The simulation of diffuse scattering from rough surfaces by Physical Optics (PO)

Regular seminar

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Outline

- Background
- Review of Physical Optics (PO)
- PO model simulation
- Comparison with ER and B-K model
- Conclusion
- Future plan

Background

- To provide higher date rate and throughput, the next generation (5G) mobile communication systems will use higher frequency bands over 6 GHz.
- As radio wavelength is much smaller in high frequency bands, the diffuse scattering waves due to the roughness of building walls and complex structures of furniture affect the propagation characteristics more significantly.
- Traditional ways of treating diffuse scattering includes rigorous methods and asymptotic methods
  - rigorous methods: finite elements, method of moments and finite-difference time domain.

    → consume excessive time
  - asymptotic methods: physical optics, etc.

    → have difficulty in treating the compound scatters

Background

- One effective and simple model - "effective roughness" (ER) model
  - proposed for modeling the diffuse scattering from buildings, based on Lambertian scattering pattern [1].
  - extended to single-lobe and double-lobe directive scattering patterns in [2].
- Research plan
  - ER model simulation. Analyze the effect of different parameters on the diffuse scattering pattern.
  - Compare with other scattering models. Take into account the influence of correlation distance of the rough surface.
  - Measurement in anechoic chamber. By measuring the scattering of different rough surface samples, we are trying to identify the relation between ER parameters and roughness parameters.

[1] V. Degli-Esposti, A Diffuse Scattering Model for Urban Propagation Prediction, *IEEE Trans. Antennas Propag.*, vol. 49, no. 7, pp. 1111–1113, Jul. 2001.
[2] V. Degli-Esposti, F. Fuschini, M. Vitucci, E., & G. Falciasecca. (2007). Measurement and Modelling of Scattering from Buildings. *IEEE Transactions on Antennas and Propagation*, 55(1), 143-153.

Review of Physical Optics (PO)

Physical optics (PO) is one of the electromagnetic simulation method for determining the scattered electromagnetic field from the scatter. PO is the method based on field equivalence principle and surface current approximation.

- Field equivalence principle

By using idea of field equivalence principle, the electric field $\boldsymbol{E}(\boldsymbol{r})$ and the magnetic field $\boldsymbol{H}(\boldsymbol{r})$ at an arbitrary observation point r can be calculated by using electric field $\boldsymbol{E}(\boldsymbol{r}')$ and magnetic field $\boldsymbol{H}(\boldsymbol{r}')$ at point r' on an arbitrary closed surface S which surrounds the observation point.

$$\boldsymbol{E}(\boldsymbol{r}) = -\int_S \left[-j\omega\mu \left\{ \hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r}') \right\} G + \left\{ \hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r}') \right\} \times \nabla G + \frac{1}{j\omega\mu} \left\{ \hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r}') \right\} \nabla\nabla G \right] dS$$

$$\boldsymbol{H}(\boldsymbol{r}) = -\int_S \left[j\omega\mu \left\{ \hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r}') \right\} G + \left\{ \hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r}') \right\} \times \nabla G - \frac{1}{j\omega\mu} \left\{ \hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r}') \right\} \nabla\nabla G \right] dS$$

μ: magnetic permeability G: free space Green's function $G = \dfrac{e^{-j\boldsymbol{k}_s \cdot (\boldsymbol{r}-\boldsymbol{r}')}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$

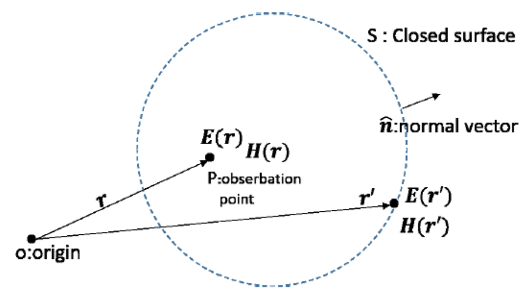
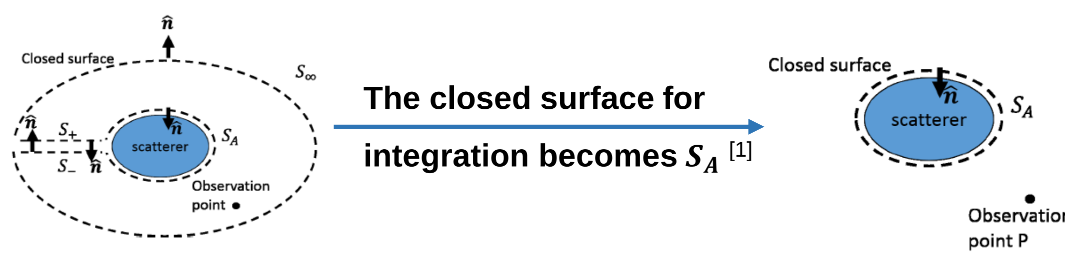


Figure: Field equivalence principle [1]

the integration becomes 0

$$S = S_A + \boxed{S_+ + S_-} + \boxed{S_\infty}$$

the integration cancels each other



[1] Tokio Miyake, “Numerical simulation of scattering from random rough surface by using physical optics”, master thesis.

The equivalent electric current J_{eq} and equivalent magnetic current M_{eq} is defined as

$$\boldsymbol{J}_{eq}(\boldsymbol{r}) = \hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r})$$

$$\boldsymbol{M}_{eq}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r}) \times \hat{\boldsymbol{n}}$$

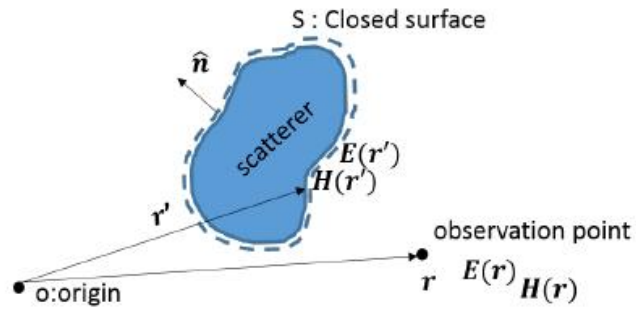


Figure: Field equivalence principle of scatterer [1]

By using vector potential A and B which defined as

$$\boldsymbol{A} = \frac{1}{4\pi}\int_S \boldsymbol{J}_{eq}\frac{e^{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{|\boldsymbol{r}-\boldsymbol{r}'|}dS$$

$$\boldsymbol{B} = \frac{1}{4\pi}\int_S \boldsymbol{M}_{eq}\frac{e^{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{|\boldsymbol{r}-\boldsymbol{r}'|}dS$$

The electric field $E(r)$ is transformed into

$$\boldsymbol{E}(\boldsymbol{r}) = -j\omega\mu\boldsymbol{A} - j\frac{\nabla\nabla\cdot\boldsymbol{A}}{\omega\epsilon} - \nabla\times\boldsymbol{B}$$

If we assume far field ($kr \gg 1$), the electric field $E(r)$ can be expressed in a simple form, which doesn't involve any Laplacian and Rotation

$$\boldsymbol{E}(\boldsymbol{r}) = j\omega\mu\boldsymbol{A}\times\hat{\boldsymbol{r}}\times\hat{\boldsymbol{r}} - j\omega\eta\epsilon\boldsymbol{B}\times\hat{\boldsymbol{r}}$$

where $\hat{r}$ is unit vector from the scatterer to the observation point and ϵ is permittivity.

- Physical optics approximation for Perfect Electric Conductor

1. **Assumption that induced current in shadow region as zero**

☐ reduced the calculation cost

☐ estimation accuracy at the boundary (diffraction happens) is low

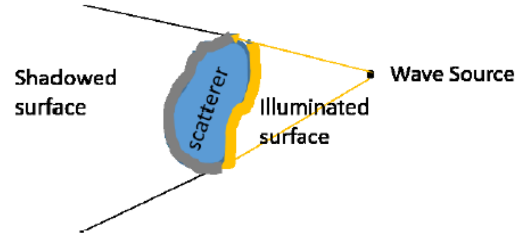


Figure: Illuminated surface and shadowed surface [1]

2. **Tangent plane approximation**

The point on the scatterer is assumed to have infinity tangent plane at that point. Assuming scatterer as Perfect Electric Conductor (PEC), then by using tangent plane approximation, the surface tangential component of the incident magnetic field H_i and the scattered magnetic field H_s on the surface are strictly same (proved by boundary condition).

Illuminated surface :

$$\boxed{\begin{gathered} \boldsymbol{M}_{eq} = 0 \\ \boldsymbol{J}_{eq} = \hat{\boldsymbol{n}} \times (\boldsymbol{H}_i + \boldsymbol{H}_s) \end{gathered}}$$

incident field scattered field

$$\Rightarrow$$

$$\boxed{\begin{gathered} \boldsymbol{J}^{PO} = 2\hat{\boldsymbol{n}} \times \boldsymbol{H}_i \\ \boldsymbol{M}^{PO} = 0 \end{gathered}}$$

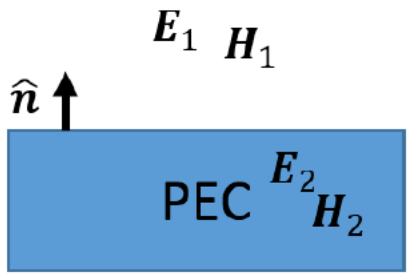


Figure: Boundary of PEC [1]

Shadowed surface :

$$\boxed{\begin{gathered} \boldsymbol{J}^{PO} = 0 \\ \boldsymbol{M}^{PO} = 0 \end{gathered}}$$

- Physical optics approximation for dielectric surface

Additional approximation : the penetrated electromagnetic field can be neglected.

By using the reflection coefficient of Fresnel equation R, the surface tangential component of the reflected electromagnetic field areas are

$$\hat{\boldsymbol{n}} \times \boldsymbol{E}_s = R\hat{\boldsymbol{n}} \times \boldsymbol{E}_i$$

$$\hat{\boldsymbol{n}} \times \boldsymbol{H}_s = -R\hat{\boldsymbol{n}} \times \boldsymbol{H}_i$$

In the dielectric scattering problem, the induced current is obtained by following

Illuminated surface :

$$\boxed{\begin{aligned} \boldsymbol{M}_{eq} &= (\boldsymbol{E}_i + \boldsymbol{E}_s) \times \hat{\boldsymbol{n}} \\ \boldsymbol{J}_{eq} &= \hat{\boldsymbol{n}} \times (\boldsymbol{H}_i + \boldsymbol{H}_s) \end{aligned}} \Rightarrow \boxed{\begin{aligned} \boldsymbol{J}^{PO} &= (1 - R)\hat{\boldsymbol{n}} \times \boldsymbol{H}_i \\ \boldsymbol{M}^{PO} &= (1 + R)\boldsymbol{E}_i \times \hat{\boldsymbol{n}} \end{aligned}}$$

Shadowed surface :

$$\boxed{\begin{aligned} \boldsymbol{J}^{PO} &= 0 \\ \boldsymbol{M}^{PO} &= 0 \end{aligned}}$$

Then the scattered electric field is obtained by

$$\boldsymbol{E}(\boldsymbol{r}) = j\omega\mu\boldsymbol{A} \times \hat{\boldsymbol{r}} \times \hat{\boldsymbol{r}} - j\omega\eta\epsilon\boldsymbol{B} \times \hat{\boldsymbol{r}}$$

$$\boldsymbol{A} = \frac{1}{4\pi}\int_S \boldsymbol{J}^{PO}\frac{e^{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{|\boldsymbol{r}-\boldsymbol{r}'|}dS$$

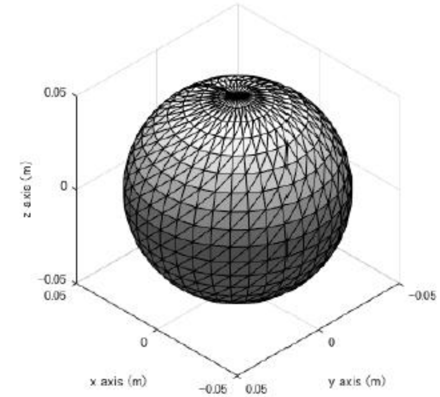
$$\boldsymbol{B} = \frac{1}{4\pi}\int_S \boldsymbol{M}^{PO}\frac{e^{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{|\boldsymbol{r}-\boldsymbol{r}'|}dS$$

• Meshed physical optics

The calculation cost of surface integral can be decreased by assuming the surface as the composition of flat surfaces. Therefore

$$\boldsymbol{A} = \frac{1}{4\pi}\int_S \boldsymbol{J}^{PO}\frac{e^{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{|\boldsymbol{r}-\boldsymbol{r}'|}dS \simeq \sum_{i=1}^{N}\frac{1}{4\pi}\int_{\Delta_i}\boldsymbol{J}^{PO}\frac{e^{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{|\boldsymbol{r}-\boldsymbol{r}'|}dS$$

$$\boldsymbol{B} = \frac{1}{4\pi}\int_S \boldsymbol{M}^{PO}\frac{e^{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{|\boldsymbol{r}-\boldsymbol{r}'|}dS \simeq \sum_{i=1}^{N}\frac{1}{4\pi}\int_{\Delta_i}\boldsymbol{M}^{PO}\frac{e^{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{|\boldsymbol{r}-\boldsymbol{r}'|}dS$$



• Surface integral calculation method for triangular face [2]

Ludwig's method calculate the surface integral by using a first-order approximation.

Assuming incident electromagnetic wave as plane wave with wave vector $\boldsymbol{k}_i$, incident electric field and magnetic field at origin as $\boldsymbol{E}_0$, $\boldsymbol{H}_0$, then the incident electromagnetic field $\boldsymbol{E}_i$, $\boldsymbol{H}_i$ are described by

$$\boldsymbol{E}_i(\boldsymbol{r}) = \boldsymbol{E}_0 \exp(-j\boldsymbol{k}_i\cdot\boldsymbol{r})$$

$$\boldsymbol{H}_i(\boldsymbol{r}) = \boldsymbol{H}_0 \exp(-j\boldsymbol{k}_i\cdot\boldsymbol{r})$$

[2] M. Dos Santos and N. Rabelo, "On the ludwig integration algorithm for triangular subregions," *Proceedings of the IEEE*, vol. 74, no. 10, pp. 1455{1456, 1986.

The vector potential is calculated as

Surface integration I

$$\boldsymbol{A}_\Delta(\boldsymbol{r}) = \frac{1}{4\pi}\int_{\Delta_i} \boldsymbol{J}^{PO}(\boldsymbol{r}')\frac{e^{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{|\boldsymbol{r}-\boldsymbol{r}'|}dS = \frac{(1-R)\hat{\boldsymbol{n}}\times\boldsymbol{H}_0}{4\pi}\int_{\Delta_i}\frac{e^{\overbrace{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')-j\boldsymbol{k}_i\cdot\boldsymbol{r}'}^{\boldsymbol{f}(\boldsymbol{r})}}}{\underbrace{|\boldsymbol{r}-\boldsymbol{r}'|}_{\boldsymbol{F}(\boldsymbol{r})}}dS$$

$$\boldsymbol{B}_\Delta(\boldsymbol{r}) = \frac{1}{4\pi}\int_{\Delta_i} \boldsymbol{M}^{PO}(\boldsymbol{r}')\frac{e^{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{|\boldsymbol{r}-\boldsymbol{r}'|}dS = \frac{(1+R)\boldsymbol{E}_0\times\hat{\boldsymbol{n}}}{4\pi}\int_{\Delta_i}\frac{e^{-j\boldsymbol{k}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')-j\boldsymbol{k}_i\cdot\boldsymbol{r}'}}{|\boldsymbol{r}-\boldsymbol{r}'|}dS$$

Then the surface integration can be expressed as

$$I = \int_{\Delta_i} F(\boldsymbol{r})e^{-jf(\boldsymbol{r})}dS$$

According to the Ludwig algorithm, the functions $F(r)$ and $f(r)$ are approximated by the linear functions

$$F(\boldsymbol{r}) = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

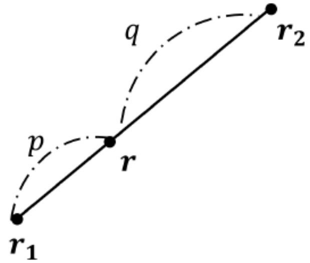
$$f(\boldsymbol{r}) = \boldsymbol{k}_s\cdot\boldsymbol{r} + (\boldsymbol{k}_i-\boldsymbol{k}_s)\cdot\boldsymbol{r}'$$

$\Rightarrow$

$$F(\boldsymbol{r}) \simeq Ax+By+Cz+D = \begin{bmatrix}A\\B\\C\end{bmatrix}\cdot\boldsymbol{r}+D$$

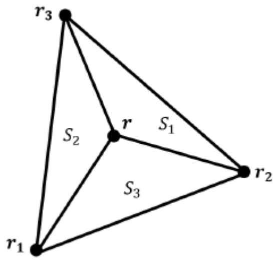
$$f(\boldsymbol{r}) \simeq \alpha x+\beta y+\gamma z+\zeta = \begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}\cdot\boldsymbol{r}+\zeta$$

first order approximation



Assume internally dividing point r which divides $r_1 r_2$ into p and q,

$$\boldsymbol{r} = \frac{q\boldsymbol{r}_1 + p\boldsymbol{r}_2}{p+q}$$



$s_1 = S_1/S$

$s_2 = S_2/S$

Similarly, r divides the total area S into S_1, S_2 and S_3, then

$$\boldsymbol{r} = \frac{S_1\boldsymbol{r}_1 + S_2\boldsymbol{r}_2 + S_3\boldsymbol{r}_3}{S} = s_1\boldsymbol{r}_1 + s_2\boldsymbol{r}_2 + (1 - s_1 - s_2)\boldsymbol{r}_3$$

Using this area coordinate, we have

$$F_a(s_1, s_2) := F(\boldsymbol{r}(s_1, s_2)) = A's_1 + B's_2 + C'$$
$$f_a(s_1, s_2) := f(\boldsymbol{r}(s_1, s_2)) = \alpha's_1 + \beta's_2 + \gamma'$$

where the coefficients A', B', C', α', β' and γ' are determined by point-matching of $F(r)$ and $f(r)$ at the triangle vertices r_1, r_2 and r_3.

The surface integration is transformed as

$$\boxed{I = \int_{\Delta_i} F(\boldsymbol{r})e^{-jf(\boldsymbol{r})}dS} \Rightarrow \boxed{I = 2S\int_0^1\int_0^{1-s_1} F_a(s_1, s_2)e^{-jf_a(s_1,s_2)}ds_2\, ds_1}$$

easier to calculate

PO simulation

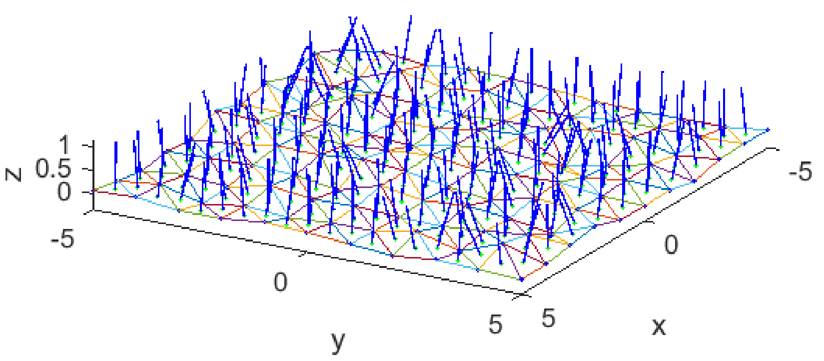
Step 1: set parameter

Step 2: generate a Gaussian surface with standard deviation and correlation length, and the surface is divided into multiple triangular surface elements. The gravity and normal vector of each surface element as well as the vertexes of the faces was obtained.

Step 3: set the observation point. For each triangular face of mesh, first do the illumination judgement. If it is illuminated, calculate the surface integration by Ludwig's method, based on which the scattered electric field at the observation point is obtained.

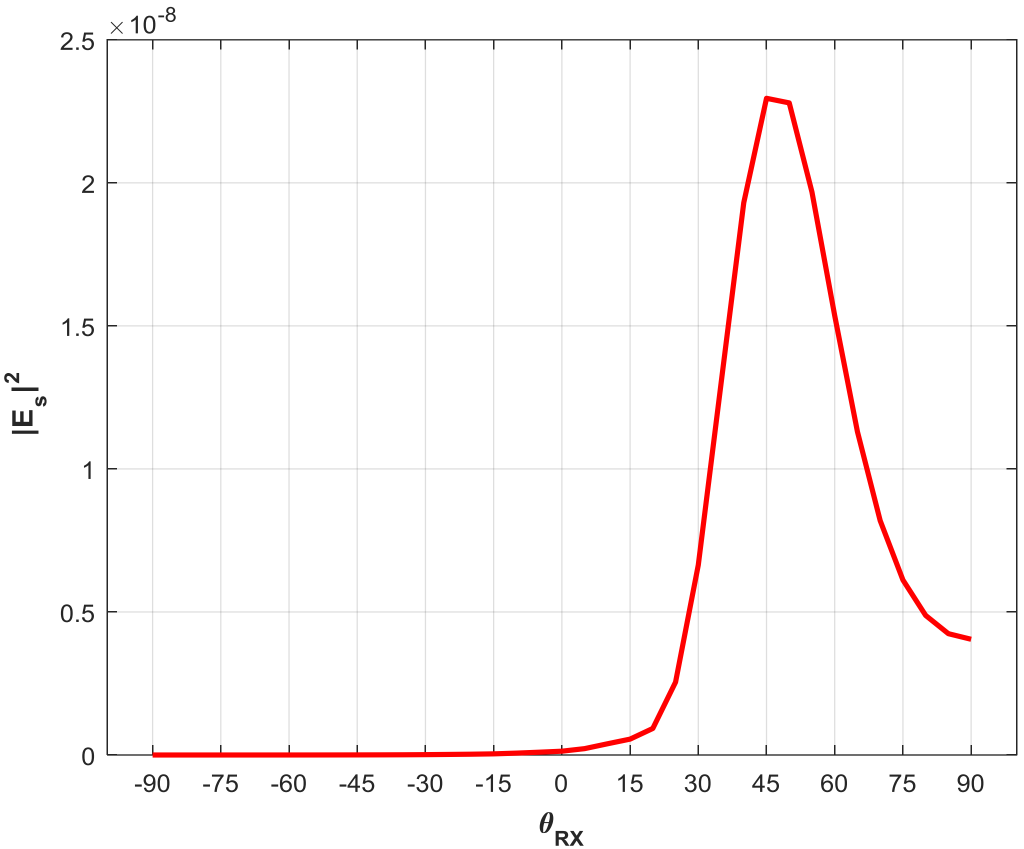
Step 4: calculate the scattered electric field for different observation points and show the scattering pattern.

The Monte Carlo approach was utilized. Scattering from deterministic surface which follows desired randomness is calculated multiple times. The scattering from random surface is calculated by taking a mean of those multiple trials.



Simulation results

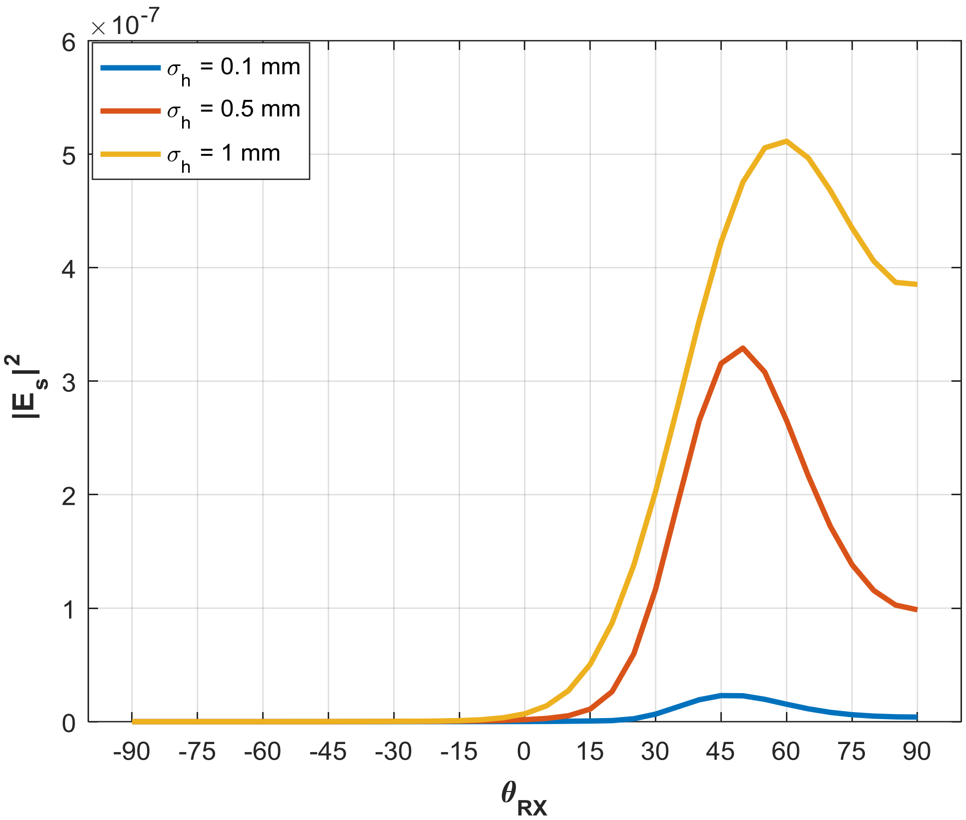
In the simulation, we generated a Gaussian surface with the size $l_x = l_y = 1m$, the value of roughness σ_h is 0.1mm, the correlation length is $l_{corr} = 5mm$, the frequency is set as $f_c = 100GHz$, the incident angle is $\theta_1 =$ -45°.



The incident angle is -45°.

From the simulation result, the maximum scattering lobe is steered to the direction of specular reflection.

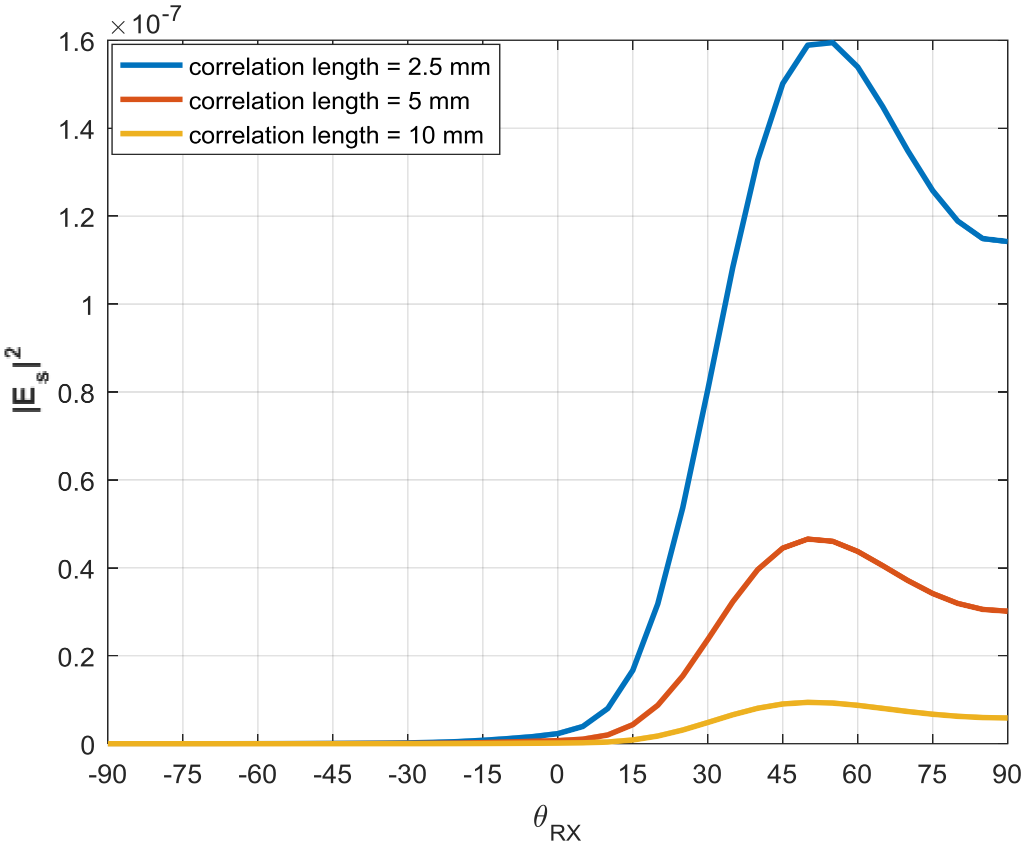
Comparison of Monte Carlo results by RMS height conditions



Surface material	PEC
Surface model	Gaussian correlated
Number of samples of Monte Carlo simulation	30
Surface size $L \times L$	$100cm \times 100cm$
Number of discrete points $N \times N$	100×100
RMS height σ_h	$0.1, 0.5, 1mm$
Correlation length l_{corr}	$10\ mm$
Frequency	$100\ GHz$
Wavelength λ	$3\ mm$

When σ_h increases, more power will be scattered, and the maximum scattering lobe will be away from the specular direction.

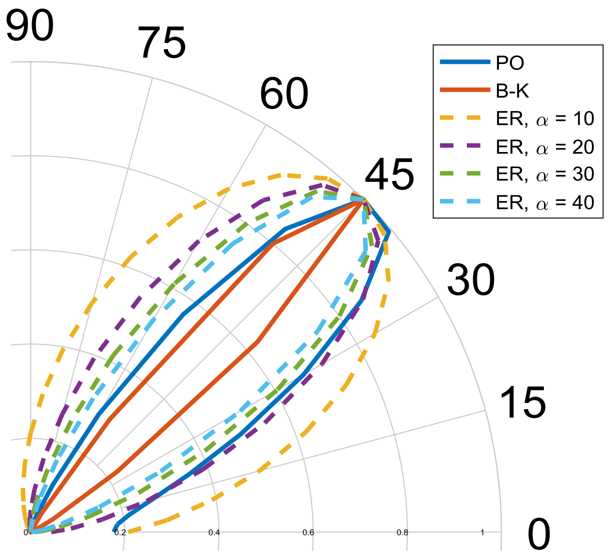
Comparison of Monte Carlo results by correlation length conditions

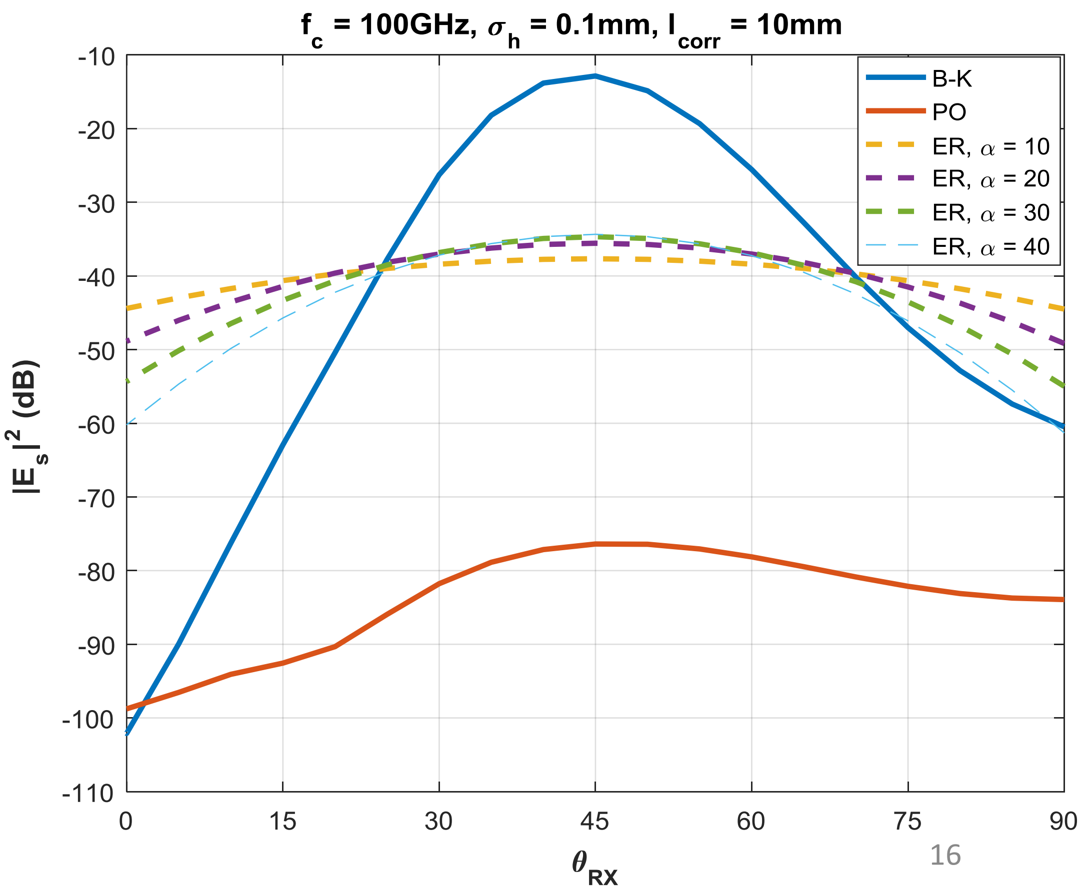


Surface material	PEC
Surface model	Gaussian correlated
Number of sample of Monte Carlo simulation	30
Surface size $L \times L$	$100cm \times 100cm$
Number of discrete points $N \times N$	100×100
RMS height σ_h	$0.1mm$
Correlation length l_{corr}	$\lambda/2$, λ, 2λ
Frequency	$60\ GHz$
Wavelength λ	$5\ mm$

When l_{corr} decreases, more power will be scattered.

PO results compared with ER and BK model

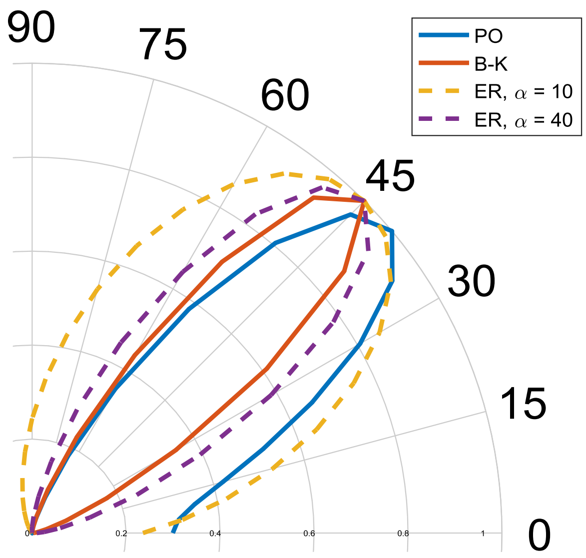


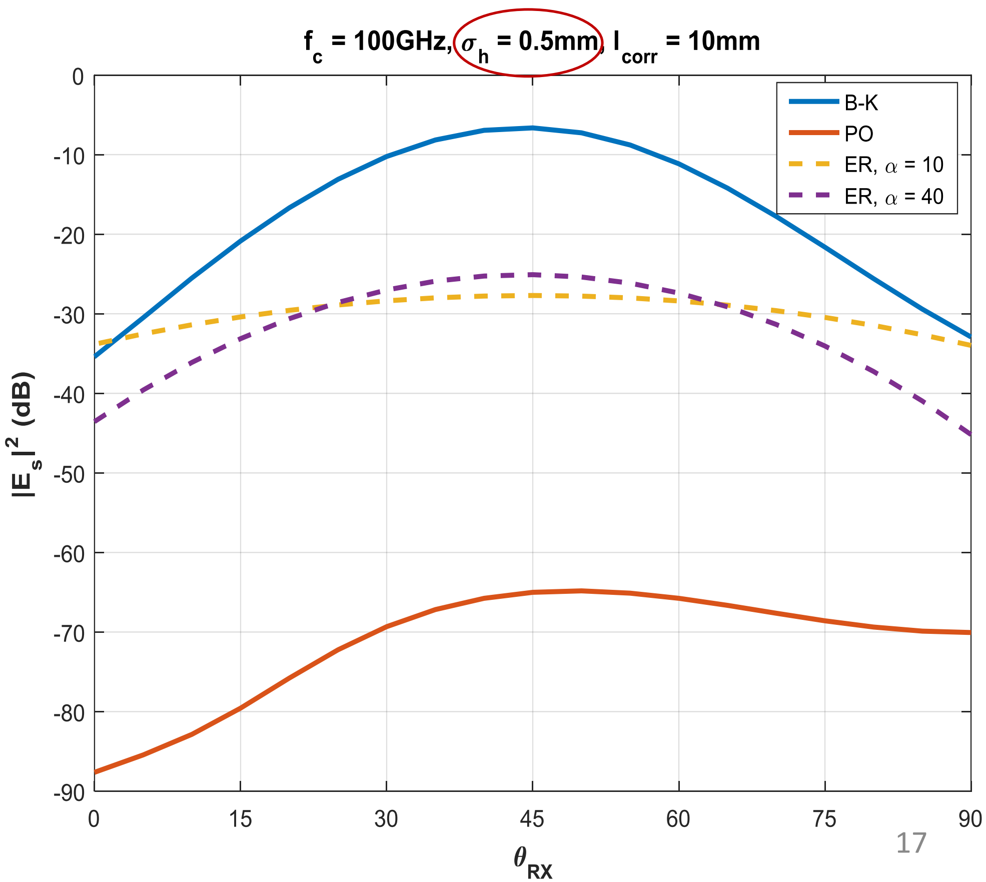


For scattering pattern, B-K model has the narrowest lobe, when α increases in ER model, it gets closer to PO.

For $|E_s|^2$, the B-K model has relatively the highest value, and PO the lowest.

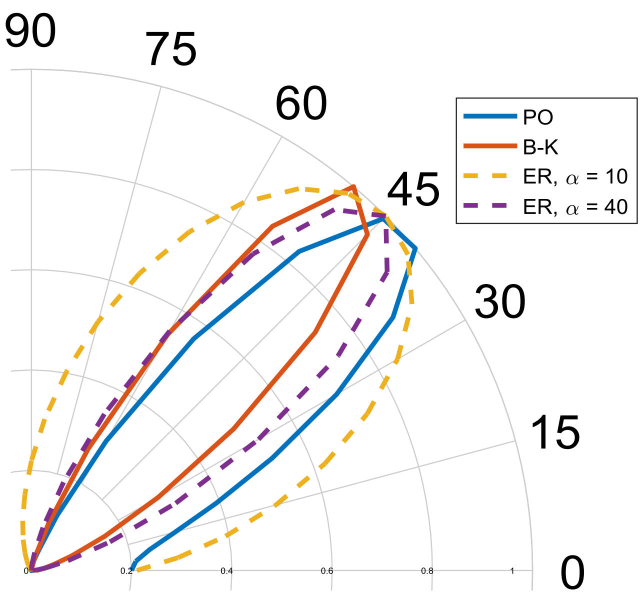
PO results compared with ER and BK model (increase σ_h)



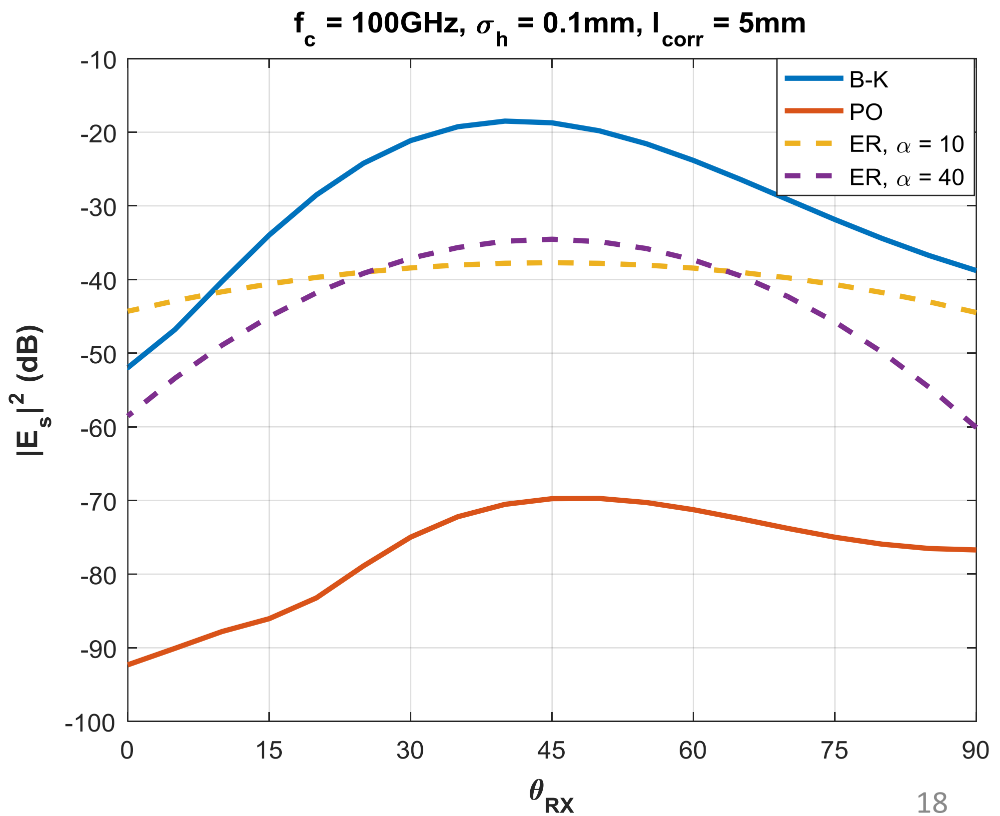


The scattering lobe of B-K model becomes wider, and the maximum lobe of PO gets away from the specular direction.

PO results compared with ER and BK model (decrease l_{corr})



For the scattering pattern, the maximum lobe of B-K model gets away from the specular direction.



Conclusion

- Review the Physical Optics (PO).
- Simulate the PO model, and analyze the impact of different parameters.
- Compare the simulation results with ER model and B-K model.

Future plan

- Recheck the simulation procedure of the three scattering models.
- Further compare the three scattering models and analyze the simulation results.
- Prepare for the measurement in the anechoic chamber.

Propagation Laboratory
Tokyo Tech.

Thank you for your attention!

Appendix

The general Beckmann-Kirchhoff (B-K) solution for scattering from rough surfaces [3]

- The rough surface is given by $\zeta = \zeta(x, y)$. The incident field is E_1 and the scattered field E_2. Assume E_1 to be linearly polarized. The quantities associated with vertical polarization will be denoted by the superscript "+" (e.g. E_1^+), and associated with horizontal polarization by the superscript "-".
- The propagation vector $\mathbf{k}_1 = \frac{2\pi}{\lambda}\frac{\mathbf{k}_1}{k_1}$, and $k_2 = |\mathbf{k}_2| = |\mathbf{k}_1| = k = 2\pi/\lambda$.
- Let P be the point of observation and R' be the distance from P to a point $(x, y, \zeta(x, y))$ on the surface S. The scattered field E_2 at point P is given by the Helmholtz integral

$$E_2(P) = \frac{1}{4\pi}\iint_S \left(E\frac{\partial\psi}{\partial n} - \psi\frac{\partial E}{\partial n}\right) dS$$

where $\psi = \dfrac{e^{ik_2R'}}{R'}$.

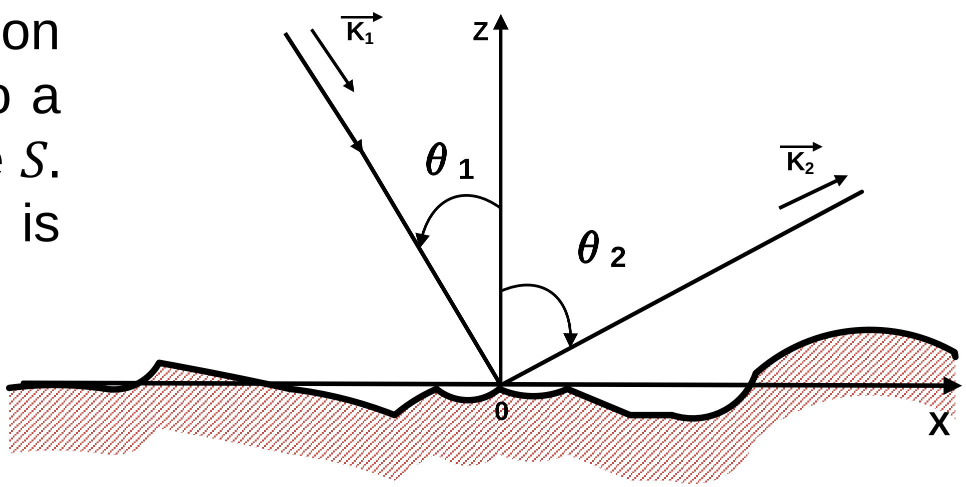


Fig . Basic notation.

[3] Beckmann, P., & Spizzichino, A. (1963). The scattering of electromagnetic waves from rough surfaces. *Norwood, MA, Artech House, Inc., 1963, 503 p.*

- In order to deal with plane scattered waves, let $R' \to \infty$, then

$$k_2 R' = k_2 R_0 - \mathbf{k}_2 \cdot \mathbf{r},$$

where R_0 is the distance of P from the origin (Fig.5), so that

$$\psi = \frac{e^{ik_2 R_0 - i\mathbf{k}_2 \cdot \mathbf{r}}}{R_0}.$$

- To know the values of E and $\partial E/\partial n$, we approximate the field at any point of the surface by the field that would be present on the tangent plane at that point, thus

$$(E)_S = (1 + R)E_1$$

and

$$\left(\frac{\partial E}{\partial n}\right)_S = (1 - R)E_1 \mathbf{k}_1 \cdot \mathbf{n}$$

where $\mathbf{n}$ is the normal to the surface and R is the reflection coefficient of a smooth plane.

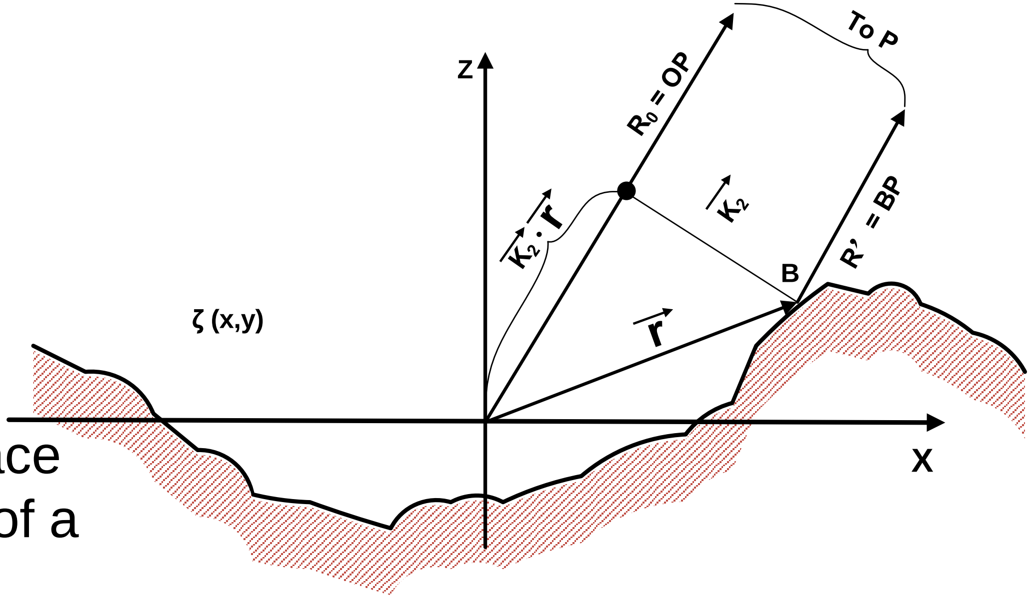


Fig . Derivation of $k_2 R'$.

- Note that the incident angle which the reflection coefficient R depends on is the “local” angle of incident (Fig 6).
- For a one-dimensionally rough surfaces $\zeta(x)$ extending from $x = -L$ to $x = L$, we may finally derive the scattered field as

$$E_2 = \frac{ike^{ikR_0}}{4\pi R_0} \int_{-L}^{L} (a\zeta' - b)e^{iv_x x + iv_z \zeta}\, dx$$

where

$$a = (1 - R)\sin\theta_1 + (1 + R)\sin\theta_2,$$

$$b = (1 + R)\cos\theta_2 - (1 - R)\cos\theta_1,$$

$$\boldsymbol{v} = \mathbf{k}_1 - \mathbf{k}_2 = v_x \mathbf{x}_0 + v_z \mathbf{z}_0.$$

- For simplicity we normalize E_2 by introducing a scattering coefficient $\rho = E_2 / E_{20}$, where E_{20} is the field reflected in the direction of specular reflection by a smooth, perfectly conducting plane of the same dimension. Hence

$$\rho = \frac{1}{4L\cos\theta_1} \int_{-L}^{L} (a\zeta' - b)e^{iv_x x + iv_z \zeta}\, dx$$

This is the general solution for one-dimensionally rough surface of finite conductivity. But a and b are in general rather complicated functions of x.

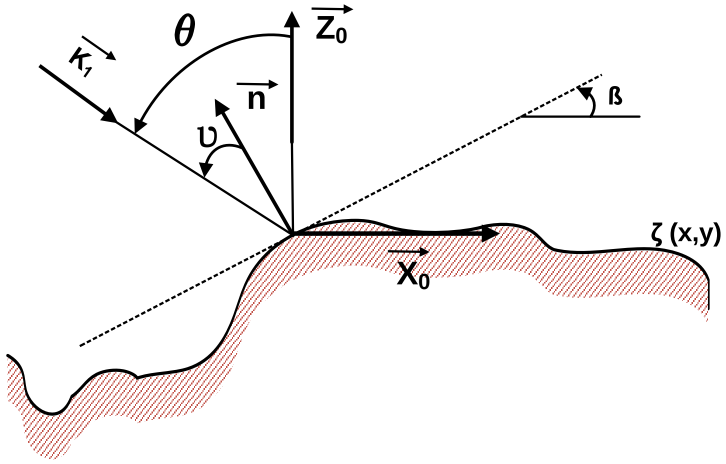


Fig . The “local” scattering geometry.

Special cases

- 1. Smooth surface ($\zeta = 0$)

$$\rho = -\frac{(1+R)\cos\theta_2 - (1-R)\cos\theta_1}{2\cos\theta_1}\,\mathrm{sinc}\, v_x L,$$

For horizontal polarization $R^- = -1$, so the scattering coefficient reduces to

$$\rho_0 = \mathrm{sinc}\, v_x L.$$

As $\lambda/L \to 0$, the side lobs will concentrate in the direction $\theta_2 = \theta_1$, making $\rho_0 = 1$ for $\theta_2 \approx \theta_1$ and $\rho_0 = 0$ in all other directions.

- 2. Perfect conducting surface ($R^+ = 1,\ R^- = -1$)

In this case, a and b are independent of x. After elementary transformations,

$$\rho^{\pm}(\theta_1, \theta_2) = \pm \sec\theta_1 \frac{1+\cos(\theta_1, \theta_2)}{\cos\theta_1 + \cos\theta_2} \frac{1}{2L} \int_{-L}^{L} e^{i\mathbf{v}\cdot\mathbf{r}} dx + \frac{e^{\pm}(L)}{2L},$$

where

$$\mathbf{v}\cdot\mathbf{r} = \frac{2\pi}{\lambda}[(\sin\theta_1 - \sin\theta_2)x - (\cos\theta_1 + \cos\theta_2)\zeta(x)],$$

$$e^{\pm}(L) = \frac{i\sec\theta_1 \sin\theta^{\pm}}{k(\cos\theta_1 + \cos\theta_2)} e^{i\mathbf{v}\cdot\mathbf{r}(x)}\Big|_{-L}^{L},$$

with

$$\theta^+ = \theta_2,\ \theta^- = \theta_1.$$

For random rough surfaces: One-dimension case

- The rough surface is given by $\zeta(x)$. Let $\zeta(x)$ be a random variable assuming values z with a probability density $w(z)$. Let the mean value be $\langle \xi \rangle = 0$.
- The characteristic function $\chi(v_z)$ associated with the distribution $w(z)$ is

$$\chi(v_z) = \left\langle e^{iv_z\xi} \right\rangle = \int_{-\infty}^{\infty} w(z)\, e^{iv_z z} dz.$$

- According to the general solution for one-dimensionally rough surface of perfect conductivity

$$\rho^{\pm}(\theta_1, \theta_2) = \pm \sec\theta_1 \frac{1+\cos(\theta_1+\theta_2)}{\cos\theta_1+\cos\theta_2} \frac{1}{2L} \int_{-L}^{L} e^{i\mathbf{v}\cdot\mathbf{r}} dx + \frac{e^{\pm}(L)}{2L}, \qquad (1)$$

We find the relation

$$\langle \rho \rangle = \chi(v_z)\rho_0,$$

where $\rho_0 = \operatorname{sinc} v_x\, L$.

$\langle \rho \rangle$ will equal zero for any direction of scattering except a narrow wedge about specular direction;

$\langle \rho \rangle = 0$ can not infer $\langle |\rho| \rangle = 0$ as ρ is complex.

- When a surface is generated by superposition of several random variables $\zeta = \xi_1 + \xi_2 + \cdots$, and $\chi_1, \chi_2, \cdots$ are the characteristic functions of $\xi_1,\ \xi_2, \cdots$ respectively, then

$$\langle\rho\rangle = \chi_1 \chi_2 \cdots \rho_0.$$

- As ρ is a complex quantity, its mean value is of little use except as a steeping stone to determine the mean value of $|\rho| = \sqrt{\rho\rho^*}$.

- Note that $\langle\rho\rho^*\rangle$ is related to the mean scattered power, the variance of ρ, and to the variance of the scattered field. Thus we can find all the quantities of interest by determining the value $\langle\rho\rho^*\rangle$.

- For $L \gg \lambda$, we can neglect the edge effect and (1) becomes $\rho = \frac{F_2}{2L}\int_{-L}^{L} e^{i\mathbf{v}\cdot\mathbf{r}} dx$, where $F_2 = \sec\theta_1 \frac{1+\cos(\theta_1+\theta_2)}{\cos\theta_1+\cos\theta_2}$. Then we have

$$\langle\rho\rho^*\rangle = \frac{F_2^2}{4L^2}\int_{-L}^{L}\int_{-L}^{L} e^{\,iv_x(x_1-x_2)}\left\langle e^{\,iv_z(\zeta_1-\zeta_2)}\right\rangle dx_1 dx_2,$$

where $\zeta_1 = \zeta(x_1), \zeta_2 = \zeta(x_2)$.

- And $\left\langle e^{\,iv_z(\zeta_1-\zeta_2)}\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(z_1, z_2) e^{\,iv_z(\zeta_1-\zeta_2)}\, dz_1 dz_2 = \chi_2(v_z, -v_z)$ is the two-dimensional characteristic function of the distribution $W(z_1, z_2)$. When the separation parameter $\tau = x_1 - x_2$ is small, ζ_1 and ζ_2 will be correlated. Therefore in addition to $w(z)$, its correlation function is also needed to determine $W(z_1, z_2)$.

- Its normalized correlation function (autocorrelation coefficient) is defined as

$$C(\tau) = \frac{\langle\zeta_1\zeta_2\rangle - \langle\zeta_1\rangle\langle\zeta_2\rangle}{\langle\zeta_1^2\rangle - \langle\zeta_1\rangle^2}.$$

- Let the distance in which $C(\tau)$ drops to e^{-1} be T, which is called "correlation distance".
- Finally we obtain the variance of ρ as

$$D\{\rho\} = \langle\rho\rho^*\rangle - \langle\rho\rangle\langle\rho^*\rangle$$

$$= \frac{F_2^2}{4L^2}\int_{-L}^{L}\int_{-L}^{L} e^{\,iv_x(x_1-x_2)}\,[\chi_2(v_z,-v_z) - \chi(v_z\,)\chi^*(v_z)]dx_1dx_2 \qquad (2)$$

- ζ_1 and ζ_2 and independent for all but very small τ, hence the square bracket in (2) will vanish for all but very small τ. After elementary manipulations we have

$$\boxed{D\{\rho\} = \frac{F_2^2}{2L}\int_{-L}^{L} e^{\,iv_x\tau}[\chi_2(v_z,-v_z) - \chi(v_z\,)\chi^*(v_z)]d\tau} \qquad (3)$$

general results

with the only significant contributions to this integral coming from the region near $\tau = 0$.

The normally distributed surface

- Let ζ be distributed normally with mean value $\langle\xi\rangle = 0$ and standard deviation σ. The characteristic function associated with the normal distribution is $\chi(v) = \exp\left(-\frac{1}{2}\sigma^2 v_z^2\right)$. We take as the general autocorrelation coefficient the function

$$C(\tau) = e^{-\tau^2/T^2} \tag{4}$$

where T is the correlation distance.

- We know $\langle\rho\rangle = \chi(v_z)\rho_0 = \rho_0 \exp\left[-\frac{2\pi\sigma^2}{\lambda^2}(\cos\theta_1 + \cos\theta_2)^2\right]$.
- The characteristic function of this distribution is

$$\chi_2(v_z, -v_z) = exp[-v_z^2\sigma^2(1 - C)]. \tag{5}$$

Substituting (4) in this expression and expanding (5) in an exponential series and let $g = v_z^2\sigma^2$ we have

$$\chi_2(v_z, -v_z) = e^{-g}\sum_{m=0}^{\infty}\frac{g^m}{m!}e^{-m\tau^2/T^2} \tag{6}$$

- Take the case of a surface rough in one dimension only. Substituting (6) in (3), we obtain

$$D\{\rho\} = \frac{F_2^2}{2L}\int_{-\infty}^{\infty} e^{\,iv_x\tau - g}\sum_{m=1}^{\infty}\frac{g^m}{m!}e^{-m\tau^2/T^2}d\tau.$$

Using the integral

$$\int_{-\infty}^{\infty} e^{-at^2}\cos bt\,dt = \sqrt{\left(\frac{\pi}{a}\right)}e^{-b^2/4a}(a>0)$$

We then find

$$D\{\rho\} = \frac{\sqrt{\pi}\cdot F_2^2 T}{2L}e^{-g}\sum_{m=1}^{\infty}\frac{g^m}{m!\sqrt{m}}e^{-v_x^2T^2/4m} \quad (7)$$

- To analyze the dependence of $D\{\rho\}$ on g, v_x and T, we therefore consider three cases: (A) $g \ll 1$, (B) $g \approx 1$, (C) $g \gg 1$. Since $\sqrt{g}$ is proportional to σ/λ, these three cases correspond to a slightly, moderate and very rough surface.

(A) $g \ll 1$: slightly rough surface

The series (7) will converge so quickly that we may take only its first term and find

$$D\{\rho\} = \frac{\sqrt{\pi}\cdot gF_2^2 T}{2L} e^{-v_x^2 T^2/4} \quad (g \ll 1).$$

(B) $g \approx 1$: moderate rough surface

$$\frac{\sqrt{\pi}\cdot F_2^2 T}{2L} e^{-g-v_x^2 T^2/4} \leq D\{\rho\} < \frac{\sqrt{\pi}\cdot F_2^2 T}{2L}.$$

(C) $g \gg 1$ very rough surface

$$D\{\rho\} = \frac{F_2^2 T\sqrt{\pi}}{L v_z \sigma} exp\left(-\frac{v_x^2 T^2}{4 v_z^2 \sigma^2}\right) \quad (g \gg 1).$$

- The mean scattered power is obtained by (7) as

$$\langle\rho\rho^*\rangle = D\{\rho\} + \langle\rho\rangle\langle\rho^*\rangle = \left(\underbrace{\rho_0^2}_{\text{Specular reflection}} + \underbrace{\frac{\sqrt{\pi}\cdot F_2^2 T}{2L}\sum_{m=1}^{\infty}\frac{g^m}{m!\sqrt{m}} e^{-v_x^2 T^2/4m}}_{\text{Diffusely scattered field}}\right) e^{-g} \quad (8).$$

Special cases are

$$\langle\rho\rho^*\rangle = e^{-g}\rho_0^2 + \frac{\sqrt{\pi}\cdot gF_2^2 T}{2L} e^{-v_x^2 T^2/4} \quad (g \ll 1) \qquad (9)$$

$$\langle\rho\rho^*\rangle = \frac{F_2^2 T\sqrt{\pi}}{L v_z \sigma} exp\left(-\frac{v_x^2 T^2}{4 v_z^2 \sigma^2}\right) \quad (g \gg 1)$$

In conclusion

B-K model

$$\mathrm{E}\{\rho\rho^*\}_\infty = \left(\frac{\pi l_{corr}^2 F^2}{dS}\sum_{m=1}^{\infty}\frac{g^m}{m!\sqrt{m}}e^{-\frac{v_{xy}^2 l_{corr}^2}{4m}}\right)e^{-g},$$

$$\mathrm{E}\{\rho\rho^*\}_{finite} = \mathrm{E}\{\Gamma\Gamma^*\}\cdot\mathrm{E}\{\rho\rho^*\}_\infty,$$

$$\mathrm{E}\{R_{power}\} = \frac{4dS^2\cos^2(\Theta_1)}{\lambda^2 r_2^2}\mathrm{E}\{\rho\rho^*\}_{finite}.$$

ER model

$$\left|E_S^{Dir}\right|^2 = \left(\frac{KS}{|\boldsymbol{r}_i||\boldsymbol{r}_s|}\right)^2 \frac{\mathrm{dS}\cos\theta_i}{F_\alpha}\left(\frac{1+\cos\Psi_R}{2}\right)^\alpha,$$

$$F_\alpha = \int_0^{\pi/2}\int_0^{2\pi}\left(\frac{1+\cos\Psi_R}{2}\right)^\alpha \sin\theta_s d\theta_s d\varphi_s,$$

$$S = \sqrt{1-\rho^2}|R|,$$

$$\rho = \exp\{-\frac{1}{2}\left(\frac{4\pi\sigma_h\cos\theta_i}{\lambda}\right)\}.$$