Analysis of Heuristics for the Number Partition Problem
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CS124 Programming Assignment 3 Writeup
https://github.com/gzhang01/cs124prog/tree/master/prog3

1 Abstract

Our goal was to implement heuristics for solving the number partition problem. We started with the deterministic Karmarkar-Karp (KK) algorithm and then moved to several randomized heuristics, including repeated random (RR), hill climbing (HC), and simulated annealing (SA). For each of the randomized algorithms, we looked at the results with two representations of the solution: as a sequence (S) of $\{-1, +1\}$ values determining the sets and as a prepartitioning (P) of the numbers. We found that the P representation produced much better results than the S representation (about a million-fold final residue improvement), though it also took about 100 times longer to run. In addition, SA performed significantly better than HC and mildly better than RR.

2 Introduction

The number partition problem can be stated as follows: given an input $A = (a_1, a_2, \ldots, a_n)$ of non-negative integers, produce a sequence $S = (s_1, s_2, \ldots, s_n)$ of signs $s_i \in \{-1, +1\}$ such that the residue

$$u = \left| \sum_{i=1}^{n} s_i a_i \right|$$

is minimized. Analogously, we are trying to split the integers in A into two sets A_1 and A_2 such that the sums of A_1 and A_2 are as similar as possible, with the residue being the absolute value of the difference between the sets. Even though this problem is NP-complete, there does exist a pseudo-polynomial time dynamic programming algorithm for it.

Dynamic Programming Solution

To start, instead of considering the optimization problem, let us consider the yes/no problem as follows: given an input A, is there a way to partition A into A_1 and A_2 such that the sums of A_1 and A_2 are equal? We let b be the sum of the sum of the elements in A, and so our question becomes whether we can produce a set A_1 such that the sum of the elements in A_1 is equal to $\lfloor b/2 \rfloor$. Note that if b is odd, the sum of A_2 will be $\lceil b/2 \rceil$, but we shall for our purposes say that this off by one answer still produces a "yes" result to our question. We shall see that with the filled array from this question, we can produce the answer to the optimization problem.

To obtain the answer to this question, we shall let $D(i, k)$ represent whether there exists a subset of $(a_1, \ldots a_i)$ whose elements sum to k. The answer we want, then, is the value $D(n, \lfloor b/2 \rfloor)$. We notice that all sets have a subset whose elements sum to zero: namely the empty subset. We also notice that in order for a set $(a_1, \ldots, a_i)$ to contain a subset whose elements sum to k, either the set $(a_1, \ldots, a_{i-1})$ contains a subset whose elements sum to k, or that set contains a subset whose elements sum to $k - a_i$. Thus mathematically, we have:

$$D(i,0) = \text{True}$$

$$D(i,k) = D(i-1,k) \text{ or } D(i-1,k-a_i)$$

Using this recurrence, we fill an array of size $n \times \lfloor b/2 \rfloor$. Our answer is the value $D(n, \lfloor b/2 \rfloor)$. We notice that each square in the array takes constant time to fill, since we are accessing two values in the array, and we iterate over the entire array. Thus the run time of this algorithm will be $O(nb)$, and correctness follows from the logic of the recurrence.

However, this does not yet give us an answer to the optimization problem we want to solve. We notice that if $D(n, \lfloor b/2 \rfloor)$ is True, then we know the optimal residue: 0 if b is even and 1 if b is odd. If $D(n, \lfloor b/2 \rfloor)$ is False, then we can look at $D(n, \lfloor b/2 \rfloor - 1)$. If this is True, then we know that the optimal residue must be 2 if b is even and 3 if b is odd, since if one set sums to $\lfloor b/2 \rfloor - 1$, the other set must sum to $\lceil b/2 \rceil + 1$, and the difference between the sum of sets is either 2 or 3 depending on the parity of b. This pattern continues, and thus we have an algorithm for finding the solution to the optimization problem based on our existing array: find the smallest $p \geq 0$ such that $D(n, \lfloor b/2 \rfloor - p)$ is True. The smallest residue is the $2p$ if b is even and $2p+1$ if b is odd. The final step at worst travels up one dimension of the array, and so it has runtime $O(b)$, and thus the solution to the optimization problem is still $O(nb)$, or pseudo-polynomial time.

Karmarkar-Karp Algorithm

The Karmarkar-Karp algorithm is a deterministic heuristic for the Number Partition problem. It repeatedly takes the two largest numbers in A and replaces them with their difference, with the intuition being that placing the two numbers in different sets is analogous to placing their difference in some set.

If we assume that arithmetic operations take constant time, we can implement the KK algorithm in $O(n \log n)$ time using a max heap. Initialization of the heap will take about $O(n \log n)$ time, since insertion is $O(\log n)$ and there are n elements to insert (of course, this is a loose bound, since insertion will really only take log of the number of elements already in the heap). Each step in KK will require popping two elements off the heap and inserting back on. Each of these actions require $O(\log n)$ time, so the total for a single step is still $O(\log n)$, since we are doing a constant number of these actions. Finally, we run the algorithm for $n-1$ steps to remove all but one element, and so the entire algorithm runs in $O(n \log n)$ time with a max heap.

Representations of Solutions and Other Heuristics

There are two representations of solutions we will consider. One is simply a set S of $+1$ and -1 values. We create a random solution by choosing n values from $\{-1, +1\}$, and we define a move from one solution to another as changing a random element's set s_i to $-s_i$, and with probability $1/2$ changing another random element's set from s_j to $-s_j$. We can also represent our solution via prepartitioning, Here a solution is of the form $P = \{p_1, \ldots, p_n\}$ where $p_i \in \{1, \cdots, n\}$ and if $p_i = p_j$, then a_i, a_j are in the same set. A random solution in this representation will assign values from 1 to n for all the p_i and then run KK on the prepartition to produce the two subsets. We define a move on this state space by selecting a p_i and assigning it a different value on $[1, n]$.

There are also three other heuristics to consider. The first is repeated random, where we repeated generate random solutions to the problem and keep the best one (where "best" is defined as the one with the lowest residue). The second is hill climbing, where we start with a random solution and then try to improve it through moves to better neighbors. Finally, we will consider simulated annealing, where we'll generate a random solution and then try to improve it by moving to neighbors which may not always be better – that is we will always move to a neighbor that is better, but we will move to a neighbor that is not better with some probability P that may be a function of how long our algorithm has run.

Our goal will be to implement the KK algorithm, as well as our three other heuristics each using both representations. We will then compare the success of these heuristics as well as their running times.

3 Implementation

We began by first implementing the Karmarkar-Karp algorithm. To maintain the $O(n \log n)$ runtime, we decided to implement a max-heap data structure (see heap.c). When that was completed, implementing KK was very straightforward, as we would simply insert all the numbers into the max-heap and pop two numbers and insert their difference at each step until there was only one number remaining (see function kk in main.c).

We then moved on to writing some functions that generated random solutions and found random neighbors with both representations (see solution.c). These were fairly straightforward with the given descriptions in the spec. We decided that we would seed our random number generator whenever someone called the function to generate a solution. Even though we needed to produce random numbers when finding neighbors, we decided it was safe to assume that a particular solution was generated first before any neighbors had to be generated.

The final step was to implement the heuristics. To do this, we started by writing functions to compute the residue of a given solution based on the solution type. The heuristics themselves were slightly repetitive, as most followed the same structure, but had different conditions on when values would be updated. Both here and during implementation of random solution generators, we decided that an object-oriented language would have made things much simpler. Since both the sequence and partition representations were solutions, it would have made sense to have a solution class, which sequence and partition would inherit from. Sequence and partition could then overwrite functions to generate solution, find random neighbors, and find residues to fit their specific requirements. Calling these functions would be simple in this case, since they would have the same name, and their implementations would depend on what type it was. Given that C is not an object oriented language, we had to make do with conditions that identified whether we had a sequence or partition and call the correct function (see mess of functions in main.c).

After the implementation was done, we wrote a few more lines that implemented the behavior expected by the spec with regards to input files and output. We also wrote a few functions that returned residues for each of the methods so we could more easily discuss our results.

4 Results

in order to compare the results, we ran 50 test trials on randomly generated numbers on the range $[1, 10^{12}]$. For each test trial, we produced the residue given by the KK algorithm, the initial residues of the randomly chosen start solution, and the final residues of each of the heuristics. In order to better compare results within a trial, we started with the same random start solution for each of the heuristics within a single trial (of course, we chose new start solutions on different trials). Because sequence and partition represent their solutions differently, we produced both a sequence random solution and a partition random solution to begin with. The complete data for all 50 trials are in appendix A. Below are the average residues over the 50 trials. KK results are shown first, followed by initial residue, final repeated random, final hill climbing, and final simulated annealing for sequence-based solutions, and finally the same four for partition-based solutions.

KK	Sequence				Partition			
	R	RR	HC	SA	R	RR	HC	SA
292029	4255641605057	252865950	353262590	248113800	9610309	240	711	212

Table 1: Average Residues for Various Heuristics

We were also interested in which heuristic performed the best given any particular trial. The table below shows the various heuristics and the number of 1st, 2nd, and last place rankings with regard to the final residue:

Place	Sequence			Partition		
	RR	HC	SA	RR	HC	SA
1st	19	16	15	18	8	24
2nd	14	15	21	21	14	15
3rd	17	19	14	11	28	11

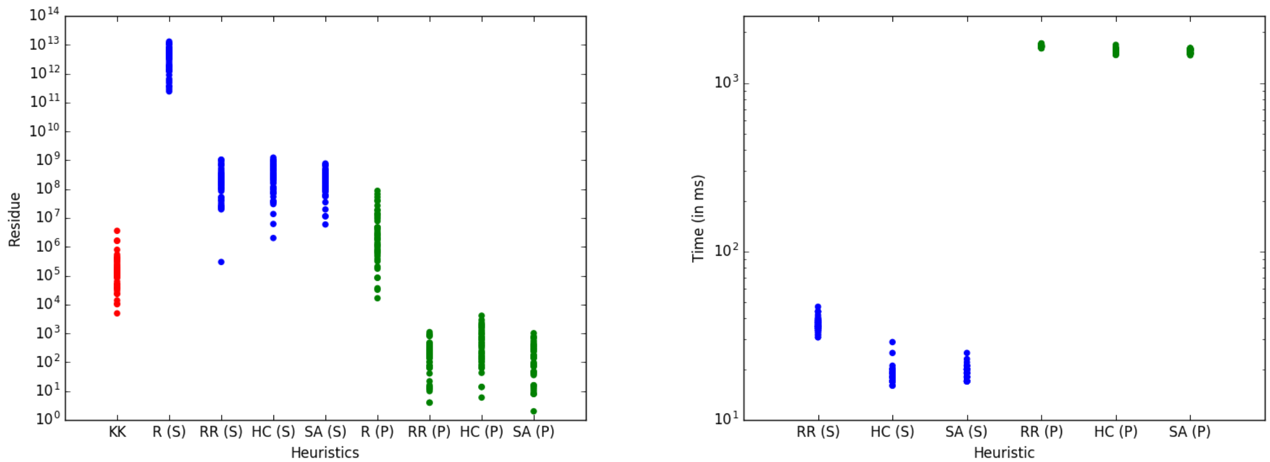
Table 2: Frequency of Rankings Among Heuristics

We also looked at the run times of each of the heuristics. The complete data for all 50 trials are in appendix B. Below are the average runtimes over the 50 trials. Data for repeated random, hill climbing, and simulated annealing are shown for sequence and partition-based data. Note that times shown are in milliseconds.

Sequence			Partition		
RR	HC	SA	RR	HC	SA
37	18	19	1654	1551	1544

Table 3: Average Runtimes for Various Heuristics

While average values and rankings are good, we also want to see trends in the full data. Below are scatter plots showing results from all 50 trials. Note that the dots are color-coded based on KK (red), sequence solutions (blue), and prepartition solutions (green), and the y-axis in both graphs are on a logarithm scale. The runtimes graph only lists runtimes for the randomized algorithms, as the time to run KK or find initial residues is negligible.



(a) Residue Values

(b) Runtimes

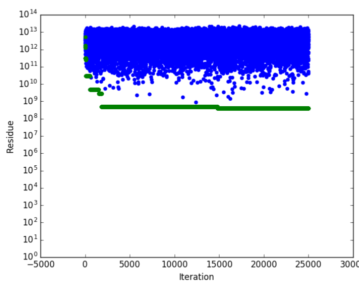
Figure 1: Heuristic Data

As we can see from the tables and the graphs, prepartition-based solutions produce better results than sequence-based solutions. Even without running any randomized algorithms, the initial residue found from a completely randomly chosen solution is better for the prepartition case than for the sequence case. This result makes sense, since the sequence solution uniquely places each number into a set, whereas the prepartition solution merely groups a few numbers together and then runs KK on the resulting set. Thus there is much more flexibility in the prepartition case than the sequence case, and so we see a lower residue. Indeed, judging from the averages, prepartitioning produces final residues that are on the order of one-million times better than the sequence residues.

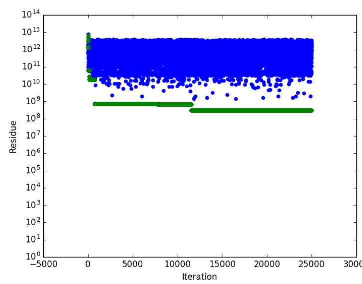
However, this flexibility and lower residue values comes at the cost of computing time. Since prepartitioning assigns groups to the numbers and does not produce a solution by itself, it must run the KK algorithm to find the residue. This is more costly than simply summing the values based on subset that we could do with a sequence solution, and so we would expect a higher runtime. This is indeed the case, as we can see in graph b. on average, prepartitioning increases the runtime of the algorithms by 100 times as compared to the sequence runtimes.

We also want to look at the heuristics themselves. From the data, we can see that hill climbing generally produced the worst residues on average among both sequence and prepartition solutions. However, the rankings of heuristics given trial is a bit more mixed for the sequence solutions than the prepartition solutions. We can see that each of the three heuristics had similar numbers of trials where they performed the best given sequence solutions. The prepartition solutions, however, generally followed a trend of SA being the best, followed by RR and then HC. The graphical data also supports this, as the data for sequence solutions are closer together than the data for prepartition solutions, suggesting a bit more variance in which solution performed best. Based on the averages, though, we can conclude that SA did better than the HC, which makes sense since HC is susceptible to getting stuck at local optima.

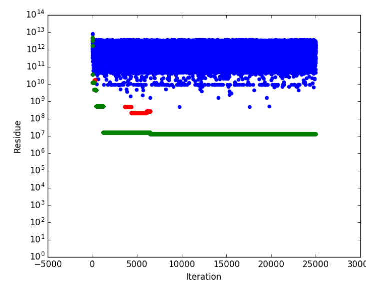
To really look at how these heuristics compare, we also collected data on residue values while the program is running. We logged the residue value of the random solution / neighbor as well as the best reside for each of the 25000 iterations (for the SA case, we also logged the neighbor we are currently on, which may not be the best one). The data is shown below, where green is best residue value, blue is random residue value, and in the case of SA, red is current neighbor residue:



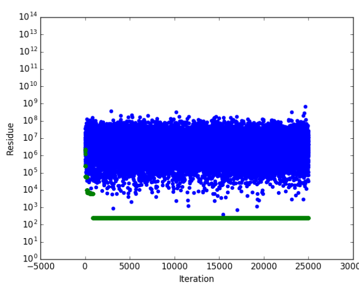
(a) RR (S)



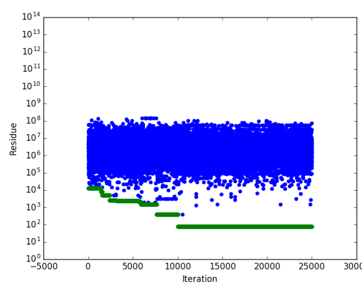
(b) HC (S)



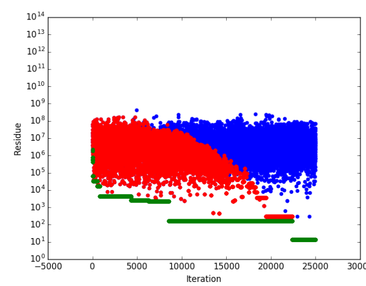
(c) SA (S)



(d) RR (P)



(e) HC (P)



(f) SA (P)

Figure 2: Residue Values During One Trial of Given Heuristic

We can draw a few conclusions from these graphs. First, the y-axis is on the same scale for all of them, so we can immediately see again that the prepartitioning produces better residues than the sequence. We also see that RR tends to fall to a minimum value early and stay there, which makes sense, since the probability of finding a lower residue by randomly selecting solutions decreases the lower our minimum residue is. We see a few more steps in HC, however. This occurs by nature of the fact that we only visit neighbors of a given solution. Thus we will not fall to a minimum quickly, but instead will gradually move toward the minimum over many iterations. SA for partition shows a gradual falling of the red dots. This corresponds to the cooling function making the algorithm less likely to choose poor solutions over time. This effect is not as obvious on the sequence graph, since the residue values are so high that the probability of shifting is almost zero. We can see that in both SA graphs, the best residue falls lower than the HC, since we allow for temporarily moving to worse neighbors in an attempt to find overall better ones.

5 Discussion

Our goal was to implement various heuristics for the number partition problem, including the Karmarkar-Karp algorithm, as well as repeated random, hill climbing, and simulated annealing heuristics. We were able to do this, and our heuristics produced the results shown above. We have already discussed some of the conclusions we could draw from the results, but there are a few other thoughts we would like to touch on.

Above, we mentioned that prepartitioned solutions did better than sequence solutions, and we attributed this to the increased flexibility that prepartitioning provides. After all, the partitions we get are not themselves solutions; we still needed to run the KK algorithm on the partitions to produce a residue. Thus a fixed solution that would be given by a random sequence would be one possible solution to some prepartition, but the KK algorithm would likely find a more optimal solution that produces a lower residue. However, because the prepartition solution requires running the KK algorithm, we would expect some relationship between our prepartition results and our KK results. From figure 1, we can see that in general, the initial residue given a random preparition is higher than the residues produced by running the KK algorithm by itself, meaning random prepartitions perform worse than no prepartitions. This suggests that it may be more advantageous to just start the random heuristics with a one-to-one prepartitioning, as the initial residue would be exactly the KK residue. Over 25000 iterations, it may not matter for the final result, but we would on average have a lower starting point.

The fact that starting from the KK prepartition may be advantageous also suggests we should consider what happens if we start from the KK solution for the sequence-based heuristics. Again from figure 1, we see that most of the final residues from the sequence-based runs are higher than the KK residues. Thus starting with the KK solution will shift all of these down to at least the level of the KK solution. Whether the heuristics would then produce better results is debatable. The repeated random heuristic will likely not move the residues down much farther, since the probability of finding an even better solution randomly appears to be very small (based on where the band is currently). With the hill climbing heuristic, we expect that the results we have now suggest that we got stuck at local optima. Starting at a lower point should allow us to move to better solutions, though if the KK solution is a local optima itself, this will not happen. The simulated annealing heuristic may have the most opportunity to find a better final solution, since we'd start with a low enough residue that the probability of taking on a worse solution is higher than it was for our current trials. This would give the algorithm more flexibility in finding a lower optimal solution.

Thus overall, we would expect that starting with a KK solution instead of a random solution would mildly benefit our sequence heuristics, since we would have a much lower starting point. For the prepartitioned solutions, we would have a lower initial residue, but the overall benefit after 25000 iterations may not be as great, since we are already able to find solutions with very low residues.

A Final Residues for Given Methods

R: initial residue; RR: repeated random; HC: hill climbing; SA: simulated annealing

KK	Sequence				Partition			
	R	RR	HC	SA	R	RR	HC	SA
34971	10822420390057	155374873	95986951	376477989	40702703	233	413	89
10730	291879606832	228472382	320260752	167217288	4625566	136	14	304
167053	6673258379789	329663447	328624371	273723935	52648453	393	247	99
259754	1376845732664	313216286	36386704	158056782	473944	22	148	180
121504	7061089758030	772136136	317551920	105527580	1887368	468	618	150
51452	10343992443280	221317106	36503542	165552008	963940	270	1914	2
107573	4614462059233	85131791	464012333	772537577	808739	101	129	397
87981	1422808104157	25221665	501411629	530321089	67005061	63	83	235
13894	1356376628822	661091942	76203024	35073816	3324460	172	1710	146
92083	479177735597	921907615	77876597	193633973	2792567	41	723	371
138088	1249545622572	175909216	70245614	246420936	1941212	344	312	138
410883	243414920825	105658165	239796979	108474771	4646317	863	369	77
37706	11430392587090	39246218	120113616	446125528	38630924	144	6	474
4959	4067787093499	24759389	542265111	233971069	18963889	77	75	45
98982	2175280376182	100821554	1236938264	688213982	2188950	940	476	148
463510	1739069504278	139869568	75036270	11502710	175152	12	158	8
1644891	3495083184471	49560039	13720169	61430319	2537213	409	63	711
139281	351765857829	53398387	988468769	81428829	1802037	237	1005	381
394078	5755470980672	21353048	157391418	5971440	745400	186	630	36
107320	3895930206648	128708358	725491620	136598406	3992690	4	202	172
50804	12983588988310	451818596	99627184	118751320	1200558	4	2356	150
324855	5748524737771	325012143	223400605	452940573	1161801	75	105	699
135242	1728004920992	704083100	676644810	332982194	16588	164	754	276
145423	5441272197069	225423987	6201619	99470517	87319175	65	2927	367
123015	663595478753	528417633	1039183345	344036915	1417147	73	369	45
176602	6613491011748	294308518	273748166	688118388	18407516	294	1558	296
36304	1856766152536	161220750	455837768	83764346	26822008	1110	136	164
301760	1899256763434	113678664	585810298	243536474	1106016	202	204	0
63108	3427430057958	344589548	950945980	256413500	36730	146	0	794
360853	4036657978237	42220127	703414819	19806529	12418945	221	65	73
218412	900346511810	232196292	661249482	60944820	637880	164	608	14
3588550	2595831900490	1050246780	263865886	11475264	4786194	10	988	164
149598	1305294397646	187019180	111840076	240987176	2614908	308	14	8

KK	Sequence				Partition			
	R	RR	HC	SA	R	RR	HC	SA
407557	3926707064809	245517357	2022865	56030691	4436253	71	1959	1019
29832	1191540834240	19895442	174864454	443830502	7770892	14	4116	16
42600	3153800750690	293365844	355409068	681390488	557082	14	158	8
1601343	8202506007583	298681	200669301	185308965	4506947	103	221	373
23445	4971984587883	263411841	439607639	308089921	9394505	67	1169	247
233750	10790631655718	390750690	223805766	125872988	317398	478	162	12
49007	4536364976903	32133863	648138619	239303795	10757717	237	735	559
23680	6828193130916	104996518	39072626	200186108	32686	12	132	162
123578	4765629557230	156272448	54308150	57226240	13887404	188	1306	158
194223	1790585216809	183999215	880380253	335239229	13997805	221	693	71
793230	3283815385864	331839424	329715906	297449466	84776	804	930	40
158231	7046013476531	1036374145	30290799	135609771	84831	255	209	85
10258	8099798512072	24022596	382573088	103978728	678944	340	484	164
188740	564919194606	140730138	281475530	683221196	379688	16	2050	10
537599	7457286511807	23262437	306643805	346288531	206375	867	43	49
83636	375693290518	26821568	802226178	359693724	1832138	168	1704	16
39541	7750497829419	156552837	35869811	95481615	2787979	223	105	399

B Runtimes for Given Methods

RR: repeated random; HC: hill climbing; SA: simulated annealing
All runtimes in milliseconds

Sequence			Partition		
RR	HC	SA	RR	HC	SA
47	16	19	1638	1576	1552
39	18	20	1658	1518	1537
38	20	19	1632	1467	1501
36	25	20	1639	1520	1518
39	29	17	1655	1525	1508
44	21	17	1605	1544	1505
41	17	17	1626	1555	1502
40	18	20	1634	1561	1596
38	18	20	1717	1505	1477
44	19	17	1614	1523	1474
38	19	20	1659	1541	1570
34	17	18	1672	1550	1560
37	19	21	1645	1470	1461
34	18	17	1651	1516	1496
36	19	20	1632	1576	1519
42	18	17	1643	1508	1498
44	17	19	1672	1568	1566
36	17	17	1661	1548	1592
36	17	18	1662	1584	1585
36	20	20	1682	1575	1582
37	19	23	1668	1559	1563
36	17	19	1649	1546	1567
37	19	18	1694	1583	1609
32	20	25	1706	1625	1605
31	17	22	1651	1622	1583

Sequence			Partition		
RR	HC	SA	RR	HC	SA
33	19	17	1648	1524	1581
35	20	21	1637	1517	1504
36	20	17	1644	1482	1500
36	19	18	1621	1518	1504
35	19	20	1632	1529	1515
39	18	18	1610	1548	1484
36	18	17	1655	1542	1493
37	17	19	1647	1534	1572
35	19	17	1635	1555	1475
38	16	20	1662	1530	1577
39	17	20	1663	1549	1530
38	17	20	1633	1544	1573
39	17	18	1678	1556	1569
40	18	19	1627	1570	1492
35	17	21	1667	1589	1512
37	18	19	1639	1501	1544
40	19	21	1647	1685	1588
32	18	20	1648	1584	1572
34	17	21	1642	1572	1605
35	18	19	1717	1584	1590
38	19	17	1679	1558	1609
31	20	20	1670	1579	1578
35	17	20	1681	1576	1612
33	19	18	1683	1596	1567
34	18	19	1698	1593	1569