University of Illinois at Urbana-Champaign

ECE486
Control Systems

Final Project: The Reaction Wheel Pendulum

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Contents

1 Introduction

In this final lab, a series of controllers is designed and implemented to control the movement of the Reaction Wheel Pendulum (RWP). RWP consists of two optical encoders which are set on the pendulum arm and wheel for the measurement of angles. (φ_p and φ_r). For the actuator, we have a 24-V, permanent magnet DC motor mounted on the pendulum. The equilibrium position is set to π (0 for initial position where pendulum arm is unforced by motor and freely hangs under gravity, π for pendulum stands vertically in air by motor force). Our use simulink module in matlab to design controllers and implement the control to the RWP.

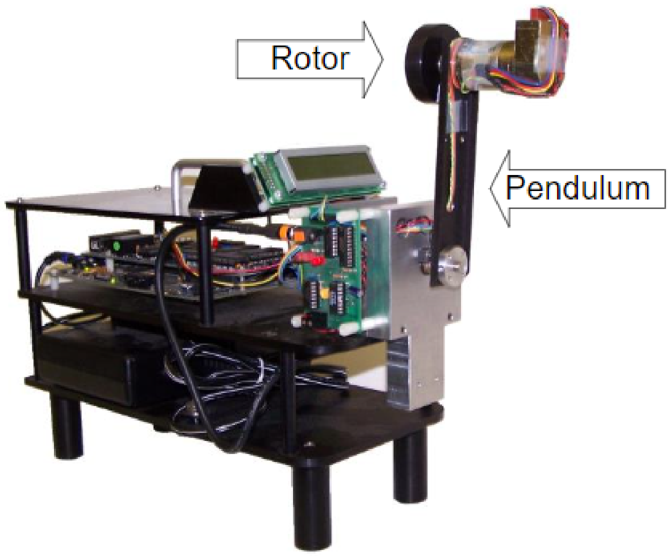

Figure 1: The Reaction Wheel Pendulum

2 Mathematical Model

2.1 Derivation of Differential Equations from Lagrangian

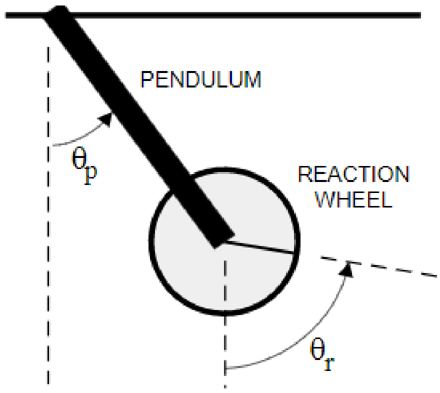

Figure 2: Schematic Diagram

We have defined several parameters from the lab manual:

m_p: mass of the pendulum and motor housing/stator

m_r: mass of the rotor

$m_{combined}$: mass of rotor and pendulum

J_p: moment of inertia of the pendulum about its center of mass

J_r: moment of inertia of the rotor about its center of mass

l_p: distance from pivot to the center of mass of the pendulum

The calculations of the kinetic energy, potential energy, Lagrangian, and Lagrange equations are shown below.

$KE_{pendulum+rotor} = \frac{1}{2}J(\frac{\mathrm{d}\theta_p}{\mathrm{d}t})^2$

$PE_{pendulum+rotor} = mgl(1-cos\theta_p)$

$KE_{rotor} = \frac{1}{2}J_r(\frac{\mathrm{d}\theta_r}{\mathrm{d}t})^2$

$PE_{rotor} = 0$

$L_{pendulum+rotor} = \frac{1}{2}J(\frac{\mathrm{d}\theta_p}{\mathrm{d}t})^2 - mgl(1-cos\theta_p)$

$L_{rotor} = \frac{1}{2}J_r(\frac{\mathrm{d}\theta_r}{\mathrm{d}t})^2$

$LagrangeEquation_{pendulum+rotor} : \ddot{\theta_p} + \frac{mgl}{J}sin\theta_p = \frac{-Ki}{J}$

$LagrangeEquation_{rotor} : J_r\ddot{\theta_r} = Ki$

2.2 Linearization into State Space Form

We rewrite the two motion equations into State Space form according to formulas. For initial condition, we only have $\vartheta_p = \pi$ for equilibrium position. Thus $x_2 = \vartheta_p{}' = 0$.

State-Space form:

$$\begin{cases} \ddot{\theta_p} = asin\theta_p = -b_p u \\ \ddot{\theta_r} = b_r u \end{cases}$$

$\dot{x} = Ax + Bu$

$x = [\theta_p \quad \ddot{\theta_p} \quad \theta_r \quad \ddot{\theta_r}]^T$

$$\dot{x} = \begin{bmatrix} \dot{x_1} \\ \ddot{\theta_p} \\ \dot{x_3} \\ \ddot{\theta_r} \end{bmatrix} = \begin{bmatrix} x_2 \\ -b_p u - asin\theta_p \\ x_4 \\ b_r u \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{\mathrm{d}x_2}{\mathrm{d}x_1} & \frac{\mathrm{d}x_2}{\mathrm{d}x_2} & \frac{\mathrm{d}x_2}{\mathrm{d}x_3} & \frac{\mathrm{d}x_2}{\mathrm{d}x_4} \\ \frac{\mathrm{d}(-b_p u - asin\theta_p)}{\mathrm{d}x_1} & 0 & 0 & 0 \\ \frac{\mathrm{d}x_4}{\mathrm{d}x_1} & \frac{\mathrm{d}x_4}{\mathrm{d}x_2} & \frac{\mathrm{d}x_4}{\mathrm{d}x_3} & \frac{\mathrm{d}x_4}{\mathrm{d}x_4} \\ \frac{\mathrm{d}(-b_r u)}{\mathrm{d}x_1} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\mathrm{d}x_2}{\mathrm{d}u} \\ \frac{\mathrm{d}(-b_p u - asin\theta_p)}{\mathrm{d}u} \\ \frac{\mathrm{d}x_4}{\mathrm{d}u} \\ \frac{\mathrm{d}(b_r u)}{\mathrm{d}u} \end{bmatrix} = \begin{bmatrix} 0 \\ -b_p \\ 0 \\ b_r \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \partial\dot{\theta_p} \\ \partial\ddot{\theta_p} \\ \partial\dot{\theta_r} \\ \partial\ddot{\theta_r} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial\theta_p \\ \partial\dot{\theta_p} \\ \partial\theta_r \\ \partial\dot{\theta_r} \end{bmatrix} + \begin{bmatrix} 0 \\ -b_p \\ 0 \\ b_r \end{bmatrix} * u$$

3 Full State Feedback Control with Friction Compensation

3.1 Development of PD Control

Friction Compensation can be calculated with tests on different radian velocity and different directions. In open loop, we keep pendulum arm fixed and give different angular velocity on wheel to collect data and linearize it. The slope of each linear function in two direction is our b+ and b-, while the cross point on y axis is our c+ and c-.

We implement velocity of 15, 20, 25, 30, 40, 50. Our positive speed is 0.9834,0.9842,1.042, 1.08, 1.179,1.256. Thus our approximation positive direction function is f(x) = 0.0089x + 0.823. Our negative direction speed is -0.933,-1.043, -1.090,-1.104,-1.216,-1.313. Our approximation negative direction function is f(x) = 0.0095x -0.824. Friction Compensators benefits our feedback controller, since under compensation, our pendulum achieve equilibrium position with less swing around equilibrium point. Compensators help controller to enforce less torque to pull pendulum back to equilibrium position when pendulum rises over equilibrium point.

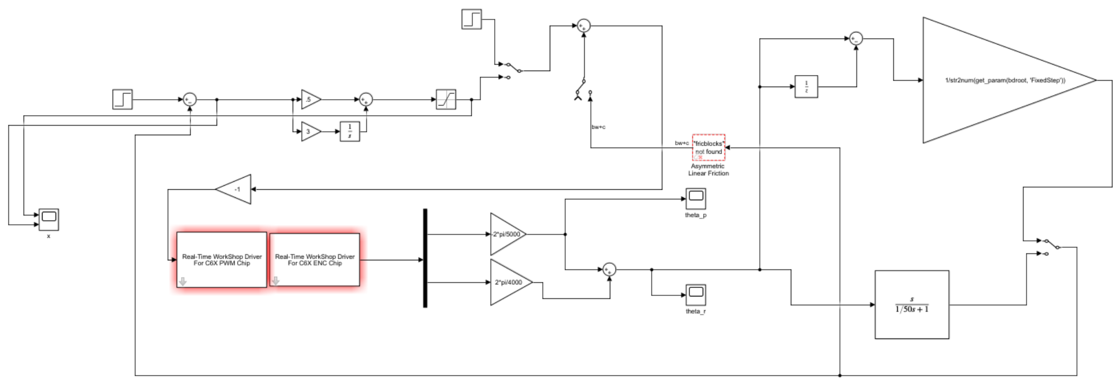

Figure 3: Simulink Diagram (2e)

3.2 Mathematical Proof

In inverted position, $\theta_p(inv) = \bar{n}$ (as target value).

$$\begin{cases} \partial\theta_p = \theta_p - \pi \\ \partial\theta_r = \theta_r \end{cases}$$

In part Mathematical Model,

we have $\dot{x} = \begin{bmatrix} \partial\dot{\theta_p} \\ \partial\ddot{\theta_p} \\ \partial\dot{\theta_r} \\ \partial\ddot{\theta_r} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial\theta_p \\ \partial\dot{\theta_p} \\ \partial\theta_r \\ \partial\dot{\theta_r} \end{bmatrix} + \begin{bmatrix} 0 \\ -b_p \\ 0 \\ b_r \end{bmatrix} * u$

while a = 78 (w_{np}^2), b_p=0.8588, b_r=137.7 (determined in lab part 3)

We input a -Kx into system to check eigenvalues in order to test stability.

In matlab, our poles are set to -6.5+6.5j, -6.5-6.5j, 0, -6.5.

$$A - BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 78 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -8.8588 \\ 0 \\ 137.7 \end{bmatrix} \begin{bmatrix} -287.6 & -30.905 & 0 & -0.051 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -168.99 & -26.54 & 0 & -0.0438 \\ 0 & 0 & 0 & 1 \\ 39602 & 4255.6 & 0 & 7.023 \end{bmatrix}$$

so eigenvalue:

$$\begin{cases} \lambda_1 = -7.172 + 7.48i \\ \lambda_2 = -7.172 - 7.48i \\ \lambda_3 = -5.173 \\ \lambda_4 = 0 \end{cases}$$

, all in LHP, so this A-BK stable.

At equilibrium state, $\begin{bmatrix} \partial\dot{\theta_p} \\ \partial\ddot{\theta_p} \\ \partial\dot{\theta_r} \\ \partial\ddot{\theta_r} \end{bmatrix} = (A - BK) \begin{bmatrix} \partial\theta_p \\ \partial\dot{\theta_p} \\ \partial\theta_r \\ \partial\dot{\theta_r} \end{bmatrix} = 0$

Thus $\begin{bmatrix} \partial\theta_p \\ \partial\dot{\theta_p} \\ \partial\theta_r \\ \partial\dot{\theta_r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

since

$$\begin{cases} \partial\theta_p = \theta_p - \pi = 0 \\ \partial\theta_r = \theta_r = 0 \end{cases}$$

(from previous proof).

$$\begin{bmatrix} \theta_p \\ \dot{\theta_p} \\ \theta_r \\ \dot{\theta_r} \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ in stable equilibrium state.

In our diagram, we input $\theta_p - \pi, \dot{\theta_p}, \theta_r, \dot{\theta_r}$ into 4-1 MUX.

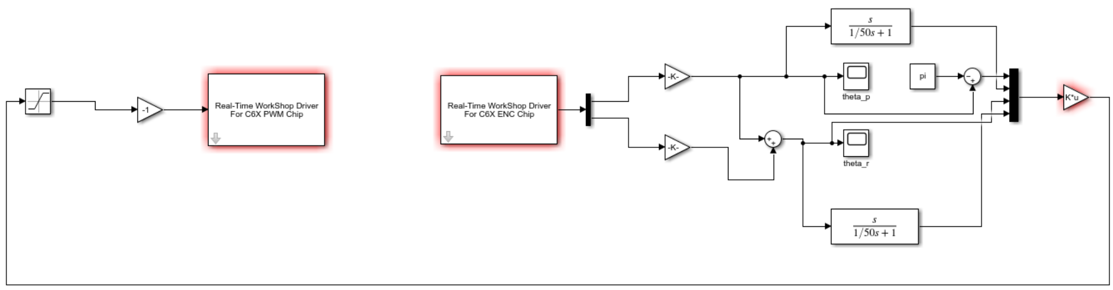

Figure 4: Simulink Diagram (4)

3.3 Simulink Simulation

The comparison between two-state feedback and three-state feedback in max IC deviations, max pulse, max disturbance is generated below by the Simulink simulations.

	Two-State Feedback	Three-State Feedback
Max IC deviations	0.91rad 0.94rad/s	0.91rad 1.02rad/s
Max pulse	4	6
Max disturbance	3.2	4.4

Table 1: Robustness Comparisons (2-state v.s. 3-state)

3.4 Windows Target Implementation

Using the Windows Target implementation and the controller we designed, the machine are able to hold its pendulum at top and keep at the equilibrium position. Also from the data from Table 1 above, we found that two-state feedback system are more robust than the three-state one due to more state to control.

4 Full State Feedback Control with Decoupled Observer

4.1 Observer Introduction

Observers are used to measure current accurate velocity and position.

From previous part, we got

$x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -b_p \\ 0 \\ b_r \end{bmatrix} * u$$

so $M = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, P = \begin{bmatrix} 0 \\ -b_p \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 0 \\ b_r \end{bmatrix}$

We can separate 4x4 matrix into multiply 1x4 matrix, which is

$$\begin{cases} \dot{x}_{1,2} = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} x_{1,2} \\ 0 \end{bmatrix} + Pu = Mx_{1,2} + Pu \\ \dot{x}_{3,4} = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} 0 \\ x_{3,4} \end{bmatrix} + Qu = Nx_{3,4} + Qu \end{cases}$$

The advantage of observer states is that it separate the system to be controlled by different poles in controller.

4.2 Mathematical Proof

Similiar to previous proof, we determine eigenvalues.

Start from $\dot{e}$=(A-LC)e, our new poles are set to -66.5+6.5j, -66.5-6.5j, -60, -66.5.

$$A - LC = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 78 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 130.017 & 0 \\ 4320.2 & 0 \\ 0 & 129.482 \\ 0 & 4211 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -130.017 & 1 & 0 & 0 \\ -4242.2 & 0 & 0 & 0 \\ 0 & 0 & -129.482 & 1 \\ 0 & 0 & -4211 & 0 \end{bmatrix}$$

so eigenvalue:

$$\begin{cases} \lambda_1 = -65 + 4i \\ \lambda_2 = -65 - 4i \\ \lambda_3 = -64.74 + 4.43i \\ \lambda_4 = -64.74 - 4.43i \end{cases}$$

, all in LHP, so this A-LC stable.

As $\dot{e}$=(A-LC)e=0 at equilibrium, since A-LC proves to be stable, e = $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ at equilibrium state.

4.3 Simulink Simulation

The comparison among two-state feedback, three-state feedback, and obserber in max IC deviations, max pulse, max disturbance is generated below by the Simulink simulations.

	Two-State Feedback		Three-State Feedback		Observer	
Max IC deviations	0.91rad	0.94rad/s	0.91rad	1.02rad/s	0.95rad	0.74rad/s
Max pulse	4		6		5	
Max disturbance	3.2		4.4		4.9	

Table 1: Robustness Comparisons (2-state v.s. 3-state v.s. observer)

4.4 Windows Target Implementation

Using the Windows Target implementation and the controller we designed, the machine are able to hold its pendulum at top and keep at the equilibrium position, just like the previous experiment. Compared to other designs, this kind of control have more sensitivity and is easier to make itself back to the equilibrium position.

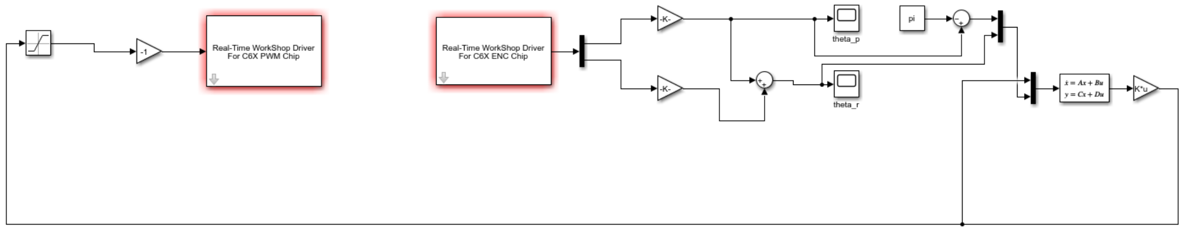

Figure 4: Simulink Diagram (5)

5 Conclusion

Throughout this final lab, we have designed several controllers (proportional control, two-state control, three-state control,observer control, up and down stabilizing control, swing-up control) according to the instructions of the lab manual. For all those controllers, our group prefer the observer control most. Although it is not as robust as other full-state feedback controllers, we still appreciate its performance based on the least responding time for it to achieve equilibrium, and the easier and methodical design process.