ECE 486 – Fall 2021
Final
Instructor: Bin Hu

Name: ______________________________

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Instructions:

- Please carefully read all questions.
- For maximum credit - attempt all questions and carefully elucidate your reasoning and answers.
- You can upload your solutions in one of three ways:
  1. You can download this PDF and write on it using a tablet (in MS OneNote for example).
  2. You can print this exam and write in the space provided by hand.
  3. You can write the solutions on your own worksheets/pages provided each question starts on a new page.
- The upload link to Gradescope will close at 10:00 p.m. C.D.T on 10th December. Make sure you allot time to scan/save and upload.
- Good luck!

1. (a) (5 points) Consider the following differential equation.

$$\dot{x}(t) + ax(t) = bu(t)$$

where $a \neq 0$. What is the general form of the free response? What is the general form of the forced response? What is the system response to the unit step input $u(t) = 1$ (for $t \geq 0$) given the initial condition $x(0) = \alpha$?

(b) (5 points) Consider a system with the transfer function:

$$H(s) = \frac{s(s+b_0)}{(s+a_0)(s+a_1)(s^2+2\zeta\omega_n s+\omega_n^2)}$$

Here, $b_0 > 0$, $a_0 > 0$, $a_1 > 0$, $0 < \zeta < 1$, $\omega_n > 0$.
What is the steady-state value of the system response to:

$$u(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Be sure to *justify your response* (e.g. why is your reasoning applicable?).

2. Consider the feedback loop in Figure 1 with the system given by $\ddot{y} - \dot{y} = u$.

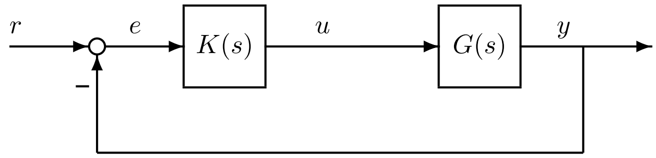


Figure 1: Feedback Loop

(a) (5 points) Consider a proportional-derivative control law $u(t) = K_p e(t) + K_d \dot{e}(t)$. What is the ODE that models the closed-loop dynamics from r to y? What is the transfer function $T(s)$ that corresponds to this ODE?

(b) (5 points) Choose K_p and K_d so that the closed-loop is stable, has a natural frequency of $\omega_n = 3$ rad/sec, and has a damping ratio of $\zeta = 0.5$.

(c) (5 points) For the gains computed in the previous part, what are the poles and zeros of the closed-loop transfer function $T(s)$?

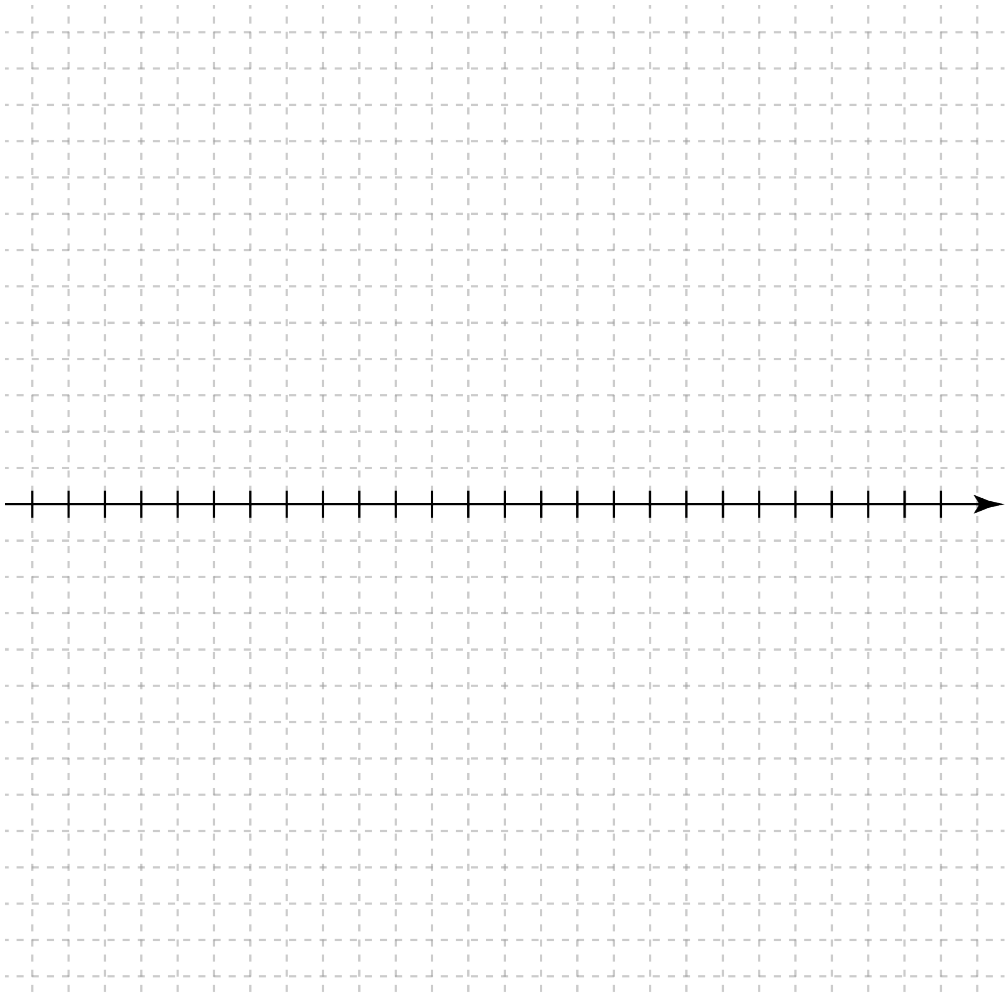
3. Consider the transfer function:

$$L(s) = \frac{s+6}{(s+2)(s+1)}$$

(a) (5 points) **On the next page**, sketch the root locus of $L(s)$ in the provided axes. In other words, plot the values of s that solve the equation below as K goes from 0 to $+\infty$.

$$1 + KL(s) = 0$$

Label all important points of your root locus, including zeros and poles of $L(s)$, the break points, and the asymptotic behavior. **Use this page for any calculations.**



(b) (5 points) Is the closed-loop system $\dfrac{KL(s)}{1+KL(s)}$ stable for all $K>0$?

(c) (5 points) Suppose $K = 3$. What is the gain margin? (Hint: Use the insight from part (b).)

4. (a) (5 points) Consider the system

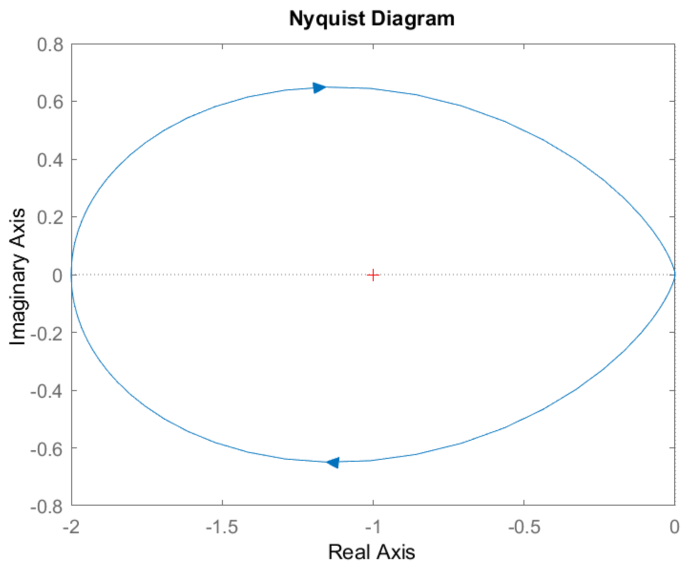
$$G(s) = \frac{10}{(5s+1)(2s+1)}$$

We wish to design a lead/lag controller that provides phase margin of at least 60°, and steady-sate tracking of constant references within 2%. Consider the lead/lag controller

$$K(s) = 4\frac{\frac{s}{0.8}+1}{\frac{s}{5}+1} \cdot \frac{s+0.05}{s+0.02},$$

compute the phase margin and steady-state tracking error. Are the design requirements met? For this question, you are allowed to generate Bode plots using `Matlab`.

(b) (5 points) Consider the open-loop transfer function $L = \frac{20}{s-10}\frac{100}{s^2+20s+100}$. The Nyquist plot for $L(s)$ is shown below. Apply the Nyquist stability theorem to predict the number of closed loop poles of the feedback system in the right-half plane. Is the closed-loop system stable or unstable?



5. Consider the feedback loop in Figure 2 where $G(s) = \dfrac{10}{s+1}$. The specifications are to design a controller so that: i) the closed-loop is stable, ii) the open loop has a crosssover frequency near 10 rad/sec.

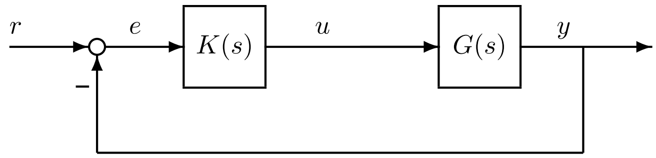


Figure 2: Feedback Loop

(a) (3 points) Sketch the Bode plot (magnitude and phase) for G(s) by hand. Label your plot with the approximate magnitude and phase at the lowest and highest frequencies. Also label the frequencies where your plot changes slope. You can use straight line approximation for this question.

(b) (2 points) Choose K_p so that $K_pG(s)$ has the desired crossover frequency. You may use your straight-line Bode sketch from Part a) to compute the approximate value of K_p.

(c) (2 points) Now you want to increase the closed-loop speed of response. Would you increase or decrease the open loop crossover frequency? Why?

(d) (3 points) The gain of a closed-loop system from noise to output is too large at high frequencies. Would you modify your controller using a proportional gain, integral boost, high frequency roll-off, or lead controller? Why?

6. All parts of this question relate to the following state-space model:

$$\dot{x} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Assume $a_1 \neq a_2$.

(a) (3 points) Calculate the controllability matrix.

(b) (2 points) Calculate the characteristic polynomial and the eigenvalues of the A matrix.

(c) (5 points) Suppose $a_1 = 2$ and $a_2 = 3$. Put the system into controllable canonical form.

(d) (5 points) Now consider the following system

$$\dot{x} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

Find the conditions on (a_1, a_2, a_3) such that the above system is controllable.

7. Consider the SISO system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 4x_3 + 3u \\
\dot{x}_3 &= 2x_1 - 3x_2 + 5x_3 + 2u
\end{aligned}$$

with $y = x_1$.

(a) (3 points) Write the above system as a state-space model. Find the $U \to Y$ transfer function.

(b) (4 points) Is the open-loop system stable? Is the system controllable? Observable? To justify your answers, calculate the controllability matrix and observability matrix by hand.

(c) (4 points) Find the gain matrix K for the full state feedback control moving the poles to -1, $-1 \pm j$. You can either perform the calculation by hand or use the `Matlab` function `place`. If you are using `Matlab`, provide the code.

(d) (4 points) Find the gains L moving the poles of the Luenberger observer to -5, -6, -7. In this question, you are required to perform the calculations by hand. All steps should be shown for full credit.

8. Consider the first-order nonlinear system:

$$\dot{x} = f(x, u) \tag{1}$$

where $f(x, u) := x + x^2 - 2\cos(u)$. Note that you can make x stay fixed at any value by appropriately choosing the input.

(a) (5 points) Find all possible equilibrium points $(\bar{x}, \bar{u})$ with $\bar{u} = 0$.

(b) (5 points) Next, linearize the nonlinear system around each equilibrium point computed in part (a). You should get a model of the form:

$$\dot{\delta}_x = a\delta_x + b\delta_u$$

where $\delta_x(t) := x(t) - \bar{x}$ and $\delta_u(t) := u(t) - \bar{u}$ measure the deviation of the state and input from the equilibrium point.