Chapter 5: Geometric Transformation

- In general, geometric transformations are used to change position, orientation, and size of objects in drawings.

- Basic feometric transformations include:
  - translation
  - rotation
  - scaling

2D Transformations

- **Translation**

  Let P(x,y) be a point. After it is moved by dx along x direction and dy along y direction, the new point is at P'(x',y'), where:

  x' = x + dx
  y' = y + dy (1)

  It can be written in vector form:

  P' = P + T (2)

  where
  $P=[x,y]^T$, $P'=[x',y']^T$, $T=[dx,dy]^T$

  NOTE: $[x,y]^T$ means "transpose". That is,

  $$[x, y]^T = \begin{bmatrix} x \\ y \end{bmatrix}.$$

  E.g.
  $$P = \begin{bmatrix} 4 \\ 5 \end{bmatrix},\ P' = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \text{then the required } T = \begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

- **Scaling** (about the origin)

  A point can be scaled by a factor S_x along x direction, and by S_y along y direction into a new point:

  $x' = S_x*x$ (3)

$y' = S_y*y$

Or, it can be written in matrix form:

$$P' = S*P \qquad (4)$$

where

$$S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

E.g. Let $P = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $S = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}$,

then, $P' = S * P = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} * \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5/4 \end{bmatrix}$

- **Rotation** (around the origin)

  A point can be rotated through an angle *a* about the origin.

  A rotation is defined mathematically by:

$$x' = x*\cos(a) - y*\sin(a) \qquad (5)$$
$$y' = x*\sin(a) + y*\cos(a)$$

This can be written in matrix form as well:

$$P' = R*P \qquad (6)$$

where

$$R = \begin{bmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{bmatrix}$$

E.g. Let $a = 45' = \pi/4$, $P = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

then

$$P' = R * P = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} * \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.1 \\ 4.9 \end{bmatrix}.$$

Homogeneous Coordinates

Why do we need homogeneous coordinates?

One of the reasons is that, translation is in vector addition form, but scaling and rotation are both in matrix-vector multiplication form. We would like to have all three in the same form (the multiplication form). The benefits will be seen shortly.

- **Homogeneous coordinates**

  For a given 2D coordinates (x,y), we introduce a third dimension:

  [x,y,1]

  In general, a homogeneous coordinates for a 2D point has the form:

  [x,y,W]

  Two homogeneous coorduinates [x,y,W] and [x',y',W'] are said to be same (or equivalent) if

  x = k*x'
  y = k*y'   for some k != 0
  W = k*W'

  E.g. [3.5, 2.6, 1] is same as [7.0, 5.2, 2].

  A valid homogeneous coordinates must have one coordinate to be non-zero. Or in other words, [0,0,0] is not allowed. In most cases, W is non-zero.

  From [x,y,W], we can normalize it by dividing each element by W:

  [x/W, y/W, 1]

  which is called the Cartesian coordinates of the homogeneous point.

- Redefine the translation by matrix multiplication:

  P' = T*P
  where

$$P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix},\ T = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix},\ \text{and}\ \ P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

  It is easy to verify that:

$$P' = \begin{bmatrix} x + dx \\ y + dy \\ 1 \end{bmatrix}.$$

Composition of Transformation Matrices

- *Additivity of successful translations*

  E.g. We want to translate a point P to P' by T(dx1,dy1) and then to p'' by another T(dx2,dy2).

  P'' = T(dx2,dy2)*P'
  = T(dx2,dy2)[T(dx1,dy1)*P]

  On the other hand, we can compute T_{21} = T(dx2,dy2)*T(dx1,dy1) first, then apply T_{21} to P:

  P'' = T_{21}*P

  We can easly verifiy that the above two methods will produce the same result. Thus, successive translations are additive.

- *Multiplicativity of successive Scalings*

  P'' = S(sx2,sy2)*[S(sx1,sy1)*P]
  = [S(sx2,sy2)*S(sx1,sy1)]*P
  = S_{21}*P

  where

$$S_{21} = S(sx2, sy2) * S(sx1, sy1)$$
$$= \begin{bmatrix} sx2 & 0 & 0 \\ 0 & sy2 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} sx1 & 0 & 0 \\ 0 & sy1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} sx2 * sx1 & 0 & 0 \\ 0 & sy2 * sy1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- *Additivity of successive rotations*

  P'' = R(*a2*)*[R(*a1*)*P]
  = [R(*a2*)*R(*a1*)]*P
  = R_{21}*P

  where

$$
\begin{aligned}
R_{21} &= R(a2) * R(a1) \\
&= \begin{bmatrix} \cos(a2) & -\sin(a2) & 0 \\ \sin(a2) & \cos(a2) & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos(a1) & -\sin(a1) & 0 \\ \sin(a1) & \cos(a1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(a2+a1) & -\sin(a2+a1) & 0 \\ \sin(a2+a1) & \cos(a2+a1) & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

- In the above, we only show the concatenations of same type of matrices. It can also be shown the different types of elementary transfomations discussed above can be contatenated as well.

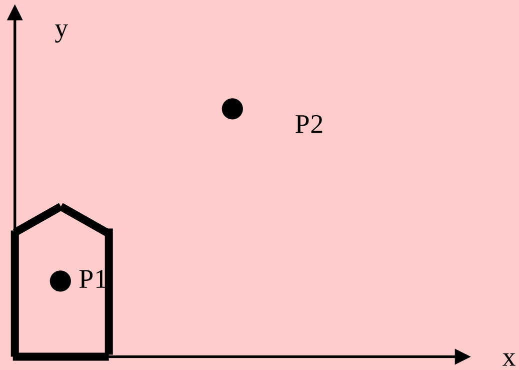
  E.g. It can be shown that:

  P' = R(a)*[T(dx,dy)*P]
  = [R(a)*T(dx,dy)]*P
  = M*P

  where M = R(a)*T(dx,dy)

- The purpose of composing transformations is to gain efficiency by applying a single composed transformation to a coordinate, instead of applying a sequence of transformations one after another.

  Example: Given the following object, reduce the size of the object by a half and rotate about its center for 90' and place it a new location at P2.



Suppose P1 = (2,3) and P2 = (6,5). The transformation consists of a sequence of elementary steps:

1. Translate its center P1 to the origin;
2. Scale the object by a half;
3. Rotate it about the origin for 90';

4. Translate the center to the destination P2.

The composed transformation matrix can be defined as:

M = T(6,5)*R(90')*S(0.5,0.5)*T(-2,-3)

3D Transformations

2D transformation matrices can be generalized to 3D cases as below:

$$T(dx, dy, dz) = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S(sx, sy, sz) = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For rotations, we rotate about each axis as below:

$$R_z(a) = \begin{bmatrix} \cos(a) & -\sin(a) & 0 & 0 \\ \sin(a) & \cos(a) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x(a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(a) & -\sin(a) & 0 \\ 0 & \sin(a) & \cos(a) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(a) = \begin{bmatrix} \cos(a) & 0 & \sin(a) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(a) & 0 & \cos(a) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Composition of 3D Transformations

The example given in Section 5.8, pages 183-186, is a very important example. The general purpose 3D view transformation to be introduced in the next chapter is based on this example.

Example: Refer to Figures 5.18 - 5.20 in the textbook.

Given three points P1, p2 and P3, two vectors are defined:

$\underline{P_1P_2} = P_2 - P_1$;
$\underline{P_1P_3} = P_3 - P_1$.

The initial positions for $\underline{P_1P_2}$ and $\underline{P_1P_3}$ are given in Figure 5.18(a).

The objective is to find a transformation such that P_1 will be at the origin, $\underline{P_1P_2}$ lies on positive z-axis, and $\underline{P_1P_3}$ is on the positive half of Y-Z plane.

This transformation can be break into a sequence of steps:

Step 1: Translate P_1 to origin;
Step 2: Rotate about the y-axis so that P_2 will be on Y-Z plane;
Step 3: Rotate about x-axis so that P_2 is on the positive z-axis;
step 4: Rotate about z-axis so that P_3 is on the Y-Z plane.

The detailed derivation can be found in the textbook.

A Different Approach: for consecutive rotations, there is an easy way to find the composite matrix. Briefly speaking,

R = Rz(c)*Rx(b)*Ry(a-90')

can be computed by using the properties of orthogonal matrices.

This approach is not covered in the lecture. It is left as an optional reading material for the students. For those who are interested in it, you may refer to page 186 for the discussion.

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