Derivatives Portfolio Optimization and Parameter Uncertainty

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Abstract

Portfolio optimization in practice almost always needs to account for parameter uncertainty. Resampled portfolio optimization is a common heuristic to tackle the parameter uncertainty issue. The recently introduced Exposure Stacking method makes the resampled approach even more attractive. While resampled optimization of cash portfolios is straightforward and well-known, resampled optimization of portfolios containing derivatives is a largely unexplored area. Derivatives introduce an additional layer of complexity, because there needs to be a logical consistency between the parameter uncertainty of the underlying, risk factors such as implied volatilities, and the derivative instrument's P&L. This article presents an elegant solution to the problem and introduces a new class of portfolio optimization with fully general risk factor parameter uncertainty.

Documented Python code that replicates the results of the case study is available in the open-source package fortitudo.tech. More information about the package can be found on https://os.fortitudo.tech.

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1 Introduction

It is well-known that portfolio optimization problems are highly sensitive to parameter estimates, or more generally the market model input, see Kristensen and Vorobets (2024). To tackle this issue, many practitioners use some variant of the resampled portfolio optimization method introduced by Michaud and Michaud (1998) and refined by Kristensen and Vorobets (2024) with the Exposure Stacking method.

Portfolio optimization for derivatives portfolios is a quite unexplored area, with a documented portfolio management framework for derivative instruments only appearing recently in Vorobets (2022a). Portfolio optimization with parameter uncertainty for portfolios containing derivatives is an even less explored area.

Derivatives introduce only a simple extra layer of complexity in general portfolio management that requires us to separate between relative market values $v \in \mathbb{R}^I$ and relative exposures $e \in \mathbb{R}^I$, see Vorobets (2022a) for a detailed explanation.

For portfolio optimization with parameter uncertainty, derivatives introduce significantly more complexity as we must ensure that the derivatives P&L is consistent with the parameter uncertainty we introduce in the underlying and risk factors such as implied volatilities. For example, the expected P&L of a European option at expiry is fully determined by the P&L of the underlying. Hence, we cannot introduce separate parameter uncertainty into the underlying and the derivative instrument if we want to maintain logical consistency.

This article is very short and sweet, focusing primarily on portfolio optimization with parameter uncertainty for portfolios containing derivatives. For references on Sequential Entropy Pooling (SeqEP), Exposure Stacking, and the portfolio management framework for derivative instruments, see Vorobets (2021), Kristensen and Vorobets (2024), and Vorobets (2022a).

The rest of this article is organized as follows: Section 2 gives a recap of the fully general Monte Carlo market representation and the Exposure Stacking method, Section 3 presents the method for consistent parameter uncertainty for portfolios containing derivatives, Section 4 is a case study performing derivatives portfolio optimization using the new approach, and Section 5 is a conclusion.

2 Market representation and Exposure Stacking recap

The market is represented by a matrix of joint market scenarios $R \in \mathbb{R}^{S \times I}$ and an associated scenario probability vector $p \in \mathbb{R}^S$, see Vorobets (2024) for a careful walkthrough of simulation and analysis methods applicable for this market representation. The Exposure Stacking method, introduced by Kristensen and Vorobets (2024), focuses on the risk-adjusted return objectives f and g that additionally contain optimization constraints $\mathcal{E}$, i.e.,

$$e^\star = \underset{e}{\operatorname{argmax}}\ f\left(R, p, \mathcal{E}, e\right) = \underset{e}{\operatorname{argmin}}\ g\left(R, p, \mathcal{E}, e\right).$$

The risk-adjusted objective can be defined for any portfolio risk measure, for example, variance or CVaR. In practice, CVaR is recommended given that empirical return distributions are usually very far from being normally or elliptically distributed, see Vorobets (2022b) for further details. When the investment universe contains options, the linear and constant dependency assumption inherent to mean-variance will also obviously be violated, see Vorobets (2022a) for a comparison between CVaR and variance optimization with options.

The Exposure Stacking method operates on samples of the market (R_b, p_b), where the optimal exposures for sample $b \in \{1, 2, \ldots, B\}$ are defined by

$$e_b^\star = \underset{e}{\operatorname{argmax}}\ f\left(R_b, p_b, \mathcal{E}, e\right) = \underset{e}{\operatorname{argmin}}\ g\left(R_b, p_b, \mathcal{E}, e\right).$$

For the mean-variance objective, the market samples consist simply of mean vectors μ_b and covariance matrices Σ_b. For the mean-CVaR objective, the market samples can be entirely new joint market scenarios R_b and associated probability vectors p_b, with p_b possibly generated using Sequential Entropy Pooling (SeqEP) as described by Vorobets (2021) and Vorobets (2023).

For the sample exposures $e_b^\star$, we have the opportunity of selecting sample weights $w_b \in [0,1]$ with $\sum_{b=1}^{B} w_b = 1$, which we use to estimate the final weighted resampled exposures vector

$$e_w^\star = \sum_{b=1}^{B} w_b e_b^\star.$$

The traditional resampled portfolio estimator introduced by Michaud and Michaud (1998) corresponds to $w_b = \frac{1}{B}$ for all $b \in \{1, 2, \dots, B\}$.

The Exposure Stacking method defines an objective function for the weights vector $w = (w_1, w_2, \dots, w_b)^T$ based on the ideas of stacked regression and cross-validation, see Breiman (1996). Let $\mathcal{B} = \{1, 2, \dots, B\}$ be the set of sample indices and suppose that we partition $\mathcal{B}$ into L nonempty sets $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_L$ for some $L \in \{2, 3, \dots, B\}$. For any choice of $l \in \{1, 2, \dots, L\}$, we consider the set of exposure vectors $\{e_k \mid k \in \mathcal{K}_l\}$ as a validation set, find sample weights w_b for the remaining $B - |\mathcal{K}_l|$ exposure vectors, and calculate the average difference ε_l between the weighted exposure and the validation set exposures, i.e., we compute

$$\varepsilon_l = \frac{1}{|\mathcal{K}_l|} \sum_{k \in \mathcal{K}_l} \left\| e_k^\star - \sum_{b \notin \mathcal{K}_l} w_b e_b^\star \right\|_2^2 .$$

If we repeat the analysis using each of the sets $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_L$ as our validation set indices, we can compute the L-fold cross-validation estimate as the average $\varepsilon = \frac{1}{L} \sum_{l=1}^{L} \varepsilon_l$, see James et al. (2023). Thus, the Exposure Stacking method solves the problem

$$w^\star = \underset{w}{\operatorname{argmin}} \frac{1}{L} \sum_{l=1}^{L} \left(\frac{1}{|\mathcal{K}_l|} \sum_{k \in \mathcal{K}_l} \left\| e_k^\star - \sum_{b \notin \mathcal{K}_l} w_b e_b^\star \right\|_2^2 \right). \tag{1}$$

Appendix B in Kristensen and Vorobets (2024) shows that (1) can be formulated as a quadratic programming problem and therefore solved in a fast and stable way.

3 Derivatives and parameter uncertainty

Derivatives in general portfolio management are fairly straightforward to handle once you split between relative market values $v \in \mathbb{R}^I$ and relative exposures $e \in \mathbb{R}^I$, see Vorobets (2022a) for a detailed explanation. When it comes to parameter uncertainty, derivatives become more challenging given that their P&L is a function of various risk factors including the underlying. Hence, we must ensure consistency between the parameter uncertainty we introduce in the underlying, the risk factors, and the derivative's P&L. Otherwise, we will make logical errors in the resampled optimization.

While Exposure Stacking allows us to introduce parameter uncertainty by generating both new market simulations R_b and scenario probability vectors p_b for $b = 1, 2, \dots, B$, we probably want to avoid the costly simulation and pricing of derivatives associated with generating R_b for each sample. Hence, we fix $R_b = R$

for all samples and introduce parameter uncertainty into the derivative instruments by adjusting p_b.

An elegant feature of the Entropy Pooling method, introduced by Meucci (2008), is that it allows us to avoid the potentially costly repricing of derivative instruments, see the case study in Vorobets (2022a). The special aspect of derivatives is that they are inseparable from the underlying and the derivative's other risk factors. Hence, parameter uncertainty related to derivatives P&L should be introduced through parameter uncertainty in the underlying and the derivative's risk factors. This is exactly what the new approach does.

For each sample $b \in \{1, 2, \dots, B\}$, the algorithm is:

1. Introduce parameter uncertainty into the non-derivative instruments and risk factors.

2. Compute Entropy Pooling posterior probability vectors p_b using the new parameters as views.

3. Compute CVaR optimal exposures $e_b^\star$ using R and p_b.

With the sample optimal exposures $e_b^\star$ at hand, we compute the final optimal exposure of the resampled estimator $e_w^\star$ using Exposure Stacking to compute the sample weights w_b. We note that using CVaR instead of variance becomes especially crucial for option portfolios that clearly have nonlinear dependencies, which the covariance matrix cannot handle.

The procedure can also be used to generate parameter uncertainty for risk factor distributions in general. For example, imagine a portfolio of US government bonds where you want to introduce uncertainty into the 10 year zero-coupon yield or a set of key interest rates. This is likely a better approach than introducing, for example, mean uncertainty into all of the bonds. An elegant feature of the framework is that it also assists us in estimating what happens to, for example, the implied volatility when we introduce uncertainty in the underlying. We do not have to specify this manually if we do not want to.

4 Case study

In this case study, we use the same P&L simulation and portfolio restrictions as in Vorobets (2022a). We introduce mean parameter uncertainty for all cash instruments similar to Kristensen and Vorobets (2024) and use EP to compute the distributions for the derivatives instruments for each sample $b = 1, 2, \dots, B$. Finally, we compare the results of the middle portfolio for the traditional resampled estimator, 2-fold Exposure Stacking, and the normal frontier portfolio. The interested reader can examine the accompanying code for more information[1].

Since CVaR optimization is computationally more intensive than variance, we only use $B = 100$ samples and $P = 5$ portfolios to span the efficient frontier. This takes about two minutes to run. Readers can experiment with more samples and portfolios, while there is no guarantee that the open-source mean-CVaR implementation will run without numerical issues. The open-source implementation of mean-CVaR optimization is not at a sufficient level to be used in production by institutional investment managers, while it should be sufficient to explore case studies like this one on moderately-sized problems.

Similar to Kristensen and Vorobets (2024), we show the efficient frontier and the resampled portfolios in Figure 1. Table 1 shows the results for the middle portfolio using the traditional equal weight resampled optimization, 2-fold Exposure Stacking, and normal efficient frontier optimization. We use 2-fold Exposure Stacking due to the results from Kristensen and Vorobets (2024), while noting that $B = 100$ samples is on the lower end of what is recommended in practice.

[1] https://github.com/fortitudo-tech/fortitudo.tech/blob/main/examples/10_DerivPortOpt_ParamUncertainty.ipynb

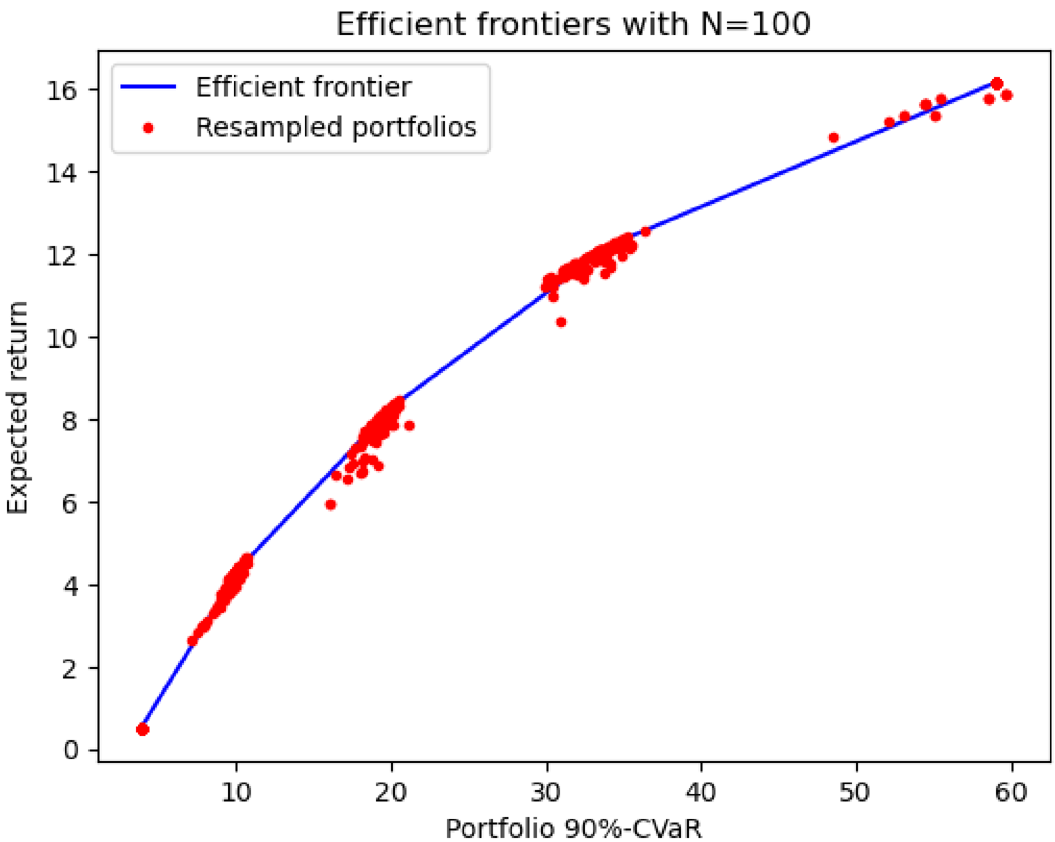

Figure 1: Mean-CVaR efficient frontier with underlying mean uncertainty.

	Resampled	2-fold Exposure Stacking	Frontier portfolio
Gov & MBS	1.36%	0.00%	0.00%
Corp IG	0.00%	0.00%	0.00%
Corp HY	0.03%	0.00%	0.00%
EM Debt	12.56%	11.50%	12.39%
DM Equity	5.30%	0.00%	0.00%
EM Equity	0.12%	0.00%	0.00%
Private Equity	12.77%	12.03%	10.28%
Infrastructure	21.32%	25.00%	25.00%
Real Estate	17.72%	21.81%	22.23%
Hedge Funds	24.17%	25.00%	25.00%
Put 90	-49.65%	-50.00%	-50.00%
Put 95	-6.87%	-14.32%	-19.04%
Put ATMF	49.76%	50.00%	50.00%
Call ATMF	3.95%	7.11%	17.55%
Call 105	47.03%	50.00%	50.00%
Call 110	50.00%	50.00%	50.00%

Table 1: Portfolio optimization results for the middle 90%-CVaR portfolio.

5 Conclusion

This article introduces a new method for handling portfolio optimization parameter uncertainty for portfolios containing derivative instruments by combining Entropy Pooling and Exposure Stacking in a clever way. It shows how to use the method to perform 90%-CVaR optimization by introducing mean uncertainty in the underlying instruments. The approach can also be used more broadly to introduce fully general risk factor uncertainty.

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