Problem1

$$\nabla_\phi E_{z\sim q_\phi(z)}\left[\log\left(\frac{h(z)}{q_\phi(z)}\right)\right] = \nabla_\phi \int q_\phi(z)\cdot\log\frac{h(z)}{q_\phi(z)}\,dz$$

$$= \int \nabla_\phi q_\phi(z)\cdot\log h(z) - \nabla_\phi\left(q_\phi(z)\log q_\phi(z)\right)dz$$

$$= \int (\nabla_\phi q_\phi(z))\log h(z) - (\nabla_\phi q_\phi(z))\cdot\log q_\phi(z) - \underbrace{\nabla_\phi q_\phi(z)}_{0}\,dz$$

$$E_{z\sim q_\phi(z)}\left[(\nabla_\phi \log q_\phi(z))\cdot\log\left(\frac{h(z)}{q_\phi(z)}\right)\right] = \int(\nabla_\phi\log q_\phi(z))\cdot\log\left(\frac{h(z)}{q_\phi(z)}\right)\cdot q_\phi(z)\,dz$$

$$= \int\left(\frac{1}{q_\phi(z)}\cdot q_\phi(z)\cdot\log\left(\frac{h(z)}{q_\phi(z)}\right)\cdot\nabla_\phi q_\phi(z)\,dz\right.$$

$$= \int(\nabla_\phi q_\phi(z))\log h(z) - (\nabla_\phi q_\phi(z))\log q_\phi(z)\,dz$$

$$\therefore \nabla_\phi E_{z\sim q_\phi(z)}\left[\log\left(\frac{h(z)}{q_\phi(z)}\right)\right] = E_{z\sim q_\phi(z)}\left[(\nabla_\phi\log q_\phi(z))\cdot\log\left(\frac{h(z)}{q_\phi(z)}\right)\right]$$

Problem 2.

$$\|x-y\|^2 = \left(\sqrt{(a-y_1)^2+(x_2-y_2)^2}\right)^2$$

$$= (a-y_1)^2 + (x_2-y_2)^2$$

$$\frac{\partial}{\partial x_2}\|x-y\|^2 = 2(x_2-y_2)$$

if $y_2<0$ :

$$\frac{\partial}{\partial x_2}\|x-y\|^2 = 2(x_2-y_2) > 0 \quad \text{for } 0\le x_2\le 1$$

$\therefore x_2=0$ 일때 최소

if $y_2>1$.

$$\frac{\partial}{\partial x_2}\|x-y\|^2 = 2(x_2-y_2) < 0 \quad \text{for } 0\le x_2\le 1$$

$\therefore x_2=1$ 일때 최소

if $0\le y_2\le 1$ 일때

$\frac{\partial}{\partial x_2}\|x-y\|^2 = 2(x_2-y_2) = 0$ 이 되는 지점 $x_2=y_2$, $x_2<y_2$면 음수이고 $x_2>y_2$면 양수임

$\therefore x_2=y_2$ 일때 최소

$\therefore x_2 = \min\{\max(y_2,0),1\}$ $\qquad \therefore \pi_C(y) = \begin{bmatrix} a \\ \min\{\max(y_2,0),1\}\end{bmatrix}$

Problem 4

(a)

$$\log\left|\frac{\partial f_1}{\partial x}\right| = \log|\det A| = \log|\det P \cdot \det L \cdot \det(U+\mathrm{diag}(s))|$$

$\det P \in \{-1, 1\}$

$\det L = 1$

$U+\mathrm{diag}(s)$ 는 upper triangular matrix 이므로

$\det(U+\mathrm{diag}(s)) = \prod_{i=1}^{c} s_i$

$\therefore \log|\det P \cdot \det L \cdot \det(U+\mathrm{diag}(s))| = \sum_{i=1}^{c} \log|s_i|$

(b) reshape이 달라져도 $\frac{\partial h(x)}{\partial x}$ 의 행과 열의 순서만 뒤바뀌므로 det를 씌우고 절댓값까지 씌우게 되면 $\left|\frac{\partial h(x)}{\partial x}\right|$ 의 값은 변하지 않는다.

(c) $X_{:,1,1}, \cdots X_{:,m,n}$ 으로 나누어서 생각해보면.

$$f_2(x|P,L,U,s)_{:,i,j} = A X_{:,i,j}$$

$f_2(x|P,L,U,s)_{:,l,m}$ 은 $X_{:,l,m}$ 외에 다른 $X_{:,i,j}$ $(i \neq l, j \neq m)$ 에 의해 영향을 받지 않는다.

즉 $\frac{\partial f_2(x|P,L,U,s)}{\partial x}$ 의 대각 성분에는 $A$가 $mn$개 배열되어있으므로 (나머지는 0)

$$\log\left|\det\left(\frac{\partial f_2(x|P,L,U,s)}{\partial x}\right)\right| = \log\left|\det(A)^{mn}\right| = mn\sum_{i=1}^{c}\log|s_i|$$

(d)

$$\frac{\partial z}{\partial x} = \begin{bmatrix} I & 0 \\ * & \frac{\partial f_2(X_{c+1:2c,:,:}|P,L,U,s)}{\partial X_{c+1:2c,:,:}} \end{bmatrix}$$

$$\log\left|\det\left(\frac{\partial z}{\partial x}\right)\right| = \log|\det I| + \log\left|\det\left(\frac{\partial f_2(X_{c+1:2c,:,:}|P,L,U,s)}{\partial X_{c+1:2c,:,:}}\right)\right| = 0 + mn\sum_{i=1}^{c}\log|s_i|$$