抽象代数2017 秋(答案)

2017.11.14

姓名____________________ 学号____________________.

1 (10 Points) 写出S_4的所有正规子群.(不必证明) List all normal subgroups of S_4. (No proof is needed.)

They are $\{e\}$, S_4, A_4, $G_1 = \{(12)(34), (13)(24), (14)(23), e\}$

2 (20 Points) 计算域 $\mathbb{F}_3 = \mathbb{Z}/3$ 上2×2矩阵环中乘法单位群的阶数. (给出过程)
Compute the order of the multiplicative group of the ring of 2×2 matrices over $\mathbb{Z}/3$. (Show your computation.)

The (2×2) matrices form a 4-dimensional space over $\mathbb{F}_3$, so there are 81 elements. It is invertible if and only if $ad - bc \neq 0$. There are many ways to compute it, the answer should be 48 elements. One is to compute non-invertible matrices: if $(a, b) = (0, 0)$, then there are 9 elements. If $(a, b) \neq (0, 0)$, there are $8 \times 3 = 24$ elements. So $81 - 9 - 24 = 48$.

3 (20 Points) 令p是一个素数,S是所有和p互素的整数的集合, 验证S是乘性子集.考虑整数对S做分式化所得环.找出其中所有非平凡理想.
Let p be a prime, and S all integers which are coprime with p. Verify that S is a multiplicative subset. Let R' be the fractional ring induced by $\mathbb{Z}$ over S. Find all its non-trivial ideals.

If $a, b \in S$, then $(p, ab) = 1$ which implies $ab \in S$, so S is a multiplicative subset. All the ideals in $S^{-1}R$ has the form (p^n).

4 (15 Points) 每一个阶为200的群有正规西罗子群.
Every order 200 group G has a normal Sylow subgroup.

$200 = 8 \times 25$. So the number of Sylow 5-group should be $(1 + 5t)|8$, so $t = 0$.

5 (20 Points) 令p, q是两个素数, G是一个阶为p^2q的群, 证明G是一个可解群.

Let p, q be prime numbers, and G a group of order p^2q, show G is solvable.

Recall that any finite group with order p, p^2 or q is commutative.

If G has a normal Sylow p-subgroup or Sylow q-subgroup G', then the quotient will be a commutative group, which implies the commutator $G^{(1)} \subset G'$. By above, this implies G is solvable. So it suffices to verify such either Sylow p-subgroup or Sylow q-subgroup G' is normal.

If not, by Sylow 3rd Theorem, we know that there are $(1+pt)|q$ $(t \in \mathbb{Z}_{\geq 0})$ Sylow $p-$subgroups, and $(1+qs)|p^2$ $(s \in \mathbb{Z}_{\geq 0})$ Sylow $q-$subgroups. If $t, s > 0$, then $p|q-1$ and $q|(p+1)(p-1)$. The only possible solutions are $q = p+1$, which implies $p = 2, q = 3$. But if there are 4 Sylow 3-subgroup, by element counting, there is only one Sylow 2-group.

6 (15 Points) 令R是交换幺元环, $f = \sum_{i=0}^n a_i z^i \in R[z]$ 是个可逆元当且仅当a_0可逆, 而且$a_1, ..., a_n$是幂零元. (提示: 先考虑R是整环的情形)

Let R be a commutative ring with indentity. An element $f = \sum_{i=0}^n a_i z^i \in R[z]$ is a unit if and only if a_0 is a unit in R and $a_1, ..., a_n$ are nilpotent. (Hint: First consider the case R is an integral domain.)

If a_0 is a unit, we can assume it is 1. If $a_1, ..., a_n$ are nilpotent, assume $a_i^{p_i} = 0$. Let $m > p_1 + \cdots + p_n$. Then let $g = \sum_{i=1}^n a_i x^i$, then

$$f(1-g+g^2-g^3+\cdots(-1)g^m) = (1+g)(1-g+g^2-g^3+\cdots(-1)g^m) = 1+(-1)^{m+1}g^{m+1}.$$

But $g^{m+1} = 0$, so f is invertible.

Now assume f is invertible. By comparing the degree 0 element, we easily see a_0 is invertible. Let p be a prime ideal of R, then the image $\bar{f}$ of f in $(R/p)[x]$ is invertible. Since R/p is integral, we know $\bar{a}_i = 0 \in R/p$. This implies a_i is in the intersection of all prime ideals, which is the ideal consisting of all nilpotent elements.

7 (15 Points) 令R是一个主理想整环. 一个理想P称为准素如果, 如果对任意$ab \in P$, $a \notin p$, 则对充分大的n有$b^n \in p$。证明任意理想P 是准素理想当且仅当P有(p^n)的形式，其中p是某个不可约元素或者0.

Let R be an principal ideal domain. An ideal P is called primary, if for any $ab \in P$ and $a \notin p$, then $b^n \in p$ for some $n \gg 0$. Prove that any ideal P is a primary ideal if and only if it has the form (p^n), where p is irreducible or 0.

If $I = (p^n)$ and $a \notin p$, then p divides b, which implies $b^n \in I$. Conversely, Any $I = (p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m})$ with prime elements $p_1, ..., p_m$ and powers $n_1, ..., n_m > 0$. If $m > 1$, then we can pick $a = p_1^{n_1}$ and $b = p_2^{n_2} \cdots p_m^{n_m}$. Clearly, $a \notin I$ and for any power n, $b^n \notin I$, so I is not primary.