

## Article

# A Proportional-Egalitarian Allocation Policy for Public Goods Problems with Complex Network

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**Abstract:** How free-riding behavior can be avoided is a constant topic in public goods problems, especially in persistent and complex resource allocation situations. In this paper, a novel allocation policy for public goods games with a complex network, called the proportional-egalitarian allocation method (PEA), is proposed. This allocation rule differs from the well-studied redistribution policies by following a two-step process without paying back into the common pool. A parameter is set up for dividing the total income into two parts, and then they are distributed by following the egalitarianism and proportional rule, respectively. The first part of total income is distributed equally, while the second part is allocated proportionally according to players' initial payoffs. In addition, a new strategy-updating mechanism is proposed by comparing the average group payoffs instead of the total payoffs. Compared with regular lattice networks, this mechanism admits the difference of cooperative abilities among players induced by the asymmetric network. Furthermore, numerical calculations show that a relatively small income for the first distribution step will promote the cooperative level, while relatively less income for the second step may harm cooperation evolution. This work thus enriches the knowledge of allocation policies for public goods games and also provides a fresh perspective for the strategy-updating mechanism.

**Keywords:** public goods game; complex network; proportional-egalitarian allocation; strategy-updating mechanism



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## 1. Introduction

Cooperation plays an extremely important role in human development, and it is difficult to provoke the backbone of science and technology without cooperative behavior [1]. Therefore, how the level of cooperation among individuals can be improved and how the social dilemma can be solved effectively have become difficult problems in the scientific community and in real life [2]. Additionally, the current society is facing many resources and environmental problems [3], such as water pollution, air pollution, marine pollution, ozone layer destruction, and so on. These problems seriously affect our quality of life. In order to deal with these problems and utilize resources efficiently, concerted actions and long-term investments should be done in the whole society [4]. However, selfish behavior and the diversity of resources lead to individuals adopting different strategies for common problems [5]. The stakeholders can be roughly divided into contributors and defectors in terms of their strategies and actions. Cooperators endeavor to keep the sustainable development of human society, while defectors take advantage of the benefit created by contributors and pay nothing. This situation may lead to public tragedy in general [6].

To solve this kind of problem, the class of dynamic evolution game models as a powerful mathematical tool is proposed [7–9]. A foundation study was done by Nowak and May (1992), where a regular lattice is considered in a public goods game and the network

consists of local groups [10]. This model promotes further researches, such as cooperative evolution with a small-world network in [11] and a snowdrift game with a spatial structure in [12]. Following this research line, some scholars have studied the positive effects of cooperative evolution in spatial public goods games and achieved many fruitful results [13–16]. However, some researchers have focused on allocation policies by means of incorporating reward or punishment strategies to guide the evolution [17–20]. These studies mentioned above are fundamental methods to resolve public tragedy by reducing the proportion of defectors in an evolutionary socioeconomic system. An alternative point of view that considers players' reputation by recording their strategies is proposed to deal with such problems [21–25]. This view actually highlights the social attribute of human beings; therefore, it is much closer to practical situations. Once a person is recognized as a betrayer, he or she is likely to be excluded from cooperation, see [26,27]. On the other hand, if a player has a good reputation for his or her altruistic behaviors or positive contributions to the class of repeated public goods games, the other players will be more willing to cooperate with this player [28–30]. Additionally, in order to avoid public tragedy, some special factors of the allocation policy are also playing important roles in dynamic evolution games, such as enhancement factors (or profit margins), distribution ratios of the incomes, logical factors to distinguish who will participate in the second-order payment, and so on [31–33]. According to some concrete simulations, appropriate values of special factors could be fixed to promote the cooperative evolution.

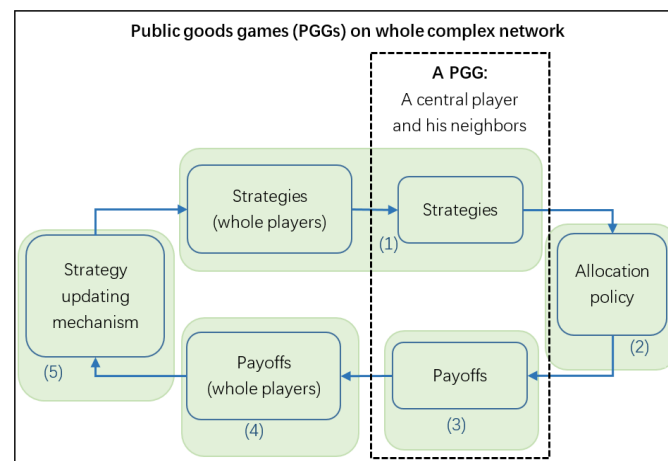
This research is motivated by [32,33], which studied the impact of income redistribution on cooperative evolution to deal with public goods games underlying a regular lattice network. Different from their works, this paper studies the public goods problems in an asymmetric social phenomenon and proposes a two-step allocation method to divide the total income. It is known that the redistribution approach means that the players have to pay back into the common pool after they get the initial payoffs, whereas the two-step method divides the total income into two parts, which may save the redistribution cost and reduce the risks in practice, such as hiring accountants and security guards to make sure the redistributing incomes are safe. In addition, by following the idea that all players are of a rational turn of mind and considering the asymmetric network structure, a novel strategy transfer mechanism is adopted in public goods problems. In the proposed allocation method, the parameter  $b$  from 0 to 1, which divides the total income into two parts, as well as the enhancement factors have a vital catalytic role in cooperative evolution. By fixing the enhancement factor, a relatively small  $b$  means that the cooperators get more payoffs, otherwise, the allocation method will benefit the defectors. From numerical simulation, the results also confirm this point of view and show that a relatively small  $b$  will promote cooperation, while a relatively bigger  $b$  hampers the evolution of cooperation.

The rest of the paper is organized as follows. In Section 2, the public goods model and the two-step allocation method are investigated. Section 3 examines analytical results and shows all findings by simulation. Section 4 provides concluding remarks.

## 2. Model and Allocation Method

A research framework regarding this section would be appropriate to be introduced before more details are provided. This research is mainly processed by following five steps (see Figure 1):

- (1) Each player selects a strategy between cooperation and betrayal;
- (2) The allocation policy is designed for public goods games, see Equations (1)–(3);
- (3) Each player in each PGG gets a payoff according to the proposed allocation rule, see Equation (4);
- (4) Each player gets the total payoff from the PGGs which they are involved in, see Equation (5);
- (5) Some players may update their strategies to pursue relatively high payoffs in the next round, see Equation (6).



**Figure 1.** Research framework.

This paper first introduces some notation and settings for public goods games (PGGs) on complex networks. The underlying complex network of the public goods problem is represented by an undirected scale-free network without weights, that is, a pair  $(N, E)$ , where  $N = \{1, \dots, n\}$  is the node set and  $E \subseteq \{\{i, j\} \mid i, j \in N, i \neq j\}$  is the set of edges (or links) on  $N$ . In a public goods problem, the nodes of a complex network represent the players or agents, and the edges model the mutual communications among players. For each player  $i \in N$ ,  $E_i = \{j \in N \mid \{i, j\} \in E\}$  is the set of neighbors of player  $i$  in a complex network  $(N, E)$ . Let  $d_i^E = |E_i|$  and  $d_i^{E,*} = d_i^E + 1$  be the degree and strong degree of player  $i$  in  $(N, E)$ , respectively, where the degree is the number of a node's neighbors and the strong degree is equal to the degree plus one by considering the player himself or herself.

Now we focus on the public goods problem by assuming that only connected players in the complex network can play such games. Thus, in the underlying complex networks, each player and his or her neighbors form a small group and take part in the PGG. At the same time, a neighbor of the former group other than the first player can also be regarded as a central player playing another PGG with his or her neighbors, and proceed in this way until all remaining players in the complex network are central players playing PGGs with their neighbors. In order to explain the allocation method in a concise way, a small group's public goods problem is considered as an example, where the small group consists of a central player and his or her neighbors underlying the complex network. It is well-known that each player can choose two strategies between cooperation and betrayal in a basic PGG. A player who chooses to betray, that is, a defector, will contribute nothing to the public pool, while the opposite strategy, called a contributor, contributes  $c$ , and the enhancement factor of the investment is given by  $r$ . For simplicity, let  $c = 1$  in this paper. In the allocation method, the resulting income of a group is distributed by two steps: first, a part of the income is split equally among all members of the group which follows the spirit of egalitarianism, and then the remaining profit is allocated proportionally according to former payoffs. Formally, taking player  $i \in N$  as the central player and considering the group  $E_i \cup \{i\}$ , the first part of the payoff for the players is

$$\epsilon_C^i = \frac{b \cdot r \cdot n_C^i}{d_i^{E,*}} - 1, \quad (1)$$

$$\epsilon_D^i = \frac{b \cdot r \cdot n_C^i}{d_i^{E,*}}, \quad (2)$$

where  $\epsilon_C^i$  and  $\epsilon_D^i$  denote the payoffs for cooperators and defectors by allocating the  $b \in [0, 1]$  percentage of the whole income, respectively.  $d_i^{E,*}$  is the size of the corresponding group.  $n_C^i = |N_C^i|$  is the number of cooperators in such a group, in which  $N_C^i \subseteq E_i \cup \{i\}$ . Note that there is some remaining income, denoted by  $T_i$ , distributed after the first step allocation

and  $T_i = (1 - b) \cdot r \cdot n_C^i$ . Moreover, the initial payoffs of some members can be negative due to the costs.

Next, the second step distributes the remaining income based on a weighted vector arising from the initial payoffs. In fact, after the first step, the players' strategies can be figured out from the initial payoffs. Therefore, the next step is to enhance the cooperative level by adapting the income gaps for the remaining income. Notation  $\varepsilon_j^i$  denotes the initial payoff that player  $j \in E_i \cup \{i\}$  gets from the group in which player  $i$  is the central player. Note that  $\varepsilon_j^i = \varepsilon_C^i$  if  $j$  is a cooperator, and  $\varepsilon_j^i = \varepsilon_D^i$  otherwise. Additionally, let  $\omega_j^i$  be the proportion of the remaining income to be allocated in the second step to player  $j$  in group  $E_i \cup \{i\}$ , which is given as

$$\omega_j^i = \frac{\exp(\varepsilon_*^i - \varepsilon_j^i)}{\sum_{h \in E_i \cup \{i\}} \exp(\varepsilon_*^i - \varepsilon_h^i)}, \quad (3)$$

where  $\varepsilon_*^i = \max_{j \in E_i \cup \{i\}} \varepsilon_j^i$  is the maximal initial payoff of such a group in the first allocation step. The proportion is defined by using the difference between players' initial payoffs and the maximal initial payoff of the corresponding group, which favors the players who get less initial payoffs. It is clear that the defectors always get relatively high payoffs than the cooperators due to the costs.

Suppose that the two-step allocation method is conducted with a fixed  $b$ , where the first allocation step follows the spirit of egalitarianism, and the second allocation step adopts the proportional approach in any group. Formally, in each group, the real payment for player  $j \in E_i \cup \{i\}$  is given as

$$\pi_j^i = \varepsilon_j^i + \omega_j^i \cdot T_i = \varepsilon_j^i + \omega_j^i \cdot (1 - b) \cdot r \cdot n_C^i, \quad (4)$$

where  $\pi_j^i$  is the actual payoff of player  $j$  in group  $E_i \cup \{i\}$  after using the two-step allocation method.  $\varepsilon_j^i$  is the initial payoff for the  $b$  percentage of the total income of the group organized by player  $i$  for playing PGG once.

In the complex network, public goods games play amongst all nodes, and all groups organized by a player or his/her neighbors interact as well. Therefore, a player obtains his or her final payoff as the sum of the payments of the groups he or she participated in, which is given by

$$\pi_i = \sum_{j \in E_i} \pi_j^i + \pi_i^i, \quad (5)$$

where the first part is the payments obtained from the groups organized by player  $i$ 's neighbors and the second part is the payment obtained from the group taking  $i$  as the central player.

Since players may be of different degrees in a complex network, it is reasonable to suppose that rational players pay more attention to the average payoffs than the total payoffs whenever they participate in playing PGGs. Therefore, the updating mechanism is designed as follows: the players prefer to change their strategies by imitating the neighbor player with the highest payoff. Precisely, taking player  $x \in N$  as an example, which is selected randomly among the player set, he or she gets the payoff  $\pi_x$  by playing the game with all his neighbors according to Equation (5); then, this player prefers to change his or her strategy if there is a player (including himself or herself) with a relatively high payoff. Hence, if the strategy of player  $x$  and the player, say,  $y \in E_x \cup \{x\}$ , with the highest average payoff are different, the player  $x$  will replicate the strategy of player  $y$  with a probability  $W_{x \rightarrow y}$  which is given as

$$W_{x \rightarrow y} = \frac{1}{1 + \exp[(\pi_x / d_x^{E,*} - \pi_y / d_y^{E,*}) / K]}, \quad (6)$$

where  $y = \arg \max_{j \in E_x \cup \{x\}} \pi_j / d_y^{E_x*}$  is the player of group  $E_x \cup \{x\}$  with the highest average payoff, and  $K > 0$  represents the noise intensity in the imitation. Therefore, the strategy-changing activity still has a possibility of inadvertently adopting the strategy of the player with a less than average payoff. Moreover, this paper sets  $K = 0.5$ .

To close this section, Algorithm 1 is proposed to employ the allocation policy for PPGs with complex networks. This algorithm also presents the main idea of the following simulation. For example, if one parameter  $r$  or  $b$  is given, one can easily find the optimal unknown parameter to guarantee all players being cooperators in the network, and vice versa.

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**Algorithm 1:** Solving PPGs with a complex network by PEA

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**Input:**  $(N, E)$ —A scale-free network,  $r$ —The enhancement factor,  $b$ —The percentage of income to be allocated, and  $K$ —The noise intensity in the imitation.  
**Initialize:**  $\{C, D\}$ —Strategies for all players in  $N$ .  
**Output:**  $P$ —Cooperative level.  
**while** strategy changes **do**  
 1. Egalitarian payoffs for players by calculating Equation (1) or Equation (2);  
 2. Proportional payoffs for players by calculating Equation (3) and Equation (4);  
 3. The final payoffs for players by calculating Equation (5); and  
 4. Updating strategies by calculating Equation (6).  
**end while**  
**return**  $P = \text{the number of cooperators} / n$ .

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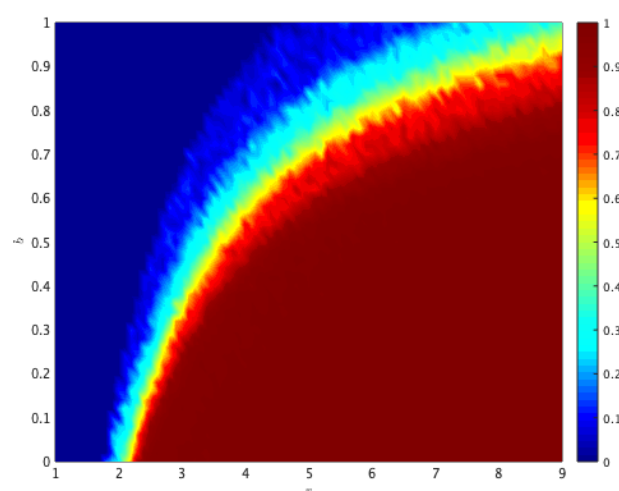
### 3. Results and Discussion

This paper focuses on the proportion of cooperators in the entire population by applying the allocation method proposed previously, and the allocation method is performed by MATLAB. The complex network size  $n = 400$  is fixed in the simulation. During the simulation process, each player has the opportunity to change his or her strategy according to the payoffs of neighbors and himself or herself. To avoid disturbances, the average frequency of cooperators is determined by the last 11,000 generations of the whole evolution process when the system reaches stability. In addition, the final results presented in this paper are also averaged by 20 independent runs.

First, a  $r$ - $b$  plane shows the macroscopic results as in Figure 2. It displays the average frequency of cooperators according to the enhancement factor  $r$  and the percentage of the whole income for the first allocation step, that is,  $b$ , respectively. From the color distribution in Figure 2, it can be observed that the lower-right corner is filled with red, which means the strategy of cooperation can be well-spread over the whole network whenever  $r$  is relatively high, while  $b$  is relatively low. However, on the other side of the pane, the cooperative level amongst the players in the network is rather low. Furthermore, if  $r < 2$ , no matter how much the whole income is distributed in the first step, the players are inclined to be free riders. On the contrary, whenever the enhancement factor is greater than 2, the proportion of cooperators in the population increases as  $b$  decreases. A further consideration is taken about the parameter  $b$  and the proportion of the second allocation step, that is,  $\omega_j^i$ , as in Equation (3). In fact, the proportion of a local group for the second allocation step  $\omega_j^i$  is fixed once the strategies are selected. For example, when all players of a small group are cooperators or defectors, the second part of the payments are distributed to the group members equally, while if both contributors and free riders coexist in a local group, the second allocation step indeed favors the contributors. In spite of this, parameter  $b$  deeply affects the final payoffs of the players, in which the variation of  $b$  leads to a marked difference in the evolution of the cooperation, as shown in Figure 2. When  $b = 0$ , the system achieves full cooperation around  $r = 2.3$ . However, as  $b$  increases to a large value, it shows a reduction in the cooperative level, that is, a higher value of  $b$  inhibits cooperation. Particularly, when  $b = 1$ , cooperators are still a minority even if the group



could earn a high return on investment ( $r = 7$ ). Therefore, a relatively small  $b$  in the first allocation step can effectively improve the level of cooperation in PGGs.

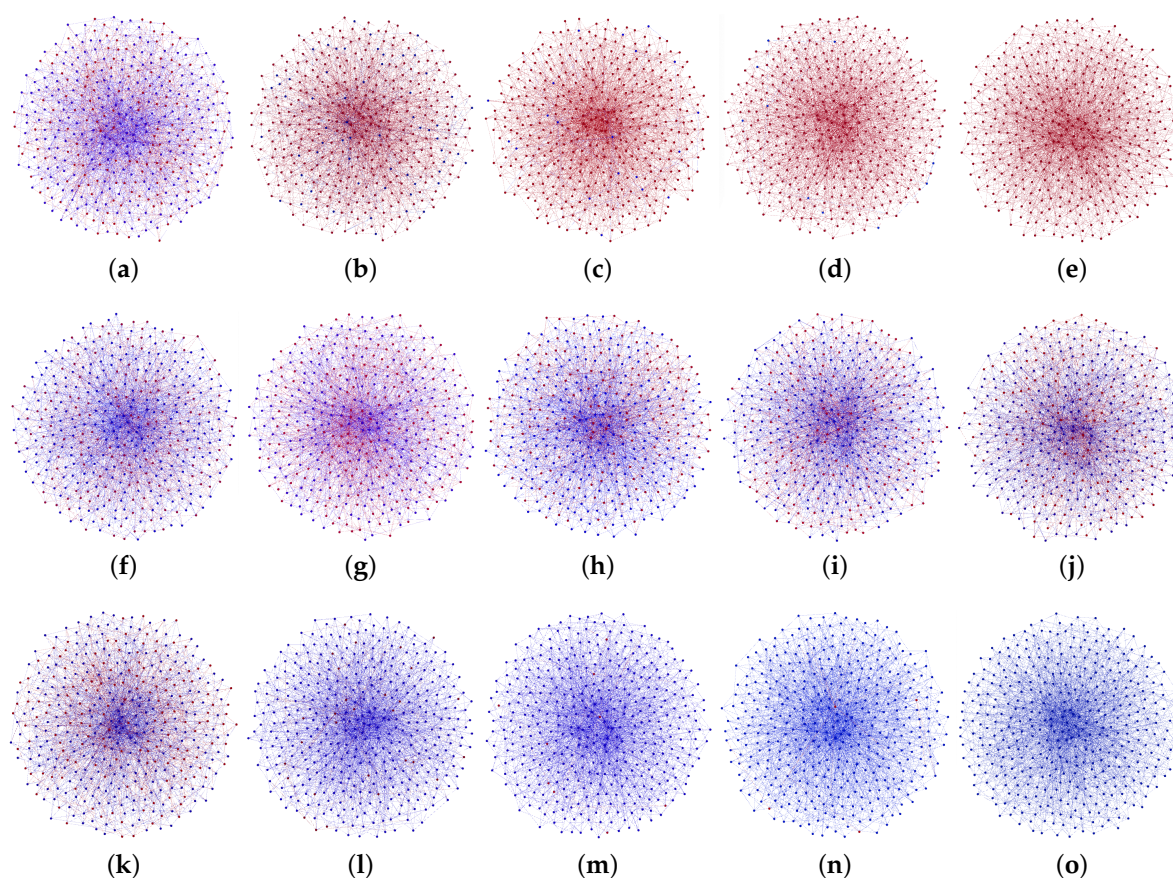


**Figure 2.** The average frequency of cooperators according to different values of enhancement factor  $r$  and parameter  $b$ . Red indicates full cooperation, blue shows full defection, and the relationship between the two cases and corresponding colors is displayed in the right bar. The simulation runs on a scale-free network with 400 nodes for  $K = 0.5$ .

Next, this paper shows the detailed results in different dimensions. In order to illustrate the process of cooperative evolution, the characteristic snapshots of cooperators and defectors are taken to demonstrate the changes of strategies by employing different values of  $b$ . The adoption of numerical values depends mainly on the result depicted in Figure 2 to show the three typical evolutionary results. Figure 3 points out the main results for  $r = 3.5$  and  $K = 0.5$ , besides the parameters of  $b$  in the three rows, which are 0.22, 0.5, and 0.9 (from top to bottom), respectively. (In order to show the results in a concise manner and to make the diverse results consistent, the settings will be used in the following discussions.) As shown in the first row of this figure, that is,  $b = 0.22$ , the pictures (a)–(e) illustrate how the cooperative strategy is rapidly replicated, and finally, the cooperators expand across the whole network. This implies that the allocation policy with a relatively small  $b$  can promote extensive cooperation. When  $b = 0.5$ , as shown in the second row of Figure 3, cooperators expand in the earlier few steps (see Figure 3g), and then the number of cooperators turns to decline slightly, and ultimately achieves the stable state (see Figure 3j), that is, both cooperators and defectors coexist in this network. It indicates that the interaction between cooperators and defectors indeed influences both of them to change their strategies, and finally reach dynamic stability. However, as  $b$  grows further, that is,  $b = 0.9$ , as shown in Figure 3k–o, the final incomes of the cooperators and defectors have a relatively large gap, then cooperators quickly change their strategies, and finally, the entire network is occupied by the defectors. These results show that, in the two-step allocation method, the relatively large remaining wealth for the second distribution will promote cooperation in the case of certain enhancement factors.

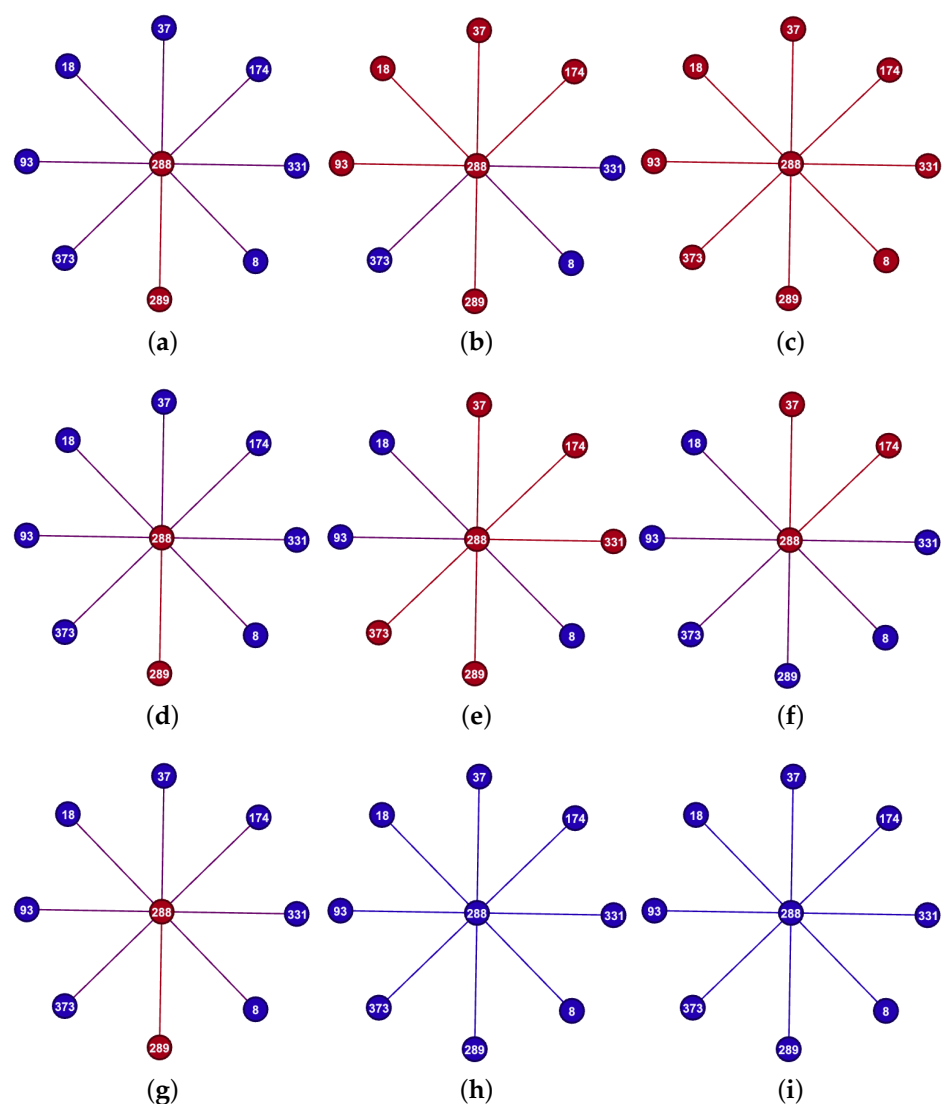
It is now time to present the process of evolution from a micro-perspective. Figure 4 shows three different evolutionary results of a randomly selected group with the same initial status, by setting different parameters. The group is a part of a complex network with nine nodes consisting of one central player No. 288 and his or her neighbors, and only two players (No. 288 and No. 289) are cooperators in the initial status. The first row (pictures (a)–(c)) in Figure 4 shows that with relatively small  $b$  ( $= 0.22$ ), defectors of this group change their strategies, even though they are the majority in the beginning. The last row (pictures (g)–(i)) presents the exact opposite when  $b = 0.9$ . However, in the second row (pictures (d)–(f)) with  $b = 0.5$ , cooperators and defectors coexist in this group and reach a dynamic and stable state, which means the population of the whole network is

split into cooperators and defectors as well. The following in this paper will further prove the validity of this result.

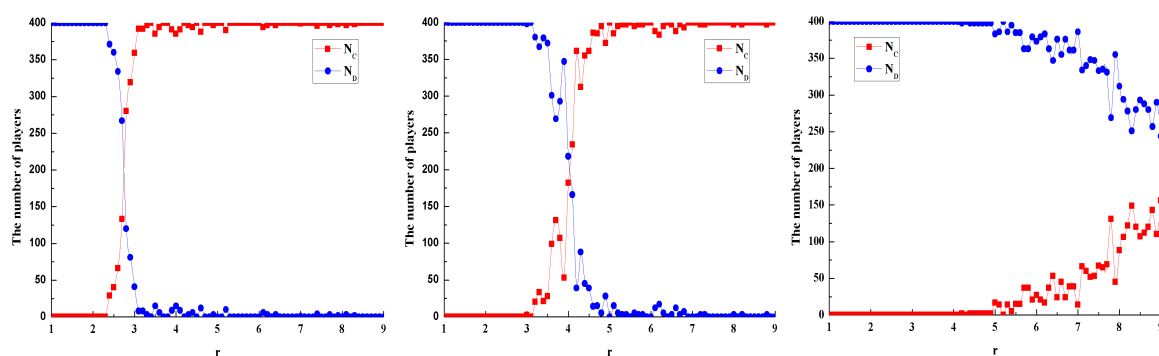


**Figure 3.** Characteristic snapshots of cooperators (red) and defectors (blue) on a scale-free network for different values of  $b$ . From top to bottom,  $b = 0.22, 0.5$ , and  $0.9$ , respectively. This result is obtained by fixing  $r = 3.5$ ,  $K = 0.5$ , and  $n = 400$ . (a) Steps = 0, (b) Steps = 1, (c) Steps = 2, (d) Steps = 3, (e) Steps = 9, (f) Steps = 0, (g) Steps = 6, (h) Steps = 11, (i) Steps = 17, (j) Steps = 30, (k) Steps = 0, (l) Steps = 1, (m) Steps = 2, (n) Steps = 3, (o) Steps = 15.

It is clear that whenever the enhancement parameter is large enough, that is, comparing with the investment cost, where the rate of return is high enough, the probability of transitions from defectors to cooperators will increase. Figure 5 demonstrates the effect on the numbers of cooperators and defectors under different values of  $r$  on the cooperative evolution. Notations  $N_c$  and  $N_d$  denote the numbers of cooperators and defectors, respectively. Note that  $N_c + N_d = n$ . The three pictures of Figure 5 (from left to right) show the number of players taking a pure strategy in the case of  $b = 0.22, 0.5$ , and  $0.9$ , respectively. When  $b = 0.22$ , cooperators outnumber defectors after  $r = 2.8$ , and then cooperators rapidly occupy the entire network as  $r$  increases. This means relatively high remaining wealth, and for the second allocation step, cooperators can effectively break through the blockade of defectors even if the reward investment is relatively small ( $r = 2.8$ ). As the value of  $b$  increases from  $0.22$  to  $0.5$  and then to  $0.9$ , the threshold  $r$ , where the number of cooperators begins to exceed the number of defectors, is also high. Meanwhile, the resistance of defectors becoming cooperators will also get stronger (see the second picture of Figure 5 around  $r = 3.9$ , and the third picture of Figure 5, the value of  $r$  from  $5$  to  $9$ ). This implies that with the increase of  $b$ , the cooperators are losing their advantage over defectors. This point has also been shown in Figure 4, for example, cooperators fill the whole network at  $b = 0.22$ , while they go extinct at  $b = 0.9$ . It is patently obvious that the value of  $b$  in a relatively lower level has a significantly positive impact on the evolution of cooperation.



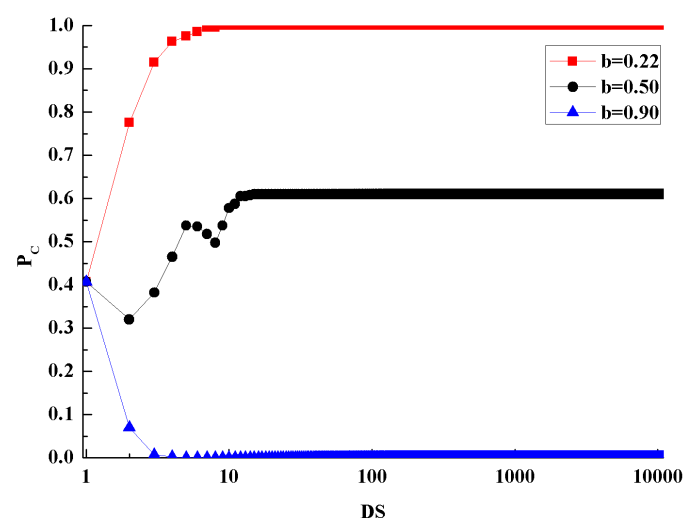
**Figure 4.** Three evolutionary results for a randomly selected group centered on node No. 288 and his or her neighbors in the complex network. Blue nodes denote defectors, while red nodes represent cooperators. The results are obtained for  $b = 0.22$ ,  $b = 0.5$ ,  $b = 0.9$ , respectively, by setting  $r = 3.5$  and  $K = 0.5$ . (a) Steps = 0, (b) Steps = 1, (c) Steps = 9, (d) Steps = 0, (e) Steps = 6, (f) Steps = 30, (g) Steps = 0, (h) Steps = 2, (i) Steps = 15.



**Figure 5.** The number of cooperators and defectors depend on  $r$  in cases of different values of  $b$  ( $b = 0.22, 0.5, 0.9$  from left to right). The results are obtained by setting  $K = 0.5$  and  $n = 400$ .

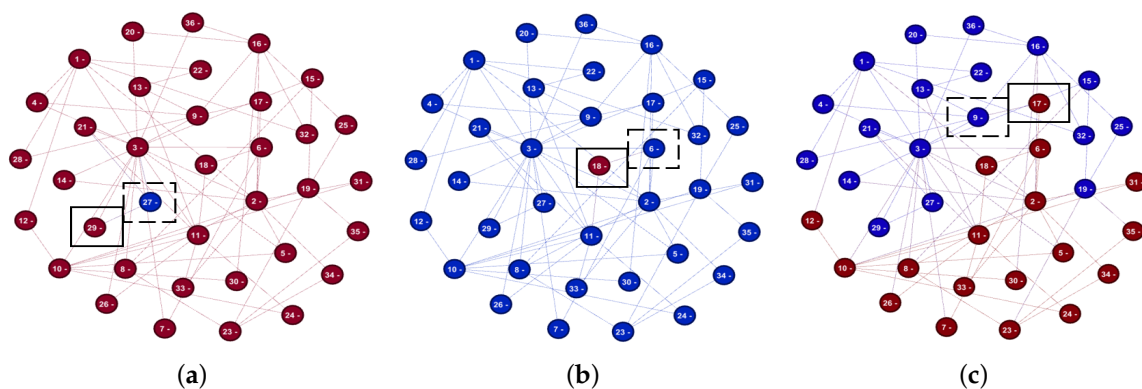


In order to present this positive impact in an intuitive way, three evolutionary tendencies are considered. The time evolution of  $P_C$  with different values of  $b$  is depicted in Figure 6, where  $P_C$  denotes the proportion of cooperators in the total population. The three lines with different colors start at the same initial state by setting  $r = 3.5$  and  $K = 0.5$ , where the strategies are randomly chosen by the players in the network. For  $b = 0.22$  (red line with squares), the players who choose cooperation have relatively low payoffs than the free-riders in the first allocation step, while in the second allocation step, the payoffs of cooperators achieve high growth due to the relatively more remaining income of the corresponding group. Therefore, the proportion of cooperators is rapidly increasing, and finally, cooperators occupy the whole network, which also implies that  $P_C$  stabilizes in the near ideal value 1. When  $b = 0.5$  (black line with dots), the total group incomes are divided equally for the two steps in the proposed allocation method. It is easy to see that the value of  $P_C$  first increases from 0.3 to 0.6, and then it turns to decrease slowly, and finally, the system reaches a stable state that cooperators and defectors coexist with a fixed proportion, that is,  $P_C$  is approximately 0.6. For  $b = 0.9$  (blue line with triangles), the majority of group incomes are distributed in the first step, which favours defectors. This leads to free-riding being imitated rapidly and spreading to the entire network in the end. From the above result, it concludes that the proposed allocation rule affects the evolutionary dynamics of cooperation in a complex network. Since only cooperators pay the investment cost while defectors contribute nothing in PGGs, defectors take advantage of cooperators by free-riding and earn more initial incomes than cooperators, which harms the evolution of cooperation in general. However, for the second allocation step, the allocation policy benefits the players who get relatively low initial incomes, that is, the policy encourages the contributors. Thus, when the value of parameter  $b$  is adjusted from 1 to 0, the contributory behavior is gradually encouraged and strengthened. In addition, it is clear that relatively more cooperators means more income for a group in one PGG, which benefits group members; meanwhile, the group member prefers to adopt the defection strategy to gain more payoffs whenever such a player is surrounded by more cooperators. However, by adjusting  $b$  appropriately, the heterogeneity of different groups is reduced, and at the same time, cooperation is strengthened, which prevents the occurrence of the former case. Therefore, the two-step allocation approach, which is similar to the redistribution mechanism, balances the payoff differences among cooperators and defectors. The proposed allocation rule is valid for promoting the cooperative level in the whole network, even if in a situation with a relatively low enhancement factor.

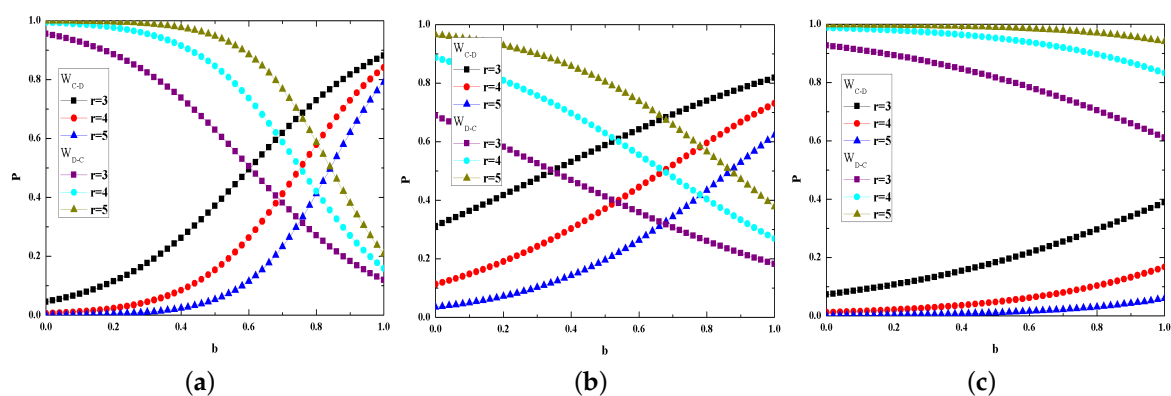


**Figure 6.** The time series of cooperative evolution correspond to  $b = 0.22$ ,  $b = 0.50$ ,  $b = 0.90$ . The three lines corresponding to this diagram are obtained for  $K = 0.5$ ,  $r = 3.5$ , and  $n = 400$ .

In the following, this paper illustrates the positive effect on evolution from the mechanism in a small scale-free network. Figure 7 displays three typical situations in the spatial game. Pattern (a) shows the distribution between cooperators and defectors in which a defector is surrounded by cooperators. Meanwhile, pattern (b) presents the distribution on the opposite side of pattern (a), that is, a cooperator surrounded by cooperators. In the last pattern, cooperators and defectors are distributed equally in the whole network, and it forms two rival camps. Further exploration is done for analyzing the possibility of strategy adoption among the circled cooperator (solid box) and circled defector (dashed box). Figure 8 shows the player's strategy transfer probability corresponding to the patterns of Figure 7 with the variation of  $b$  by selecting some specific values of  $r$ . Notation  $W_{C \rightarrow D}$  and  $W_{D \rightarrow C}$ , respectively, denote the possibility that a player changes his or her strategy from the former one to the latter. It is well-known that the fitness among players with different strategies plays an important role in the adoption of strategies. This point of view can also be seen in Figure 8: when the value of  $b$  is relatively small, such as when close to 0, a defector is more likely to become a cooperator by changing his or her strategy even when in a situation where defectors are a majority; and with the increase of  $b$ , the players are tending to adopt the free-riding strategy. In addition, if the value of  $b$  is large enough, the value of  $W_{D \rightarrow C}$  arrives at the minimum. Therefore, raising the income for the first allocation step weakens the advantage of cooperation in general, and this will also harm cooperation evolution. Additionally, although in the different patterns as well as in the corresponding transition, the possibilities are different, the tendencies to change strategies, which can be seen in Figure 8, are consistent with the increase of  $b$ .



**Figure 7.** Three typical sub-patterns in the spatial game: (a) a defector (blue) besieged by collaborators (red); (b) a collaborator besieged by defectors; (c) cooperators and defectors were well-matched.



**Figure 8.** The transition possibility between the selected cooperator and defector in Figure 7. Patterns (a–c) correspond to Figure 7a–c, respectively. The players are distributed in a small scale-free network with 36 nodes, and the result is obtained for  $K = 0.5$ .

#### 4. Conclusions

This paper proposed a two-step allocation method for public goods games on complex networks, and studied the impact of such a distribution policy on cooperative behavior. In the proposed allocation method, a parameter was set up for dividing the total income into two parts, and then distributed by following the spirits of the egalitarianism and proportional rule. For a given enhancement factor, by adjusting the parameter  $b$ , the optimal ratio of incomes for the two-step allocation method can be found to improve cooperation for the entire network. As it was shown previously that a relatively small value of  $b$ , that is, more remaining income for the second allocation step, benefits the evolution of cooperation and makes the system a reciprocity network, as for the opposite side, that is, employing a relatively large value of  $b$ , will enhance the payoffs gap between cooperators and defectors, which will obstruct the development of cooperation.

Furthermore, the proposed allocation method in this paper differs from the well-studied redistribution policies. As for the redistribution approaches, for the payments of a group, the members have to pay them back to the group, and this may lead to costs and risks of action, such as hiring extra accountants and security staff to keep taxes safe. However, the two-step method can effectively prevent the aforementioned events as the initiative for allocation is always in the organizers' hands. In fact, this kind of allocation method can be found in many situations; for example, the salary is usually divided into two parts—basic and performance—and the well-field system consists of nine squares: eight outer squares are the individual field, and the central square is the public field, which can be used for the protection against natural calamities and emergencies. This paper has proposed an allocation rule for public goods games underlying a complex network to deal with “tragedy of the commons” problems in different scenarios. This study enriches the knowledge of two-step allocation methods for public goods games, which can also be viewed as a complement of redistribution approaches.

This paper studied the allocation policy for public goods problems where the players only have two strategies (cooperation and betrayal) on a given fixed complex network. Therefore, some research directions could be considered for further study. Within this two-step allocation framework, the flexible complex network and multiple strategies situations could be considered for public goods games. The flexible complex network means that players have coercive power over others by refusing to join PGG(s), that is, deleting the link(s) among this player with neighbor(s). However, as for the strategies of players, besides cooperation and betrayal, some players could also choose other strategies, such as the refusal to play any PGG. Both situations and the combined situation would be worthwhile to study in the future.

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