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Mean-risk models using two risk measures: a multi-objective approach

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This paper proposes a model for portfolio optimization, in which distributions are characterized and compared on the basis of three statistics: the expected value, the variance and the CVaR at a specified confidence level. The problem is multi-objective and transformed into a single objective problem in which variance is minimized while constraints are imposed on the expected value and CVaR. In the case of discrete random variables, the problem is a quadratic program. The mean-variance (mean-CVaR) efficient solutions that are not dominated with respect to CVaR (variance) are particular efficient solutions of the proposed model. In addition, the model has efficient solutions that are discarded by both mean-variance and mean-CVaR models, although they may improve the return distribution. The model is tested on real data drawn from the FTSE 100 index. An analysis of the return distribution of the chosen portfolios is presented.

Keywords: Portfolio selection; Multi-objective optimisation; Risk measures; Conditional Value at Risk

1. Introduction and motivation

Mean-risk models are still the most widely used approach in the practice of portfolio selection. With mean-risk models, return distributions are characterized and compared using two statistics: the expected value and the value of a risk measure. Thus, mean-risk models have a ready interpretation of results and in most cases are convenient from a computational point of view. Sceptics on the other hand may question these advantages since the practice of describing a distribution by just two parameters involves great loss of information.

It is evident that the risk measure used plays an important role in making the decisions. Variance was the first risk measure used in mean-risk models (Markowitz 1952) and, in spite of criticism and many proposals of new risk measures (see e.g. Fishburn 1977, Yitzhaki 1982, Konno and Yamazaki 1991, Ogryczak and Ruszczynski 1999, 2001, Rockafellar and Uryasev 2000, 2002), variance is still the most widely used measure of risk in the practice of portfolio selection. For regulatory and reporting purposes, risk measures concerned with the left tails of distributions (extremely unfavourable outcomes) are used. The most widely used risk measure for such purposes is Value-at-Risk (VaR). However, it is known that VaR has undesirable theoretical properties (it is not subadditive, as shown, for example, in Tasche (2002) and thus fails to reward diversification). In addition, optimization of VaR leads to a non-convex NP-hard problem which is computationally intractable. In spite of a considerable amount of research, optimizing VaR is still an open problem (see e.g. Larsen *et al*. 2002, Leyffer *et al*. 2005 and references therein). For these reasons, another risk measure concerned with the left tail, the Conditional Value-at-Risk (CVaR), is gaining more popularity. CVaR has attractive theoretical properties: it controls the magnitude of losses beyond VaR and it is coherent (see e.g. Artzner *et al*. 1999, Pflug 2000, Acerbi and Tasche 2002, Rockafellar and Uryasev 2002, Tasche 2002). In addition, CVaR is easy to optimize.

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Optimizing CVaR is a convex programming problem. In the case when the random variables under consideration are discrete, with a finite number of outcomes, represented by various outcomes under different scenarios, optimizing CVaR leads to a linear programming model of finite dimension (Rockafellar and Uryasev 2000, 2002).

Variance and CVaR quantify risk from different perspectives. Variance measures the spread around the expected value of a random variable, while CVaR measures the expected loss corresponding to a number of worst cases, depending on the chosen confidence level. Thus, the mean-variance and the mean-CVaR models may lead to very different solutions. A portfolio obtained as a solution in the mean-variance model may be considered unacceptable by a regulator, since it may have an excessively large CVaR, leading to big losses under unfavourable scenarios. On the other hand, traditional fund managers may consider a portfolio obtained with the mean-CVaR model unacceptable since it may have an excessively large variance and thus an excessively small Sharpe index (see Luenberger 1998)

In this paper, we seek to address the requirements of the traditional fund manager and the regime imposed by the regulator. We propose a model for portfolio selection that uses both variance and CVaR in order to make decisions. We call this model the mean-variance-CVaR model. Random variables are described and compared using three statistics: the expected value, variance and CVaR. Thus, the model may be considered as belonging to the family of mean-risk models.

We formally define the preference relation for random variables in this model. The efficient solutions with respect to this preference relation are such that we cannot improve on one statistic (of the three: expected value, variance and CVaR) without worsening another. Mathematically, the problem is multi-objective (maximize expected return, minimize variance, minimize CVaR) and the efficient solutions of the mean-variance-CVaR model are the Pareto optimal solutions of the multi-objective problem.

We prove that the efficient solutions of this model may be found by solving a single objective optimization problem in which variance is minimized while constraints are imposed on the expected return and the CVaR level. The practical importance of this approach is twofold. Firstly, a solution obtained in this way has an intuitive appeal. For example, if the CVaR of a mean-variance efficient portfolio is considered as unacceptably large, a constraint could be imposed on the CVaR level and a new portfolio obtained, which has a minimal variance under these conditions. Secondly, the problem is tractable from a computational point of view. In the case where the random variables under consideration are discrete and described by their realizations under various scenarios, the problem is one of quadratic programming.

Generally, the mean-variance and mean-CVaR efficient portfolios are particular efficient solutions of the proposed model.† However, most of the efficient portfolios in the mean-variance-CVaR model are dominated in both mean-variance and mean-CVaR models, although they may represent improved distributions: a compromise between the classical fund managers' and the regulators' points of view.

The rest of this paper is structured as follows. In section 2 the portfolio selection problem is described. Section 3 is concerned with mean-risk models, in particular with the mean-variance and the mean-CVaR models. In section 4 we present the mean-variance-CVaR model. Firstly, the preference relation among random variables is defined. The efficient solutions of the proposed model are Pareto non-dominated solutions of a multi-objective problem. Secondly, an optimization approach for solving the multi-objective problem is proposed. With this approach, the efficient solutions of the proposed model are found by solving a single optimization problem, in which variance is minimized and constraints are imposed on the expected value and the CVaR level. Thirdly, we describe how all the efficient solutions of the model may be obtained. Finally, the algebraic form of the mean-variance-CVaR model for the case of scenario models is presented. Section 5 presents the computational results. A dataset, drawn from the FTSE 100 index is used to evaluate the performance of the proposed model. For several fixed levels of expected return, we consider the mean-variance and the mean-CVaR efficient portfolios together with other portfolios, efficient only in the mean-variance-CVaR model. We evaluate their performances using both in-sample and out-of-sample analysis. Section 6 presents the conclusions.

2. The portfolio selection problem

The problem of portfolio selection with one investment period is an example of the general problem of deciding between random variables when larger outcomes are preferred. Decisions are required on the amount (proportion) of capital to be invested in each of a number of available assets such that at the end of the investment period the return is as high as possible. Consider a set of n assets, with asset j in $\{1, \ldots, n\}$ giving a return R_j at the end of the investment period. R_j is a random variable, since the future price of the asset is not known. Let x_j be the proportion of capital invested in asset j ($x_j = w_j/w$ where w_j is the capital invested in asset j and w is the total amount of capital to be invested), and let $x = (x_1, \ldots, x_n)$ represent the portfolio resulting from this choice. This portfolio's return is the random variable: $R_x = x_1R_1 + \cdots + x_nR_n$, with distribution function

†There may be a situation when several mean-CVaR efficient portfolios have the same mean return and the same (optimal) CVaR, but different variances. Only the portfolio with the minimal variance is efficient in the proposed model. The same discussion applies for mean-variance efficient portfolios. We reconsider the issue in section 4.4.

$F(r) = P(R_x \le r)$ that depends on the choice $x = (x_1, \ldots, x_n)$.

To represent a portfolio, the weights $(x_1, \ldots, x_n)$ must satisfy a set of constraints that forms a feasible set $\mathcal{A}$ of decision vectors. The simplest way to define a feasible set is by the requirement that the weights must sum to 1 and short selling is not allowed. For this basic version of the problem, the set of feasible decision vectors is

$$\mathcal{A} = \left\{ (x_1, \ldots, x_n) \Big/ \sum_{j=1}^{n} x_j = 1,\ x_j \ge 0,\ \forall j \in \{1, \ldots, n\} \right\}. \tag{1}$$

Consider a different portfolio defined by the decision vector $y = (y_1, \ldots, y_n) \in \mathcal{A}$, where y_j is the proportion of capital invested in asset j. The return of this portfolio is given by the random variable $R_y = y_1 R_1 + \cdots + y_n R_n$.

The problem of choosing between portfolio $x = (x_1, \ldots, x_n)$ and portfolio $y = (y_1, \ldots, y_n)$ becomes the problem of choosing between random variables R_x and R_y. The criteria by which one random variable is considered 'better' than another random variable need to be specified and models for choosing between random variables (models for preference) are required. The purpose of such models is firstly, to define a preference relation among random variables and secondly, to identify random variables that are non-dominated with respect to that preference relation.

The next issue is to consider a practical representation for the random variables that describe asset and portfolio returns. We treat these random variables as discrete and described by realizations under T states of the world, generated using scenario generation or finite sampling of historical data. For any $i \in 1, \ldots, T$, let state ω_i occur with probability p_i, $\sum_{i=1}^{T} p_i = 1$. Thus, the random returns are defined on a discrete probability space $\{\Omega, \mathcal{F}, P\}$ with $\Omega = \{\omega_1, \ldots, \omega_T\}$, $\mathcal{F}$ a σ-field and $P(\omega_i) = p_i$.

Let r_{ij} be the return of asset j under scenario i, $i \in \{1, \ldots, T\}$, $j \in \{1, \ldots, n\}$. Thus, the random variable R_j representing the return of asset j is finitely distributed over $\{r_{1j}, \ldots, r_{Tj}\}$ with probabilities $p_1, \ldots, p_T$. The random variable R_x representing the return of portfolio $x = (x_1, \ldots, x_n)$ is finitely distributed over $\{R_{x1}, \ldots, R_{xT}\}$, where $R_{xi} = x_1 r_{i1} + \cdots + x_n r_{in}$, $\forall i \in \{1, \ldots, T\}$.

3. Mean-risk models

3.1. The general case

Mean-risk models were developed in the early 1950s for the portfolio selection problem. In his seminal work 'Portfolio selection', Markowitz (1952) proposed variance as a risk measure. Since then, many alternative risk measures have been proposed. The question of which risk measure is most appropriate is still the subject of much debate.

In mean-risk models, two scalars are attached to each random variable: the expected value (mean) and the value of a risk measure. Preference is then defined using a trade-off between the mean where a larger value is desirable and risk where a smaller value is desirable:

> In the mean-risk approach with the risk measure denoted by ρ, random variable R_x dominates (is preferred to) random variable R_y if and only if: $E(R_x) \ge E(R_y)$ and $\rho(R_x) \le \rho(R_y)$ with at least one strict inequality. Alternatively, we can say that portfolio x dominates portfolio y.

In this approach, the choice x (or the random variable R_x) is efficient (non-dominated) if and only if there is no other choice y such that R_y has higher expected value and less risk than R_x. This means that, for a given level of minimum expected return, R_x has the lowest possible risk, and, for a given level of risk, it has the highest possible expected return. Plotting the efficient portfolios in a mean-risk space gives *the efficient frontier*.

Thus, the efficient solutions in a mean-risk model are Pareto efficient solutions of a multi-objective problem, in which the expected return is maximized and the risk is minimized:

$$\max\{(E(R_x),\ -\rho(R_x)) : x \in A\}.$$

Generally, for a multi-objective problem:

$$\max\{f(x) = (f_1(x), \ldots, f_T(x)) : x \in A\}, \tag{2}$$

the Pareto preference relation is defined as follows:

> A feasible solution $x^1 \in \mathcal{A}$ *Pareto dominates* another feasible solution $x^2 \in \mathcal{A}$ if $f_i(x^1) \ge f_i(x^2)$ for all i with at least one strict inequality.

x_0 is a Pareto efficient (non-dominated) solution of (2) if and only if there does not exist a feasible x such that x Pareto dominates x_0. In other words, a Pareto efficient solution is a feasible solution such that, in order to improve upon one objective function, at least one other objective function must assume a worse value.

In order to find an efficient portfolio, we solve an optimization problem with decision variables $x_1, \ldots x_n$:

$$\begin{aligned}&\text{Minimize } \rho(R_x)\\ &\text{Subject to: } E(R_x) \ge d \quad \text{and} \quad (x_1, \ldots x_n\} \in \mathcal{A},\end{aligned}$$

where d represents the desired level of expected return for the portfolio.

Varying d and repeatedly solving the corresponding optimization problem identifies the minimum risk portfolio for each value of d. These are the efficient portfolios that compose the efficient set. By plotting the corresponding values of the objective function and of the expected return respectively in a return-risk space, we trace out *the efficient frontier*.

An alternative formulation, which explicitly trades risk against return in the objective function, is

$$\begin{aligned}&\text{Maximize } E(R_x) - \tau\rho(R_X)\ (\tau \ge 0)\\ &\text{Subject to: } (x_1, \ldots x_n\} \in A.\end{aligned}$$

Varying the trade-off coefficient τ and repeatedly solving the corresponding optimization problems traces out the efficient frontier.

3.2. The mean-variance model

The variance of a random variable R_x is defined as its second central moment:

$$\sigma^2(R_x) = E[(R_x - E(R_x))^2].$$

An important property is that the variance of the portfolio return $R_x = x_1R_1 + \cdots + x_nR_n$, resulting from choice $(x_1, \ldots, x_n)$, can be expressed as:

$$\sigma^2(R_x) = \sum_{k=1}^{n}\sum_{j=1}^{n} x_k x_j \sigma_{kj},$$

where σ_{kj} is the covariance of R_k and R_j, and thus variance is expressed as a quadratic function of $x_1, \ldots, x_n$.

The mean-variance model can be formulated for the portfolio selection problem as follows:

$$\begin{aligned} &\text{Minimize} \sum_{k=1}^{n}\sum_{j=1}^{n} x_k x_j \sigma_{kj} \\ &\text{Subject to} \\ &\qquad \sum_{j=1}^{n} \mu_j x_j \geq d \\ &\qquad \sum_{i=1}^{n} x_j = 1 \\ &\qquad x_j \geq 0, \quad \forall j = 1, \ldots, n, \end{aligned}$$

where μ_j is the expected rate of return of asset j, $j \in \{1, \ldots, n\}$; σ_{kj} is the covariance between returns of asset k and asset j, with k, $j \in \{1, \ldots, n\}$; and d is the desired expected value of the portfolio return.

3.3. The mean-CVaR model

Let R_x be a random variable representing the return of a portfolio x over a given holding period and $A\% = \alpha \in (0,1)$ a percentage which represents a sample of 'worst cases' for the outcomes of R_x (usually, $\alpha = 0.01$ or $\alpha = 0.05$).

The definition of CVaR at the specified level α is the mathematical transcription of the concept 'average of losses in the worst A% of cases'† (Acerbi and Tasche 2002), where a 'loss' is a negative outcome of R_x (thus the loss associated with R_x is described by the random variable $-R_x$).

Formally, the Conditional Value-at-Risk at level α of R_x is defined as minus the mean of the α-tail distribution of R_x, where the α-tail distribution is obtained by taking the lower α part of the distribution of R_x (corresponding to extreme unfavourable outcomes) and rescaling its distribution function to span [0,1]:

$$\begin{aligned} &\mathrm{CVaR}_\alpha(R_x) \\ &= -\frac{1}{\alpha}\left\{E(R_X 1_{\{R_X \leq q^\alpha(R_x)\}}) - q^\alpha(R_x)[P(R_X \leq q^\alpha(R_x)) - \alpha]\right\}, \end{aligned} \tag{3}$$

where q^α is an α-quantile of R_x, meaning a real number r such that $P(R_x < r) \leq \alpha \leq P(R_x \leq r)$ (see Laurent (2003) for more details on α-quantiles), and

$$\begin{aligned} 1_{\{\text{Relation}\}} = 1,&\ \text{if Relation is true} \\ 0,&\ \text{if Relation is false} \end{aligned}$$

(see Rockafellar and Uryasev (2000, 2002) for more details).

An important result, proved by Rockafellar and Uryasev (2000, 2002), and independently by Ogryczak and Ruszczynski (2002), is that the CVaR of a random variable R_x can be calculated by solving a convex optimization problem. Moreover, CVaR can be minimized over the set of feasible decision vectors. These results are summarized below:

Proposition 1: (CVaR calculation and optimization). *Let R_x be a random variable depending on a decision vector x that belongs to a feasible set $\mathcal{A}$, and $\alpha \in (0,1)$. Consider the function*:

$$F_\alpha(x, v) = \frac{1}{\alpha} E\{[-R_x + v]^+\} - v,$$

where

$$\begin{aligned} [u]^+ &= u \quad \text{for } u \geq 0, \\ [u]^+ &= 0 \quad \text{for } u < 0. \end{aligned}$$

Then:

(*a*) *As a function of v, F_α is finite and continuous* (*hence convex*) *and*

$$\mathrm{CVaR}_\alpha(R_x) = \min_{v \in R} F_\alpha(x, v).$$

In addition, the set consisting of the values of v for which the minimum is attained, denoted by $A_\alpha(x)$, is a non-empty, closed and bounded interval (*possibly formed by just one point*).

(*b*) *Minimizing* CVaR_α *with respect to* $x \in \mathcal{A}$ *is equivalent to minimizing F_α with respect to* $(x, v) \in \mathcal{A}xR$:

$$\min_{x \in A} \mathrm{CVaR}_\alpha(R_x) = \min_{(x, v) \in AxR} F_\alpha(x, v).$$

In addition, a pair (x^*, v^*) minimizes the right-hand side if and only if x^* *minimizes the left-hand side and* $v^* \in A_\alpha(x^*)$.

(*c*) $\mathrm{CVaR}_\alpha\ (R_x)$ *is convex with respect to x and $F_\alpha\ (x, v)$ is convex with respect to* (x, v).

†This is not necessarily the same as 'the expected value of losses exceeding VaR at confidence level α', as it is defined in earlier papers on CVaR. The two definitions lead to the same results when the distribution of the random variable under consideration is continuous, but differences may appear when the considered distribution has discontinuities—see Acerbi and Tasche (2002), and Rockafellar and Uryasev (2002) for more details.

Thus, if the set $\mathcal{A}$ of feasible decision vectors is convex (which is the case for the basic version of the portfolio selection problem), and even if we impose a further lower limit on the expected return, minimizing CVaR is a convex optimization problem.

In the case when R_x is a discrete random variable (as described in section 2), calculating and optimizing CVaR are linear programming problems. Suppose that R_x has T possible outcomes $R_{x1}, \ldots, R_{xT}$ with probabilities $p_1, \ldots, p_T$. Then:

$$F_\alpha(x, v) = \frac{1}{\alpha}\sum_{i=1}^{T} p_i[v - R_{xi}]^+ - v.$$

For the portfolio selection problem, as presented in section 2, where $R_{xi} = \sum_{j=1}^{n} x_j r_{ij}$ with r_{ij} the return of asset j under scenario i,

$$F_\alpha(x, v) = \frac{1}{\alpha}\sum_{i=1}^{T} p_i\left[v - \sum_{j=1}^{n} x_j r_{ij}\right]^+ - v.$$

Thus, the mean-CVaR$_\alpha$ model can be formulated for the portfolio selection problem as follows:

$$\text{Minimize} - v + \frac{1}{\alpha}\sum_{i=1}^{T} p_i y_i$$

Subject to

$$\sum_{j=1}^{n} -r_{ij}x_j + v \leq y_i, \quad \forall i = 1, \ldots, T$$

$$y_i \geq 0, \quad \forall i = 1, \ldots, T$$

$$\sum_{j=1}^{n} \mu_j x_j \geq d$$

$$\sum_{j=1}^{n} x_j = 1$$

$$x_j \geq 0, \quad \forall j = 1, \ldots, n$$

4. The mean-variance-CVaR model

4.1. The theoretical background

In this section, a model for portfolio selection is proposed, in which random variables are described by three statistics: the expected value, the variance and the CVaR at a specified confidence level $\alpha \in (0,1)$. We claim that taking three parameters into consideration, instead of two, gives a better modelling power. The proposed model may bring an improvement in the solution, in the case where a mean-variance efficient portfolio has an excessively large CVaR, or a mean-CVaR efficient portfolio has an excessively large variance.

The idea of restricting the risk of a distribution from two different perspectives has been used before in various contexts.

Konno *et al*. (1993) proposed a 'mean-absolute deviation skewness portfolio optimization model', in which the lower semi-third moment of the portfolio return is maximized subject to constraints on the mean and on the absolute deviation of the portfolio return. A 'mean-variance-skewness portfolio optimization model' was proposed by Konno *et al*. (1995): they maximized the third moment of the portfolio return subject to constraints on the mean and on the variance of the portfolio return. Optimization approaches are provided, in which the corresponding cubic and quadratic functions are approximated by linear functions.

Wang (2000) proposed a model in which the portfolio return has constraints on both variance and Value-at-Risk (VaR), and a maximum expected return under these conditions. However, no practical optimization approach is provided.

Harvey *et al*. (2003) proposed a model in which random variables are chosen with respect to their expected value, variance and skewness. Thus, it may be considered that they use two risk measures in order to control the selection of a solution: the variance and the negative of skewness. Their model has a distributional assumption for portfolio returns and uses an expected utility maximization approach, with the utility function depending on the expected value, variance and skewness.

Jorion (2003) proposed that a portfolio return distribution should have constraints on both variance and 'tracking error volatility', which is 'the volatility of the deviation of the active portfolio from the benchmark', with a maximum expected return under these conditions. Thus, this approach may also fall into the category of index-tracking models.

There have been various formulations of portfolio selection problems as multiple criteria models (see e.g. Ogryczak 2000, 2002). However, to the best of our knowledge, the use of CVaR together with variance within a multi-attribute model is novel. A categorized bibliography on the applications of multiple criteria decision-making techniques in finance is provided in Steuer and Na (2003).

The model proposed in this paper does not assume a particular distribution for the returns and, in addition, is convenient from a computational point of view. We define a preference relation for random variables and provide an optimization approach for finding the efficient solutions with respect to this preference relation.

Consider again the portfolio selection problem described in section 2, with the random variable R_x and R_y describing the returns of portfolios x and y respectively, with $x, y \in \mathcal{A}$.

We consider a model for choice under risk that we refer to as *the mean-variance*-CVaR *model*, in which the preference relation among random variables is defined as follows:

> In the mean-variance-CVaR model, a random variable R_x is preferred to a random variable R_y (or, similarly, the portfolio x is preferred to portfolio y) if and only if $E(R_x) \geq E(R_y)$, $\sigma^2(R_x) \leq \sigma^2(R_y)$ and $\text{CVaR}_\alpha(R_x) \leq \text{CVaR}_\alpha(R_y)$, with at least one strict inequality.

Thus, the non-dominated (efficient) solutions in the mean-variance-CVaR model are the Pareto efficient solutions of a multi-objective problem in which the expected value is maximized while the variance and the CVaR are minimized:

$$\text{(MVC): } \max(E(R_x), -\sigma^2(R_x), -\text{CVaR}_\alpha(R_x))$$
$$\text{Subject to: } x \in \mathcal{A}.$$

When plotting the efficient solutions in a mean-variance-CVaR space, a surface is obtained; we refer to this surface as 'the efficient frontier' of the mean-variance-CVaR model.

4.2. An optimization approach

The next issue to address is how to obtain the efficient solutions of the mean-variance-CVaR model.

Firstly, the multi-objective problem (MVC) is transformed into a single objective problem in which one objective function is optimized while lower limits are imposed on the remaining objective functions and transformed into constraints. This method, known in multi-objective optimization as the 'ε-constraint method' (Haimes *et al.* 1971, see also Steuer 1986) generally requires some regularization in order to guarantee that an optimal solution of the single-objective problem obtained is a Pareto optimal solution of the original multi-objective problem.

We choose to minimize variance for two reasons. Firstly, it is more intuitively appealing to impose limits on the expected value and CVaR, rather than on variance. Secondly, we show that minimizing variance is more convenient from a computational point of view. In either case, a convex optimization problem would be obtained† irrespective of which statistic we choose for the objective function, but, when optimizing variance, a quadratic programming problem is obtained, as shown below.

In what follows, for a random variable R_x that depends on the decision vector x, the variance of R_x is denoted alternatively by $\sigma^2(x)$ or $\sigma^2(R_x)$. Similarly, the Conditional Value-at-Risk at level α of R_x is denoted by $\text{CVaR}_\alpha(x)$ or $\text{CVaR}_\alpha(R_x)$, and the expected value of R_x by $E(x)$ or $E(R_x)$.

We consider the following optimization problem:

$$\text{(P1): } \min \sigma^2(x)$$
$$\text{Subject to: } \text{CVaR}_\alpha(x) \leq z$$
$$E(x) \geq d$$
$$x \in \mathcal{A}$$

where z and d are real numbers.

It is easy to prove that: if x^* is a Pareto optimal solution of (MVC) then x^* is also an optimal solution of (P1) with $z = \text{CVaR}_\alpha(x^*$ and $d = E(x^*)$.

Indeed, assume that x^* is not an optimal solution of (P1). Obviously x^* is a feasible solution of (P1). Denote by x' an optimal solution of (P1). It follows that $\sigma^2(x') \leq \sigma^2(x^*)$, $\text{CVaR}_\alpha(x') \leq \text{CVaR}_\alpha(x^*)$ and $E(x') \geq E(x^*)$, which means that x' Pareto dominates x^* and we have a contradiction.

The converse is also true, with the additional assumption of uniqueness of the optimal solution:

> If x^* is the unique optimal solution of (P1), then x^* is also a Pareto optimal solution of (MVC).

Indeed, assume that x^* is Pareto dominated in (MVC) and denote by x' a point that Pareto dominates x^*. This means that $\sigma^2(x') \leq \sigma^2(x^*)$, $\text{CVaR}_\alpha(x') \leq \text{CVaR}_\alpha(x^*) \leq z$ and $E(x') \geq E(x^*) \geq d$ with at least one strict inequality. Thus x' is another feasible solution of (P1) such that $\sigma^2(x') \leq \sigma^2(x^*)$, which is a contradiction.

Remark 1: If the covariance matrix of returns is positive definite, then variance is a strictly convex function of x. In this case, minimizing variance over a convex set has at most one optimal solution; thus, the possibility of multiple optimal solutions for (P1) is eliminated. This is usually the case; if there are no redundant assets (ones that can be replicated by the remaining of the assets) or risk-free assets in the collection of assets considered, then the covariance matrix is positive definite.

We summarize these results below:

Proposition 2: *If the covariance matrix is positive definite, a point x^* is a Pareto efficient solution of (MVC) if and only if x^* is an optimal solution of* (P1) *with* $z = \text{CVaR}_\alpha(x^*)$ *and* $d = E(x^*)$.

Thus, in the case of a positive definite covariance matrix of returns, the Pareto efficient solutions of (MVC) can be fully characterized as optimal solutions in (P1) with active constraints on mean and on CVaR.

In appendix A we treat the general case of a positive semi-definite covariance matrix.

The next issue that arises is how to represent the CVaR constraint in (P1). As presented in Proposition (1), the function $F_\alpha(x, v) = (1/\alpha)E\{[v - R_x]^+\} - v$ may be used both for calculating the CVaR of a given random variable and for optimizing CVaR with respect to all feasible decisions vectors.

Furthermore, Krokhmal *et al.* (2002) proved that the same function $F_\alpha(x, v)$ may be used for imposing an upper limit on the CVaR of a random variable, while maximizing its expected value. Their result may be

†As stated in Proposition 1, CVaR is a convex function of x. Variance is also a convex function of x, since the variance-covariance matrix is positive semi-definite. The expected value is a linear function of x.

extended to a much more general case. In fact, the constraint '$\mathrm{CVaR}_\alpha(x) \le z$' can be replaced with the constraint '$F_\alpha(x, v) \le z$' in all optimization problems, irrespective of the form of the objective function or the feasible set.

Proposition 3: *Consider two optimization problems* (P) *and* (P′) *with* $A \subset R^n$ *a feasible set of decision vectors and the objective function* $f: R^n \to R$ *of any form*:

$$\begin{aligned}&(\mathrm{P}):\ \min f(x)\\ &\text{Subject to}:\ \mathrm{CVaR}_\alpha(x) \le z\\ &\qquad x \in \mathcal{A}\end{aligned}$$

$$\begin{aligned}&(\mathrm{P}'):\ \min f(x)\\ &\text{Subject to}:\ F_\alpha(x, v) \le z\\ &\qquad x \in \mathcal{A}, \quad v \in R.\end{aligned}$$

In (P), *the variables are* $x_1, \ldots, x_n$ *while in* (P′), *the variables are* $x_1, \ldots, x_n$ *and* v.

Then: (P) *and* (P′) *achieve the same optimal value. Moreover, a point* $x^* \in A$ *is an optimal solution for* (P) *iff there exists* $v^* \in R$ *such that* (x^*, v^*) *is an optimal solution for* (P′). *If, in addition, the constraint* $\mathrm{CVaR}_\alpha(x) \le z$ *in* (P) *is active, then* $v^* \in A_\alpha(x^*)$ *(meaning that* $F(x^*, v^*) = \min_{v\in R} F_\alpha(x^*, v)$*)*.

Proof: As stated in Proposition 1, $\mathrm{CVaR}_\alpha(x) = \min_{v\in R} F_\alpha(x, v)$. Thus, the problem (P) may be written as:

$$\begin{aligned}&(\mathrm{P}):\ \min f(x)\\ &\text{Subject to:}\quad \min_{v\in R} F_\alpha(x, v) \le z\\ &\qquad x \in \mathcal{A}.\end{aligned}$$

Suppose now that x^* is an optimal solution for (P). Obviously (x^*, v^*) is a feasible solution for (P′), where v^* is such that $F(x^*, v^*) = \min_{v\in R} F_\alpha(x^*, v)$. Assume that there exists (x', v') another feasible solution for (P′) such that $f(x') < f(x^*)$. Since $F_\alpha(x', v') \le z$ it follows that $\min_{v\in R} F_\alpha(x', v) \le z$; thus, x' is a feasible solution of (P1) which improves the objective function as compared to x^*, which is a contradiction.

Similarly, in a straightforward way, the converse may be proven; the last part of the proposition is obvious. □

Thus, we consider another optimization problem, with variables $x = (x_1, \ldots, x_n) \in A \subset R^n$ and $v \in R$:

$$\begin{aligned}&(\mathrm{P2}):\ \min \sigma^2(x)\\ &\text{Subject to:}\ F_\alpha(x, v) \le z\\ &\qquad E(x) \ge d\\ &\qquad x \in \mathcal{A}, \quad v \in R,\end{aligned}$$

where $\mathcal{A}$ is the (convex) set of feasible decision vectors, as given, for example, by (1).

The result below follows from Propositions 2 and 3:

Proposition 4: *If the covariance matrix of returns is positive definite, the Pareto efficient solutions of* (*MVC*) *are fully characterized as optimal solutions of* (P2) *with active constraints on mean and on* CVaR.

In other words, x^* is a Pareto efficient solution of (MVC) if and only if there exists $v^* \in R$ such that (x^*, v^*) is an optimal solution to (P2) with $z = F_\alpha(x^*, v^*)$ and $d = E(x^*)$.

Therefore, varying d and z in the problem (P2) such that the constraints on CVaR and on the expected value are active produces all the efficient solutions of the mean-variance-CVaR model. As shown in section 4.4, this means varying d and z between some finite limits that can be easily determined.

4.3. Alternative optimization approaches

The optimization approach described in the previous subsection is not unique. A commonly used method of obtaining a Pareto efficient solution of a multi-objective optimization problem is to use a scalarizing function, meaning a real-valued function that is a composite of all objective functions. When optimized, the scalarizing function produces a Pareto efficient solution of the multi-objective optimization problem. Thus, the problem is reduced to a single objective optimization problem. We give below two examples of scalarizing functions, leading to two alternative optimization approaches for the mean-variance-CVaR model.

The most common scalarizing function is a weighted sum of the objective functions in the original multi-objective optimization problem. The general requirement on weights is that they should be strictly positive but usually they are normalized such they sum to 1. In our case, the single objective optimization problem that results is:

$$\begin{aligned}&\max w_1 E(x) - w_2\sigma^2(x) - w_3 \mathrm{CVaR}_\alpha(x) \qquad (\mathrm{P3})\\ &\text{Subject to: } x \in \mathcal{A}\end{aligned}$$

where w_1, w_2, w_3 are strictly positive†.

It is clear that every optimal solution of (P3) is a Pareto efficient solution of (MVC).

The converse is not always true, in the sense that there may be Pareto optimal solutions of (MVC) that cannot be obtained as optimal solutions of a problem (P3) with strictly positive w_1, w_2 and w_3 (for example, the Pareto optimal solution of (MVC) that globally minimizes variance).

However, due to the convexity of all objective functions on (MVC), every Pareto optimal solution of (MVC) can be obtained as an optimal solution of (P3) with non-negative weights (see Jahn 1985). For example, the Pareto optimal solution of (MVC) that globally minimizes variance is obtained as an optimal solution of (P3) with $w_1 = w_3 = 0$, $w_2 = 1$.

This approach has several disadvantages (see Das and Dennis 1997), one of them being the fact that the weights w_1, w_2, w_3 are rather difficult to interpret. It is more

†If additionally there is the assumption of unique optimal solutions of (P3) when some of the weights are zero, then only the non-negativity condition is required for w_1, w_2 and w_3.

meaningful to set desired levels of expected return and of CVaR and solve (P2).

Another example of a scalarizing function is obtained by considering target values (called reference points or aspiration points) for the values of the objective functions. This technique for multi-objective optimization, named The Reference Point Method is fully described in Wierzbicki (1998). Consider the general multi-objective problem

$$(\mathrm{MO}'): \quad \max(f_1(x), f_2(x), \ldots, f_T(x)) \\ \text{Subject to: } x \in X,$$

and let $w_1^*, w_2^*, \ldots, w_T^*$ be the user-defined aspiration points for the objective functions. The simplest form of scalarizing function is:

$$\gamma_{w^*}(x) = \min_{1 \le k \le T}(f_k(x) - w_k^*) + \varepsilon \sum_{k=1}^{T}(f_k(x) - w_k^*), \quad (4)$$

where $\varepsilon > 0$ is an arbitrary small parameter.

The terms $f_k(x) - w_k^*$ in (4) are usually replaced by more complicated functions of x and $w_k^*, \gamma_k(x, w_k^*)$, which must satisfy certain properties (see e.g. Wierzbicki 1998, Makowski and Wierzbicki 2003). These functions are called partial achievement functions since they measure the actual achievement of the kth objective function with respect to its corresponding aspiration level w_k^*.

Various functions $\gamma_k(x, w_k^*)$ provide a wide modelling environment for measuring individual achievements. Other examples of such functions may be found in Wierzbicki (1998), and Makowski and Wierzbicki (2003).

Provided that all the reference points lie between the lower and the upper bound of the corresponding objective function, the maximization of (4) provides a Pareto efficient solution of (MO′). The converse is true, in the sense that for every Pareto efficient solution of (MO′), there exist aspiration levels such that this efficient solution maximizes the corresponding achievement function (see Wierzbicki 1998). In our case, the scalarizing achievement function to maximize is:

$$\gamma_{w^*}(x) = \min\{E(x) - w_1^*, w_2^* - \sigma^2(x), w_3^* - \mathrm{CVaR}_\alpha(x)\} \\ + \varepsilon[E(x) - w_1^* + w_2^* - \sigma^2(x) + w_3^* - \mathrm{CVaR}_\alpha(x)\}],$$

where $\varepsilon > 0$ is an arbitrary small parameter.

The Reference Point Method is primarily designed for obtaining a specific solution of a multi-objective problem rather than the whole set of efficient solutions. Although all the efficient solutions may be obtained with this method by choosing appropriate reference points, care must be taken in choosing the reference points between the lower and upper bound of each objective function. The lower bounds for the objective functions are difficult to find and often approximations are used.

In contrast, the optimization method described in section 4.2 produces the entire set of efficient solutions of the mean-variance-CVaR model with no difficulty, as described in the next section.

4.4. The efficient frontier of the mean-variance-CVaR model

We consider the case when the covariance matrix of returns is positive definite; the general case of a positive semi-definite covariance matrix is treated in appendix A.

As presented in section 4.2, varying the right-hand sides d and z in (P2) such that the corresponding constraints on mean and CVaR are active produces all the efficient solutions of (MVC).

Thus, the level d for the expected value must lie in the interval $[d_{\min}, d_{\max}]$. We define $d_{\min} = \max\{d_{\mathrm{minvar}}, d_{\mathrm{minCVaR}}\}$, where d_{minvar} and d_{minCVaR} are the expected returns of the minimum variance portfolio (mean-variance efficient) and minimum CVaR portfolio (mean-CVaR efficient) respectively. d_{minvar} may be found as the optimal value of the variable d_0 in the problem:

$$\min \sigma^2(x) \\ \text{Subject to: } E(x) \ge d_0 \\ x \in A, \quad d_0 \in R.$$

d_{minCVaR} may be found as the optimal value of the variable d_1 in the problem:

$$\min F_\alpha(x, v) \\ \text{Subject to: } E(x) \ge d_1 \\ x \in \mathcal{A}, \quad v \in R, \quad d_1 \in R.$$

To be more precise, d_{minCVaR} may be found as above only when the minimization of $F_\alpha(x, v)$ with respect to (x, v) over $\mathcal{A}xR$ provides a unique optimal solution. In the case of non-unique optimal solutions, we can obtain portfolios having the same minimal CVaR but different expected returns; among these, we are interested in the portfolio with the maximum expected return. To obtain this portfolio, we denote by $\mathrm{CVaR}_{\min}$ the optimal value of the above problem and solve another optimization problem:

$$\max E(x) \\ \text{Subject to: } F_\alpha(x, v) \le \mathrm{CVaR}_{\min} \\ x \in \mathcal{A}, \quad v \in R.$$

We define $d_{\max}$ as the maximum possible expected return:† the optimal value of the objective function in the problem:

$$\max E(x) \\ \text{Subject to: } x \in \mathcal{A}.$$

Furthermore, for a specific $d^* \in [d_{\min}, d_{\max}]$, the level z of CVaR_α must lie in the interval $[z_{d^*,\min}, z_{d^*,\max}]$, where

†$d_{\max}$ is also equal to the highest expected return of the component assets in the portfolio selection problem.

$z_{d^*,\min}$ is the best (minimum) CVaR_α level for the expected return d^* and $z_{d^*,\max}$ is the CVaR_α level of the (unique) portfolio that minimizes variance for the expected return d^*. $z_{d^*,\min}$ is the optimal value of the objective function in the problem:

$$\begin{aligned} &\min F_\alpha(x, v) \\ &\quad \text{Subject to: } E(x) \geq d^* \\ &\qquad\qquad x \in \mathcal{A}, \quad v \in R. \end{aligned}$$

$z_{d^*,\max}$ may be found as the optimal value of the objective function in the problem:

$$\begin{aligned} &\min F_\alpha(x^*, v) \\ &\quad \text{Subject to: } v \in R, \end{aligned}$$

where $x^* = (x_1^*, \ldots, x_n^*)$ is the (unique) portfolio that minimizes variance for the mean return d^*.

The fact that the imposed limit z on CVaR_α is greater than or equal to $z_{d^*,\min}$ ensures that the problem (P2) is not infeasible, while z being less than or equal to $z_{d^*,\max}$ ensures that the constraint on CVaR in (P2) is active. When solving problem (P2) for a level of expected return equal to d^* and a CVaR level equal to $z_{d^*,\min}$, we obtain a mean-CVaR efficient portfolio; more precisely, the mean-CVaR efficient portfolio with the lowest variance for expected return d^*.

When solving problem (P2) for a level of expected return equal to d^* and a CVaR level equal to $z_{d^*,\max}$, we obtain the mean-variance efficient portfolio with expected return d^*.

For a fixed level of expected return, the efficient solutions in the mean-variance-CVaR model form a curve when plotted in a variance-CVaR space, where the lower end of this curve is represented by the mean-CVaR efficient solution (with the lowest variance) and the upper end is represented by the mean-variance efficient solution (see figure 1). The other points of this curve are not efficient in either the mean-variance or the mean-CVaR model.

For the maximum level of expected return $d_{\max}$, this curve degenerates into just one point, with the coordinates equal to the variance and CVaR of the (only) efficient portfolio obtained for $d_{\max}$, consisting of the asset with the highest expected return.

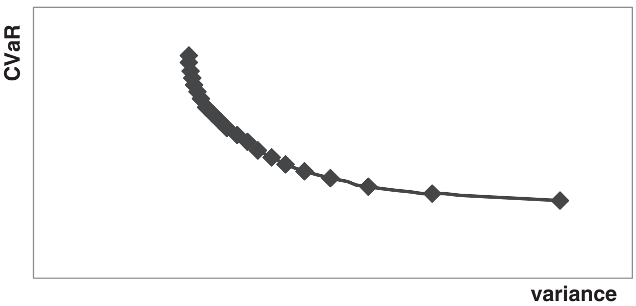

Figure 1. The efficient solutions of the mean-variance-CVaR model, for a fixed level of expected value, plotted in a variance-CVaR space.

4.5. *The formulation of the mean-variance-CVaR model for scenario models*

For the portfolio selection problem, as presented in section 2, consider T scenarios and n assets with

r_{ij} = the return of asset j under scenario i, for $i = 1, \ldots, T$ and $j = 1, \ldots, n$;
p_i = the probability of scenario i occurring, for $i = 1 \ldots T$;
μ_j = the expected return of asset j, $j = 1, \ldots, n$;
σ_{jk} = the covariance between the returns of assets j and k, for $j, k = 1, \ldots, n$.

As presented in section 3.3, the function F_α can be written as:

$$F_\alpha(x, v) = \frac{1}{\alpha} \sum_{i=1}^{T} p_i \left[v - \sum_{j=1}^{n} x_j r_{ij} \right]^+ - v.$$

Thus, we write the mean-variance-CVaR model as:

$$\begin{aligned} &\min \sum_{j,k=1}^{n} x_j x_k \sigma_{jk} \\ &\text{Subject to:} \\ &\sum_{j=1}^{n} x_j \mu_j \geq d \\ &\frac{1}{\alpha} \sum_{i=1}^{T} p_i y_i - v \leq z \\ &y_i \geq v - \sum_{j=1}^{n} x_j r_{ij}, \\ &\forall i \in 1, \ldots, T \\ &y_i \geq 0, \\ &\forall i \in 1, \ldots, T \\ &\sum_{j=1}^{n} x_j = 1 \\ &x_j \geq 0 \\ &\forall j \in 1, \ldots, n. \end{aligned}$$

The minimization is over $v, x_1, \ldots, x_n, y_1, \ldots, y_T$.

5. Computational results

5.1. *The data set and methodology*

The purpose of this section is to investigate the practical performance of the mean-variance-CVaR model as compared to that of the mean-variance or mean-CVaR model. Precisely, for several levels of expected return, we select portfolios that are efficient in the mean-variance-CVaR model, but dominated in the mean-variance or mean-CVaR model, and we also consider the corresponding mean-variance efficient portfolio and the mean-CVaR efficient portfolio. We compare their in-sample and out-of-sample performances.

We use CVaR at 0.01 confidence level.

A dataset, drawn from the FTSE 100 index, was used for this analysis. The returns of the 76 stocks that belonged to the index throughout the period January 1993–December 2003 were considered (for each of the remaining 24 stocks data there is at least one missing data item in the specified period). The dataset consists of monthly returns and has 132 time periods, considered as equally probable scenarios ($n = 76$, $T = 132$). For the out-of sample analysis, the behaviour of the portfolios obtained was examined over the eighteen months following the date of selection (January 2004–June 2005). The models were written in the MPL modelling language (Maximal Software Inc. 2000) and processed using CPLEX 9.0 optimization solver (ILOG 2003). The matrix of covariances of the returns is computed from historical data.

5.2. *In-sample analysis*

We consider six levels of expected return, which divide the interval $[d_{\min}, d_{\max}]$ (see section 4.4) into 5 equal parts: $d_1 = d_{\min} = 0.009268$, $d_2 = 0.014034$, $d_3 = 0.018801$, $d_4 = 0.023567$, $d_5 = 0.028334$, $d_6 = d_{\max} = 0.0331$. For each level of expected return d_i, with $i = 1, \ldots, 5$, we determine $z_{d_i,\min}$: the minimum level of CVaR (corresponding to the mean-CVaR efficient portfolio) and $z_{d_i,\max}$: the maximum level of CVaR (the lowest CVaR of a mean-variance efficient portfolio with expected return d_i) and, between them, another 3 equally spaced levels of CVaR. Thus, the interval $[z_{d_i,\min}, z_{d_i,\max}]$ for CVaR is divided into 4 equal parts. For a specific level of expected return, when solving the mean-variance-CVaR model with these CVaR levels, we obtain 5 portfolios, denoted by: P_{CVaR}, $P_{1/4\mathrm{CVaR}}$, $P_{1/2\mathrm{CVaR}}$, $P_{3/4\mathrm{CVaR}}$ and P_{var} respectively. Thus, P_{CVaR} is the mean-CVaR efficient portfolio (with the lowest variance, for the specified expected return) and P_{var} is the (unique) mean-variance efficient portfolio for the specified expected return (see figure 2).†

We first investigate the composition of the considered portfolios. For all levels of expected return, the mean-variance efficient portfolios have considerably more assets in their composition than the mean-CVaR efficient portfolios. This was expected, since the 'diversification effect' is the basis of the mean-variance theory.

The other three portfolios $P_{1/4\mathrm{CVaR}}$, $P_{1/2\mathrm{CVaR}}$, $P_{3/4\mathrm{CVaR}}$ have usually a number of assets in composition significantly higher than mean-CVaR efficient portfolios, but usually smaller than mean-variance efficient portfolios. There are cases in which these portfolios are as well as or even more diversified than the mean-variance efficient portfolios (see table 1) we notice that this happens when the expected return of the portfolio is high, thus, at high levels of risk. However, in most cases, the number of assets in the composition increases while the level of variance decreases (and the level of CVaR increases). Generally, the assets there are in the composition of mean-CVaR efficient portfolios are also in the composition of portfolios with a higher CVaR level. However, there are assets in the composition of the mean-CVaR portfolios but not in the composition of portfolios with a higher CVaR level. This aspect happens for small portfolio expected returns, thus, at low levels of risk. It may be noticed that, while the expected portfolio return (and thus the risk) increases, those assets are no longer in the composition of any efficient portfolio.

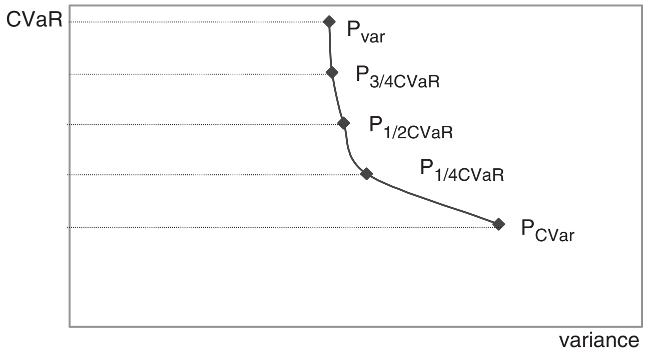

Figure 2. The efficient frontier for a fixed level of expected return, in a variance-CVaR space. The interval for CVaR is divided into 4 equal parts.

Table 1. The number of assets in the composition of mean-variance-CVaR efficient portfolios.

	P_{CVaR}	$P_{1/4\mathrm{CVaR}}$	$P_{1/2\mathrm{CVaR}}$	$P_{3/4\mathrm{CVaR}}$	P_{var}
$d_1 = 0.00927$	10	18	21	21	23
$d_2 = 0.01403$	7	17	20	22	21
$d_3 = 0.01880$	7	12	14	13	13
$d_4 = 0.02357$	6	9	9	8	7
$d_5 = 0.02833$	4	5	5	6	6

The portfolio weights of the efficient portfolios considered are presented in appendix B.

We next investigate the in-sample performances of $P_{1/4\mathrm{CVaR}}$, $P_{1/2\mathrm{CVaR}}$, $P_{3/4\mathrm{CVaR}}$, as compared with those of P_{CVaR} and P_{var}. We analyse their return distributions using common in sample parameters. Obviously, the CVaR levels of $P_{1/4\mathrm{CVaR}}$, $P_{1/2\mathrm{CVaR}}$, $P_{3/4\mathrm{CVaR}}$ are better than the CVaR of P_{var}. On the other hand, their variance is generally significantly smaller than that of P_{CVaR}. All the other in-sample parameters are between those of P_{CVaR} and P_{var}. In most cases, P_{CVaR} has the return distribution with the best skewness, kurtosis and minimum of returns but also with the worst variance.

In contrast, P_{var} has the return distribution with the best variance but usually the worst skewness, kurtosis and minimum of returns. This is in line with the modelling paradigm since minimization of CVaR leads to reduction in the (weighted) tail of the resulting portfolio return distribution. The other portfolios $P_{1/4\mathrm{CVaR}}$, $P_{1/2\mathrm{CVaR}}$, $P_{3/4\mathrm{CVaR}}$ represent a compromise in between these two 'extremes'. Their return distribution improves in the left tail, as compared with P_{var} and also has a significantly

†The CVaR level of $P_{1/2\mathrm{CVaR}}$ is the arithmetic mean of the CVaR levels of P_{CVaR} and P_{var}. Similarly, the CVaR level of $P_{1/4\mathrm{CVaR}}$ is the arithmetic mean of the CVaR levels of P_{CVaR} and $P_{1/2\mathrm{CVaR}}$.

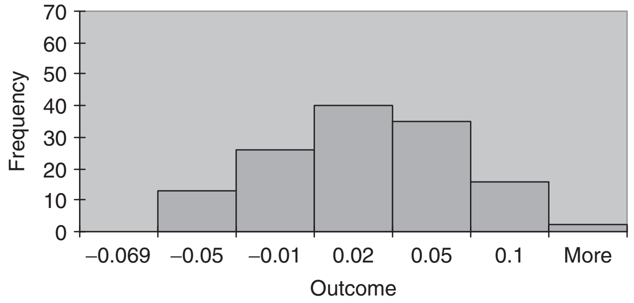

Figure 3. The histogram of the return distribution of P_{CVaR} for expected return $d_1 = 0.00927$.

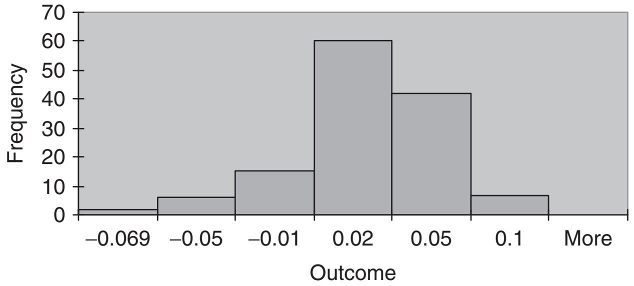

Figure 4. The histogram of the return distribution of P_{var} for expected return $d_1 = 0.00927$.

smaller spread around the mean, as compared with P_{CVaR}. In particular, $P_{1/4\mathrm{CVaR}}$ has return distributions with the variance significantly smaller than that of P_{CVaR} at the expense of a relatively small increase in CVaR. This aspect can be seen from tables 9–13 (with the best values in italic bold and the worst values enclosed by rectangles) in appendix C and is also illustrated in figure 2.

In figure 3 below the histogram of the return distribution of P_{CVaR} for expected return $d_1 = 0.009\,27$ is presented. This distribution is positively skewed, with a short left tail, a long right tail and a large probability of outcomes below the expected value. Therefore, the probability of large losses is very small, but there is a large probability of small losses. In addition, this distribution is particularly 'flat', that is, not concentrated around the expected value.

In figure 4 the histogram of the return distribution of P_{var} for the same expected return $d_1 = 0.00927$ is presented. This distribution is negatively skewed, with a long left tail, a short right tail and also a large probability of outcomes above the expected value; thus, there is a large probability of small gains. This distribution is concentrated around the expected value.

In figure 5 the histogram of the return distribution of $P_{1/4\mathrm{CVaR}}$ for the same expected return $d_1 = 0.00927$ is presented. This distribution has approximately the same shape as the return distribution of P_{var}: concentrated around the expected value and with a large probability of outcomes just above the expected value. However, its left tail is shorter, due to the constraint imposed on the CVaR level, and thus the probability of large losses is reduced.

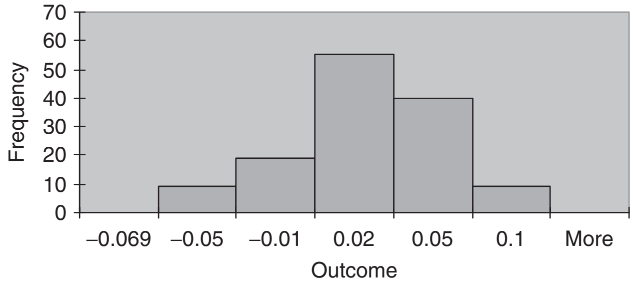

Figure 5. The histogram of the return distribution of $P_{1/4\mathrm{CVaR}}$ for expected return $d_1 = 0.00927$.

Table 2. Ex-post parameters of the mean-variance-CVaR efficient portfolios with in-sample mean return $d_1 = 0.009\,268$.

	P_{CVaR}	$P_{1/4\mathrm{CVaR}}$	$P_{1/2\mathrm{CVaR}}$	$P_{3/4\mathrm{CVaR}}$	P_{var}
Mean	***0.016294***	0.01472	0.013835	0.013556	$\boxed{0.01345}$
Median	0.013106	***0.015918***	0.01456	0.012515	$\boxed{0.011549}$
Standard Deviation	$\boxed{0.029173}$	0.026514	0.025082	0.023893	***0.022882***
Minimum	$\boxed{-0.03494}$	−0.03156	−0.03316	−0.02945	−0.02491
Maximum	$\boxed{0.052624}$	***0.07282***	0.071134	0.068515	0.066001

Table 3. Ex-post parameters of the mean-variance-CVaR efficient portfolios with in-sample mean return $d_3 = 0.01880$.

	P_{CVaR}	$P_{1/4\mathrm{CVaR}}$	$P_{1/2\mathrm{CVaR}}$	$P_{3/4\mathrm{CVaR}}$	P_{var}
Mean	$\boxed{0.01133}$	***0.012532***	0.012342	0.012352	0.012342
Median	$\boxed{0.010171}$	***0.013783***	0.013118	0.012365	0.01231
Standard Deviation	***0.028682***	0.031943	0.03221	0.032159	$\boxed{0.032581}$
Minimum	$\boxed{-0.04247}$	***−0.03263***	−0.03614	−0.03817	−0.04004
Maximum	0.081765	***0.08752***	0.082737	0.078024	$\boxed{0.072908}$

5.3. *Out-of-sample analysis*

We analyse the performance of the portfolios described in the previous section over the next 18 time periods following the date of selection (January 2004–June 2005).

The portfolios that are non-efficient in either the mean-variance or the mean-CVaR model, denoted by $P_{1/4\mathrm{CVaR}}$, $P_{1/2\mathrm{CVaR}}$ and $P_{3/4\mathrm{CVaR}}$, have an out-of-sample performance comparable to that of the mean-variance and the mean-CVaR efficient portfolios. It may be noted the generally good out-of-sample performance of the mean-CVaR portfolios and the somewhat poorer performance of the mean-variance portfolios, although the differences were not significant.

In general, the best out-of-sample parameters correspond to mean-CVaR portfolios, but for some levels of expected return, $P_{1/4\mathrm{CVaR}}$ had equally good or even better out-of-sample parameters (see tables 2, 3, with the best values in italic bold and the worse values enclosed by rectangles).

Figure 6 presents the compound out-of-sample returns of the mean-variance-CVaR efficient portfolios with in-sample mean return $d_1 = 0.009268$. $P_{1/4\mathrm{CVaR}}$ had a better out-of-sample performance than P_{CVaR} in the first eight out-of-sample periods (January–August 2004)

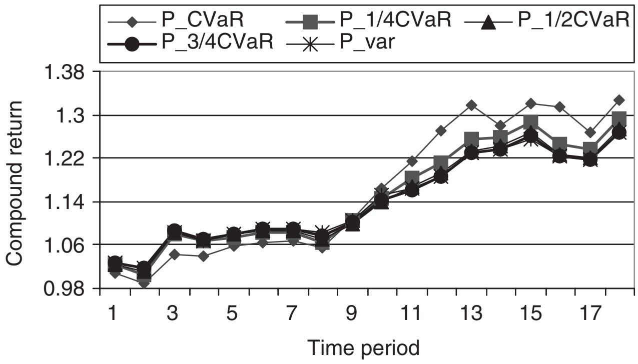

Figure 6. Ex-post compounded returns of the mean-variance-CVaR efficient portfolios with in-sample mean return $d_1 = 0.009268$.

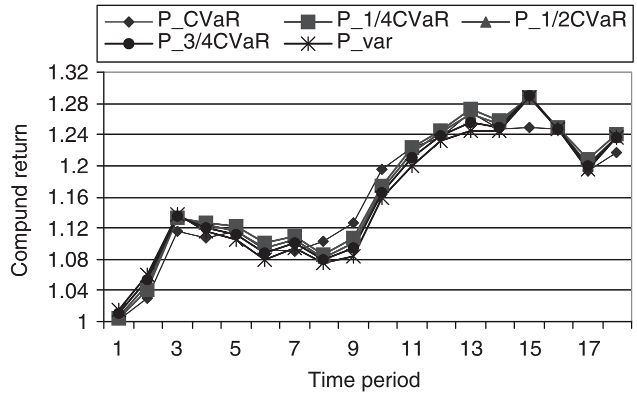

Figure 7. Ex-post compounded returns of the mean-variance-CVaR efficient portfolios with in-sample mean return $d_3 = 0.01880$.

(moreover, P_{CVaR} had a compound return less than one in February 2004, which means that its value fell below the amount invested). At the same time, $P_{1/4\mathrm{CVaR}}$ had a better out-of-sample performance than P_{var} in the last ten out-of-sample periods (September 2004–June 2005).

Figure 7 presents the compounded out-of-sample returns of the mean-variance-CVaR efficient portfolios with in-sample mean return $d_3 = 0.018\,80$. $P_{1/4\mathrm{CVaR}}$ had a better out-of-sample performance than both P_{CVaR} and P_{var}, although the differences are small.

6. Summary and conclusions

In this paper, we presented a model for portfolio selection, which selects a solution (distribution) on the basis of three parameters: the expected value, the variance and the CVaR at a specified confidence level. We called this model the mean-variance-CVaR model. The problem of selecting an efficient solution of this model is multi-objective: the expected value is maximized, while the variance and CVaR are minimized. We chose variance and CVaR mainly because they are well established risk measures that quantify risk from different perspectives: variance measures the deviation around the expected value while CVaR measures the average loss over a specified number of worst cases.

Computationally, the problem reduces to solving a single objective problem in which variance is minimized, while constraints are imposed on the expected value and CVaR. In the practice of portfolio selection, the random variables under consideration are usually represented as discrete and described by realizations under various scenarios. In this case, the problem is one of quadratic programming, thus routinely solved by standard available software. Having a constraint on CVaR rather than on the variance has advantages not only from a computational point of view. It is more natural to impose a maximum CVaR level than a maximum variance level, since CVaR represents the mean of the worst outcomes of a distribution.

Varying the right-hand side of the constraints on the expected value and on CVaR such that these constraints are active produces all the efficient solutions of the mean-variance-CVaR model.

When solving the model for a fixed level of expected return, there is a range of efficient solutions. Plotted in a variance-CVaR space, they form a curve, with one end represented by the minimum variance portfolio (with the lowest CVaR), the other represented by the minimum CVaR portfolio (with the lowest variance).

The model was tested on a dataset drawn from the FTSE 100 index. Several levels of expected return were considered, and, for each level of expected return, five portfolios that were efficient in the mean-variance-CVaR model, were analysed: the minimum variance portfolio, the minimum CVaR portfolio and other three portfolios that were dominated in both mean-variance and mean-CVaR models. As expected, the best in-sample parameters concerning the left tail of distributions corresponded to mean-CVaR efficient portfolios: highest skewness, lowest kurtosis and highest maximum. However, the return distributions of mean-CVaR efficient portfolios have also the highest variances. In contrast, the mean-variance efficient portfolios have the return distributions with the lowest variance, but also with the 'worst' left tail (as described by skewness, kurtosis, minimum and CVaR). The other portfolios, efficient only in the mean-variance-CVaR model, improve on the left tail of the mean-variance efficient distributions: they have higher skewness, lower kurtosis higher maximum and higher CVaR. In some cases, this improvement comes at the expense of only a marginal increase in variance. The out-of-sample performances of these portfolios are comparable to those of the mean-variance and mean-CVaR efficient portfolios. In two out of five cases, such a portfolio achieved the highest mean of out-of-sample returns and in almost all cases led to the highest maximum of out-of-sample returns.

As a final remark, it may be noted that the proposed model does not dismiss mean-variance or mean-CVaR models, but on the contrary, it 'embeds' them. Most of the mean-variance and the mean-CVaR efficient solutions are particular solutions of the proposed model. For example, a mean-variance efficient solution is not a solution of the proposed model only if there is another mean-variance efficient solution with the same mean and variance but with lower CVaR. Likewise, from the set of mean-CVaR efficient solutions with a specified mean return, only the one(s) with the lowest variance is solution of the proposed model. Thus, the proposed model makes a 'positive' discrimination between mean-variance and mean-CVaR efficient solutions. In addition, the mean-variance-CVaR model has a range of solutions that are normally discarded by both mean-variance and mean-CVaR model. These solutions may bring an improvement in the distribution, in the case when the CVaR of a mean-variance efficient portfolio is considered to be unacceptably large. They represent a compromise between regulators' requirements for short tails and classical fund managers' requirements for small variance. In making the final choice, the personal preference of the decision-maker plays a key role.

References

Acerbi, C. and Tasche, D., On the coherence of expected shortfall. *J. Bank. Finance*, 2002, **26**, 1487–1503.

Artzner, P., Delbaen, F., Eber, J.M. and Heath, D., Coherent measures of risk. *Math. Finance*, 1999, **9**(3), 203–228.

Das, I. and Dennis, J., A closer look at the drawbacks of minimising weighted sums of objectives for Pareto set generation in multicriteria optimisation problems. *Struct. Optimisation*, 1997, **14**(1), 63–69.

Fishburn, P.C., Mean-risk analysis with risk associated with below target returns. *Am. Econ. Rev.*, 1977, **67**, 116–126.

Haimes, Y.Y., Lasdon, L.S. and Wismer, D.A., On a bicriterion formulation of the problems of integrated system identification and system optimization. *IEEE Trans. Syst. Man Cybern.*, 1971, **1**, 296–297.

Harvey, C., Liechty, R.J., Liechty, M.W. and Mueller, P., *Portfolio selection with higher moments*, 2003 (Duke University). Working Paper (Available online at: http://ssrn.com/abstract = 634141).

ILOG, *CPLEX 9.0*, *User's Manual*, 2003 (ILOG SA: Gentilly, France).

Jahn, J., Some characterizations of the optimal solutions of a vector optimization problem. *OR Spektrum*, 1985, **7**, 7–17.

Jorion, P., Portfolio optimisation with tracking-error constraints. *Financ. Analysts J.*, 2003, **59**(5), 70–82.

Konno, H. and Yamazaki, H., Mean absolute deviation portfolio optimization model and its applications to Tokyo stock market. *Manag. Sci.*, 1991, **37**, 519–531.

Konno, H., Shirakawa, H. and Yamazaki, H., A mean-absolute deviation-skewness portfolio optimisation model. *Ann. Oper. Res.*, 1993, **45**, 205–220.

Konno, H. and Suzuki, K., A mean-variance-skewness portfolio optimisation model. *J. Oper. Res. Soc. Jpn*, 1995, **38**(2), 173–187.

Krokhmal, P., Palmquist, J. and Uryasev, S., Portfolio optimisation with conditional value-at-risk objective and constraints. *J. Risk*, 2002, **4**(2), 43–68.

Larsen, N., Mausser, H. and Uryasev, S., Algorithms for optimisation of value-at-risk. In *Financial Engineering, E-Commerce and Supply Chain*, edited by P. Pardalos and V.K. Tsitsiringos, pp. 129–157, 2002 (Kluwer Academic Publishers: Norwell).

Laurent, J.P., *Sensitivity analysis of risk measures for discrete distributions*, 2003. Available online at: http://laurent.jeanpaul.free.fr/var_risk_measure_sensitivity.pdf

Leyffer, S. and Pang, J.S., *On the global minimization of the value-at-risk*, 2005. Available online at: http://www.citebase.org/cgi-bin/citations?id = oai:arXiv.org:math/0401063

Luenberger, D.G., *Investment Science*, 1998 (Oxford University Press: New York).

Makowski, M. and Wierzbicki, A.P., Modelling knowledge, *Model Based Decision Support and Soft Computations, Applied Decision Support with Soft Computing*, Vol. 124, pp. 3–60, 2003 (Springer-Verlag: Berlin, New York) also Available at: www.iiasa.ac.at/ marek/pubs

Markowitz, H.M., Portfolio selection. *J. Finance*, 1952, **7**(1), 77–91.

Maximal Software Incorporation, *MPL Modelling System*, 2000Arlington, Virginia, USA. Release 4.11.

Ogryczak, W., Multiple criteria linear programming model for portfolio selection. *Ann. Oper. Res.*, 2000, **97**, 143–162.

Ogryczak, W., Multiple criteria optimization and decisions under risk. *Control and Cybernetics*, 2002, **31**, 975–1003.

Ogryczak, W. and Ruszczynski, A., From stochastic dominance to mean-risk models: semideviations as risk measures. *Eur. J. Oper. Res.*, 1999, **116**, 33–50.

Ogryczak, W. and Ruszczynski, A., On consistency of stochastic dominance and mean-semideviations models. *Math. Program.*, 2001, **89**, 217–232.

Ogryczak, W. and Ruszczynski, A., Dual stochastic dominance and related mean-risk models. *SIAM J. Optimiz.*, 2002, **13**, 60–78.

Pflug, G., Some remarks on the value-at-risk and the conditional value-at-risk. In *Probabilistic Constrained Optimisation: Methodology and Applications*, edited by S. Uryasev, 2000 (Available online at: www.gloriamundi.org)

Rockafeller, RT. and Uryasev, S., Optimization of conditional value-at-risk. *J. Risk*, 2000, **2**, 21–42.

Rockafeller, R.T. and Uryasev, S., Conditional value-at-risk for general loss distributions,. *J. Bank. Finance*, 2002, **26**(7), 1443–1471.

Steuer, R., *Multiple Criteria Optimization—Theory, Computation and Application*, 1986 (John Wiley & Sons: New York).

Steuer, R. and Na, P., Multiple criteria decision making combined with finance: a categorized bibliographic study. *Eur. J. Oper. Res.*, 2003, **150**, 496–515.

Tasche, D., Expected shortfall and beyond. *J. Bank. Finance*, 2002, **26**(7), 1519–1533.

Wang, J., *Mean-variance-VaR based portfolio optimisation*, 2000. Available online at: www.gloriamundi.org

Wierzbicki, A., Reference point methods in vector optimization and decision support, *IIASA Interim Report IR-98-017*, 1998 (IIASA).

Yitzhaki, S., Stochastic dominance, mean variance and Gini's mean difference. *Am. Econ. Rev.*, 1982, **72**, 178–185.

Appendix A: The general case of a positive semi-definite covariance matrix

In the general case, when the covariance matrix of returns is positive semi-definite, the minimization of variance over a convex set may not have a unique optimal solution. Thus, when using the optimization problem (P2) as described in section 4.4, we may obtain solutions that are Pareto dominated in (MVC)†. However, we can still use (P2) for obtaining efficient solutions of (MVC), provided the right-hand sides d and z for the mean and CVaR constraints are chosen as described below.

The level d for the expected value must lie in the interval $[d'_{\min}, d_{\max}]$, where $d_{\max}$ is the maximum possible expected return (as presented in section 4.4). We define $d'_{\min} = \max\{d'_{\text{minvar}}, d_{\text{minCVaR}}\}$, where d'_{minvar} and d_{minCVaR} are the expected returns of the minimum variance portfolio (mean-variance efficient) and minimum CVaR portfolio (mean-CVaR efficient) respectively. d_{minCVaR} may be found as described in section 4.4. The expected return of the minimum variance portfolio d'_{minvar} cannot be determined so straightforward as for the case of a positive definite covariance matrix. We cannot just minimize variance over the whole feasible set $\mathcal{A}$ (with no constraints on the mean) since there may be different optimal solutions to this problem, with the same (optimal) variance but with different expected returns. Among these solutions that globally minimize variance, we consider only the one with the maximum expected return. To obtain this solution, we first solve the problem:

$$\begin{aligned} &\min \sigma^2(x) \\ &\text{Subject to: } x \in \mathcal{A}. \end{aligned}$$

Denote the optimum value of this problem by $\sigma_{\min}$. In order to find the specific optimal solution of this problem with the maximum possible expected return, we propose a convex program with quadratic constraint:

$$\begin{aligned} &\max E(x) \\ &\text{Subject to: } \sigma^2(x) \le \sigma_{\min} \\ &\qquad\qquad\quad x \in \mathcal{A}. \end{aligned}$$

The optimal value of the above optimization problem is d'_{minvar}.

Furthermore, for a specific $d \in [d'_{\min}, d_{\max}]$, the right-hand side for the CVaR constraint z must lie in the interval $[z_{d,\min}, z'_{d,\max}]$; $z_{d,\min}$ is the best (minimum) CVaR_α level for the expected return d and may be found as described in section 4.4. $z'_{d,\max}$ is the *minimum* CVaR_α level of the mean-variance efficient portfolios with expected return d‡.

In order to determine $z'_{d,\max}$, one may solve two optimization problems. Firstly, the optimal variance for the expected return d (denoted by σ_d^2) may be found as the optimal value of the objective function in the problem:

$$\begin{aligned} &\min \sigma^2(x) \\ &\text{Subject to: } \quad E(x) \ge d \\ &\qquad\qquad\qquad x \in \mathcal{A}. \end{aligned}$$

Secondly, $z'_{d,\max}$ may be found as the optimal value of the objective function in the problem:

$$\begin{aligned} &\min F_\alpha(x, v) \\ &\text{Subject to: } E(x) \ge d \\ &\qquad\qquad\quad \sigma^2(x) \le \sigma_d^2 \\ &\qquad\qquad\quad x \in \mathcal{A}, \quad v \in R. \end{aligned}$$

Proposition 5: *Consider the optimization problem*

$$\begin{aligned} (\text{P1}): \ &\min \sigma^2(x) \\ &\text{Subject to: } \text{CVaR}_\alpha(x) \le z \\ &\qquad\qquad\quad E(x) \ge d \\ &\qquad\qquad\quad x \in \mathcal{A}. \end{aligned}$$

If x^* is an optimal solution of (P1) *for* $d \in [d'_{\min}, d_{\max}]$ *and* $z \in [z_{d,\min}, z'_{d,\max}]$ *(as described above), then x^* is Pareto efficient in* (MVC).

Proof: Assume that x^* is not Pareto efficient in (MVC). Denote by x' a feasible solution of (MVC) that Pareto dominates x^*. This means that $\sigma^2(x') \le \sigma^2(x^*)$, $\text{CVaR}_\alpha(x') \le \text{CVaR}_\alpha(x^*) \le z$ and $E(x') \ge E(x^*) \ge d$ with at least one strict inequality. Thus, x' is a feasible solution of (P1). The case $\sigma^2(x') < \sigma^2(x^*)$ is excluded since this contradicts the fact that x^* is an optimal solution of (P1). It only remains the possibility that x' and x^* are both optimal solutions of (P1) and $\text{CVaR}_\alpha(x') < \text{CVaR}_\alpha(x^*) \le z$ or $E(x') > E(x^*) \ge d$.

Consider first the case. $\text{CVaR}_\alpha(x') < \text{CVaR}_\alpha(x^*) \le z$; thus, x' is an optimal solution of (P1) and the constraint $\text{CVaR}_\alpha(x) \le z$; is not binding. Since (P1) is a convex optimization problem, it follows that x' is an optimal solution of the 'reduced' problem, obtained from (P1) by removing the constraint on CVaR:

$$\begin{aligned} (\text{P1}_{\text{red}}): \ &\min \sigma^2(x) \\ &\text{Subject to: } E(x) \ge d \\ &\qquad\qquad\quad x \in \mathcal{A}. \end{aligned}$$

This means that both x' and x^* are mean-variance efficient portfolios with expected return $d \in [d'_{\min}, d_{\max}]$. Thus, we have two mean-variance efficient solutions with the same variance, the same expected return d but different CVaRs.

$\text{CVaR}_\alpha(x') < \text{CVaR}_\alpha(x^*) \le z \le z'_{d,\max}$. However, $z'_{d,\max}$ is, by construction, the lowest possible CVaR of a mean-variance efficient portfolio with mean return d and we have a contradiction.

†For example, multiple optimal solutions of (P2) may have the same variance, the same expected return but different CVaRs; only the one with the lowest CVaR is Pareto efficient in (MVC).

‡In case there are several mean-variance efficient portfolios with expected return d, with different CVaR levels, only the portfolio with the lowest CVaR is efficient in the (MVC) model; its CVaR level is denoted by $z_{d,\max}$.

Obviously the constraint $E(x) \geq d$ in (P1) is binding for $d \in [d'_{\min}, d_{\max}]$; thus, the case $E(x') > E(x^*) \geq d$ is also impossible and this ends the proof. □

Thus, when the right-hand sides d and v are chosen as above, the constraints on CVaR and on mean are active.

It was shown in section 4.2 that the constraint $\mathrm{CVaR}_\alpha(x) \leq z$ can be replaced with the constraint $F_\alpha(x, v) \leq z$, $v \in R$ and thus the problem (P2), equivalent to (P1), is obtained:

$$\begin{aligned}
&\text{(P2):}\ \min \sigma^2(x)\\
&\text{Subject to: } F_\alpha(x, v) \leq z\\
&\qquad E(x) \geq d\\
&\qquad x \in \mathcal{A}, \quad v \in R
\end{aligned}$$

Solving problem (P2) with d varying between $d'_{\min}$ and $d_{\max}$ and z varying between $z_{d,\min}$ and $z'_{d,\max}$ as described above, gives an efficient solution of the mean-variance-CVaR model.

Appendix B: The composition of efficient portfolios

The composition of efficient portfolios are given in tables 4–8.

For the highest level of expected return $d_6 = d_{\max} = 0.0331$, the efficient portfolio consists of the asset no. 58.

Table 4. The portfolio weights of the efficient portfolios for $d_1 = 0.009268$.

Asset No.	P_{CVaR}	$P_{1/4\mathrm{CVaR}}$	$P_{1/2\mathrm{CVaR}}$	$P_{3/4\mathrm{CVaR}}$	P_{var}
4	0	0.050	0.050	0.048	0.042
5	0.182	0.060	0.043	0.028	0.021
11	0	0	0.023	0.047	0.068
13	0	0.002	0.028	0.052	0.068
16	0	0.063	0.052	0.037	0.019
17	0	0.071	0.064	0.054	0.046
21	0	0	0	0	0.001
24	0	0.011	0.016	0.019	0.023
25	0	0.000	0.020	0.034	0.044
27	0	0.029	0.054	0.070	0.077
29	0.026	0	0	0	0
40	0.222	0.097	0.081	0.075	0.07
42	0	0.075	0.075	0.065	0.055
43	0.036	0.066	0.062	0.062	0.061
44	0.016	0.088	0.083	0.066	0.05
45	0.053	0	0.010	0.022	0.033
48	0	0	0	0	0.004
52	0	0.001	0	0	0
63	0	0.017	0.002	0.006	0.006
64	0	0.050	0.045	0.028	0.011
65	0	0	0	0	0.04
66	0	0.039	0.060	0.065	0.064
69	0.186	0.050	0.021	0.008	0
70	0	0	0	0.035	0.059
71	0.019	0	0	0	0
72	0.045	0.104	0.115	0.107	0.081
73	0.215	0.129	0.096	0.072	0.057

Table 5. The portfolio weights of the efficient portfolios for $d_2 = 0.01403$.

Asset No.	P_{CVaR}	$P_{1/4\mathrm{CVaR}}$	$P_{1/2\mathrm{CVaR}}$	$P_{3/4\mathrm{CVaR}}$	P_{var}
4	0.008	0.058	0.069	0.075	0.079
5	0	0.073	0.062	0.058	0.057
13	0	0.037	0.063	0.078	0.086
16	0	0.004	0.027	0.021	0.010
17	0.289	0.096	0.081	0.075	0.070
20	0.086	0	0	0	0
21	0	0.070	0.064	0.052	0.043
24	0	0.014	0.022	0.019	0.018
25	0	0	0.009	0.019	0.026
27	0	0.001	0.011	0.022	0.026
29	0.04	0.030	0	0	0
40	0	0.081	0.092	0.066	0.043
42	0	0	0.005	0.012	0.013
43	0	0.020	0.006	0.004	0
44	0	0.121	0.118	0.097	0.077
45	0.258	0.106	0.093	0.093	0.097
48	0	0	0	0.005	0.019
56	0	0.041	0.042	0.037	0.031
58	0	0	0.002	0.014	0.027
63	0	0.098	0.068	0.059	0.045
65	0	0	0	0.002	0.031
66	0	0.009	0.034	0.047	0.053
69	0.05	0	0	0	0
70	0	0	0.013	0.045	0.066
73	0.269	0.139	0.120	0.100	0.085

Table 6. The portfolio weights of the efficient portfolios for $d_3 = 0.0188$.

Asset No.	P_{CVaR}	$P_{1/4\mathrm{CVaR}}$	$P_{1/2\mathrm{CVaR}}$	$P_{3/4\mathrm{CVaR}}$	P_{var}
4	0	0.041	0.065	0.088	0.116
5	0	8.1E-05	0.029	0.048	0.066
13	0	0	0.009	0.023	0.032
16	0	0	0.002	0.017	0.035
17	0	0.113	0.119	0.102	0.065
20	0.016	0	0	0	0
21	0.116	0.183	0.163	0.144	0.124
28	0.02	0	0	0	0
29	0.134	0.023	0	0	0
40	0	0.052	0.028	0	0
44	0	0.121	0.126	0.125	0.125
45	0.303	0.179	0.173	0.168	0.161
56	0.209	0.145	0.137	0.124	0.112
58	0	0.021	0.034	0.048	0.066
63	0	0	0.001	0.009	0.012
73	0.202	0.102	0.095	0.084	0.068
76	0	0.022	0.019	0.018	0.019

Table 7. The portfolio weights of the efficient portfolios for $d_4 = 0.02357$.

Asset No.	P_{CVaR}	$P_{1/4\mathrm{CVaR}}$	$P_{1/2\mathrm{CVaR}}$	$P_{3/4\mathrm{CVaR}}$	P_{var}
4	0	0	0.046	0.09	0.134
17	0	4E-05	0.007	0	0
21	0.49	0.298	0.278	0.247	0.214
28	0.072	0	0	0	0
29	0	0.0138	0	0	0
44	0	0.027	0.041	0.053	0.059
45	0.373	0.269	0.225	0.222	0.217
56	0	0.249	0.248	0.225	0.2
58	6E-04	0.057	0.087	0.112	0.139
73	9E-04	0.007	0.016	0.007	0
76	0.063	0.079	0.054	0.045	0.036

Table 8. The portfolio weights of the efficient portfolios for $d_5 = 0.02833$.

asset no	P_{CVaR}	$P_{1/4\text{CVaR}}$	$P_{1/2\text{CVaR}}$	$P_{3/4\text{CVaR}}$	P_{var}
4	0	0	0	0.003	0.018
21	0.472	0.418	0.367	0.325	0.324
45	0.119	0.105	0.066	0.032	0.022
56	0	0.065	0.138	0.197	0.196
58	0.29	0.342	0.370	0.394	0.397
76	0.119	0.069	0.059	0.049	0.043

Appendix C: The in-sample parameters for the return distributions of efficient portfolios

The in-sample parameters for the return distributions of efficient portfolios are given in tables 9–13.

Table 9. In-sample parameters for the return distributions of efficient portfolios in the mean-variance-0.01CVaR model with expected return $d_1 = 0.009268$.

	P_{CVaR}	$P_{1/4\text{CVaR}}$	$P_{1/2\text{CVaR}}$	$P_{3/4\text{CVaR}}$	P_{var}
Median	0.010905	$\boxed{0.009989}$	0.010678	***0.011774***	0.011348
Standard Deviation	$\boxed{0.039557}$	0.032288	0.030899	0.030186	***0.030006***
Skewness	***0.175763***	−0.43318	−0.59261	−0.75996	$\boxed{-0.89894}$
Kurtosis	***−0.16328***	0.214433	0.763715	1.35481	$\boxed{1.964419}$
Minimum	***−0.05813***	−0.06857	−0.08198	−0.09601	$\boxed{-0.10946}$
Maximum	***0.128209***	0.085995	0.084375	0.081927	$\boxed{0.077194}$

Table 10. In-sample parameters for the return distributions of efficient portfolios in the mean-variance-0.01CVaR model with expected return $d_2 = 0.014034$.

	P_{CVaR}	$P_{1/4\text{CVaR}}$	$P_{1/2\text{CVaR}}$	$P_{3/4\text{CVaR}}$	P_{var}
Median	$\boxed{0.009982}$	0.016801	0.016398	0.017359	***0.0176***
Standard Deviation	$\boxed{0.043277}$	0.035516	0.034453	0.03398	***0.033852***
Skewness	***0.238317***	−0.5367	−0.64824	−0.75897	$\boxed{-0.87193}$
Kurtosis	***0.100689***	0.329636	0.799505	1.213484	$\boxed{1.633637}$
Minimum	***−0.07056***	−0.07906	−0.08756	−0.09606	$\boxed{-0.10498}$
Maximum	***0.149618***	0.095584	0.093019	0.090123	$\boxed{0.087926}$

Table 11. In-sample parameters for the return distributions of efficient portfolios in the mean-variance-0.01CVaR model with expected return $d_3 = 0.018801$.

	P_{CVaR}	$P_{1/4\text{CVaR}}$	$P_{1/2\text{CVaR}}$	$P_{3/4\text{CVaR}}$	P_{var}
Median	$\boxed{0.019982}$	0.021909	0.021945	***0.022453***	0.02225
Standard Deviation	$\boxed{0.051467}$	0.045116	0.043917	0.043138	***0.042869***
Skewness	***0.105138***	−0.27928	−0.35782	−0.44374	$\boxed{-0.50531}$
Kurtosis	0.816632	***0.588582***	0.748811	1.016336	$\boxed{1.309189}$
Minimum	***−0.09186***	−0.10046	−0.11094	−0.12183	$\boxed{-0.13216}$
Maximum	***0.188287***	0.139995	0.132851	0.127387	$\boxed{0.12672}$

Table 12. In-sample parameters for the return distributions of efficient portfolios in the mean-variance-0.01CVaR model with the expected return $d_4 = 0.023567$.

	P_{CVaR}	$P_{1/4\text{CVaR}}$	$P_{1/2\text{CVaR}}$	$P_{3/4\text{CVaR}}$	P_{var}
Median	***0.026665***	$\boxed{0.02185}$	0.023582	0.022484	0.023786
Standard Deviation	$\boxed{0.071333}$	0.061135	0.059382	0.058374	***0.058031***
Skewness	***0.595438***	−0.12047	−0.23122	−0.30692	$\boxed{-0.36555}$
Kurtosis	$\boxed{3.354617}$	0.816052	***0.797705***	0.808283	0.834841
Minimum	***−0.12247***	−0.13142	−0.14231	−0.1528	$\boxed{-0.16327}$
Maximum	***0.367729***	0.204922	0.181086	0.162425	$\boxed{0.159635}$

Table 13. In-sample parameters for the return distributions of efficient portfolios in the mean-variance-0.01CVaR model with expected return $d_5 = 0.028334$.

	P_{CVaR}	$P_{1/4\text{CVaR}}$	$P_{1/2\text{CVaR}}$	$P_{3/4\text{CVaR}}$	P_{var}
Median	***0.035256***	0.032523	0.027021	0.023606	$\boxed{0.022036}$
Standard Deviation	$\boxed{0.091039}$	0.088892	0.087699	0.087357	***0.087337***
Skewness	***0.319572***	0.215952	0.112308	0.050204	$\boxed{0.041352}$
Kurtosis	$\boxed{1.470049}$	1.069079	0.885207	***0.817357***	0.841093
Minimum	***−0.19129***	−0.19541	−0.19749	−0.19974	$\boxed{-0.20228}$
Maximum	***0.358639***	0.308329	0.26884	$\boxed{0.266499}$	$\boxed{0.267819}$