# STAT 450 TUTORIAL · WEEK 2

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# Information

- Tutorials: Thu. 9:30 – 10:20
- Office hours: Thu. ???
- Emails: gca58@sfu.ca
- Notes: https://github.com/gz-chen/STAT450-Notes-Fall21

- Probability and probability space $(\Omega, P)$ [or $(\Omega, \mathcal{F}, P)$]
  1. $P(A) \geq 0$;
  2. $P(\Omega) = 1$;
  3. $P(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} P(A_j)$ for any disjoint $\{A_j\}_{j=1}^{\infty}$.
- Conditional probability: Assume $P(B) > 0$,

$$P(A|B) = \frac{P(A \cap B)}{P(B)};$$

- Some useful formulas:
  - Total probability formula: $P(B) = \sum_{j=1}^{n} P(B|A_j)P(A_j)$, where $\{A_j\}_{j=1}^{n}$ is a mutually exclusive and exhaustive sequence;
  - Bayes formula:

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$$

**Problem 1.**

There are three cards. The first is green on both sides, the second is red on both sides and the third is green on one side and red on the other. We choose a card at random and we see one side (also chosen at random). If the side we see is green, what is the probability that the other side is also green? Many people intuitively answer $1/2$. Show that the correct answer is $2/3$.

**Solution.** Denote the three cards by $GG$, $RR$ and $GR$ and two sides by $h$ and $t$. We use symbol $Xy$ to denote the outcome that card $X$ is selected and side $y$ is observed, where $X \in \{GG, RR, GR\}$ and $y \in \{h, t\}$. For example, $GRh$ means the card $GR$ is selected and its head ($G$ side) is observed.

# Problem 1, Ctd.

Then the set of all possible outcomes

$$\Omega = \{GGh, GGt, RRh, RRt, GRh, GRt\}.$$

The event

$$\begin{aligned} \{\text{the side we see is green}\} &= \{GGh, GGt, GRh\} =: A \\ \{\text{the other side we don't see is green}\} &= \{GGh, GGt, GRt\} =: B \end{aligned}$$

Thus $A \cap B = \{GGh, GGt\}$. The conditional probability of interest

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{2/6}{3/6} = \frac{2}{3}.$$

**Problem 2.** Two people take turns trying to sink a basketball into a net (independently). Person 1 succeeds with prob. $1/3$ while person 2 succeeds with prob. $1/4$. What is the probability that person 1 succeeds before person 2?

**Solution.** Let $E$ denote the event of interest and $A_j$ denote the event that the first success is by person 1 and it occurs on the $j$-th trial. Note that $A_1, A_2, \ldots$ are disjoint and that $E = \cup_{j=1}^{\infty} A_j$. Hence,

$$P(E) = P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j).$$

# Problem 2, Ctd.

Note that $P(A_1) = 1/3$, $P(A_2) = (2/3)(3/4)(1/3) = (1/2)(1/3)$ and we can derive that

$$P(A_j) = (1/2)^{j-1}(1/3).$$

Hence,

$$P(E) = \sum_{j=1}^{\infty} P(A_j) = \sum_{j=1}^{\infty} \frac{1}{3}\left(\frac{1}{2}\right)^{j-1} = \frac{2}{3}.$$

Here we use a fact from calculus that for $0 < r < 1$, we have

$$\sum_{j=k}^{\infty} r^j = r^k/(1-r).$$

**Problem 3.** Show that

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} P(A_j)$$

**Solution.** Define

$$\begin{aligned} B_1 &= A_1, \\ B_2 &= A_2 \setminus A_1 \subseteq A_2, \\ B_3 &= A_3 \setminus (A_1 \cup A_2) \subseteq A_3, \\ &\vdots \end{aligned}$$

Then $\cup_{j=1}^{\infty} A_j = \cup_{j=1}^{\infty} B_j$ and $B_j$'s are disjoint, apply the third rule of probability and note that $P(B_j) \leq P(A_j)$, we have the result.

**Problem 4.** Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, show that

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} P(A_j)$$

**Solution.** Using the same trick as in problem 3, we have

$$\begin{aligned} P\left(\bigcup_{j=1}^{\infty} A_j\right) &= P(A_1) + \sum_{j=1}^{\infty} P(A_{j+1} \backslash A_j) \\ &= P(A_1) + \sum_{j=1}^{\infty} [P(A_{j+1}) - P(A_j)] \\ &= \lim_{j\to\infty} P(A_j). \end{aligned}$$

**Exercise.** Let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$, show that

$$P\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} P(A_j)$$

Hint: De Morgan's laws.

Application: show that distribution function $F$ is right continuous.

## Problem 5.

Suppose that 30 percent of computer owners use a Macintosh, 50 percent use Windows, and 20 percent use Linux. Suppose that 65 percent of the Mac users have succumbed to a computer virus, 82 percent of the Windows users get the virus, and 50 percent of the Linux users get the virus. We select a person at random and learn that her system was infected with the virus. What is the probability that she is a Windows user?

Define events

$$
\begin{aligned}
A_1 &= \{\text{The user uses Macintosh}\} \\
A_2 &= \{\text{The user uses Windows}\} \\
A_3 &= \{\text{The user uses Linux}\} \\
B &= \{\text{The user has succumed to virus}\}.
\end{aligned}
$$

Then by Bayes' rule,

$$
\begin{aligned}
\mathbb{P}(A_2|B) &= \frac{\mathbb{P}(B|A_2)\mathbb{P}(A_2)}{\mathbb{P}(B|A_1)\mathbb{P}(A_1) + \mathbb{P}(B|A_2)\mathbb{P}(A_2) + \mathbb{P}(B|A_3)\mathbb{P}(A_3)} \\
&= \frac{0.82 \times 0.5}{0.65 \times 0.3 + 0.82 \times 0.5 + 0.5 \times 0.2} \\
&\approx 58.2\%.
\end{aligned}
$$

Therefore, the probability that she is a Windows user is approximately 58.2%.

# Bibliography


Casella, G. and Berger, R. L.
Statistical Inference, 2ed.
*Duxbury*, 2002.

Wasserman, L.
All of Statistics.
*Springer Science & Business Media*, 2013.