STAT 403 Tutorial · Week 5

For this week's tutorial, I'm going to take a review of (i) F-test and ANOVA table; (ii) contrasts and their inferences using a numerical example.

Example. A researcher is asked to compare 4 fertilizers (labelled 1,2,3,4) with 11 plots. So he conducted a randomized experiment and obtained a dataset as follows.

Unit	1	2	3	4	5	6	7	8	9	10	11
Treatment	2	1	3	4	1	2	1	4	3	2	1
Yield	8	16	10	7	14	12	19	9	13	15	20

→

trt	obs (Y_{ij})	sample mean ($\bar{Y}_{i.}$)
1	16, 14, 19 20	17.25
2	8, 12, 15	11.67
3	10, 13	11.5
4	7, 9	8

(grand mean $\bar{Y}_{..} = 13$)

Denote the j-th yield under treatment i by Y_{ij} $(i=1,\dots,4;\ j=1,\dots,n_i)$ and assume that $Y_{ij} = \mu_i + \varepsilon_{ij}$, where $\varepsilon_{ij} \overset{iid}{\sim} N(0,\sigma^2)$.

TASK I: Assess whether the four fertilizers have the same effect, i.e. assess

$H_0: \mu_1=\mu_2=\mu_3=\mu_4$ v.s. $H_a: \mu_i$ not the same.

TASK II: Suppose that fertilizer 1 is new and we want to compare its effect with the average effect of the other three, i.e., assess

$H_0: \mu_1 = \frac{\mu_2+\mu_3+\mu_4}{3}$ v.s. $H_a: \mu_1 \neq \frac{1}{3}(\mu_2+\mu_3+\mu_4)$.

and obtain a 95% confidence interval for $3\mu_1-(\mu_2+\mu_3+\mu_4)$.

Solutions:

TASK I. If H_0 is true, then $F = \dfrac{SS_{trt}/df_{trt}}{SS_{err}/df_{err}} \sim F_{df_{trt},\,df_{err}}$.

Here $df_{trt} = t-1 = 4-1 = 3$, $df_{err} = N-t = 11-4 = 7$

$$SS_{trt} = \sum_{i=1}^{t} n_i(\bar{Y}_{i.}-\bar{Y}_{..})^2 = 4\times(17.25-13)^2 + 3\times(11.67-13)^2 + 2\times(11.5-13)^2 + 2\times(8-13)^2 = 132.06$$

$$SS_{err} = \sum_{i=1}^{4}\sum_{j=1}^{n_i}(Y_{ij}-\bar{Y}_{i.})^2 = (16-17.25)^2+(14-17.25)^2+(19-17.25)^2+(20-17.25)^2$$
$$+(8-11.67)^2+(12-11.67)^2+(15-11.67)^2$$
$$+(10-11.5)^2+(13-11.5)^2$$
$$+(7-8)^2+(9-8)^2 = 53.92$$

Then we can calculate observed F value:

$$F_{obs} = \frac{132.06/3}{53.92/7} = 5.71$$

You choose to calculate p-value $= P(F \geq F_{obs}) = 0.027 < 0.05$

or compare F_{obs} with the upper 5% cut-point of $F_{3,7}$: $F_{0.05,3,7} = 4.34 < F_{obs}$

So we reject the null hypothesis at 5%.

We can summarize these calculations in an ANOVA table.

Source	sum of squares	degrees of freedom	F
trt	132.06	3	5.71
err	53.92	7	
total	185.98	10	

TASK II.

We are interested in the contrast $\theta = 3\mu_1 - (\mu_2 + \mu_3 + \mu_4)$.

You've learned how to make inferences for contrasts when $n_1 = \cdots = n_t = n$:

$\theta = c_1\mu_1 + \cdots + c_t\mu_t$, $\hat{\theta} = c_1 \bar{Y}_{1.} + \cdots + c_t \bar{Y}_{t.}$,

$var(\hat{\theta}) = (c_1^2 + \cdots + c_t^2) \cdot \frac{\sigma^2}{n}$, $\hat{\sigma}^2 = \frac{SS_{err}}{df_{err}} = \frac{SS_{err}}{N-t}$, $se(\hat{\theta}) = \sqrt{(c_1^2 + \cdots + c_t^2)\frac{\hat{\sigma}^2}{n}}$.

Then a CI is given by $\hat{\theta} \pm t_{\alpha/2, N-t} \cdot se(\hat{\theta})$.

To test hypothesis $H_0: \theta = 0$ v.s. $H_a: \theta \neq 0$, reject H_0 if $|T| \geq t_{\alpha/2, N-t}$,

where $T = \frac{\hat{\theta}}{se(\hat{\theta})}$.

Here the situation is slightly more complicated because n_i's are different.

So in the calculations above, $var(\hat{\theta})$ & $se(\hat{\theta})$ need to be adjusted:

$$var(\hat{\theta}) = \left(\frac{c_1^2}{n_1} + \cdots + \frac{c_t^2}{n_t}\right)\sigma^2, \quad se(\hat{\theta}) = \sqrt{\left(\frac{c_1^2}{n_1} + \cdots + \frac{c_t^2}{n_t}\right)\hat{\sigma}^2}$$

Now we're ready to infer about $\theta = 3\mu_1 - \mu_2 - \mu_3 - \mu_4$.

$$\hat{\theta} = 3\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot} - \bar{Y}_{3\cdot} - \bar{Y}_{4\cdot} = 3 \times 17.25 - 11.67 - 11.5 - 8 = 20.58.$$

$$\hat{\sigma}^2 = \frac{SS_{err}}{N-t} = \frac{53.92}{7} = 7.70.$$

$$se(\hat{\theta}) = \sqrt{\left[\frac{3^2}{4} + \frac{(-1)^2}{3} + \frac{(-1)^2}{2} + \frac{(-1)^2}{2}\right] \times 7.70} = 5.25$$

Therefore a 95% CI for θ is given by $\hat{\theta} \pm t_{0.025,7} \cdot se(\hat{\theta}) = 20.58 \pm 2.36 \times 5.25$
$= [8.17, 32.99]$

For hypothesis testing problem: $H_0: \theta = 0$ v.s. $H_a: \theta \neq 0$.

$$T = \frac{\hat{\theta}}{se(\hat{\theta})} = \frac{20.58}{5.25} = 3.92$$

The upper 2.5% cut-point of t_7 is $2.36 < 3.92$, therefore we reject the null at 5% level.