Problem Set 0

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Problem 0-1.

(a) $A \cap B = \{6, 12\}$

(b) $|A \cup B| = 7$

(c) $|A - B| = 3$

Problem 0-2.

(a) $E[X] = 0.5 + 0.5 + 0.5 = 1.5$

(b) $E[Y] = 7/2 \times 7/2 = 12.25$

(c) $E[X + Y] = E[X] + E[Y] = 13.75$ (by linearity of expectation).

Problem 0-3.

(a) $A \equiv B \pmod 2$ is True.

(b) $A \equiv B \pmod 3$ is False.

(c) $A \equiv B \pmod 4$ is False.

Problem 0-4. *Proof.* Let $P(n)$ be the predicate as described in the question, where n is an integer and $n \geq 1$. We proceed by induction on n.

Base case. Our Base case occurs when $n = 1$. The left-hand side of the equation is then evaluated as $\sum_{i=1}^{n} i^3 = 1^3 = 1$. The right-hand side of the equation is also evaluated as $\left[\frac{1(1+1)}{2}\right]^2 = (2/2)^2 = 1$. Thus, the Base case is satisfied.

Inductive step. Assume that the predicate is true for all integers $1, \ldots, n$. For $P(n+1)$, we have

$$\sum_{i=1}^{n+1} i^3 = (n+1)^3 + \sum_{i=1}^{n} i^3$$

$$= (n+1)^3 + \left[\frac{n(n+1)}{2}\right]^2 \quad \text{(by the Inductive Hypothesis)}$$

$$= \frac{(n+1)^2}{2^2}\left(2^2(n+1) + n^2\right)$$

$$= \frac{(n+1)^2(n+2)^2}{2^2} \quad \text{(by factoring the right term from the previous step).}$$

And the desired result that $\sum_{i=1}^{n+1} i^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2$ naturally follows. This concludes the proof.

Problem 0-5. *Proof.* Let $P(n)$ be the predicate that every connected undirected graph $G = (V, E)$ for which $|E| = |V| - 1$ is acyclic, where n is the number of vertices.

Base case. Our Base case occurs when $n = 1$ since we cannot have a negative number of edges. A graph G with 1 vertex and no edges trivially satisfies the predicate. Thus, $P(0)$ holds.

Inductive step. Assume that the predicate is true for all graphs with number of edges from $0, \ldots, n$. In the case that $P(n + 1)$, graph G has $n + 1$ vertices and n edges. Since G is connected, every vertex connects to at least one edge. Since the number of edges connecting the vertices is one less than the total number of vertices, there must be one vertex with degree 1. Suppose we remove that degree-1 vertex (and the accompanying edge), say v_0. The resultant graph, G' then is connected, and has exactly n vertices and $n - 1$ edges. v_0 could not have been part of a cycle since it only has a degree of 1, so the only way G has a cycle is if G' has a cycle. But we know G' is acyclic by the Inductive Hypothesis, therefore G must be acyclic. This concludes the proof.

Problem 0-6. Submit your implementation to `alg.mit.edu`.

```python
def count_long_subarray(A):
    '''
    Input:  A     | Python Tuple of positive integers
    Output: count | number of longest increasing subarrays of A
    '''
    count = 1
    N = len(A)
    curLen, curMax = 1, 1

    for i in range(1, N):
        if A[i] > A[i - 1]:
            curLen += 1
        else:
            curLen = 1

        if curLen == curMax:
            count += 1
        elif curLen > curMax:
            curMax = curLen
            count = 1

    return count
```