Method 2:

Background: *inner-product*

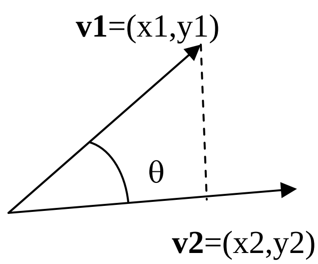
Given two vectors v1=(x1, y1) and v2=(x2, y2), the *inner-product* between them is defined as:

$$v1 \bullet v2 = x1 \cdot x2 + y1 \cdot y2 \qquad (1)$$

or

$$v1 \bullet v2 = |v1||v2|\cos(\theta) \qquad (2)$$

where $|v1| = \sqrt{x1^2 + y1^2}$ and $|v2| = \sqrt{x2^2 + y2^2}$

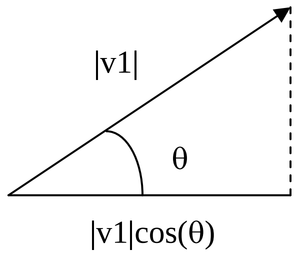


Remarks:

- Calculation of inner-product is often using the definition 1 above.
- If you want to find the angle between the two vectors, both formulas are used:

$$\theta = \cos^{-1}\left(\frac{x1 \cdot x2 + y1 \cdot y2}{|v1||v2|}\right)$$

- The definition 2 gives a geometric meaning of the inner-product. Assume v2 is a unit-length vector, i.e. |v2| = 1.
  Then, the definition 2 becomes: $v1 \bullet v2 = |v1|\cos(\theta)$.
  It means that, the inner-product between a vector (e.g. v1) and a unit-length vector (e.g. v2) gives the projected length of v1 on v2 as shown below:

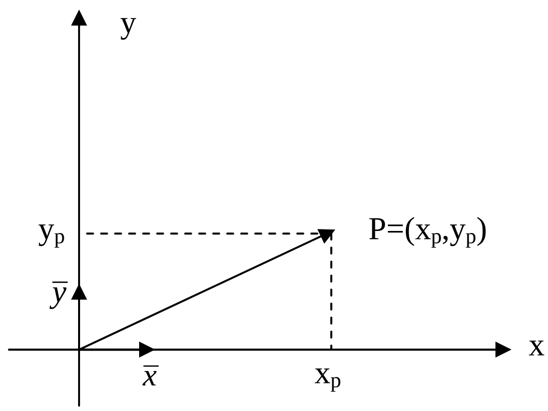


[Question: We often want to find the distance from a point to a line. Think about how you would calculate it.

Later, you will find that the above definition is also useful to find the distance from a point to a plane.]

Cartesian Coordinates in Vector Forms

- Given a point P = (x, y), find its coordinates in x and y axis respectively.
- This question sounds very silly. You will see my point shortly.



- Let $\bar{x}$ = (1, 0) and $\bar{y}$ = (0, 1) be the unit-length vectors along x and y-asix respectively.
- Consider the vector from the origin O to the point P:

$$(P - O) = [(x_p, y_p) - (0, 0)] = (x_p, y_p) = P$$

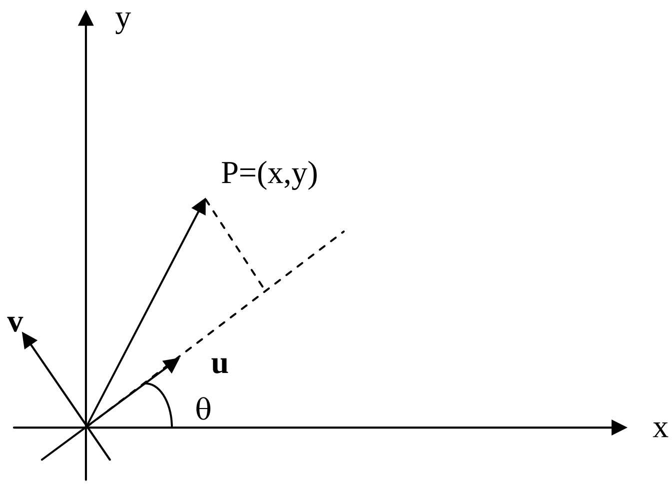
- Based on the geometric meaning of inner-product given above, the inner-product:

$$(P - O) \bullet \bar{x} = P \bullet \bar{x} = (x_p, y_p) \bullet (1,0) = x_p$$

is the projected length of vector (P-O) on $\bar{x}$. It is x_p, the coordinate in x-axis.

- In other words, if an axis is defined by a unit-length vector, the coordinate of a point on that axis can be calculated by the above inner-product.
- Similarly, you can calculate the y coordinate.

Next, you will find the above "silly" definition useful for rotation.



- Let **u** = (u_x, u_y) and **v** = (v_x, v_y) be two orthogonal unit-length vectors that represents a new Cartesian coordinate system rotated from the x-y coordinate system by an angle θ.
- Given a point P = (x, y) in the x-y coordinate system, find the P's coordinates in the u-v coordinate system.

Without the above knowledge, one would use rotation method with (-θ). (Why negative?)

- Now, we can use inner-product to find the projected length of P onto **u** and **v**.
- For example,

  $P \bullet \mathbf{u} = [(x, y) \bullet (u_x, u_y)] = x\, u_x + y\, u_y = u$

  will be P measured on **u** direction, thus the coordinate in u-axis.
- Similarly, the coordinate on v-axis can be find by:

  $P \bullet \mathbf{v} = [(x, y) \bullet (v_x, v_y)] = x\, v_x + y\, v_y = v$

- Are these much simpler than the rotation matrix? Well, we can put the above calculation into matrix form too:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Summary:

- If a new coordinate system is defined by a pair of orthogonal unit-length vectors:

$$\begin{cases} \bar{u} = (u_x, u_y) \\ \bar{v} = (v_x, v_y) \end{cases}$$

then, the rotation matrix that converts a point from x-y coordinates to u-v coordinates is simply:

$$R = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

This method can be extended to 3D case.

- Assume three orthogonal unit-length vectors define a new coordinate system:

$$\begin{cases} \bar{u} = (u_x, u_y, u_z) \\ \bar{v} = (v_x, v_y, v_z) \\ \bar{n} = (n_x, n_y, n_z) \end{cases}$$

then, the rotation matrix that converts a point from x-y-z coordinates to u-v-n coordinates is simply:

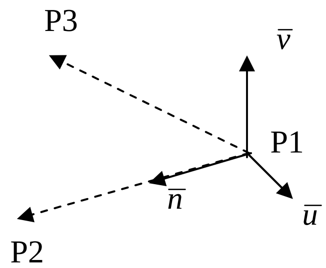
$$R = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ n_x & n_y & n_z \end{bmatrix}$$

Remark:

- The above method applies to rotation only. It implies that the origins of the two coordinate systems are aligned at the same point.

Now, let's back to the problem to be solved by method 1.

- Once P1 is translated to the origin, the remaining steps are rotations. The calculations of these rotations were certainly not simple.
- Now, we can try to find such rotation by using the new method. In order to do this, we must first find the three unit-length vectors.



- Let (P2-P1) be the vector from P1 to p2; and (P3-P1) be the vector from P1 to P3. Let's define three unit-length orthogonal vectors as follows:

  $\bar{n}$ is the unit-length vector along (P2-P1);
  $\bar{u}$ is the unit-length vector perpendicular to both (P2-P1) and (P3-P1);
  $\bar{v}$ is the unit-length vector perpendicular to both $\bar{n}$ and $\bar{u}$.

- These three vectors can be calculated by cross-products:

$$\bar{n} = \frac{(P2 - P1)}{|P2 - P1|}$$
$$\bar{u} = \frac{(P3 - P1) \times (P2 - P1)}{|(P3 - P1) \times (P2 - P1)|}$$
$$\bar{v} = \bar{n} \times \bar{u}$$

- Once these three vectors are available, the rotation matrix R is obtained (shown above).
- The final transformation matrix is:

  M = R*T

  Where T is the translation to move p1 to the origin.

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