Analysis of Modified Strassen's Algorithm for Matrix Multiplication

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CS124 Programming Assignment 2 Writeup

https://github.com/gzhang01/cs124prog/tree/master/prog2

Abstract

We implemented a modified version of Strassen's algorithm, where instead of recursing to a base case of a 1 by 1 matrix, we establish a base case of an n_0 by n_0 matrix and compute the product using standard matrix multiplication at that point. Our goal was to examine the runtime of this modified algorithm in order to determine the best threshold n_0. Analytically, we found that the optimal n_0 assuming constant time basic operations (addition, subtraction, multiplication, etc.) was about 9-15. Experimentally, we saw that a threshold of 30-50 produced the optimal run time.

Introduction

The standard matrix multiplication algorithm most people learn requires computing each cell of the product matrix by taking the dot product of the corresponding row / column of the input matrices. This is a $O(n^3)$ algorithm, since it takes n steps to compute each of n^2 cells. Strassen's algorithm uses a divide and conquer strategy that requires only 7 multiplications of matrices of size $n/2$, which results in a $O(n^{\log_2 7}) \approx O(n^{2.81})$ algorithm. Asymptotically, Strassen's algorithm is an improvement over the conventional method. However, for small n, the conventional method is faster.

Our task was to analyze a modified version of Strassen's algorithm, where instead of recursing to a 1 by 1 matrix, we use a base case of an n_0 by n_0 matrix and compute the product using conventional matrix multiplication. We wanted to find the threshold n_0 that would optimize this algorithm. We also wanted to compare the performances of these three algorithms–conventional multiplication, Strassen's, and our modified Strassen's–to see which performs best for matrices of different sizes. To determine this threshold, we will start by calculating it analytically assuming constant cost of arithmetic operations (addition, subtraction, multiplication, and division of real numbers) and no cost for all other operations.

We start by looking at Strassen's algorithm itself. Below is a summary of the operations needed to compute a matrix product using Strassen's, where A through H represent the matrices' quadrants.

Variable	Expression	Variable	Expression
P_1	$A(F-H)$	Q_1	$P_5+P_4-P_2+P_6$
P_2	$(A+B)H$	Q_2	P_1+P_2
P_3	$(C+D)E$	Q_3	P_3+P_4
P_4	$D(G-E)$	Q_4	$P_5+P_1-P_3-P_7$
P_5	$(A+D)(E+H)$		
P_6	$(B-D)(G+H)$		
P_7	$(A-C)(E+F)$		

Table 1: Summary of Strassen's Algorithm

We can see that for a given matrix of dimension n, we require 7 multiplications of matrices of dimension $n/2$. In addition, there are 18 additions/subtractions of matrices of size $n/2$, which contain $(n/2)^2$ elements. Thus we can write a recurrence equation describing the run time of standard Strassen's:

$$f(n) = 7f\left(\frac{n}{2}\right) + 18\left(\frac{n}{2}\right)^2$$

$$f(1) = 1$$

For conventional matrix multiplication, where we compute the dot product of rows and columns, we have a total of n^3 multiplications and $n^2(n-1)$ additions. As a result, the run time for this algorithm can be expressed in closed form easily:

$$s(n) = n^2(2n-1)$$

Our modified algorithm will combine these two procedures. We know that there is a threshold where conventional multiplication becomes cheaper than Strassen's. We call this threshold n_0. For $n > n_0$, we perform Strassen's. If $n < n_0$, we have reached our base case and we perform standard matrix multiplication. Thus our recurrence equation is:

$$f(n) = 7\min\left[f\left(\frac{n}{2}\right), s\left(\frac{n}{2}\right)\right] + 18\left(\frac{n}{2}\right)^2$$

We've defined n_0 to be the threshold dimension at which Strassen's begins to be more expensive than conventional multiplication. We can make the following assertions:

$$f(n_0) = s(n_0)$$

$$\min\left[f\left(\frac{n_0}{2}\right), s\left(\frac{n_0}{2}\right)\right] = s\left(\frac{n_0}{2}\right)$$

These assertions allow us to solve for n_0:

$$f(n_0) = 7s\left(\frac{n_0}{2}\right) + 18\left(\frac{n_0}{2}\right)^2 = s(n_0)$$

$$\longrightarrow n_0 = 15$$

However, this method only holds for n that are powers of 2, since they can be continually divided by 2. We decided to find the thresholds for other dimensions numerically by plugging in various values of n_0 and n into the recurrence relations. The table below shows an abbreviated sample of the results. The displayed values for n were chosen to display a combination of prime numbers, odd numbers, even numbers, and powers of 2.

However, the data shows that the cutoff point for n as powers of two is in the range of 8-15, as predicted by our solution above. For n as other values, the threshold seems to vary around 33, with very occasional exceptions such as $n = 777$, which has a threshold between 25-48.

Table 3 presents the dimensions that will be tested in our implementation. The values are intentionally not powers of two in order to examine how different dimensions affect the runtime.

n_0	n						
	128	233	413	512	777	950	1024
4	3397552	23651014	156939994	169724080	1095552088	1171249090	1192787152
8	3128640	21768630	105762679	156547392	737310883	1079012274	1100550336
12	3128640	21768630	105762679	156547392	737310883	1079012274	1100550336
16	3150592	18534483	87006067	157623040	606014599	920539071	1108079872
20	3150592	18534483	87006067	157623040	606014599	920539071	1108079872
24	3150592	18534483	87006067	157623040	606014599	920539071	1108079872
28	3150592	18534483	91469526	157623040	572535055	920539071	1108079872
32	3363840	19692108	91469526	168072192	572535055	977262696	1181223936
36	3363840	19692108	91469526	168072192	572535055	977262696	1181223936
40	3363840	19692108	91469526	168072192	572535055	977262696	1181223936

Table 2: Theoretical Run Time of Modified Strassen's

Dimension	Numerically Solved n_0
600	19-37
800	25-49
1000	8-15
1200	19-37

Table 3: Predicted Threshold Ranges from Numeric Calculations

Implementation

We found the threshold experimentally by implementing our modified Strassen's algorithm. To start, we created a new matrix type to encapsulate the information we would need. This type includes a 2D array of ints (representing the matrix itself), the dimension of the matrix, and the start row and start column, which are used to avoid copying large amounts of data when splitting matrices in Strassen's algorithm. In addition, we created basic functions to create a matrix, free the matrix, get / set elements, get the rows / columns, and add matrices (see matrix.c). We also included a function to pretty print the matrix to simplify debugging.

We then moved on to implementing conventional matrix multiplication (see matrixMultiplication.c). This process was also relatively straightforward. Using three nested for loops, we can iterate through all the necessary sums and products to calculate the matrix product. From prior knowledge of caching behavior, we used a column-first method that runs slightly faster than the standard dot product method. This speedup is due to the fact that we have more cache hits in the former than the latter, so we touch memory less frequently, which speeds up our algorithm.

Implementing Strassen's algorithm was a bit more technical. We wrote a function to split a given matrix into four matrices of size $n/2$ (see matrix.c). With our matrix type, this was relatively simple, as we could just point to the same array of numbers but assign different start rows and columns. After splitting, we simply implemented the various additions and multiplications necessary to produce the quadrants we want in our product matrix. Finally, we populated the product matrix quadrant by quadrant, again by taking advantage of the start column and row variables. Our modified Strassen's algorithm simply changed the base case of Strassen's to use the n_0 by n_0 base case instead of the 1 by 1 case.

However, this approach would only work if we did not run into a matrix with odd dimensionality while splitting our matrices. To resolve this issue, we added some padding to our matrix to begin with, so that we would never have to split a matrix with odd dimensions. Instead of simply padding to the next power of two, which we realized would be expensive for large n, we realized that once we hit our threshold and started performing conventional matrix multiplication, the dimensions of our matrix no longer mattered. Thus our padding function divided our dimension in half (taking the ceiling if necessary) until we were below our threshold and then multiplied back up to ensure that we could divide evenly when splitting our matrix.

Testing

We evaluated our algorithm by inputting matrices with known results, using the identity matrix for example. Once we verified the correctness of our conventional matrix multiplication, we used it to confirm the correctness of our Strassen's implementation by comparing the output of the conventional matrix multiplication function against the result of the Strassen's algorithm.

Results

We ran and timed our program on matrices of dimension 600, 800, 1000, and 1200 with threshold values ranging from 5 to 395 at interval 5. The entire data set is available in appendix A. Below, we have reproduced the data at intervals of 10 up until 150. Note that for each threshold-dimension pair, the run time is calculated from the average of 5 runs, and all run times are given in seconds.

threshold	d = 600	d = 800	d = 1000	d = 1200
10	12.4	35.0	48.4	84.4
20	10.0	24.8	43.2	68.0
30	10.0	21.4	43.0	68.0
40	10.0	21.6	43.4	69.4
50	10.0	22.6	43.4	69.0
60	10.2	22.2	43.0	69.0
70	10.2	22.2	43.8	69.0
80	10.4	22.0	44.0	72.0
90	10.6	22.6	44.0	71.6
100	10.6	24.6	44.0	71.6
110	10.4	24.6	44.0	71.4
120	10.2	24.4	43.8	71.2
130	10.6	24.4	47.2	71.4
140	10.4	24.6	47.4	71.4
150	11.6	24.4	47.6	79.2

Table 4: Run Times of Modified Strassen's for Various Thresholds and Dimensions

The trend is difficult to see in a table, so the results are summarized in the table below. We also plotted all our data on graphs to better visualize the trends (see Figure 1 on next page).

Dimension	Experimental n_0
600	20-60
800	30-50
1000	20-60
1200	20-40

Table 5: Experimental Thresholds

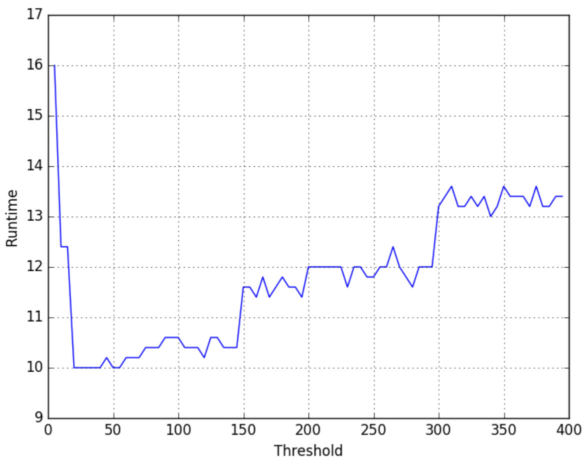
From these plots, we can see that the run times do in fact dip early on, roughly around 20-60 for dimension 600 and 1000 matrices, about 30-50 for dimension 800, and about 20-40 for dimension 1200. Since these times only came from the average of 5 runs, and since run time measurements are inherently slightly variable (resource allocation may easily produce noisy results) we are unable to provide a specific numerical threshold. However, the range 30-50 appears to be roughly optimal for each of these test cases. Much lower or higher than this results in significantly higher run times.

We also decided to investigate how our modified Strassen's algorithm compared to standard Strassen's (with base case 1 by 1) and conventional matrix multiplication. The results are shown in Table 4 below. Note that each of these were run only once, the threshold used for modified Strassen's was 40, and standard Strassen's was not run for dim = 800, 1000.

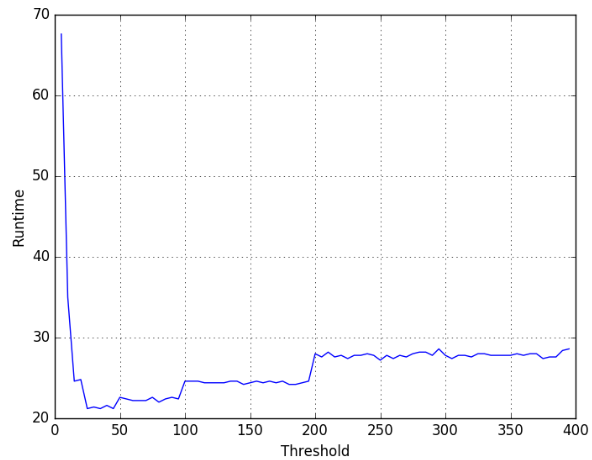
dimension	conventional	standard Strassen's	modified Strassen's
200	1	8	1
400	4	56	3
600	15	398	10
800	35	—	21
1000	67	—	43

Table 6: Comparison of Matrix Multiplication Algorithms

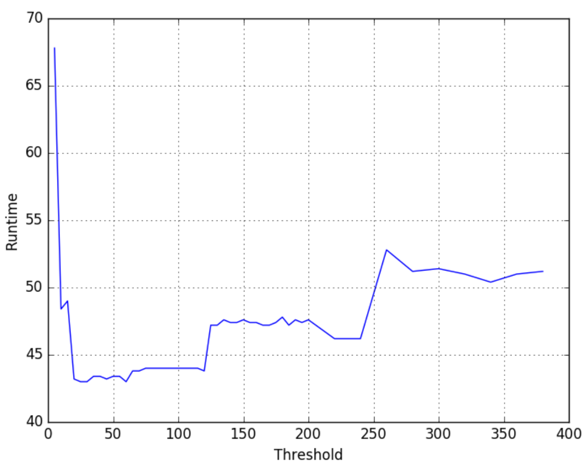
As we can see, our modified Strassen's algorithm runs faster than the conventional algorithm and the standard Strassen's algorithm when n is large.



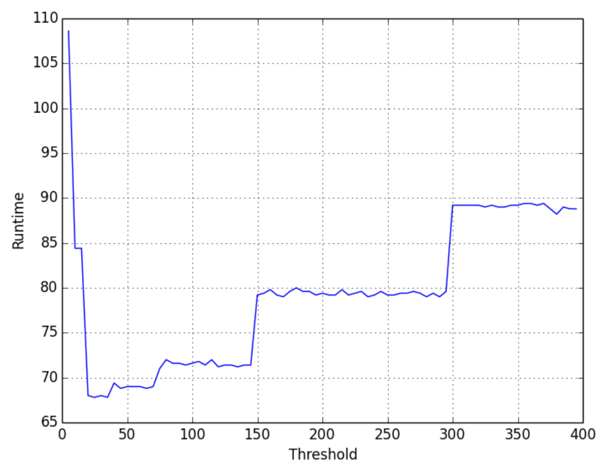
(a) Run Time vs. Threshold (dim = 600)



(b) Run Time vs. Threshold (dim = 800)



(c) Run Time vs. Threshold (dim = 1000)



(d) Run Time vs. Threshold (dim = 1200)

Figure 1: Plots of Results

Discussion

Our goal was to implement and find the threshold for our modified Strassen's algorithm. Analytically, we rewrote the recurrence equation for our modified algorithm and found the cutoff to be around 9-15 for dimensions of powers of two. Using dynamic programming, we wrote a script to calculate the number of operations needed to find the matrix product given different dimensions and thresholds. Using this method, we confirmed the threshold for powers of 2 and also calculated the thresholds for other dimensions. Experimentally, we implemented our modified algorithm and tested with various dimensions and values for thresholds. We found that a threshold of about 30-50 was optimal for our implementation.

We need to consider why there is a slight discrepancy between our analytical result and our experimental result. First, our analytical result assumed certain costs that may not be consistent with what is true in an experimental setting. For example, we assumed that non-arithmetic operations were free. However, in our implementation, we needed to allocated memory in order to create our matrices. Allocating memory is definitely not a free operation, even though our analytical analysis assumed it was. As a result, we would expect the Strassen's part of our algorithm to take longer experimentally than we would predict given our assumptions. This longer run time would encourage us to switch over to conventional matrix multiplication sooner in our calculations, which is analogous to a higher threshold.

Along the same lines, our arithmetic operations may not have cost 1. In reality, calculating basic arithmetic processes would require getting the numbers, which may need to touch memory if not in the cache, perform the operation, and assign the resulting value to some variable. Especially if we have a cache miss, this operation may not always have the same cost. As a result, we would again expect the actual run time to be slower than our prediction, and so we would have a higher threshold.

Our calculations were done with matrices of relatively high dimensions. We made this choice because we wanted several iterations of Strassen's before we reached our base case. If we ran with a low-dimension matrix, such as dim = 100, then we would not see as much variation in the run times. Indeed, we collected some data for dim = 400 and saw that the run times were generally not that different for various thresholds, and so we decided to use larger matrices to more effectively show the effects of the threshold on run time.

Finally, we want to address the behavior of the graphs that we produced from our results. Because of the way we implemented our padding, the amount of padding needed does not strictly increase as the threshold increases. Therefore, given increasing dimensionality, the dimensions of the matrices we actually work with in our algorithm are not strictly increasing. In addition, even in areas where it is strictly increasing, it does not necessarily increase linearly. Because of the variable sizes of the matrices we actually work with, we expect run time to increase in the long run, but we also expect local variability. This is indeed what we see in our graphs.

Appendix A: Run Times of Modified Strassen's for Various Thresholds and Dimensions

threshold	d = 600	d = 800	d = 1000	d = 1200
5	16.0	67.6	67.8	108.6
10	12.4	35.0	48.4	84.4
15	12.4	24.6	49.0	84.4
20	10.0	24.8	43.2	68.0
25	10.0	21.2	43.0	67.8
30	10.0	21.4	43.0	68.0
35	10.0	21.2	43.4	67.8
40	10.0	21.6	43.4	69.4
45	10.2	21.2	43.2	68.8
50	10.0	22.6	43.4	69.0
55	10.0	22.4	43.4	69.0
60	10.2	22.2	43.0	69.0
65	10.2	22.2	43.8	68.8
70	10.2	22.2	43.8	69.0
75	10.4	22.6	44.0	71.0
80	10.4	22.0	44.0	72.0
85	10.4	22.4	44.0	71.6
90	10.6	22.6	44.0	71.6
95	10.6	22.4	44.0	71.4
100	10.6	24.6	44.0	71.6
105	10.4	24.6	44.0	71.8
110	10.4	24.6	44.0	71.4
115	10.4	24.4	44.0	72.0
120	10.2	24.4	43.8	71.2
125	10.6	24.4	47.2	71.4
130	10.6	24.4	47.2	71.4
135	10.4	24.6	47.6	71.2
140	10.4	24.6	47.4	71.4
145	10.4	24.2	47.4	71.4
150	11.6	24.4	47.6	79.2
155	11.6	24.6	47.4	79.4
160	11.4	24.4	47.4	79.8
165	11.8	24.6	47.2	79.2
170	11.4	24.4	47.2	79.0
175	11.6	24.6	47.4	79.6
180	11.8	24.2	47.8	80.0
185	11.6	24.2	47.2	79.6
190	11.6	24.4	47.6	79.6
195	11.4	24.6	47.4	79.2

threshold	d = 600	d = 800	d = 1000	d = 1200
200	12.0	28.0	47.2	79.4
205	12.0	27.6	46.6	79.2
210	12.0	28.2	46.0	79.2
215	12.0	27.6	46.2	79.8
220	12.0	27.8	46.0	79.2
225	12.0	27.4	46.0	79.4
230	11.6	27.8	46.2	79.6
235	12.0	27.8	46.0	79.0
240	12.0	28.0	46.0	79.2
245	11.8	27.8	46.4	79.6
250	11.8	27.2	51.8	79.2
255	12.0	27.8	51.8	79.2
260	12.0	27.4	51.4	79.4
265	12.4	27.8	51.8	79.4
270	12.0	27.6	51.4	79.6
275	11.8	28.0	51.8	79.4
280	11.6	28.2	51.6	79.0
285	12.0	28.2	51.6	79.4
290	12.0	27.8	51.8	79.0
295	12.0	28.6	51.8	79.6
300	13.2	27.8	51.8	89.2
305	13.4	27.4	51.4	89.2
310	13.6	27.8	51.6	89.2
315	13.2	27.8	52.0	89.2
320	13.2	27.6	51.6	89.2
325	13.4	28.0	51.2	89.0
330	13.2	28.0	52.0	89.2
335	13.4	27.8	51.4	89.0
340	13.0	27.8	51.6	89.0
345	13.2	27.8	51.6	89.2
350	13.6	27.8	51.6	89.2
355	13.4	28.0	51.6	89.4
360	13.4	27.8	51.8	89.4
365	13.4	28.0	51.6	89.2
370	13.2	28.0	51.8	89.4
375	13.6	27.4	51.6	88.8
380	13.2	27.6	51.4	88.2
385	13.2	27.6	51.6	89.0
390	13.4	28.4	51.8	88.8
395	13.4	28.6	51.6	88.8