Conditional Value-at-Risk for Log-Distributions

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Abstract

Conditional Value-at-Risk (CVaR) represents a significant improvement over the Value-at-Risk (VaR) in the area of risk measurement, as it catches the risk beyond the VaR threshold. CVaR is also theoretically more solid, being a coherent risk measure, which enables building more robust risk assessment and management systems. This paper addresses the derivation of the closed-form CVaR formulas for log-normal, log-logistic, log-Laplace and log-hyperbolic secant distributions, which are relevant for modeling the returns of financial assets. In many cases financial risk managers assume that not the returns themselves but their logarithms adhere to a particular distribution, and the samples of log-returns are considered. In such cases, the appropriate way to assess risk would be to use a log-distribution for CVaR and VaR estimation. We show how to use the log-normal and the log-logistic distributions for assessing the risk for different asset classes in 2003-2007 and 2007-2009. The log-Laplace and the log-hyperbolic secant distributions have fatter tails, and they may provide a higher accuracy for the tail risk assessment during the crisis periods.

Keywords:

CVaR; Conditional Value at Risk; risk management; log-normal distribution; log-logistic distribution; Laplace distribution; hyperbolic secant distribution.

Introduction

Conditional value-at-risk (CVaR) has recently become a popular measure of risk. While less known than value-at-risk (VaR), CVaR has an important advantage of being a coherent risk measure as defined by Artzner (1999). Another serious drawback of VaR is its inability to quantify the expected losses beyond the threshold amount, i.e. VaR only allows identifying the threshold itself but says nothing about the worst-case scenarios below it. However, VaR remains the most popular risk measure and is widely covered by academics and practitioners, such as Jorion (1996), Stambaugh (1996), Linsmeier (2000). CVaR, on the other hand, has been mostly overlooked by practitioners because it lacks an easy-to-use straightforward formula.

Starting with Rockafellar and Uryasev (2000) there were attempts of simplifying the generic CVaR formula for some distributions and deriving it in a closed form. Rockafellar and Uryasev (2000) did this for the normal distribution using the error function, which unfortunately is not present in most non-mathematical software (e.g. Microsoft Excel™). Andreev et al. (2005) derived closed-form CVaR formulas for some of the most widely-used distributions, and Khokhlov (2016) provided the closed-form CVaR formulas derivation for elliptical distributions. However, in many practical cases financial risk managers consider not the samples of the raw returns, but sample of the logarithms of returns (log-returns). In such cases, for VaR or CVaR estimation it is required to account for the fact that not the random variable (return or loss) but its logarithm follows a particular distribution. Alternatively, the log-distribution approach can be utilized, e.g. when the logarithm of a random variable is normally distributed the variable itself follows the log-normal distribution, and the VaR or CVaR formulas derived for the latter can be used with the parameters estimated from the sample of log-returns.

The purpose of this paper is to derive analytically closed-form CVaR formulas for several log-distributions so that the risk managers can easily apply CVaR concept when using samples

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of log-returns. The log-normal and the log-logistic distributions, which are covered by this paper, are widely used in practice and included into distribution fitting software, such as EasyFit. The log-Laplace distribution is less known, but its usability is emphasized by the fact that the Laplace distribution is one of the best for modeling financial assets log-returns. Finally, log-hyperbolic secant distribution is almost unknown, but the simplicity of the hyperbolic secant distribution and its ability to model the "fat tails" much better than the normal distribution opens possibilities for its usage in practice.

Our methodology is based on the variance-covariance definition of CVaR, so all the formulas are derived under this approach. However, for some assets we also compare the results of our CVaR estimation with CVaR calculated under the empirical (historical) approach.

The definitions of CVaR

CVaR is defined as the conditional expectation of losses above VaR. Under the theoretical (variance-covariance) approach, we assume that the loss is a random variable that follows a certain distribution, and the CVaR is calculated as

$$c_\alpha = E\left[x \middle| x > v_\alpha\right] = \frac{1}{1-\alpha}\int_{v_\alpha}^{\infty} x\,\mathrm{f}\left(x\right)dx\,, \tag{1a}$$

where c_α is the CVaR @ α,

v_α is the VaR @ α,

x is the loss (random variable),

$\mathrm{f}\left(x\right)$ is the probability density function (PDF) for the distribution of losses.

Formula (1a) basically means CVaR is the conditional expectation in the right tail of the distribution of losses, where the VaR is the threshold. This formula is quite common in many fields of risk management, however it is not the most useful for financial applications. In finance, we are working with the distributions of returns as the key random variable, and the loss in its inverse. Therefore, we are mostly interested in negative returns in the left tail of their distributions. When the return is a random variable that follows a certain distribution, the CVaR per $1 of the investment, an assets or a portfolio is calculated as

$$-c_\alpha = E\left[x \middle| x < -v_\alpha\right] = \frac{1}{\alpha}\int_{-\infty}^{-v_\alpha} x\,\mathrm{f}\left(x\right)dx\,, \tag{1b}$$

where c_α is the CVaR @ α per $1 of the selected asset,

v_α is the VaR @ α per $1 of the selected asset,

x is the return on the selected asset (random variable),

$\mathrm{f}\left(x\right)$ is the probability density function (PDF) for the distribution of return.

The second possible way of calculating CVaR under the theoretical approach would be integrating the quantile (inverse CDF) over the range of relevant probabilities instead of integrating the random variable multiplied by the PDF over the range of its relevant values. As $F\left(x\right) = p \Leftrightarrow x = \mathrm{F}^{-1}\left(p\right)$, $\mathrm{f}\left(x\right)dx = d\mathrm{F}\left(x\right) = dp$, and $\mathrm{F}\left(-\infty\right) = 0, \mathrm{F}\left(-v_\alpha\right) = \alpha, \mathrm{F}\left(+\infty\right) = 1$, we can rewrite (1a) or (1b) accordingly. Considering that for continuous distributions $-v_\alpha = \mathrm{F}^{-1}\left(\alpha\right)$, (1b) is equivalent to

$$-c_\alpha = \frac{1}{\alpha}\int_0^\alpha \mathrm{F}^{-1}\left(p\right)dp = \frac{1}{\alpha}\int_0^\alpha \left(-v_p\right)dp\,. \tag{1c}$$

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Under the empirical (historical) approach, when working with samples of the asset returns, the CVaR is calculated as the conditional sample mean for the values that are less than the VaR:

$$-c_\alpha^{hist} = \frac{1}{n} \sum_{r_i < -v_\alpha^{hist}} r_i \,, \tag{2}$$

where c_α^{hist} is the historical CVaR @ α per $1,

v_α^{hist} is the historical VaR @ α per $1, which is the α-th percentile,

r_i is the i-th value in the sample of historical returns ($i = 1,\ldots,N$).

n is the amount of returns that are less than $-v_\alpha^{hist}$, i.e. $n = \lceil \alpha N \rceil$.

CVaR formula for the log-normal distribution

The asset return r is distributed log-normally when the random variable $x = \ln(1+r)$ follows the normal (Gaussian) distribution with the PDF

$$\mathrm{f}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,,$$

where μ is the location parameter (expected value),

σ is the scale parameter, which equals to the standard deviation.

It should be noted that we hereafter work with the parameters μ, σ of the underlying normal distribution, so the unbiased estimators for those are the sample mean and the sample standard deviation of the log-returns. They should not be confused with the sample statistics of the raw returns. Instead of working with PDF or CDF of the log-distributions, we will express the VaR $v_\alpha = -r_\alpha$, where r_α is the α-quantile of the log-distribution for r, in terms of the inverse CDF of underlying distribution for x: $\mathrm{F}(x) = \mathrm{F}(\ln(1+r)) = \alpha \Rightarrow 1 - v_\alpha = 1 + r_\alpha = e^{\mathrm{F}^{-1}(\alpha)}$. Finally,

$$\frac{1}{\alpha}\int_0^\alpha e^{\mathrm{F}^{-1}(p)} dp = \frac{1}{\alpha}\int_0^\alpha (1 - v_p)\, dp = \frac{1}{\alpha}\int_0^\alpha dp + \frac{1}{\alpha}\int_0^\alpha (-v_p)\, dp = 1 - c_\alpha \,. \tag{3}$$

To derive the VaR for the log-normal returns, let's set $x = \ln(1+r) = \mu + \sigma z$, where z follows the standard normal distribution with CDF Φ, so we can use it to express the probability of getting the return lower than the given level: $\Pr\left\{ z < \frac{\ln(1+r) - \mu}{\sigma} \right\} = \Phi\left(\frac{\ln(1+r) - \mu}{\sigma} \right) = \alpha$.

The corresponding return is $r_\alpha = \exp(\mu + \sigma\phi_\alpha) - 1$, where $\phi_\alpha = \Phi^{-1}(\alpha)$ is the α-quantile of the standard normal distribution, and the VaR is $v_\alpha = -r_\alpha = 1 - \exp(\mu + \sigma\phi_\alpha)$.

Proposition L1. For the log-normal distribution the CVaR @ α can be calculated as

$$c_\alpha = 1 - \exp\left(\mu + \frac{\sigma^2}{2} \right) \frac{\Phi(\phi_\alpha - \sigma)}{\alpha} \,, \tag{4}$$

where μ, σ are the location and the scale parameter of the distribution of the log-return,

Φ is the CDF of the standard normal distribution,

$\phi_\alpha = \Phi^{-1}(\alpha)$ is the α-quantile of the standard normal distribution.

Proof: The proof is based on the derivation of the partial expected value for the log-normal distribution provided at https://en.wikipedia.org/wiki/Talk:Log-normal_distribution:

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$$E[X \mid X < k] = \exp\left(\mu + \frac{\sigma^2}{2}\right) \frac{\Phi\left(\frac{\ln(k) - \mu - \sigma^2}{\sigma}\right)}{\Phi\left(\frac{\ln(k) - \mu}{\sigma}\right)},$$

taking into account that in our notation $X = 1 + r$ setting k to the VaR threshold $k = 1 + r_\alpha$, we immediately arrive to 1 + the expected return in the left tail below the VaR threshold, which is the inverse of the CVaR for the log-normal distribution:

$$1 + (-c_\alpha) = \exp\left(\mu + \frac{\sigma^2}{2}\right) \frac{\Phi\left(\frac{\ln(1 + r_\alpha) - \mu - \sigma^2}{\sigma}\right)}{\Phi\left(\frac{\ln(1 + r_\alpha) - \mu}{\sigma}\right)} = \exp\left(\mu + \frac{\sigma^2}{2}\right) \frac{\Phi\left(\frac{\mu + \sigma\phi_\alpha - \mu - \sigma^2}{\sigma}\right)}{\Phi\left(\frac{\mu + \sigma\phi_\alpha - \mu}{\sigma}\right)},$$

$$c_\alpha = 1 - \exp\left(\mu + \frac{\sigma^2}{2}\right) \frac{\Phi(\phi_\alpha - \sigma)}{\Phi(\phi_\alpha)} = 1 - \exp\left(\mu + \frac{\sigma^2}{2}\right) \frac{\Phi(\phi_\alpha - \sigma)}{\alpha}. \ \square$$

Formula (4) can easily be implemented in spreadsheets, for example in Microsoft Excel™ ϕ_α is NORMSINV(α) and $\Phi(x)$ is NORMSDIST(x).

CVaR formula for the log-logistic distribution

The asset return r follows the log-logistic distribution when the random variable $x = \ln(1 + r)$ follows the logistic distribution with the PDF and the CDF respectively

$$\mathrm{f}(x) = \frac{e^{-\frac{x-\mu}{s}}}{s\left(1 + e^{-\frac{x-\mu}{s}}\right)^2} \text{ and } \mathrm{F}(x) = \frac{1}{1 + e^{-\frac{x-\mu}{s}}},$$

where μ is the location parameter (expected value),

s is the scale parameter (note that the standard deviation is $\sigma = \pi s / \sqrt{3}$).

To derive the VaR for the log-logistic returns, let's express the probability of getting the return lower than the given level: $\Pr\{r < r_\alpha\} = \dfrac{1}{1 + \exp\left(-\dfrac{\ln(1 + r_\alpha) - \mu}{s}\right)} = \alpha$, from which we get

$r_\alpha = \exp\left[\mu + s \ln \dfrac{\alpha}{1-\alpha}\right] - 1 = e^\mu \left(\dfrac{\alpha}{1-\alpha}\right)^s - 1$, and the VaR is the inverse value $v_\alpha = -r_\alpha$.

Proposition L2. For the log-logistic distribution the CVaR @ α can be calculated as

$$c_\alpha = 1 - \frac{e^\mu}{\alpha} I_\alpha(1 + s, 1 - s) \frac{\pi s}{\sin \pi s} = 1 - \frac{e^\mu \alpha^s}{s + 1} {}_2F_1(s, s + 1; s + 2; \alpha), \tag{5}$$

where μ, s are the location and the scale parameter of the distribution of the log-return,

I_α is the regularized incomplete beta function: $I_\alpha(a, b) = \dfrac{\mathrm{B}_\alpha(a, b)}{\mathrm{B}(a, b)}$,

${}_2F_1$ is the hypergeometric function.

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Proof: Using (3), $1+(-c_\alpha)=\frac{1}{\alpha}\int_0^\alpha (1-v_p)\,dp=\frac{1}{\alpha}\int_0^\alpha e^\mu\left(\frac{p}{1-p}\right)^s dp$ and, by definition of the incomplete beta function, $\frac{1}{\alpha}e^\mu\int_0^\alpha p^s(1-p)^{-s}dp=\frac{1}{\alpha}e^\mu \mathrm{B}_\alpha(1+s,1-s)$. We can expand it as $\mathrm{B}_\alpha(1+s,1-s)=I_\alpha(1+s,1-s)\mathrm{B}(1+s,1-s)=I_\alpha(1+s,1-s)\frac{\pi s}{\sin \pi s}$.

However, as the beta function is defined only for positive arguments, the formula derived above works only for $-1<s<1$, and in most cases for financial assets their standard deviation falls into the corresponding range $\sigma<\pi/\sqrt{3}\approx 181\%$. A more generic case requires the hypergeometric function, as it can be shown that $\int_0^\alpha p^s(1-p)^{-s}dp=\frac{\alpha^{s+1}}{s+1}\,{}_2F_1(s,s+1;s+2;\alpha)$.□

To use formula (5) in Microsoft Excel™, $I_\alpha(1+s,1-s)$ can be calculated as BETADIST(α, 1+*s*, 1–*s*), and $B_\alpha(1+s,1-s)$ as BETADIST(α, 1+*s*, 1–*s*)*PI()**s*/SIN(PI()**s*).

When using EasyFit distribution fitting package, note that it has an alternate definition of the log-logistic distribution with parameters α,β,γ, where $\alpha=s^{-1},\beta=e^\mu$. The VaR and the CVaR at level ζ can be expressed as $-r_\zeta=\beta\left(\frac{\zeta}{1-\zeta}\right)^{1/\alpha}+\gamma;-c_\zeta=\frac{1}{\zeta}\beta \mathrm{B}_\zeta\left(1+\frac{1}{\alpha},1-\frac{1}{\alpha}\right)+\gamma$.

CVaR formula for the log-Laplace distribution

The asset return r follows the log-Laplace distribution when the random variable $x=\ln(1+r)$ follows the Laplace distribution with the PDF and the CDF

$$\mathrm{f}(x)=\frac{1}{2b}e^{-\frac{|x-\mu|}{b}},\mathrm{F}(x)=\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(x-\mu)\left(1-e^{-\frac{|x-\mu|}{b}}\right),$$

where μ is the location parameter (expected value),

b is the scale parameter, where the standard deviation $\sigma=b\sqrt{2}$.

To derive the VaR for the returns that adhere to the Laplace distribution, we need to consider two cases separately. If $\ln(1+r)\le\mu$, $\Pr\{r<r_\alpha\}=\frac{1}{2}\exp\left(\frac{\ln(1+r_\alpha)-\mu}{b}\right)=\alpha$, so that $r_\alpha=e^{\mu+b\ln(2\alpha)}-1=e^\mu(2\alpha)^b-1$. If $\ln(1+r)>\mu$, $\Pr\{r<r_\alpha\}=1-\frac{1}{2}\exp\left(-\frac{\ln(1+r_\alpha)-\mu}{b}\right)=\alpha$, and $r_\alpha=e^{\mu-b\ln(2-2\alpha)}-1=e^\mu(2-2\alpha)^{-b}-1$. The VaR is the inverse: $v_\alpha=-r_\alpha$.

Proposition L3. For the log-Laplace distribution the CVaR @ α can be calculated as

$$c_\alpha=\begin{cases}1-\frac{e^\mu(2\alpha)^b}{b+1}=1-\frac{1-v_\alpha}{b+1},\alpha\le 1/2,\\ 1-\frac{e^\mu 2^{-b}}{\alpha(b-1)}\left[(1-\alpha)^{1-b}-1\right]=1-\left(\frac{1-\alpha}{\alpha}\right)\frac{1-v_\alpha}{b-1}+\frac{e^\mu 2^{-b}}{\alpha(b-1)},\alpha>1/2,\end{cases}\qquad(6)$$

where μ,b are the location and the scale parameter of the distribution of return,

v_α is the VaR @ α per \$1 of the asset's value.

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Proof: Using (3), when $\alpha \le 1/2$ we can expand the expression as

$$1+(-c_\alpha)=\frac{1}{\alpha}\int_0^\alpha (1-v_p)\,dp=\frac{1}{\alpha}\int_0^\alpha e^{\mu}(2p)^b\,dp=\frac{e^{\mu}}{2\alpha(b+1)}(2\alpha)^{b+1}=\frac{e^{\mu}}{b+1}(2\alpha)^b.$$ When $\alpha > 1/2$,

$$1+(-c_\alpha)=\frac{1}{\alpha}\int_0^\alpha (1-v_p)\,dp=\frac{1}{\alpha}\int_0^\alpha e^{\mu}(2-2p)^{-b}\,dp=\frac{e^{\mu}2^{-b}}{\alpha}\int_0^\alpha (1-p)^{-b}\,dp=$$

$$=\frac{e^{\mu}2^{-b}}{\alpha(b-1)}\left[(1-\alpha)^{1-b}-1\right]=\frac{(1-\alpha)e^{\mu}}{\alpha(b-1)}\left[(2-2\alpha)^{-b}-\frac{2^{-b}}{1-\alpha}\right]=\frac{(1-\alpha)(1-v_\alpha)}{\alpha(b-1)}-\frac{e^{\mu}2^{-b}}{\alpha(b-1)}.\ \square$$

CVaR formula for the generalized log-hyperbolic secant (log-GHS) distribution

The asset return r follows the generalized log-hyperbolic secant (log-GHS) distribution when the random variable $x=\ln(1+r)$ follows the GHS distribution with the PDF and the CDF

$$\mathrm{f}(x)=\frac{1}{2\sigma}\operatorname{sech}\left(\frac{\pi}{2}\frac{x-\mu}{\sigma}\right);\mathrm{F}(x)=\frac{2}{\pi}\arctan\left[\exp\left(\frac{\pi}{2}\frac{x-\mu}{\sigma}\right)\right],$$

where μ is the location parameter (expected value),

σ is the scale parameter (standard deviation).

To derive the VaR for the log-GHS returns, let's express the probability of getting the return lower than the given level: $\Pr\{r<r_\alpha\}=\frac{2}{\pi}\arctan\left[\exp\left(\frac{\pi}{2}\frac{\ln(1+r_\alpha)-\mu}{\sigma}\right)\right]=\alpha$, from which we get $r_\alpha=\left(\tan\frac{\pi\alpha}{2}\exp\frac{\pi\mu}{2\sigma}\right)^{2\sigma/\pi}-1$, and the VaR is the inverse value $v_\alpha=-r_\alpha$.

Proposition L4. For the log-GHS distribution CVaR @ α can be calculated as

$$c_\alpha=1-\frac{1}{\alpha(\sigma+\pi/2)}\left(\tan\frac{\pi\alpha}{2}\exp\frac{\pi\mu}{2\sigma}\right)^{2\sigma/\pi}\tan\frac{\pi\alpha}{2}\,{}_2F_1\left(1,\frac{1}{2}+\frac{\sigma}{\pi};\frac{3}{2}+\frac{\sigma}{\pi};-\left(\tan\frac{\pi\alpha}{2}\right)^2\right),\qquad(7)$$

where μ,σ are the location and the scale parameter of the distribution of the log-return.

${}_2F_1$ is the hypergeometric function.

Proof: Using (3),

$$1+(-c_\alpha)=\frac{1}{\alpha}\int_0^\alpha (1-v_p)\,dp=\frac{1}{\alpha}\int_0^\alpha\left(\tan\frac{\pi p}{2}\exp\frac{\pi\mu}{2\sigma}\right)^{2\sigma/\pi}dp=\frac{1}{\alpha}\left(\exp\frac{\pi\mu}{2\sigma}\right)^{2\sigma/\pi}\int_0^\alpha\left(\tan\frac{\pi p}{2}\right)^{2\sigma/\pi}dp,$$

as per http://functions.wolfram.com/ElementaryFunctions/Tan/introductions/Tan/ShowAll.html, the indefinite integral of $\tan^v(az)\,dz$ can be expressed using the hypergeometric function, so

$$1-c_\alpha=\frac{1}{\alpha}\left(\exp\frac{\pi\mu}{2\sigma}\right)^{2\sigma/\pi}\frac{1}{\sigma+\pi/2}\left(\tan\frac{\pi\alpha}{2}\right)^{2\sigma/\pi+1}{}_2F_1\left(1,\frac{1}{2}+\frac{\sigma}{\pi};\frac{3}{2}+\frac{\sigma}{\pi};-\left(\tan\frac{\pi\alpha}{2}\right)^2\right)=$$

$$=\frac{1-v_\alpha}{\alpha(\sigma+\pi/2)}\left(\tan\frac{\pi\alpha}{2}\right){}_2F_1\left(1,\frac{1}{2}+\frac{\sigma}{\pi};\frac{3}{2}+\frac{\sigma}{\pi};-\left(\tan\frac{\pi\alpha}{2}\right)^2\right).\square$$

Monte-Carlo simulation of the derived formulas

Monte-Carlo simulations were used to test the formulas (4)–(7), as in the ideal case when a reasonably large sample is known to follow a particular distribution, the empirical (sample) VaR and CVaR should be located close enough to the theoretical (variance-covariance) VaR and

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CVaR. While we do not operate with confidence intervals for VaR and CVaR in this research, it should be really straightforward to see whether those values are "close enough" or not. So our methodology is generating a sample of 10,000 random returns that follow a particular distribution with known parameters, estimate the sample CVaR with (2) and compare it with the theoretical CVaR calculated with (4)–(7). For VaR a similar methodology is utilized by comparing the sample percentile with the quantile for the respective distribution. All samples were generated to model an asset with the expected return of 5% and its standard deviation of 20%. The results are presented in Table 1 and confirm that the derived formulas are correct.

Table 1. Theoretical and sample VaR and CVaR comparison for Monte-Carlo simulated data

Distribution		Log-normal	Log-logistic	Log-Laplace	Log-GHS
Parameters		$\mu = 0.05$, $\sigma = 0.2$	$\mu = 0.05$, $s = 0.2\sqrt{3}/\pi$	$\mu = 0.05$, $b = 0.2/\sqrt{2}$	$\mu = 0.05$, $\sigma = 0.2$
VaR @ 5%	sample	0.23994	0.23907	0.24016	0.23340
	theoretical	0.24344	0.24018	0.24091	0.23942
CVaR @ 5%	sample	0.29335	0.31453	0.33500	0.32171
	theoretical	0.30224	0.31749	0.33496	0.32543
VaR @ 1%	sample	0.33396	0.36504	0.39084	0.38148
	theoretical	0.33984	0.36662	0.39542	0.38050
CVaR @ 1%	sample	0.37960	0.42220	0.47790	0.44339
	theoretical	0.38194	0.42982	0.47034	0.45047

(Source: author's calculations based on Monte-Carlo simulated samples)

The simulated results clearly indicate the fact that the log-logistic distribution has fatter tails than the log-normal, the log-GHS distribution has fatter tails, and the log-Laplace distribution has the fattest tails of all considered in this paper. While the difference is virtually nonexistent for VaR @ 5%, it becomes more and more essential as we move towards the end of the tail.

Risk assessment for asset classes (log-normal and log-logistic distributions)

This section presents VaR and CVaR estimation for real-life asset classes, such as equities, bonds, real estate, gold, and so on. It is based on findings of the original research conducted by the author and professor A. Kaminsky from Kyiv National University. Exchange-traded funds tracking the relevant indices are used to represent the following asset classes:

1. SPY — large-cap U.S. equities (tracking S&P 500 index)
2. MDY — mid-cap U.S. equities (tracking S&P 400 index)
3. IJR — small-cap U.S. equities (tracking S & P 600 index)
4. IEF — U.S. Treasury bonds
5. LQD — investment-grade corporate bonds
6. TIP — inflation-linked U.S. Treasury bonds
7. VNQ — real estate (tracking US REIT index)
8. GLD — gold
9. PSP — private equity investments
10. MBB — mortgage-backed bonds

Only the log-normal and the log-logistic distributions are analyzed as the fitting to the returns sample was performed using EasyFit package that does not support log-Laplace or log-GHS distributions. Table 2 presents the VaR estimation results for the asset classes 1–8 using the daily returns from March 2003 to October 2007, a very long growth period on the stock market. The log-logistic distribution proved to be a much better fit to the data according to several

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statistical tests, and in most cases it also provides a better VaR estimation accuracy (with the exception of TIP that is much better modeled with the log-normal distribution).

Table 2. VaR @ 5% and @ 1% per $1 estimation for selected stocks in 2003–2007

Ticker	VaR @ 5%			VaR @ 1%		
	historical	log-normal	log-logistic	historical	log-normal	log-logistic
SPY	1.2362%	1.1912%	1.2287%	1.9488%	1.6926%	1.9176%
MDY	1.5054%	1.3759%	1.4691%	2.1468%	1.9483%	2.2927%
IJR	1.7213%	1.6068%	1.7433%	2.3828%	2.2791%	2.7206%
IEF	0.5999%	0.5775%	0.5901%	0.8893%	0.8134%	0.9209%
LQD	0.5359%	0.5490%	0.5800%	0.9185%	0.7755%	0.9051%
TIP	0.5017%	0.5094%	0.5425%	0.7237%	0.7168%	0.8466%
VNQ	1.7522%	1.7743%	1.8440%	3.1119%	2.5107%	2.8778%
GLD	1.7833%	1.7562%	1.7786%	3.3492%	2.5017%	2.7757%

(Source: author's calculations based on historical prices from Yahoo! Finance)

Table 3 presents the CVaR estimation results for the asset classes 1–8 using the daily returns from March 2003 to October 2007. Here we can see a significant underestimation of the tail risk when the log-normal distribution is used. That signifies one of the key differences between the VaR and the CVaR — the former ignores the tail structure beyond the threshold, whereas the latter takes it into account.

Table 3. CVaR @ 5% and @ 1% per $1 estimation for selected stocks in 2003–2007

Ticker	CVaR @ 5%			CVaR @ 1%		
	historical	log-normal	log-logistic	historical	log-normal	log-logistic
SPY	1.7152%	1.4984%	1.6569%	2.5874%	1.9377%	2.3370%
MDY	1.9204%	1.7265%	1.9810%	2.5903%	2.2273%	2.7942%
IJR	2.1644%	2.0186%	2.3507%	2.9386%	2.6070%	3.3156%
IEF	0.7965%	0.7220%	0.7957%	1.1970%	0.9284%	1.1223%
LQD	0.7633%	0.6878%	0.7820%	1.1572%	0.8861%	1.1031%
TIP	0.6652%	0.6364%	0.7315%	0.9314%	0.8175%	1.0318%
VNQ	2.5917%	2.2254%	2.4865%	3.6080%	2.8700%	3.5073%
GLD	2.7026%	2.2130%	2.3983%	4.3250%	2.8670%	3.3828%

(Source: author's calculations based on historical prices from Yahoo! Finance)

Table 4 presents the VaR estimation results for all asset classes using the daily returns from November 2008 to February 2009, the period of the global financial crisis. We can note the substantial increase in the VaR and CVaR values comparing to the previous period. Both the log-normal and the log-logistic distributions greatly underestimate the tail risk. However, the log-normal distribution appears to be much more accurate in VaR estimation despite the fact it scored worse according to statistical goodness-of-fit tests.

Table 4. VaR @ 5% and @ 1% per $1 estimation for selected stocks in 2007–2009

Ticker	VaR @ 5%			VaR @ 1%		
	historical	log-normal	log-logistic	historical	log-normal	log-logistic
SPY	4.3265%	4.0965%	3.6767%	7.2827%	5.6254%	5.5913%
MDY	4.4549%	4.5775%	4.0001%	8.3208%	6.3186%	6.2426%
IJR	4.2273%	4.4216%	4.0642%	7.7014%	6.0964%	6.3427%
IEF	0.9189%	0.9426%	0.9716%	1.3144%	1.3287%	1.5204%
LQD	1.1136%	1.9622%	1.3702%	3.9800%	2.7523%	2.1383%
TIP	1.0606%	1.1401%	1.1397%	1.9225%	1.6015%	1.7786%
VNQ	7.4898%	7.3659%	6.9357%	12.3331%	10.2258%	10.5710%
GLD	3.0697%	3.1149%	3.0142%	4.9604%	4.3582%	4.7320%
PSP	6.9305%	6.5071%	6.1612%	11.2783%	8.9102%	6.9305%
MBB	0.6040%	0.6475%	0.6350%	1.0433%	0.9144%	0.6040%

(Source: author's calculations based on historical prices from Yahoo! Finance)

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Table 5 presents the CVaR estimation results for all asset classes using the daily returns from November 2007 to February 2009. We can observe another difference between VaR and CVaR — despite the fact that the log-normal distribution provided much better estimates for VaR in this case, the CVaR estimates are slightly more accurate when the log-logistic distribution is used. Nevertheless, as both distributions underestimate the tail risk, we may presume that the distributions with fatter tails (log-Laplace, log-GHS) would be more relevant in risk assessment during the crisis periods.

Table 5. CVaR @ 5% and @ 1% per $1 estimation for selected stocks in 2007–2009

Ticker	CVaR @ 5%			CVaR @ 1%		
	historical	log-normal	log-logistic	historical	log-normal	log-logistic
SPY	5.9699%	5.0327%	4.8645%	8.4905%	6.3658%	6.7367%
MDY	6.6207%	5.6440%	5.3938%	9.8727%	7.1673%	7.6080%
IJR	6.4184%	5.4475%	5.4802%	9.0201%	6.9129%	7.7299%
IEF	1.1953%	1.1790%	1.3117%	1.5741%	1.5152%	1.8454%
LQD	2.6583%	2.4464%	1.8476%	6.3857%	3.1404%	2.6060%
TIP	1.6434%	1.4227%	1.5368%	2.3255%	1.8269%	2.1676%
VNQ	10.3353%	9.1178%	9.1903%	15.0129%	11.6211%	12.7393%
GLD	4.2483%	3.8763%	4.0809%	5.9847%	4.9604%	5.7699%
PSP	9.1081%	7.9789%	8.1550%	12.4078%	10.0770%	11.3130%
MBB	0.8963%	0.8109%	0.8552%	1.2613%	1.0438%	1.2005%

(Source: author's calculations based on historical prices from Yahoo! Finance)

Conclusion

Conditional value-at-risk (CVaR) was considered a theoretically appealing coherent risk measure with limited practical implications because of the computational complexity. The closed-form CVaR formulas have already been derived for the elliptical distributions. However, for practical applications many risk managers are using the log-returns rather than the raw returns. In this paper the closed-form CVaR formulas have been derived for several log-distributions — the log-normal, the log-logistic, the log-Laplace, and the log-hyperbolic secant. The first two ones are quite popular and present in special distribution fitting software. However, the latter two may be better when the "fat tails" is an issue.

The CVaR for the log-normal, the log-logistic, and the log-Laplace distributions can be easily calculated with Microsoft Excel™, except for a very volatile assets the log-logistic distribution may require a hypergeometric function that is available only in special mathematical packages. The CVaR for the log-Laplace can be calculated using merely a financial or scientific calculator, as it doesn't require any special functions.

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Appendix — The distributional parameters used for modeling the asset classes

The theoretical VaR and CVaR estimates presented in Tables 2–5 were calculated using the following distributional parameters, which were obtained using the EasyFit package. They are provided in Table A1 in the EasyFit notation.

Table A1. EasyFit distributional parameters representing the best fit to the asset class returns

Ticker	2003-2007 sample		2007-2009 sample	
	log-normal	log-logistic	log-normal	log-logistic
SPY	σ=0.026229; μ=-1.21913; γ=-0.294922	α=6.60986e+8; β=2.75837e+6; γ=-2.75837e+6	σ=0.040364; μ=-0.507234; γ=-0.604443	α=103.837; β =1.2489; γ=-1.25075
MDY	σ=0.030941; μ=-1.24242; γ=-0.288119	α=3.07237e+8; β =1.53296e+6; γ=-1.53296e+6	σ=0.030411; μ=-0.113907; γ=-0.894578	α=1.03874e+8; β =1.41116e+6; γ=-1.41116e+6
IJR	σ=0.029134; μ=-1.0252; γ=-0.358008	α=2.47568e+6; β =14657.6; γ=-14657.6	σ=0.030278; μ=-0.148586; γ=-0.864267	α=2.68810e+8; β=3.71039e+6; γ=-3.71039e+6
IEF	σ=0.029093; μ=-2.07121; γ=-0.125919	α=4.78626e+8; β=959175.0; γ=-959175.0	σ=0.044515; μ=-1.97302; γ=-0.138646	α=66.2582; β=0.23317; γ=-0.232751
LQD	σ=0.028369; μ=-2.0878; μ=-0.123798	α=9.17154e+7; β=180652.0; γ=-180652.0	σ=0.018585; μ=-0.435081; γ=-0.647349	α=3.09772e+8; β=1.44151e+6; γ=-1.44151e+6
TIP	σ=0.037858; μ=-2.44572; γ=-0.086526	α=5.03870e+7; β=92834.7; γ=-92834.7	σ=0.026488; μ=-1.31163; γ=-0.269297	α=7.85358e+8; β=3.03982e+6; γ=-3.03982e+6
VNQ	σ=0.028935; μ=-0.927513; γ=-0.394895	α=3.22987e+8; β=2.02281e+6; γ=-2.02281e+6	σ=0.029168; μ=0.421663; γ=-1.52674	α=88.7309; β=2.03887; γ=-2.04168
GLD	σ=0.022275; μ=-0.667014; γ=-0.512337	α=4.44111e+8; β=2.68267e+6; γ=-2.68267e+6	σ=0.03997; μ=-0.705045; γ=-0.493797	α=236.177; β=2.49739; γ=-2.49659
PSP			σ=0.036627; μ=0.034753; γ=-1.0399	α=266.298; β=5.25303; γ=-5.25688
MBB			σ=0.037649; μ=-2.18856; γ=-0.111823	α=61.6967; β=0.141211; γ=-0.14098

(Source: EasyFit distribution fitting for the returns based on historical prices from Yahoo! Finance)

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