Categorification of Negative Information using Enrichment

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In many applications of category theory it is useful to reason about "negative information". For example, in planning problems, providing an optimal solution is the same as giving a feasible solution (the "positive" information) together with a proof of the fact that there cannot be feasible solutions better than the one given (the "negative" information). We model negative information by introducing the concept of "norphisms", as opposed to the positive information of morphisms. A "nategory" is a category that has "Nom"-sets in addition to hom-sets, and specifies the compatibility rules between norphisms and morphisms. We derive the composition rules for norphisms; we show that norphisms do not compose by themselves, but rather they need to use morphisms as catalysts. We have two distinct rules of the type morphism + norphism → norphism. We then show that those complex rules for norphism inference are actually as natural as the ones for morphisms, from the perspective of enriched category theory. Every small category is enriched over $\mathbf{P} = \langle \mathbf{Set}, \times, 1\rangle$. We show that we can derive the machinery of norphisms by considering an enrichment over a certain monoidal category called $\mathbf{PN}$ (for "positive"/"negative"). In summary, we show that an alternative to considering negative information using logic on top of the categorical formalization is to "categorify" the negative information, obtaining negative arrows that live *at the same level* as the positive arrows, and suggest that the new inference rules are born of the same substance from the perspective of enriched category theory.

1 Introduction

1.1 Manipulation of negative information is important in applications of category theory

Our background is in robotics and systems theory. In our fields, we have found that category theory can describe well a lot of the structures in our problems, but something is missing: we often find ourselves in the position of reasoning and writing algorithms that manipulate "negative information", but we do not know what is an appropriate categorical concept for it. We give some examples.

Robot motion planning can be formalized as the problem of finding a trajectory through an environment, respecting some constraint (e.g., avoiding obstacles). One can think of the robot configuration manifold $\mathbb{M}$ as a category where the objects are elements of the tangent bundle and the morphisms are the feasible paths according to the problem constraints. The output of planning problems has an intuitive representation in category theory, if the problem is feasible. A *path* planning algorithm is given two objects and must compute a *morphism* as a solution. A *motion* planning algorithm would compute a trajectory, which could be seen as a *functor* F from the manifold $[0, T]$ to M with $F(0) = A$ and $F(T) = B$. However, if the problem is infeasible–if no morphisms between two

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points can be found—if the algorithm must present a *certificate of infeasibility*–what is the equivalent concept in category theory?

In many cases, the problems are not binary (either a solution exists or not, either a proposition is true or not) but we care about the performance of solutions. For example, consider the case of the **weighted shortest path problem in dynamic programming**. The problem is to find a path through a graph that minimizes the sum of the weights of the edges on the path. In robotics, this can be used for planning problems, where the weights could represent the time, the distance, or the energy required by a robot to traverse an edge, and the nodes are either regions of space or, more generally, joint states of the world and environment. Proving that a path is optimal means producing the path *together with* a proof that there are no shorter paths. This is called a "certificate of optimality" and like certificates of infeasibility is negative information as it consists in negating the existence of a certain class of paths. Interestingly, one can see algorithms such as Dijkstra's algorithm as constructing both positive and negative information at the same time, such that when a path is finally found, we are sure that there are no shorter ones [4].

In some cases, the negative information is a first-class citizen which is critical to the efficiency. Algorithms such as A* require the definition of *heuristic* functions, which is negative information: they provide a *lower bound* on the cost of a path between two points. And better heuristics make the algorithm faster. Again, we ask, what could be the categorical counterpart of heuristics?

In **co-design** [5, 2], a morphism $\mathbf{F} \to \mathbf{R}$ describes what functionality can be achieved with which resources. They are characterized as boolean profunctors, that is, monotone functions $\mathbf{F}^{\mathrm{op}} \times \mathbf{R} \to \mathbf{Bool}$. The negative information would be a "nesign" problem that characterizes an impossibility. For example, if $\mathbf{F} = \mathbf{R} = \text{Energy}$, we expect that in this universe we cannot find a realizable morphism d that satisfies $d(2J, 1J)$ (obtaining 2 Joules from 1 Joule). The dual information would be a function $\mathbf{F} \times \mathbf{R}^{\mathrm{op}} \to \mathbf{Bool}$. Is this a morphism? In which category does it live?

1.2 Our approach: "Categorification" of negative information

We briefly describe our thought process in finding a formalization for dealing with negative information.

One approach could have been to build structure on top of a category, at a higher level, using logic. We eschew this approach because of the belief that we should find a duality between positive and negative information that puts them "at the same level", but on the opposite sides of a mirror.

Our approach has been one in the spirit of "categorification": representing the negative information with a concrete structure for which to find axioms and inference rules.

An early influence in our thinking was the paper of Shulman about "proofs and refutations" [9]. What follows is a simplified explanation of one of the concepts of the paper. Consider a category where objects are propositions and morphisms $X \to Y$ are propositions $X \Rightarrow Y$ (with the particular case of $X \simeq \top \to X$). We can then consider the type $P(X \to Y)$ of *proofs* and the type $R(X \to Y)$ of *refutations*, which correspond to *positive* and *negative* information. According to intuitionist logic, $P(X \to Y) = (P(X) \to P(Y)) \times (R(Y) \to R(X))$: a proof of $X \Rightarrow Y$ is a way to convert a proof of X into a proof of Y together with a way to convert a refutation of Y into a refutation of X.

In that paper, proofs and refutations, positive and negative information, are treated *at the same level* but not symmetrically—proof and refutations have different semantics, and P and R map products and coproducts $(\vee, \wedge)$ to different linear logic operators. This led to the idea that negative information should be at the same level of positive information: if positive information is represented by morphisms, then also the negative information should be described as “negative arrows” between objects, which we called *norphisms* (for negative morphisms).

We also realized that the positive/negative information duality we are looking for is richer than the structure of proofs/refutations in logic. In (classical/intuitionistic) logic, one expects the existence of either a proof of a proposition A, a refutation of A, or neither, but not both. Instead, in our formalization, norphisms are a more general notion, which can coexist with morphisms and give complementary information, as in the planning examples in the introduction.

An initial idea was to consider for each category a “twin” category, whose morphisms would be the norphisms we were looking for to represent the negative information; however, this idea failed. By the end of the paper, it will be clear that positive/negative information cannot be decoupled, because negative information cannot be composed independently. The norphisms *cannot be* morphisms in an auxiliary category associated to the original category because the inference rules are fundamentally different.
In the end, we will show that morphisms and norphisms are “twins” in the sense that they are both born of the same enrichment structure.

1.3 Plan of the paper

This paper follows an inductive exposition. We consider some categories and work out what is “negative information” in each case, and what are inference rules that we expect to hold. By the end of the paper, we show that all the particular notions can be subsumed into saying that the category is **PN**-enriched.

This paper is divided in two parts. In the **first part** we provide the **motivation and several examples of representing negative information with “norphism” structure**. In Section 2 we consider the case of a thin category. In this simple setting we can already show that norphisms compose differently from morphisms, and that we need two composition formulas for them. In Section 3 we define the concept of a “nategory”. This is a category with additional structure: a set of norphisms and a compatibility relation between morphisms and norphisms. In Section 5 and Section 6 we discuss the categories **Berg** and **DP**, which have non-trivial norphism structure, in which norphisms and morphisms are not exclusive, as in the case of a thin category.

In the **second part** our goal is to provide **an elegant way to think of norphisms and their composition by using enriched category theory**. By doing so, we show that the additional structure of norphisms and their composition rules which may appear “funky” is not an arbitrary structure, but rather it is as “natural” as the positive information of morphisms. In Section 7 we recall the notion of enrichment, and that “any small category” is “enriched” in $\mathbf{P} = \langle \mathbf{Set}, \times, 1 \rangle$. In **??** we define a category **PN**, and in Section 9 we show how the machinery related to the general case of norphisms can be derived by considering enrichment in **PN**.

2 Building intuition: the case of thin categories

To build an intuition about norphisms, we look at the case of “thin” categories, in which each homset contains at most one morphism. Thin categories are essentially pre-orders. To aid the interpretation, one can think of a pre-order as defining a reachability relation, in which a morphism $X \to Y$ represents “I can reach Y from X”. Or, we can think of morphisms as (proof-irrelevant) implications: $X \to Y$ represents “I can prove Y from X”. In a thin category, negative information is limited to indicate the refutation of positive information. Therefore, a norphism $n : X \dashrightarrow Y$ is equivalent to "There are no morphisms from X to Y". Particularly, this means "I cannot reach Y from X" or "I cannot prove Y from X".

We will later see that, in general, norphisms are not necessarily mutually exclusive with morphisms, and that the thin category case is a trivial case. Still, this example is sufficient to get us started to appreciate how morphisms and norphisms compose differently. The composition rule for morphisms reads:

$$\frac{f : X \to Y \quad g : Y \to Z}{(f \mathbin{⨟} g) : X \to Z}. \tag{1}$$

By mimicking what one does for categories, one could start with two norphisms $n : X \dashrightarrow Y$ and $m : Y \dashrightarrow Z$ and expect to be able to say something about a norphism $X \dashrightarrow Z$, with a composition rule of the form:

$$\frac{n : X \dashrightarrow Y \quad m : Y \dashrightarrow Z}{??? : X \dashrightarrow Z}. \tag{2}$$

However, norphisms do not compose this way. In fact, one can derive the following rule:

$$\frac{o : X \dashrightarrow Z \qquad Y : \mathrm{Ob}_{\mathbf{C}}}{(n : X \dashrightarrow Y) \vee (m : Y \dashrightarrow Z)}. \tag{3}$$

This rule is "the dual" of Equation (1) in the same sense as these two axioms are dual:

$$\frac{\top}{X \to X}, \qquad \frac{X \dashrightarrow X}{\bot}, \tag{4}$$

that is, in the sense of switching orders and negating the propositions.

The expression in Equation (3) means that if there is no morphism $X \to Z$, it is because, for every possible intermediate Y, there cannot be a morphism $X \to Y$ or $Y \to Z$. Note that composition goes in the "opposite" direction meaning that from one norphism, we get some information about the existence of one or two in a pair. The composition is not constructive: from the "$\vee$", we do not know which side we can create. Indeed, this composition highlights the asymmetry between morphisms and norphisms: morphisms compose constructively by themselves (i.e., without taking into account norphisms); norphisms, instead, do not "compose", but rather "decompose" by themselves. To construct norphisms, we need to start from a norphism *and* a morphism that acts as a "catalyst".

When interpreting a thin category as a graph, if there is a norphism $n : X \dashrightarrow Y$, it means that for any Y, the path $X \to Y \to Z$ must be interrupted in either part. What we cannot have, is a contradiction. Indeed, if we know that morphisms $f : X \to Y$ and $g : Y \to Z$ exist, then their composition $f \mathbin{⨟} g : X \to Z$ must exist, and therefore no norphism $n : X \dashrightarrow Z$ can exist. This observation can be turned around in a constructive way. Starting from a morphism $f : X \to Y$ and a norphism $n : X \dashrightarrow Z$ (i.e., morphisms and norphisms with the same source), we can infer a norphism $f \bullet\!\!-\!\!\bullet n : Y \dashrightarrow Z$ (i.e., there cannot be a morphism $Y \to Z$):

$$\begin{tikzcd} Z & \\ & Y \\ X \arrow[uu, dashed, "n"] \arrow[ur, "f"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} Z & \\ & Y \arrow[ul, dashed, "f \bullet\!\!-\!\!\bullet n"'] \\ X \arrow[uu, dashed, "n"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \frac{Y \xleftarrow{f} X \overset{n}{\dashrightarrow} Z}{Y \overset{f \bullet\!\!-\!\!\bullet n}{\dashrightarrow} Z}. \tag{5}$$

Symmetrically, starting from a morphism $g : Y \to Z$ and a norphism $n : X \dashrightarrow Z$ (i.e., morphisms and norphisms with the same target), we can infer a norphism $n \mathbin{\bullet\!\!-} f : X \dashrightarrow Y$:

$$
\begin{tikzcd}
Z & \\
& Y \arrow[ul, "g"'] \\
X \arrow[uu, dashed, "n"] &
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
Z & \\
& Y \arrow[ul, "g"'] \\
X \arrow[uu, dashed, "n"] \arrow[ur, dashed, "n \mathbin{\bullet\!\!-} g"'] &
\end{tikzcd}
\qquad
\frac{X \overset{n}{\dashrightarrow} Z \overset{g}{\leftarrow} Y}{X \overset{n \mathbin{\bullet\!\!-} g}{\dashrightarrow} Y}\,. \tag{6}
$$

Note that the new norphism is pointing in the “same direction” as the starting one, meaning that either source or target are preserved.

3 Describing negative information: *nategories*

In this section we start making the notion of norphisms more precise, by concretely defining the additional structure which a category must have.

Definition 1 (Nategory)**.** A locally small *nategory* $\mathbf{C}$ is a locally small category with the following additional structure. For each pair of objects $X, Y \in \mathrm{Ob}_{\mathbf{C}}$, in addition to the set of morphisms $\mathrm{Hom}_{\mathbf{C}}(X;Y)$, we also specify:

- A set of norphisms $\mathrm{Nom}_{\mathbf{C}}(X;Y)$.
- An *incompatibility relation*, which we write as a binary function

$$i_{XY} : \mathrm{Nom}_{\mathbf{C}}(X;Y) \times \mathrm{Hom}_{\mathbf{C}}(X;Y) \to \mathbf{Bool}. \tag{7}$$

For all triples X, Y, Z, in addition to the morphism composition function

$${}_{9}^{\circ}{}_{XYZ} : \mathrm{Hom}_{\mathbf{C}}(X;Y) \times \mathrm{Hom}_{\mathbf{C}}(Y;Z) \to \mathrm{Hom}_{\mathbf{C}}(X;Z), \tag{8}$$

we require the existence of two norphism composition functions

$$\begin{aligned} \bullet\!\!\multimap_{XYZ} &: \mathrm{Hom}_{\mathbf{C}}(X;Y) \times \mathrm{Nom}_{\mathbf{C}}(X;Z) \to \mathrm{Nom}_{\mathbf{C}}(Y;Z), \\ \multimap\!\!\bullet_{XYZ} &: \mathrm{Nom}_{\mathbf{C}}(X;Z) \times \mathrm{Hom}_{\mathbf{C}}(Y;Z) \to \mathrm{Nom}_{\mathbf{C}}(X;Y). \end{aligned} \tag{9}$$

and we ask that they satisfy two "equivariance" conditions:

$$i_{YZ}(f \bullet\!\!\multimap n, g) \Rightarrow i_{XZ}(n, f \,{}_{9}^{\circ}\, g), \tag{equiv-1}$$

$$i_{XY}(n \multimap\!\!\bullet g, f) \Rightarrow i_{XZ}(n, f \,{}_{9}^{\circ}\, g). \tag{equiv-2}$$

We write $n : X \dashrightarrow Y$ to say that $n \in \mathrm{Nom}_{\mathbf{C}}(X;Y)$.

Definition 2 (Exact nategory)**.** If the two conditions Eqs. (equiv-1) and (equiv-2) are satisfied with "$\Leftrightarrow$" instead of just "$\Rightarrow$", we say that the nategory is *exact*.

We give the intuition behind the two conditions Eqs. (equiv-1) and (equiv-2). Intuitively, we want to make sure that the norphism composition operations are not looking at structure other than the categorical structure. For example Equation (equiv-1) says that the norphism $f \bullet\!\!\multimap n$ can exclude the morphism g only if $f \,{}_{9}^{\circ}\, g$ is excluded by n. No additional information can be used to exclude other morphisms.

We can draw some figures (Fig. 1) to develop further intuition. We define the function J_{XY} which maps a norphism to the set of incompatible morphisms:

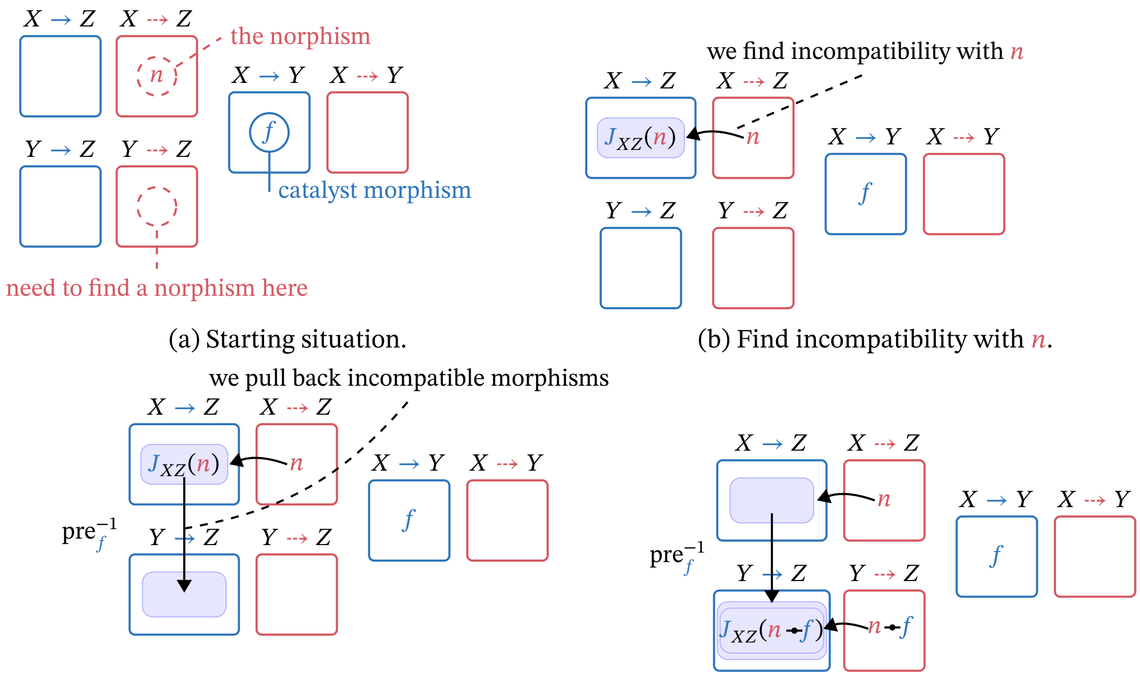
$$\begin{aligned} J_{XY} : \mathrm{Nom}_{\mathbf{C}}(X;Y) &\to \mathsf{Pow}(\mathrm{Hom}_{\mathbf{C}}(X;Y)), \\ n &\mapsto \{f \in \mathrm{Hom}_{\mathbf{C}}(X;Y) : i_{XY}(n, f)\}. \end{aligned} \tag{10}$$

We start in Fig. 1a with a norphism $n : X \dashrightarrow Z$ and a morphism $f : X \to Y$. In Fig. 1b we apply J_{XZ} to find the set of incompatible morphisms $J_{XZ}(n)$. In Fig. 1c we apply the precomposition map

$$\begin{aligned} \mathrm{pre}_f : \mathrm{Hom}_{\mathbf{C}}(Y;Z) &\to \mathrm{Hom}_{\mathbf{C}}(X;Z), \\ g &\mapsto f \,{}_{9}^{\circ}\, g, \end{aligned} \tag{11}$$

to obtain the set of morphisms

$$\mathrm{pre}_f^{-1}(J_{XZ}(n)). \tag{12}$$



(a) Starting situation. (b) Find incompatibility with n.

(c) Pulling back incompatible morphisms. (d) We have now found $n \multimapdotboth f$.

Figure 1: Systematic composition of norphisms.

These are to be prohibited because when pre-composed with f they give a morphism that is forbidden by n. Now, in principle, it could be that our norphism inference is so powerful that $f \multimapdotboth n$ manages to exclude all of these:

$$J_{YZ}(f \multimapdotboth n) = \mathrm{pre}_f^{-1}(J_{XZ}(n)). \tag{13}$$

In general, we are happy with the composition operation if it excludes part of those (but not more):

$$J_{YZ}(f \multimapdotboth n) \subseteq \mathrm{pre}_f^{-1}(J_{XZ}(n)). \tag{14}$$

We can see that this condition is equivalent to Equation (equiv-1). It is an inclusion of two sets $S_1 \subseteq S_2$. Call the generic element of the two sets g. Then inclusion means that the indicator function of the first implies the indicator function of the second: $S_1(g) \Rightarrow S_2(g)$. The indicator function of the first set is:

$$S_1(g) = g \in J_{YZ}(f \multimapdotboth n) = i_{YZ}(f \multimapdotboth n, g). \tag{15}$$

For the second set it is:

$$S_2(g) = g \in \mathrm{pre}_f^{-1}(J_{XZ}(n)) = i_{XZ}(n, f \mathbin{;} g). \tag{16}$$

4 Canonical nategory constructions

Here are three canonical constructions that allow to get a nategory out of a category in a more or less trivial way:

1. Setting the norphisms to be the empty set (Example 3);
2. Setting the norphisms to be a singleton that negates any morphism (Example 4);
3. Setting the norphisms to be the powerset of the set of morphisms (Example 5).

Example 3 (A nategory with no norphisms)**.** For any category $\mathbf{C}$, let

$$\mathrm{Nom}_{\mathbf{C}}(X;Y) := \emptyset. \tag{17}$$

For all pairs X, Y the function i_{XY} is uniquely defined as it has an empty domain. The functions $\multimapdot$, $\multimapdotboth$ also have empty domains. The conditions Eqs. (equiv-1) and (equiv-2) are trivially verified. A nategory with no norphisms is just a category.

Example 4 (A nategory with one norphism negating every morphism)**.** In this construction, we turn a category into a nategory by making the choice that a norphism is a witness for the fact that the corresponding hom-set is empty. For any category $\mathbf{C}$, let

$$\mathrm{Nom}_{\mathbf{C}}(X;Y) := \{\bullet\} \tag{18}$$

and for any pair X, Y and any $f : X \to Y$ let

$$i_{XY}(\bullet, f) = \top, \tag{19}$$

except for the special case $X = Y$, $f = \mathrm{id}_X$, where we set

$$i_{XX}(\bullet, \mathrm{id}_X) = \bot. \tag{20}$$

In this case, the element $\bullet$ is a witness for "$\mathrm{Hom}_{\mathbf{C}}(X;Y)$ there is no other morphism other than the identity". (If working with semicategories, we can remove the special case **??** and let the norphism exclude the identity as well. Or, if working in a category, we could use the semantics that excluding the identity at the object X means excluding the entire object X.)

Next, we need to define the two maps:

$$\multimapdotboth : \mathrm{Hom}_{\mathbf{C}}(X;Y) \times \mathrm{Nom}_{\mathbf{C}}(X;Z) \to \mathrm{Nom}_{\mathbf{C}}(Y;Z), \tag{21}$$

$$\multimapdot : \mathrm{Nom}_{\mathbf{C}}(X;Z) \times \mathrm{Hom}_{\mathbf{C}}(Y;Z) \to \mathrm{Nom}_{\mathbf{C}}(X;Y). \tag{22}$$

The choice is forced, as there is only one norphism in the codomains! We obtain:

$$\begin{aligned} f \multimapdotboth \bullet &= \bullet, \\ \bullet \multimapdot g &= \bullet. \end{aligned} \tag{23}$$

The conditions Eqs. (equiv-1) and (equiv-2) are trivially verified because i_{XY} always evaluates to $\top$.

Example 5 (Setting the norphism to be subsets of morphisms)**.** For any category $\mathbf{C}$, let

$$\mathrm{Nom}_{\mathbf{C}}(X;Y) = \mathsf{Pow}(\mathrm{Hom}_{\mathbf{C}}(X;Y)) \tag{24}$$

Set the incompatibility relation as

$$i_{XY}(n, f) = f \in n. \tag{25}$$

Define the composition operations as

$$f \multimapdotbothA n = \mathrm{pre}_f^{-1}(n), \tag{26}$$

$$n \multimapdotbothB g = \mathrm{post}_g^{-1}(n), \tag{27}$$

where pre_f and post_g are the pre- and post-composition maps

$$\begin{aligned} \mathrm{pre}_f : \mathrm{Hom}_{\mathbf{C}}(Y;Z) &\to \mathrm{Hom}_{\mathbf{C}}(X;Z), \\ g &\mapsto f \mathbin{;} g, \end{aligned} \tag{28}$$

$$\begin{aligned} \mathrm{post}_g : \mathrm{Hom}_{\mathbf{C}}(X;Y) &\to \mathrm{Hom}_{\mathbf{C}}(X;Z), \\ f &\mapsto f \mathbin{;} g. \end{aligned} \tag{29}$$

Let's check the condition Equation (equiv-1):

$$i_{YZ}(f \multimapdotbothA n, g) \Rightarrow i_{XZ}(n, f \mathbin{;} g). \tag{30}$$

Using our definitions, we get

$$g \in f \multimapdotbothA n \quad \Rightarrow \quad f \mathbin{;} g \in n. \tag{31}$$

Expanding the left-hand side we get

$$g \in \mathrm{pre}_f^{-1}(n) \quad \Rightarrow \quad f \mathbin{;} g \in n. \tag{32}$$

Another expansion shows that both sides are the same:

$$f \mathbin{;} g \in n \quad \Rightarrow \quad f \mathbin{;} g \in n. \tag{33}$$

Checking the condition Equation (equiv-2) is symmetric. We note that this nategory is exact.

Finally, we provide an example of a nategory that we are going to use later as a counter-example.

Example 6 (Very weak composition operations)**.** For any category **C**, as in the previous example, use subsets of morphisms as the norphisms:

$$\mathrm{Nom}_{\mathbf{C}}(X;Y) = \mathsf{Pow}(\mathrm{Hom}_{\mathbf{C}}(X;Y)), \tag{34}$$

and set the incompatibility relation as

$$i_{XY}(n, f) = f \in n. \tag{35}$$

However, define the composition operations as

$$f \multimapdotbothA n = \emptyset, \tag{36}$$

$$n \multimapdotbothB g = \emptyset. \tag{37}$$

The conditions are still satisfied. For example condition Equation (equiv-1):

$$i_{YZ}(f \multimapdotbothA n, g) \Rightarrow i_{XZ}(n, f \mathbin{;} g) \tag{38}$$

becomes

$$g \in \emptyset \ \Rightarrow \ f \mathbin{;} g \in \emptyset, \tag{39}$$

which is vacuously satisfied as the premise is always false.

5 Example: hiking on the Swiss mountains

In this section we present an example of planning, giving a more concrete description of the path planning problems mentioned in the introduction. We describe **Berg**, a category whose morphisms are hiking paths of various difficulty on a mountain. We then consider the problem of finding paths of minimum length.

Definition 7 (**Berg**). Let $h : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a C^1 function, describing the elevation of a mountain. The set with elements $\langle a, b, h(a,b)\rangle$ is a manifold $\mathbb{M}$ that is embedded in $\mathbb{R}^3$. Let $\sigma = [\sigma_{\mathrm{L}}, \sigma_{\mathrm{U}}] \subset \mathbb{R}$ be a closed interval of real numbers. The category $\mathbf{Berg}_{h,\sigma}$ is specified as follows:

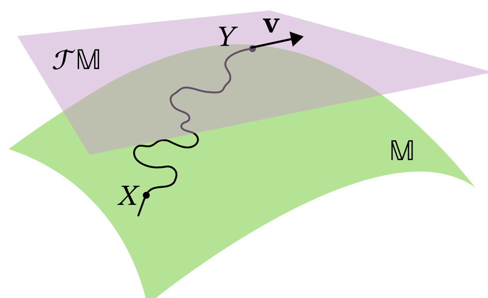
1. An object X is a pair $\langle \mathbf{p}, \mathbf{v}\rangle \in \mathcal{T}\mathbb{M}$, where $\mathbf{p} = \langle \mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\rangle$ is the position, $\mathbf{v}$ is the velocity, and $\mathcal{T}\mathbb{M}$ is the tangent bundle of the manifold.
2. Morphisms are C^1 paths on the manifold.At each point of a path we define the *steepness* as:

$$s(\langle \mathbf{p}, \mathbf{v}\rangle) := \mathbf{v}_z / \sqrt{\mathbf{v}_x^2 + \mathbf{v}_y^2}. \tag{40}$$

   We choose as morphisms only the paths that have the steepness values contained in the interval σ:

$$\mathrm{Hom}_{\mathbf{Berg}_{h,\sigma}}(X;Y) = \{f \text{ is a } C^1 \text{ path from } X \text{ to } Y \text{ and } s(f) \subseteq \sigma\}, \tag{41}$$

3. Morphism composition is given by concatenation of paths.
4. Given any object, the identity morphism is the trivial self path with only one point.



For the complete proof that **Berg** is a category, we refer the reader to Lemma 20.

The steepness interval σ allows considering different categories on the same mountain, with possible hikes varying in difficulty, measured as minimum/maximum steepness. For example, a good hiker has $\sigma = [-0.57, 0.57]$ (positive/negative 30° slope). If $\sigma = [-0.57, 0]$, we are only allowed to climb down. If $\sigma = [0, 0]$, we can only walk along isoclines.

Interpretation of norphisms in Berg What should a norphism be in this case?

One possibility is to let a norphism $n : X \dashrightarrow Y$ mean "there exists no path from X to Y". This is a trivial choice that is similar to Example 4 and that makes morphisms and norphisms mutually exclusive.

We can obtain a more useful theory by letting norphisms carry more information that is *complementary* to morphisms by interpreting them as *lower bounds* on distances. Let the set of norphism be the real numbers completed by plus infinity:

$$\mathrm{Nom}_{\mathbf{Berg}}(X;Y) := \mathbb{R}_{\geq 0} \cup \{+\infty\}. \tag{42}$$

(a) (b)

Figure 2: Composition of morphisms and norphisms in the case of paths and lengths.

Let $\mathrm{length}(f)$ be the length of the path (according to the manifold metric). A norphism $n : X \dashrightarrow Y$ is a witness of "for all paths $f : X \to Y$, we have $\mathrm{length}(f) \geq n$". This is negative, complementary information to morphisms, providing a lower bound on the length of the paths. The case in which $n = \infty$ means that there is no path from X to Y. The incompatibility relation i_{XY} can be written as follows:

$$i_{XY}(n, f) = \mathrm{length}(f) < n \tag{43}$$

To say that a path f is optimal means saying that f is feasible *and* that $\mathrm{length}(f)$ is a norphism:

$$\frac{f : X \to Y \quad \mathrm{length}(f) : X \dashrightarrow Y}{f \text{ is optimal}}. \tag{44}$$

Composition rules for norphisms Next, we derive the two composition rules that are the equivalent of Equation (5) and Equation (6). In this case, we obtain that $n \multimap f$ and $f \multimapboth n$, while having different signatures, have equal value:

$$n \multimap f \;=\; \max\{n - \mathrm{length}(f), 0\} \;=\; f \multimapboth n. \tag{45}$$

The reasoning follows Fig. 2: if f is a path from Z to Y, and we know that going from X to Y takes at least n, then any path from X to Z must be at least $n - \mathrm{length}(f)$ long. For the other direction: if there is a path f from X to Y and we know that going from X to Z takes at least n, then any path from Y to Z must be at least $n - \mathrm{length}(f)$ long. In this case, one has

$$J_{XZ}(f \multimapboth n) = \{g : \mathrm{length}(g) < \max\{n - \mathrm{length}(f), 0\}\}. \tag{46}$$

In case $n < \mathrm{length}(f)$, this corresponds to the empty set $\emptyset$, which differs from

$$\mathrm{pre}_f^{-1}(J_{XY}(n)) = \{g : \mathrm{length}(g) + \mathrm{length}(f) < n\}, \tag{47}$$

The nategory is not exact. However, since

$$\{g : \mathrm{length}(g) < \max\{n - \mathrm{length}(f), 0\}\} \subseteq= \{g : \mathrm{length}(g) + \mathrm{length}(f) < n\}, \tag{48}$$

the nategory satisfies Equation (equiv-1). The check for Equation (equiv-2) is analogous.

Exact version By considering norphisms to be real numbers

$$\mathrm{Nom}_{\mathbf{Berg}}(X; Y) \;:=\; \mathbb{R} \cup \{+\infty\}, \tag{49}$$

and defining the composition operations as

$$n \multimap f \;=\; n - \mathrm{length}(f) = \; f \multimapboth n, \tag{50}$$

the nategory becomes exact. Indeed, for this case one has

$$\begin{aligned} J_{XZ}(f \multimapboth n) &= \{g : \mathrm{length}(g) < n - \mathrm{length}(f)\}, \\ &= \{g : \mathrm{length}(g) + \mathrm{length}(f) < n\} \\ &= \mathrm{pre}_f^{-1}(J_{XY}(n)), \end{aligned} \tag{51}$$

Integer version We also think of a variation in which the norphisms are integers:

$$\mathrm{Nom}_{\mathbf{Berg}}(X;Y) \ := \ \mathbb{Z} \cup \{+\infty\}. \tag{52}$$

In this case we are limited to express constraints of the type

$$\begin{aligned} \mathrm{length}(f) &\geq 0, & (53)\\ \mathrm{length}(f) &\geq 1, & (54)\\ \mathrm{length}(f) &\geq 2, & (55)\\ \mathrm{length}(f) &\geq ... & (56) \end{aligned}$$

The composition rule needs to be redefined as:

$$n \multimap f \ = \mathsf{floor}(n - \mathrm{length}(f)) = \ f \multimap n. \tag{57}$$

This makes the considered nategory satisfy Equation (equiv-1), however non exact, because

$$\begin{aligned} J_{XZ}(f \multimap n) &= \{g : \ \mathrm{length}(g) < \mathsf{floor}(n - \mathrm{length}(f))\} \\ &\subseteq \{g : \ \mathrm{length}(g) < n - \mathrm{length}(f)\} \\ &= \mathrm{pre}_f^{-1}(J_{XY}(n)), \end{aligned} \tag{58}$$

since $\mathsf{floor}(n - \mathrm{length}(f)) + \mathrm{length}(f) \leq n$. An analogous reasoning follows for Equation (equiv-2). Note that using $\mathsf{round}(-)$ or $\mathsf{ceil}(-)$ in Equation (57) would violate Eqs. (equiv-1) and (equiv-2).

Norphisms axioms Finally, we need to specify the set of axioms for the norphisms. So far, we said that norphisms are nonnegative numbers plus infinity, but we did not say how exactly we associate a set Nom to each pair of objects. We obtain different nategories by choosing more or less axioms.

1. **Trivial norphism**: since lengths cannot be negative, for all pair of objects we have the norphism $0 : X \dashrightarrow X$. Having this as an axiom is not very useful, as the composition rules just generate other zeros as norphisms.
2. **Bound based on distance in** $\mathbb{R}^3$. Any path along the mountain cannot be shorter than the distance of a straight line ("as the crow flies"). Therefore, for two objects $\langle \mathbf{p}^1, \mathbf{v}^1 \rangle$, $\langle \mathbf{p}^2, \mathbf{v}^2 \rangle$, we have the distance in $\mathbb{R}^3$ $\|\mathbf{p}^1 - \mathbf{p}^2\|$ as a valid norphism:

$$\|\mathbf{p}^1 - \mathbf{p}^2\| : \ \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle. \tag{59}$$

3. **Bound based on geodesic distance**. A better bound is based on the geodesic distance. This is well defined because the points live on a smooth manifold:

$$d_{\mathbb{M}}(\mathbf{p}^1, \mathbf{p}^2) : \ \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle. \tag{60}$$

4. **Bound based on steepness interval**. Finally, we can use the bound on steepness interval. Given two objects $\langle \mathbf{p}^1, \mathbf{v}^1 \rangle$, $\langle \mathbf{p}^2, \mathbf{v}^2 \rangle$, we can use one of the following bounds

$$|\mathbf{p}_z^1 - \mathbf{p}_z^2|/\sigma_{\mathrm{U}} : \ \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle, \qquad |\mathbf{p}_z^1 - \mathbf{p}_z^2|/\sigma_{\mathrm{L}} : \ \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle, \tag{61}$$

depending on the case (if $\mathbf{p}_z^1 - \mathbf{p}_z^2 < 0$ we use the first, and if $\mathbf{p}_z^1 - \mathbf{p}_z^2 > 0$ the second).

6 Example: co-design

The next example revolves around the construction of norphisms for the category of design problems **DP** [2, 5]; this is called $\mathbf{Feas_{Bool}}$ in [5]. The objects of **DP** are posets. The morphisms are design problems (also referred to as feasibility relations or boolean profunctors). A *design problem* (DP) $d : \mathbf{P} \nrightarrow \mathbf{Q}$ is a monotone map of the form $d : \mathbf{P}^{\mathrm{op}} \times \mathbf{Q} \to_{\mathbf{Pos}} \mathbf{Bool}$, where $\mathbf{P}, \mathbf{Q}$ are arbitrary posets.

The semantics for a DP is that it describes a process which provides a certain functionality, by requiring certain resources. d is a monotone map, since lowering the requested functionalities will not require more resources, and increasing the available resources will not provide less functionalities.

Morphism composition is defined as follows. Given DPs $d : \mathbf{P} \nrightarrow \mathbf{Q}$ and $e : \mathbf{Q} \nrightarrow \mathbf{R}$, they compose into a DP $(d \fatsemi e) : \mathbf{P} \nrightarrow \mathbf{R}$ as:

$$
\begin{aligned}
(d \fatsemi e) : \mathbf{P}^{\mathrm{op}} \times \mathbf{R} &\to_{\mathbf{Pos}} \mathbf{Bool}, \\
\langle p, r \rangle &\mapsto \bigvee_{q \in \mathbf{Q}} d(p,q) \wedge e(q,r).
\end{aligned}
\tag{62}
$$

For any poset $\mathbf{P}$, the identity DP $\mathrm{id}_{\mathbf{P}} : \mathbf{P} \nrightarrow \mathbf{P}$ is a monotone map

$$
\begin{aligned}
\mathrm{id}_{\mathbf{P}} : \mathbf{P}^{\mathrm{op}} \times \mathbf{P} &\to_{\mathbf{Pos}} \mathbf{Bool}, \\
\langle p_1, p_2 \rangle &\mapsto p_1 \preceq_{\mathbf{P}} p_2.
\end{aligned}
\tag{63}
$$

Interpretation of norphisms in DP Given that the morphisms of **DP** are feasibility relations, we expect that the norphisms of **DP** ("nesign problems"), should be *infeasibility* relations. A nesign problem (NP) $n : \mathbf{F} \nrightarrow \mathbf{R}$ should be a boolean map $n : \mathbf{F} \times \mathbf{R}^{\mathrm{op}} \to \mathbf{Bool}$, such that $n(f,r) = \top$ means that it is *not* possible to produce f from r. The semantics of an NP make it so this map should also be monotone:

$$
n : \mathbf{F} \times \mathbf{R}^{\mathrm{op}} \to_{\mathbf{Pos}} \mathbf{Bool}. \tag{64}
$$

In fact, if $\langle f_1, r_1 \rangle$ is not feasible, and $f_2 \geq f_1$, this implies that $\langle f_2, r_1 \rangle$ should not be feasible.

Note that the source poset of a nesign problem is the $^{\mathrm{op}}$ of the source poset for a design problem:

$$
d : \mathbf{F}^{\mathrm{op}} \times \mathbf{R} \to_{\mathbf{Pos}} \mathbf{Bool}. \tag{65}
$$

Compatibility of morphisms and norphisms Consider a DP $d : \mathbf{F} \nrightarrow \mathbf{R}$ and a NP $n : \mathbf{F} \nrightarrow \mathbf{R}$. The compatibility relation between DP and NP should ensure that there are no contradictions. We ask that, for any pair of functionality/resources $\langle f, r \rangle$, it cannot happen that they are declared feasible by the DP ($d(f,r)$) *and* declared infeasible by the NP ($n(f,r)$).

$$
i_{\mathbf{FR}}(n, d) = \exists f \in \mathbf{F}, r \in \mathbf{R} : d(f,r) \wedge n(f,r) \tag{66}
$$

Composition rules for norphisms We can recover the composition rules presented in Equation (5) and Equation (6). Given a NP $n : \mathbf{P} \nrightarrow \mathbf{Q}$ and a DP $d : \mathbf{R} \nrightarrow \mathbf{Q}$, one can compose them to get a NP $n \multimap d : \mathbf{P} \nrightarrow \mathbf{R}$:

$$
(n \multimap d)(p,r) = \bigvee_{q \in \mathbf{Q}} n(p,q) \wedge d(r,q). \tag{67}
$$

Given a DP $d: \mathbf{Q} \to\!\!\!\!\!\!\to \mathbf{P}$ and a NP $n: \mathbf{Q} \nrightarrow \mathbf{R}$, one can compose them to get a NP $d \multimap n: \mathbf{P} \nrightarrow \mathbf{R}$:

$$(d \multimap n)(p,r) = \bigvee_{q \in \mathbf{Q}} d(q,p) \wedge n(q,r). \tag{68}$$

The composition rules satisfy Equation (equiv-1) and Equation (equiv-2). We prove Equation (equiv-2) (Equation (equiv-1) is analogous). One has

$$\begin{aligned} J_{\mathbf{PR}}(d \multimap n) &= \{e: \mathbf{P} \to\!\!\!\!\!\!\to \mathbf{R} \mid \exists p \in \mathbf{P}, r \in \mathbf{R}: e(p,r) \wedge d \multimap n(p,r)\} \\ &= \{e: \mathbf{P} \to\!\!\!\!\!\!\to \mathbf{R} \mid \exists p \in \mathbf{P}, r \in \mathbf{R}: e(p,r) \wedge \bigvee_{q \in \mathbf{Q}} d(q,p) \wedge n(q,r)\} \\ &= \{e: \mathbf{P} \to\!\!\!\!\!\!\to \mathbf{R} \mid \exists p \in \mathbf{P}, q \in \mathbf{Q}, r \in \mathbf{R}: e(p,r) \wedge d(q,p) \wedge n(q,r)\}, \end{aligned} \tag{69}$$

$$J_{\mathbf{QR}}(n) = \{g: \mathbf{Q} \to\!\!\!\!\!\!\to \mathbf{R} \mid \exists q \in \mathbf{Q}, r \in \mathbf{R}: g(q,r) \wedge n(q,r)\}, \tag{70}$$

and

$$\begin{aligned} \mathrm{pre}_d^{-1}(J_{\mathbf{PR}}(n)) &= \{e: \mathbf{P} \to\!\!\!\!\!\!\to \mathbf{R} \mid d \mathbin{⨾} e \in J_{\mathbf{QR}}(n)\} \\ &= \{e: \mathbf{P} \to\!\!\!\!\!\!\to \mathbf{R} \mid \exists q \in \mathbf{Q}, r \in \mathbf{R}: \left(\bigvee_{p \in \mathbf{P}} d(q,p) \wedge e(p,r)\right) \wedge n(q,r)\} \\ &= \{e: \mathbf{P} \to\!\!\!\!\!\!\to \mathbf{R} \mid \exists p \in \mathbf{P}, q \in \mathbf{Q}, r \in \mathbf{R}: d(q,p) \wedge e(p,r) \wedge n(q,r)\}. \end{aligned} \tag{71}$$

Sets 69 and 71 are equal, so the nategory is exact.

Example 8. Consider the posets $\mathbf{P} = \langle \mathbb{N}_{[\text{kg pears}]}, \leq \rangle$, $\mathbf{Q} = \langle \mathbb{R}_{\geq 0,[\text{CHF}]}, \leq \rangle$, and $\mathbf{R} = \langle \mathbb{N}_{[\text{kg raisins}]}, \leq \rangle$. Consider the design problem $d: \mathbf{R} \to\!\!\!\!\!\!\to \mathbf{Q}$ and the nesign problem $n: \mathbf{P} \nrightarrow \mathbf{Q}$. The (in)feasibility relations are given by:

$$\frac{d(r,q)}{\overline{\overline{r \cdot 10 \leq q}}}, \qquad \frac{n(p,q)}{\overline{\overline{p \cdot 5 > q}}}.$$

In other words, it is possible to buy raisins at 10 CHF/kg or more, and never possible to buy pears at less than 5 CHF/kg. We can evaluate the composition in a particular point to understand its meaning. First, the nesign problem $(n \multimapinv d): \mathbf{P} \nrightarrow \mathbf{R}$ describes the possibility to obtain pears from raisins. For instance:

$$\begin{aligned} (n \multimapinv d)(10,4) &= \bigvee_{q \in \mathbf{Q}} n(10,q) \wedge d(4,q) \\ &= \bigvee_{q \in \mathbf{Q}} (40 \leq q < 50) = \top. \end{aligned}$$

The translation is as follows. Can I get 10 kg of pears from 4 kg of raisins? No. Why? If I could, I would need to buy the 4 kg of raisins using d, incurring at least in a cost of 40 CHF. In others words, I would pay 40 CHF for 10 kg of pears, which is impossible as per nesign problem n.

For further explanations please refer to Remark 21.

Norphisms axioms Norphims axioms could follow some knowledge about particular designs we know are (in)feasible. Every engineering discipline has some fundamental limits in the performance of its designs that come from physics or information theory.

Interestingly, we can also formulate a very general axiom that is valid across all fields: in this universe, physically realizable designs can never produce strictly more resources that one starts with.

This axiom can be encoded as a norphism. For each object $\mathbf{P}$, we postulate a NP $n_{\mathbf{P}} : \mathbf{P} \nrightarrow \mathbf{P}$ such that

$$n_{\mathbf{P}}(q, p) = p \prec_{\mathbf{P}} q, \tag{72}$$

where $p \prec_{\mathbf{P}} q = (p \preceq_{\mathbf{P}} q) \wedge (p \neq q)$. Interestingly, starting from a morphism $d : \mathbf{F} \nrightarrow \mathbf{R}$, one can directly obtain two NPs in $\mathbf{R} \nrightarrow \mathbf{F}$ that go in the opposite direction. These are

$$(n_{\mathbf{R}} \multimap d)(r, f) = \bigvee_{r' \in \mathbf{R}} n_{\mathbf{R}}(r, r') \wedge d(f, r'), \qquad (d \multimapdot n_{\mathbf{F}})(r, f) = \bigvee_{f' \in \mathbf{F}} d(f', r) \wedge n_{\mathbf{F}}(f', f),$$

which gives two impossibility results. The first states infeasibility because, while it is possible to get f from r' via d for a certain r', it is not possible to obtain r from r'. The second states infeasibility because, while it is possible to get f' from r via d for a certain f', it is not possible to obtain f' from f. Therefore, for this nategory, *every positive information induces negative information* in the other direction.

7 Enrichment

We recall a standard definition of enrichment [6].

Definition 9 (Enriched category)**.** Let $\langle \mathbf{V}, \otimes, \mathbf{1}, as, lu, ru \rangle$ be a monoidal category, where as is the associator, lu is the left unitor, and ru is the right unitor.
A $\mathbf{V}$-enriched category $\mathbf{E}$ is given by a tuple $\langle \mathrm{Ob}_{\mathbf{E}}, \alpha_{\mathbf{E}}, \beta_{\mathbf{E}}, \gamma_{\mathbf{E}} \rangle$, where

1. $\mathrm{Ob}_{\mathbf{E}}$ is a set of "objects".
2. $\alpha_{\mathbf{E}}$ is a function such that, for all pairs of objects $X, Y \in \mathrm{Ob}_{\mathbf{E}}$, the value $\alpha_{\mathbf{E}}(X, Y)$ is an object of $\mathbf{V}$.
3. $\beta_{\mathbf{E}}$ is a function such that, for all $X, Y, Z \in \mathrm{Ob}_{\mathbf{E}}$, there exists a morphism $\beta_{\mathbf{E}}(X, Y, Z)$ of $\mathbf{V}$, called *composition morphism*:
$$\beta_{\mathbf{E}}(X, Y, Z) : \alpha_{\mathbf{E}}(X, Y) \otimes \alpha_{\mathbf{E}}(Y, Z) \to_{\mathbf{V}} \alpha_{\mathbf{E}}(X, Z). \tag{73}$$
4. $\gamma_{\mathbf{E}}$ is a function such that, for each $X \in \mathrm{Ob}_{\mathbf{E}}$, there exists a morphism of $\mathbf{V}$:
$$\gamma_{\mathbf{E}}(X) : \mathbf{1} \to_{\mathbf{V}} \alpha_{\mathbf{E}}(X, X). \tag{74}$$

Moreover, for any $X, Y, Z, U \in \mathrm{Ob}_{\mathbf{E}}$, the following diagrams must commute.

$$\begin{tikzcd}
\alpha_{\mathbf{E}}(X,Y) \otimes (\alpha_{\mathbf{E}}(Y,Z) \otimes \alpha_{\mathbf{E}}(Z,U)) \arrow[rr, "as"] \arrow[d, "\mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)} \otimes \beta_{\mathbf{E}}(Y,Z,U)"'] & & (\alpha_{\mathbf{E}}(X,Y) \otimes \alpha_{\mathbf{E}}(Y,Z)) \otimes \alpha_{\mathbf{E}}(Z,U) \arrow[d, "\beta_{\mathbf{E}}(X,Y,Z) \otimes \mathrm{id}_{\alpha_{\mathbf{E}}(Z,U)}"] \\
\alpha_{\mathbf{E}}(X,Y) \otimes \alpha_{\mathbf{E}}(Y,U) \arrow[r, "\beta_{\mathbf{E}}(X,Y,U)"'] & \alpha_{\mathbf{E}}(X,U) & \alpha_{\mathbf{E}}(X,Z) \otimes \alpha_{\mathbf{E}}(Z,U) \arrow[l, "\beta_{\mathbf{E}}(X,Z,U)"]
\end{tikzcd} \tag{75}$$

$$\begin{tikzcd}
\alpha_{\mathbf{E}}(X,Y) \otimes \alpha_{\mathbf{E}}(Y,Y) \arrow[r, "\beta_{\mathbf{E}}(X,Y,Y)"] & \alpha_{\mathbf{E}}(X,Y) & \alpha_{\mathbf{E}}(X,X) \otimes \alpha_{\mathbf{E}}(X,Y) \arrow[l, "\beta_{\mathbf{E}}(X,X,Y)"'] \\
\alpha_{\mathbf{E}}(X,Y) \otimes \mathbf{1} \arrow[u, "\mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)} \otimes \gamma_{\mathbf{E}}(Y)"] \arrow[ur, "ru"'] & & \mathbf{1} \otimes \alpha_{\mathbf{E}}(X,Y) \arrow[u, "\gamma_{\mathbf{E}}(X) \otimes \mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)}"'] \arrow[ul, "lu"]
\end{tikzcd} \tag{76}$$

Recall that enriched categories generalize ordinary categories as follows. Consider the monoidal category
$$\mathbf{P} := \langle \mathbf{Set}, \times, 1 \rangle, \tag{77}$$
where $\times$ is the Cartesian product and 1 is the one-element set $\{\bullet\}$.

Lemma 10. A category enriched in $\mathbf{P}$ gives the data necessary to define a small category, and vice versa.

Proof. We show one direction. Suppose that we are given a $\mathbf{P}$-enriched category as a tuple $\langle \mathrm{Ob}_{\mathbf{E}}, \alpha_{\mathbf{E}}, \beta_{\mathbf{E}}, \gamma_{\mathbf{E}} \rangle$. We can define a small category $\mathbf{C}$ as follows:

- Set $\mathrm{Ob}_{\mathbf{C}} := \mathrm{Ob}_{\mathbf{E}}$.
- For each $X, Y \in \mathrm{Ob}_{\mathbf{C}}$, let $\mathrm{Hom}_{\mathbf{C}}(X; Y) := \alpha_{\mathbf{E}}(X, Y)$.
- For each $X, Y, Z \in \mathrm{Ob}_{\mathbf{C}}$, we know a function
$$\beta_{\mathbf{E}}(X, Y, Z) : \mathrm{Hom}_{\mathbf{C}}(X; Y) \otimes \mathrm{Hom}_{\mathbf{C}}(Y; Z) \to_{\mathbf{Set}} \mathrm{Hom}_{\mathbf{C}}(X; Z). \tag{78}$$
The diagrams constraints imply that this function is associative.
Therefore, we use it to define morphism composition in $\mathbf{C}$, setting $⨟_{X,Y,Z} := \beta_{\mathbf{E}}(X, Y, Z)$.

- For each $X \in \mathrm{Ob}_{\mathbf{C}}$ we know a function $\gamma_{\mathbf{E}}(X) \colon 1 \to_{\mathbf{Set}} \mathrm{Hom}_{\mathbf{C}}(X; X)$ that selects a morphism. The diagrams constraints imply that such morphism satisfies unitality with respect to $\mathbin{;}_{X,Y,Z}$. Therefore, we can use it to define the identity at each object:

$$\mathrm{id}_X := \gamma_{\mathbf{E}}(X)(\bullet). \tag{79}$$

$\square$

8 The categories $\mathbf{G}(\mathbf{C})$

The $\mathbf{G}(\mathbf{C})$ construction is due to De Paiva [8, 3]

References: Blass [1]; Valeria; Niu [7]

We begin by the simple presentation of $\mathbf{G}(\mathbf{Set})$.

Definition 11 ($\mathbf{G}(\mathbf{Set})$). An object of $\mathbf{G}(\mathbf{Set})$ is a tuple

$$\langle Q, A, C\rangle, \tag{80}$$

where: Q is a set, A is a set, $C : Q \to_{\mathbf{Rel}} A$ is a relation.

A morphism $\mathbf{r} : \langle Q_1, A_1, C_1\rangle \to_{\mathbf{GC}} \langle Q_2, A_2, C_2\rangle$ is a pair of maps

$$\mathbf{r} = \langle r_\flat, r^\sharp\rangle, \tag{81}$$

$$r_\flat : Q_1 \leftarrow_{\mathbf{Set}} Q_2, \tag{82}$$

$$r^\sharp : A_1 \to_{\mathbf{Set}} A_2, \tag{83}$$

that satisfy the property

$$\forall q_2 : Q_2 \quad \forall a_1 : A_1 \quad r_\flat(q_2) C_1 a_1 \ \Rightarrow\ q_2 C_2 r^\sharp(a_1). \tag{84}$$

Morphism composition is defined component-wise:

$$(\mathbf{r} \mathbin{⨟} \mathbf{s})_\flat = s_\flat \mathbin{⨟} r_\flat, \tag{85}$$

$$(\mathbf{r} \mathbin{⨟} \mathbf{s})^\sharp = r^\sharp \mathbin{⨟} s^\sharp, \tag{86}$$

and satisfies Equation (84) via composition of implications.

The identity at $\langle Q, A, C\rangle$ is given by $\mathrm{id}_{\langle Q, A, C\rangle} = \langle \mathrm{id}_Q, \mathrm{id}_A\rangle$.

We chose the notation as to facilitate the "questions and answers" interpretation [1]. In this interpretation, an object of $\mathbf{G}(\mathbf{Set})$ is a "problem": a relation C between a set of questions Q and a set of answers A. For a particular question $q \in Q$ and answer $a \in A$, $q\,C\,a$ means that the answer is correct for the question. A morphism $\mathbf{r} : \langle Q_1, A_1, C_1\rangle \to_{\mathbf{GC}} \langle Q_2, A_2, C_2\rangle$ is a *reduction* of problem 2 to problem 1, in the sense that we can use a solution to problem 1 to solve problem 2. We start from a question q_2. We can transform it to a question $q_1 = r_\flat(q_2)$ of the first problem. Assuming we can find an answer a_1 to q_1, then we can transform it in an answer of the second problem $a_2 = r^\sharp(a_1)$. The condition Equation (84) ensures that the answer so produced is correct for the second problem.

8.1 A slight reformulation

We rewrite the definition in a slightly different way. A relation can be represented as a boolean function. Instead of giving

$$C : Q \to_{\mathbf{Rel}} A \tag{87}$$

we can choose to consider a boolean function

$$\varkappa : Q \times A \to \{\bot, \top\}. \tag{88}$$

Let $\mathbf{Bool}$ be the category with two objects "$\bot$" and "$\top$" and the one non-identity morphism $\Rightarrow : \bot \to_{\mathbf{Bool}} \top$. Recall that in $\mathbf{Bool}$ the categorical product $\times$ is the conjunction $\wedge$ and the coproduct $+$ is the disjunction $\vee$. We can rewrite Equation (88) again as

$$\varkappa : Q \times A \to \mathrm{Ob}_{\mathbf{Bool}} \tag{89}$$

It is interesting for us to introduce some form of proof relevance, thereby replacing **Bool** with a generic category **B**, which we need to have all finite products and coproducts (and, in particular, initial and final object). So we can use

$$\kappa : Q \times A \to \mathrm{Ob}_{\mathbf{B}} \tag{90}$$

Definition 12 (Category $\mathbf{G}(\mathbf{Set}, \mathbf{B})$)**.** Let **B** be a category with finite products and coproducts. An object of the category $\mathbf{G}(\mathbf{Set}, \mathbf{B})$ is a tuple

$$\langle Q, A, \kappa \rangle, \tag{91}$$

where Q is a set; A is a set, κ is a function

$$\kappa : Q \times A \to \mathrm{Ob}_{\mathbf{B}}. \tag{92}$$

A morphism $\mathbf{r} : \langle Q_1, A_1, \kappa_1 \rangle \to \langle Q_2, A_2, \kappa_2 \rangle$ is a tuple of three functions

$$\mathbf{r} = \langle r_\flat, r^\sharp, r^* \rangle, \tag{93}$$

$$r_\flat : Q_1 \leftarrow_{\mathbf{Set}} Q_2, \tag{94}$$

$$r^\sharp : A_1 \to_{\mathbf{Set}} A_2, \tag{95}$$

$$r^* : \{q_2 : Q_2, a_1 : A_1\} \to \kappa_1(r_\flat(q_2), a_1) \to_{\mathbf{B}} \kappa_2(q_2, r^\sharp(a_1)). \tag{96}$$

The composition of the above morphism $\mathbf{r}$ with $\mathbf{s} : \langle Q_2, A_2, \kappa_2 \rangle \to \langle Q_3, A_3, \kappa_3 \rangle$ is defined as follows:

$$(\mathbf{r} \fatsemi \mathbf{s})_\flat = s_\flat \fatsemi r_\flat, \tag{97}$$

$$(\mathbf{r} \fatsemi \mathbf{s})^\sharp = r^\sharp \fatsemi s^\sharp, \tag{98}$$

$$(\mathbf{r} \fatsemi \mathbf{s})^* : \langle q_3, a_1 \rangle \mapsto r^*(s_\flat(q_3), a_1) \fatsemi_{\mathbf{B}} s^*(q_3, r^\sharp(a_1)). \tag{99}$$

More explicitly,

$$(\mathbf{r} \fatsemi \mathbf{s})^* : \langle q_3, a_1 \rangle \mapsto$$
$$\kappa_1((s_\flat \fatsemi r_\flat)(q_3), a_1) \xrightarrow{r^*(s_\flat(q_3), a_1)} \kappa_2(s_\flat(q_3), r^\sharp(a_1)) \xrightarrow{s^*(q_3, r^\sharp(a_1))} \kappa_3(q_3, (r^\sharp \fatsemi s^\sharp)(a_1)). \tag{100}$$

The identity at $\langle Q, A, \kappa \rangle$ is given by $\langle \mathrm{id}_Q, \mathrm{id}_A, \langle q, a \rangle \mapsto \mathrm{id}_{\kappa(q,a)} \rangle$.

In the case of $\mathbf{B} = \mathbf{Bool}$, Equation (96) is a way to say constructively that for all $q_2 : Q_2, a_1 : A_1$, we have that $\kappa_1(r_\flat(q_2), a_1) \Rightarrow \kappa_2(q_2, r^\sharp(a_1))$. The value $r^*(q_2, a_1)$ is a morphism in **Bool** that witnesses this fact. In the following, it is easier to think about morphism composition rather than logic.

8.2 The many monoidal structures of G(Set)

GC has an extremely rich structure. It provides a non-trivial model of linear logic, which means that it is possible to produce 4 monoidal products and 4 distinct units. Of interest here are the multiplicatives $\otimes$ and $⅋$. We recall their definitions here; then we define another one that we need.

Table 1: Linear logic binary-operators

	connective	unit
multiplicatives	$\otimes$	$\mathbf{1}$
	$\invamp$	$\bot$
additives	$\oplus$	$\mathbf{0}$
	$\&$	$\top$

8.3 Monoidal product $*$

Definition 13 (Monoidal product $*$)**.** The action on the objects is defined as follows:

$$\langle Q_1, A_1, \kappa_1\rangle \mathbin{\dot{*}} \langle Q_2, A_2, \kappa_2\rangle = \langle Q_1 \times Q_2, A_1 \times A_2, \kappa_1 * \kappa_2\rangle \tag{101}$$

$$\kappa_1 * \kappa_2 : \langle\langle q_1, q_2\rangle, \langle a_1, a_2\rangle\rangle \mapsto \kappa_1(q_1, a_1) \mathbin{\dot{\times}_{\mathbf{B}}} \kappa_2(q_2, a_2), \tag{102}$$

where $\dot{\times}_{\mathbf{B}}$ is the product of two objects in $\mathbf{B}$. The monoidal unit is

$$1_* = \langle\{\bullet\}, \{\bullet\}, \top\rangle, \qquad \top : \langle\bullet, \bullet\rangle \mapsto 1_{\mathbf{B}}. \tag{103}$$

The product of $\mathbf{r} : \langle Q_1, A_1, \kappa_1\rangle \to \langle Q_3, A_3, \kappa_3\rangle$ and $\mathbf{s} : \langle Q_2, A_2, \kappa_2\rangle \to \langle Q_4, A_4, \kappa_4\rangle$ is

$$\mathbf{r} * \mathbf{s} : \langle Q_1 \times Q_2, A_1 \times A_2, \kappa_1 * \kappa_2\rangle \to \langle Q_3 \times Q_4, A_3 \times A_4, \kappa_3 * \kappa_4\rangle \tag{104}$$

$$(\mathbf{r} \mathbin{\vec{*}} \mathbf{s})_\flat = r_\flat \mathbin{\vec{\times}} s_\flat, \tag{105}$$

$$(\mathbf{r} \mathbin{\vec{*}} \mathbf{s})^\sharp = r^\sharp \mathbin{\vec{\times}} s^\sharp, \tag{106}$$

$$(\mathbf{r} \mathbin{\vec{*}} \mathbf{s})^* : \langle\langle q_3, q_4\rangle, \langle a_1, a_2\rangle\rangle \mapsto r^*(q_3, a_1) \mathbin{\vec{\times}_{\mathbf{B}}} s^*(q_4, a_2). \tag{107}$$

8.4 Monoidal product $\otimes$

Definition 14 (Monoidal product $\otimes$)**.** The action on the objects is defined as follows:

$$\langle Q_1, A_1, \kappa_1\rangle \otimes \langle Q_2, A_2, \kappa_2\rangle = \langle Q_1^{A_2} \times Q_2^{A_1}, A_1 \times A_2, \kappa_1 \otimes \kappa_2\rangle \tag{108}$$

$$\kappa_1 \otimes \kappa_2 : \langle\langle q_1, q_2\rangle, \langle a_1, a_2\rangle\rangle \mapsto \kappa_1(q_1(a_2), a_1) \mathbin{\dot{\times}_{\mathbf{B}}} \kappa_2(q_2(a_1), a_2), \tag{109}$$

where $\dot{\times}_{\mathbf{B}}$ is the product of two objects in $\mathbf{B}$. The monoidal unit is

$$1_\otimes = \langle\{\bullet\}, \{\bullet\}, \top\rangle, \qquad \top : \langle\bullet, \bullet\rangle \mapsto 1_{\mathbf{B}}. \tag{110}$$

The product of $\mathbf{r} : \langle Q_1, A_1, \kappa_1\rangle \to \langle Q_3, A_3, \kappa_3\rangle$ and $\mathbf{s} : \langle Q_2, A_2, \kappa_2\rangle \to \langle Q_4, A_4, \kappa_4\rangle$. is

$$\mathbf{r} \otimes \mathbf{s} : \langle Q_1^{A_2} \times Q_2^{A_1}, A_1 \times A_2, \kappa_1 \otimes \kappa_2\rangle \to \langle Q_3^{A_4} \times Q_4^{A_3}, A_3 \times A_4, \kappa_3 \otimes \kappa_4\rangle \tag{111}$$

$$(\mathbf{r} \mathbin{\vec{\otimes}} \mathbf{s})_\flat = \langle s^\sharp \mathbin{⨟} - \mathbin{⨟} r_\flat, r^\sharp \mathbin{⨟} - \mathbin{⨟} s_\flat\rangle, \tag{112}$$

$$(\mathbf{r} \mathbin{\vec{\otimes}} \mathbf{s})^\sharp = r^\sharp \mathbin{\vec{\times}} s^\sharp, \tag{113}$$

$$(\mathbf{r} \mathbin{\vec{\otimes}} \mathbf{s})^* : \langle\langle q_3, q_4\rangle, \langle a_1, a_2\rangle\rangle \mapsto r^*((s^\sharp \mathbin{⨟} q_3)(a_2), a_1) \mathbin{\vec{\times}_{\mathbf{B}}} s^*((r^\sharp \mathbin{⨟} q_3)(a_1), a_2). \tag{114}$$

The "$\vec{\times}_{\mathbf{B}}$" in Equation (114) is the product of the two morphisms in $\mathbf{B}$.

8.4.1 Derivation of Equation (114)

The function $(\mathbf{r} \otimes^{\to} \mathbf{s})^*$ needs to have type

$$
\begin{aligned}
(\mathbf{r} \otimes^{\to} \mathbf{s})^* \; &: \{\langle q_3, q_4\rangle : Q_3^{A_4} \times Q_4^{A_3}, \langle a_1, a_2\rangle : A_1 \times A_2\} \to && (115)\\
&\kappa_1 \otimes \kappa_2\big((\mathbf{r} \otimes^{\to} \mathbf{s})_\flat(\langle q_3, q_4\rangle), \langle a_1, a_2\rangle\big) \to_{\mathbf{B}} && (116)\\
&\kappa_3 \otimes \kappa_4\Big(\langle q_3, q_4\rangle, r^\sharp \times^{\to} s^\sharp(\langle a_1, a_2\rangle)\Big) && (117)
\end{aligned}
$$

Simplifying, we get that we should find a map

$$
\begin{aligned}
(\mathbf{r} \otimes^{\to} \mathbf{s})^* : &\; \langle\langle q_3, q_4\rangle, \langle a_1, a_2\rangle\rangle \mapsto && (118)\\
&\kappa_1 \otimes \kappa_2\Big(\langle s^\sharp \mathbin{⨟} q_3 \mathbin{⨟} r_\flat, r^\sharp \mathbin{⨟} q_4 \mathbin{⨟} s_\flat\rangle, \langle a_1, a_2\rangle\Big) \to_{\mathbf{B}} && (119)\\
&\kappa_3 \otimes \kappa_4\Big(\langle q_3, q_4\rangle, \langle r^\sharp(a_1), s^\sharp(a_2)\rangle\Big) && (120)
\end{aligned}
$$

Expanding $\kappa_1 \otimes \kappa_2$ and $\kappa_3 \otimes \kappa_4$ we have

$$
\begin{aligned}
(\mathbf{r} \otimes^{\to} \mathbf{s})^* : &\; \langle\langle q_3, q_4\rangle, \langle a_1, a_2\rangle\rangle \mapsto && (121)\\
&\kappa_1(r_\flat(s^\sharp \mathbin{⨟} q_3(a_2)), a_1) \times^{\cdot}_{\mathbf{B}} \kappa_2(s_\flat(r^\sharp \mathbin{⨟} q_4(a_1)), a_2) \to_{\mathbf{B}} && (122)\\
&\kappa_3(s^\sharp \mathbin{⨟} q_3(a_2), r^\sharp(a_1)) \times^{\cdot}_{\mathbf{B}} \kappa_4(r^\sharp \mathbin{⨟} q_4(a_1), s^\sharp(a_2)) && (123)
\end{aligned}
$$

Now notice that we can use r^* and s^* to obtain the two morphisms

$$
\begin{aligned}
r^*((s^\sharp \mathbin{⨟} q_3)(a_2), a_1) &: \kappa_1(r_\flat(s^\sharp \mathbin{⨟} q_3(a_2)), a_1) \to_{\mathbf{B}} \kappa_3(s^\sharp \mathbin{⨟} q_3(a_2), r^\sharp(a_1)) && (124)\\
s^*((r^\sharp \mathbin{⨟} q_3)(a_1), a_2) &: \kappa_2(s_\flat(r^\sharp \mathbin{⨟} q_4(a_1)), a_2) \to_{\mathbf{B}} \kappa_4(r^\sharp \mathbin{⨟} q_4(a_1), s^\sharp(a_2)) && (125)
\end{aligned}
$$

Therefore, we can take the product of the two morphisms and obtain a morphism of the right type:

$$
(\mathbf{r} \otimes^{\to} \mathbf{s})^* : \langle\langle q_3, q_4\rangle, \langle a_1, a_2\rangle\rangle \mapsto r^*((s^\sharp \mathbin{⨟} q_3)(a_2), a_1) \times^{\to} s^*((r^\sharp \mathbin{⨟} q_3)(a_1), a_2). \tag{126}
$$

8.5 Monoidal product ⅋

Definition 15 (Monoidal product ⅋)**.** The action on the objects is defined as follows:

$$
\begin{aligned}
\langle Q_1, A_1, \kappa_1\rangle \mathbin{⅋^{\cdot}} \langle Q_2, A_2, \kappa_2\rangle &= \langle Q_1 \times Q_2, A_1^{Q_2} \times A_2^{Q_1}, \kappa_1 \mathbin{⅋} \kappa_2\rangle && (127)\\
\kappa_1 \mathbin{⅋} \kappa_2 : \langle\langle q_1, q_2\rangle, \langle a_1, a_2\rangle\rangle &\mapsto \kappa_1(q_1(a_2), a_1) +^{\cdot}_{\mathbf{B}} \kappa_2(q_2(a_1), a_2), && (128)
\end{aligned}
$$

where $+^{\cdot}_{\mathbf{B}}$ is the coproduct of two objects in $\mathbf{B}$. The monoidal unit is

$$
1_{⅋} = \langle\{\bullet\}, \{\bullet\}, \bot\rangle, \qquad \bot : \langle \bullet, \bullet\rangle \mapsto 0_{\mathbf{B}}. \tag{129}
$$

The product of $\mathbf{r} : \langle Q_1, A_1, \kappa_1\rangle \to \langle Q_3, A_3, \kappa_3\rangle$, $\mathbf{s} : \langle Q_2, A_2, \kappa_2\rangle \to \langle Q_4, A_4, \kappa_4\rangle$ is

$$
\mathbf{r} \mathbin{⅋^{\to}} \mathbf{s} : \langle Q_1 \times Q_2, A_1^{Q_2} \times A_2^{Q_1}, \kappa_1 \mathbin{⅋} \kappa_2\rangle \to \langle Q_3 \times Q_4, A_3^{Q_4} \times A_4^{Q_3}, \kappa_3 \mathbin{⅋} \kappa_4\rangle \tag{130}
$$

$$
\begin{aligned}
(\mathbf{r} \mathbin{⅋^{\to}} \mathbf{s})_\flat &= r_\flat \times^{\to} s_\flat, && (131)\\
(\mathbf{r} \mathbin{⅋^{\to}} \mathbf{s})^\sharp &= \langle s_\flat \mathbin{⨟} - \mathbin{⨟} r^\sharp, r_\flat \mathbin{⨟} - \mathbin{⨟} s^\sharp\rangle, && (132)\\
(\mathbf{r} \mathbin{⅋^{\to}} \mathbf{s})^* &: \langle\langle q_3, q_4\rangle, \langle a_1, a_2\rangle\rangle \mapsto r^*(\dots, a_1) +^{\to}_{\mathbf{B}} s^*(\dots, a_2). && (133)
\end{aligned}
$$

where $+^{\to}_{\mathbf{B}}$ is the coproduct of two morphisms in $\mathbf{B}$.

8.6 Monoidal product $\sqcup$

We now define a monoidal product $\sqcup$. It is very similar to $\otimes$, but conveys an idea of "or" instead of "and" when combining two relations for forming the monoidal product of two objects.

Definition 16 (Monoidal product $\sqcup$)**.** The action on the objects is defined as follows:

$$\langle Q_1, A_1, \kappa_1 \rangle \,\dot{\sqcup}\, \langle Q_2, A_2, \kappa_2 \rangle = \langle Q_1^{A_2} \times Q_2^{A_1}, A_1 \times A_2, \kappa_1 \sqcup \kappa_2 \rangle \tag{134}$$

$$\kappa_1 \sqcup \kappa_2 : \langle \langle q_1, q_2 \rangle, \langle a_1, a_2 \rangle \rangle \mapsto \kappa_1(q_1(a_2), a_1) \mathbin{\dot{+}_{\mathbf{B}}} \kappa_2(q_2(a_1), a_2) \tag{135}$$

The monoidal unit is

$$1_{\sqcup} = \langle \{\bullet\}, \{\bullet\}, \bot \rangle, \qquad \bot : \langle \bullet, \bullet \rangle \mapsto 0_{\mathbf{B}}. \tag{136}$$

The product of $\mathbf{r} : \langle Q_1, A_1, \kappa_1 \rangle \to \langle Q_3, A_3, \kappa_3 \rangle$ and $\mathbf{s} : \langle Q_2, A_2, \kappa_2 \rangle \to \langle Q_4, A_4, \kappa_4 \rangle$ is

$$\mathbf{r} \sqcup \mathbf{s} : \langle Q_1^{A_2} \times Q_2^{A_1}, A_1 \times A_2, \kappa_1 \sqcup \kappa_2 \rangle \to \langle Q_3^{A_4} \times Q_4^{A_3}, A_3 \times A_4, \kappa_3 \sqcup \kappa_4 \rangle \tag{137}$$

$$(\mathbf{r} \,\vec{\sqcup}\, \mathbf{s})_\flat = \langle s^\sharp \fatsemi - \fatsemi r_\flat, r^\sharp \fatsemi - \fatsemi s_\flat \rangle, \tag{138}$$

$$(\mathbf{r} \,\vec{\sqcup}\, \mathbf{s})^\sharp = r^\sharp \,\vec{\times}\, s^\sharp, \tag{139}$$

$$(\mathbf{r} \,\vec{\sqcup}\, \mathbf{s})^* : \langle \langle q_3, q_4 \rangle, \langle a_1, a_2 \rangle \rangle \mapsto r^*(s^\sharp \fatsemi q_3(a_2), a_1) \mathbin{\vec{+}_{\mathbf{B}}} s^*(r^\sharp \fatsemi q_3(a_1), a_2). \tag{140}$$

The "$\vec{+}_{\mathbf{B}}$" in Equation (140) is the coproduct of two morphisms in $\mathbf{B}$.

One can show that $\langle \mathbf{G}(\mathbf{Set}, \mathbf{B}), \sqcup \rangle$ is a monoidal category.

Lemma 17. $\langle \mathbf{G}(\mathbf{Set}, \mathbf{B}), \sqcup \rangle$ can be equipped to be a monoidal category.

9 Describing nategories using enrichment in PN

Definition 18 (**PN**)**.** We call **PN** the monoidal category $\langle \mathbf{G}(\mathbf{Set}, \mathbf{Bool}), \sqcup \rangle$.

Proposition 19. A **PN**-enriched category provides the data necessary to specify a nategory. However, not all nategories can be specified by the data of a **PN**-enriched category, because the nategory produced has two additional neutrality properties:

$$\mathrm{id}_X \mathbin{\bullet\!\!-\!\!\bullet} n = n, \tag{neut-1}$$

$$n \mathbin{-\!\bullet\!-} \mathrm{id}_Y = n, \tag{neut-2}$$

two "distributivity" conditions:

$$(f \mathbin{;} g) \mathbin{\bullet\!\!-\!\!\bullet} n = g \mathbin{\bullet\!\!-\!\!\bullet} (f \mathbin{\bullet\!\!-\!\!\bullet} n), \tag{dist-1}$$

$$n \mathbin{-\!\bullet\!-} (g \mathbin{;} h) = (n \mathbin{-\!\bullet\!-} h) \mathbin{-\!\bullet\!-} g, \tag{dist-2}$$

and a "mixed associativity" condition

$$f \mathbin{\bullet\!\!-\!\!\bullet} (n \mathbin{-\!\bullet\!-} h) = (f \mathbin{\bullet\!\!-\!\!\bullet} n) \mathbin{-\!\bullet\!-} h, \tag{assoc}$$

which are not necessarily satisfied by all nategories.

Proof. Suppose somebody has provided us with a **PN**-enriched category $\mathbf{E}$ as a tuple $\langle \mathrm{Ob}_{\mathbf{E}}, \alpha_{\mathbf{E}}, \beta_{\mathbf{E}}, \gamma_{\mathbf{E}} \rangle$. Using this data we will describe a coherent subnategory $\mathbf{C}$.

As for the objects of $\mathbf{C}$, we set $\mathrm{Ob}_{\mathbf{C}} := \mathrm{Ob}_{\mathbf{E}}$.

For every pair of objects $X, Y \in \mathrm{Ob}_{\mathbf{C}}$, we have an object of **PN** $\alpha_{\mathbf{E}}(X, Y)$. This is a tuple

$$\langle Q, A, \varkappa \rangle, \tag{141}$$

that we interpret as

$$\langle \mathrm{Nom}_{\mathbf{C}}(X; Y), \mathrm{Hom}_{\mathbf{C}}(X; Y), i_{XY} \rangle, \tag{142}$$

thereby setting $\mathrm{Nom}_{\mathbf{C}}(X; Y) := Q$, $\mathrm{Hom}_{\mathbf{C}}(X; Y) := A$, and $i_{XY} = \varkappa$.

Next, for each $X \in \mathrm{Ob}_{\mathbf{E}}$ we have the morphism $\gamma_{\mathbf{E}}(X) : \mathbf{1}_{\mathbf{PN}} \to_{\mathbf{PN}} \alpha_{\mathbf{E}}(X, X)$. We unroll the definition. Because $\mathbf{1}_{\mathbf{PN}} = \langle \{\bullet\}, \{\bullet\}, \bot \rangle$, this is a morphism of **PN**

$$\mathbf{r} = \gamma_{\mathbf{E}}(X) : \langle \{\bullet\}, \{\bullet\}, \bot \rangle \to_{\mathbf{PN}} \langle \mathrm{Hom}_{\mathbf{C}}(X; X), \mathrm{Nom}_{\mathbf{C}}(X; X), i_{XX} \rangle \tag{143}$$

which corresponds to three maps $\langle r_\flat, r^\sharp, r^* \rangle$.

The map $r^\sharp$ provides the same information as in the construction of Lemma 10. As in the previous derivations, $r^\sharp$ picks up a morphism that, given the diagram conditions (discussed further below), is constrained to be the identity at X ($r^\sharp(\bullet) = \mathrm{id}_X$). The backward map $r_\flat$ has type

$$r_\flat : \mathrm{Nom}_{\mathbf{C}}(X; X) \to \{\bullet\}. \tag{144}$$

This map is unique and does not carry information. As for r^*, it has type

$$r^* : \{q_2 : Q_2, a_1 : A_1\} \to \varkappa_1(r_\flat(q_2), a_1) \to_{\mathbf{Bool}} \varkappa_2(q_2, r^\sharp(a_1)) \tag{145}$$

with our choices, this becomes

$$r^*(n, \bullet) : \bot \to_{\mathbf{Bool}} i_{XX}(n, \mathrm{id}_X) \tag{146}$$

Because $\bot$ is an initial object in **Bool** such morphism always exists, no matter what the right-hand side is. Therefore, this constraint does not carry any additional information.

For any three objects X, Y, Z, we know the morphism of type

$$\beta_{\mathbf{E}}(X, Y, Z) : \alpha_{\mathbf{E}}(X, Y) \otimes_{\mathbf{PN}} \alpha_{\mathbf{E}}(Y, Z) \to_{\mathbf{PN}} \alpha_{\mathbf{E}}(X, Z). \tag{147}$$

Substituting our choice of $\alpha_{\mathbf{E}}(-, -)$, the morphism has type

$$\langle \mathrm{Nom}_{\mathbf{C}}(X; Y), \mathrm{Hom}_{\mathbf{C}}(X; Y), i_{XY} \rangle \otimes_{\mathbf{PN}} \langle \mathrm{Nom}_{\mathbf{C}}(Y; Z), \mathrm{Hom}_{\mathbf{C}}(Y; Z), i_{YZ} \rangle \to_{\mathbf{PN}} \langle \mathrm{Nom}_{\mathbf{C}}(X; Z), \mathrm{Hom}_{\mathbf{C}}(X; Z), i_{XZ} \rangle. \tag{148}$$

We rewrite using abbreviated notation as

$$\langle \mathrm{N}_{XY}, \mathrm{H}_{XY}, i_{XY} \rangle \otimes_{\mathbf{PN}} \langle \mathrm{N}_{YZ}, \mathrm{H}_{YZ}, i_{YZ} \rangle \to_{\mathbf{PN}} \langle \mathrm{N}_{XZ}, \mathrm{H}_{XZ}, i_{XZ} \rangle. \tag{149}$$

Expanding using the definition of $\otimes_{\mathbf{PN}}$ gives a morphism

$$\mathbf{s}_{XYZ} : \langle {\mathrm{N}_{XY}}^{\mathrm{H}_{YZ}} \times {\mathrm{N}_{YZ}}^{\mathrm{H}_{XY}}, \mathrm{H}_{XY} \times \mathrm{H}_{YZ}, i_{XY} \sqcup i_{YZ} \rangle \to_{\mathbf{PN}} \langle \mathrm{N}_{XZ}, \mathrm{H}_{XZ}, i_{XZ} \rangle. \tag{150}$$

Such morphism corresponds to three maps $\langle s_\flat, s^\sharp, s^* \rangle$. As in Lemma 10, we obtain that $s^\sharp$ has type

$$s^\sharp : \mathrm{Hom}_{\mathbf{C}}(X; Y) \times \mathrm{Hom}_{\mathbf{C}}(Y; Z) \to \mathrm{Hom}_{\mathbf{C}}(X; Z) \tag{151}$$

and the diagrams, which are discussed below, imply that it is associative and unital. Thus, we use it to define morphism composition by setting $\fatsemi_{\mathbf{C}} := s^\sharp$.

At this point, we have recovered the structure of $\mathbf{C}$ as a category (hom-sets, identities, and composition) and we have defined the nom-sets $\mathrm{Nom}_{\mathbf{C}}(X; Y)$ and the incompatibility relation i_{XY}.

We now use the map $s_\flat$ to derive the norphism composition operations. The map $s_\flat$ has type

$$s_\flat : \mathrm{N}_{XZ} \to {\mathrm{N}_{XY}}^{\mathrm{H}_{YZ}} \times {\mathrm{N}_{YZ}}^{\mathrm{H}_{XY}}, \tag{152}$$

which means that it specifies two maps, which we can write as:

$$\mathbin{-\!\bullet\!-} : \mathrm{N}_{XZ} \times \mathrm{H}_{YZ} \to \mathrm{N}_{XY}, \tag{153}$$

$$\mathbin{\bullet\!-\!\bullet} : \mathrm{H}_{XY} \times \mathrm{N}_{XZ} \to \mathrm{N}_{YZ}. \tag{154}$$

These are our candidate norphism composition operations. For any $n : \mathrm{N}_{XZ}$, $f : \mathrm{H}_{XY}$, $g : \mathrm{H}_{YZ}$ we can evaluate the map s^* to obtain

$$s^*(n, \langle f, g \rangle) : (i_{XY} \sqcup i_{YZ})(\langle (n \mathbin{-\!\bullet\!-} -), (- \mathbin{\bullet\!-\!\bullet} n) \rangle, \langle f, g \rangle) \to_{\mathbf{Bool}} i_{XZ}(n, f \fatsemi g). \tag{155}$$

Expanding more:

$$s^*(n, \langle f, g \rangle) : i_{XY}(n \mathbin{-\!\bullet\!-} g, f) + i_{YZ}(f \mathbin{\bullet\!-\!\bullet} n, g) \to_{\mathbf{Bool}} i_{XZ}(n, f \fatsemi g) \tag{156}$$

which is equivalent to having two maps:

$$s_1^*(n, \langle f, g \rangle) : i_{XY}(n \mathbin{-\!\bullet\!-} g, f) \to_{\mathbf{Bool}} i_{XZ}(n, f \fatsemi g), \tag{157}$$

$$s_2^*(n, \langle f, g \rangle) : i_{YZ}(f \mathbin{\bullet\!-\!\bullet} n, g) \to_{\mathbf{Bool}} i_{XZ}(n, f \fatsemi g). \tag{158}$$

These witness the implications required by Equation (equiv-1) and Equation (equiv-2).

Conditions from unitality diagram for enriched categories We want to infer conditions on the constitutents from the following commutative diagram (the case for right unitality is analogous):

$$\begin{tikzcd}
1_{\mathbf{PN}} \sqcup \alpha_{\mathbf{E}}(X,Y) \arrow[r, "lu"] \arrow[d, "\gamma_{\mathbf{E}}(X) \sqcup \mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)}"'] & \alpha_{\mathbf{E}}(X,Y) \arrow[d, "\mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)}"] \\
\alpha_{\mathbf{E}}(X,X) \sqcup \alpha_{\mathbf{E}}(X,Y) \arrow[r, "\beta_{\mathbf{E}}(X,X,Y)"'] & \alpha_{\mathbf{E}}(X,Y)
\end{tikzcd}$$

We have:

$$\begin{aligned}
(\gamma_{\mathbf{E}}(X) \sqcup \mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)})_{\flat} :\ & Q_{XX}^{A_{XY}} \times Q_{XY}^{A_{XX}} \to 1^{A_{XY}} \times Q_{XY}^{1} \\
& \langle \varphi_{XX}, \varphi_{XY} \rangle \mapsto \langle !, \gamma_{\mathbf{E}}(X)^{\sharp} ⨾ \varphi_{XY} ⨾ \mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)\flat} \rangle = \langle !, \bullet \mapsto \varphi_{XX}(\gamma_{\mathbf{E}}(X)^{\sharp}(\bullet)) \rangle,
\end{aligned} \tag{159}$$

and

$$\begin{aligned}
(\gamma_{\mathbf{E}}(X) \sqcup \mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)})^{\sharp} :\ & 1 \times A_{XY} \to A_{XX} \times A_{XY} \\
& \langle \bullet, a \rangle \mapsto \langle \gamma_{\mathbf{E}}(X)^{\sharp}(\bullet), a \rangle.
\end{aligned} \tag{160}$$

Therefore:

$$\begin{aligned}
((\gamma_{\mathbf{E}}(X) \sqcup \mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)}) ⨾ \beta_{\mathbf{E}}(X,X,Y))^{\sharp} :\ & 1 \times A_{XY} \to A_{XX} \times A_{XY} \to A_{XY} \\
& \langle \bullet, a \rangle \mapsto \langle \gamma_{\mathbf{E}}(X)^{\sharp}(\bullet), a \rangle \mapsto \gamma_{\mathbf{E}}(X)^{\sharp}(a) ⨾ a,
\end{aligned} \tag{161}$$

which, compared with $lu^{\sharp}$ gives the condition

$$\begin{aligned}
\gamma_{\mathbf{E}}(X)^{\sharp}(a) ⨾ a &= a \\
\mathrm{id}_X ⨾ a &= a,
\end{aligned} \tag{162}$$

and

$$\begin{aligned}
((\gamma_{\mathbf{E}}(X) \sqcup \mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)}) ⨾ \beta_{\mathbf{E}}(X,X,Y))_{\flat} :\ & Q_{XY} \to Q_{XX}^{A_{XY}} \times Q_{XY}^{A_{XX}} \to 1^{A_{XY}} \times Q_{XY}^{1} \\
& q \mapsto \beta_{\mathbf{E}}(X,X,Y)^{\sharp}(q) = \langle q \multimapdotinv (-), (-) \multimapboth q \rangle \\
& \mapsto \langle \mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)}{}^{\sharp} ⨾ q \multimapdotinv (-) ⨾ \gamma_{\mathbf{E}}(X)_{\flat}, \gamma_{\mathbf{E}}(X)^{\sharp} ⨾ (-) \multimapboth q ⨾ \mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)\flat} \rangle \\
& = \langle !, \gamma_{\mathbf{E}}(X)^{\sharp}(\bullet) \multimapboth q \rangle,
\end{aligned} \tag{163}$$

which, compared with $lu^{\sharp}$ gives the condition

$$\mathrm{id}_X \multimapboth q = q. \tag{164}$$

From the right-unitality diagram one gets the analogous conditions:

$$a ⨾ \mathrm{id}_Y = a, \tag{165}$$

and

$$q \multimapdotinv \mathrm{id}_Y = q. \tag{166}$$

Conditions from associativity diagram for enriched categories We want to infer conditions on the constituents from the following commutative diagram:

$$\begin{tikzcd}
(\alpha_{\mathbf{E}}(X,Y) \sqcup \alpha_{\mathbf{E}}(Y,Z)) \sqcup \alpha_{\mathbf{E}}(Z,U) \arrow[r, "\beta_{\mathbf{E}}(X,Y,Z) \sqcup \mathrm{id}_{\alpha_{\mathbf{E}}(Z,U)}"] \arrow[d, "as"'] & \alpha_{\mathbf{E}}(X,Z) \sqcup \alpha_{\mathbf{E}}(Z,U) \arrow[dd, "\beta_{\mathbf{E}}(X,Z,U)"] \\
\alpha_{\mathbf{E}}(X,Y) \sqcup (\alpha_{\mathbf{E}}(Y,Z) \sqcup \alpha_{\mathbf{E}}(Z,U)) \arrow[d, "\mathrm{id}_{\alpha_{\mathbf{E}}(X,Y)} \sqcup \beta_{\mathbf{E}}(Y,Z,U)"'] & \\
\alpha_{\mathbf{E}}(X,Y) \sqcup \alpha_{\mathbf{E}}(Y,U) \arrow[r, "\beta_{\mathbf{E}}(X,Y,U)"'] & \alpha_{\mathbf{E}}(X,U)
\end{tikzcd}$$

On the level of "forward" maps we get that β must be associative (here, the forward part represents category composition). For the level of "backward" maps, the following diagram must commute:

$$\begin{tikzcd}
\left(Q_{XY}^{A_{YZ}} \times Q_{YZ}^{A_{XY}}\right)^{A_{ZU}} \times Q_{ZU}^{A_{XY}\times A_{YZ}} & Q_{XZ}^{A_{ZU}} \times Q_{ZU}^{A_{XZ}} \arrow[l, "(\beta_{\mathbf{E}}(X,Y,Z) \sqcup \mathrm{id}_{ZU})_\flat"'] \\
Q_{XY}^{A_{YZ}\times A_{ZU}} \times \left(Q_{YZ}^{A_{ZU}} \times Q_{ZU}^{A_{YZ}}\right)^{A_{XY}} \arrow[u, "as_\flat"] & \\
Q_{XY}^{A_{YU}} \times Q_{YU}^{A_{XY}} \arrow[u, "(\mathrm{id}_{XY} \sqcup \beta_{\mathbf{E}}(Y,Z,U))_\flat"] & Q_{XU} \arrow[l, "\beta_{\mathbf{E}}(X,Y,U)_\flat"] \arrow[uu, "\beta_{\mathbf{E}}(X,Z,U)_\flat"']
\end{tikzcd}$$

Let us look at the different routes.
Right-hand route: First, one has:

$$\begin{aligned}
(\beta_{\mathbf{E}}(X,Y,Z) \sqcup \mathrm{id}_{ZU})_\flat &= \langle {\mathrm{id}_U}^\sharp \fatsemi (-) \fatsemi \beta_{\mathbf{E}}(X,Y,Z)_\flat, \beta_{\mathbf{E}}(X,Y,Z)^\sharp \fatsemi (-) \fatsemi \mathrm{id}_{ZU\flat} \rangle \\
&= \langle (-) \fatsemi \beta_{\mathbf{E}}(X,Y,Z)_\flat, \beta_{\mathbf{E}}(X,Y,Z)^\sharp \fatsemi (-) \rangle.
\end{aligned} \tag{167}$$

Furthermore:

$$\begin{aligned}
\beta_{\mathbf{E}}(Y,Z,U)_\flat \fatsemi (\beta_{\mathbf{E}}(X,Y,Z) \sqcup \mathrm{id}_{ZU})_\flat :\ & Q_{XU} \to Q_{XZ}^{A_{ZU}} \times Q_{ZU}^{A_{XZ}} \to \left(Q_{XY}^{A_{YZ}} \times Q_{YZ}^{A_{XY}}\right)^{A_{ZU}} \times Q_{ZU}^{A_{XY}\times A_{YZ}} \\
& q \mapsto \langle q \mathbin{-\!\!\bullet\!\!-} (-), (-) \mathbin{\bullet\!\!-\!\!\bullet} q \rangle \mapsto \langle q \mathbin{-\!\!\bullet\!\!-} (-) \fatsemi \beta_{\mathbf{E}}(X,Y,Z)_\flat, \beta_{\mathbf{E}}(X,Y,Z)^\sharp \fatsemi (-) \mathbin{\bullet\!\!-\!\!\bullet} q \rangle.
\end{aligned} \tag{168}$$

Now, given $\langle f, g, h \rangle \in A_{XY} \times A_{YZ} \times A_{ZU}$, we have:

$$(q \mathbin{-\!\!\bullet\!\!-} (-) \fatsemi \beta_{\mathbf{E}}(X,Y,Z)_\flat)(h) = \beta_{\mathbf{E}}(X,Y,Z)_\flat(q \mathbin{-\!\!\bullet\!\!-} h) : \langle g, f \rangle \mapsto \langle (q \mathbin{-\!\!\bullet\!\!-} h) \mathbin{-\!\!\bullet\!\!-} g, (q \mathbin{-\!\!\bullet\!\!-} h) \mathbin{\bullet\!\!-\!\!\bullet} f \rangle, \tag{169}$$

and

$$(\beta_{\mathbf{E}}(X,Y,Z)^\sharp \fatsemi (-) \mathbin{\bullet\!\!-\!\!\bullet} q)(\langle f, g \rangle) = (f \fatsemi g) \mathbin{\bullet\!\!-\!\!\bullet} q. \tag{170}$$

Left-hand route: First, we have:

$$\begin{aligned}
(\mathrm{id}_{XY} \sqcup \beta_{\mathbf{E}}(Y,Z,U))_\flat &= \langle \beta_{\mathbf{E}}(Y,Z,U)^\sharp \fatsemi (-) \fatsemi \mathrm{id}_{XY\flat}, {\mathrm{id}_{XY}}^\sharp \fatsemi (-) \fatsemi \beta_{\mathbf{E}}(Y,Z,U)_\flat \rangle \\
&= \langle \beta_{\mathbf{E}}(Y,Z,U)^\sharp \fatsemi (-), (-) \fatsemi \beta_{\mathbf{E}}(Y,Z,U)_\flat \rangle.
\end{aligned} \tag{171}$$

Furthermore:

$$\begin{aligned}\beta_{\mathbf{E}}(X,Y,U)_\flat \mathbin{⨟} (\mathrm{id}_{XY} \sqcup \beta_{\mathbf{E}}(Y,Z,U))_\flat :\ & Q_{XU} \to Q_{XY}^{A_{YU}} \times Q_{YU}^{A_{XY}} \to Q_{XY}^{A_{YZ}\times A_{ZU}} \times \left(Q_{YZ}^{A_{ZU}} \times Q_{ZU}^{A_{YZ}}\right)^{A_{XY}} \\ & q \mapsto \langle q \multimapinv (-), (-) \multimap q\rangle \mapsto \langle \beta_{\mathbf{E}}(Y,Z,U)^\sharp \mathbin{⨟} (-) \multimapinv q, (-) \multimap q \mathbin{⨟} \beta_{\mathbf{E}}(Y,Z,U)_\flat\rangle.\end{aligned} \tag{172}$$

Now, given $\langle f, g, h\rangle \in A_{XY} \times A_{YZ} \times A_{ZU}$, we have:

$$(\beta_{\mathbf{E}}(Y,Z,U)^\sharp \mathbin{⨟} (-) \multimapinv q)(\langle g, h\rangle) = (g \mathbin{⨟} h) \multimapinv q, \tag{173}$$

and

$$((-) \multimap q \mathbin{⨟} \beta_{\mathbf{E}}(Y,Z,U)_\flat)(f) = \beta_{\mathbf{E}}(Y,Z,U)_\flat(f \multimap q) : \langle h, g\rangle \mapsto \langle i \multimapinv (g \multimap q), g \multimap (f \multimap q)\rangle. \tag{174}$$

By comparing the two routes, we obtain the rules:

$$\begin{aligned}(f \mathbin{⨟} g) \multimap q &= g \multimap (f \multimap q), \\ q \multimapinv (g \mathbin{⨟} h) &= (q \multimapinv h) \multimapinv g, \\ (q \multimapinv h) \multimap f &= h \multimapinv (f \multimap q).\end{aligned} \tag{175}$$

$\square$

10 Conclusions

This work showed that we can encode negative information using "norphisms", negative arrows, as opposed to the positive arrows of morphisms.

In general categories, morphisms and norphisms between two objects can live along each other, and are not exclusive. Norphisms can give complementary information to morphisms. We have seen how, in the category **Berg**, norphisms can represent negative results, such as lower bounds on distances between two locations. A path planning algorithm must construct a morphism to give a path *and* a norphism to prove that the path is optimal. Furthermore, we have seen how, in the category **DP**, norphisms can represent design impossibility results.

We have described "nategories" as categories that have the norphism structure. For each pair of objects there is a set $\mathrm{Nom}_{\mathbf{C}}(X;Y)$, along with $\mathrm{Hom}_{\mathbf{C}}(X;Y)$, and a relation which describes the compatibility of morphisms and norphisms. Two norphisms cannot be "composed". Rather, there are rules allowing one to derive norphisms using morphisms as "catalysts", presented in Equation (5) and Equation (6).

Finally, we showed that this series of new definitions and baroque composition operators can be described using enriched category theory. We defined a monoidal category **PN** (which stands for "positive" and "negative") and we have shown that a **PN**-enriched category gives the data necessary to define a nategory. This generalizes the fact that a $\langle\mathbf{Set}, \times, 1\rangle$-enriched category provides the data for a small category.

Future work includes:

- Surveying known categories for natural norphism structures.
- As suggested by a reviewer, defining the relation between **PN** and **Poly**.
- Understanding how the norphism structure generalizes to higher-level construction. For example, what is the natural definition of a "nunctor" (a "negative" functor)?

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References

[1] Andreas Blass (1995): *Questions and answers—a category arising in linear logic, complexity theory, and set theory*. *Advances in linear logic* 222, pp. 61–81.

[2] Andrea Censi, Jonathan Lorand & Gioele Zardini (2022): *Applied Compositional Thinking for Engineers*. Available at `https://bit.ly/3qQNrdR`. Work in progress book.

[3] Valeria CV De Paiva (1989): *A dialectica-like model of linear logic*. In: *Category Theory and Computer Science*, Springer, pp. 341–356.

[4] Daniel Delling, Peter Sanders, Dominik Schultes & Dorothea Wagner (2009): *Engineering route planning algorithms*. In: *Algorithmics of large and complex networks*, Springer, pp. 117–139.

[5] Brendan Fong & David I Spivak (2018): *Seven sketches in compositionality: An invitation to applied category theory*. *arXiv preprint arXiv:1803.05316*.

[6] Gregory Maxwell Kelly, Max Kelly et al. (1982): *Basic concepts of enriched category theory*. 64, CUP Archive.

[7] Nelson Niu (2022): *Dialectica categories and polynomial functors (Part 1)*. Available at `https://bit.ly/3PMKLHB`. Topos Institute Blog Post.

[8] Valeria de Paiva & JW Gray (1989): *The dialectica categories*. *Categories in Computer Science and Logic* 92, pp. 47–62.

[9] Michael Shulman (2018): *Affine logic for constructive mathematics*. *arXiv preprint arXiv:1805.07518*.

A Proofs, examples, and explanations

Lemma 20. $\mathbf{Berg}_{h,\sigma}$ is indeed a category.

Proof. We can start clarifying what a morphism in this category is. A morphism $\langle \mathbf{p}_1, \mathbf{v}_1 \rangle \to \langle \mathbf{p}_2, \mathbf{v}_2 \rangle$ is a path on the manifold. One way to define a path on the manifold concretely is as a pair $\langle \gamma, T \rangle$, where

- $T \in \mathbb{R}_{\geq 0}$, which we think of the "time" taken to travel from $\mathbf{p}_1$ to $\mathbf{p}_2$.
- $\gamma : [0, T] \to \mathbb{M}$ is a C^1 function with $\gamma(0) = \mathbf{p}_1$ and $\gamma(T) = \mathbf{p}_2$, as well as $\dot{\gamma}(0) = \mathbf{v}_1$ and $\dot{\gamma}(T) = \mathbf{v}_2$ (we take one-sided derivatives at the boundaries).

Technically, composition of morphisms works as follows. Given morphisms

$$\langle \gamma_1, T_1 \rangle : \langle \mathbf{p}_1, \mathbf{v}_1 \rangle \to \langle \mathbf{p}_2, \mathbf{v}_2 \rangle \tag{176}$$

and

$$\langle \gamma_2, T_2 \rangle : \langle \mathbf{p}_2, \mathbf{v}_2 \rangle \to \langle \mathbf{p}_3, \mathbf{v}_3 \rangle, \tag{177}$$

their composition is $\langle \gamma, T \rangle$ with $T = T_1 + T_2$ and

$$\gamma(t) = \begin{cases} \gamma_1(t) & 0 \leq t \leq T_1 \\ \gamma_2(t - T_1) & T_1 \leq t \leq T_1 + T_2, \end{cases} \tag{178}$$

expressing concatenation of paths. Furthermore, we can express identity morphisms explicitly. For any object $\langle \mathbf{p}, \mathbf{v} \rangle$, we define the identity morphism

$$\mathrm{id}_{\langle \mathbf{p}, \mathbf{v} \rangle} = \langle \gamma, 0 \rangle \tag{179}$$

formally: its path γ is defined on the closed interval $[0,\ 0]$, (with $T = 0$ and $\gamma(0) = \mathbf{p}$). We declare this path to be C^1 by convention, and declare its derivative at 0 to be $\mathbf{v}$. Note that composition of intervals is associative, because $s(f \mathbin{⨟} g) = s(f) \cup s(g)$. From these constituents and the fact that the composition of steepness intervals is associative, it is clear that morphism composition is associative. Furthermore, the identity morphism satisfies unitality. $\square$

Remark 21 (Explanation for composition of design and nesign problems)**.** Recall that given a NP $n : \mathbf{P} \nrightarrow \mathbf{Q}$ and a DP $d : \mathbf{R} \nrightarrow \mathbf{Q}$, one can compose them to get a NP $n \multimap d : \mathbf{P} \nrightarrow \mathbf{R}$:

$$(n \multimap d)(p, r) = \bigvee_{q \in \mathbf{Q}} n(p, q) \wedge d(r, q). \tag{180}$$

The derivation of the rule is as follows. Can I get p from r? No, if I cannot get p from q, for any q, and I can get r from q. Similarly, given a DP $d : \mathbf{Q} \nrightarrow \mathbf{P}$ and a NP $n : \mathbf{Q} \nrightarrow \mathbf{R}$, one can compose them to get a NP $d \multimap n : \mathbf{P} \nrightarrow \mathbf{R}$:

$$(d \multimap n)(p, r) = \bigvee_{q \in \mathbf{Q}} d(q, p) \wedge n(q, r). \tag{181}$$

Can I get p from r? No, if I cannot get q from r, for any q, and I can get q from p.

A.1 GCat

Definition 22 (**G**(**Cat**, **B**))**.** Given a category **B** with all finite products and coproducts, an object of **G**(**Cat**, **B**) is a tuple

$$\langle Q, A, \kappa \rangle, \tag{182}$$

where Q is a category, A is a category, κ is a profunctor

$$\kappa : Q^{\mathrm{op}} \times A \to \mathbf{B}. \tag{183}$$

A morphism $\mathbf{r} : \langle Q_1, A_1, \kappa_1 \rangle \to_{\mathbf{GCat}} \langle Q_2, A_2, \kappa_2 \rangle$ is a tuple

$$\mathbf{r} = \langle r_\flat, r^\sharp, r^* \rangle, \tag{184}$$

where

- $r_\flat : Q_2 \to_{\mathbf{Cat}} Q_1$ is a functor,
- $r^\sharp : A_1 \to_{\mathbf{Cat}} A_2$ is a functor,
- r^* is a natural transformation between two functors

$$F, G : Q_2^{\mathrm{op}} \times A_1 \to \mathbf{B}, \tag{185}$$

  defined as

$$F = (r_\flat \times \mathrm{id}_{A_1}) \mathbin{⨾} \kappa_1, \tag{186}$$

$$G = (\mathrm{id}_{Q_2^{\mathrm{op}}} \times r^\sharp) \mathbin{⨾} \kappa_2. \tag{187}$$

The composition of the above morphism $\mathbf{r}$ with $\mathbf{s} : \langle Q_2, A_2, \kappa_2 \rangle \to \langle Q_3, A_3, \kappa_3 \rangle$ is defined as follows:

$$(\mathbf{r} \mathbin{⨾} \mathbf{s})_\flat = s_\flat \mathbin{⨾} r_\flat, \tag{188}$$

$$(\mathbf{r} \mathbin{⨾} \mathbf{s})^\sharp = r^\sharp \mathbin{⨾} s^\sharp, \tag{189}$$

$$(\mathbf{r} \mathbin{⨾} \mathbf{s})^* : \langle q_3, a_1 \rangle \mapsto r^*(s_\flat(q_3), a_1) \mathbin{⨾}_{\mathbf{B}} s^*(q_3, r^\sharp(a_1)). \tag{190}$$

More explicitly,

$$(\mathbf{r} \mathbin{⨾} \mathbf{s})^* : \langle q_3, a_1 \rangle \mapsto$$

$$\kappa_1((s_\flat \mathbin{⨾} r_\flat)(q_3), a_1) \xrightarrow{r^*(s_\flat(q_3), a_1)} \kappa_2(s_\flat(q_3), r^\sharp(a_1)) \xrightarrow{s^*(q_3, r^\sharp(a_1))} \kappa_3(q_3, (r^\sharp \mathbin{⨾} s^\sharp)(a_1)). \tag{191}$$

The identity at $\langle Q, A, \kappa \rangle$ is given by $\langle \mathrm{id}_Q, \mathrm{id}_A, \langle q, a \rangle \mapsto \mathrm{id}_{\kappa(q,a)} \rangle$.