Project #5: FEM for Diffusion Equations

Use the **finite element method** to solve the following 1-D Poisson equation

$$-u''(x) = f(x), \quad u(0) = 0,\, u(1) = 0, \quad x \in [0, 1]$$

where $f(x) = 1$. The exact solution is $u_{\text{exact}}(x) = 0.5x(1 - x)$ which can be used to compare with the numerical solution.

Use a uniform grid

$$x_i = ih, \quad i = 0, 1, \cdots, N + 1, \text{ where } h = \frac{1}{N + 1}$$

to discretize the computational domain. Obviously, $N + 1$ is the number of elements.

1. Even if this is a simple 1-D problem, it would be beneficial to use our general data structures `xyz[nn][NSD]`, `ien[ne][NEN]` and `rng[ne][NEF]` to express the grid. Here `nn` is the number of nodes, `ne` is the number of elements, `NSD` is the number of spatial dimensions, `NEN (=2 here)` is the number of local vertices of the element, `NEF (=2 here)` is the number of local edges of the element. Using this universal data structure minimizes the modifications needed for higher-dimension problems.

2. Use linear basis functions (hat functions) to approximate the solution

$$u^h = \sum_{j=1}^{N} c_j \phi_j(x)$$

3. Calculate the local 2×2 stiffness matrix and local 2×1 load vector and assemble it to form the global stiffness matrix and global load vector.

4. Use the Thomas algorithm to solve the system on four grids with $n = 4$, 8, 16, and 32. Plot the results $u^h(x)$ vs. x. Also plot the exact solution $u_{\text{exact}}(x)$ vs. x on the same figure for comparison purpose.

5. You need to include your source code in your submission.