Categorification of Negative Information using Enrichment

Andrea Censi Emilio Frazzoli Jonathan Lorand Gioele Zardini
Institute for Dynamic Systems and Control
Department of Mechanical and Process Engineering
ETH Zurich, Switzerland
{acensi,efrazzoli,jlorand,gzardini}@ethz.ch

In many applications of category theory it is useful to reason about "negative information". For example, in planning problems, providing an optimal solution is the same as giving a feasible solution (the "positive" information) together with a proof of the fact that there cannot be feasible solutions better than the one given (the "negative" information). We model negative information by introducing the concept of "norphisms", as opposed to the positive information of morphisms. A "nategory" is a category that has "Nom"-sets in addition to hom-sets, and specifies the compatibility rules between norphisms and morphisms. With this setup we can choose to work in "coherent" "subnategories": subcategories that describe a potential instantiation of the world in which all morphisms and norphisms are compatible. We derive the composition rules for norphisms in a coherent subnategory; we show that norphisms do not compose by themselves, but rather they need to use morphisms as catalysts. We have two distinct rules of the type morphism $+$ norphism $\to$ norphism. We then show that those complex rules for norphism inference are actually as natural as the ones for morphisms, from the perspective of enriched category theory. Every small category is enriched over $\mathbf{P} = \langle \mathbf{Set}, \times, 1 \rangle$. We show that we can derive the machinery of norphisms by considering an enrichment over a certain monoidal category called **PN** (for "positive"/"negative"). In summary, we show that an alternative to considering negative information using logic on top of the categorical formalization is to "categorify" the negative information, obtaining negative arrows that live *at the same level* as the positive arrows, and suggest that the new inference rules are born of the same substance from the perspective of enriched category theory.

1 Introduction

1.1 Manipulation of negative information is important in applications of category theory

Our background is in robotics and systems theory. In our fields, we have found that category theory can describe well a lot of the structures in our problems, but something is missing: we often find ourselves in the position of reasoning and writing algorithms that manipulate "negative information", but we do not know what is an appropriate categorical concept for it. We give some examples.

Robot motion planning can be formalized as the problem of finding a trajectory through an environment, respecting some constraint (e.g., avoiding obstacles). One can think of the robot configuration manifold $\mathbb{M}$ as a category where the objects are elements of the tangent bundle and the morphisms are the feasible paths according to the problem constraints. The output of planning problems has an intuitive representation in category theory, if the problem is feasible. A *path* planning algorithm is given two objects and must compute a *morphism* as a solution. A *motion* planning algorithm would compute a trajectory, which could be seen as a *functor* F from the manifold $[0,T]$ to M with $F(0) = A$ and $F(T) = B$. However, if the problem is infeasible–if no morphisms between two points can be found—if the algorithm must present a *certificate of infeasibility*–what is the equivalent concept in category theory?

Submitted to:
ACT 2022

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In many cases, the problems are not binary (either a solution exists or not, either a proposition is true or not) but we care about the performance of solutions. For example, consider the case of the **weighted shortest path problem in dynamic programming**. The problem is to find a path through a graph that minimizes the sum of the weights of the edges on the path. In robotics, this can be used for planning problems, where the weights could represent the time, the distance, or the energy required by a robot to traverse an edge, and the nodes are either regions of space or, more generally, joint states of the world and environment. Proving that a path is optimal means producing the path *together with* a proof that there are no shorter paths. This is called a "certificate of optimality" and like certificates of infeasibility is negative information as it consists in negating the existence of a certain class of paths. Interestingly, one can see algorithms such as Dijkstra's algorithm as constructing both positive and negative information at the same time, such that when a path is finally found, we are sure that there are no shorter ones [2].

In some cases, the negative information is a first-class citizen which is critical to the efficiency. Algorithms such as A* require the definition of *heuristic* functions, which is negative information: they provide a *lower bound* on the cost of a path between two points. And better heuristics make the algorithm faster. Again, we ask, what could be the categorical counterpart of heuristics?

In **co-design** [3, 1], a morphism $\mathbf{F} \to \mathbf{R}$ describes what functionality can be achieved with which resources. They are characterized as boolean profunctors, that is, monotone functions $\mathbf{F}^{\mathrm{op}} \times \mathbf{R} \to \mathbf{Bool}$. The negative information would be a "nesign" problem that characterizes an impossibility. For example, if $\mathbf{F} = \mathbf{R} =$ Energy, we expect that in this universe we cannot find a realizable morphism d that satisfies $d(2J, 1J)$ (obtaining 2 Joules from 1 Joule). The dual information would be a function $\mathbf{F} \times \mathbf{R}^{\mathrm{op}} \to \mathbf{Bool}$. Is this a morphism? In which category does it live?

1.2 Our approach: "Categorification" of negative information

We briefly describe our thought process in finding a formalization for dealing with negative information.

One approach could have been to build structure on top of a category, at a higher level, using logic. We eschew this approach because of the belief that we should find a duality between positive and negative information that puts them "at the same level", but on the opposite sides of a mirror.

Our approach has been one in the spirit of "categorification": representing the negative information with a concrete structure for which to find axioms and inference rules.

An early influence in our thinking was the paper of Shulman about "proofs and refutations" [5]. What follows is a simplified explanation of one of the concepts of the paper. Consider a category where objects are propositions and morphisms $X \to Y$ are propositions $X \Rightarrow Y$ (with the particular case of $X \simeq \top \to X$). We can then consider the type $P(X \to Y)$ of *proofs* and the type $R(X \to Y)$ of *refutations*, which correspond to *positive* and *negative* information. According to intuitionist logic, $P(X \to Y) = (P(X) \to P(Y)) \times (R(Y) \to R(X))$: a proof of $X \Rightarrow Y$ is a way to convert a proof of X into a proof of Y together with a way to convert a refutation of Y into a refutation of X.

In that paper, proofs and refutations, positive and negative information, are treated *at the same level* but not symmetrically—proof and refutations have different semantics, and P and R map products and coproducts ($\vee$, $\wedge$) to different linear logic operators. This led to the idea that negative information should be at the same level of positive information: if positive information is represented by morphisms, then also the negative information should be described as "negative arrows" between objects, which we called *norphisms* (for negative morphisms).

We also realized that the positive/negative information duality we are looking for is richer than the structure of proofs/refutations in logic. In (classical/intuitionistic) logic, one expects the existence of either a proof of a proposition A, a refutation of A, or neither, but not both. Instead, in our formalization, norphisms are a more general notion, which can coexist with morphisms and give complementary information, as in the planning examples in the introduction.

An initial idea was to consider for each category a "twin" category, whose morphisms would be the norphisms we were looking for to represent the negative information; however, this idea failed. By the end of the paper, it will be clear that positive/negative information cannot be decoupled, because negative information cannot be composed independently. The norphisms *cannot be* morphisms in an auxiliary category associated to the original category because the inference rules are fundamentally different.
In the end, we will show that morphisms and norphisms are "twins" in the sense that they are both born of the same enrichment structure.

1.3 Plan of the paper

This paper follows an inductive exposition. We consider some categories and work out what is "negative information" in each case, and what are inference rules that we expect to hold. By the end of the paper, we show that all the particular notions can be subsumed into saying that the category is **PN**-enriched.

This paper is divided in two parts. In the **first part** we provide the **motivation and several examples of representing negative information with "norphism" structure**. In Section 2 we consider the case of a thin category. In this simple setting we can already show that norphisms compose differently from morphisms, and that we need two composition formulas for them. In Section 3 we define the concept of a "nategory". This is a category with additional structure: a set of norphisms and a compatibility relation between morphisms and norphisms. We define "coherent subnategories" as subcategories that "do not contain any contradiction" between morphisms and norphisms. We work out the generic formulas for obtaining the norphisms. In Section 4 and Section 5 we discuss the categories **Berg** and **DP**, which have non-trivial norphism structure, in which norphisms and morphisms are not exclusive, as in the case of a thin category.

In the **second part** our goal is to provide **an elegant way to think of norphisms and their composition by using enriched category theory**. By doing so, we show that the additional structure of norphisms and their composition rules which may appear "funky" is not an arbitrary structure, but rather it is as "natural" as the positive information of morphisms. In Section 6 we recall the notion of enrichment, and that "any small category" is "enriched" in $\mathbf{P} = \langle \mathbf{Set}, \times, 1 \rangle$. In Section 7 we define a category **PN**, and in Section 8 we show how the machinery related to the general case of norphisms can be derived by considering enrichment in **PN**.

2 Building intuition: the case of thin categories

To build an intuition about norphisms, we look at the case of "thin" categories, in which each hom-set contains at most one morphism. Thin categories are essentially pre-orders. To aid the interpretation, one can think of a pre-order as defining a reachability relation, in which a morphism $X \to Y$ represents "I can reach Y from X". Or, we can think of morphisms as (proof-irrelevant) implications: $X \to Y$ represents "I can prove Y from X". In a thin category, negative information is limited to indicate the refutation of positive information. Therefore, a norphism $n\colon X \dashrightarrow Y$ is equivalent to "There are no morphisms from X to Y". Particularly, this means "I cannot reach Y from X" or "I cannot prove Y from X".

We will later see that, in general, norphisms are not necessarily mutually exclusive with morphisms, and that the thin category case is a trivial case. Still, this example is sufficient to get us started to appreciate how morphisms and norphisms compose differently. The composition rule for morphisms reads:

$$\frac{f\colon X \to Y \quad g\colon Y \to Z}{(f \mathbin{⨾} g)\colon X \to Z}\,. \tag{1}$$

By mimicking what one does for categories, one could start with two norphisms $n\colon X \dashrightarrow Y$ and $m\colon Y \dashrightarrow Z$ and expect to be able to say something about a norphism $X \dashrightarrow Z$, with a composition rule of the form:

$$\frac{n\colon X \dashrightarrow Y \quad m\colon Y \dashrightarrow Z}{???\colon X \dashrightarrow Z}\,. \tag{2}$$

However, norphisms do not compose this way. In fact, one can derive the following rule:

$$\frac{o\colon X \dashrightarrow Z \qquad Y\colon \mathrm{Ob}_{\mathbf{C}}}{(n\colon X \dashrightarrow Y) \vee (m\colon Y \dashrightarrow Z)}\,. \tag{3}$$

This rule is "the dual" of Equation (1) in the same sense as these two axioms are dual:

$$\frac{\top}{X \to X}\,, \qquad \frac{X \dashrightarrow X}{\bot}\,, \tag{4}$$

that is, in the sense of switching orders and negating the propositions.

The expression in Equation (3) means that if there is no morphism $X \to Z$, it is because, for every possible intermediate Y, there cannot be a morphism $X \to Y$ or $Y \to Z$. Note that composition goes in the "opposite" direction meaning that from one norphism, we get some information about the existence of one or two in a pair. The composition is not constructive: from the "$\vee$", we do not know which side we can create. Indeed, this composition highlights the asymmetry between morphisms and norphisms: morphisms compose constructively by themselves (i.e., without taking into account norphisms); norphisms, instead, do not "compose", but rather "decompose" by themselves. To construct norphisms, we need to start from a norphism *and* a morphism that acts as a "catalyst".

When interpreting a thin category as a graph, if there is a norphism $n\colon X \dashrightarrow Y$, it means that for any Y, the path $X \to Y \to Z$ must be interrupted in either part. What we cannot have, is a contradiction. Indeed, if we know that morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ exist, then their composition $f \mathbin{⨾} g\colon X \to Z$ must exist, and therefore no norphism $n\colon X \dashrightarrow Z$ can exist. This observation can be turned around in a constructive way. Starting from a morphism $f\colon X \to Y$ and a norphism $n\colon X \dashrightarrow Z$ (i.e., morphisms and norphisms with the same source), we can infer a norphism $f \mathbin{\bullet\!\!-\!\!\bullet} n\colon Y \dashrightarrow Z$ (i.e., there cannot be a morphism $Y \to Z$):

$$\begin{tikzcd} Z & \\ & Y \\ X \arrow[uu, dashed, "n"] \arrow[ur, "f"'] & \end{tikzcd} \implies \begin{tikzcd} Z & \\ & Y \arrow[ul, dashed, "f \bullet\!-\!\bullet n"'] \\ X \arrow[uu, dashed, "n"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \frac{Y \xleftarrow{f} X \overset{n}{\dashrightarrow} Z}{Y \overset{f \bullet\!-\!\bullet n}{\dashrightarrow} Z}\,. \tag{5}$$

Symmetrically, starting from a morphism $g\colon Y \to Z$ and a norphism $n\colon X \dashrightarrow Z$ (i.e., morphisms and norphisms with the same target), we can infer a norphism $n \mathbin{-\!\!\bullet} f\colon X \dashrightarrow Y$:

$$\begin{tikzcd} Z & \\ & Y \arrow[ul, "g"'] \\ X \arrow[uu, dashed, "n"] & \end{tikzcd} \implies \begin{tikzcd} Z & \\ & Y \arrow[ul, "g"'] \\ X \arrow[uu, dashed, "n"] \arrow[ur, dashed, "n -\!\bullet g"'] & \end{tikzcd} \qquad \frac{X \overset{n}{\dashrightarrow} Z \xleftarrow{g} Y}{X \overset{n -\!\bullet g}{\dashrightarrow} Y}\,. \tag{6}$$

Note that the new norphism is pointing in the "same direction" as the starting one, meaning that either source or target are preserved.

3 Describing negative information: *nategories* and *coherence*

In this section we start making the notion of norphisms more precise, by concretely defining the additional structure which a category must have.

Definition 1 (Nategory)**.** A small *nategory* $\mathbf{C}$ is a small category with the following additional structure. For each pair of objects $X, Y \in \mathrm{Ob}_{\mathbf{C}}$, in addition to the set of morphisms $\mathrm{Hom}_{\mathbf{C}}(X;Y)$, we also specify:

- A set of norphisms $\mathrm{Nom}_{\mathbf{C}}(X;Y)$. We write $n : X \dashrightarrow Y$ to say that a norphism belongs to that set.
- A *compatibility relation* between the two sets:

$$R_{X,Y} : \mathrm{Hom}_{\mathbf{C}}(X;Y) \to_{\mathbf{Rel}} \mathrm{Nom}_{\mathbf{C}}(X;Y), \tag{7}$$

  where $(f R_{X,Y} n)$ means that $f : X \to Y$ is "compatible" with the norphism $n : X \dashrightarrow Y$.

Definition 2 (Subnategory)**.** A *subnategory* $\mathbf{D}$ of $\mathbf{C}$ is a nategory $\mathbf{D}$ that is a subcategory of $\mathbf{C}$ in the usual sense, and for which $\mathrm{Nom}_{\mathbf{D}}(X;Y) \subseteq \mathrm{Nom}_{\mathbf{C}}(X;Y)$.

Definition 3 (Coherent subnategory)**.** A subnategory $\mathbf{D}$ of $\mathbf{C}$ is *coherent* if all morphisms and norphisms are compatible:

$$\frac{f : \mathrm{Hom}_{\mathbf{D}}(X;Y) \quad n : \mathrm{Nom}_{\mathbf{D}}(X;Y)}{f(R_{X,Y})n}. \tag{8}$$

The interpretation is as follows. The ambient category $\mathbf{C}$ describes all morphisms and norphisms and their compatibility rules. A coherent subnategory is a particular instantiation of the world in which some things are possible, some impossible, and the consequences are coherent.

There is a canonical construction that turns a category in a nategory by making the choice that a norphism is a witness for the fact that the corresponding hom-set is empty.

Example 4. For any category $\mathbf{C}$, let $\mathrm{Nom}_{\mathbf{C}}(X;Y) = \{\bullet\}$ and $R_{X,Y} = \emptyset$. In this case, the element $\bullet$ is a witness for "$\mathrm{Hom}_{\mathbf{C}}(X;Y)$ is empty". In fact, if $\bullet \in \mathrm{Nom}_{\mathbf{C}}(X;Y)$, then because of $R_{X,Y}$ there cannot exist any morphism. Vice versa, if there is a morphism $f \in \mathrm{Hom}_{\mathbf{C}}(X;Y)$, then $\mathrm{Nom}_{\mathbf{C}}(X;Y)$ must be empty.

In the general case, we do not expect morphisms and norphisms to be exclusive. Indeed, they are both useful as characterizing different types of information.

3.1 Inference rules for norphisms in coherent subnategories

Given the structure of a nategory we can find inference rules for the norphisms of a coherent subnategory. The two types of compositions are obtained by the image/preimage of the compatibility relation by the pre- and post-composition action of the morphisms. The binary relation $R_{X,Y}$ induces the two maps

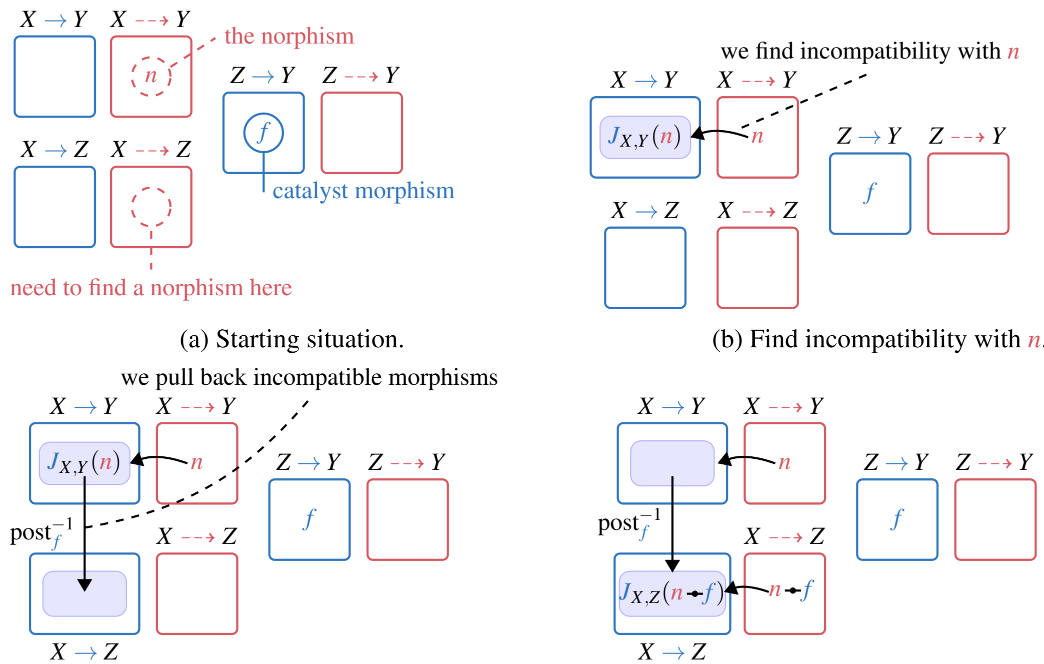
$$\begin{aligned} I_{X,Y} : \mathrm{Hom}_{\mathbf{C}}(X;Y) &\to \mathsf{Pow}(\mathrm{Nom}_{\mathbf{C}}(X;Y)), \\ f &\mapsto \{n \in \mathrm{Nom}_{\mathbf{C}}(X;Y) : \neg f R_{X,Y} n\}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} J_{X,Y} : \mathrm{Nom}_{\mathbf{C}}(X;Y) &\to \mathsf{Pow}(\mathrm{Hom}_{\mathbf{C}}(X;Y)), \\ n &\mapsto \{f \in \mathrm{Hom}_{\mathbf{C}}(X;Y) : \neg f R_{X,Y} n\}. \end{aligned} \tag{10}$$

Equation (9) answers the question "given a morphism, which norphisms are incompatible with it?". Equation (10) answers the question "given a norphism, which morphisms are incompatible with it?". Giving a norphism $n : X \dashrightarrow Y$ is equivalent to giving the value $J_{X,Y}(n)$.

These two maps can be used to construct new norphisms. Suppose we are in a coherent subnategory $\mathbf{D}$ of the ambient nategory $\mathbf{C}$. We define the two maps $I^{\mathbf{D}}_{X,Y}$ and $J^{\mathbf{D}}_{X,Y}$ as the restriction of $I_{X,Y}$ and $J_{X,Y}$ on the hom-set $\mathrm{Hom}_{\mathbf{D}}(X;Y) \subseteq \mathrm{Hom}_{\mathbf{C}}(X;Y)$ and $\mathrm{Nom}_{\mathbf{D}}(X;Y) \subseteq \mathrm{Nom}_{\mathbf{C}}(X;Y)$. For clarity, with abuse of



(a) Starting situation.

(b) Find incompatibility with n.

(c) Pulling back incompatible morphisms.

(d) We have now found $n \mathbin{-\!\bullet} f$.

Figure 1: Systematic composition of norphisms.

notation, we omit the indices and continue to use $I_{X,Y}$ and $J_{X,Y}$, being clear that we are working in a certain coherent subnategory $\mathbf{D}$.

We start from the case of Equation (6) and follow the steps illustrated in Fig. 1a. First, we can use $J_{X,Y}$ to find the morphisms which are incompatible with n, written as $J_{X,Y}(n)$ (Fig. 1b). Second, note that the morphism $f\colon Z \to Y$ induces the post-composition map

$$\begin{aligned} \mathrm{post}_f\colon \mathrm{Hom}_{\mathbf{C}}(X;Z) &\to \mathrm{Hom}_{\mathbf{C}}(X;Y), \\ g &\mapsto g \mathbin{⨾} f, \end{aligned} \tag{11}$$

We can now obtain the "bad" morphisms using the pre-image post_f^{-1} and obtain $\mathrm{post}_f^{-1}(J_{X,Y}(n))$ (Fig. 1c). We have now found $n \mathbin{-\!\bullet} f$, since we have found its incompatible elements (Fig. 1d):

$$J_{X,Z}(n \mathbin{-\!\bullet} f) = \mathrm{post}_f^{-1}(J_{X,Y}(n)). \tag{12}$$

We repeat the same procedure for the case of Equation (5), in which the morphism has type $f\colon X \to Z$. In that case we use the pre-composition map

$$\begin{aligned} \mathrm{pre}_f\colon \mathrm{Hom}_{\mathbf{C}}(X;Y) &\to \mathrm{Hom}_{\mathbf{C}}(X;Y), \\ h &\mapsto f \mathbin{⨾} h. \end{aligned} \tag{13}$$

We obtain this expression for $n \mathbin{\bullet\!\!-\!\!\bullet} f$:

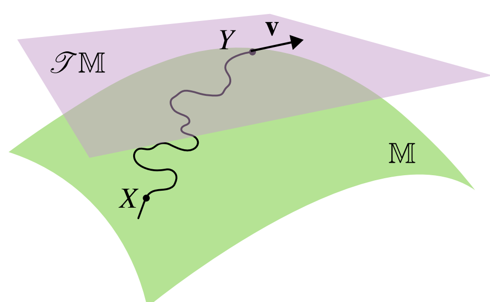
$$J_{X,Z}(n \mathbin{\bullet\!\!-\!\!\bullet} f) = \mathrm{pre}_f^{-1}(J_{X,Y}(n)). \tag{14}$$

4 Example: hiking on the Swiss mountains

In this section we present an example of planning, giving a more concrete description of the path planning problems mentioned in the introduction. We describe **Berg**, a category whose morphisms are hiking paths of various difficulty on a mountain. We then consider the problem of finding paths of minimum length.

Definition 5 (Berg). Let $h\colon \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a C^1 function, describing the elevation of a mountain. The set with elements $\langle a, b, h(a, b)\rangle$ is a manifold $\mathbb{M}$ that is embedded in $\mathbb{R}^3$. Let $\sigma = [\sigma_{\mathrm{L}}, \sigma_{\mathrm{U}}] \subset \mathbb{R}$ be a closed interval of real numbers. The category $\mathbf{Berg}_{h,\sigma}$ is specified as follows:

1. An object X is a pair $\langle \mathbf{p}, \mathbf{v}\rangle \in \mathscr{T}\mathbb{M}$, where $\mathbf{p} = \langle \mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\rangle$ is the position, $\mathbf{v}$ is the velocity, and $\mathscr{T}\mathbb{M}$ is the tangent bundle of the manifold.
2. Morphisms are C^1 paths on the manifold.At each point of a path we define the *steepness* as:
$$s(\langle \mathbf{p}, \mathbf{v}\rangle) := \mathbf{v}_z / \sqrt{\mathbf{v}_x^2 + \mathbf{v}_y^2}. \tag{15}$$
We choose as morphisms only the paths that have the steepness values contained in the interval σ:
$$\mathrm{Hom}_{\mathbf{Berg}_{h,\sigma}}(X;Y) = \{f \text{ is a } C^1 \text{ path from } X \text{ to } Y \text{ and } s(f) \subseteq \sigma\}, \tag{16}$$
3. Morphism composition is given by concatenation of paths.
4. Given any object, the identity morphism is the trivial self path with only one point.



For the complete proof that **Berg** is a category, we refer the reader to Lemma 13.

The steepness interval σ allows considering different categories on the same mountain, with possible hikes varying in difficulty, measured as minimum/maximum steepness. For example, a good hiker has $\sigma = [-0.57, 0.57]$ (positive/negative 30° slope). If $\sigma = [-0.57, 0]$, we are only allowed to climb down. If $\sigma = [0, 0]$, we can only walk along isoclines.

Interpretation of norphisms in Berg What should a norphism be in this case?

One possibility is to let a norphism $n\colon X \dashrightarrow Y$ mean "there exists no path from X to Y". This is a trivial choice that is similar to Example 4 and that makes morphisms and norphisms mutually exclusive.

We can obtain a more useful theory by letting norphisms carry more information that is *complementary* to morphisms by interpreting them as *lower bounds* on distances. Let the set of norphism be the nonnegative numbers plus infinity:
$$\mathrm{Nom}_{\mathbf{Berg}_{h,\sigma}}(X;Y) \subseteq \mathbb{R}_{\geq 0} \cup \{\infty\}. \tag{17}$$

Let $\mathrm{length}(f)$ be the length of the path (according to the manifold metric). A norphism $n\colon X \dashrightarrow Y$ is a witness of "for all paths $f\colon X \to Y$, we have $\mathrm{length}(f) \geq n$". This is negative, complementary information to morphisms, providing a lower bound on the length of the paths. The case in which $n = \infty$ means that there is no path from X to Y. The compatibility relation $R_{X,Y}$ can be written as follows:

$$\frac{f R_{X,Y} n}{\text{length}(f) \geq n}. \tag{18}$$

To say that a path f is optimal means saying that f is feasible *and* that length(f) is a norphism:

$$\frac{f\colon X \to Y \quad \text{length}(f)\colon X \dashrightarrow Y}{f \text{ is optimal}}. \tag{19}$$

Composition rules for norphisms Next, we derive the two composition rules that are the equivalent of Equation (5) and Equation (6). In this case, we obtain that $n \mathbin{-\!\!\bullet} f$ and $f \mathbin{\bullet\!\!-\!\!\bullet} n$ are equal:

$$n \mathbin{-\!\!\bullet} f \;=\; \max\{n - \text{length}(f), 0\} \;=\; f \mathbin{\bullet\!\!-\!\!\bullet} n. \tag{20}$$

The reasoning follows Fig. 2: if f is a path from Z to Y, and we know that going from X to Y takes at least n, then any path from X to Z must be at least $n - \text{length}(f)$ long. For the other direction: if there is a path f from X to Y and we know that going from X to Z takes at least n, then any path from Y to Z must be at least $n - \text{length}(f)$ long.

(a) (b)

Figure 2: Composition of morphisms and norphisms in the case of paths and lengths.

Norphisms axioms Finally, we need to specify the set of axioms for the norphisms. So far, we said that norphisms are nonnegative numbers plus infinity, but we did not say how exactly we associate a set Nom to each pair of objects. We obtain different subnategories by choosing more or less axioms.

1. **Trivial norphism**: since lengths cannot be negative, for all pair of objects we have the norphism $0\colon X \dashrightarrow X$. Having this as an axiom is not very useful, as the composition rules just generate other zeros as norphisms.
2. **Bound based on distance in $\mathbb{R}^3$**. Any path along the mountain cannot be shorter than the distance of a straight line ("as the crow flies"). Therefore, for two objects $\langle \mathbf{p}^1, \mathbf{v}^1 \rangle$, $\langle \mathbf{p}^2, \mathbf{v}^2 \rangle$, we have the distance in $\mathbb{R}^3$ $\|\mathbf{p}^1 - \mathbf{p}^2\|$ as a valid norphism:
$$\|\mathbf{p}^1 - \mathbf{p}^2\|\colon \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle. \tag{21}$$
3. **Bound based on geodesic distance**. A better bound is based on the geodesic distance. This is well defined because the points live on a smooth manifold:
$$d_{\mathbb{M}}(\mathbf{p}^1, \mathbf{p}^2)\colon \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle. \tag{22}$$
4. **Bound based on steepness interval**. Finally, we can use the bound on steepness interval. Given two objects $\langle \mathbf{p}^1, \mathbf{v}^1 \rangle$, $\langle \mathbf{p}^2, \mathbf{v}^2 \rangle$, we can use one of the following bounds
$$|\mathbf{p}^1_z - \mathbf{p}^2_z|/\sigma_{\text{U}}\colon \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle, \qquad |\mathbf{p}^1_z - \mathbf{p}^2_z|/\sigma_{\text{L}}\colon \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle, \tag{23}$$
depending on the case (if $\mathbf{p}^1_z - \mathbf{p}^2_z < 0$ we use the first, and if $\mathbf{p}^1_z - \mathbf{p}^2_z > 0$ the second).

5 Example: co-design

The next example revolves around the construction of norphisms for the category of design problems $\mathbf{DP}$ [1, 3]; this is called $\mathbf{Feas_{Bool}}$ in [3]. The objects of $\mathbf{DP}$ are posets. The morphisms are design problems (also referred to as feasibility relations or boolean profunctors). A *design problem* (DP) $d\colon \mathbf{P} \nrightarrow \mathbf{Q}$ is a monotone map of the form $d\colon \mathbf{P}^{\mathrm{op}} \times \mathbf{Q} \to_{\mathbf{Pos}} \mathbf{Bool}$, where $\mathbf{P}, \mathbf{Q}$ are arbitrary posets.

The semantics for a DP is that it describes a process which provides a certain functionality, by requiring certain resources. d is a monotone map, since lowering the requested functionalities will not require more resources, and increasing the available resources will not provide less functionalities.

Morphism composition is defined as follows. Given DPs $d\colon \mathbf{P} \nrightarrow \mathbf{Q}$ and $e\colon \mathbf{Q} \nrightarrow \mathbf{R}$, they compose into a DP $(d \mathbin{⨾} e)\colon \mathbf{P} \nrightarrow \mathbf{R}$ as:

$$\begin{aligned} (d \mathbin{⨾} e)\colon \mathbf{P}^{\mathrm{op}} \times \mathbf{R} &\to_{\mathbf{Pos}} \mathbf{Bool}, \\ \langle p, r\rangle &\mapsto \bigvee_{q \in \mathbf{Q}} d(p,q) \wedge e(q,r). \end{aligned} \tag{24}$$

For any poset $\mathbf{P}$, the identity DP $\mathrm{id}_{\mathbf{P}}\colon \mathbf{P} \nrightarrow \mathbf{P}$ is a monotone map

$$\begin{aligned} \mathrm{id}_{\mathbf{P}}\colon \mathbf{P}^{\mathrm{op}} \times \mathbf{P} &\to_{\mathbf{Pos}} \mathbf{Bool}, \\ \langle p_1, p_2\rangle &\mapsto p_1 \preceq_{\mathbf{P}} p_2. \end{aligned} \tag{25}$$

Interpretation of norphisms in DP Given that the morphisms of $\mathbf{DP}$ are feasibility relations, we expect that the norphisms of $\mathbf{DP}$ ("nesign problems"), should be *infeasibility* relations. A nesign problem (NP) $n\colon \mathbf{F} \nrightarrow \mathbf{R}$ should be a boolean map $n\colon \mathbf{F} \times \mathbf{R}^{\mathrm{op}} \to \mathbf{Bool}$, such that $n(f,r) = \top$ means that it is *not* possible to produce f from r. The semantics of an NP make it so this map should also be monotone:

$$n\colon \mathbf{F} \times \mathbf{R}^{\mathrm{op}} \to_{\mathbf{Pos}} \mathbf{Bool}. \tag{26}$$

In fact, if $\langle f_1, r_1\rangle$ is not feasible, and $f_2 \geq f_1$, this implies that $\langle f_2, r_1\rangle$ should not be feasible.

Note that the source poset of a nesign problem is the $^{\mathrm{op}}$ of the source poset for a design problem:

$$d\colon \mathbf{F}^{\mathrm{op}} \times \mathbf{R} \to_{\mathbf{Pos}} \mathbf{Bool}. \tag{27}$$

Compatibility of morphisms and norphisms Consider a DP $d\colon \mathbf{F} \nrightarrow \mathbf{R}$ and a NP $n\colon \mathbf{F} \nrightarrow \mathbf{R}$. The compatibility relation between DP and NP should ensure that there are no contradictions. We ask that, for any pair of functionality/resources $\langle f, r\rangle$, it cannot happen that they are declared feasible by the DP ($d(f,r)$) *and* declared infeasible by the NP ($n(f,r)$).

$$\frac{d R_{\mathbf{F},\mathbf{R}} n}{\forall f \in \mathbf{F}, r \in \mathbf{R}\colon \neg(d(f,r) \wedge n(f,r))}. \tag{28}$$

Composition rules for norphisms We can recover the composition rules presented in Equation (5) and Equation (6). Given a NP $n\colon \mathbf{P} \nrightarrow \mathbf{Q}$ and a DP $d\colon \mathbf{R} \nrightarrow \mathbf{Q}$, one can compose them to get a NP $n \multimapdotinv d\colon \mathbf{P} \nrightarrow \mathbf{R}$:

$$(n \multimapdotinv d)(p,r) = \bigvee_{q \in \mathbf{Q}} n(p,q) \wedge d(r,q). \tag{29}$$

Given a DP $d\colon \mathbf{Q} \nrightarrow \mathbf{P}$ and a NP $n\colon \mathbf{Q} \nrightarrow \mathbf{R}$, one can compose them to get a NP $d \multimapboth n\colon \mathbf{P} \nrightarrow \mathbf{R}$:

$$(d \multimapboth n)(p,r) = \bigvee_{q \in \mathbf{Q}} d(q,p) \wedge n(q,r). \tag{30}$$

Example 6. Consider the posets $\mathbf{P} = \langle \mathbb{N}_{[\text{kg pears}]}, \leq \rangle$, $\mathbf{Q} = \langle \mathbb{R}_{\geq 0,[\text{CHF}]}, \leq \rangle$, and $\mathbf{R} = \langle \mathbb{N}_{[\text{kg raisins}]}, \leq \rangle$. Consider the design problem $d\colon \mathbf{R} \nrightarrow \mathbf{Q}$ and the nesign problem $n\colon \mathbf{P} \nrightarrow \mathbf{Q}$. The (in)feasibility relations are given by:

$$\frac{d(r,q)}{r \cdot 10 \leq q}, \qquad \frac{n(p,q)}{p \cdot 5 > q}.$$

In other words, it is possible to buy raisins at 10 CHF/kg or more, and never possible to buy pears at less than 5 CHF/kg. We can evaluate the composition in a particular point to understand its meaning. First, the nesign problem $(n \multimap d)\colon \mathbf{P} \nrightarrow \mathbf{R}$ describes the possibility to obtain pears from raisins. For instance:

$$\begin{aligned}(n \multimap d)(10,4) &= \bigvee_{q \in \mathbf{Q}} n(10,q) \wedge d(5,q) \\ &= \bigvee_{q \in \mathbf{Q}} (40 \leq q < 50) = \top.\end{aligned}$$

The translation is as follows. Can I get 10 kg of pears from 4 kg of raisins? No. Why? If I could, I would need to buy the 4 kg of raisins using d, incurring at least in a cost of 40 CHF. In others words, I would pay 40 CHF for 10 kg of pears, which is impossible as per nesign problem n. ◁

For further explanations please refer to Remark 14.

Norphisms axioms Norphims axioms could follow some knowledge about particular designs we know are (in)feasible. Every engineering discipline has some fundamental limits in the performance of its designs that come from physics or information theory.

Interestingly, we can also formulate a very general axiom that is valid across all fields: in this universe, physically realizable designs can never produce strictly more resources that one starts with. This axiom can be encoded as a norphism. For each object $\mathbf{P}$, we postulate a NP $n_{\mathbf{P}}\colon \mathbf{P} \nrightarrow \mathbf{P}$ such that

$$n_{\mathbf{P}}(q,p) = p \prec_{\mathbf{P}} q, \tag{31}$$

where $p \prec_{\mathbf{P}} q = (p \preceq_{\mathbf{P}} q) \wedge (p \neq q)$. Interestingly, starting from a morphism $d\colon \mathbf{F} \nrightarrow \mathbf{R}$, one can directly obtain two NPs in $\mathbf{R} \nrightarrow \mathbf{F}$ that go in the opposite direction. These are

$$(n_{\mathbf{R}} \multimap d)(r,f) = \bigvee_{r' \in \mathbf{R}} n_{\mathbf{R}}(r,r') \wedge d(f,r'), \qquad (d \multimapdot n_{\mathbf{F}})(r,f) = \bigvee_{f' \in \mathbf{F}} d(f',r) \wedge n_{\mathbf{F}}(f',f),$$

which gives two impossibility results. The first states infeasibility because, while it is possible to get f from r' via d for a certain r', it is not possible to obtain r from r'. The second states infeasibility because, while it is possible to get f' from r via d for a certain f', it is not possible to obtain f' from f. Therefore, for this nategory, *every positive information induces negative information* in the other direction.

6 Enrichment

We recall a standard definition of enrichment [4].

Definition 7 (Enriched category)**.** Let $\langle \mathbf{V}, \otimes, \mathbf{1}, as, lu, ru\rangle$ be a monoidal category, where *as* is the associator, *lu* is the left unitor, and *ru* is the right unitor.
A $\mathbf{V}$-enriched category $\mathbf{E}$ is given by a tuple $\langle \mathrm{Ob}_{\mathbf{E}}, \alpha_{\mathbf{E}}, \beta_{\mathbf{E}}, \gamma_{\mathbf{E}}\rangle$, where

1. $\mathrm{Ob}_{\mathbf{E}}$ is a set of "objects".
2. $\alpha_{\mathbf{E}}$ is a function such that, for all pairs of objects $X,Y \in \mathrm{Ob}_{\mathbf{E}}$, the value $\alpha_{\mathbf{E}}(X,Y)$ is an object of $\mathbf{V}$.
3. $\beta_{\mathbf{E}}$ is a function such that, for all $X,Y,Z \in \mathrm{Ob}_{\mathbf{E}}$, there exists a morphism $\beta_{\mathbf{E}}(X,Y,Z)$ of $\mathbf{V}$, called *composition morphism*:
$$\beta_{\mathbf{E}}(X,Y,Z)\colon \alpha_{\mathbf{E}}(X,Y) \otimes \alpha_{\mathbf{E}}(Y,Z) \to_{\mathbf{V}} \alpha_{\mathbf{E}}(X,Z). \tag{32}$$
4. $\gamma_{\mathbf{E}}$ is a function such that, for each $X \in \mathrm{Ob}_{\mathbf{E}}$, there exists a morphism of $\mathbf{V}$:
$$\gamma_{\mathbf{E}}(X)\colon \mathbf{1} \to_{\mathbf{V}} \alpha_{\mathbf{E}}(X,X). \tag{33}$$

Moreover, for any $X,Y,Z,U \in \mathrm{Ob}_{\mathbf{E}}$, the diagrams reported in Appendix A must commute.

We recall a known construction that we generalize later. Consider the monoidal category $\mathbf{P} := \langle \mathbf{Set}, \times, 1\rangle$, where $\times$ is the Cartesian product and 1 is the one-element set $\{\bullet\}$.

Lemma 8. A category enriched in $\mathbf{P}$ gives the data necessary to define a small category, and vice versa.

Proof. We show one direction. Suppose that we are given a $\mathbf{P}$-enriched category as a tuple $\langle \mathrm{Ob}_{\mathbf{E}}, \alpha_{\mathbf{E}}, \beta_{\mathbf{E}}, \gamma_{\mathbf{E}}\rangle$. We can define a small category $\mathbf{C}$ as follows:

- Set $\mathrm{Ob}_{\mathbf{C}} := \mathrm{Ob}_{\mathbf{E}}$.
- For each $X,Y \in \mathrm{Ob}_{\mathbf{C}}$, let $\mathrm{Hom}_{\mathbf{C}}(X;Y) := \alpha_{\mathbf{E}}(X,Y)$.
- For each $X,Y,Z \in \mathrm{Ob}_{\mathbf{C}}$, we know a function
$$\beta_{\mathbf{E}}(X,Y,Z) : \mathrm{Hom}_{\mathbf{C}}(X;Y) \otimes \mathrm{Hom}_{\mathbf{C}}(Y;Z) \to_{\mathbf{Set}} \mathrm{Hom}_{\mathbf{C}}(X;Z). \tag{34}$$
The diagrams constraints imply that this function is associative.
Therefore, we use it to define morphism composition in $\mathbf{C}$, setting $\fatsemi_{X,Y,Z} := \beta_{\mathbf{E}}(X,Y,Z)$.
- For each $X \in \mathrm{Ob}_{\mathbf{C}}$ we know a function $\gamma_{\mathbf{E}}(X)\colon 1 \to_{\mathbf{Set}} \mathrm{Hom}_{\mathbf{C}}(X;X)$ that selects a morphism. The diagrams constraints imply that such morphism satisfies unitality with respect to $\fatsemi_{X,Y,Z}$. Therefore, we can use it to define the identity at each object:
$$\mathrm{id}_X := \gamma_{\mathbf{E}}(X)(\bullet). \tag{35}$$

□

7 The category PN

In this section, we will use dependent type notation. For instance, when we write $f\colon (a\colon A) \to (g(a) \to B)$, we mean that $a \in A$ is a particular element of a set, and $g(a)$ is a set which depends on a. So, given an a, $f(a)$ is a map from $g(a)$ to B. Another example of dependent-type notation would be writing:
$$\fatsemi\colon (X\colon \mathrm{Ob}_{\mathbf{C}}, Y\colon \mathrm{Ob}_{\mathbf{C}}, Z\colon \mathrm{Ob}_{\mathbf{C}}) \to (\mathrm{Hom}_{\mathbf{C}}(X;Y) \times \mathrm{Hom}_{\mathbf{C}}(Y;Z) \to \mathrm{Hom}_{\mathbf{C}}(X;Z)) \tag{36}$$

Definition 9 (Category **PN**)**.** The category **PN** is defined as follows.

1. The objects of **PN** are dependent pairs $\langle H, m\colon H \to \mathsf{Pow}(N)\rangle$, where H, N are sets, and m is a map that associates to an element of H a subset of N.

2. A morphism $f\colon \langle H_1, m_1\rangle \to \langle H_2, m_2\rangle$ is a pair of functions $\langle \varphi, \psi\rangle$ where

$$\begin{aligned} \varphi &\colon H_1 \to H_2, \\ \psi &\colon (h_1 : H_1) \to (m_2(\varphi(h_1)) \to m_1(h_1)). \end{aligned} \tag{37}$$

3. Given morphisms $f\colon \langle H_1, m_1\rangle \to \langle H_2, m_2\rangle$ and $g\colon \langle H_2, m_2\rangle \to \langle H_3, m_3\rangle$, their composition is a morphism $f \mathbin{⨟} g$, where

$$\begin{aligned} \varphi_{f⨟g} &= \varphi_f ⨟ \varphi_g, \\ \psi_{f⨟g}(h_1) &= \psi_g(\varphi_f(h_1)) ⨟ \psi_f(h_1). \end{aligned} \tag{38}$$

4. An identity for an object $\langle H, m\rangle$ is given by

$$\varphi = \mathrm{id}_H, \quad \psi(h) = \mathrm{id}_{m(h)}, \tag{39}$$

   where id_H is the identity function on the set H and $\mathrm{id}_{m(h)}$ is the identity function on the set $m(h)$.

Lemmas 15 to 17 check that this definition is well-posed.

7.1 Monoidal structure on PN

We define a monoidal structure on **PN** so that we can use it as a target of enrichment.
We start by defining a useful composition.

Definition 10 ("$\triangle$"). Given two maps $f\colon H_1 \to \mathsf{Pow}(N_1)$ and $g\colon H_2 \to \mathsf{Pow}(N_2)$, we define

$$\begin{aligned} (f \triangle g)\colon H_1 \times H_2 &\to \mathsf{Pow}(N_1 + N_2), \\ \langle h_1, h_2\rangle &\mapsto \mathsf{in}_1(f(h_1)) \cup \mathsf{in}_2(g(h_2)), \end{aligned} \tag{40}$$

where $\mathsf{in}_1, \mathsf{in}_2$ are the injections in the disjoint union lifted to sets.

The operation just defined has a neutral element (up to set isomorphism) given by the map

$$\begin{aligned} \mathrm{id}_\triangle : 1 &\to \mathsf{Pow}(\emptyset), \\ \bullet &\mapsto \emptyset. \end{aligned} \tag{41}$$

We can now proceed to define the monoidal structure on **PN**.

Lemma 11. $\langle \mathbf{PN}, \otimes_{\mathbf{PN}}, \langle 1, \mathrm{id}_\triangle\rangle\rangle$ is a monoidal category, defining the product of two objects as

$$\langle H_1, m_1\rangle \otimes_{\mathbf{PN}} \langle H_2, m_2\rangle := \langle H_1 \times H_2, m_1 \triangle m_2\rangle, \tag{42}$$

and the product of two morphisms $f\colon \langle H_1, m_1\rangle \to \langle K_1, l_1\rangle$, $g\colon \langle H_2, m_2\rangle \to \langle K_2, l_2\rangle$

$$f \otimes_{\mathbf{PN}} g\colon \langle H_1 \times H_2, m_1 \triangle m_2\rangle \to \langle K_1 \times K_2, l_1 \triangle l_2\rangle \tag{43}$$

as the morphism defined by the two functions $\varphi_{f\otimes_{\mathbf{PN}} g}$ and $\psi_{f\otimes_{\mathbf{PN}} g}$ defined as

$$\varphi_{f\otimes_{\mathbf{PN}} g} = \varphi_f \times \varphi_g, \tag{44}$$

$$\psi_{f\otimes_{\mathbf{PN}} g} : (\langle h_1, h_2\rangle : H_1 \times H_2) \to \psi_f(h_1) + \psi_g(h_2), \tag{45}$$

where $\times$ is the product of functions and $+$ is the direct sum of functions.

The proof of this lemma is in the appendix.

8 Describing coherent subnategories using enrichment in PN

We have seen in Lemma 8 that a **P**-enriched category provides the data necessary to define a small category. Here, we show the generalization to subnategories using **PN**-enrichment.

Proposition 12. A category enriched in **PN** provides the data necessary to specify a coherent subnategory.

Proof. Suppose somebody has provided us with a **PN**-enriched category **E** as a tuple $\langle \mathrm{Ob}_{\mathbf{E}}, \alpha_{\mathbf{E}}, \beta_{\mathbf{E}}, \gamma_{\mathbf{E}} \rangle$. Using this data we will describe a coherent subnategory **C**.

As for the objects of **C**, we set $\mathrm{Ob}_{\mathbf{C}} \coloneqq \mathrm{Ob}_{\mathbf{E}}$.

For every pair of objects $X, Y \in \mathrm{Ob}_{\mathbf{C}}$, we know an object of **PN** $\alpha_{\mathbf{E}}(X,Y)$. This is a dependent pair $\langle H, m \colon H \to \mathsf{Pow}(N) \rangle$. We set $\mathrm{Hom}_{\mathbf{C}}(X;Y) \coloneqq H$ and $\mathrm{Nom}_{\mathbf{C}}(X;Y) \coloneqq N$. We let the function $m \colon H \to \mathsf{Pow}(N)$ be the incompatibility function $I_{X,Y}$ defined in Equation (9). From that, we can define the compatibility relation $R_{X,Y}$.

Next, for each $X \in \mathrm{Ob}_{\mathbf{E}}$ we know the morphism $\gamma_{\mathbf{E}}(X) \colon \mathbf{1}_{\mathbf{PN}} \to_{\mathbf{PN}} \alpha_{\mathbf{E}}(X,X)$. We unroll the definition. Because $\mathbf{1}_{\mathbf{PN}} = \langle 1, \mathrm{id}_{\triangle} \rangle$, this is a morphism of **PN**

$$\gamma_{\mathbf{E}}(X) \colon \langle 1, \mathrm{id}_{\triangle} \rangle \to_{\mathbf{PN}} \langle \mathrm{Hom}_{\mathbf{C}}(X;X), I_{X,X} \rangle \tag{46}$$

which corresponds to two maps φ, ψ. The map φ provides the same information as in the construction of Lemma 8. As in the previous derivations, φ picks up a morphism that, given the diagrams conditions, is constrained to be the identity at X ($\varphi(\bullet) = \mathrm{id}_X$). The other map has type

$$\psi \colon (h \colon 1) \to (I_{X,X}(\varphi(h)) \to \mathrm{id}_{\triangle}(h)). \tag{47}$$

Simplifying, we get:

$$\psi(\bullet) \colon I_{X,X}(\mathrm{id}_X) \to \emptyset. \tag{48}$$

A map $I_{X,X}(\mathrm{id}_X) \to \emptyset$ exists (and is unique) only if $I_{X,X}(\mathrm{id}_X) = \emptyset$. That is, there cannot be norphisms incompatible with the identity.

For any three objects X, Y, Z, we know the morphism of type

$$\beta_{\mathbf{E}}(X,Y,Z) \colon \alpha_{\mathbf{E}}(X,Y) \otimes_{\mathbf{PN}} \alpha_{\mathbf{E}}(Y,Z) \to_{\mathbf{PN}} \alpha_{\mathbf{E}}(X,Z). \tag{49}$$

Substituting our choice of $\alpha_{\mathbf{E}}(-,-)$, the morphism has type

$$\langle \mathrm{Hom}_{\mathbf{C}}(X;Y), I_{X,Y} \rangle \otimes_{\mathbf{PN}} \langle \mathrm{Hom}_{\mathbf{C}}(Y;Z), I_{Y,Z} \rangle \to_{\mathbf{PN}} \langle \mathrm{Hom}_{\mathbf{C}}(X;Z), I_{X,Z} \rangle. \tag{50}$$

Expanding using the definition of $\otimes_{\mathbf{PN}}$ gives

$$\langle \mathrm{Hom}_{\mathbf{C}}(X;Y) \times \mathrm{Hom}_{\mathbf{C}}(Y;Z), I_{X,Y} \triangle I_{Y,Z} \rangle \to_{\mathbf{PN}} \langle \mathrm{Hom}_{\mathbf{C}}(X;Z), I_{X,Z} \rangle. \tag{51}$$

Such morphism is given by two maps φ, ψ. As in Lemma 8, we obtain that φ has type

$$\varphi \colon \mathrm{Hom}_{\mathbf{C}}(X;Y) \times \mathrm{Hom}_{\mathbf{C}}(Y;Z) \to \mathrm{Hom}_{\mathbf{C}}(X;Z) \tag{52}$$

and the diagrams imply that it is associative and unital. Thus, we use it to define morphism composition $\fatsemi \coloneqq \varphi$.

At this point, we have recover the structure of **C** as a category (hom-sets, identities, and composition) and we have defined the nom-sets $\mathrm{Nom}_{\mathbf{C}}(X;Y)$ and the compatibility relation $R_{X,Y}$. We now use the map ψ to derive the norphism composition operations.

The map ψ has the following dependent type:

$$\psi \colon (\langle f, g \rangle \colon \mathrm{Hom}_{\mathbf{C}}(X;Y) \times \mathrm{Hom}_{\mathbf{C}}(Y;Z)) \to (I_{X,Z}(\varphi(f,g)) \to (I_{X,Y} \triangle I_{Y,Z})(f,g)). \tag{53}$$

For a specific pair of compatible morphisms $\langle f, g\rangle$, we have:

$$\psi(f,g)\colon I_{X,Z}(f \fatsemi g) \to (\mathsf{in}_1(I_{X,Y}(f)) \cup \mathsf{in}_2(I_{Y,Z}(g))). \tag{54}$$

We consider two cases, according to whether $I_{X,Z}(f \fatsemi g)$ is empty or not:
1) If $I_{X,Z}(f \fatsemi g) = \emptyset$ it means that there is no norphism that forbids $f \fatsemi g$. The choice of $\psi(f,g)$ is unique as there is only one function out of $\emptyset$.
2) If $I_{X,Z}(f \fatsemi g)$ is not empty, it means that $f \fatsemi g$ is forbidden by one or more norphisms. For each norphism $n \in I_{X,Z}(f \fatsemi g)$ we can continue by evaluating the function to obtain

$$(\psi(f,g))(n) \in \mathsf{in}_1(I_{X,Y}(f)) \cup \mathsf{in}_2(I_{Y,Z}(g)). \tag{55}$$

This says that an explanation must be given. Indeed, $f \fatsemi g$ is forbidden because either f or g (or both) are forbidden. Now, suppose f is not forbidden, i.e.: $I_{X,Y}(f) = \emptyset$. This implies that we can pick out a norphism in $I_{Y,Z}(g)$. Call this norphism $f \multimap n$.

Alternatively, by supposing that g is not forbidden ($I_{Y,Z}(g) = \emptyset$), we are able to pick up a norphism in $I_{X,Y}(f)$; call it $n \multimapinv g$. We have recovered the norphism composition operations.

□

9 Conclusions

This work showed that we can encode negative information using "norphisms", negative arrows, as opposed to the positive arrows of morphisms.

In the case of thin categories, norphisms have the interpretation of being witnesses that there is no morphism between two objects. In more general categories, morphisms and norphisms between two objects can live along each other, and are not exclusive. Norphisms can give complementary information to morphisms. We have seen how, in the category **Berg**, norphisms can represent negative results, such as lower bounds on distances between two locations. A path planning algorithm must construct a morphism to give a path *and* a norphism to prove that the path is optimal. Furthermore, we have seen how, in the category **DP**, norphisms can represent design impossibility results.

We have described "nategories" as categories that have the norphism structure. For each pair of objects there is a set $\mathrm{Nom}_{\mathbf{C}}(X;Y)$, along with $\mathrm{Hom}_{\mathbf{C}}(X;Y)$, and a relation which describes the compatibility of morphisms and norphisms. One can then ask if a subnategory is *coherent*, meaning that there are no pairs of morphisms and norphisms that are incompatible. In a coherent subnategory one can derive rules to obtain new norphisms. Two norphisms cannot be "composed". Rather, there are rules allowing one to derive norphisms using morphisms as "catalysts", presented in Equation (5) and Equation (6).

Finally, we showed that this series of new definitions and baroque composition operators can be described using enriched category theory. We defined a monoidal category **PN** (which stands for "positive" and "negative") and we have shown that a **PN**-enriched category gives the data necessary to define a coherent subnategory. This generalizes the fact that a $\langle \mathbf{Set}, \times, 1\rangle$-enriched category provides the data for a small category.

Future work includes:

- Surveying known categories for natural norphism structures.
- As suggested by a reviewer, defining the relation between **PN** and **Poly**.
  We conjecture that **PN** is a submonoidal category of **Poly**.
- Understanding how the norphism structure generalizes to higher-level construction.
  For example, what is the natural definition of a "nunctor" (a "negative" functor)?

Acknowledgments

The authors would like to thank Dr. David Spivak for fruitful discussions. Gioele Zardini was supported by the Swiss National Science Foundation, under NCCR Automation, grant agreement 51NF40_180545.

References

[1] Andrea Censi, Jonathan Lorand & Gioele Zardini (2022): *Applied Compositional Thinking for Engineers*. Available at `https://bit.ly/3qQNrdR`. Work in progress book.

[2] Daniel Delling, Peter Sanders, Dominik Schultes & Dorothea Wagner (2009): *Engineering route planning algorithms*. In: Algorithmics of large and complex networks, Springer, pp. 117–139.

[3] Brendan Fong & David I Spivak (2018): *Seven sketches in compositionality: An invitation to applied category theory*. arXiv preprint arXiv:1803.05316.

[4] Gregory Maxwell Kelly, Max Kelly et al. (1982): *Basic concepts of enriched category theory*. 64, CUP Archive.

[5] Michael Shulman (2018): *Affine logic for constructive mathematics*. arXiv preprint arXiv:1805.07518.

A Definitions and diagrams

In Definition 7, for any $X,Y,Z,U \in \mathrm{Ob}_{\mathbf{E}}$, the following diagrams must commute.

$$\begin{tikzcd}
\alpha_{\mathbf{E}}(X,Y)\otimes(\alpha_{\mathbf{E}}(Y,Z)\otimes\alpha_{\mathbf{E}}(Z,U)) \arrow[rr, "as"] \arrow[d, "\mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)}\otimes\beta_{\mathbf{E}}(Y,Z,U)"'] & & (\alpha_{\mathbf{E}}(X,Y)\otimes\alpha_{\mathbf{E}}(Y,Z))\otimes\alpha_{\mathbf{E}}(Z,U) \arrow[d, "\beta_{\mathbf{E}}(X,Y,Z)\otimes\mathrm{id}_{\alpha_{\mathbf{E}}(Z,U)}"] \\
\alpha_{\mathbf{E}}(X,Y)\otimes\alpha_{\mathbf{E}}(Y,U) \arrow[r, "\beta_{\mathbf{E}}(X,Y,U)"'] & \alpha_{\mathbf{E}}(X,U) & \alpha_{\mathbf{E}}(X,Z)\otimes\alpha_{\mathbf{E}}(Z,U) \arrow[l, "\beta_{\mathbf{E}}(X,Z,U)"]
\end{tikzcd} \tag{56}$$

$$\begin{tikzcd}
\alpha_{\mathbf{E}}(X,Y)\otimes\alpha_{\mathbf{E}}(Y,Y) \arrow[r, "\beta_{\mathbf{E}}(X,Y,Y)"] & \alpha_{\mathbf{E}}(X,Y) & \alpha_{\mathbf{E}}(X,X)\otimes\alpha_{\mathbf{E}}(X,Y) \arrow[l, "\beta_{\mathbf{E}}(X,X,Y)"'] \\
\alpha_{\mathbf{E}}(X,Y)\otimes\mathbf{1} \arrow[u, "\mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)}\otimes\gamma_{\mathbf{E}}(Y)"] \arrow[ur, "ru"'] & & \mathbf{1}\otimes\alpha_{\mathbf{E}}(X,Y) \arrow[u, "\gamma_{\mathbf{E}}(X)\otimes\mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)}"'] \arrow[ul, "lu"]
\end{tikzcd} \tag{57}$$

B Proofs, examples, and explanations

Lemma 13. $\mathbf{Berg}_{h,\sigma}$ is indeed a category.

Proof. We can start clarifying what a morphism in this category is. A morphism $\langle \mathbf{p}_1, \mathbf{v}_1\rangle \to \langle \mathbf{p}_2, \mathbf{v}_2\rangle$ is a path on the manifold. One way to define a path on the manifold concretely is as a pair $\langle \gamma, T\rangle$, where

- $T \in \mathbb{R}_{\geq 0}$, which we think of the "time" taken to travel from $\mathbf{p}_1$ to $\mathbf{p}_2$.
- $\gamma\colon [0,T] \to \mathbb{M}$ is a C^1 function with $\gamma(0) = \mathbf{p}_1$ and $\gamma(T) = \mathbf{p}_2$, as well as $\dot{\gamma}(0) = \mathbf{v}_1$ and $\dot{\gamma}(T) = \mathbf{v}_2$ (we take one-sided derivatives at the boundaries).

Technically, composition of morphisms works as follows. Given morphisms

$$\langle \gamma_1, T_1 \rangle \colon \langle \mathbf{p}_1, \mathbf{v}_1 \rangle \to \langle \mathbf{p}_2, \mathbf{v}_2 \rangle \tag{58}$$

and

$$\langle \gamma_2, T_2 \rangle \colon \langle \mathbf{p}_2, \mathbf{v}_2 \rangle \to \langle \mathbf{p}_3, \mathbf{v}_3 \rangle, \tag{59}$$

their composition is $\langle \gamma, T \rangle$ with $T = T_1 + T_2$ and

$$\gamma(t) = \begin{cases} \gamma_1(t) & 0 \leq t \leq T_1 \\ \gamma_2(t - T_1) & T_1 \leq t \leq T_1 + T_2, \end{cases} \tag{60}$$

expressing concatenation of paths. Furthermore, we can express identity morphisms explicitly. For any object $\langle \mathbf{p}, \mathbf{v} \rangle$, we define the identity morphism

$$\mathrm{id}_{\langle \mathbf{p}, \mathbf{v} \rangle} = \langle \gamma, 0 \rangle \tag{61}$$

formally: its path γ is defined on the closed interval $[0, 0]$, (with $T = 0$ and $\gamma(0) = \mathbf{p}$). We declare this path to be C^1 by convention, and declare its derivative at 0 to be $\mathbf{v}$. Note that composition of intervals is associative, because $s(f \mathbin{⨾} g) = s(f) \cup s(g)$. From these constituents and the fact that the composition of steepness intervals is associative, it is clear that morphism composition is associative. Furthermore, the identity morphism satisfies unitality. □

Remark 14 (Explanation for composition of design and nesign problems)**.** Recall that given a NP $n \colon \mathbf{P} \nrightarrow \mathbf{Q}$ and a DP $d \colon \mathbf{R} \nrightarrow \mathbf{Q}$, one can compose them to get a NP $n \multimapinv d \colon \mathbf{P} \nrightarrow \mathbf{R}$:

$$(n \multimapinv d)(p, r) = \bigvee_{q \in \mathbf{Q}} n(p, q) \wedge d(r, q). \tag{62}$$

The derivation of the rule is as follows. Can I get p from r? No, if I cannot get p from q, for any q, and I can get r from q. Similarly, given a DP $d \colon \mathbf{Q} \nrightarrow \mathbf{P}$ and a NP $n \colon \mathbf{Q} \nrightarrow \mathbf{R}$, one can compose them to get a NP $d \multimapboth n \colon \mathbf{P} \nrightarrow \mathbf{R}$:

$$(d \multimapboth n)(p, r) = \bigvee_{q \in \mathbf{Q}} d(q, p) \wedge n(q, r). \tag{63}$$

Can I get p from r? No, if I cannot get q from r, for any q, and I can get q from p.

Lemma 15. The composition of morphisms in **PN** is well defined.

Proof. Consider morphisms f, g, h as in the definition. Clearly, $\varphi_h = \varphi_f \mathbin{⨾} \varphi_g$ is well defined. We expect ψ_h to be of type:

$$\psi_h \colon (h_1 \colon H_1) \to (m_3(\varphi_h(h_1)) \to m_1(h_1)). \tag{64}$$

Let's check this. We have:

$$\psi_f(h_1) \colon m_2(\varphi_f(h_1)) \to m_1(h_1) \tag{65}$$

Expanding, we get:

$$\psi_g(h_2) \colon m_3(\varphi_g(h_2)) \to m_2(h_2). \tag{66}$$

Let $h_2 = \varphi_f(h_1)$. Then, we have:

$$\psi_g(\varphi_f(h_1)) \colon m_3(\varphi_g(\varphi_f(h_1))) \to m_2(\varphi_f(h_1)), \tag{67}$$

which expanded becomes:

$$\psi_g(\varphi_f(h_1)) \colon m_3(\varphi_h(h_1)) \to m_2(\varphi_f(h_1)) \tag{68}$$

Now, from $\psi_h(h_1) = \psi_g(\varphi_f(h_1)) \mathbin{⨾} \psi_f(h_1)$, we see that the composition is well defined. □

Lemma 16. **PN** satisfies associativity.

Proof. Consider composable morphisms f, g, h. Clearly $(\varphi_f \mathbin{⨾} \varphi_g) \mathbin{⨾} \varphi_h = \varphi_f \mathbin{⨾} (\varphi_g \mathbin{⨾} \varphi_h)$. Furthermore, we have:

$$
\begin{aligned}
\psi_{(f ⨾ g) ⨾ h}(h_1) &= \psi_h(\varphi_{f ⨾ g}(h_1)) ⨾ \psi_{f ⨾ g}(h_1) \\
&= \psi_h((\varphi_f ⨾ \varphi_g)(h_1)) ⨾ \psi_g(\varphi_f(h_1)) ⨾ \psi_f(h_1),
\end{aligned}
\tag{69}
$$

and

$$
\begin{aligned}
\psi_{f ⨾ (g ⨾ h)}(h_1) &= \psi_{g ⨾ h}(\varphi_f(h_1)) ⨾ \psi_f(h_1) \\
&= \psi_h((\varphi_f ⨾ \varphi_g)(h_1)) ⨾ \psi_g(\varphi_f(h_1)) ⨾ \psi_f(h_1),
\end{aligned}
\tag{70}
$$

proving associativity. □

Lemma 17. **PN** satisfies unitality.

Proof. Consider a morphism f. Clearly $\varphi_{\mathrm{id} ⨾ f} = \varphi_{\mathrm{id}} ⨾ \varphi_f = f$ and $\varphi_{f ⨾ \mathrm{id}} = \varphi_f ⨾ \varphi_{\mathrm{id}} = f$. Similarly, we have

$$
\begin{aligned}
\psi_{\mathrm{id} ⨾ f}(h) &= \psi_f(\varphi_{\mathrm{id}}(h)) ⨾ \psi_{\mathrm{id}}(h) \\
&= \psi_f(h) ⨾ \mathrm{id}_{m(h)} \\
&= \psi_f(h),
\end{aligned}
\tag{71}
$$

and

$$
\begin{aligned}
\psi_{f ⨾ \mathrm{id}}(h) &= \psi_{\mathrm{id}}(\varphi_f(h)) ⨾ \psi_f(h) \\
&= \mathrm{id}_{m(h)} ⨾ \psi_f(h) \\
&= \psi_f(h),
\end{aligned}
\tag{72}
$$

proving unitality. □

Remark 18. Note that the map

$$
\begin{aligned}
\mathrm{id}_\triangle : 1 &\to \mathsf{Pow}(\emptyset), \\
\bullet &\mapsto \emptyset,
\end{aligned}
\tag{73}
$$

is neutral for $\triangle$, in the sense that, starting from $f \colon H_1 \to \mathsf{Pow}(N_1)$, we have:

$$
\begin{aligned}
(f \triangle \mathrm{id}_\triangle) \colon H_1 \times 1 &\to \mathsf{Pow}(N_1 + \emptyset) \\
\langle h_1, \bullet \rangle &\mapsto \mathsf{in}_1(f(h_1)) \cup \mathsf{in}_2(\emptyset)
\end{aligned}
\qquad
\begin{aligned}
(\mathrm{id}_\triangle \triangle f) \colon 1 \times H_1 &\to \mathsf{Pow}(\emptyset + N_1) \\
\langle \bullet, h_1 \rangle &\mapsto \mathsf{in}_1(\emptyset) \cup \mathsf{in}_2(f(h_1)),
\end{aligned}
$$

which obey the following commutative diagram:

$$
\begin{tikzcd}
H_1 \times 1 \arrow[r, "ru_{H_1}^{\langle \mathbf{Set}, \times, 1 \rangle}"] \arrow[d, "f \triangle \mathrm{id}_\triangle"'] & H_1 \arrow[d, "f"] & 1 \times H_1 \arrow[l, "lu_{H_1}^{\langle \mathbf{Set}, \times, 1 \rangle}"'] \arrow[d, "\mathrm{id}_\triangle \triangle f"] \\
\mathsf{Pow}(N_1 + \emptyset) \arrow[r, "\mathsf{Pow}(ru_{N_1}^{\langle \mathbf{Set}^{\mathrm{op}}, +, \emptyset \rangle})"'] & N_1 & \mathsf{Pow}(\emptyset + N_1) \arrow[l, "\mathsf{Pow}(lu_{N_1}^{\langle \mathbf{Set}^{\mathrm{op}}, +, \emptyset \rangle})"]
\end{tikzcd}
$$

where we leverage the unitors from the previously defined categories $\langle \mathbf{Set}, \times, 1 \rangle$ and $\langle \mathbf{Set}^{\mathrm{op}}, +, \emptyset \rangle$, and Pow represents the powerset functor. ◁

Proof of Lemma 11. Consider objects $X = \langle H_1, m_1 \rangle$, $Y = \langle H_2, m_2 \rangle$, and $Z = \langle H_3, m_3 \rangle$. The associator is given by the isomorphism:

$$
as_{X,Y,Z} \colon \langle (H_1 \times H_2) \times H_3, (m_1 \triangle m_2) \triangle m_3 \rangle \to \langle H_1 \times (H_2 \times H_3), m_1 \triangle (m_2 \triangle m_3) \rangle,
\tag{74}
$$

which is specified by two maps

$$
\begin{aligned}
as^{\varphi}_{X,Y,Z} \colon (H_1 \times H_2) \times H_3 &\to H_1 \times (H_2 \times H_3) \\
\langle \langle h_1, h_2 \rangle, h_3 \rangle &\mapsto \langle h_1, \langle h_2, h_3 \rangle \rangle,
\end{aligned}
\tag{75}
$$

which is a natural isomorphism (as seen in $\langle \mathbf{Set}, \times, 1\rangle$), and

$$as^{\psi}_{X,Y,Z} \colon (\langle\langle h_1, h_2\rangle, h_3\rangle \colon (H_1 \times H_2) \times H_3) \to ((m_1 \triangle (m_2 \triangle m_3))(\langle h_1, \langle h_2, h_3\rangle\rangle) \to ((m_1 \triangle m_2) \triangle m_3)(\langle\langle h_1, h_2\rangle, h_3\rangle)). \tag{76}$$

We can specify the latter map, by fixing:

$$as^{\psi}_{X,Y,Z}(\langle\langle h_1, h_2\rangle, h_3\rangle) = as^{\langle \mathbf{Set}^{\mathrm{op}}, +, \emptyset\rangle}_{m_1(h_1), m_2(h_2), m_3(h_3)} \tag{77}$$

The left unitor is given by the morphism:

$$lu_{\langle H, m\rangle} \colon \langle 1 \times H, \mathrm{id}_{\triangle} \triangle m\rangle \to \langle H, m\rangle, \tag{78}$$

which is given by maps

$$\begin{aligned} lu^{\varphi}_{\langle H, m\rangle} \colon 1 \times H &\to H \\ \langle \bullet, h\rangle &\mapsto h, \end{aligned} \tag{79}$$

which is a natural isomorphism (as seen in $\langle \mathbf{Set}, \times, 1\rangle$), and

$$lu^{\psi}_{\langle H, m\rangle} \colon (\langle \bullet, h\rangle \colon 1 \times H) \to (m(h) \to (\mathrm{id}_{\triangle} \triangle m)(\langle \bullet, h\rangle)). \tag{80}$$

We can specify the latter map, by fixing:

$$lu^{\psi}_{X}(h) = lu^{\langle \mathbf{Set}^{\mathrm{op}}, +, \emptyset\rangle}_{m(h)} \tag{81}$$

The right unitor is given by the morphism:

$$ru_{\langle H, m\rangle} \colon \langle H \times 1, m \triangle \mathrm{id}_{\triangle}\rangle \to \langle H, m\rangle, \tag{82}$$

which is given by maps

$$\begin{aligned} ru^{\varphi}_{\langle H, m\rangle} \colon H \times 1 &\to H \\ \langle h, \bullet\rangle &\mapsto h, \end{aligned} \tag{83}$$

which is a natural isomorphism (as seen in $\langle \mathbf{Set}, \times, 1\rangle$), and

$$ru^{\psi}_{\langle H, m\rangle} \colon (\langle h, \bullet\rangle \colon H \times 1) \to (m(h) \to (m \triangle \mathrm{id}_{\triangle})(\langle h, \bullet\rangle)). \tag{84}$$

We can specify the latter map, by fixing:

$$ru^{\psi}_{X}(h) = ru^{\langle \mathbf{Set}^{\mathrm{op}}, +, \emptyset\rangle}_{m(h)}. \tag{85}$$

We have already proved that the "φ" part of the morphisms satifies the triangle and pentagon rules. With the choices we made for the "ψ" part of the morphisms, we know that they satisfy the triangle and pentagon rules for every evaluation. □