APPENDICES

The eigenmode equation is written as

$$C_4\psi^{''''} + C_3\psi^{'''} + C_2\psi^{''} + C_1\psi' + C_0\psi = 0, \tag{A.1}$$

where

$$C_4 = \omega\chi\frac{N_5}{D} \times A, \tag{A.2}$$

$$C_3 = \omega\chi\left[\frac{N_4}{D} \times A - \frac{\left(-\frac{2ikB_\theta}{r} + \ iG' + G'\omega\eta_0 + \frac{3G\omega\eta_0 B_\theta^2}{r} - \frac{2mA_1\omega\chi}{r^2}\right)}{\frac{k_0^2 G}{kr^3} - \frac{i2\omega\chi m A_1^2}{kr^3} - \frac{iG^3\omega\eta_0}{F} + 2\ iFG\omega\eta_0} \times \frac{1}{rF}\right], \tag{A.3}$$

$$C_2 = \frac{N_3}{D} \times A - \frac{\left(-\frac{2\ ikB_\theta}{r} + G'i + G'\omega\eta_0 + \frac{3G\omega\eta_0 B_\theta^2}{r} - \frac{2mA_1\omega\chi}{r^2}\right)}{-M} \times \frac{-\ i\omega\chi}{r^2}\left(\frac{i}{r^2} + \frac{2\ mA_2'}{r} + 2kB_2'\right)$$
$$-\frac{(1 -\ i\omega\eta_0 -\ i\omega\chi)}{r}, \tag{A.4}$$

$$C_1 = \frac{N_2}{D} \times A$$
$$- \frac{\left(-\frac{2ikB_\theta}{r} + iG' + G\omega\eta_0 + \frac{3G\omega\eta_0 B_\theta^2}{r} - \frac{2mA_1\omega\chi}{r^2}\right)}{-M} \times$$
$$\left[\left(\frac{i\omega^2}{rF} - \frac{2F\omega\eta_0}{r} + \frac{iG^2}{Fr} + \frac{G^2\omega\eta_0}{Fr}\right) + \frac{i\omega\chi}{F}\left(\frac{ik_0^2 + mA_2}{r}\right) - \frac{i\omega\chi m}{rF}\left(\frac{A_2'}{r} + A_2''\right) - \frac{i\omega\chi k}{F}\left(\frac{B_2'}{r} + B_2''\right)\right]$$
$$+ \frac{\left[1 -\ i\omega\eta_0 + 2\ i\omega\eta_0 B_\theta^2 -\ i\omega\chi\left(1 + 2\ imA_2\right)\right]}{r^2}, \tag{A.5}$$

$$C_0 = \frac{N_1}{D} \times A \tag{A.6}$$
$$+ \frac{\left(-\frac{2ikB_\theta}{r} + \ iG' + G'\omega\eta_0 + \frac{3G\omega\eta_0 B_\theta^2}{r} - \frac{2mA_1\omega\chi}{r^2}\right)}{MFr^2} \times \left(2\ iGB_\theta k + 2F^2\omega\eta_0 B_\theta^2 - G^2\omega\eta_0 B_\theta^2 + \frac{2\omega\chi m^2}{r^2}\right)$$
$$+ \left[-\frac{\left(\omega^2 - \omega_a^2\right)}{r} + 2\frac{B_\theta^2}{r^3} - 2\frac{B_\theta B_\theta'}{r^2} + 2\left(\frac{B_\theta^2}{r^2}\right)' - \left(\frac{i\omega\eta_0 B_\theta^2}{r^2}\right)' - \frac{3\ i\omega\eta_0 B_\theta^4}{r^3} - \frac{i\omega\chi k_0^2}{r}\right],$$

$$N_3 = H \times \frac{-i\omega\chi}{F}\left(\frac{i}{r^2} + \frac{2mA_2'}{r} + 2kB_2'\right) + M\times \tag{A.7}$$

$$\left\{\begin{array}{l}\left(\frac{i\omega^2}{rF} - \frac{2\ F\omega\eta_0}{r} + \frac{iG^2}{Fr} + \frac{G^2\omega\eta_0}{Fr}\right) \\ -\frac{i\omega\chi m}{rF}\left(\frac{-2A_2}{r^2} + 3A_2''\right) - \left(\frac{i\omega\chi k}{F}\right)'\left(\frac{B_2}{r} + 2B_2'\right) - \frac{i\omega\chi k}{F}\left(\frac{2rB_2' - B_2}{r^2} + 3B_2''\right)\end{array}\right\}$$

$$-2\left(\frac{m_1^2}{r^2k}\right) \times \left\{\begin{array}{l}\frac{\left(\ i\omega^3\chi - k_0^2\omega^2\chi^2\right)}{F}\left[\frac{m}{r}\left(\frac{A_2}{r} + 2A_2\right) + k\left(\frac{B_2}{r} + 2B_2'\right)\right] \\ -\left(i\omega\chi k_0^2 + G^2\omega^2\rho\chi\eta_0\right)\left[B_\theta\left(\frac{A_2}{r} + 2A_2\right) + B_{:}\left(\frac{B_2}{r} + 2B_2\right)\right] \\ -2FG\omega^2\chi\eta_0\left(2B_zA_2 - 2B_\theta B_2'\right) + \omega^2\chi^2\left(\frac{A_1'}{r} + A_1'' - \frac{A_1}{r^2}\right)\left(\frac{B_2}{r} + 2B_2'\right) \\ -\omega^2\chi^2\left(\frac{B_1'}{r} + B_1''\right)\left(\frac{A_2}{r} + 2A_2\right)\end{array}\right\},$$

$$N_2 = H \times \frac{1}{rF}[(\ i\omega^2 - 2F^2\omega\eta_0 +\ iG^2 + G^2\omega\eta_0) \tag{A.8}$$

$$+ i\omega\chi\left(\ ik_0^2 + \frac{mA_2}{r^2}\right) -\ i\omega\chi m\left(\frac{A_2'}{r} + A_2''\right) -\ i\omega\chi k\left(B_2' + rB_2''\right)] + M\times$$

$$\left\{\begin{array}{l}\left[\left(\frac{i\omega^2}{rF} - \frac{2F\omega\eta_0}{r} + \frac{iG^2}{Fr} + \frac{G^2\omega\eta_0}{Fr}\right) + \frac{i\omega\chi}{F}\left(\frac{ik_0^2}{r} + \frac{mA_2}{r^3}\right)\right]' \\ +\frac{1}{Fr^2}\left(2F^2\omega\eta_0B_\theta^2 + 2\ iGB_\theta k - G^2\omega\eta_0B_\theta^2 + \frac{2\omega\chi m^2}{r^2}\right) - \left(\frac{i\omega\chi m}{rF}\right)'\left(\frac{A_2'}{r} + A_2''\right) \\ -\frac{i\omega\chi m}{r^3F}\left(rA_2'' - A_2' + r^2A_2''\right) - \left(\frac{i\omega\chi k}{F}\right)'\left(\frac{B_2'}{r} + B_2''\right) - \frac{i\omega\chi k}{r^2F}\left(rB_2'' - B_2' + r^2B_2'''\right)\end{array}\right\}$$

$$-2\left(\frac{mA_1^2}{r^2k}\right) \times \left\{\begin{array}{l}\frac{\left(i\omega^3\chi - k_0^2\omega^2\chi^2\right)}{F}\left[\frac{m}{r}\left(\frac{A_2'}{r} + A_2''\right) + k\left(\frac{B_2'}{r} + B_2''\right) - \frac{ik_0^2}{r} - \frac{mA_2}{r^3}\right] \\ -\frac{ik_0^2\omega\chi}{Fr}\left(i\omega^2 - 2F^2\omega\eta_0 +\ iG^2 + G^2\omega\eta_0\right) \\ -\left(\ i\omega\chi k_0^2B + G^2\omega^2\chi\eta_0\right)\left[B_\theta\left(\frac{A_2'}{r} + A_2''\right) + B_z\left(\frac{B_2'}{r} + B_2''\right) - \frac{ik_0^2}{rF} - \frac{B_\theta A_2}{r^2}\right] \\ -2FG\omega^2\chi\eta_0\left[B_z\left(\frac{A_2'}{r} + A_2''\right) - B_\theta\left(\frac{B_2'}{r} + B_2''\right) - \frac{B_zA_2}{r^2}\right] \\ +\left(-\omega^2 + k_0^2 -\ iG^2\omega\eta_0\right)\left(-\frac{2F\omega\eta_0}{r} + \frac{i\omega^2}{rF}\right) \\ +\ i\omega\chi\left(\frac{A_1'}{r} + A_1'' - \frac{A_1}{r^2}\right)\left(\frac{iB_z\rho\omega^2}{rF} - \frac{2FB_z\omega\eta_0}{r} - \frac{iGB_\theta}{r} - \frac{GB_\theta\omega\eta_0}{r}\right) \\ -\left(2\ iFG_\omega\eta_0 + \frac{G\omega^2}{F}\right)\left(\frac{iG + G\omega\eta_0}{r}\right) + \omega^2\chi^2\left(\frac{A_1'}{r} + A_1'' - \frac{A_1}{r^2}\right)\left(\frac{B_2'}{r} + B_2' - \frac{ik_0^2B_z}{rF}\right) \\ +\ i\omega\chi\left(\frac{B_1'}{r} + B_1''\right)\left(-\frac{iGB_z}{r} - \frac{GB_z\omega\eta_0}{r} - \frac{iB_\theta\omega^2}{rF} + \frac{2FB_\theta\omega\eta_0}{r}\right) \\ \omega^2\chi^2\left(\frac{B_1'}{r} + B_1''\right)\left(\frac{A_2'}{r} + A_2'' - \frac{ik_0^2B_\theta}{rF} - \frac{A_2}{r^2}\right)\end{array}\right\},$$

$$N_1 = H \times \frac{1}{Fr^2}\left(2F^2\omega\eta_0 B_\theta^2 + 2iGB_\theta k - G^2\omega\eta_0 B_\theta^2 + \frac{2\omega\chi m^2}{r^2}\right) \tag{A.9}$$

$$+ M \times \left[\frac{1}{Fr^2}\left(2F^2\omega\eta_0 B_\theta^2 + 2iGB_\theta k - G^2\omega\eta_0 B_\theta^2 + \frac{2\omega\chi m^2}{r^2}\right)\right]' - \frac{2mA_1^2}{r^2 k}\times$$

$$\left\{\begin{array}{l} -\dfrac{2\omega^3\chi m^2}{r^4 F} - \dfrac{ik_0^2\omega\chi}{Fr^2}\left(\dfrac{2\ iGB_\theta k}{Fr^2} + \dfrac{2\ F\omega\eta_0 B_\theta^2}{r^2} - \dfrac{G^2\omega\eta_0 B_\theta^2}{Fr^2}\right) \\ -\left(2iFG\omega\eta_0 + \dfrac{G\omega^2}{F}\right)\left(\dfrac{2\ iB_\theta k - G\omega\eta_0 B_\theta^2}{r^2}\right) \\ -\dfrac{2ik_0^2\omega^2\chi^2 m^2}{r^4 F} - \dfrac{2imB_\theta}{r^3}\left(\ i\omega\chi k_o^2 + G^2\omega^2\chi\eta_0\right) - \dfrac{4\ iF^2\omega^2\chi\eta_0 GmB_z}{r^3 F} \\ +\dfrac{i\omega\chi}{r^2}\left(\dfrac{A_1'}{r} + A_1'' - \dfrac{A_1}{r^2}\right)\left(2F\omega\eta_0 B_\theta^2 B_z - 2iB_\theta^2 k + G\omega\eta_0 B_\theta^3\right) + \dfrac{2F\omega\eta_0 B_\theta^2}{r^2}\left(-\omega^2 + k_0^2 -\ iG^2\omega\eta_0\right) \\ -\dfrac{i\omega\chi}{r^2}\left(\dfrac{B_1'}{r} + B_1''\right)\left(2ikB_\theta B_z - G\omega\eta_0 B_\theta^2 B_z + 2F\omega\eta_0 B_\theta^3\right) - \dfrac{2i\omega^2\chi^2 m}{r^3}\left(\dfrac{B_1'}{r} + B_1''\right) \end{array}\right\},$$

$$N_4 = \frac{H}{rF} + M \times \left[-\frac{i}{F}\left(\frac{2mA_2'}{r} + 2kB_2'\right) + \frac{1}{Fr^2} + \left(\frac{1}{rF}\right)'\right] \tag{A.10}$$

$$-\frac{2mA_1^2}{r^2 k}\left\{\frac{-\ ik_0^2\omega\chi -\ iG^2\omega\eta_0}{Fr} + \frac{k_0^2 - \omega^2}{Fr} + \omega\chi B_2\left[\left(\frac{A_1}{r}\right)' + A_1''\right] - \omega\chi A_2\left[\left(\frac{B_1'}{r} + B_1'\right)\right]\right\}$$

$$N_5 = \frac{M}{rF}, \tag{A.11}$$

$$A = -\left(iG + G\omega\eta_0\right) + \frac{2\omega\chi m A_1 \times \left[\frac{2ikB_\theta}{r} + -iG' - G'\omega\eta_0 - \frac{3G\omega\eta_0 B_\theta^2}{r} + \frac{2mA_1\omega\chi}{r^2}\right]}{ik_0^2 Gr^2 + \frac{2\omega\chi mFA_1^2}{kr} + G^3\omega\eta_0 r^2 - 2F^2 r^2 G\omega\eta_0}, \tag{A.12}$$

$$H = \left[\frac{i\omega\chi m}{k}\left(\frac{A_1^2}{r^2}\right)' - \left(\frac{iG^3\omega\eta_0}{F} - 2\ iFG\omega\eta_0 - \frac{k_0^2 G}{F}\right)\right]', \tag{A.13}$$

$$M = -\frac{i\omega\chi m}{k}\left(\frac{A_1^2}{r^2}\right)' + \left(\frac{iG^3\omega\eta_0}{F} - 2\ iFG\omega\eta_0 - \frac{k_0^2 G}{F}\right), \tag{A.14}$$

$$D = -\frac{2\omega\chi m A_1^2}{r^2 k} \times \left\{\frac{iM}{r} - 2\ i\left[\left(B_\theta A_1' + B_z B_1'\right)\left(\omega^2 - k_o^2 +\ iG^2\omega\eta_0\right) + \left(B_z A_1' - B_\theta B_1'\right)\left(\frac{G\omega^2}{F} + 2\ iFG\omega\eta_0\right)\right.\right.$$

$$\left.\left. -\ i\omega\chi\left(A_1'' B_1' - A_1' B_1'' - \frac{k_0^2 m A_1^2}{r^2 k} - \frac{A_1 B_1'}{r^2}\right)\right]\right\} - M^2 + \frac{2\ iH\omega\chi m A_1^2}{r^2 k} +\ iM\left(\frac{2\omega\chi m A_1^2}{r^2 k}\right)', \tag{A.15}$$

where $G = mB_z/r - kB_\theta$, $F = kB_z + mB_\theta/r$, $A_1 = k/F$, $A_2 = iB_\theta/rF$, $B_1 = -m/rF$, $\omega_a = F/\sqrt{\mu_0\rho}$ and $B_2 = iB_z/rF$.