Hermite Polynomial-based Valuation of American Options with General Jump-Diffusion Process*

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Abstract

We present a new approximation scheme for both of the price and exercise policy of American options. The scheme is based on Hermite polynomials expansion of the transition density of the underlying asset dynamics and the early exercise premium (EEP) representation of the American option price. The advantage of the proposed approach is threefold. First, our approach does not require the transition density and characteristic functions of the underlying asset dynamics to be attainable in closed form. Second, our approach is shown to be fast and accurate, while the prices and exercise policy could be jointly produced. Third, our approach has a wide range of application scopes, and can be easily extended to higher dimensional cases and jump-diffusion models. We show that the proposed approximations of the price and optimal exercise boundary will be convergent to the true ones. We also provide a numerical method based on a step function for the implementation of our proposed approach. Examples such as nonlinear mean-reverting model, double mean-reverting model, Merton's and Kou's jump-diffusion models are presented and discussed.

Keywords: Hermite polynomials, American option, early exercise premium, jump-diffusion process

JEL codes: C22, C41, G12, G13.

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1 Introduction

The valuation of American-style options poses a challenge for both academic and industrial professionals. One of the difficulties comes from the fact that such valuation process relies on the identification of optimal exercise policy. Due to this inherent feature, considerable efforts have been put to simple settings where the underlying asset price follows a log-normal process and the interest rate is constant (i.e., the standard model, or the Black-Scholes model). Within this context, Kim (1990), have decomposed the American option price to two parts: the corresponding European option price and an Early Exercise Premium (EEP) which captures the gains from exercising the option prior to its maturity. Similar results can also be found in Jacka (1991) and Carr et al. (1992). The EEP representation of American option price has proved extremely useful, because such representation provides a recursive integral equation for the optimal exercise boundary. Solving the integral equations is key to the valuation process: it identifies the optimal exercise policy, and hence provides a parametric formula for the option price. Such approach, based on the integral equation, is straightforward to be implemented and shows significant advantages over other numerical procedures such as methods based on binomial lattices, Monte Carlo simulation, and Partial Differential Equations (PDE). See Brodie and Detemple (2004) for a survey of methods on the valuation of American option.

While the valuation of American options in the standard model has been resolved, empirical evidence suggests that the log-normality assumption does not hold in reality. For example, the "volatility smile" phenomenon is a well- known pattern in option pricing practice. To allow for the consistency of models with empirical regularities, non-constant, or even non-deterministic model parameters should be considered. Unfortunately, analytical results in the standard model could not be generalized to models with stochastic parameters in a straightforward way. Efforts have been put to solve diffusion models with non-constant parameters. For example, Jacka and Lynn (1992) considers general contingent claims written on diffusion processes. Detemple and Tian (2002) presents an integral equation approach for the valuation of American-style derivatives when the underlying asset price follows a general diffusion process and the interest rate is stochastic. See Rutkowski (1994), and Gukhal (2001) for valuation of American option on other non standard models.

All the above mentioned methods rely on the fact that the transition density of the underlying asset dynamics preserves a closed functional form. Such condition has limited the scope of stochastic process that could fit into these frameworks. Even when the transition density exists in closed-form, the structure may be quite complex that it may be difficult to be implemented. To overcome these difficulties, this article presents a systematic treatment of valuation of American options based on Hermite polynomial expansions and the EEP formula. Our contribution is threefold.

The first contribution is that, our method does not reply on the existence of analytical solutions to the transition density or the characteristic function of the distribution of the underlying asset price. Moreover, we have no requirements for affine structures. An established body of research has been focused on the option pricing problem with a closed-form transition density. Another line of effort has been put to consider affine dynamics and closed form characteristic functions. Departing from these option pricing routines, we propose to apply the Hermite polynomials to approximate the transition density for a given jump-diffusion model. The Hermite polynomial approximation is based on Ait-Sahalia (2002, 2008), and Yu (2007). This approach gives an explicit sequence of closed-form solution to the transition density and is shown to be convergent to the turn density. See Ait-Sahalia (1999), Egorov, Li and Xu (2003), Ait-Sahalia and Kimmel (2010, 2010), and Xiu (2014) for studies related to this approach.

The second contribution is that, our method is fast and accurate, while the price and exercise policy could be jointly approximated by our approximation scheme. Due to the inherent nature of the EEP approach, by solving the integral equations with the Hermite polynomial-based approximated transition density, we could generate an approximation of the optimal exercise boundary. We provide a theorem (Theorem 2 in Section 2.3) on the convergence of our proposed approximation. When we increase the order of Hermite polynomial in the approximation of the transition density, the proposed approximations of the price and exercise boundary of the American option. We could actually control the smoothness and accuracy of the exercise boundary with varying degree of orders in expansion of the transition density.

Third, our method could be easily extended to jump-diffusion models and multi-dimensional cases. The extension is straightforward to implement without introducing extra theoretical/modeling efforts. Kou (2002) established the analytical solutions for European option pricing in a jump-diffusion model. However, American option pricing with jump-diffusion process remains challenging. Gukhal (2001) derives an EEP formula for American option in a jump-diffusion model. Despite all these efforts, one major drawback is that jump-diffusion models usually comes without closed-form transition densities. Even if such density exists, its functional form may be quite complicated in structure and the implementation requires significant amount of human and computer power. Our method, on the other hand, could overcomes these difficulties. By using Hermite polynomials expansion, we could control the computational cost of the pricing algorithm by specifying the degree of orders in the expansion. Moreover, our method could be easily extended to higher dimensional cases. Unlike conventional approaches like PDE-based method (finite difference, for example), Hermite polynomial expansion could be applied to a vector of stochastic processes and the results could be directly applied to multi-dimensional models.

The layout of the article is as follows. Section 2 describes the approach to American option valuation when the underlying asset prices follows a general diffusion process. Section 3 describes the generalization of the method to jump-diffusion process. Section 4 presents a numerical algorithm for the implementation of our proposed approach. Section 5 provides several examples to show the efficiency of our approach. Section 6 concludes the paper. All proofs are collected in appendix.

2 Valuation of American Option in Diffusion Models

2.1 American Option

We consider the stock price S defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}^*)$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions and following:

$$dS_t = (r(S_t; \theta) - \delta(S_t; \theta))\, dt + \sigma(S_t; \theta)\, dW_t \tag{1}$$

We also denote $\mu(S_t; \theta) = r(S_t; \theta) - \delta(S_t; \theta)$. Let $D_S = (\underline{s}, \bar{s})$ be the domain of the diffusion S.

The arbitrage-free price of an American put option with finite expiration time $T > 0$ and strike price K can be expressed as the expected value of its discounted payoff:

$$P(t, S_t) = \sup_{t \le \tau \le T} \mathbb{E}^* \left[e^{-(\tau - t) r} (K - S_\tau)^+ | S_t \right] \tag{2}$$

under the risk-neutral probability measure $\mathbb{P}^*$. Here τ is the stopping time.

2.2 Early Exercise Boundary

Let $\mathcal{B} = \{B_t : B_t \ge 0, t \in [0, T]\}$ denote the optimal early exercise boundary of the American option. Then the arbitrage-free put price, $P(t, S_t)$, solves the following free boundary problem:

$$\mathcal{L}P = 0,$$

$$P(T, S_T) = (K - S_T)^+,$$

$$\lim_{S_t \uparrow \infty} P(t, S_t) = 0,$$

$$\lim_{S_t \downarrow B_t} P(t, S_t) = K - B_t,$$

$$\lim_{S_t \downarrow B_t} \frac{\partial P(t, S_t)}{\partial S_t} = -1,$$

where $\mathcal{L}f = \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} + (r - \delta) S_t \frac{\partial f}{\partial S_t} - rf + \frac{\partial f}{\partial t}$.

Theorem 1 (Exercise Premium Representation) *We assume r, δ, and σ are continuously differentiable and (1) has a unique strong solution. Then on the continuation region $\mathcal{C}$*[1]*, the American put value $P_0 \equiv P(0, S_0 = s_0)$ has the following early exercise premium representation:*

$$P_0 = p_0 + e_0 \tag{3}$$

[1] The continuation region is the set of pairs (S, t) at which immediate exercise is sub-optimal.

where p_0 represents the price of a European put, that is,

$$p_0 \equiv p(0, S_0 = s_0) = \int_0^K e^{-rT}(K - S_T)\psi(S_T; S_0 = s_0)\, dS_T \tag{4}$$

and e_0 is the early exercise premium given by

$$e_0 \equiv e(0, S_0 = s_0, B(\cdot)) = \int_0^T \int_0^{B_t} (rK - \delta S_t) e^{-rt}\psi(S_t; S_0 = s_0)\, dS_t dt \tag{5}$$

and $\psi(S_t; S_0 = s_0)$ denotes the risk neutral transitional density function of S_t given $S_0 = s_0$. The exercise boundary B_t solves the recursive nonlinear integral equation

$$K - B_t = p(t, B_t) + e(t, B_t, B(\cdot)) \quad \forall t \in [0, T), \tag{6}$$

subject to the boundary condition $B_{T-} \equiv \lim_{t\uparrow T} B_t = \min\left\{K, \frac{r(B_T;\theta)}{\delta(B_T;\theta)}K\right\}$. At maturity $B_T = K \geq B_{T-}$. The functions p and e in (6) are defined as following:

$$p(t, B_t) \equiv \int_0^K (K - S_T)\psi(S_T; S_t = B_t)\, dS_T, \tag{7}$$

$$e(t, B_t, B(\cdot)) \equiv \int_t^T \int_0^{B_s} (rK - \delta S_s) e^{-r(s-t)}\psi(S_s; S_t = B_t)\, dS_s ds. \tag{8}$$

(4)-(6) in Theorem 1 for the valuation of American option can be simplified if we make further assumptions on the model. For example, if we assume the stock price follows Geometric Brownian Motion (GBM), i.e., $r(S_t;\theta) = r$, $\sigma(S_t;\theta) = \sigma$ for constants r and σ, and $\delta(S_t;\theta) = 0$, then we have a Black-Scholes style formula for the valuation of American option. We summarize this result in the following lemma.

Lemma 1 (Exercise Premium Representation under GBM) *If the stock price S follows Geometric Brownian Motion, then the American put value P_0 can be written as:*

$$P_0 = Ke^{-rT}N(k_2(S_0, K, T)) - S_0 N(k_1(S_0, K, T)) + rk\int_0^T e^{-rt}N(b_2(S_0, B_t, t))\, dt, \tag{9}$$

where $k_1(S_0, K, T) \equiv \frac{\log(K/S_0) - \rho_1 T}{\sigma\sqrt{T}}$, $k_2(S_0, K, T) \equiv \frac{\log(K/S_0) - \rho_2 T}{\sigma\sqrt{T}}$, $\rho_1 \equiv \rho_2 + \sigma^2 = r + \frac{\sigma^2}{2}$, $b_2(S_0, B_t, t) \equiv \frac{\log(B_t/S_0) - \rho_2 t}{\sigma\sqrt{t}}$, and B_t solves the following integral equation:

$$\begin{aligned} K - B_t = Ke^{-r(T-t)}N(k_2(B_t, K, T-t)) - B_t N(k_1(B_t, K, T-t)) \\ + rk\int_t^T e^{-r(s-t)}N(b_2(B_t, B_s, s-t))\, ds. \end{aligned} \tag{10}$$

2.3 Hermite Polynomial-based Approximation

Theorem 1 provides an intuitive approach to the valuation of American option in diffusion models, however, we still have two difficulties. First, most of the diffusion models does not admit a closed-form solution for the transition density. Second, the exercise boundary $\mathcal{B}$ is unknown in (5), and we need to solve the integral equation (6) recursively to compute $\mathcal{B}$.

In this study, we propose to use the Hermite polynomials to approximate the transition density and valuation of American option in Theorem 1. Our approach is based on the work by Ait-Sahalia (2002, 2006) and Yu (2007). The Hermite polynomial approach by Ait-Sahalia (2002) provides an explicit sequence of closed form functions to approximate the unknown transition density. Ait-Sahalia (2006) and Yu (2007) extends the approach to multivariate case and jump-diffusion models.

To approximate the transition density of the stock price S, we first transform S into a new random variable Y by defining $Y \equiv \gamma(S) = \int^S du/\sigma(u)$. We know Y has a unit diffusion, that is

$$dY_t = \mu_Y(Y_t;\theta)\,dt + dW_t,$$

where

$$\mu_Y(y;\theta) = \frac{\mu\left(\gamma^{-1}(y;\theta);\theta\right)}{\sigma\left(\gamma^{-1}(y;\theta);\theta\right)} - \frac{1}{2}\frac{\partial\sigma}{\partial S}\left(\gamma^{-1}(y;\theta);\theta\right). \tag{11}$$

We denote the domain of Y as $D_Y = \left(\underline{y}, \bar{y}\right)$. According to Ait-Sahalia (2002), the transition density of Y can be approximated using Hermite polynomials, and then the transition density of S can be derived from that of Y. More specifically, the transition density of S with time interval Δ can be approximated up to order m as following:

$$\begin{aligned}\widetilde{\psi}^{(m)}\left(S_{t+\Delta}=S';S_t=S\right) = \sigma^{-1}\left(S';\theta\right)\Delta^{-\frac{1}{2}}\phi\left(\frac{\gamma(S';\theta)-\gamma(S;\theta)}{\Delta^{\frac{1}{2}}}\right)\times \\ \exp\left(\int_{\gamma(S;\theta)}^{\gamma(S';\theta)}\mu(w;\theta)\,dw\right)\times\sum_{k=0}^{m}c_k\left(\gamma\left(S';\theta\right)|\gamma(S;\theta);\theta\right)\frac{\Delta^k}{k!},\end{aligned} \tag{12}$$

where $\phi(z) \equiv \exp\left(-z^2/2\right)/\sqrt{2\pi}$ denotes the density function of standard normal distribution, and for all $j \geq 1$,

$$\begin{aligned}c_j\left(\gamma\left(S';\theta\right)|\gamma(S;\theta);\theta\right) = j\left(S'-S\right)^{-j}\int_{\gamma(S;\theta)}^{\gamma(S';\theta)}(w-\gamma(S;\theta))^{j-1} \\ \times\left\{\lambda(w;\theta)\,c_{j-1}(w|\gamma(S;\theta);\theta) + \left(\partial^2 c_{j-1}(w|\gamma(S;\theta);\theta)/\partial w^2\right)/2\right\}dw\end{aligned} \tag{13}$$

where $\lambda(x;\theta) \equiv -\left(\mu_Y^2(x;\theta) + \partial\mu_Y(x;\theta)/\partial x\right)/2$ with μ_Y defined in (11), and $c_0 = 1$.

Once we get the approximation of the transition density of S in (12), we can plug $\widetilde{\psi}^{(m)}$ into Theorem 1 and get the approximation of the valuation of American option. More specifically, we have the following approximated early exercise premium representation for the value of American put up to order m:

$$\widetilde{P}_0^{(m)} = \widetilde{p}_0^{(m)} + \widetilde{e}_0^{(m)} \tag{14}$$

where $\widetilde{p}_0^{(m)} \equiv \widetilde{p}^{(m)}(0, S_0 = s_0) = \int_0^K e^{-rT}(K - S_T)\widetilde{\psi}^{(m)}(S_T; S_0 = s_0)\,dS_T$ represents the approximated price of a European put and $\widetilde{e}_0^{(m)}$ is the approximated early exercise premium given by

$$\widetilde{e}_0^{(m)} \equiv \widetilde{e}^{(m)}\left(0, S_0 = s_0, \widetilde{B}^{(m)}(\cdot)\right) = \int_0^T \int_0^{\widetilde{B}_t^{(m)}} (rK - \delta S_t)\, e^{-rt}\widetilde{\psi}^{(m)}(S_t; S_0 = s_0)\, dS_t dt. \tag{15}$$

The approximated exercise boundary up to order m, $\widetilde{B}_t^{(m)}$ solves the following recursive nonlinear integral equation:

$$K - \widetilde{B}_t^{(m)} = \widetilde{p}^{(m)}\left(t, \widetilde{B}_t^{(m)}\right) + \widetilde{e}^{(m)}\left(t, \widetilde{B}_t^{(m)}, \widetilde{B}^{(m)}(\cdot)\right) \quad \forall t \in [0, T). \tag{16}$$

Similarly to (7)-(8), $\widetilde{p}^{(m)}$ and $\widetilde{e}^{(m)}$ are defined as:

$$\widetilde{p}^{(m)}\left(t, \widetilde{B}_t^{(m)}\right) \equiv \int_0^K (K - S_T)\,\widetilde{\psi}^{(m)}\left(S_T; S_t = \widetilde{B}_t^{(m)}\right) dS_T, \tag{17}$$

$$\widetilde{e}^{(m)}\left(t, \widetilde{B}_t^{(m)}, \widetilde{B}^{(m)}(\cdot)\right) \equiv \int_t^T \int_0^{\widetilde{B}_s^{(m)}} (rK - \delta S_s)\, e^{-r(s-t)}\widetilde{\psi}^{(m)}\left(S_s; S_t = \widetilde{B}_t^{(m)}\right) dS_s ds, \tag{18}$$

subject to the boundary condition $\widetilde{B}_{T-}^{(m)} \equiv \lim_{t\uparrow T} \widetilde{B}_t^{(m)} = \min\left\{K, \frac{r(B_T;\theta)}{\delta(B_T;\theta)}K\right\}$, and $\widetilde{B}_T^{(m)} = B_T = K \geq \widetilde{B}_{T-}^{(m)}$.

The following theorem guarantees that proposed approach in (14)-(18) is a well-behaved approximation of the American put value P_0.

Theorem 2 *Under Assumptions 1-3 given in Appendix A, as* $m \to \infty$, *we have*

1. $\widetilde{p}_0^{(m)} \to p_0$,
2. $\widetilde{B}_t^{(m)} \to B_t$ *for any* $t \in [0, T]$,
3. $\widetilde{e}_0^{(m)} \to e_0$,
4. $\widetilde{P}_0^{(m)} \to P_0$.

Proof. *In Appendix B.* ■

3 Valuation of American Option in Jump-Diffusion Models

3.1 Valuation of American Option

In this section, we discuss the approximation of the value of American option when the underlying asset price follows a jump-diffusion process. Due to the discontinuous nature of the asset price path, the exercise premium representation is different from the one without jump. More specifically, we consider the stock price under risk-neutral measure follows:

$$d\log S_t = \left(r\left(S_t;\theta\right) - \delta\left(S_t;\theta\right) - \rho j\right)dt + \sigma\left(S_t;\theta\right)dW_t + (J-1)\,dq_t \tag{19}$$

where dq is a Poisson process with rate ρt, $J-1$ is the proportional change in the price due to a jump with density function ν as a function of jump size with support D_J, and $j = E\left(J-1\right)$. We assume $r\left(S_t;\theta\right)$, $\delta\left(S_t;\theta\right)$, and $\sigma\left(S_t;\theta\right)$ are smooth functions of S_t. Then based on Gukhal (2001), the American put value $P_0 \equiv P\left(0, S_0 = s_0\right)$ has the following representation:

$$P_0 = p_0 + e_0 + g_0 \tag{20}$$

where

$$p_0 \equiv p\left(0, S_0 = s_0\right) = \int_0^K e^{-rT}\left(K - S_T\right)\psi\left(S_T; S_0 = s_0\right)dS_T \tag{21}$$

$$e_0 \equiv e\left(0, S_0 = s_0, B\left(\cdot\right)\right) = \int_0^T\int_0^{B_t}\left(rK - \delta S_t\right)e^{-rt}\psi\left(S_t; S_0 = s_0\right)dS_t dt \tag{22}$$

and

$$\begin{aligned} g_0 \equiv g\left(0, S_0 = s_0, B\left(\cdot\right)\right) = \int_0^T\int_0^{B_{t-}}\int_{B_t}^{\infty} & e^{-rt}\rho\left(P\left(t, J_t S_{t-}\right) - \left(K - J_t S_{t-}\right)\right) \\ & \times\psi\left(S_{t-}; S_0 = s_0\right)\psi\left(J_t S_{t-}; S_{t-}\right)d\left(J_t S_{t-}\right)dS_{t-}dt. \end{aligned} \tag{23}$$

The exercise boundary B_t solves the following integral equation

$$K - B_t = p\left(t, B_t\right) + e\left(t, B_t, B\left(\cdot\right)\right) - g\left(t, B_t, B\left(\cdot\right)\right) \tag{24}$$

where

$$p\left(t, B_t\right) = \int_0^K\left(K - S_T\right)\psi\left(S_T; B_t\right)dS_T$$

$$e\left(t, B_t, B\left(\cdot\right)\right) = \int_t^T\int_0^{B_s}\left(rK - \delta S_s\right)e^{-r(s-t)}\psi\left(S_s; B_t\right)dS_s ds$$

$$\begin{aligned} g\left(t, B_t, B\left(\cdot\right)\right) = \int_t^T\int_0^{B_{s-}}\int_{B_s}^{\infty} & e^{-r(s-t)}\rho\left(P\left(s, J_s S_{s-}\right) - \left(K - J_s S_{s-}\right)\right) \\ & \times\psi\left(S_{s-}; B_t\right)\psi\left(J_s S_{s-}; S_{s-}\right)d\left(J_s S_{s-}\right)dS_{s-}ds. \end{aligned}$$

The representation in (20) has a straightforward interpretation. As in the case without jump, p_0 represents the price of a European put, e_0 is the early exercise premium. g_0 is the re-balancing cost due to jumps from the exercise region (the stock price is below the exercise boundary) into the continuation region (the stock price is above the exercise boundary).

3.2 Hermite Polynomial-based Approximation

Our approach to jump-diffusion models is similar to our approach in Section 2.3. We first approximate the transition density using Hermite polynomials. According to Yu (2007), an approximation of order $m > 0$ is obtained as following:

$$\widetilde{\psi}^{(m)}\left(S_{t+\Delta} = S'; S_t = S\right) = \Delta^{-\frac{1}{2}} \exp\left[-\frac{C^{(-1)}\left(S, S'\right)}{\Delta}\right] \sum_{k=0}^{m} C^{(k)}\left(S, S'\right) \Delta^k + \sum_{k=1}^{m} D^{(k)}\left(S, S'\right) \Delta^k \quad (25)$$

where

$$C^{(-1)}\left(S, S'\right) = \frac{1}{2}\left[\int_S^{S'} \sigma(s)^{-1} ds\right]^2, \quad (26)$$

$$C^{(0)}\left(S, S'\right) = \frac{1}{\sqrt{2\pi}\sigma\left(S'\right)} \exp\left[\int_S^{S'} \frac{\mu(s)}{\sigma^2(s)} - \frac{\sigma'(s)}{2\sigma(s)} ds\right], \quad (27)$$

$$C^{(k+1)}\left(S, S'\right) = -\left[\int_S^{S'} \sigma(s)^{-1} ds\right]^{-k+1} \int_S^{S'} \left\{\exp\left[\int_u^{S} \frac{\mu(u)}{\sigma^2(u)} - \frac{\sigma'(u)}{2\sigma(u)} du\right] \times \sigma(s)^{-1} \left[\int_s^{S'} \sigma(u)^{-1} du\right]^k \left[\rho(s) - \mathcal{L}\right] C^{(k)}\left(s, S'\right)\right\} ds, \quad \text{for } k \geq 0, \quad (28)$$

$$D^{(1)}\left(S, S'\right) = \rho(S) - \upsilon\left(S' - S\right), \quad (29)$$

$$D^{(k+1)}\left(S, S'\right) = \frac{1}{1+k}\left[\mathfrak{L} D^{(k)}\left(S, S'\right) + \sqrt{2\pi}\rho(S) \sum_{r=0}^{k} \frac{M_{2r}^1}{(2r)!} \frac{\partial^{2r}}{\partial w^{2r}} m_{k-r}\left(S, S', w\right)|_{w=0}\right], \quad \text{for } k \geq 0, \quad (30)$$

where

$$m_k\left(S, S', w\right) \equiv C^{(k)}\left(w_B^{-1}(w), S'\right) \upsilon\left(w_B^{-1}(w) - S\right) \sigma\left(w_B^{-1}(w)\right), \quad (31)$$

$$M_{2r}^1 \equiv 1/\sqrt{2\pi} \int_{\mathbb{R}} \exp\left(-s^2/2\right) s^{2r} ds, \quad (32)$$

$$w_B\left(S, S'\right) = \int_{S'}^{S} \sigma(s)^{-1} ds, \quad (33)$$

$$\mathscr{L}f\left(s,s'\right)=\frac{1}{2}\sigma^2 s^2\frac{\partial^2 f}{\partial s^2}\left(s,s'\right)+\left(r-\delta-\rho j\right)s\frac{\partial f}{\partial s}\left(s,s'\right),\tag{34}$$

and

$$\mathfrak{L}f\left(s,s'\right)=\mathscr{L}f\left(s,s'\right)+\rho\int_{D_J}\left[f\left(s+c,s'\right)-f\left(s,s'\right)\right]\upsilon\left(c\right)dc.\tag{35}$$

Once we get the Hermite polynomial approximation of the transition density as above, we plug the approximation into (21)-(23), and solve the integral equation (24) recursively. More specifically, we have the approximated American put value up to order m, $\widetilde{P}_0^{(m)}\equiv\widetilde{P}^{(m)}\left(0,S_0=s_0\right)$ by:

$$\widetilde{P}_0^{(m)}=\widetilde{p}_0^{(m)}+\widetilde{e}_0^{(m)}+\widetilde{g}_0^{(m)}\tag{36}$$

where

$$\widetilde{p}_0^{(m)}\equiv\widetilde{p}^{(m)}\left(0,S_0=s_0\right)=\int_0^K e^{-rT}\left(K-S_T\right)\widetilde{\psi}^{(m)}\left(S_T;S_0=s_0\right)dS_T,\tag{37}$$

$$\widetilde{e}_0^{(m)}\equiv\widetilde{e}^{(m)}\left(0,S_0=s_0,\widetilde{B}^{(m)}\left(\cdot\right)\right)=\int_0^T\int_0^{\widetilde{B}_t^{(m)}}\left(rK-\delta S_t\right)e^{-rt}\widetilde{\psi}^{(m)}\left(S_t;S_0=s_0\right)dS_t dt,\tag{38}$$

and

$$\widetilde{g}_0^{(m)}\equiv\widetilde{g}^{(m)}\left(0,S_0=s_0,\widetilde{B}^{(m)}\left(\cdot\right)\right)=\int_0^T\int_0^{\widetilde{B}_{t-}^{(m)}}\int_{\widetilde{B}_t^{(m)}}^{\infty}e^{-rt}\rho\left(\widetilde{P}^{(m)}\left(t,J_tS_{t-}\right)-\left(K-J_tS_{t-}\right)\right)$$
$$\times\widetilde{\psi}^{(m)}\left(S_{t-};S_0=s_0\right)\widetilde{\psi}^{(m)}\left(J_tS_{t-};S_{t-}\right)d\left(J_tS_{t-}\right)dS_{t-}dt.\tag{39}$$

The approximated exercise boundary up to order m, $\widetilde{B}_t^{(m)}$ solves the following recursive nonlinear integral equation:

$$K-\widetilde{B}_t^{(m)}=\widetilde{p}^{(m)}\left(t,\widetilde{B}_t^{(m)}\right)+\widetilde{e}^{(m)}\left(t,\widetilde{B}_t^{(m)},\widetilde{B}^{(m)}\left(\cdot\right)\right)-\widetilde{g}^{(m)}\left(t,\widetilde{B}_t^{(m)},\widetilde{B}^{(m)}\left(\cdot\right)\right)\tag{40}$$

where

$$\widetilde{p}^{(m)}\left(t,\widetilde{B}_t^{(m)}\right)\equiv\int_0^K\left(K-S_T\right)\widetilde{\psi}^{(m)}\left(S_T;S_t=\widetilde{B}_t^{(m)}\right)dS_T,\tag{41}$$

$$\widetilde{e}^{(m)}\left(t,\widetilde{B}_t^{(m)},\widetilde{B}^{(m)}\left(\cdot\right)\right)\equiv\int_t^T\int_0^{\widetilde{B}_s^{(m)}}\left(rK-\delta S_s\right)e^{-r(s-t)}\widetilde{\psi}^{(m)}\left(S_s;S_t=\widetilde{B}_t^{(m)}\right)dS_s ds,\tag{42}$$

$$\widetilde{g}^{(m)}\left(t,\widetilde{B}_t^{(m)},\widetilde{B}^{(m)}\left(\cdot\right)\right)\equiv\int_t^T\int_0^{\widetilde{B}_{s-}^{(m)}}\int_{\widetilde{B}_s^{(m)}}^{\infty}e^{-r(s-t)}\rho\left(\widetilde{P}^{(m)}\left(s,J_sS_{s-}\right)-\left(K-J_sS_{s-}\right)\right)$$
$$\times\widetilde{\psi}^{(m)}\left(S_{s-};\widetilde{B}_t^{(m)}\right)\widetilde{\psi}^{(m)}\left(J_sS_{s-};S_{s-}\right)d\left(J_sS_{s-}\right)dS_{s-}ds.\tag{43}$$

4 Numerical Method and Algorithm

Following Detemple (2006), we divide the period $[0, T]$ into N equal subintervals and let $\Delta = T/N$. We then use a step function to recursively compute the exercise boundary. The algorithm works as follows. Suppose our step function approximation of the exercise boundary is $\left\{\widetilde{B}^{(m,N)}_{n\Delta}, n = 0, \ldots, N\right\}$[2]. The terminal condition tells us that $\widetilde{B}^{(m,N)}_{N\Delta} = \min\left\{K, \frac{r(B_T;\theta)}{\delta(B_T;\theta)}K\right\}$. Suppose that $\widetilde{B}^{(m,N)}_{l\Delta}$ is known for all $l > n$, then $\left\{\widetilde{B}^{(m,N)}_{l\Delta}, l = 0, \ldots, n\right\}$ can be obtained by discretizing the integral in (16) for a diffusion model or (40) for a jump-diffusion model using trapezoidal rule. For example, we obtain the following equation for diffusion models:

$$\begin{aligned} K - \widetilde{B}^{(m,N)}_{l\Delta} = \widetilde{p}^{(m)}\left(l\Delta, \widetilde{B}^{(m,N)}_{l\Delta}\right) + \sum_{q=l+1}^{N-l} \widetilde{\epsilon}^{(m)}\left((q-l)\Delta, \widetilde{B}^{(m,N)}_{l\Delta}, \widetilde{B}^{(m,N)}_{q\Delta}\right)\Delta \\ + \left[\widetilde{\epsilon}^{(m)}\left(0, \widetilde{B}^{(m,N)}_{l\Delta}, \widetilde{B}^{(m,N)}_{l\Delta}\right) + \widetilde{\epsilon}^{(m)}\left((N-l)\Delta, \widetilde{B}^{(m,N)}_{l\Delta}, \widetilde{B}^{(m,N)}_{N\Delta}\right)\right]\frac{\Delta}{2} \end{aligned} \tag{44}$$

where

$$\widetilde{\epsilon}^{(m)}\left(s\Delta, \widetilde{B}^{(m)}_{t}, \widetilde{B}^{(m)}_{t+s\Delta}\right) \equiv \int_0^{\widetilde{B}^{(m)}_{t+s\Delta}} \left(rK - \delta S_{t+s\Delta}\right) e^{-rs\Delta}\widetilde{\psi}^{(m)}\left(S_{t+s\Delta}; S_t = \widetilde{B}^{(m)}_{t}\right) dS_{t+s\Delta} \tag{45}$$

And for jump-diffusion models, we have

$$\begin{aligned} K - \widetilde{B}^{(m,N)}_{l\Delta} = \widetilde{p}^{(m)}\left(l\Delta, \widetilde{B}^{(m,N)}_{l\Delta}\right) + \sum_{q=l+1}^{N-l} \widetilde{\epsilon}^{(m)}\left((q-l)\Delta, \widetilde{B}^{(m,N)}_{l\Delta}, \widetilde{B}^{(m,N)}_{q\Delta}\right)\Delta \\ + \left[\widetilde{\epsilon}^{(m)}\left(0, \widetilde{B}^{(m,N)}_{l\Delta}, \widetilde{B}^{(m,N)}_{l\Delta}\right) + \widetilde{\epsilon}^{(m)}\left((N-l)\Delta, \widetilde{B}^{(m,N)}_{l\Delta}, \widetilde{B}^{(m,N)}_{N\Delta}\right)\right]\frac{\Delta}{2} \\ + \left[\widetilde{\eta}^{(m)}\left(0, \widetilde{B}^{(m,N)}_{l\Delta}, \widetilde{B}^{(m,N)}_{l\Delta}\right) + \widetilde{\eta}^{(m)}\left((N-l)\Delta, \widetilde{B}^{(m,N)}_{l\Delta}, \widetilde{B}^{(m,N)}_{N\Delta}\right)\right]\frac{\Delta}{2} \\ + \sum_{q=l+1}^{N-l} \widetilde{\eta}^{(m)}\left((q-l)\Delta, \widetilde{B}^{(m,N)}_{l\Delta}, \widetilde{B}^{(m,N)}_{q\Delta}\right)\Delta \end{aligned} \tag{46}$$

where $\widetilde{\epsilon}^{(m)}$ is defined as in (45), and $\widetilde{\eta}^{(m)}$ is defined by

$$\begin{aligned} &\widetilde{\eta}^{(m)}\left(s\Delta, \widetilde{B}^{(m)}_{t}, \widetilde{B}^{(m)}_{t+s\Delta}\right) \equiv \\ &\int_0^{\widetilde{B}^{(m)}_{t+s\Delta-}} \int_{\widetilde{B}^{(m)}_{t+s\Delta}}^{\infty} e^{-rs\Delta}\rho\left(\widetilde{P}^{(m)}\left(s\Delta, J_{t+s\Delta}S_{t+s\Delta s-}\right) - \left(K - J_{t+s\Delta}S_{t+s\Delta s-}\right)\right) \\ &\times \widetilde{\psi}^{(m)}\left(S_{t+s\Delta-}; \widetilde{B}^{(m)}_{t}\right)\widetilde{\psi}^{(m)}\left(J_{t+s\Delta}S_{t+s\Delta-}; S_{t+s\Delta-}\right) d\left(J_{t+s\Delta}S_{t+s\Delta-}\right) dS_{t+s\Delta-} \end{aligned} \tag{47}$$

We run the above procedure recursively, and obtain the exercise boundary $\left\{\widetilde{B}^{(m,N)}_{n\Delta}, n = 0, \ldots, N\right\}$. Finally, the American put value can be computed by plug the exercise boundary into (14) for diffusion models or (36) for jump-diffusion models.

[2]Our step function approximation is based on Huang, Subrahmanyan and Yu (1996). A exponential function approximation can also be found in Ju (1998)

5 Applications

In this section, we provide several examples of approximation of American put value and the responding exercise boundary for diffusion models and jump-diffusion models respectively. We use these examples to illustrate the accuracy and speed of our proposed approach. We approximate the transition density up to order $m = 2$ through all the examples in this section.

5.1 Applications in Diffusion Models

5.1.1 Geometric Brownian Motion (GBM) Model

In the Geometric Brownian Motion model, the stock price S follows

$$dS_t = (r - \delta) S_t dt + \sigma S_t dW_t,$$

where r, δ, and σ are constants. To test the efficiency of our recursive algorithm in Section 2, we compare the results of American put value from our approach with those from three widely used methods: the binomial method by Cox, Ross and Rubinstein (1979), accelerated binomial methods by Breen (1991), the finite difference method, and the analytical approximation by Geske and Johnson (1984). We use the results from the binomial method with 10,000 time steps as a benchmark to measure the accuracy. Following Huang, Subrahmanyan and Yu (1996) and Geske and Johnson (1984), we set $S_0 = 40$, $r = 4.88\%$, and $\delta = 0$.

Table 1 reports the results of American put value from 6 approaches. Column 1 through 3 represent the values of the parameters, K (strike price), σ (volatility), and T (maturity), respectively. Column 4 gives the numerical results from the binomial method with 10,000 time steps, and we take this approach as benchmark. Column 5 includes the results of Table I of Geske and Johnson (1984). Columns 6 through 8 report the results from the binomial method with 150 time steps, the finite difference method with 200 steps, the accelerated binomial method with 150 time steps. Column 9 provides the results of our proposed approach with 100 time steps. The accuracy is measured by its root of the mean squared error as shown in the last row. It's clear in this table that our approach achieve the best performance in accuracy compared with other methods.

We report the approximated exercise boundary in Figure A1 for several combinations of strike price and volatility. Parameters values are the same as those in Table 1 and T is 0.5833. This figure shows the marginal effect of strike price and volatility to the exercise boundary. For example, as the volatility turns to be small, the responding exercise boundary will turn to be flat. The intuition for this result is when the volatility is small, the return from withholding the American option is limited, thus, the American put option will be exercised at a higher boundary instead of a lower one. This result can also be confirmed by checking the partial difference of the exercise boundary with respect to the volatility in (10).

Table 1: Value of American Put Option Based on Different Numerical Methods

K	σ	T (yr)	Binomial	G&J	Binomial (150)	Accelerated	FD	Hermite
35	0.2	0.0833	0.0062	0.0062	0.0061	0.0061	0.0278	0.0062
35	0.2	0.3333	0.2004	0.1999	0.1995	0.1994	0.2382	0.2004
35	0.2	0.5833	0.4328	0.4321	0.4340	0.4331	0.4624	0.4329
40	0.2	0.0833	0.8522	0.8528	0.8512	0.8517	0.9874	0.8523
40	0.2	0.3333	1.5798	1.5807	1.5783	1.5752	1.6244	1.5800
40	0.2	0.5833	1.9904	1.9905	1.9886	1.9856	2.0177	1.9906
45	0.2	0.0833	5.0000	4.9985	5.0000	4.9200	5.0052	5.0000
45	0.2	0.3333	5.0883	5.0951	5.0886	4.9253	5.1327	5.0886
45	0.2	0.5833	5.2670	5.2719	5.2677	5.2844	5.2699	5.2673
35	0.3	0.0833	0.0774	0.0744	0.0775	0.0772	0.1216	0.0774
35	0.3	0.3333	0.6975	0.6969	0.6993	0.6977	0.7300	0.6976
35	0.3	0.5833	1.2198	1.2194	1.2239	1.2218	1.2407	1.2199
40	0.3	0.0833	1.3099	1.3100	1.3083	1.3095	1.3860	1.3100
40	0.3	0.3333	2.4825	2.4817	2.4799	2.4781	2.5068	2.4828
40	0.3	0.5833	3.1696	3.1733	3.1665	3.1622	3.1819	3.1699
45	0.3	0.0833	5.0597	5.0599	5.0600	5.0632	5.1016	5.0598
45	0.3	0.3333	5.7056	5.7012	5.7065	5.6978	5.7193	5.7059
45	0.3	0.5833	6.2436	6.2365	6.2448	6.2395	6.2477	6.2440
35	0.4	0.0833	0.2466	0.2466	0.2454	0.2456	0.2949	0.2466
35	0.4	0.3333	1.3460	1.3450	1.3505	1.3481	1.3696	1.3461
35	0.4	0.5833	2.1549	2.1568	2.1602	2.1569	2.1676	2.1551
40	0.4	0.0833	1.7681	1.7679	1.7658	1.7674	1.8198	1.7683
40	0.4	0.3333	3.3874	3.3632	3.3835	3.3863	3.4011	3.3877
40	0.4	0.5833	4.3526	4.3556	4.3480	4.3426	4.3567	4.3530
45	0.4	0.0833	5.2868	5.2855	5.2875	5.2863	5.3289	5.2870
45	0.4	0.3333	6.5099	6.5093	6.5103	6.5054	6.5147	6.5101
45	0.4	0.5833	7.3830	7.3831	7.3897	7.3785	7.3792	7.3833
RMSE			0.0000	5.3383e-03	2.6380e-03	3.5302e-02	4.1041e-02	2.1602e-04

The GBM model is one of the limited cases that we know the true transition density, and to examine the accuracy of our approximated exercise boundary, we plug in the true transition density into the numerical algorithm in Section 4, and compare it's result with our approximated exercise boundary. We report this comparison for different strike prices and volatilities in Figure A2.

In Figure A3, we compare the results of our proposed approach with the finite difference method in approximating the exercise boundary.

To further investigate the performance of our approach, we report the approximated value of American put with respect to strike prices through 10 to 70 in Figure A4a. Also, We computed the approximated American put value for different strikes based on various orders of the Hermite polynomial-based approximation of the transition density. More specifically, first order means $m = 1$; second order means $m = 2$; third order means $m = 3$. We use the results from binomial method as benchmark for comparison, and report the relative error in approximation in Figure A4b.

5.1.2 Constant Elasticity Volatility (CEV) Model

The Constant Elasticity Volatility model assumes that the stock price S follows

$$dS_t = (r - \delta) S_t dt + \sigma S_t^{\alpha/2} dW_t,$$

where r, δ, σ, and α are constants. Detemple and Kitapbayev (2018) apply this model to study the pricing of American VIX option. Further extension of this model on the valuation of VIX option can also be found in Goard and Mazur (2013).

We set $K = 100$, $r = 6/100$, $\delta = r/2$, $\sigma = \sqrt{10}/5$, $S_0 = 40$, and $T = 1$. We report in Figure A5 the approximated exercise boundary of CEV model for $\alpha = 1.9$ and $\alpha = 1.7$ respectively. From Figure A5 we can find as α becomes smaller, the volatility of the stock price becomes smaller, and thus the optimal exercise boundary will be higher. This is the same result as that we find in the GBM model.

5.1.3 Nonlinear Mean Reversion (NMR) Model

The Nonlinear Mean Reversion model assumes that

$$dS_t = \left(\frac{a}{S_t} + b + cS_t + vS_t^2 \right) dt + \sigma S_t^{\gamma} dW_t,$$

where a, b, c, v, σ, and γ are constants. This model has been discussed in Ait-Sahalia (1996, 1999), and Gallant and Tauchen (1998) for modelling the interest. Eraker and Wang (2012) propose a similar model for VIX option.

In the NMR model, we set $a = 500$, $b = 5$, $c = 0.05$, $v = -0.05$, $\sigma = 0.2$, $\gamma = 3/2$, $K = 20$, $r = 5/100$, $\delta = 0$, $S_0 = 20$, and $T = 0.0833$. We report the approximated exercise boundary in Figure A6.

5.1.4 Double Mean Reversion (DMR) Model

The Double Mean Reversion model assumes that

$$dS_t = \beta\left(y_t - S_t\right)dt + \sigma\sqrt{S_t}dW_t$$

$$dy_t = \xi\left(\alpha - y_t\right)dt + \kappa\sqrt{y_t}dU_t$$

where W and U are two independent Brownian motions, and α, β, ξ, κ, and σ are constants.

Based on the usual square root model, this DMR model includes an additional stochastic factor for the mean level of the stock price. In this model, the speed of mean-reversion towards the short-run stochastic mean level of the stock price is controlled by β, and the speed of mean-reversion towards the long-run mean level of the short-run stochastic mean is controlled by ξ. This model is discussed in Amengual (2008), Mencia and Sentana (2009) and Egloff et al. (2010).

In the DMR model, the optimal exercise boundary will be a function of time, t, and y. We set $K = 40/100$, $r = 4.88/100$, $\delta = 0$, $\sigma = 0.25$, $\kappa = 0.2$, $\beta = 2.5$, $\xi = 4$, $\alpha = 0.25$, $T = 0.5$. We report in Figure A7 the approximated exercise boundary in the DMR model. The boundary is approximated with 20 steps on time, and 100 steps on y for y in $[0, 1]$.

5.2 Applications in Jump-Diffusion Models

5.2.1 Merton's Jump-Diffusion Model

Merton (1976) proposed a jump diffusion model to incorporate discontinuous returns, and derived the closed-form vanilla option pricing formula. This Merton's jump-diffusion model assume that

$$d\log S_t = \left(r - \delta - \lambda j\right)dt + \sigma dW_t + \left(J - 1\right)dq_t$$

where dq is a Poisson process with rate λt, J has a lognormal distribution with mean μ_J and variance σ_J^2, and $j = E\left[J - 1\right] = \exp\left(\mu_J + \sigma_J^2/2\right) - 1$.

We set $K = 40$, $r = 4.88/100$, $\sigma = 0.2$, $\mu_J = 0$, $\sigma_J = 0.2$, $S_0 = 40$, and $T = 0.5$. We approximate the exercise boundary for different values of ρ. More specifically, we try $\lambda = 1/100, 10/100$, and $25/100$, and present the result in Figure A8. By comparing the exercise boundaries in Figure A8, we can find when λ is smaller, the exercise boundary will be higher. The intuition for this result is that when λ is smaller, the jump of the return will happen less frequently, and thus the return will be less volatile. Similar to the models without jumps in Section 5.1, when the stock price or the return is less volatile, the exercise boundary will be higher.

5.2.2 Kou's Jump-Diffusion Model

To incorporate the leptokurtic feature of the return distribution and "volatility smile" phenomenon in option market, Kou (2002) proposes a double exponential jump-diffusion model. This model assumes

$$d\log S_t = (r-\delta)\,dt + \sigma dW_t + Jdq_t$$

where J has an asymmetric double exponential distribution with density:

$$\upsilon(z) = p * \eta_1 e^{-\eta_1 z}\mathbf{1}_{\{z\geq 0\}} + q * \eta_2 e^{-\eta_2 z}\mathbf{1}_{\{z<0\}},$$

where $\eta_1 > 1$, $\eta_2 > 0$, $p+q=1$, and $0 \leq p,q \leq 1$. The mean, variance, and skewness of the jump size in log returns are:

$$\varphi_1 = \frac{p}{\eta_1} - \frac{q}{\eta_2},$$

$$\varphi_2 = pq\left(\frac{1}{\eta_1}+\frac{1}{\eta_2}\right)^2 + \frac{p}{\eta_1^2} + \frac{q}{\eta_2^2},$$

$$\varphi_3 = \frac{2\left(p^3-1\right)\eta_1^3 - 2\left(q^3-1\right)\eta_2^3 + 6pq\eta_1\eta_2\left(q\eta_2 - p\eta_1\right)}{\left(p\eta_2^2 + q\eta_1^2 + pq\left(\eta_1+\eta_2\right)^2\right)^{\frac{3}{2}}}.$$

Also, dq is a Poisson process with rate λt.

In Figure A9, we report the approximated exercise boundary for Kou's jump-diffusion model for $\lambda = 1/100, 10/100$, and $20/100$. We set $K=40$, $r=4.88/100$, $\delta=0$, $\sigma=0.2$, $p=0.04$, $q=0.96$, $\eta_1=3.7$, $\eta_2=1.8$, $S_0=40$, and $T=0.5$. We approximate the boundary with 50 steps on time. Similarly, we find that the smaller the intensity of the jump is, the higher the exercise boundary will be.

6 Conclusion

In this study, we develop a new approach to approximate the exercise boundary and value of the American put option based on Hermite polynomials. We also provide a numerical scheme to implement our approach. We show theoretically that our approximation will converge to the true exercise boundary and value of the American put option, and provide evidence of the efficiency of our approach through several numerical examples including diffusion processes and jump-diffusion processes. We only discuss the case of American put option, however, the value of the American call option can be approximated similarly.

A drawback of our approach is the computational complexity. Although we have a closed-form approximation of the transition density for a given jump-diffusion model, we need to evaluate the integral of the transition density for the valuation of the American option, and the integral usually does not admit a closed-form solution. This incurs a heavy computational burden to our numerical implementation. Other approaches in approximating the transition density of a given jump-diffusion model, such as finite mixture models, can simplify the integral equation and thus reduce the computational complexity. We leave this for future research.

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A Assumptions

Assumption 1 (Smoothness of Coefficients) *: The functions $r(S_t;\theta)$, $\delta(S_t;\theta)$ and $\sigma(S_t;\theta)$ are infinitely differentiable in S, and three times continuously differentiable in θ, for all $S \in D_S$ and $\theta \in \Theta$.*

Assumption 2 (Non-Degeneracy of the Diffusion) *:*

1. *If $D_S = (-\infty, +\infty)$, there exists a constant c such that $\sigma(S_t;\theta) > c > 0$ for all $S \in D_S$ and $\theta \in \Theta$.*
2. *If $D_S = (0, +\infty)$, there exists constants $\zeta_0 > 0$, $\omega > 0$, $\eta \geq 0$ such that $\sigma(S_t;\theta) \geq \omega S^\eta$ for all $0 < S \leq \zeta_0$ and $\theta \in \Theta$.*

Assumption 3 (Boundary Behavior) *: For all $\theta \in \Theta$, $\mu_Y(y;\theta)$ in (11) and its derivatives with respect to y and θ have at most polynomial growth near the boundaries and $\lim_{y \to \underline{y}^+ \text{or} \bar{y}^-} \lambda(y;\theta) < +\infty$ where $\lambda(y;\theta) \equiv -\left(\mu_Y^2(y;\theta) + \partial\mu_Y(y;\theta)/\partial y\right)/2$.*

1. *Left Boundary: If $\underline{y} = 0$, there exist constants ε_0, χ, ς such that for all $0 < y \leq \varepsilon_0$ and $\theta \in \Theta$, $\mu_Y(y;\theta) \geq \chi y^{-\varsigma}$ where either $\varsigma > 1$ and $\chi > 0$, or $\varsigma = 1$ and $\chi \geq 1$. If $\underline{y} = -\infty$, there exist constants $E_0 > 0$ and $K_0 > 0$ such that for all $y \leq -E_0$ and $\theta \in \Theta$, $\mu_Y(y;\theta) \geq K_0 y$.*
2. *Right Boundary: If $\bar{y} = +\infty$, there exist constants $E_0 > 0$ and $K_0 > 0$ such that for all $y \geq E_0$ and $\theta \in \Theta$, $\mu_Y(y;\theta) \leq K_0 y$. If $\bar{y} = 0$, there exist constants ε_0, χ, ς such that for all $0 > y \geq -\varepsilon_0$ and $\theta \in \Theta$, $\mu_Y(y;\theta) \leq -\chi|y|^{-\varsigma}$ where either $\varsigma > 1$ and $\chi > 0$, or $\varsigma = 1$ and $\chi \geq 1/2$.*

B Proof

B.1 Proof of Theorem 2

Step 1: According to Ait-Sahalia (2002), we apply the following transform to S by $S \to Y \to Z$:

$$Y \equiv \gamma(S) = \int^S du/\sigma(u),$$

and

$$Z \equiv \Delta^{-\frac{1}{2}}(Y - y_0).$$

We approximate the transition density of Z by the following Hermite polynomial construction up to order m:

$$\widetilde{\psi}_Z^{(m)}\left(Z_{t+\Delta} = z'; Z_t = z\right) \equiv \phi\left(z'\right) \sum_{j=0}^{m} \eta_Z^{(j)}(\Delta, z;\theta) H_j(z),$$

where ϕ is the density of standard normal distribution, H is the Hermite polynomials and η_Z is the coefficient in the approximation. We have

$$|\eta_Z^{(j)}(\Delta, z; \theta) H_j(z)| \leq Q\left\{1 + |z^{5/2}/2^{5/4}|\right\} e^{z^2/4} \times \left\{j^{-1/2}(j+1)^{-1} + (j+1)! v_{j+1}^2(\Delta, z)\right\}/2,$$

where

$$v_{j+1}(\Delta, z) = (j!)^{-1} \int_{-\infty}^{+\infty} H_j(w) \left\{\frac{\partial p_Z(\Delta, w|z; \theta)}{\partial w}\right\} dw,$$

and Q is a constant. $p_Z(\Delta, z'|z; \theta)$ is defined as the true transition density of Z. It's easy to verify that $j^{-1/2}(j+1)^{-1} \leq \varrho$ (a constant), and

$$\begin{aligned}\sum_{j=0}^{m}(j)! v_j^2(\Delta, z) &\leq \int_{-\infty}^{+\infty} e^{w^2/2}\left\{\frac{\partial p_Z(\Delta, w|z; \theta)}{\partial w}\right\}^2 dw \\ &\leq \int_{-\infty}^{+\infty} e^{w^2/2}\left(b_0 e^{-3w^2/8} R(|w|, |z|) e^{b_1|w||z| + b_2|w| + b_3|z| + b_4 z^2}\right) dw\end{aligned}$$

where R is a polynomial of finite order in $(|w|, |z|)$ with coefficients uniform in $\theta \in \Theta$, and where the constants $b_i, i = 0, \ldots 4$, are uniform in $\theta \in \Theta$.

By Lebesgue's Dominant Convergence Theorem, $\sum_{j=0}^{m}(j)! v_j^2(\Delta, z)$ is convergent, and thus bounded.

Then,

$$|\widetilde{\psi}_Z^{(m)}| \leq \phi(z)\left\{Q\left\{1 + |z^{5/2}/2^{5/4}|\right\} e^{z^2/4}\varrho' + \sum_{j=0}^{m}(j+1)! v_{j+1}^2(\Delta, z)\right\}/2$$

where $\varrho' = m\varrho$.

Notice that $Q\left\{1 + |z^{5/2}/2^{5/4}|\right\} e^{z^2/4}\varrho'$ and $\sum_{j=0}^{m}(j+1)! v_{j+1}^2(\Delta, z)$ are integrable. It follows from above that $\widetilde{\psi}_Z^{(m)}$ is also integrable.

By our assumption, $\sigma$is globally nondegenerate, i.e., there exists a constant ξ such that $\sigma^{-1}(s) < \xi^{-1} < \infty$. We can recover the transition density of S from that of Z by

$$\widetilde{\psi}_S^{(m)}(S'; S) = \sigma^{-1}\Delta^{-1/2}\widetilde{\psi}_Z^{(m)}\left(\Delta^{-1/2}(\gamma(S') - \gamma(S)), \gamma(S)\right).$$

It's easy to verify that $\widetilde{\psi}_S^{(m)}$ is integrable. From Ait-Sahalia (2002), we know $\widetilde{\psi}_S^{(m)}$ is convergent to the true transition density ψ as $m \to \infty$. Then, by DCT, we have the integral of $\widetilde{\psi}_S^{(m)}$ will also be convergent to the integral of ψ. That is $\widetilde{p}_0^{(m)} \to p_0$ as $m \to \infty$. We complete the proof of the first part of Theorem 2.

Step 2: By our construction in Section 2.3 for approximating the exercise boundary, we have $\widetilde{B}_T^{(m)} = B_T$ and $\widetilde{B}_{T-}^{(m)} = B_{T-}$. Now, we assume that $\widetilde{B}_s^{(m)} = B_s$ for $s > t$, then $\widetilde{e}^{(m)}\left(t, \widetilde{B}_t^{(m)}, \widetilde{B}^{(m)}(\cdot)\right) =$

$\widetilde{e}^{(m)}\left(t, \widetilde{B}_t^{(m)}, B(\cdot)\right)$. Based on our analysis in Step 1, we know that $\widetilde{e}^{(m)}$ and $\widetilde{p}^{(m)}$ are well defined smooth function of $\widetilde{B}_t^{(m)}$.

Let $\widetilde{F}^{(m)}\left(\widetilde{B}_t^{(m)}\right)$ be a smooth function of $\widetilde{B}_t^{(m)}$ such that

$$\widetilde{F}^{(m)}\left(\widetilde{B}_t^{(m)}\right) \equiv \widetilde{p}^{(m)}\left(t, \widetilde{B}_t^{(m)}\right) + \widetilde{e}^{(m)}\left(t, \widetilde{B}_t^{(m)}, B(\cdot)\right) + \widetilde{B}_t^{(m)},$$

and let $F(B_t)$ be a smooth function of B_t such that

$$F(B_t) \equiv p(t, B_t) + e(t, B_t, B(\cdot)) + B_t.$$

By this construction, we have $\widetilde{B}_t^{(m)} = \left(\widetilde{F}^{(m)}\right)^{-1}(K)$ and $B_t = F^{-1}(K)$ where we denote $\left(\widetilde{F}^{(m)}\right)^{-1}$ as the inverse function of $\widetilde{F}^{(m)}$.

Since $\widetilde{\psi}_S^{(m)} \to \psi$ as $m \to \infty$, we know $\widetilde{p}^{(m)}(t, \cdot) \to p(t, \cdot)$ and $\widetilde{e}^{(m)}(t, \cdot, B(\cdot)) \to e(t, \cdot, B(\cdot))$ as $m \to \infty$. It follows that $\widetilde{F}^{(m)} \to F$ and so does the inverse. This implies $\left(\widetilde{F}^{(m)}\right)^{-1}(K) \to F^{-1}(K)$, and thus $\widetilde{B}_t^{(m)} \to B_t$ as $m \to \infty$.

Next, we assume that $\widetilde{B}_s^{(m)} \to B_s$ for $s > t$. We denote $\delta_t^{(m)} = \widetilde{B}_t^{(m)} - B_t$. It's easy to verify that we still have $\widetilde{p}^{(m)}(t, \cdot) \to p(t, \cdot)$ in this case. For $\widetilde{e}^{(m)}\left(t, \widetilde{B}_t^{(m)}, \widetilde{B}^{(m)}(\cdot)\right)$, we have

$$\begin{aligned}
\widetilde{e}^{(m)}\left(t, \widetilde{B}_t^{(m)}, \widetilde{B}^{(m)}(\cdot)\right) &= \int_t^T \int_0^{\widetilde{B}_s^{(m)}} (rK - \delta S_s)\, e^{-r(s-t)} \widetilde{\psi}^{(m)}\left(S_s; S_t = \widetilde{B}_t^{(m)}\right) dS_s ds \\
&= \int_t^T \int_0^{B_s} (rK - \delta S_s)\, e^{-r(s-t)} \widetilde{\psi}^{(m)}\left(S_s; S_t = \widetilde{B}_t^{(m)}\right) dS_s ds \\
&\qquad + \int_t^T \int_0^{\delta_t^{(m)}} (rK - \delta S_s)\, e^{-r(s-t)} \widetilde{\psi}^{(m)}\left(S_s; S_t = \widetilde{B}_t^{(m)}\right) dS_s ds
\end{aligned}$$

As $\widetilde{\psi}^{(m)}$ is uniformly integrable, and $\delta_t^{(m)} \to 0$, we have

$$\int_t^T \int_0^{\delta_t^{(m)}} (rK - \delta S_s)\, e^{-r(s-t)} \widetilde{\psi}^{(m)}\left(S_s; S_t = \widetilde{B}_t^{(m)}\right) dS_s ds \to 0$$

Then it follows that

$$\begin{aligned}
\widetilde{e}^{(m)}\left(t, \widetilde{B}_t^{(m)}, \widetilde{B}^{(m)}(\cdot)\right) &\to \int_t^T \int_0^{B_s} (rK - \delta S_s)\, e^{-r(s-t)} \widetilde{\psi}^{(m)}\left(S_s; S_t = \widetilde{B}_t^{(m)}\right) dS_s ds \\
&\to \int_t^T \int_0^{B_s} (rK - \delta S_s)\, e^{-r(s-t)} \psi\left(S_s; S_t = \widetilde{B}_t^{(m)}\right) dS_s ds = e\left(t, \widetilde{B}_t^{(m)}, B(\cdot)\right)
\end{aligned}$$

We define

$$\widetilde{\mathcal{F}}^{(m)}\left(\widetilde{B}_t^{(m)}\right) \equiv \widetilde{p}^{(m)}\left(t, \widetilde{B}_t^{(m)}\right) + \widetilde{e}^{(m)}\left(t, \widetilde{B}_t^{(m)}, \widetilde{B}^{(m)}(\cdot)\right) + \widetilde{B}_t^{(m)}.$$

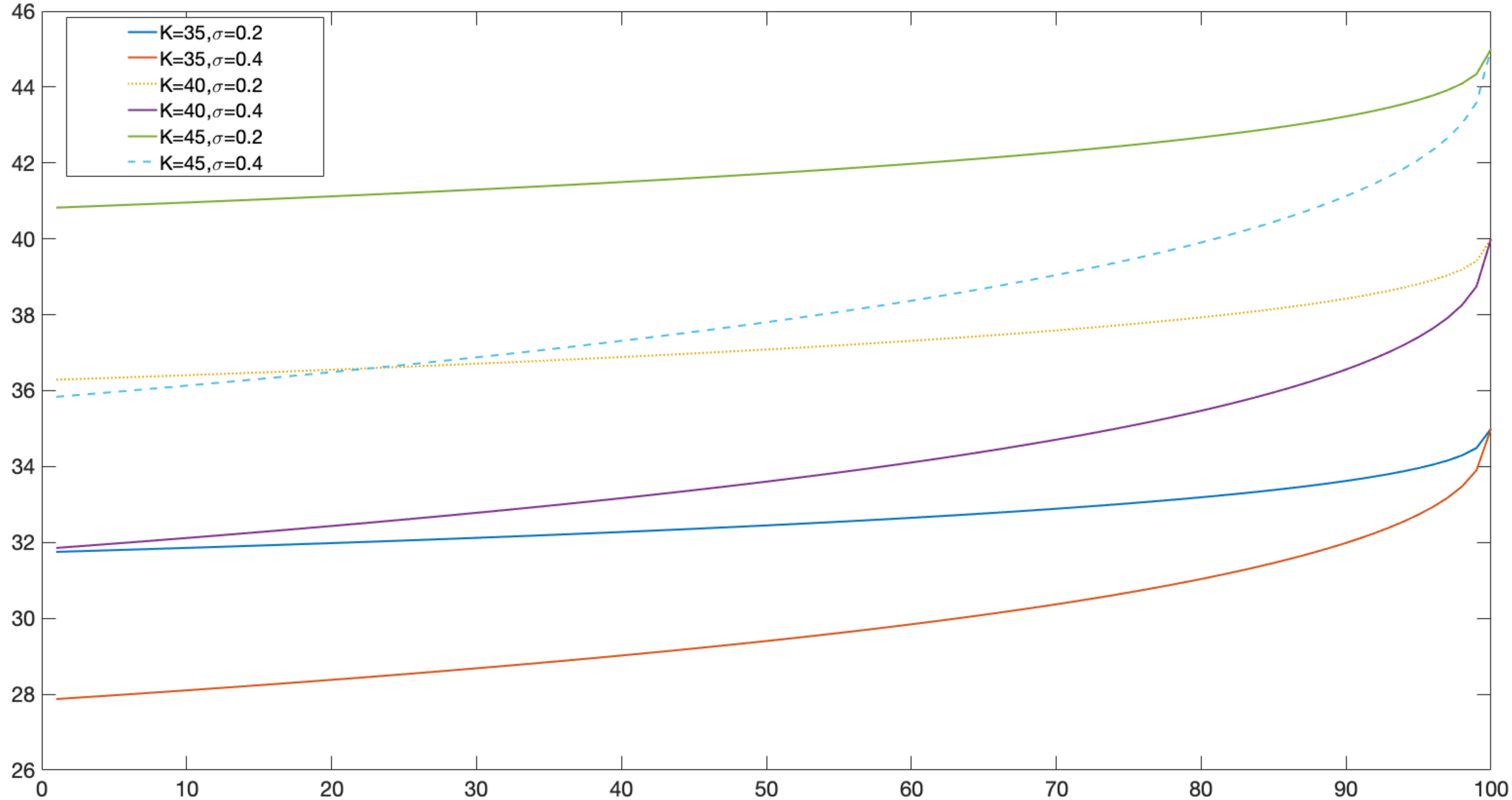
Based on the above analysis, we know $\widetilde{\mathcal{F}}^{(m)} \to F$ as $m \to \infty$, and thus $\left(\widetilde{\mathcal{F}}^{(m)}\right)^{-1}(K) \to F^{-1}(K)$. This means $\widetilde{B}_t^{(m)} \to B_t$ as $m \to \infty$ for any $t \in [0, T]$, which establish the second part of Theorem 2.

Step 3: As we already proved in Step 2, $\widetilde{B}_t^{(m)} \to B_t$ as $m \to \infty$, and since $\widetilde{\psi}^{(m)}$ is uniformly integrable, it's straightforward to have $\widetilde{e}_0^{(m)} \to e_0$ as $m \to \infty$.

Step 4: It's elemental to prove that $\widetilde{P}_0^{(m)} \to P_0$ as $m \to \infty$ based on our results in Steps 1 and 3.

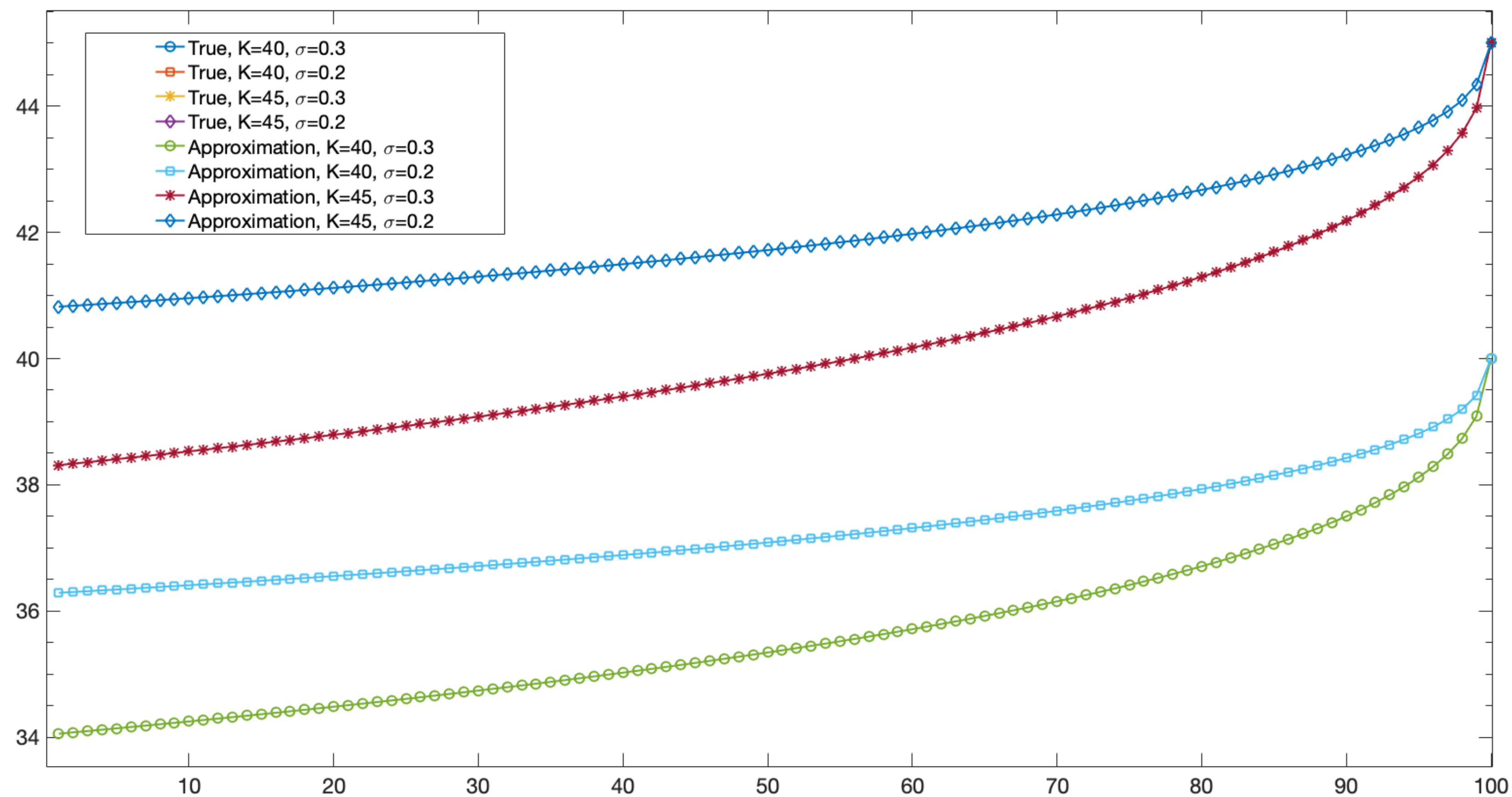
C Figures

Figure A1: Approximated Exercise Boundary for Different Strikes and Volatilities



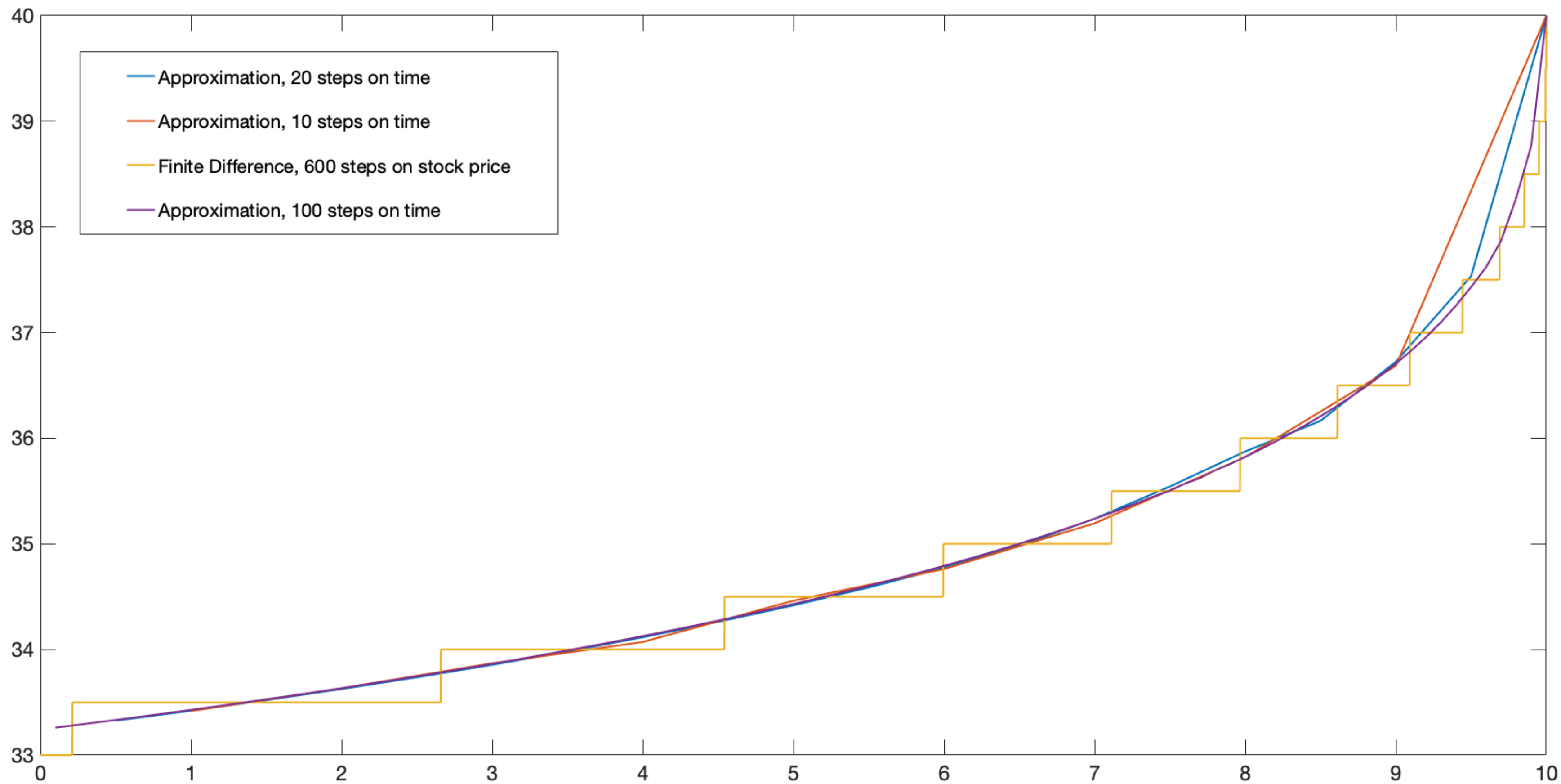
Note: The horizontal axis represents the 100 steps. That is, 100 in the horizontal axis means the time at maturity. The vertical axis represents the price.

Figure A2: The Exercise Boundary: True vs. Approximation



Note: The horizontal axis represents the 100 steps. That is, 100 in the horizontal axis means the time at maturity. The vertical axis represents the price. We use the same marker to represent the same set of parameter values. $\diamond$ represents the boundary when $K = 45$, and $\sigma = 0.2$; $\star$ represents the boundary when $K = 45$, and $\sigma = 0.3$; $\square$ represents the boundary when $K = 40$, and $\sigma = 0.2$;; and $\circ$ represents the boundary when $K = 40$, and $\sigma = 0.3$;

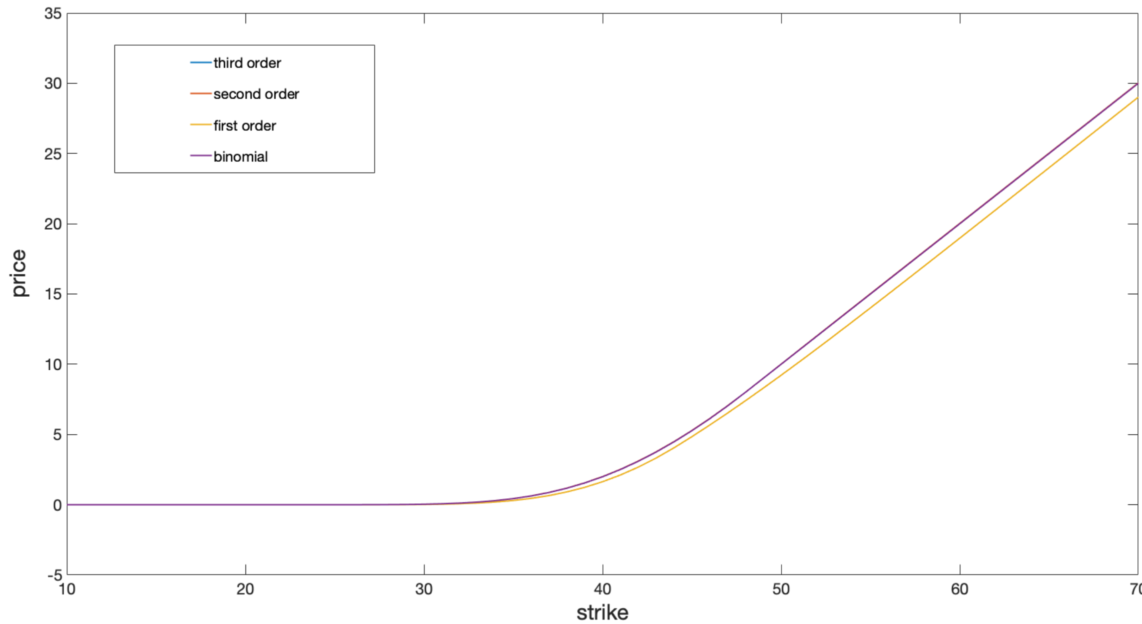
Figure A3: The Exercise Boundary: Hermite polynomial Approximation vs. Finite Difference



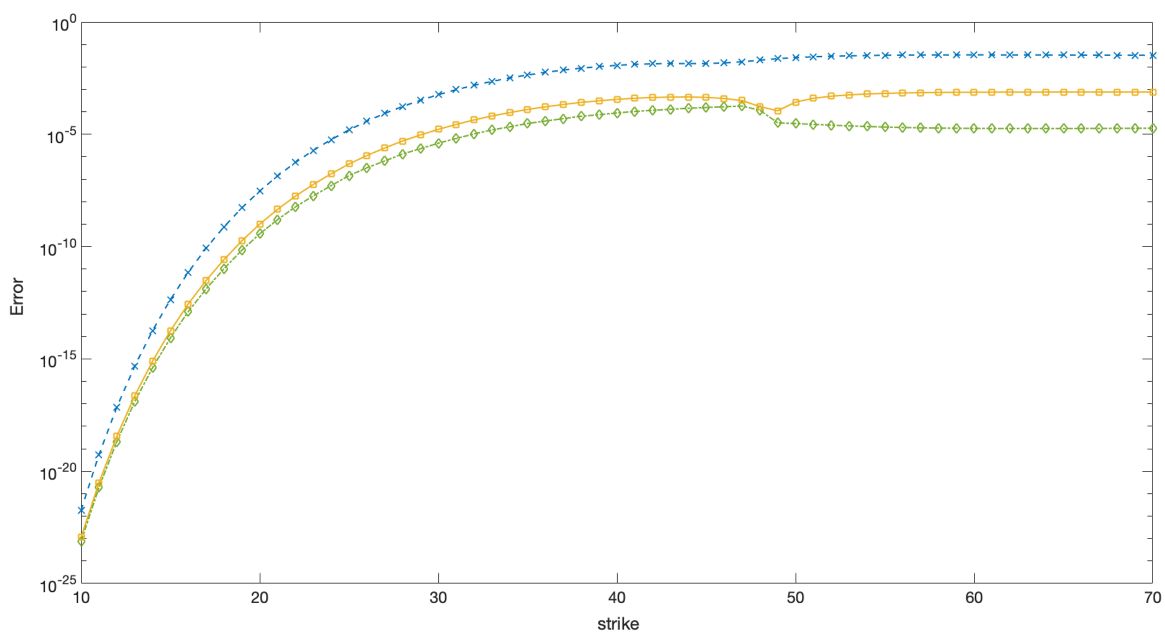
Note: The horizontal axis represents the 10 steps. That is, 10 in the horizontal axis means the time at maturity. The vertical axis represents the price. 1.932 seconds spent for our approach when we have 20 steps on time; 0.898 second spent when we have 10 steps on time; and 1.247 seconds spend for the finite difference approach when we have 600 steps on the support of stock price. We consider our approach with 100 steps on time as benchmark for comparison.

Figure A4: The Value of American Put and Strikes

(a) The value of American put with respect to different strikes for various orders of approximation

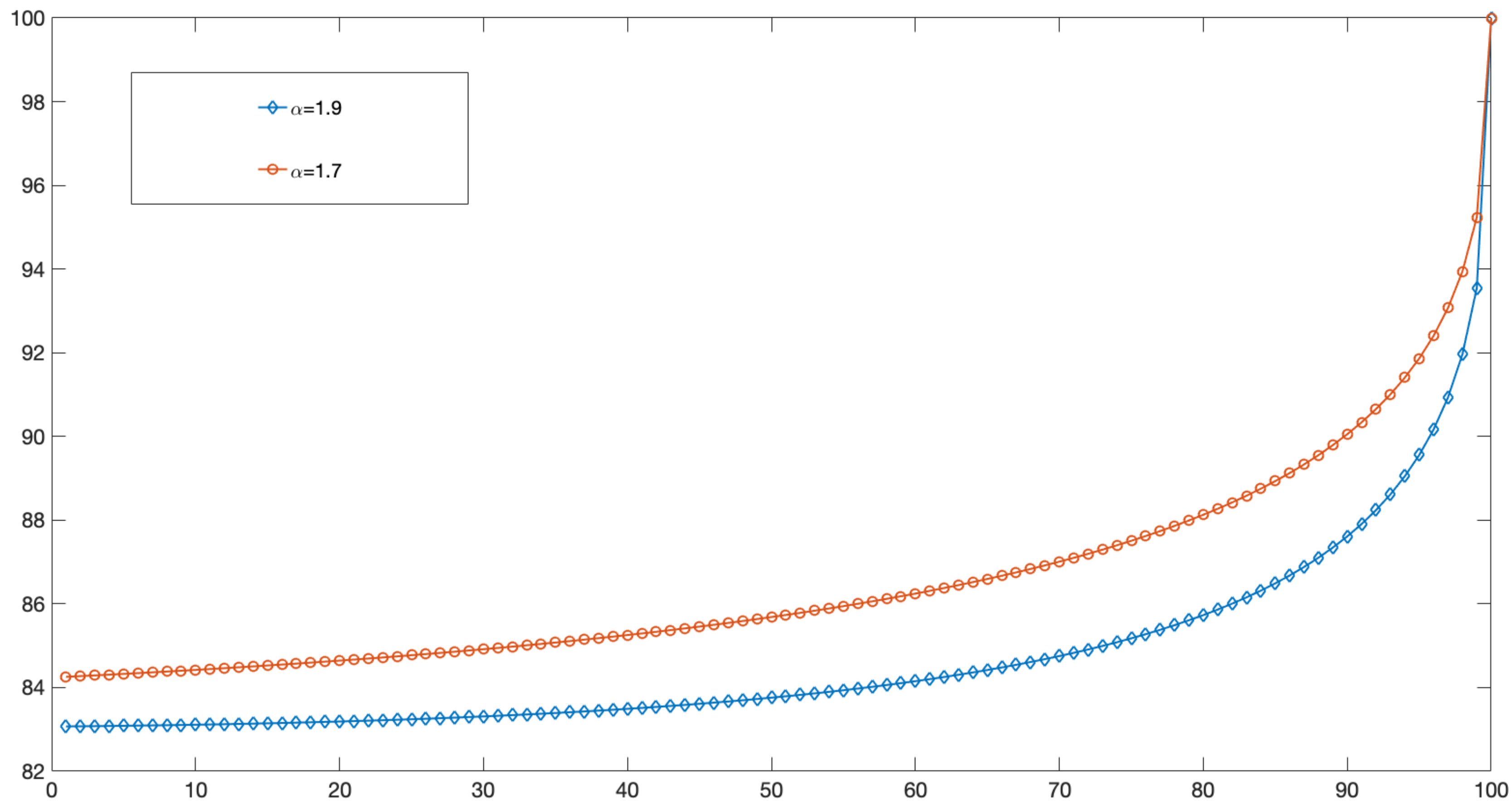


(b) Approximation error of the American put value with respect to different strikes for various orders of approximation



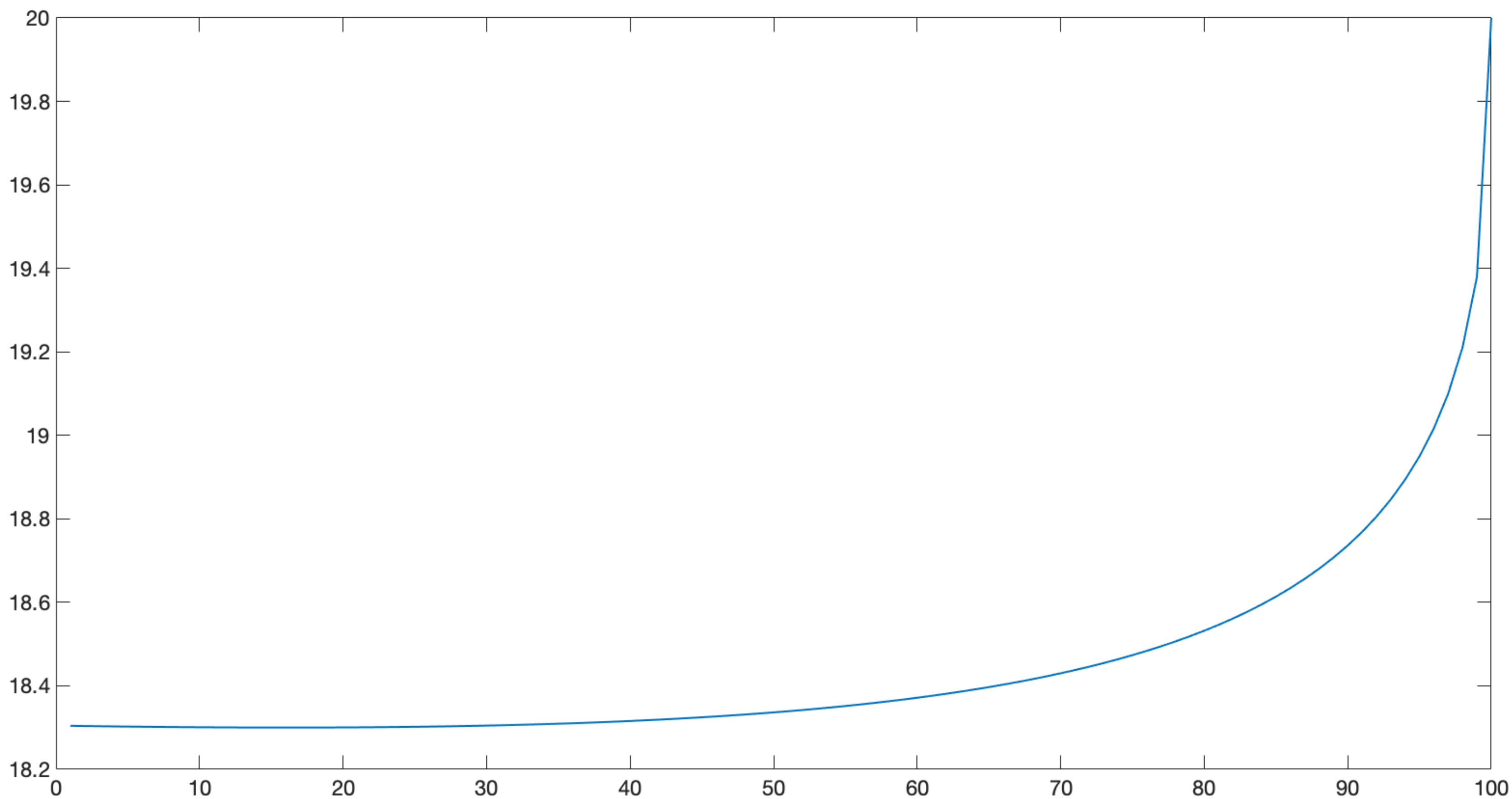
Note: (a) first order represents $m = 1$ in the Hermite polynomial approximation of the transition density; second order represents $m = 2$; third order represents $m = 3$. (b) the blue curve is the relative error in approximating the price with first order accuracy of the approximation of the transition density; yellow curve is that for second order accuracy; green curve is that for third order accuracy

Figure A5: The Exercise Boundary of American Put in the CEV Model



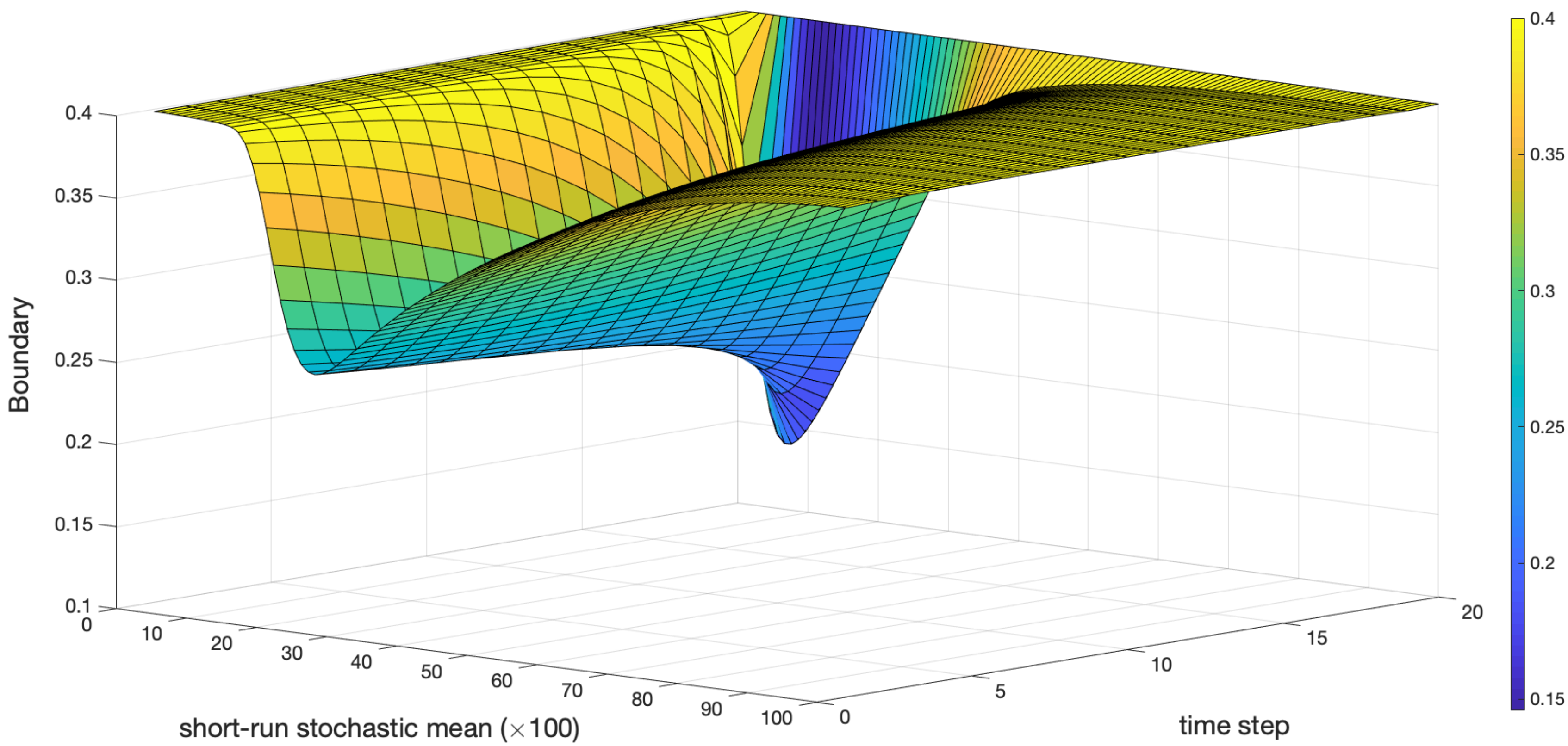
Note: The horizontal axis represents the 100 steps. That is, 100 in the horizontal axis means the time at maturity. The vertical axis represents the price. $K = 100$, $r = 6/100$, $\delta = r/2$, $\sigma = \sqrt{10}/5$, $S_0 = 40$, and $T = 1$.

Figure A6: The Exercise Boundary of American Put in the NMR Model



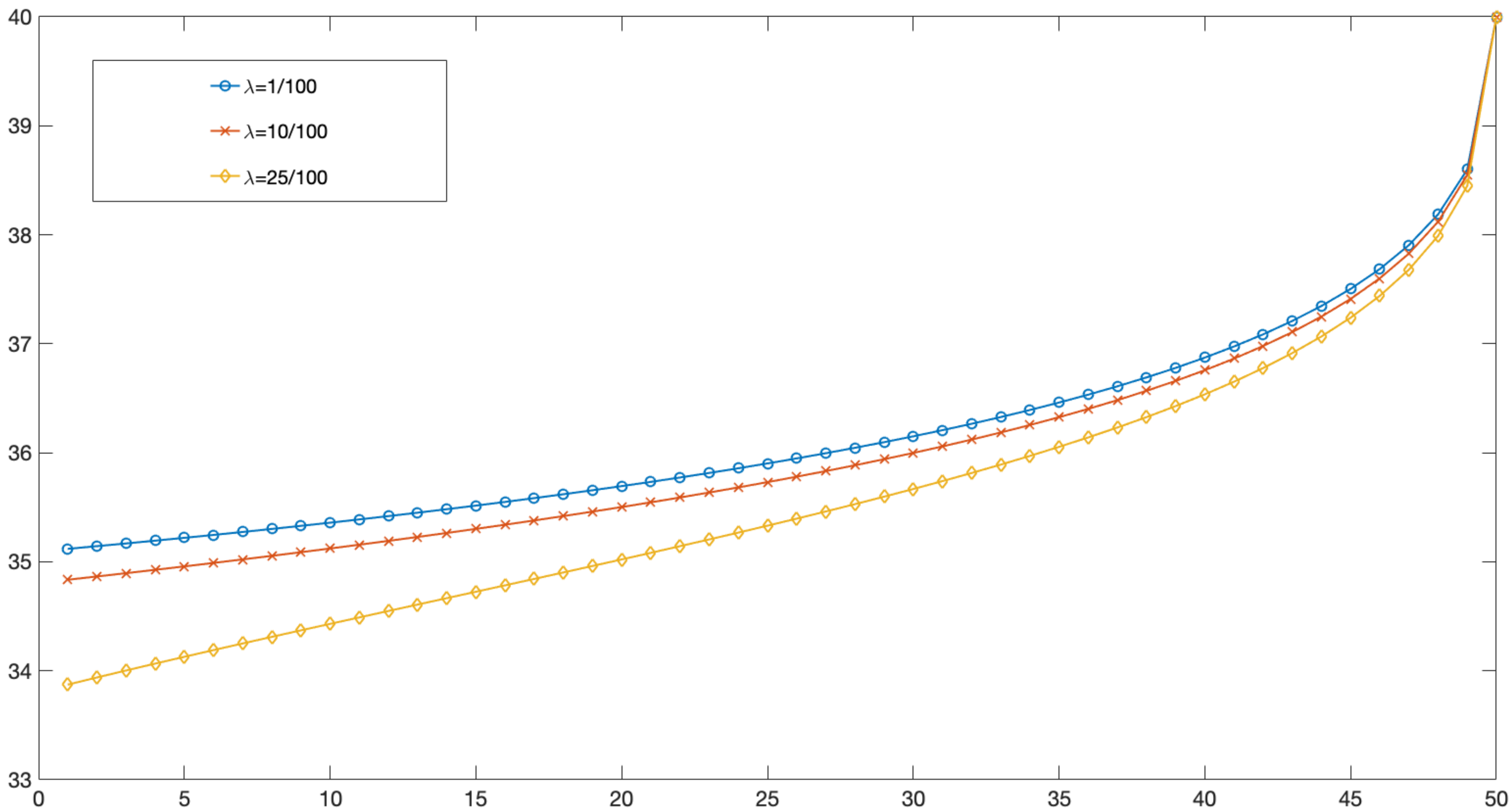
Note: The horizontal axis represents the 100 steps. That is, 100 in the horizontal axis means the time at maturity. The vertical axis represents the price. $a = 500$, $b = 5$, $c = 0.05$, $v = -0.05$, $\sigma = 0.2$, $\gamma = 3/2$, $K = 20$, $r = 5/100$, $\delta = 0$, $S_0 = 20$, and $T = 0.0833$.

Figure A7: The Exercise Boundary of American Put in the DMR Model



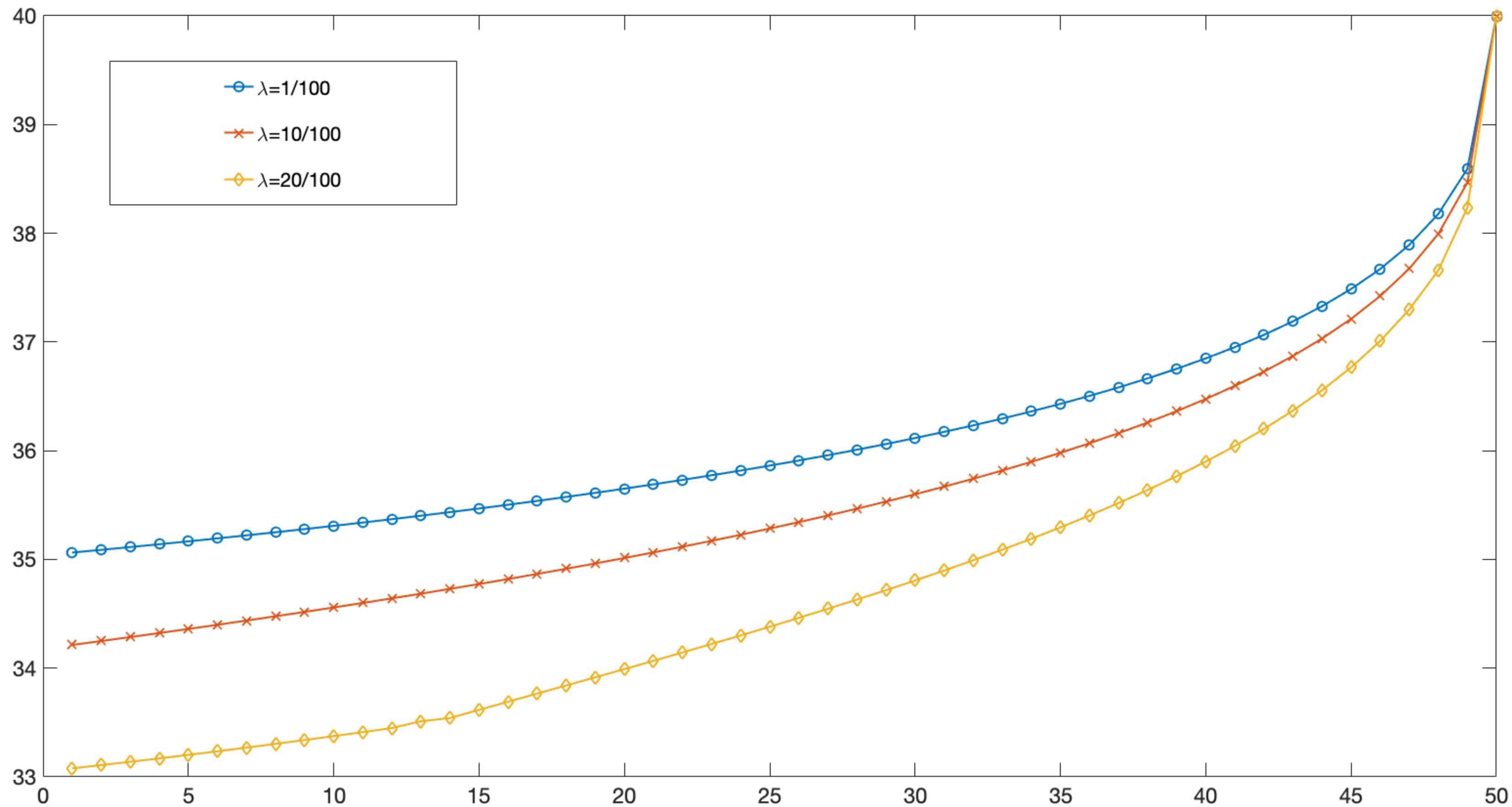
Note: The exercise boundary is approximated with 20 steps on time and 100 steps on y for y in $[0, 1]$. $K = 40/100$, $r = 4.88/100$, $\delta = 0$, $\sigma = 0.25$, $\kappa = 0.2$, $\beta = 2.5$, $\xi = 4$, $\alpha = 0.25$, $T = 0.5$

Figure A8: The Exercise Boundary of American Put in Merton's Jump-Diffusion Model for $\lambda = 1/100, 10/100$, and $25/100$



Note: The exercise boundary is approximated with 50 steps. $K = 40$, $r = 4.88/100$, $\sigma = 0.2$, $\mu_J = 0$, $\sigma_J = 0.2$, $S_0 = 40$, and $T = 0.5$.

Figure A9: The Exercise Boundary of American Put in Kou's Jump-Diffusion Model for $\lambda = 1/100, 10/100$, and $20/100$



Note: The exercise boundary is approximated with 50 steps. $K = 40$, $r = 4.88/100$, $\delta = 0$, $\sigma = 0.2$, $p = 0.04$, $q = 0.96$, $\eta_1 = 3.7$, $\eta_2 = 1.8$, $S_0 = 40$, and $T = 0.5$.