STAT 450 TUTORIAL · WEEK 3

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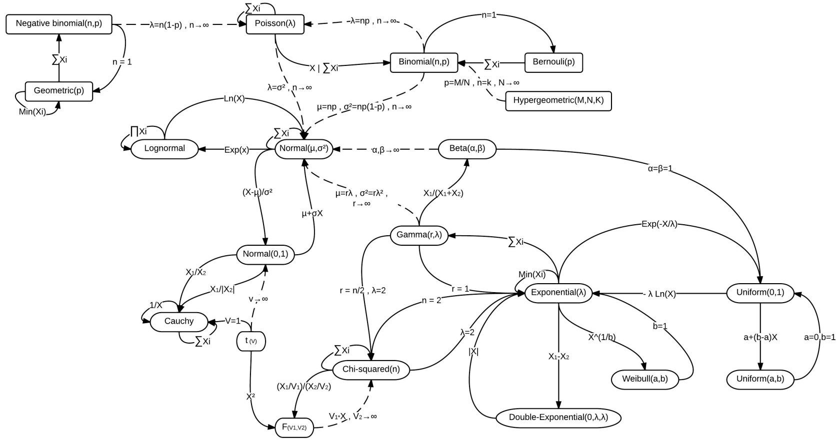
- A random variable X is a function from sample space Ω to the set of real numbers $\mathbb{R}$.
- Given a random variable X, we are interested in its cumulative distribution function (cdf) F_X:

$$F_X(x) = P(X \leq x) = \begin{cases} \sum_{i:x_i \leq x} p(x_i), & \text{if } X \text{ has pmf } p(t); \\ \int_{-\infty}^{x} f(t)dt, & \text{if } X \text{ has pdf } f(t). \end{cases}$$

- Expectation, variance and the moment generating function

$$M_X(t) = E(e^{tX}).$$

  - The mgf of the r.v. $aX + b$ is $M_{aX+b}(t) = e^{tb} M_X(at)$;
  - If $X_1, \dots, X_n$ are iid, then $M_{X_1+\dots+X_n}(t) = [M_X(t)]^n$.



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- joint pmf $p(x_1, x_2)$; joint pdf $f(x_1, x_2)$;
- marginal distribution, conditional distribution, independence;
- covariance and correlation:

$$\text{Cov}(X_1, X_2) = E[\{X_1 - E(X_1)\}\{X_2 - E(X_2)\}]$$

  - $\text{Cov}(X_1, X_2) = \text{Cov}(X_2, X_1)$;
  - $\text{Cov}(aX_1, X_2) = a\text{Cov}(X_1, X_2)$;
  - $\text{Cov}(X_1 + X_2, X_3) = \text{Cov}(X_1, X_3) + \text{Cov}(X_2, X_3)$.

  From above one can show

$$\begin{aligned} &\text{Cov}(aX_1 + bX_2, cX_1 + dX_2) \\ &= ac\text{Var}(X_1) + bd\text{Var}(X_2) + (ad + bc)\text{Cov}(X_1, X_2) \end{aligned}$$

Given a sequence of r.v.'s $\{X_n\}$ and a r.v. X, then

- X_n's converge to X in probability (or $X_n \xrightarrow{P} X$):

  $$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0 \text{ for any } \epsilon > 0.$$

  e.g. WLLN.

- X_n's converge to X in distribution (or $X_n \xrightarrow{D} X$):

  $$\lim_{n\to\infty} F_{X_n}(t) = F_X(t) \text{ for any } t \in C_F.$$

  e.g. CLT.

Let (X, Y) have density

$$f(x,y) = \begin{cases} \frac{21}{4}x^2 y & \text{if } x^2 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the pdf of X and the conditional pdf/mean of Y given $X = 0.5$.

Let $X_1, X_2, \ldots$ be a sequence of random variables such that

$$\mathbb{P}\left(X_n = \frac{1}{n}\right) = 1 - \frac{1}{n^2} \quad \text{and} \quad \mathbb{P}(X_n = n) = \frac{1}{n^2}.$$

Does X_n converge in probability?

Suppose that $X_n \sim N(0, 1/n)$ and let X be a random variable with distribution $F(x) = 0$ if $x < 0$ and $F(x) = 1$ if $x \geq 0$. Does X_n converge to X in probability? (Prove or disprove). Does X_n converge to X in distribution? (Prove or disprove).

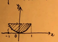
Bibliography

Casella, G. and Berger, R. L.
Statistical Inference, 2ed.
Duxbury, 2002.

Wasserman, L.
All of Statistics.
Springer Science & Business Media, 2013.

Tutorial Solution (Week3).

Problem 1.



(i) $f_X(x) = \int_{x^2}^{1} f(x,y)\,dy = \int_{x^2}^{1} \frac{21}{4}x^2 y \,dy = \frac{21}{4}x^2 \int_{x^2}^{1} y\,dy = \frac{21}{8}x^2(1-x^4)$, $(-1 \le x \le 1)$.

$$\Rightarrow \quad f_X(x) = \begin{cases} \frac{21}{8}x^2(1-x^4), & -1 \le x \le 1; \\ 0, & \text{otherwise}. \end{cases}$$

(ii). Given $X=x$, the conditional pdf of Y given $X=x$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{21}{4}x^2 y \Big/ \frac{21}{8}x^2(1-x^4) = 2y \big/ 1-x^4 \ .(x^2 \le y \le 1).$$

$$\Rightarrow \quad f_{Y|X}(y|x) = \begin{cases} \frac{2y}{1-x^4}, & x^2 \le y \le 1; \\ 0, & \text{otherwise}. \end{cases} \qquad \Rightarrow f_{Y|X=0.5}(y|0.5) = \begin{cases} \frac{32}{15}y, & \frac{1}{4} \le y \le 1; \\ 0, & \text{o.w.} \end{cases}$$

(iii) $$E(Y|X=\tfrac{1}{2}) = \int_{x^2}^{1} y f_{Y|X}(y|0.5)\,dy = \int_{\frac{1}{4}}^{1} \frac{32}{15}y^2\,dy = \frac{32}{15} \times \frac{1}{3}\left(1-\frac{1}{4^3}\right) = \frac{7}{10}.$$

Problem 2. We show that $X_n \xrightarrow{P} 0$.

For any given $\varepsilon > 0$, and sufficiently large n $P(|X_n - 0| > \varepsilon) = P(X_n = n) = \frac{1}{n^2} \to 0$

as n goes to ∞. Therefore, $\lim_{n\to\infty} P(|X_n - 0| > \varepsilon) = 0$.

Therefore $X_n \xrightarrow{P} 0$.

Problem 3. For any given $\varepsilon > 0$.

$$P(|X_n - X| > \varepsilon) = P(|X_n - 0| > \varepsilon)$$

$$= P\left(\left|\frac{X_n - 0}{\sqrt{1/n}}\right| > \sqrt{n}\,\varepsilon\right)$$

$$= P(|Z| > \sqrt{n}\,\varepsilon) \quad \text{where } Z \sim N(0,1).$$

$$= 2\Phi(-\sqrt{n}\,\varepsilon) \longrightarrow 0.$$

Therefore $X_n \xrightarrow{P} X$. $\Rightarrow$ $X_n \xrightarrow{D} X$.

The fact that $X_n \xrightarrow{D} X$ can also be verified directly:

$$F_{X_n}(x) = P(X_n \le x) = P\left(X_n/\sqrt{1/n} \le x/\sqrt{1/n}\right)$$

$$= P\left(Z \le x/\sqrt{\tfrac{1}{n}}\right) = \Phi(\sqrt{n}\,x) \longrightarrow \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x > 0. \end{cases}$$

$F_n(x) \longrightarrow F(x)$ for all $x \neq 0$.

Therefore $X_n \xrightarrow{D} X$.