$\vec{F} = m\vec{a} = m \cdot \frac{D\vec{u}}{Dt} = m\vec{g} - V \cdot \nabla p + V \mu \nabla^2 \vec{u}$, where V = volume, P = pressure, μ = viscosity, $\frac{D\vec{u}}{Dt}$ = material derivative

Dividing each side by V (volume),

$$\frac{m}{V} \cdot \frac{D\vec{u}}{Dt} = \frac{m}{V}\vec{g} - \nabla p + \mu \nabla^2 \vec{u}$$

Because $\frac{m}{V} = \rho$, (ρ = density)

$$\rho \cdot \frac{D\vec{u}}{Dt} = \rho\vec{g} - \nabla p + \mu \nabla^2 \vec{u}$$

Dividing each side by ρ.

$$\frac{D\vec{u}}{Dt} = \vec{g} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \vec{u}$$

Considering that $\frac{D\vec{u}}{Dt}$ is a material derivative in Lagrangian (particle-based) viewpoint,

$$\frac{Dq}{Dt} = \underbrace{\frac{\partial q}{\partial t}}_{\text{change of } q \text{ in fixed position}} + \underbrace{\nabla q}_{\text{change of } q} \cdot \vec{u} \qquad \left(\frac{\partial q}{\partial t} \text{ is in Eulerian viewpoint} \right)$$

due to advection, think it as a dot product of the velocity with the gradient of the quantity q.

So we get $\frac{\partial q}{\partial t} = -\nabla q \cdot \vec{u} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \vec{u} + \vec{g}$, (which is same as Navier-Stoke Equation)

Navier-Stokes equation itself doesn't say anything about 'incompressible fluid'

For that, we also need the continuity equation.

$\nabla \cdot \vec{u} = 0$, which also means zero-divergence vector field

Many times, we get viscosity-effect due to how we interpolate things.

Therefore, we can just drop the viscosity term, and still get the viscosous look.

By ignoring viscosity, we get

$$\boxed{\begin{aligned} \frac{\partial \vec{u}}{\partial t} &= -\vec{u} \cdot \nabla \vec{u} - \frac{1}{\rho}\nabla p + \vec{g} \\ \nabla \cdot \vec{u} &= 0 \end{aligned}}$$

Boundary Conditions

For solid walls, we say $\vec{u} \cdot \hat{n} = 0$, wher $\hat{n}$ is the normal of the wall. $(\vec{u} - \vec{u}_{wall}) \cdot \hat{n} = 0$, if the wall was moving.

This means we preserve the tangential part, and only remove the normal part.

If the fluid has viscosity, we simply set the velocity to 0 (or the object)

$$\vec{u} = 0$$
$$\vec{u} = \vec{u}_{wall}$$

Forward Euler

In N-S equation, we don't do every step in once, but try to split it and do one operation at a time.

This method is valid thanks to Forward Euler.

Let's say we have: $\frac{dq}{dt} = f(q) + g(q)$.

we can: $\tilde{q} = q^n + \Delta t\, f(q^n)$ where q^n and q^{n+1} is a value of q in n, $n+1$ timesteps

$q^{n+1} = \tilde{q} + \Delta t\, g(\tilde{q})$.

This can also be applied to: $\frac{dr}{dt} = f(r)$, $\frac{ds}{dt} = g(s)$.

$\tilde{q} = f(q^n) + \Delta t f(q^n)$ so we do operation on f and plug the result to g

$q^{n+1} = \tilde{q} + \Delta t g(\tilde{q})$

Now, we can split our equation into:

Advection, $\frac{Dq}{Dt} = 0$ Advection only works in a divergence-free field.

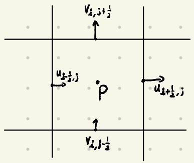
Add force, $\frac{\partial \vec{u}}{\partial t} = \vec{g}$

Projection, $\frac{\partial \vec{u}}{\partial t} = -\frac{1}{\rho}\nabla p$.

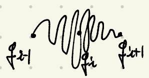
MAC grid (staggered grid)

Marker And Cell

Basic idea is to store different variables in different locations of the grid.



This removes the problem that we had with the center-based positions. (especially with high frequency vector fields) i.e. $(-1)^i$



$\frac{\partial q_i}{\partial x} \approx \frac{q_{i+1} - q_{i-1}}{2\Delta x}$, so it doesn't capture any of q_i. (This is called null-space)

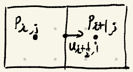
but with MAC, we can get $\frac{\partial q_i}{\partial x} \approx \frac{q_{i+\frac{1}{2}} - q_{i-\frac{1}{2}}}{\Delta x}$

But this also means that whenever we wanna sample the velocity,
we need to interpolate between two values.

For example, $\vec{u}_{i,j} = \left(\frac{\vec{u}_{i-\frac{1}{2},j} + \vec{u}_{i+\frac{1}{2},j}}{2}, \frac{\vec{v}_{i,j-\frac{1}{2}} + \vec{v}_{i,j+\frac{1}{2}}}{2} \right)$

Not only that, it has some really nice feature with the velocity and pressure.
Recall that the projection looked like

$$\frac{\partial q}{\partial t} = -\frac{1}{\rho} \nabla P$$



This means when we modify velocity with pressure to remove divergence, the velocity is defined exactly where the lerp result of $P_{i,j}$ & $P_{i+1,j}$ is.

Advection - 1

$\frac{Dq}{Dt} = 0$, which is same as $\frac{\partial q}{\partial t} = -\vec{u} \nabla q$

Advection should only happen in divergence-free vector field.

The natural thing to do is to move forward in time

$$\frac{\partial q}{\partial t} + \vec{u} \nabla q = \frac{\underbrace{q_i^{n+1}}_{\text{we went forward in time}} - q_i^n}{\Delta t} + \vec{u} \cdot \boxed{\frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}}$$

∇ Gradient in 1D, in 2D, it should be $\left(\frac{q_{i+1,j} - q_{i-1,j}}{2\Delta x}, \frac{q_{i,j+1} - q_{i,j-1}}{2\Delta y} \right)$

which can be re-arranged.

$$q_i^{n+1} = q_i^n - \Delta t\, u_i^n \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}$$

=> But F. euler is unstable by it's nature.
what if we miss the frame and Δt become too large?
Even if we keep Δt small (offline simulation), it will eventually blow up. (a lot of math here that I don't understand)

The solution is to go backwards in time. (Semi-Lagrangian method: we use the idea of fluid being particles for Eulerian view point)

Let's say we want to get q_G^{n+1}. We can conceptually think of a particle located at P, which will end up at G in the next frame.

So $q_G^{n+1} = q_P^n$ + advection.

which means that we can get the position x_P with.

$x_P = x_G - \Delta t\, \vec{u}_G$ (of course this is not exact as $\vec{u}$ should change along the vector field, but it's good enough)

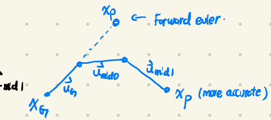
For better precision, we can divide this into multiple steps (Runge-Kutta method)

$$x_{mid} = x_G - \frac{1}{2} \Delta t\, \vec{u}_G$$
$$x_P = x_G - \Delta t\, \vec{u}_{mid}$$

or even better, use RK3 (or even higher based on $\vec{u}$ or the size of Δt)

This being x_G is not a typo.
All the mid points are there only to get the velocities!

$$x_{mid0} = \boxed{x_G} - \frac{1}{2} \Delta t\, \vec{u}_G$$
$$x_{mid1} = \boxed{x_G} - \frac{3}{4} \Delta t\, \vec{u}_{x_{mid0}}$$
$$x_P = \boxed{x_G} - \frac{2}{9} \Delta t\, \vec{u}_G - \frac{3}{9} \Delta t\, \vec{u}_{mid0} - \frac{4}{9} \Delta t\, \vec{u}_{mid1}$$



How do we do the forward advection with this?

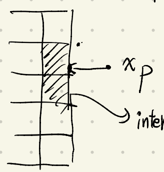
Advect-2

If x_p is outside the grid.

1. If x_p is inside the new fluid region (new fluid was pouring into the old fluid)
   $\Rightarrow$ use the value from the new fluid.

2. If x_p is just outside the grid (This can only happen due to numerical errors (Δt being too large, RK3 being not accurate))
   we extrapolate back to the grid by finding the closest point,
   (signed distance function can easily give us that)
   and interpolate based on that point.

special care for 'outside' being solid wall or ~~free~~-space (air).



x_p

interpolate using these two cells.

Although semi-Lagrangian method is 100% stable, it doesn't mean it's 100% accurate.
We can also limit the size of Δt to make it more plausible.

Taylor Series

We can approximate any function with a polynomial function. ($f(x)$)

ex) $\cos(x)$

$\cos(0)=1$, so $f(0)=1$

$\cos'(0)=0$, so $f'(0)=0$

$\cos''(0)=-1$, so $f''(0)=-1$.

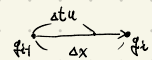
And the more we do, the more accurate $f(x)$ becomes.

$$\Rightarrow \text{we say } f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^n = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots$$

We can also approximate $u(x+\Delta x) = u(x)+\frac{\Delta x}{1!}\frac{\partial u}{\partial x}+\frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}+\cdots$

Numerical Diffusion (AKA why we can drop viscosity)

Notice that we are _always_ interpolating when performing advection.



$$q_i^{n+1} = \frac{\Delta t u}{\Delta x}q_{i-1}^n + \left(1-\frac{\Delta t u}{\Delta x}\right)q_i^n, \text{ where } \Delta t \cdot \vec{u} < \Delta x \text{ (for simplification)}$$

$$= q_i^n - \frac{\Delta t u}{\Delta x}\left(q_i^n - q_{i-1}^n\right)$$

using the taylor series (plug $-\Delta x$ instead of Δx in $u(x+\Delta x) = u(x)+\cdots$)

$$q_{i-1}^n = q_i^n - \Delta x\frac{\partial q}{\partial x}+\frac{(\Delta x)^2}{2!}\frac{\partial^2 q}{\partial x^2}+\cdots$$

plugging this to the equation above,

$$q_i^{n+1} = q_i^n - \frac{\Delta t u}{\Delta x}\Big(\underbrace{q_i^n - q_i^n + \Delta x\frac{\partial q}{\partial x} - \frac{(\Delta x)^2}{2!}\frac{\partial^2 q}{\partial x^2}+\cdots}_{\text{plugged } q_{i-1}^n}\Big)$$

$$= q_i^n - \Delta t u\frac{\partial q}{\partial x} + \Delta t u \Delta x\frac{\partial^2 q}{\partial x^2}+\cdots$$

Ignoring the last part, we can see that this is actually

$$\frac{\partial q}{\partial t}+u\frac{\partial q}{\partial x} = u\Delta x\frac{\partial^2 q}{\partial x^2}$$

in other words, $\frac{\partial q}{\partial t}+u\nabla q = u\Delta x\nabla^2 q$. $\Rightarrow$ but we started with $\frac{\partial q}{\partial t} = -u\nabla q$!!

So when we use semi-L method while ignoring the viscosity (inviscous fluid),
we still get a viscous look. This becomes a problem when we simulate any inviscous fluid.

Reducing Numerical Diffusion

Instead of linear interpolation, we can use other type of interpolation for better result.

For example, we can use cubic interpolation, which uses 4 points $\Rightarrow x_{i-1}, x_i, x_{i+1}, x_{i+2}$.

$$\begin{aligned} q \approx\ & \left(-\tfrac{1}{3}s+\tfrac{1}{2}s^2-\tfrac{1}{6}s^3\right)q_{i-1} + \\ & \left(1-s^2+\tfrac{1}{2}(s^3-s)\right)q_i + \\ & \left(s+\tfrac{1}{2}(s^2-s^3)\right)q_{i+1} + \\ & \left(\tfrac{1}{6}(s^3-s)\right)q_{i+2} \end{aligned}$$

This also works for 2D & 3D!

This might make the quantity negative, so clamp to 0.

Projection (literally)

Recall the momentum equation with the viscosity term

$$\frac{\partial \vec{u}}{\partial t} = -\vec{u}\cdot\nabla\vec{u} - \frac{1}{\rho}\nabla p + \vec{g}$$

There is a pressure term inside the equation, and we want P that will make the vector field divergence-free,

Luckily, we know one condition → continuity equation

$$\nabla\cdot\vec{u} = 0$$

Using the finite difference, (df means divergence-free, yf means yes-divergence)

$$\nabla\cdot\vec{u}^{df} = \frac{u^{df}_{i+\frac{1}{2},j,k} - u^{df}_{i-\frac{1}{2},j,k}}{\Delta x} + \frac{v^{df}_{i,j+\frac{1}{2},k} - v^{df}_{i,j-\frac{1}{2},k}}{\Delta y} + \frac{w^{df}_{i,j,k+\frac{1}{2}} - w^{df}_{i,j,k-\frac{1}{2}}}{\Delta z} \qquad (\because \Delta x = \Delta y = \Delta z)$$

$$= \frac{1}{\Delta x}\Big(u^{yd}_{i+\frac{1}{2},j,k} - \underbrace{\frac{\Delta t}{\rho}\nabla p - \big(u^{yd}_{i-\frac{1}{2},j,k} - \frac{\Delta t}{\rho}\nabla p\big) + \big(v_{i,j+\frac{1}{2},k} - \frac{\Delta t}{\rho}\nabla p}_{\text{All these } \nabla p \text{ are not the same!}}\big) - \big(v_{i,j-\frac{1}{2},k} - \frac{\Delta t}{\rho}\nabla p\big) + \big(w_{i,j,k+\frac{1}{2}} - \frac{\Delta t}{\rho}\nabla p\big) - \big(w_{i,j,k-\frac{1}{2}} - \frac{\Delta t}{\rho}\nabla p\big)\Big)$$

$$= 0.$$

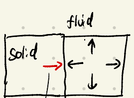
For example, the first $\nabla p = \frac{p_{i+1,j,k} - p_{i,j,k}}{\Delta x}$

while the second $\nabla p = \frac{p_{i,j,k} - p_{i-1,j,k}}{\Delta x}$ and so on.

Simplifying the equation, we get

$$\boxed{\frac{\Delta t}{\rho}\frac{\left(6p_{i,j,k} - (\text{All 6 neighboring pressures})\right)}{\Delta x^2} = -\nabla\vec{u}}$$

But what if there was no neighboring pressures? (The cell was at the boundary, or there was a solid wall inside the grid)



⇒ To keep divergence free we need the pressure that contracts it.

we need this 'ghost' pressure to keep the right cell divergence-free.

We know that the only way to keep divergence-free is by moving along with the solid wall, or more generally speaking,

$$\vec{u}_{fluid}\cdot\hat{n} = \vec{u}_{solid}\cdot\hat{n}$$ (which also means that the fluid flows tangentially)

so in $\boxed{\text{solid}}\boxed{\text{fluid}}$ case, we can say that

$$\underbrace{u^{n+1}_{i+\frac{1}{2},j,k}}_{\text{fluid}} - \underbrace{u^{solid}_{i+\frac{1}{2},j,k}}_{\text{solid}} = 0,\quad u^{n+1}_{i+\frac{1}{2},j,k} = u^{solid}_{i+\frac{1}{2},j,k}$$ where u^{n+1} is divergence-free.

↳ Note that only the velocity at the boundary is same.

$$u^{n+1}_{i+\frac{1}{2},j,k} = u^{n}_{i+\frac{1}{2},j,k} - \frac{\Delta t}{\rho}\nabla p = u^{n}_{i+\frac{1}{2},j,k} - \frac{\Delta t}{\rho}\left(\frac{\overset{\text{fluid}}{p_{i+1,j,k}} - \overset{\text{pressure}}{p_{i,j,k}}}{\Delta x}\right) = u^{solid}_{i+\frac{1}{2},j,k}.$$

Simplifying the equation,

$$\boxed{p^{solid}_{i,j,k} = p_{i+1,j,k} - \frac{\rho\Delta x}{\Delta t}\left(u^{n}_{i+\frac{1}{2},j,k} - u^{solid}_{i+\frac{1}{2},j,k}\right)}$$

As I mentioned, pressure doesn't actually exist in solid wall (it does, but it's barely effective on fluids) so this is purely a conceptual pressure, or a 'ghost' pressure.

Returning to our projection,

$$\frac{\Delta t}{\rho}\frac{\left(6p_{i,j,k} - (\text{All 6 neighboring pressures})\right)}{\Delta x^2} = -\nabla\vec{u}$$

Let's say we have one solid wall on $(i-1,j,k)$. using the ghost pressure,

$$\frac{\Delta t}{\rho}\left(6p_{i,j,k} - \left(p_{i,j,k} - \frac{\rho\Delta x}{\Delta t}\left(u^{n}_{i+\frac{1}{2},j,k} - u^{solid}_{i+\frac{1}{2},j,k}\right)\right) - (\text{All the other } \overset{5}{\vee} \text{ pressures})\right) = -\Delta x^2\nabla\vec{u}$$

$$\frac{\Delta t}{\rho}\left(5p_{i,j,k} - (\text{All 5 neighboring pressures})\right) = -\Delta x^2\nabla\vec{u} - \Delta x\left(u^{n}_{i+\frac{1}{2},j,k} - u^{solid}_{i+\frac{1}{2},j,k}\right)$$

This tells us that whenever we have a neighboring solid wall, we set the pressure (ghost) to $P_{i,j,k}$. which will effectively make $N P_{i,j,k}$ to $(N-1)P_{i,j,k}$, and add to the right side.

So the equation (in code) can be

$$P_{i,j,k} = \left(\frac{-\Delta x^2\cdot\rho}{\Delta t}\nabla\vec{u} - \frac{\Delta x\cdot\rho}{\Delta t}\left(u^{n}_{i+\frac{1}{2},j,k} - u^{solid}_{i+\frac{1}{2},j,k}\right) + \text{All 6 pressures}\right)\cdot\frac{1}{6}$$

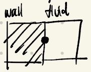
(one of them equals to $P_{i,j,k}$ in this case)

This can be generalized as:

$$\boxed{P_{i,j,k} = \frac{1}{6}\cdot\frac{\rho}{\Delta t}\cdot\Big(-\Delta x^2\nabla\vec{u} - \Delta x\big(\underbrace{u-u}_{u^{solid}-u^{fluid},\ \text{or}\ u^{fluid}-u^{solid}\ \text{depending on the location of solid wall}}\big) + \text{All 6 neighboring pressures}\Big)}$$

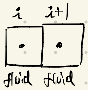
So in conclusion,

if the velocity is located on the boundary



set it to $u^{\text{solid wall}}$ (0 if solid wall is stationary)

else



$$u^{n+1}_{i+\frac{1}{2}} = u^{n}_{i+\frac{1}{2}} - \frac{\rho}{\Delta t}\nabla P$$
$$= u^{n}_{i+\frac{1}{2}} - \frac{\rho}{\Delta t}\left(\frac{P_{i+1}-P_i}{\Delta x}\right)$$