbf{l}''$과 같이 구할 수 있으므로

$$\begin{aligned} \mathbf{F}_{21} &= \mathbf{e}'^{\wedge}\mathbf{H} \\ &= \mathbf{e}'^{\wedge}\begin{bmatrix}\mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3\end{bmatrix}\mathbf{l}'' \quad {}^{\exists}\mathbf{l}'' \end{aligned} \tag{418}$$

이 된다. 이 때 $\mathbf{l}''$은 $\mathbf{T}_i$의 Null Space에 존재하면 안된다. 즉 $\mathbf{T}_i\mathbf{l}'' \neq 0$이어야 한다. 다시 말하면 $\begin{bmatrix}\mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3\end{bmatrix}\mathbf{l}''$의 rank가 3이어야 한다. 이 때, Epipole $\mathbf{e}''$은 $\mathbf{l}''$의 Null Space에 존재하므로 $\mathbf{e}''^{\intercal}\mathbf{l}'' = 0$을 만족한다. 따라서

$$\mathbf{e}'' \perp \text{Nul } \mathbf{T}_i \quad {}^{\forall}i \tag{419}$$

를 만족하므로 **$\mathbf{l}''$ 대신 $\mathbf{e}''$을 대입하면 항상 행렬의 rank가 3이 된다. 결론적으로 Fundamental Matrix $\mathbf{F}_{21}$은**

$$\mathbf{F}_{21} = \mathbf{e}'^{\wedge}\begin{bmatrix}\mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3\end{bmatrix}\mathbf{e}'' \tag{420}$$

을 통해 구할 수 있다. 동일한 방법을 사용하여 $\mathbf{F}_{31} = \mathbf{e}''^{\wedge}\begin{bmatrix}\mathbf{T}_1^{\intercal} & \mathbf{T}_2^{\intercal} & \mathbf{T}_3^{\intercal}\end{bmatrix}\mathbf{e}'$을 구할 수 있다.

Retrieving the camera matrices

Two-view Geometry에서는 Fundamental Matrix $\mathbf{F}$가 주어지면 카메라 행렬 대응쌍 $(\mathbf{P}, \mathbf{P}')$을 사영모호성을 포함하여(up to projectivity) 구할 수 있었다.

$$\begin{aligned} &\mathbf{P} = [\mathbf{I} \mid 0] \\ &\mathbf{P}' = [\mathbf{A} \mid \mathbf{e}'] \\ &\text{where, } \mathbf{F} = \mathbf{e}'^{\wedge}\mathbf{A} \quad \text{in two-view.} \end{aligned} \tag{421}$$

Three-view Geoemtry에서는 Trifocal Tensor $\mathcal{T}$를 통해 $\mathbf{F}_{21} = \mathbf{e}'^{\wedge}\left[\mathbf{T}_1 \quad \mathbf{T}_2 \quad \mathbf{T}_3\right]\mathbf{e}''$ 그리고 $\mathbf{F}_{31} = \mathbf{e}''^{\wedge}\left[\mathbf{T}_1^{\intercal} \quad \mathbf{T}_2^{\intercal} \quad \mathbf{T}_3^{\intercal}\right]\mathbf{e}'$을 구했을 때

$$\begin{aligned} &\mathbf{P} = [\mathbf{I} \mid 0] \\ &\mathbf{P}' = [\left[\mathbf{T}_1 \quad \mathbf{T}_2 \quad \mathbf{T}_3\right]\mathbf{e}'' \mid \mathbf{e}'] \\ &\text{but, } \mathbf{P}'' \neq [\left[\mathbf{T}_1^{\intercal} \quad \mathbf{T}_2^{\intercal} \quad \mathbf{T}_3^{\intercal}\right]\mathbf{e}' \mid \mathbf{e}''] \quad \text{in three-view.} \end{aligned} \tag{422}$$

관계가 성립한다. 다시 말하면 **$(\mathbf{P}, \mathbf{P}')$을 계산하면 월드좌표계가 고정되고 따라서 $\mathbf{P}''$은 고정된 월드좌표계에 대해서 다시 표현해야 한다는 것을 의미한다.** p256 공식 (9.10)에 의해 Canonical Form에 대한 가장 일반적인 카메라 행렬은 다음과 같이 나타낼 수 있다.

$$\mathbf{P}'' = [\mathbf{H} + \mathbf{e}''\mathbf{v}^{\intercal} \mid \lambda\mathbf{e}''] \tag{423}$$

이를 전개하면

$$\begin{aligned} \mathbf{P}'' &= [\mathbf{H} + \mathbf{e}''\mathbf{v}^{\intercal} \mid \lambda\mathbf{e}''] \\ &= [\underbrace{\left[\mathbf{T}_1^{\intercal} \quad \mathbf{T}_2^{\intercal} \quad \mathbf{T}_3^{\intercal}\right]\mathbf{e}' + \mathbf{e}''\mathbf{v}^{\intercal}}_{\mathbf{B}} \mid \underbrace{\lambda\mathbf{e}''}_{\mathbf{b}_4}] \end{aligned} \tag{424}$$

이 되고 $\mathbf{P}' = [\mathbf{A} \mid \mathbf{a}_4]$은

$$\begin{aligned} \mathbf{P}' &= [\mathbf{A} \mid \mathbf{a}_4] \\ &= [\underbrace{\left[\mathbf{T}_1 \quad \mathbf{T}_2 \quad \mathbf{T}_3\right]\mathbf{e}''}_{\mathbf{A}} \mid \underbrace{\mathbf{e}'}_{\mathbf{a}_4}] \end{aligned} \tag{425}$$

이므로 $\mathbf{T}_i$의 정의에 의해

$$\begin{aligned} \mathbf{T}_i &= \mathbf{a}_i\mathbf{b}_4^{\intercal} + \mathbf{a}_4\mathbf{b}_i^{\intercal} \\ &= \mathbf{T}_i\mathbf{e}''\mathbf{e}''^{\intercal} - \mathbf{e}'\mathbf{b}_i^{\intercal} \end{aligned} \tag{426}$$

이 된다. 이를 정리하면

$$\mathbf{T}_i(\mathbf{I} - \mathbf{e}''\mathbf{e}''^{\intercal}) = -\mathbf{e}'\mathbf{b}_i^{\intercal} \tag{427}$$

이 된다. 이 때 $\|\mathbf{e}'\| = 1$이라고 가정하고 양 변에 $\mathbf{e}'^{\intercal}$을 곱하면

$$\mathbf{b}_i^{\intercal} = \mathbf{e}'^{\intercal}\mathbf{T}_i(\mathbf{e}''\mathbf{e}'' - \mathbf{I}) \tag{428}$$

가 된다. **최종적으로 $\mathbf{P}''$은 다음과 같다.**

$$\mathbf{P}'' = [(\mathbf{e}''\mathbf{e}''^{\intercal} - \mathbf{I})\left[\mathbf{T}_1^{\intercal} \quad \mathbf{T}_2^{\intercal} \quad \mathbf{T}_3^{\intercal}\right]\mathbf{e}' \mid \mathbf{e}''] \tag{429}$$

The trifocal tensor and tensor notation

Trifocal Tensor $\mathcal{T}$를 Tensor 표현법으로 나타내면 다음과 같다.

$$\begin{aligned} \mathcal{T}_i^{jk} &= (j,k) \text{ entry of } \mathbf{T}_i \\ &= a_i^j b_4^k - a_4^j b_i^k \end{aligned} \tag{430}$$

행렬로 표현한 직선 $\mathbf{l}$의 i번째 좌표는 $l_i = \mathbf{l}'^{\intercal}\mathbf{T}_i\mathbf{l}''$과 같이 나타낼 수 있고 이를 Tensor 표현법으로 나타내면

$$
\begin{aligned}
l_i &= l'_i \mathcal{T}_i^{jk} l''_k \\
&= l'_i l''_k \mathcal{T}_i^{jk}
\end{aligned} \tag{431}
$$

가 된다. Tensor 표현법을 통해 첫 번째 카메라의 직선 $\mathbf{l}$을 세 번째 카메라의 직선 $\mathbf{l}''$으로 변환하는 Homography $\mathbf{H} : \pi_{\mathbf{P}} \mapsto \pi_{\mathbf{P}''}$을 구할 수 있다.

$$
l_i = l''_k (l'_j \mathcal{T}_i^{jk}) = l''_k h_i^k \tag{432}
$$

이 때, $h_i^k = l'_j \mathcal{T}_i^{jk}$를 의미한다. Homography $\mathbf{H}$를 사용하여 점을 변환하는 경우

$$
x''^k = h_i^k x^i \tag{433}
$$

가 된다.

Tensor 표현법을 나타내는데 유용한 ϵ_{ijk}에 대해 설명하면

$$
\epsilon_{ijk} = \begin{cases} 0 & \text{unless i,j,k are all distinct} \\ +1 & \text{i,j,k are even permutation of 1,2,3} \\ -1 & \text{i,j,k are odd permutation of 1,2,3} \end{cases} \tag{434}
$$

과 같다. 다시 말하면 ϵ_{ijk}는 i,j,k가 전부 다르지 않은 이상 0의 값을 가지며 i,j,k가 $(1,2,3),(3,1,2)$ 또는 $(2,3,1)$과 같이 순차적일 때는 $+1$의 값을 가지고 아닌 경우에는 -1의 값을 가진다. 이를 통해 3×3 벡터들의 Cross Product를 표현해보면 $\mathbf{c} = \mathbf{a} \times \mathbf{b}$일 때

$$
c_i = \epsilon_{ijk} a^j b^k \tag{435}
$$

와 같이 나타낼 수 있다. 이는 $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$을 Tensor 표현법으로 나타낸 것이다. 따라서 $(\mathbf{a}^\wedge)_{ik}$는 Tensor 표현법으로

$$
(\mathbf{a}^\wedge)_{ik} = \epsilon_{ijk} a^j \tag{436}
$$

같이 나타낼 수 있다.

The trilinearities

Tensor 표현법을 사용해서 이전 섹션에서 설명한 여러 점들과 직선의 관계를 다시 표현할 수 있다. 예를 들어 두 점과 한 직선 사이(point-line-point)의 관계 공식을 일반적인 형태로 나타내면 다음과 같다.

$$
\mathbf{l}'^\intercal \Big(\sum_i x^i \mathbf{T}_i\Big) \mathbf{x}''^\wedge = \mathbf{0}^\intercal \tag{437}
$$

이를 Tensor 표현법으로 나타내면 $(\mathbf{x}''^\wedge)_{qs} = -x''^k \epsilon_{kqs}$가 되므로 이를 다시 정리하면

$$
l'_j x^i \mathcal{T}_i^{jq} x''^k \epsilon_{kqs} = 0_s \tag{438}
$$

과 같이 나타낼 수 있다. 이는 **세 개의(tri-) 서로 다른 이미지에 존재하는 점 또는 직선을 사용하여 선형방정식(linear)을 도출했으므로 다른 용어로 Trilinearities라고 한다.**

Transfer

세 개의 카메라가 주어졌을 때 이 중 두 개의 이미지 평면에서 점 또는 직선의 위치를 알고 있을 때 Trifocal Tensor $\mathcal{T}$를 사용하여 나머지 한 개의 이미지 평면 상의 점 또는 직선의 위치를 결정하는 방법을 Transfer 라고 한다.

Point transfer using fundamental matrices

Point Transfer는 세 개의 이미지 평면 $\pi_{\mathbf{P}}, \pi_{\mathbf{P}'}, \pi_{\mathbf{P}''}$에 대한 Fundamental Matrix $\mathbf{F}_{21}, \mathbf{F}_{31}, \mathbf{F}_{32}$가 주어졌을 때 이미 알고 있는 $\mathbf{x}, \mathbf{x}'$의 위치를 통해 $\mathbf{x}''$의 위치를 결정하는 것을 말한다. 이는 Epipolar Geometry를 통해 결정할 수 있다. 세 번째 이미지 평면 $\pi_{\mathbf{P}''}$ 상에 존재하는 Epipolar Line $\mathbf{l}''$은

$$\begin{aligned} \mathbf{l}''_{31} &= \mathbf{F}_{31}\mathbf{x} \\ \mathbf{l}''_{32} &= \mathbf{F}_{32}\mathbf{x}' \end{aligned} \tag{439}$$

이므로 $\pi_{\mathbf{P}''}$ 위에 존재하는 한 점 $\mathbf{x}''$은

$$\mathbf{x}'' \in \mathbf{l}''_{31} \text{ and } \mathbf{l}''_{32} \tag{440}$$

을 만족해야 한다. 따라서 $\mathbf{x}''$은 위 두 직선의 교차점으로 결정된다.

$$\mathbf{x}'' = (\mathbf{F}_{31}\mathbf{x}) \times (\mathbf{F}_{32}\mathbf{x}') \tag{441}$$

이 때, **위 공식에서 $\mathbf{F}_{21}$은 사용되지 않았는데 실제로는 대응점 쌍 $(\mathbf{x}, \mathbf{x}')$에 노이즈가 존재하기 때문에 이를 개선시키기 위해 사용된다.** 대응점 쌍의 노이즈로 인해 $\mathbf{x}^\intercal\mathbf{F}_{21}\mathbf{x}' = 0$ 공식을 만족하지 않으므로 이전 섹션에서 설명한 Optimal Triangulation 방법을 사용하여 $d(\mathbf{x}, \mathbf{l}(t))^2 + d(\mathbf{x}', \mathbf{l}'(t))^2$ 값이 최소가 되는 $\hat{\mathbf{x}} \leftrightarrow \hat{\mathbf{x}}'$를 구한 후 이를 통해 $\mathbf{x}''$을 계산한다.

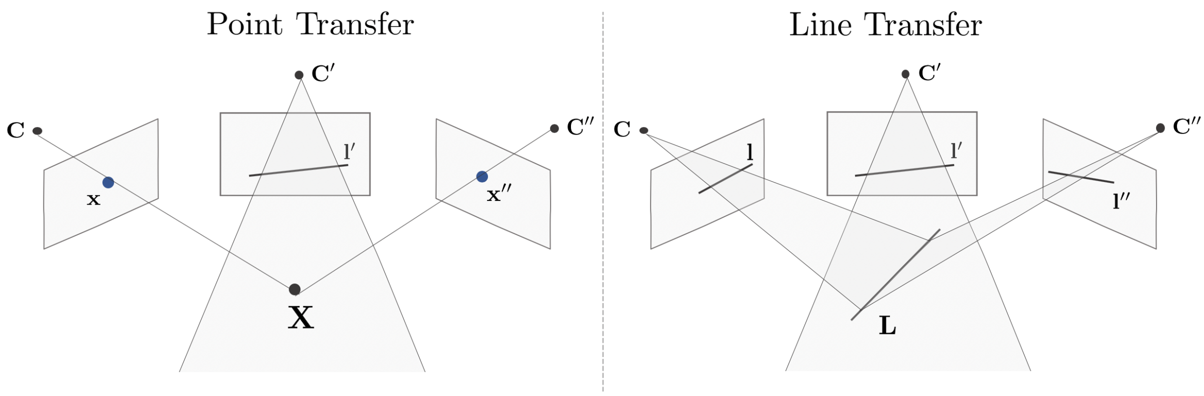
$$\mathbf{x}'' = (\mathbf{F}_{31}\hat{\mathbf{x}}) \times (\mathbf{F}_{32}\hat{\mathbf{x}}') \tag{442}$$

하지만 **해당 Point Transfer는 세 개의 카메라의 중심점 $\mathbf{C}, \mathbf{C}', \mathbf{C}''$이 이루는 Trifocal 평면에 대응점 쌍 $\mathbf{x}, \mathbf{x}'$이 존재하는 경우 $\pi_{\mathbf{P}''}$에 프로젝션되는 Epipolar Line이 다음과 같이 동일하게 생성되어**

$$\mathbf{F}_{31}\mathbf{x} = \mathbf{F}_{32}\mathbf{x}' \tag{443}$$

이를 통해 $\mathbf{x}''$의 유일한 위치를 결정할 수 없다. 이는 Fundamental Matrix를 이용한 Point Transfer의 한계점으로 볼 수 있다.

Point transfer using the trifocal tensor



Fundamental Matrix $\mathbf{F}_{ij}$가 아닌 Trifocal Tensor $\mathcal{T}$를 사용하면 더욱 다양한 경우에 Transfer를 수행할 수 있다. $\mathcal{T}$를 이용해 Point Transfer를 하는 방법은 다음과 같다.

- $\mathcal{T}$를 이용해 $\mathbf{F}_{21}$를 계산한다.

$$\mathbf{F}_{21} = \mathbf{e}'^\intercal \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3 \end{bmatrix} \mathbf{e}'' \tag{444}$$

- Optimal Triangulation 방법을 사용하여 노이즈를 제거한 최적의 대응점 쌍을 계산한다.

$$(\mathbf{x}, \mathbf{x}') \rightarrow (\hat{\mathbf{x}}, \hat{\mathbf{x}}') \tag{445}$$

- 두 번째 이미지 평면 $\pi_{\mathbf{P}'}$의 Epipolar Line $\mathbf{l}'_{\mathbf{e}}$을 계산한다. $\mathbf{l}'_{\mathbf{e}} = \mathbf{F}_{21}\hat{\mathbf{x}}$. 다음으로 $\mathbf{l}'_{\mathbf{e}}$과 수직이고 $\hat{\mathbf{x}}'$를 통과하는 직선 $\mathbf{l}'$을 계산한다.

$$\hat{\mathbf{x}}' = (x'_1, x'_2, 1)^\intercal \qquad \mathbf{l}'_\mathbf{e} = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^\intercal$$
$$\Rightarrow \mathbf{l}' = \begin{bmatrix} l_2 \\ -l_1 \\ -l_2 x'_1 + l_1 x'_2 \end{bmatrix} \tag{446}$$

- $\mathbf{l}'$을 사용하여 $\mathbf{x}'$을 Point Transfer한다.

$$x''^k = x'^i l'_j \mathcal{T}_i^{jk} \tag{447}$$

Degenerate configurations

Trifocal Tensor $\mathcal{T}$를 사용하는 경우에도 두 카메라의 중심점 $\mathbf{C}, \mathbf{C}'$을 잇는 baseline 직선 상 위에 3차원 공간 상의 점 $\mathbf{X}$가 존재하는 경우 이를 통해 $\mathbf{x}''$을 결정할 수 없다.

Line transfer using the trifocal tensor

Trifocal Trnsor $\mathcal{T}$를 사용하면 점 뿐만 아니라 직선 또한 Transfer할 수 있다. 세 이미지 평면 상의 직선 $\mathbf{l} \leftrightarrow \mathbf{l}' \leftrightarrow \mathbf{l}''$이 대응관계를 가지는 경우

$$l_i = l'_j l''_k \mathcal{T}_i^{jk} \tag{448}$$

와 같이 $\mathcal{T}$를 통해 표현할 수 있고 이는 곧 직선 $\mathbf{l}$이 벡터 $[l'_j l''_k \mathcal{T}_i^{jk}]$과 평행하다는 의미이다.

$$\mathbf{l} \parallel [l'_j l''_k \mathcal{T}_i^{jk}]_{3\times 1}, \ i = 1, 2, 3 \tag{449}$$

평행한 직선들은 Cross Product가 0이 되어야 하므로

$$\begin{aligned} (\mathbf{l}^\wedge)_{si} l'_j l''_k \mathcal{T}_i^{jk} = 0 \\ (l_s \epsilon^{ris}) l'_j l''_k \mathcal{T}_i^{jk} = 0 \end{aligned} \tag{450}$$

이 성립한다. 이는 곧 $\mathbf{l}''$에 대한 선형방정식이므로 이를 다시 정리하면

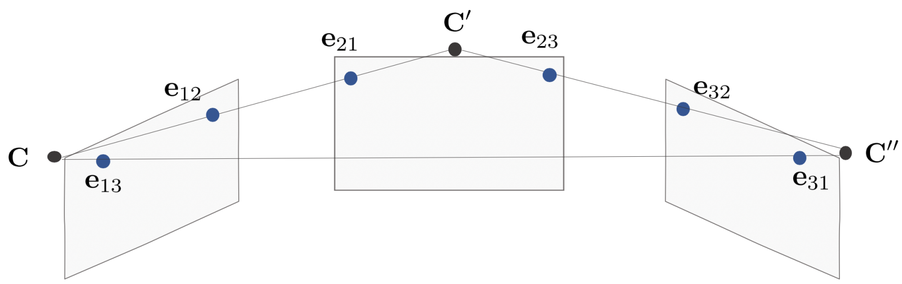
$$(l_s \epsilon^{ris} l'_j \mathcal{T}_i^{jk}) l''_k = 0 \tag{451}$$

선형방정식을 풂으로써 $\mathbf{l} \leftrightarrow \mathbf{l}'$이 주어진 경우 $\mathbf{l}''$을 계산할 수 있다.

Degeneracies

직선 $\mathbf{l}, \mathbf{l}'$이 Epipolar Line인 경우 이를 Back-projection한 평면 π는 π'과 동일한 Epipolar Plane이 되고 이 경우 세 번째 이미지 평면 $\pi_{\mathbf{P}''}$ 상의 직선 $\mathbf{l}''$을 유일하게 결정할 수 없다.

The fundamental matrices for three views



세 개의 카메라에 대한 Fundamental Matrix $\mathbf{F}_{21}, \mathbf{F}_{31}, \mathbf{F}_{32}$가 주어졌을 때 이들은 서로 독립적이지 않다. 세 개의 이미지 평면에 대해 총 6개의 Epipole이 생성되고

$$\mathbf{e}_{23}^\intercal \mathbf{F}_{21} \mathbf{e}_{13} = \mathbf{e}_{31}^\intercal \mathbf{F}_{32} \mathbf{e}_{21} = \mathbf{e}_{32}^\intercal \mathbf{F}_{31} \mathbf{e}_{12} = 0 \tag{452}$$

공식을 만족한다.

Definition 15.5

세 개의 Fundamental Matrix $\mathbf{F}_{21}, \mathbf{F}_{31}, \mathbf{F}_{32}$가 독립적이지 않고 서로 상호호환(compatible)하기 위해서는 위 공식이 성립해야 한다.

Uniqueness of camera matrices given three fundamental matrices

만약 세 개의 Fundamental Matrix가 상호호환인 경우 이를 생성하는 세 개의 카메라 행렬 $(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$이 사영모호성을 포함하여(up to projectivity) 유일하게 결정된다.

$$\begin{aligned} &^{\exists}(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3) \ \text{ such that fundamental matrix of } (\mathbf{P}_i, \mathbf{P}_j) \ \text{ is } \ \mathbf{F}_{ij} \\ &(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3) \ \text{ are unique up to projectivity.} \end{aligned} \tag{453}$$

이를 증명하기 위해 다음 순서대로 진행한다.

- Two-view Geometry의 원리를 사용하면 $\mathbf{F}_{21} = \mathbf{e}_{21}^{\wedge}\mathbf{A}$ 일 때 카메라 행렬 $\mathbf{P}_1, \mathbf{P}_2$는 다음과 같이 구할 수 있다.

$$\begin{aligned} \mathbf{P}_1 &= [\mathbf{I} \mid 0] \\ \mathbf{P}_2 &= [\mathbf{A} \mid \mathbf{e}_{21}] \end{aligned} \tag{454}$$

- $\mathbf{x}'^{\intercal}\mathbf{F}_{21}\mathbf{x} = 0$을 만족하는 대응점 쌍 $(\mathbf{x}_i, \mathbf{x}'_i)$을 생성한다. 이를 통해 월드 공간 상의 점 $\mathbf{X}_i$를 Triangulation한다.
- 세 점 사이(point-point-point)의 관계 공식을 사용하여 $\mathbf{x}''_i$을 구한다.

$$\mathbf{x}''_i = (\mathbf{F}_{31}\mathbf{x}_i) \times (\mathbf{F}_{32}\mathbf{x}'_i) \tag{455}$$

- $\mathbf{P}_3\mathbf{X}_i = \mathbf{x}''_i$ 공식을 사용하여 $\mathbf{P}_3$을 계산할 수 있다.

단, 월드 공간 상의 점 $\mathbf{X}_i$가 세 카메라의 중심점 $\mathbf{C}, \mathbf{C}', \mathbf{C}''$을 포함하는 Trifocal 평면에 존재하는 경우 $\mathbf{x}''$을 유일하게 결정할 수 없다.

13 Revision log

- 1st: 2020-05-12
- 2nd: 2020-06-06
- 3rd: 2020-06-07
- 4th: 2020-06-09
- 5th: 2020-06-10
- 6th: 2020-06-11
- 7th: 2020-06-12
- 8th: 2020-06-14
- 9th: 2020-06-15
- 10th: 2020-06-16
- 11th: 2020-06-20
- 12th: 2020-06-22
- 13th: 2020-06-23
- 14th: 2022-06-28
- 15th: 2022-12-20

- 16th: 2023-01-01
- 17th: 2023-01-21
- 18th: 2024-03-20
- 19th: 2024-03-23

14 References

Hartley, Richard, and Andrew Zisserman. Multiple view geometry in computer vision. Cambridge university press, 2003

15 Closure

Check out alida.tistory.com for the web version posting.