STAT 403/603 Midterm, Spring 2020

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1. (10 marks) The following are data collected from a randomized paired comparison design that aims at finding out if treatment B produces high response than treatment A.

A	B	A	B	A	B
13	16	12	11	8	10

H_0: A & B same;

H_a: B is better than A.

(a) (5 marks) Find the p-value from the paired randomization test.
(b) (5 marks) If this were actually a completely randomized design, obtain the p-value from an appropriate randomization test.

Solution:

For randomization tests, the testing statistic D is defined to be the difference of two treatment means, i.e. $D = \bar{Y}_B - \bar{Y}_A$, where $\bar{Y}_B$ is the average of obs. from treatment B, and $\bar{Y}_A$ is the average of obs. from treatment A.

Obviously, $D_{obs} = \frac{1}{3}(16+11+10) - \frac{1}{3}(13+12+8) = \frac{4}{3}$.

(a). In a paired design, randomization is done by switching treatment labels "A" and "B" in each block. In our case, there are $2^3 = 8$ possible treatment assignments. If H_0 is true, then one of them yields a D value larger than D_{obs}: $\boxed{A\ B \mid B\ A \mid A\ B}$; and one of them yields a D value equal to D_{obs}: $\boxed{A\ B \mid A\ B \mid A\ B}$ (same as given). Thus,

$p\text{-value} = P(D > D_{obs}) + \frac{1}{2}P(D = D_{obs}) = \frac{1}{8} + \frac{1}{8} \times \frac{1}{2} = \frac{3}{16} \approx 0.187$.

(b). If the design is not paired, then the randomization is done by randomly picking 3 to assign "A" and the rest to assign "B". Among the $\binom{6}{3} = 20$ treatment assignments, 5 of them give D values larger than D_{obs}: ABBAAB, BBAAAB, AABBAB, BBABAA, BBBAAA; two of them give D values identical to D_{obs}: BBAABA, ABABAB. Therefore, the

$p\text{-value} = P(D > D_{obs}) + \frac{1}{2}P(D = D_{obs}) = \frac{5}{20} + \frac{1}{2} \times \frac{2}{20} = 0.3$.

2. (10 marks) Data were collected to compare two treatments A and B with the following summary results:

A	$m = 5$	$\bar{Y}_A = 8$	$S_1^2 = 1.5$
B	$n = 3$	$\bar{Y}_B = 6$	$S_2^2 = 1.0$

(a) (5 marks) At level $\alpha = 0.01$, conduct an appropriate test to compare the two treatments.
(b) (5 marks) Suppose there is another observation $Y = 7$ from treatment B. Re-do part(a) using all available data.

Solution:

Denote the 5 observations from treatment A by $Y_A^{(1)}, \dots, Y_A^{(m)} \overset{iid}{\sim} N(\mu_A, \sigma^2)$, $m = 5$;

and the 3 observations from treatment B by $Y_B^{(1)}, \dots, Y_B^{(n)} \overset{iid}{\sim} N(\mu_B, \sigma^2)$, $n = 3$.

We wish to test $H_0: \mu_A = \mu_B$ v.s. $H_a: \mu_A \neq \mu_B$.

(a)(i) If we use t-test, the testing statistic $T = \dfrac{\bar{Y}_B - \bar{Y}_A}{\sqrt{(\frac{1}{m} + \frac{1}{n}) S^2}} \overset{H_0}{\sim} t_{n+m-2}$,

where $S^2 = \frac{1}{n+m-2}[(m-1)S_1^2 + (n-1)S_2^2]$ and we reject the null hypothesis

if $|T_{obs}| > t_{0.005, n+m-2}$. We have $T_{obs} = \dfrac{6-8}{\sqrt{(\frac{1}{5}+\frac{1}{3}) \times (4 \times 1.5 + 2 \times 1)/6}} = -2.37$ and

$t_{0.005, 6} = 3.707$. Thus $|T_{obs}| < t_{0.005, 6}$ and we cannot reject H_0 at 1% level;

(ii) If we choose to use F-test, the testing statistic becomes

$F = \dfrac{SS_{trt}/df_{trt}}{SS_{err}/df_{err}} \overset{H_0}{\sim} F_{df_{trt}, df_{err}}$, where $SS_{trt} = m(\bar{Y}_A - \bar{Y})^2 + n(\bar{Y}_B - \bar{Y})^2$,

$\bar{Y} = \frac{1}{m+n}[m\bar{Y}_A + n\bar{Y}_B]$, $df_{trt} = \underset{\text{(number of trt's)}}{2} - 1 = 1$; $SS_{error} = (m-1)S_1^2 + (n-1)S_2^2$,

$df_{err} = m + n - 2$. We have $F_{obs} = \dfrac{[5 \times (8-7.25)^2 + 3 \times (6-7.25)^2]/1}{(4 \times 1.5 + 2 \times 1.0)/6} = 5.625$,

which is less than $F_{0.01,1,6} = 13.75$. We cannot reject H_0 that the two treatments are the same at 1% level.

(b). If there is an additional observation $Y=7$ from treatment B.

Then $\bar{Y}_B' = \frac{1}{n+1}\left[n\bar{Y}_B + Y \right] = \frac{1}{4}\left[3\times 6 + 7 \right] = 6.25$;

$$(S_2^2)' = \frac{1}{n+1-1}\left[\sum_{i=1}^{n} (Y_B^{(i)} - \bar{Y}_B')^2 + (Y - \bar{Y}_B')^2 \right] = \frac{1}{n}\left[\sum_{i=1}^{n} Y_B^{(i)2} + Y^2 - (n+1)(\bar{Y}_B')^2 \right] \qquad (*)$$

Note that
$$S_2^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} (Y_B^{(i)} - \bar{Y}_B)^2 \right] = \frac{1}{n-1}\left[\sum_{i=1}^{n} Y_B^{(i)2} - n\bar{Y}_B^2 \right]$$

$$\Rightarrow \sum_{i=1}^{n} Y_B^{(i)2} = (n-1)S_2^2 + n\bar{Y}_B^2 = 2\times 1 + 3\times 6^2 = 110.$$

Thus by $(*)$ $(S_2^2)' = \frac{1}{3}\left[110 + 49 - 4\times 6.25^2 \right] = 0.9167.$

(i). If we use t-test,

$$T'_{obs} = \frac{6.25 - 8}{\sqrt{(\frac{1}{5}+\frac{1}{4})\times(4\times 1.5 + 3\times 0.9167)/7}} = -2.33,$$

hence $|T'_{obs}| < t_{0.005,7} = 3.50$. $\Rightarrow$ We still cannot reject H_0 at 1% level.

(ii). If we choose F-test, note that $\bar{Y}' = \frac{5}{9}\times 8 + \frac{4}{9}\times 6.25 = 7.22$

$$F'_{obs} = \frac{[5\times(8-7.22)^2 + 4\times(6.25-7.22)^2]/1}{[4\times 1.5 + 3\times 0.9167]/7} = \frac{6.81}{1.25} = 5.448$$

which is less than $F_{0.01,1,7} = 12.25$.

Therefore, we still cannot reject H_0 at 1% level.

3. (10 marks) The following data were colllected from an interlaboratory calibration check. Each of the laboratories (A, B and C) was sent a sample of some standard wire. The results obtained from their horizontal tension testing machines were as follows:

A	48	50		
B	51	52	50	
C	50	50	48	52

(a) (5 marks) At 5% level, compare the three laboratories.

(b) (5 marks) Another objective of the study is to compare laboratory A with the average of laboratories B and C. Answer this question using a suitable contrast. You may use $\alpha = 0.05$.

Solution.

(a). Label the three laboratories A, B, C by 1, 2, 3, respectively. Denote the j-th observation in the i-th laboratory by Y_{ij} $\left(\begin{smallmatrix} i=1,2,3=t \\ j=1,\dots,n_i \end{smallmatrix}\right)$. Suppose $Y_{ij} \sim N(\mu_i, \sigma^2)$.

To assess $H_0: \mu_1 = \mu_2 = \mu_3$ v.s. $H_a: \mu_i$ not the same. We else F-test.

$$F = \frac{SS_{trt} / df_{trt}}{SS_{err} / df_{err}}, \quad \text{where} \quad \begin{cases} SS_{trt} = \sum_{i=1}^{t} n_i (\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 \\ df_{trt} = t-1 \\ SS_{err} = \sum_{i=1}^{t} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i\cdot})^2 \\ df_{err} = N - t \end{cases}$$

$$F_{obs} = \frac{[2\times(49-50.11)^2 + 3\times(51-50.11)^2 + 4\times(50-50.11)^2]/(3-1)}{[(48-50.11)^2 + (50-50.11)^2 + \dots + (52-50.11)^2]/(9-3)} = 1.22 < F_{0.05,2,6} = 5.14$$

Therefore, we cannot say the three laboratories are different at 5% level.

(b). The objective of the study is equivalent to assess whether $\theta = \mu_1 - \frac{1}{2}(\mu_2 + \mu_3)$ equals to 0. Then θ can be estimated by $\hat{\theta} = \bar{Y}_{1\cdot} - \frac{1}{2}(\bar{Y}_{2\cdot} + \bar{Y}_{3\cdot}) = -1.5$

(See Tutorial Notes of Week 5)

with variance $var(\hat{\theta}) = [1^2 \cdot \frac{1}{2} + (-\frac{1}{2})^2 \frac{1}{3} + (-\frac{1}{2})^2 \frac{1}{4}] \sigma^2 = \frac{3}{2} \sigma^2$, where σ^2 can be estimated by $\hat{\sigma}^2 = \frac{SS_{err}}{df_{err}} = 2$, and stand error $se(\hat{\theta}) = \sqrt{[1 \cdot \frac{1}{2} + \frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{4}] \times 2} = 1.137$

For hypothesis testing problem $H_0: \theta = 0$ v.s. $H_a: \theta \neq 0$, we reject H_0 at 5% if $|T_{obs}| > t_{0.025, df_{err}}$, where T_{obs} is the observed value for $T = \hat{\theta} / se(\hat{\theta})$.

Here $T_{obs} = -1.5 / 1.137 = -1.32 \Rightarrow |T_{obs}| = +1.32 < t_{0.025,6} = 2.45$, do not reject H_0 at 5% level.

4. (10 marks) A randomized block design was used to compare several treatments. Suppose that the treatment averages are 43, 45, 49, 47, and 51.

(a) (5 marks) Complete the following ANOVA table. Show your work.

source	SS	df
treatment	240	4
block	850	5
error	160	20
total	1250	29

Solution : $\bar{Y}_{..} = \frac{1}{5}(43+45+49+47+51) = 47$, number of blocks $b = 5+1 = 6$

$SS_{trt} = b[(43-47)^2 + (45-47)^2 + \cdots + (51-47)^2] = 240$; $SS_{blk} = SS_{total} - SS_{trt} - SS_{err} = 850$.

$df_{trt} = t - 1 = 5 - 1 = 4$ (t = number of treatments)

$df_{error} = df_{treatment} \times df_{block} = 20$.

$df_{total} = df_{trt} + df_{blk} + df_{err} = 29$.

(b) At 5% level, is there any difference in the treatments? How would your analysis have been different if the experiment had not been blocked?

Solution. (i) $F = \dfrac{SS_{treatment} / df_{treatment}}{SS_{error} / df_{error}} \overset{H_0}{\sim} F_{df_{treatment},\, df_{error}}$

$F_{obs} = \dfrac{240/4}{160/20} = 7.5 > F_{0.05,4,20} = 2.87$. Thus at 5% level, there are some differences in the treatments.

(ii) If the experiment had not been blocked,

$SS_{error}' = SS_{block} + SS_{error} = 850 + 160 = 1010$;

$df_{error}' = df_{block} + df_{error} = 5 + 20 = 25$.

Thus, $F_{obs}' = \dfrac{240/4}{1010/25} = 1.49 < F_{0.05,4,25} = 2.76$.

Therefore, at 5% level we cannot say there are differences in treatments.