portfolio_2

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Factor Analysis and Independent Component Analysis

Task 1

For this task we are going to use the communities and crime dataset:

```
library(mogavs)
data(crimeData)
```

Using the factor model on this dataset is a reasonable assumption as the number of samples will be larger than the number of factors and there is no perfect multicolinearity between any of the variables (ie. the covariance matrix will be full rank). Let's first analyse the difference between the number of constraints and the number of free parameters in the factor model for different values of k (the number of factors):

```
p <- ncol(crimeData) -1
for(k in 1:p){
  print( paste( k, ": ", ((p-k)**2 /2 ) - ((p+k)/2) )  )
}
```

```
## [1] "1 :  7259"
## [1] "2 :  7138"
## [1] "3 :  7018"
## [1] "4 :  6899"
## [1] "5 :  6781"
## [1] "6 :  6664"
## [1] "7 :  6548"
## [1] "8 :  6433"
## [1] "9 :  6319"
## [1] "10 :  6206"
## [1] "11 :  6094"
## [1] "12 :  5983"
## [1] "13 :  5873"
## [1] "14 :  5764"
## [1] "15 :  5656"
## [1] "16 :  5549"
## [1] "17 :  5443"
## [1] "18 :  5338"
## [1] "19 :  5234"
## [1] "20 :  5131"
## [1] "21 :  5029"
## [1] "22 :  4928"
## [1] "23 :  4828"
## [1] "24 :  4729"
## [1] "25 :  4631"
```

```
## [1] "26 :    4534"
## [1] "27 :    4438"
## [1] "28 :    4343"
## [1] "29 :    4249"
## [1] "30 :    4156"
## [1] "31 :    4064"
## [1] "32 :    3973"
## [1] "33 :    3883"
## [1] "34 :    3794"
## [1] "35 :    3706"
## [1] "36 :    3619"
## [1] "37 :    3533"
## [1] "38 :    3448"
## [1] "39 :    3364"
## [1] "40 :    3281"
## [1] "41 :    3199"
## [1] "42 :    3118"
## [1] "43 :    3038"
## [1] "44 :    2959"
## [1] "45 :    2881"
## [1] "46 :    2804"
## [1] "47 :    2728"
## [1] "48 :    2653"
## [1] "49 :    2579"
## [1] "50 :    2506"
## [1] "51 :    2434"
## [1] "52 :    2363"
## [1] "53 :    2293"
## [1] "54 :    2224"
## [1] "55 :    2156"
## [1] "56 :    2089"
## [1] "57 :    2023"
## [1] "58 :    1958"
## [1] "59 :    1894"
## [1] "60 :    1831"
## [1] "61 :    1769"
## [1] "62 :    1708"
## [1] "63 :    1648"
## [1] "64 :    1589"
## [1] "65 :    1531"
## [1] "66 :    1474"
## [1] "67 :    1418"
## [1] "68 :    1363"
## [1] "69 :    1309"
## [1] "70 :    1256"
## [1] "71 :    1204"
## [1] "72 :    1153"
## [1] "73 :    1103"
## [1] "74 :    1054"
## [1] "75 :    1006"
## [1] "76 :    959"
## [1] "77 :    913"
## [1] "78 :    868"
## [1] "79 :    824"
```

```
## [1] "80 :   781"
## [1] "81 :   739"
## [1] "82 :   698"
## [1] "83 :   658"
## [1] "84 :   619"
## [1] "85 :   581"
## [1] "86 :   544"
## [1] "87 :   508"
## [1] "88 :   473"
## [1] "89 :   439"
## [1] "90 :   406"
## [1] "91 :   374"
## [1] "92 :   343"
## [1] "93 :   313"
## [1] "94 :   284"
## [1] "95 :   256"
## [1] "96 :   229"
## [1] "97 :   203"
## [1] "98 :   178"
## [1] "99 :   154"
## [1] "100 :   131"
## [1] "101 :   109"
## [1] "102 :   88"
## [1] "103 :   68"
## [1] "104 :   49"
## [1] "105 :   31"
## [1] "106 :   14"
## [1] "107 :   -2"
## [1] "108 :   -17"
## [1] "109 :   -31"
## [1] "110 :   -44"
## [1] "111 :   -56"
## [1] "112 :   -67"
## [1] "113 :   -77"
## [1] "114 :   -86"
## [1] "115 :   -94"
## [1] "116 :   -101"
## [1] "117 :   -107"
## [1] "118 :   -112"
## [1] "119 :   -116"
## [1] "120 :   -119"
## [1] "121 :   -121"
## [1] "122 :   -122"
```

For $k < 108$ we have the difference is positive which means there exists no solution but we can find an approximate solution. However, for $k \geq 108$ we have that the difference is negative indicating there are infinitely many solutions and the problem is therefore not well-defined. So we must select a $k < 108$. Let's now compute the correlation matrix which we will use in our further analysis:

```
R <- cor(crimeData[,-ncol(crimeData)])
corrplot(R, method="shade", tl.pos='n')
```
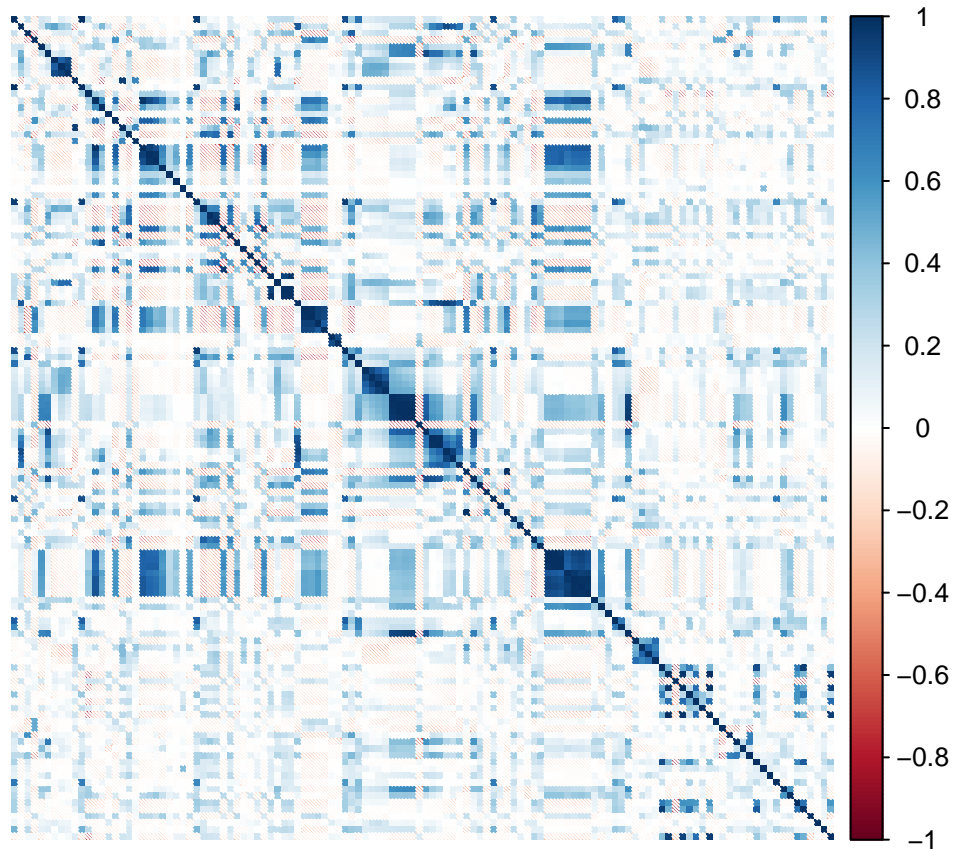
Using the help of the **nFactors** package we can create a plot that shows both the Scree plot and parallel analysis for the number of factors:

```r
ev <- eigen(R) # Get eigenvalues
ap <- parallel(subject=nrow(crimeData),var=ncol(crimeData)-1, rep=100, model = "factors") # Conduct par
nS <- nScree(x=ev$values, aparallel=ap$eigen$qevpea)
plotnScree(nS)
```

Non Graphical Solutions to Scree Test
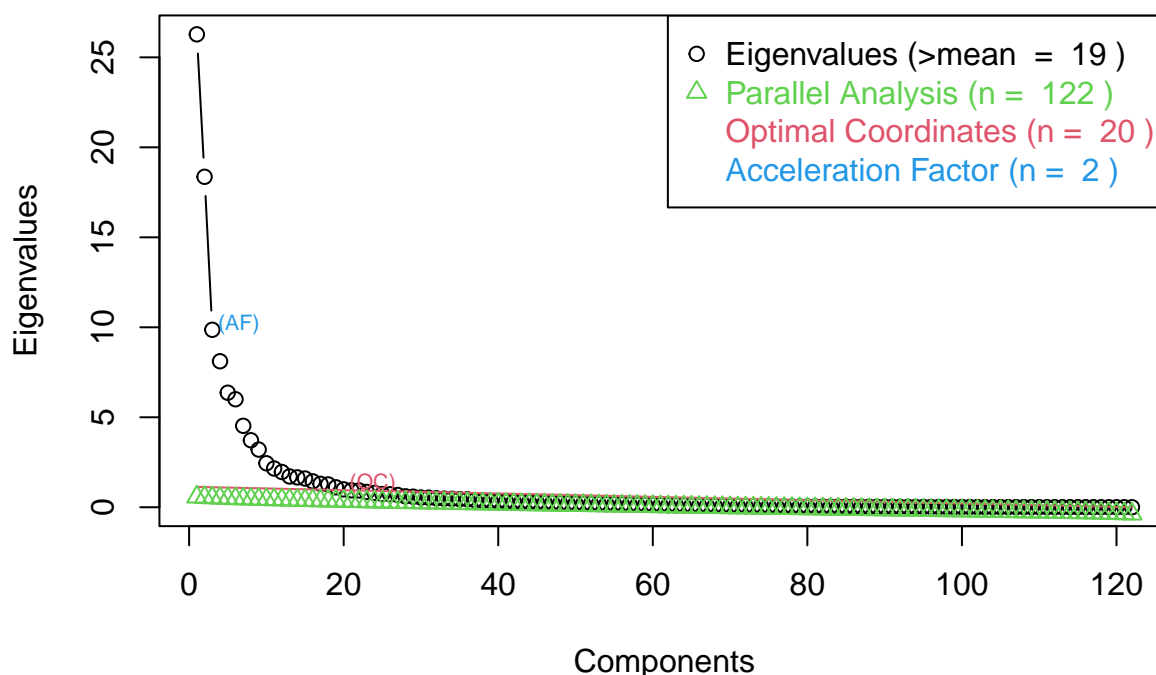
The acceleration factor tells us where the elbow of the Scree plot is (n=2), and the optimal coordinates corresponds to an extrapolation if the preceding eigenvalue by a regression line between the eigenvalue coordinates and the last eigenvalue coordinates. We can also find the Kaiser criterion:

```
nS$Components$nkaiser
```

```
## [1] 19
```

Both the visual and analytical analyses support choosing a $k \approx 19$, so we will use k=19 for our factor analysis. We will now carry out our factor analysis:

```
fit <- fapa(R, numFactors = 19)
```

Note that the initial matrix of specific variances used is the squared multiple correlation which we calculate as $1/\text{diag}(R^{-1})$.

Let's now print out our estimated loading matrix

```
fit$loadings
```

```
##               [,1]         [,2]         [,3]         [,4]         [,5]
##  [1,] -0.269380288  0.379537758  0.553014111  0.256565132 -0.218150521
##  [2,]  0.072522440  0.468461664 -0.668836201 -0.070555792 -0.318186309
##  [3,] -0.542206097  0.074464710  0.229719572  0.029879885  0.123965853
##  [4,]  0.567137628 -0.533543164 -0.012358987 -0.070614362 -0.071754092
##  [5,]  0.254235602  0.660034580  0.108174301 -0.048901065  0.015922091
##  [6,] -0.315528520  0.674976796 -0.376252281  0.155167892 -0.107274383
##  [7,] -0.288841723  0.149412734 -0.275940959 -0.572583891 -0.227416377
##  [8,] -0.341016014  0.321682186 -0.060333821 -0.675896488 -0.184625546
##  [9,] -0.296373116  0.201203748 -0.003128732 -0.666802050 -0.153900326
```

```
## [10,]  -0.189891656 -0.443181476  0.247077902  0.390018365  0.391940753
## [11,]  -0.229151117  0.405077726  0.557746831  0.268260679 -0.161927355
## [12,]   0.315381560  0.386160477  0.229708848  0.142107916  0.427706059
## [13,]   0.909760783  0.231987197  0.031508946  0.169461632 -0.107210725
## [14,]   0.573880987  0.339916839 -0.121710160 -0.292181003 -0.323635685
## [15,]  -0.003054331 -0.101867531 -0.247246967 -0.244429413 -0.294147371
## [16,]   0.861920639 -0.130420473  0.238296847  0.019025225 -0.008098138
## [17,]  -0.341041067 -0.472433360  0.068348126  0.409958008  0.352864087
## [18,]  -0.816728454  0.172009386 -0.172574056  0.186456480  0.091949967
## [19,]   0.039061353 -0.391535851  0.121165895  0.356889342  0.276899701
## [20,]   0.914191254  0.194699981  0.126995981  0.104388261 -0.076785445
## [21,]   0.837689519  0.147503190  0.322768537  0.150480014  0.007400224
## [22,]   0.728399122  0.229899173  0.391574624  0.166325632  0.012561892
## [23,]   0.624566470  0.214293208  0.086258764  0.148044659  0.017344886
## [24,]   0.259823117  0.104317324  0.062519370  0.063740217  0.045744041
## [25,]   0.406495657 -0.004955469  0.084183107  0.227834443 -0.065897721
## [26,]  -0.019149547 -0.012906965  0.012595991  0.019137839 -0.011331618
## [27,]   0.610883455  0.040671755  0.182452271  0.171201688  0.015248076
## [28,]  -0.415941249  0.349081526  0.495059704  0.252251254 -0.202810303
## [29,]  -0.865253456  0.081529039 -0.073001160 -0.162602928 -0.002936539
## [30,]  -0.723157980  0.145174605 -0.371695500  0.231946742  0.044960574
## [31,]  -0.807154135  0.050965544 -0.329276750  0.293816299  0.104421910
## [32,]   0.679788100  0.191408258  0.362241997 -0.265106639 -0.103760926
## [33,]  -0.783319589  0.147559948 -0.217497522  0.182243476  0.066202019
## [34,]   0.644971974  0.178252552  0.060105861 -0.176402709 -0.157746516
## [35,]  -0.082298323 -0.148148994 -0.271592988  0.187607280  0.109025105
## [36,]   0.094979271 -0.052975903  0.282894877 -0.299096957 -0.003877692
## [37,]  -0.661781455 -0.148779412 -0.375375211  0.157730459  0.128275885
## [38,]   0.727722397  0.119185878  0.356709920 -0.128456327 -0.089623141
## [39,]  -0.628874870 -0.067576832  0.349441494  0.015782345  0.214585098
## [40,]  -0.287973606  0.453553210  0.231887184 -0.369561522  0.028542688
## [41,]  -0.638519254  0.086151115  0.315597992 -0.029190679  0.176482697
## [42,]  -0.651339763  0.007075829  0.340021582 -0.004567703  0.201648116
## [43,]  -0.088767933  0.548380478 -0.654944072  0.176264156 -0.228353044
## [44,]   0.864453531 -0.057716971 -0.287223180  0.005862793 -0.236330469
## [45,]   0.879336379 -0.104764568 -0.226098269 -0.042841463 -0.205432579
## [46,]   0.854581349 -0.004400842 -0.160644049 -0.021426311 -0.215038906
## [47,]   0.752055683 -0.088472587 -0.321675705  0.039612063 -0.231676142
## [48,]  -0.050746575 -0.266095520  0.145733987 -0.264066437  0.074145160
## [49,]   0.091085488 -0.350265364  0.195772138 -0.274326006  0.048562950
## [50,]  -0.389524811  0.331460783  0.494551477  0.303025322 -0.176771613
## [51,]  -0.718841103  0.296293791  0.172511395  0.061297785  0.132836288
## [52,]  -0.183619356  0.571225742  0.325238665  0.275141680 -0.205858638
## [53,]  -0.223114705  0.311850355  0.211432429 -0.543904967 -0.088896576
## [54,]  -0.262127271  0.377947222  0.203623816 -0.571912139 -0.055266041
## [55,]  -0.261668690  0.437020472  0.203005381 -0.566557027 -0.034013535
## [56,]  -0.296550867  0.479879751  0.161850446 -0.517008610 -0.034376789
## [57,]  -0.046561204  0.864257275  0.001557305 -0.054617573  0.049796076
## [58,]  -0.058621672  0.875563179 -0.017844714 -0.024285487  0.054194717
## [59,]  -0.046172654  0.886594912 -0.023698180  0.003189753  0.058655756
## [60,]  -0.056510940  0.889943797 -0.045259510  0.028911416  0.065460240
## [61,]   0.155511757 -0.789317835  0.282142419 -0.226478891  0.002922799
## [62,]  -0.268826599  0.774762947 -0.299098697  0.194713141 -0.002016907
## [63,]  -0.373717868  0.660242349 -0.480055294  0.181699119 -0.070588024
```

```
## [64,]  -0.280076214  0.633184675 -0.576677474  0.185440027 -0.100912259
## [65,]   0.148924101  0.465677224 -0.713716577  0.178458778 -0.282551339
## [66,]   0.249650391  0.461011785 -0.626626510  0.126933796 -0.241381248
## [67,]  -0.271293061  0.499822728 -0.577812613  0.078672273 -0.226185060
## [68,]   0.688041193 -0.353911218 -0.245314592  0.342488850 -0.126553892
## [69,]  -0.465164056  0.735213782 -0.342372522  0.084387798 -0.078344813
## [70,]  -0.623260836  0.270527781  0.295057862 -0.134674005  0.254373464
## [71,]   0.478623402 -0.254073841 -0.252663323  0.134950247 -0.139263175
## [72,]  -0.326957132  0.257818341  0.595903937  0.230852606 -0.155744824
## [73,]   0.402142298  0.098254361 -0.181860964  0.007044780 -0.003556256
## [74,]   0.626301795 -0.323897221 -0.294236590  0.363489163 -0.152320719
## [75,]  -0.521393263  0.091364095  0.047226222  0.243469196  0.047398826
## [76,]  -0.210783386 -0.364852051 -0.140423824  0.262276649  0.015671801
## [77,]   0.192668239  0.097829876 -0.144977435 -0.338266926 -0.310157368
## [78,]  -0.826627053 -0.092256040 -0.150942499 -0.024872229 -0.021208111
## [79,]  -0.554254142  0.163044023 -0.161555933  0.166646198 -0.084919847
## [80,]   0.684034208  0.510469583  0.152211694  0.160584480  0.037847143
## [81,]   0.669948333  0.518607687  0.172742654  0.154118914  0.033434464
## [82,]   0.654222139  0.510362278  0.209180621  0.146783771  0.023342039
## [83,]   0.722966749  0.491931260  0.154271954  0.098967035 -0.038584986
## [84,]   0.745154351  0.494010438  0.148590813  0.152112808 -0.023051776
## [85,]   0.740978260  0.502811340  0.155907965  0.162429742 -0.028889565
## [86,]   0.749736233  0.458269463  0.131872283  0.177139453 -0.046760087
## [87,]  -0.309394899  0.355385235  0.078510745 -0.069518035  0.014865900
## [88,]   0.252869119  0.591677235  0.020724770  0.099202403  0.007297273
## [89,]  -0.087833566 -0.092286402  0.039533743  0.259782776  0.092704148
## [90,]  -0.285535084  0.311013029  0.522540703  0.249676526 -0.189763804
## [91,]  -0.222167843  0.322736516  0.413378933  0.214607263 -0.176960790
## [92,]   0.055290819  0.863330165 -0.078120128  0.176844165  0.072182233
## [93,]  -0.181766277 -0.495127805 -0.257766558  0.176724095  0.089957567
## [94,]   0.216571523 -0.275341453 -0.189189182  0.710478621  0.179391321
## [95,]  -0.072912374 -0.180311772 -0.206391581  0.703993531  0.305866518
## [96,]  -0.080315366 -0.259354635 -0.295645706  0.507025767  0.153184835
## [97,]  -0.337791769 -0.157339314  0.309594311  0.364344333 -0.616451259
## [98,]  -0.347319949 -0.428964272 -0.066192868 -0.006104840 -0.575807226
## [99,]   0.302441833  0.163334314 -0.289236011 -0.371637328  0.608188595
## [100,] -0.312738318 -0.429112778 -0.019296635  0.034598190 -0.583756921
## [101,] -0.224268049 -0.013363739  0.422670060  0.298883908 -0.471756012
## [102,] -0.349277595 -0.292346288  0.119653541  0.012098991 -0.497344959
## [103,] -0.025480629  0.236299753  0.208634040 -0.032542286  0.013462391
## [104,] -0.347611171 -0.428543681 -0.065998465 -0.006436086 -0.575745917
## [105,]  0.172651332 -0.173962057  0.066608414  0.013527301 -0.035683406
## [106,]  0.057084182 -0.489399342  0.045084359 -0.218718591  0.043084925
## [107,] -0.236545343  0.151637950  0.146890019  0.146056848  0.099054669
## [108,] -0.182171545  0.497180493 -0.200217564  0.153609160 -0.157015837
## [109,] -0.020406433  0.326486684  0.206874646 -0.110811504  0.060018463
## [110,] -0.268985803  0.488549816 -0.015697817  0.222498777 -0.019414491
## [111,] -0.289506034 -0.147619415  0.310666122  0.368672214 -0.552145526
## [112,] -0.197749236 -0.016551695  0.129167353  0.079768563 -0.092246766
## [113,] -0.215423016  0.103712947 -0.015940659  0.153622129  0.121460198
## [114,] -0.143836310  0.126476871  0.374220768  0.145306269 -0.281563700
## [115,] -0.133337668  0.566075455  0.151649055  0.163747661  0.264936203
## [116,]  0.114905248  0.421956449  0.314229394  0.219240768  0.182156348
## [117,] -0.434372796 -0.182873889  0.319098803  0.199776329 -0.555693481
```

```
## [118,] -0.315574155 -0.135024833  0.328538225  0.340706265 -0.622857525
## [119,]  0.299265031  0.027583939  0.056080824  0.120352886  0.097538721
## [120,] -0.121552930  0.050994318 -0.001836264 -0.147906305  0.027151623
## [121,] -0.232613719  0.269542237  0.331998518  0.095018354  0.064876850
## [122,] -0.279289769 -0.358991478 -0.036163510 -0.025058818 -0.584819527
##                 [,6]         [,7]         [,8]         [,9]        [,10]
##   [1,] -0.487812772  0.041071709  0.2156686568 -0.0458199861  0.074705134
##   [2,] -0.250834057 -0.203789166 -0.0180767072 -0.0314411717 -0.042554080
##   [3,] -0.178885625 -0.379186909 -0.5055396710  0.1182237416  0.083357109
##   [4,]  0.065477954  0.239761001  0.4034659739 -0.0459930120 -0.085638185
##   [5,]  0.155039585  0.089589236 -0.0014125158  0.0606864299  0.092927042
##   [6,]  0.148118017  0.171938217  0.0892049986 -0.1676161906 -0.017397727
##   [7,] -0.210794225 -0.404211840  0.2179747239 -0.0035823169 -0.094051584
##   [8,] -0.158986045 -0.179582084  0.1773527827  0.1706445762 -0.269658049
##   [9,] -0.069032139 -0.348970325  0.2838667109  0.0964619781 -0.219906103
##  [10,]  0.435715936 -0.058638826  0.2275780260 -0.1749129349  0.125059274
##  [11,] -0.510716673  0.051264337  0.2166001180 -0.0420029589  0.059311338
##  [12,] -0.336379605  0.105639059  0.0524163450 -0.0004513231 -0.117697987
##  [13,] -0.011993897 -0.102431771 -0.1647482356  0.0675651811  0.077796545
##  [14,] -0.351245452  0.136586110 -0.1991584162  0.2737268330 -0.148051824
##  [15,]  0.012929880 -0.103099694  0.0664353868 -0.1482149038  0.192115906
##  [16,]  0.107189592 -0.142130418  0.1904141545 -0.0198019210  0.031482373
##  [17,]  0.359398391 -0.134757222  0.2468846593 -0.1503414763  0.106820008
##  [18,]  0.067226603 -0.146120179 -0.0481862373 -0.0419079247  0.049164550
##  [19,]  0.151759048 -0.020659932  0.1810226781 -0.2211557471 -0.069673610
##  [20,]  0.047950642 -0.169773054 -0.1053879736  0.0698719662  0.071248968
##  [21,]  0.158744372 -0.148254948 -0.1200825980 -0.0047263699  0.148428249
##  [22,]  0.134981614 -0.221683434 -0.2085231143  0.0010741975  0.188778738
##  [23,]  0.032252649  0.017083705 -0.0868718767  0.0725143669  0.052486439
##  [24,]  0.020374813  0.005717151 -0.1350542250 -0.0129146802  0.039757001
##  [25,]  0.173840786 -0.136901016 -0.1117271798  0.0036133682  0.097689000
##  [26,]  0.022370795 -0.050788532 -0.0880872501 -0.0423404256 -0.023674462
##  [27,]  0.066881492 -0.111793286 -0.1300153429  0.0386926940  0.041913793
##  [28,] -0.465195421 -0.051885029  0.2172878906 -0.0602634863  0.080926010
##  [29,]  0.040552686 -0.279280911  0.0645153299 -0.1590013784  0.058995534
##  [30,]  0.172479731 -0.025277269  0.0876315724  0.0726911578  0.054604926
##  [31,]  0.121669091  0.027100480  0.0368114147  0.0955736827  0.029792746
##  [32,]  0.063190601 -0.388005604 -0.0046298818 -0.0807249298  0.096774727
##  [33,]  0.047157370 -0.174394446  0.0400042258 -0.1059870377 -0.030470952
##  [34,] -0.291710252  0.314548852 -0.2544573687  0.3334594734 -0.118483666
##  [35,] -0.054497787  0.195720669 -0.0425768927  0.4786354356  0.058070354
##  [36,]  0.057124877 -0.655282704  0.2135933894 -0.1546112954 -0.021844773
##  [37,] -0.036704934  0.199446941  0.0002962446  0.3258492622  0.015530336
##  [38,]  0.068500500 -0.366289013 -0.0353403853 -0.1260604288  0.119006941
##  [39,]  0.014965567  0.381236770 -0.3566780975 -0.0217541435  0.030390808
##  [40,]  0.043928585 -0.437134240  0.1752783662  0.2449682946 -0.326761073
##  [41,]  0.013640728  0.361564400 -0.4038182800 -0.0220317683  0.029560166
##  [42,]  0.016616504  0.372358115 -0.3939941488 -0.0195113611  0.034312400
##  [43,] -0.303059915 -0.090453668 -0.0872876318  0.0865645862  0.031615812
##  [44,] -0.056993056  0.107650095  0.2198684858 -0.0702412208  0.047804027
##  [45,] -0.046094889  0.118332434  0.2519567770 -0.0505351945  0.023635407
##  [46,] -0.009933006  0.131247245  0.2071073733 -0.0319299143 -0.029986365
##  [47,] -0.034950599  0.132573021  0.2432409783 -0.0696411688  0.026031421
##  [48,] -0.162032845  0.166135993 -0.2358172572  0.2738210610 -0.162063932
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##  [49,]  -0.146735099   0.177156140  -0.2149285541   0.2941101396  -0.234509481
##  [50,]  -0.433499300  -0.098951506   0.1152144513   0.0390292810   0.069612150
##  [51,]  -0.087576225  -0.276684257  -0.3099769206   0.1671790412  -0.022668833
##  [52,]  -0.174312344   0.118279560   0.2442455142   0.0229883701   0.028590551
##  [53,]   0.107308640  -0.138407154   0.1192042036   0.1463650172   0.258341430
##  [54,]   0.096449533  -0.078671270   0.0983934447   0.1874241569   0.375233302
##  [55,]   0.067195424  -0.045840534   0.0443248841   0.1836617710   0.412740387
##  [56,]   0.044627145  -0.002302801   0.0030995531   0.1605193614   0.385906714
##  [57,]   0.320756984   0.088045879   0.1701341649   0.0930582774   0.077389716
##  [58,]   0.316547838   0.117392468   0.1472139438   0.0973446116   0.083475856
##  [59,]   0.305440527   0.129262556   0.1243403733   0.0950959145   0.085055199
##  [60,]   0.292436054   0.143440859   0.1117350419   0.0819825651   0.077369253
##  [61,]  -0.244742474  -0.098217240  -0.1677975650   0.0742726107   0.051197494
##  [62,]   0.224851391   0.157375379   0.1680235145  -0.0136727932   0.012172745
##  [63,]  -0.101209814  -0.050876027  -0.0256506995  -0.0249648454   0.037588001
##  [64,]  -0.125011041  -0.039082334  -0.0294744934  -0.0480422573   0.047737347
##  [65,]  -0.293084044  -0.022932290  -0.1227138126   0.0031523817   0.062768646
##  [66,]  -0.352364509  -0.065649694  -0.0244732944   0.1383524008  -0.053127918
##  [67,]  -0.101797279   0.103271404  -0.1954106374  -0.1295800854   0.128042803
##  [68,]  -0.185696881  -0.089165425  -0.0203652727  -0.1372493331   0.145300795
##  [69,]   0.098619938   0.140039192  -0.0206234646  -0.0916359878   0.031130799
##  [70,]   0.318249620   0.214375141   0.1246947843  -0.0309923896  -0.193445260
##  [71,]  -0.234741248  -0.139123608  -0.1443048384   0.0291301421   0.140163252
##  [72,]  -0.404468001   0.044787685   0.1439815239  -0.1987097437   0.092300903
##  [73,]  -0.109432083  -0.033403260   0.1382681519   0.4690036061  -0.078071060
##  [74,]  -0.161261263  -0.070325069  -0.0664518908  -0.1984469903   0.196766358
##  [75,]  -0.243442720  -0.266994891  -0.0942866465  -0.0065053439   0.017819791
##  [76,]  -0.059829286  -0.362892852   0.0502883056  -0.0541399348   0.039044073
##  [77,]  -0.192870701   0.333482603  -0.3088212310  -0.3953504018   0.095371035
##  [78,]   0.042902915  -0.093235684  -0.1400428365  -0.0951723842   0.189257248
##  [79,]   0.099613558  -0.128991445  -0.0151926597   0.0341661634  -0.007362662
##  [80,]   0.273179241  -0.108494680  -0.0939797406   0.0837752350  -0.022005870
##  [81,]   0.284187950  -0.128862776  -0.0971329917   0.0580326939  -0.001861792
##  [82,]   0.289010431  -0.153590392  -0.1002018949   0.0226671101   0.024616863
##  [83,]   0.173215894   0.095666771  -0.1473453581   0.0621353854  -0.089848949
##  [84,]   0.192323740  -0.017041453  -0.1507551288   0.0500112420  -0.072557760
##  [85,]   0.188416510  -0.059129336  -0.1365993051   0.0307950446  -0.088406028
##  [86,]   0.166478987  -0.029433406  -0.1904109238   0.0400765602  -0.062107029
##  [87,]   0.227850632  -0.213846687   0.0313373074  -0.2372626261  -0.145302359
##  [88,]   0.166709382   0.080315128  -0.2222618298  -0.1336010562  -0.185175389
##  [89,]   0.093879762  -0.298310357  -0.0241283924   0.2161449662  -0.132586644
##  [90,]  -0.348163058  -0.023510717   0.2023837751   0.0238921192   0.071185516
##  [91,]  -0.230516074   0.056762758   0.1510086141  -0.0146235370   0.050637407
##  [92,]   0.330712886   0.122226027   0.1243157977   0.0249283585  -0.009468462
##  [93,]  -0.221343799  -0.260911760   0.1239704462   0.3589268284   0.016833730
##  [94,]  -0.008749670  -0.293793696   0.1371774316   0.2717766255   0.070112715
##  [95,]  -0.080949834  -0.102758391   0.0476464290   0.2756593062   0.109979662
##  [96,]  -0.068236936  -0.155055363   0.0860204797   0.2968073866   0.051802961
##  [97,]   0.107495736   0.109794572   0.0586556013   0.1342877101  -0.133213532
##  [98,]   0.491142862  -0.127087950  -0.1032473264   0.0745850156   0.008861305
##  [99,]  -0.097217554  -0.112153597  -0.0687880277  -0.1403461499   0.128332055
## [100,]   0.487082716  -0.127921187  -0.0788073295   0.1117351142  -0.009213317
## [101,]   0.006901750   0.180997001  -0.0357197529   0.0415197466  -0.082736547
## [102,]   0.402215279  -0.056940762  -0.1971379623  -0.0144070205   0.107814539
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## [103,]  -0.150002746   0.046316341  -0.1397406391  -0.1691762548   0.049753993
## [104,]   0.491572318  -0.127358040  -0.1033092166   0.0740045901   0.008634990
## [105,]   0.039690868   0.126268344   0.2818583202  -0.1607366636  -0.530139268
## [106,]   0.052973895   0.321853287   0.3129468878   0.2587774767   0.202114084
## [107,]  -0.210612854  -0.440286937  -0.4175190846   0.0099581430  -0.196461866
## [108,]   0.158354097  -0.021300364   0.0365781722  -0.3807075948  -0.230905589
## [109,]   0.181075157   0.180289264   0.0399516756  -0.0478841389  -0.143066693
## [110,]  -0.059772129  -0.316996758  -0.3009544228  -0.2712393679  -0.364214414
## [111,]   0.084159086   0.032951159   0.0421837809   0.1338540642  -0.042061627
## [112,]   0.061783425   0.040172440  -0.0120144536   0.0549808584  -0.051351293
## [113,]   0.063328465   0.017129523   0.0400739592   0.0120637269   0.149461613
## [114,]  -0.456164376   0.048370890   0.0748974989  -0.1850148536   0.081722693
## [115,]   0.194271803   0.023133466   0.1707748512   0.3113763151  -0.145516229
## [116,]   0.057267679  -0.254106796   0.0955109053   0.2365080683  -0.059348830
## [117,]   0.154932853   0.071732679  -0.0212039111   0.0668105307   0.068939243
## [118,]   0.127751078   0.143517574   0.0230152715   0.1529606903  -0.097176001
## [119,]  -0.153048525   0.098713916   0.1718656068   0.0927826893   0.045173878
## [120,]  -0.064374999   0.046585862  -0.0676812473  -0.0513986147   0.222792749
## [121,]  -0.185598942  -0.013076736   0.0332270135  -0.0086532789   0.003950948
## [122,]   0.507915406  -0.066544106  -0.1284998344   0.0904077150   0.010934975
##               [,11]         [,12]         [,13]          [,14]          [,15]
##    [1,]  -0.072975647   0.151031324  -0.0888377413   4.678570e-02  -0.0659915948
##    [2,]   0.120389984  -0.005184226  -0.1320391302  -1.076873e-02  -0.0891354247
##    [3,]   0.020975003  -0.127724092  -0.1135359857   6.532285e-03   0.0750933907
##    [4,]   0.069316745   0.120232637   0.0807863546  -2.217888e-03   0.0117783355
##    [5,]  -0.037769214  -0.006492038  -0.0417306406  -2.859321e-02  -0.2590145980
##    [6,]  -0.114829193   0.014579288   0.0873018073   1.914304e-02   0.0768192556
##    [7,]   0.072654751  -0.035327241  -0.0700583211  -1.132816e-02  -0.1279125212
##    [8,]   0.118116448   0.013147886  -0.0274311007   5.415936e-02  -0.0454950212
##    [9,]   0.110040925  -0.030306820  -0.1060172906   5.375900e-05  -0.1028531454
##   [10,]  -0.060436881  -0.039732603  -0.2004020058  -1.136700e-01   0.0251444202
##   [11,]  -0.067344633   0.117020407  -0.0750723017   3.844108e-02  -0.0563626683
##   [12,]   0.086849946  -0.244939712   0.0967792312  -5.991461e-02   0.0517669341
##   [13,]   0.009158433   0.011498943   0.0208975852   2.759386e-02  -0.0008079496
##   [14,]  -0.002280894   0.028654445   0.1287594264   3.326340e-02   0.0847680228
##   [15,]  -0.276065630   0.187174542   0.0823375473  -2.590751e-02   0.1731341794
##   [16,]  -0.092253550   0.030070893  -0.0170404294  -3.801024e-02  -0.0341841697
##   [17,]  -0.006907547  -0.033445024  -0.2292973552  -1.216932e-01  -0.0007222016
##   [18,]   0.059794860   0.037647735   0.0926046798   1.590855e-02  -0.1850144192
##   [19,]   0.087584688  -0.170816696  -0.2190572925  -1.110942e-01  -0.2380909056
##   [20,]  -0.015576609   0.020228648   0.0415046882   2.563794e-02  -0.0092537846
##   [21,]  -0.067956549   0.006686470   0.1000997064   7.181653e-02   0.0276917639
##   [22,]  -0.097112796  -0.006463319   0.0824440194   7.452156e-02   0.0402473473
##   [23,]   0.107402243  -0.058925545   0.1322865304  -2.451044e-03  -0.0054862830
##   [24,]  -0.006894290   0.070448274   0.0302707268  -2.028266e-02   0.0143091424
##   [25,]  -0.037672127   0.090977883   0.0921366294   1.286591e-01  -0.0274218086
##   [26,]   0.089400068   0.270127342   0.1339077989  -2.850457e-01   0.0020276275
##   [27,]   0.048644971  -0.019118625   0.0927698547   3.482275e-02   0.0336083293
##   [28,]  -0.057271545   0.134445722  -0.0466272936   6.701622e-02  -0.0440480217
##   [29,]   0.020326449  -0.019950365   0.1210759688  -6.281603e-03  -0.0207952503
##   [30,]  -0.091888285   0.030098086  -0.0003237166  -2.502087e-02   0.1191107374
##   [31,]   0.019083330  -0.014959526  -0.0451964176  -2.068419e-02   0.0914811443
##   [32,]  -0.119847240  -0.002058753   0.1460497689   1.637025e-02   0.0121823831
##   [33,]   0.123616626  -0.016894983   0.0569629258   2.218608e-02  -0.0140753823
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10

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## [34,] -0.087443537  0.078423201  0.0763895244  4.590153e-02  0.1487265996
## [35,] -0.042114521  0.102125236 -0.0144596185 -4.521427e-02 -0.2103683952
## [36,] -0.144491726 -0.045429646  0.0621548409 -1.343209e-02 -0.0344983138
## [37,]  0.012040887  0.089889522 -0.0658186326 -4.530968e-02 -0.1198030231
## [38,] -0.141492006 -0.031360398  0.1730682145  4.007769e-02 -0.0065636637
## [39,]  0.023111124  0.076277510  0.1314506377  7.347503e-02 -0.0891186861
## [40,]  0.063934997  0.068603647 -0.0477305334  2.435271e-02 -0.0225785722
## [41,] -0.034117404  0.013776562  0.1710483148  6.672663e-02 -0.1407198809
## [42,] -0.009038418  0.040402326  0.1629142008  6.896872e-02 -0.1208465690
## [43,]  0.061227293  0.052922151  0.0318031507  2.524022e-02  0.0240070077
## [44,] -0.042944484  0.025405208 -0.0198095113 -2.425322e-02  0.0466634182
## [45,] -0.042568526  0.042438336  0.0109269307 -5.334040e-03  0.0316675092
## [46,] -0.067658744 -0.019504950 -0.0048137502 -4.910350e-02  0.0714449734
## [47,] -0.066809622  0.001989415 -0.0260816158 -2.735638e-02 -0.0182587675
## [48,] -0.416806256  0.065349646 -0.3870433969 -2.235715e-01  0.1534216552
## [49,] -0.349887376  0.063235185 -0.3924329708 -2.557579e-01  0.1695754296
## [50,] -0.020765646  0.106902771 -0.0810426026  9.863138e-02 -0.0205503409
## [51,]  0.070857927 -0.027276971 -0.0549467897  4.601190e-02  0.0408376179
## [52,] -0.102743414  0.113790131 -0.1182032327  1.476642e-02 -0.0828438091
## [53,] -0.053637802 -0.127943306  0.0888128500 -2.364379e-01 -0.0012157470
## [54,]  0.017529506 -0.139229832  0.0413844014 -2.498650e-01  0.0397925231
## [55,] -0.002603083 -0.127645065  0.0243309054 -2.249461e-01  0.0578463431
## [56,] -0.049373262 -0.145106918  0.0063344304 -1.359349e-01  0.0813180887
## [57,] -0.071291438 -0.020036824 -0.0277380187 -4.257981e-02  0.0223610622
## [58,] -0.058408485 -0.020427038 -0.0505777541 -3.171412e-02  0.0398285059
## [59,] -0.060636609 -0.015071058 -0.0618333231 -2.783026e-02  0.0336696373
## [60,] -0.071712809 -0.010142565 -0.0677993832 -1.203892e-02  0.0383770728
## [61,]  0.099912332 -0.029374550 -0.0285841283 -1.912943e-02 -0.1227414280
## [62,] -0.100419503  0.016966817  0.0155654573 -8.362045e-05  0.1045750395
## [63,]  0.013227410  0.024265768 -0.0196987036 -1.398959e-02  0.0015446203
## [64,]  0.018449265  0.008301079 -0.0321010204 -2.543104e-02 -0.0119183227
## [65,]  0.095037862 -0.016271263 -0.0406947455 -1.150556e-02 -0.0247899256
## [66,]  0.043232777  0.046483615  0.0497429251  2.131533e-02  0.0240771344
## [67,]  0.137977816 -0.091105840 -0.1248811983 -1.731820e-02 -0.0453248767
## [68,] -0.046025574 -0.072447057 -0.0956029288 -1.728470e-01  0.0112441302
## [69,] -0.082850592 -0.019971363  0.0026775760 -1.401449e-02 -0.0078028581
## [70,] -0.064936689  0.079510217  0.0186716738  4.121010e-02  0.1285937422
## [71,]  0.047104330 -0.017782055 -0.0395998696  1.173808e-02 -0.1154364545
## [72,] -0.004951967  0.137778171 -0.1058503137  5.618107e-02  0.1097495895
## [73,] -0.151188975 -0.044357938  0.1053982630 -5.267496e-02 -0.3630664334
## [74,] -0.013230565 -0.098212906 -0.1396562756 -1.891619e-01 -0.0028672258
## [75,]  0.087987194  0.041467672  0.1157799647  2.883415e-02  0.0589400177
## [76,]  0.138053456  0.087953444  0.0275574461 -4.251445e-02  0.2354139826
## [77,] -0.068224256 -0.083995019 -0.0910579647 -1.807793e-02  0.0695093559
## [78,] -0.019224007  0.003439288  0.1078801846  8.143643e-02  0.0969502315
## [79,] -0.073170787  0.021992277  0.0860514607  3.726178e-02  0.1288092387
## [80,]  0.030465981  0.098852234 -0.0451981124  1.998132e-02 -0.1071189111
## [81,]  0.016603887  0.104803500 -0.0331466411  3.247809e-02 -0.1042940064
## [82,] -0.002741883  0.101224629 -0.0219868913  4.005427e-02 -0.0964436685
## [83,]  0.117562077 -0.024604253 -0.0897031639 -1.484465e-02 -0.0235624448
## [84,]  0.115731966  0.022136628 -0.1067538867 -4.672543e-03 -0.0231796479
## [85,]  0.120556066  0.036215389 -0.1035268838 -3.075690e-02 -0.0037322648
## [86,]  0.151208026 -0.002172218 -0.1239403108 -3.400066e-02  0.0047916409
## [87,]  0.260883492 -0.004630522 -0.2044357772 -1.137090e-01 -0.0947837269
```

11

```
## [88,]  0.185306679  0.141071652 -0.2173771977 -8.031760e-03  0.1531770442
## [89,]  0.268672949  0.071010387 -0.0788857148  3.316391e-02  0.4241353219
## [90,] -0.042484955  0.135564130 -0.0677626414  1.076820e-01 -0.0342508201
## [91,] -0.047534982  0.106036236 -0.0760068136  5.700405e-02 -0.0579095828
## [92,] -0.068408446 -0.005454338 -0.0766650423  8.434455e-04  0.0463196371
## [93,] -0.067507790  0.097427976  0.0955622744 -7.238316e-02 -0.1043596037
## [94,] -0.054637855 -0.026492221  0.0157162635 -1.138669e-01  0.0652793623
## [95,] -0.157001789  0.047415668  0.0838887507 -7.976463e-02 -0.0500921295
## [96,] -0.101133725  0.085607942  0.0566376937 -1.187756e-01 -0.1273280294
## [97,]  0.017711346 -0.382503879  0.0867876834 -5.446192e-02  0.0489357023
## [98,] -0.089634161  0.188520337 -0.0833394708  1.251126e-01 -0.0358800027
## [99,] -0.052845040  0.362838680 -0.1183705283  9.709321e-02 -0.0710264345
## [100,] -0.043403255  0.149985792 -0.0398040913  8.400429e-02  0.0132382743
## [101,]  0.292615138 -0.105482814  0.0656972343 -2.951909e-01 -0.0197685031
## [102,]  0.199149315  0.305349390 -0.0139375857 -2.230533e-01 -0.0389971558
## [103,]  0.387711080  0.327138130  0.0126051623 -5.092683e-01 -0.0901259545
## [104,] -0.089695193  0.188727833 -0.0832456752  1.247149e-01 -0.0356554306
## [105,] -0.147766494  0.144980388  0.2093083580 -1.868871e-01 -0.0078163538
## [106,]  0.270360164  0.176062137 -0.0339251635  1.491090e-01  0.1113783857
## [107,] -0.213851584 -0.072894301 -0.0978184649  9.701353e-02 -0.0254728396
## [108,] -0.279126770  0.013304327  0.2187294046 -9.202192e-02  0.0027660132
## [109,]  0.057206824 -0.015728272  0.0483538250 -7.329326e-02 -0.3083114752
## [110,] -0.372572130 -0.094507078  0.1335130077 -1.906842e-01 -0.0309827910
## [111,] -0.036626671 -0.321507991  0.0557373484 -8.559129e-02  0.0491349533
## [112,] -0.043164869  0.421377777  0.3523145920 -3.215146e-01  0.1001574942
## [113,] -0.285014225 -0.035954223 -0.0670881127  2.114363e-02 -0.0899931195
## [114,] -0.042771865  0.130039246 -0.0969062199  1.679445e-02  0.0216910514
## [115,]  0.047434092 -0.008590321  0.0543708542  5.473437e-02  0.0503313393
## [116,]  0.112167342  0.048614964  0.0605833398  1.210025e-01  0.1685667444
## [117,]  0.028163558 -0.084879922 -0.0652703827  5.996119e-02 -0.0143666913
## [118,]  0.034879625 -0.312426722  0.0345230490 -2.937931e-02 -0.0377834404
## [119,]  0.140329225  0.008072236  0.2018906996 -1.513287e-01  0.1520701895
## [120,] -0.181221754  0.106550741 -0.0240117902 -7.677997e-02 -0.0500328469
## [121,]  0.038468436  0.044408463  0.0235276746 -4.465411e-02  0.0565679237
## [122,] -0.089544708  0.223513263 -0.0642668808  7.827901e-02 -0.1259730903
##              [,16]        [,17]        [,18]        [,19]
##   [1,] -0.0450981188  0.0342794989 -0.014548518 -0.0136999151
##   [2,] -0.0104226028  0.0192195531  0.035728980  0.0150326907
##   [3,]  0.0371533152  0.0792244587  0.198891040 -0.1289408204
##   [4,] -0.0526997425 -0.0182784103 -0.182691733  0.0984214330
##   [5,]  0.1006491159 -0.0165933052  0.063330508  0.0023152239
##   [6,]  0.0110946757 -0.1443001975 -0.015049044  0.0272576624
##   [7,]  0.0542003808 -0.1425444483 -0.063442618 -0.1169986841
##   [8,]  0.0201018457 -0.1155241107 -0.066081158 -0.1019247200
##   [9,]  0.0424398569 -0.1532663018 -0.082115974 -0.1607242487
##  [10,]  0.0389293046  0.0584457139  0.008865127 -0.0579861077
##  [11,] -0.0296158173  0.0364907039 -0.016080584 -0.0188244358
##  [12,]  0.1004362217  0.0307171311  0.013866923 -0.0313973469
##  [13,] -0.0242605616  0.0019370266 -0.044262383 -0.0269605553
##  [14,] -0.0263059755 -0.0143754922  0.043187381  0.0053140216
##  [15,]  0.1261539943 -0.1348073833 -0.179406315  0.0308641633
##  [16,]  0.0319273593  0.0072671630  0.058723547 -0.0003050702
##  [17,]  0.0310743286  0.0433089831 -0.013302724 -0.0387066771
##  [18,]  0.0285330702 -0.0253601109 -0.069666309  0.0218478359
```

```
## [19,]   0.1099377981   0.1343389689   0.053181281   0.0342310901
## [20,]  -0.0356868305  -0.0271491207  -0.049133297  -0.0243549062
## [21,]  -0.0319204699  -0.0530404047  -0.030062457  -0.0404179196
## [22,]  -0.0151341756  -0.0467090142  -0.005152240  -0.0590288286
## [23,]  -0.1234065967  -0.1667885981   0.104073797  -0.1248189117
## [24,]   0.0463111910   0.0382831415  -0.028776537  -0.0591529192
## [25,]  -0.0037883037  -0.0105589563   0.059769362  -0.0845208542
## [26,]   0.0942265100   0.0904190711   0.175173844  -0.1684706654
## [27,]   0.0016825229  -0.0145870516   0.045009469  -0.1125896710
## [28,]  -0.0481238894   0.0167613600  -0.003817107  -0.0100165471
## [29,]   0.0005707901  -0.0876475346  -0.049525010  -0.0754700923
## [30,]  -0.1384445534  -0.1091168649  -0.023167810  -0.1938203665
## [31,]  -0.1374856628  -0.0593833914  -0.033797965  -0.1604767045
## [32,]   0.0163488907  -0.1038101479   0.070138485   0.0132069554
## [33,]   0.0247181413   0.0058432981  -0.089840948   0.0109153764
## [34,]  -0.0047270385  -0.0212282399  -0.010696041   0.0486133153
## [35,]  -0.2809352058   0.0163522498  -0.028855876  -0.1903562427
## [36,]   0.1392328098  -0.1058802039   0.093521654   0.1552180889
## [37,]  -0.2303297358   0.0567835074  -0.037944520  -0.1039366057
## [38,]   0.0326172807  -0.0996135892   0.089109767   0.0296326111
## [39,]  -0.0172289066  -0.0610411057  -0.081080421   0.1241918738
## [40,]   0.0259690715  -0.0707852836  -0.062996806  -0.0470737641
## [41,]   0.0410845740  -0.0715841029  -0.098522856   0.1562454890
## [42,]   0.0179153851  -0.0737474377  -0.093158758   0.1505915452
## [43,]   0.0866886029   0.0580704934   0.002958346   0.1135326774
## [44,]  -0.0667961436   0.0222663897   0.020297650   0.0079370665
## [45,]  -0.0504804011   0.0286021893   0.013430563   0.0449605961
## [46,]  -0.0216151390   0.0057825868   0.030503864   0.0141562025
## [47,]  -0.1356781363   0.0332946165   0.041894256   0.0472727653
## [48,]   0.0742676208  -0.1833942757   0.083446966   0.0451896126
## [49,]   0.0656481691  -0.1437951438   0.046402103   0.0358189637
## [50,]  -0.0221191648   0.0474904577   0.021391733  -0.0062355009
## [51,]   0.0316062330   0.0536599265   0.060269420  -0.0880314223
## [52,]  -0.0190980755  -0.0113201838   0.054916890  -0.0234669287
## [53,]  -0.1589029966   0.2471828432  -0.094432893   0.0894185131
## [54,]  -0.1603095172   0.1684297425  -0.109445989   0.0909309477
## [55,]  -0.1260377994   0.0972980021  -0.104832366   0.0545798018
## [56,]  -0.1159336100   0.0760188906  -0.106315703   0.0509140763
## [57,]   0.0102967778   0.0576662844   0.090805056   0.0286959856
## [58,]   0.0247130977   0.0373437411   0.088657889   0.0220834565
## [59,]   0.0361103399   0.0092936368   0.099763430   0.0167288486
## [60,]   0.0347027246   0.0020955284   0.103216490   0.0171344171
## [61,]   0.0098779696   0.1292888534  -0.046561875   0.0145567809
## [62,]   0.0189989733  -0.0908991058   0.041167225  -0.0092955353
## [63,]   0.1486518257   0.0553835203  -0.009060055   0.1096939019
## [64,]   0.1364780240   0.0642942460  -0.008672165   0.0979785323
## [65,]   0.0654559847   0.0727964741  -0.005071320   0.0475514004
## [66,]   0.0456903662   0.0619524311   0.023671956   0.0918703354
## [67,]   0.0570567677   0.0239785425  -0.046268122   0.0137689449
## [68,]  -0.0065221773   0.0126139678   0.003549017   0.0116204762
## [69,]   0.1046278172  -0.0525549952   0.014636425   0.0106940206
## [70,]  -0.0451886348  -0.0076556356  -0.045944028  -0.0451305765
## [71,]  -0.0059355686   0.0806653228   0.064991268   0.0298834001
## [72,]  -0.0577138898   0.0192497879  -0.038254928  -0.0015849293
```

```
## [73,]  0.0805525957  0.0550936225  0.072183583 -0.0305011626
## [74,]  0.0023527942  0.0117796304 -0.008551379 -0.0097009910
## [75,] -0.0191172648  0.0134352131 -0.004878690  0.1140063109
## [76,] -0.1414039975 -0.0356686275 -0.015306659  0.1153378009
## [77,]  0.0302273693  0.0431198511 -0.059228248 -0.2061059670
## [78,] -0.0904470871 -0.1277841287 -0.033829217 -0.0884829849
## [79,] -0.1039064422 -0.0190325614 -0.003407940 -0.0359454403
## [80,] -0.0215892460 -0.0619953381 -0.194215691 -0.0332443536
## [81,] -0.0340616831 -0.0654044464 -0.186266169 -0.0367810323
## [82,] -0.0376343132 -0.0694643036 -0.175980383 -0.0375670249
## [83,]  0.0210065525  0.0085083789 -0.116680447 -0.0241029027
## [84,] -0.0226981739  0.0049937940 -0.147361562 -0.0324981762
## [85,] -0.0355280843  0.0095135882 -0.147502276 -0.0149467840
## [86,] -0.0313487391 -0.0008326408 -0.147317381 -0.0387078244
## [87,]  0.0120806966 -0.0078505296 -0.154779657  0.0385214539
## [88,] -0.0789668091  0.0820173696 -0.183167354  0.0032597997
## [89,] -0.1625084087  0.1018321292 -0.030026813  0.0607701047
## [90,] -0.0033058282  0.0263454033  0.002860547  0.0342124216
## [91,] -0.0148333549  0.0058674527  0.019225306  0.0128670470
## [92,]  0.0273845912 -0.0272620886  0.101077365 -0.0142772458
## [93,] -0.0905446979 -0.1066418761 -0.109804352  0.0215393395
## [94,]  0.0100006926 -0.0510245583  0.015077447  0.0322109952
## [95,] -0.0227980139 -0.0917199924 -0.066972454  0.0617707825
## [96,] -0.0112413916 -0.1462717791 -0.158021640  0.0480439923
## [97,]  0.0737830165 -0.0058393415 -0.093265053  0.0031636564
## [98,]  0.0059008354  0.1057316866  0.026704910  0.0695571738
## [99,] -0.1084340516  0.0406703691  0.081499724 -0.0154318234
## [100,]  0.0564036547  0.0642143192  0.019648796  0.0609201025
## [101,] -0.0414009412 -0.1441593539  0.144185484  0.0006118322
## [102,] -0.0156406392 -0.2026230238  0.102900550  0.0989939674
## [103,]  0.0158003677 -0.2779090416  0.093227658  0.1624590741
## [104,]  0.0053953156  0.1058589499  0.026686483  0.0695077441
## [105,] -0.1493509670  0.2145151745 -0.207766070  0.1582131936
## [106,]  0.2031374077  0.0311153555 -0.065635100  0.0761401167
## [107,]  0.0550114328  0.2026030999 -0.112745290  0.1182074023
## [108,] -0.2442516521 -0.0953832293  0.070965695 -0.0466598786
## [109,] -0.1611542874  0.1037051647  0.253102757  0.0035225611
## [110,] -0.1712427075  0.1046887884  0.014724515  0.0501204021
## [111,]  0.0789920769 -0.0714064315 -0.062541892 -0.0545776271
## [112,]  0.3895726909  0.3914261714 -0.058493811 -0.3664601032
## [113,]  0.3406165411 -0.0740382506 -0.319577536 -0.0012882795
## [114,] -0.1292320782  0.0270919486 -0.069738640 -0.0823017571
## [115,]  0.1049182584  0.0341404074  0.146241419  0.1400579114
## [116,]  0.0453254005  0.0857532219  0.168707739  0.1738918035
## [117,] -0.0201551402 -0.0425589411  0.043508080 -0.1031887607
## [118,]  0.0265836598 -0.0162039768 -0.057515120  0.0116600248
## [119,]  0.1834727524 -0.1517166801 -0.017913507 -0.0559871027
## [120,]  0.0175356754 -0.0830375810  0.005330511  0.0380596771
## [121,]  0.0453760450 -0.0376710333  0.010273746 -0.0059764371
## [122,] -0.0156774940  0.0828149725  0.111045552  0.0495818524
```

Now let's find the specific variances:

```
# Since the variance of each variable is one (as we are using the correlation matrix) we have the speci
1-fit$h2
```

```
##    [1]  0.024123830  0.072151132  0.079991967  0.077286881  0.355450393
##    [6]  0.130637683  0.105444336  0.046465359  0.045857350  0.042121224
##   [11]  0.027137762  0.259226695  0.024224360  0.043165679  0.492227893
##   [16]  0.096699512  0.047886028  0.143159105  0.328460136  0.033102817
##   [21]  0.039011373  0.059579629  0.415442500  0.876630542  0.654064774
##   [26]  0.729350779  0.496608964  0.046121629  0.068572132  0.122316053
##   [31]  0.052583682  0.066287001  0.204999313  0.071795458  0.399908103
##   [36]  0.224395459  0.112642796  0.061865983  0.094370131  0.105633104
##   [41]  0.062130049  0.038961177  0.030534009  0.029526997  0.027579632
##   [46]  0.120898958  0.154639372  0.169965020  0.109458203  0.121575036
##   [51]  0.107713095  0.262130572  0.185184606  0.056008444  0.054359921
##   [56]  0.128661279  0.070127658  0.054675459  0.044806013  0.042735235
##   [61]  0.069761868  0.064250215  0.101809354  0.086231894  0.010888925
##   [66]  0.084606960  0.141513449  0.078195511  0.051911336  0.131918507
##   [71]  0.459919012  0.109061540  0.351032021  0.082043946  0.475508124
##   [76]  0.471858954  0.238569523  0.147768973  0.533437118  0.050646130
##   [81]  0.051554188  0.067134535  0.089482068  0.038026921  0.029330129
##   [86]  0.043035244  0.433286228  0.268325499  0.437627313  0.242548886
##   [91]  0.489614878  0.043894532  0.283089890  0.095413333  0.162906830
##   [96]  0.331575853  0.016001556  0.001486096  0.044037764  0.058723709
##  [101]  0.184456634  0.068953593  0.185932787  0.001351661  0.272015913
##  [106]  0.201135241  0.275668362  0.195787813  0.537577123 -0.007919846
##  [111]  0.191678851  0.058182313  0.556630765  0.407982125  0.292221548
##  [116]  0.362551155  0.257713782  0.093201320  0.642501842  0.834984953
##  [121]  0.701146076  0.055964140
```

Let's now find and print the varimax rotation of the loadings which will be more easily interpretable:

```
rotated_loadings <- varimax(fit$loadings)$loadings
rotated_loadings
```

```
##
## Loadings:
##        [,1]   [,2]   [,3]   [,4]   [,5]   [,6]   [,7]   [,8]   [,9]   [,10]
##   [1,]                             -0.111 -0.965
##   [2,]          0.401 -0.551 -0.210  0.138         0.122 -0.214         0.133
##   [3,] -0.244                      -0.199        -0.491         0.594
##   [4,]   0.209 -0.420                      0.208         0.508        -0.509
##   [5,]   0.458  0.492               -0.102
##   [6,] -0.176  0.845 -0.165                             -0.116
##   [7,] -0.353        -0.256 -0.721  0.111
##   [8,] -0.289  0.132 -0.347 -0.778                             -0.194
##   [9,] -0.245        -0.115 -0.872                             -0.163
##  [10,] -0.121 -0.104  0.906  0.213                              0.154
##  [11,]                             -0.107 -0.952
##  [12,]   0.304  0.104               -0.172
##  [13,]   0.886        -0.228  0.147
##  [14,]   0.439        -0.788                                    -0.129
##  [15,] -0.104        -0.161               -0.138
##  [16,]   0.729 -0.264  0.162                      -0.212  0.173        -0.156
##  [17,] -0.311         0.835  0.148                             0.242
##  [18,] -0.568  0.326  0.104 -0.106        -0.118        -0.263  0.226  0.154
##  [19,]         -0.207  0.619  0.149
##  [20,]   0.916        -0.140
##  [21,]   0.933 -0.104         0.148               -0.130
```

15

```
## [22,]  0.915              0.136              -0.150              0.122
## [23,]  0.635        -0.117
## [24,]  0.302
## [25,]  0.480              0.134
## [26,]
## [27,]  0.641 -0.130       0.108
## [28,] -0.148  0.102              -0.111 -0.932       -0.101
## [29,] -0.665  0.172  0.131 -0.326       -0.110       -0.249              0.148
## [30,] -0.583  0.481  0.122                     0.222 -0.141  0.263  0.132
## [31,] -0.669  0.389  0.158       -0.103        0.285 -0.132  0.274  0.164
## [32,]  0.752 -0.165       -0.195  0.108       -0.389       -0.158
## [33,] -0.602  0.323  0.116 -0.138              -0.245  0.128  0.103
## [34,]  0.528        -0.625  0.126               0.184       -0.111
## [35,] -0.141        -0.123  0.118         0.430  0.103  0.516
## [36,]  0.151 -0.216  0.229 -0.433  0.105       -0.587 -0.196
## [37,] -0.672  0.135                     0.418        0.411
## [38,]  0.782 -0.201                    -0.408       -0.120
## [39,] -0.363              0.217       -0.180
## [40,]         0.232       -0.777       -0.188 -0.134 -0.144
## [41,] -0.331              0.177       -0.168       -0.104
## [42,] -0.355              0.202       -0.173
## [43,] -0.144  0.536 -0.616                    -0.198  0.168  0.145
## [44,]  0.498        -0.226  0.172        0.197        0.253       -0.231
## [45,]  0.507 -0.130 -0.206  0.149        0.201        0.302       -0.282
## [46,]  0.548        -0.191  0.126        0.183        0.244       -0.256
## [47,]  0.385        -0.192  0.194        0.196        0.217       -0.266
## [48,] -0.127 -0.228
## [49,]        -0.337
## [50,]                          -0.134 -0.878       -0.127        0.125
## [51,] -0.339  0.189       -0.200       -0.274       -0.385        0.405
## [52,]         0.444       -0.126 -0.704
## [53,]         0.107 -0.222                    -0.131
## [54,]         0.168 -0.204                    -0.135
## [55,]         0.214 -0.193                    -0.149  0.126
## [56,]         0.277 -0.111 -0.158       -0.119       -0.169  0.148
## [57,]  0.218  0.844       -0.110       -0.103
## [58,]  0.210  0.870              -0.103
## [59,]  0.226  0.882              -0.109
## [60,]  0.211  0.897              -0.114
## [61,]        -0.934
## [62,]         0.945
## [63,] -0.265  0.723 -0.307              -0.111       -0.238        0.141
## [64,] -0.239  0.705 -0.354                           -0.235        0.137
## [65,]         0.445 -0.610                     0.164 -0.169        0.152
## [66,]         0.386 -0.675        0.101              -0.108  0.166
## [67,] -0.256  0.559 -0.386                     0.260 -0.193 -0.104  0.240
## [68,]  0.357 -0.333        0.370        0.159              0.152
## [69,] -0.261  0.852 -0.183                           -0.183        0.101
## [70,] -0.325  0.392  0.323 -0.182       -0.149       -0.177 -0.172
## [71,]  0.235 -0.329 -0.236  0.207        0.134        0.147  0.135
## [72,]                          -0.103 -0.889       -0.165
## [73,]  0.278        -0.265 -0.130        0.123        0.438
## [74,]  0.314 -0.280        0.400        0.171        0.108
## [75,] -0.347                    -0.311       -0.288  0.189  0.148
```

16

```
## [76,] -0.275 -0.229  0.131                               -0.126  0.240
## [77,]               -0.336  0.149                   0.248       -0.610
## [78,] -0.668                                              -0.178        0.247
## [79,] -0.375  0.321                                        -0.218  0.135
## [80,]  0.906  0.288
## [81,]  0.909  0.289
## [82,]  0.908  0.272
## [83,]  0.853  0.253 -0.119                           0.135       -0.130
## [84,]  0.904  0.244                                  0.145
## [85,]  0.904  0.246                                  0.125
## [86,]  0.895  0.206                                  0.169
## [87,]         0.309  0.237 -0.359                           -0.214 -0.202
## [88,]  0.460  0.441              0.107               0.289 -0.137 -0.278
## [89,]                                                        0.165
## [90,]                            -0.126 -0.838
## [91,]         0.163             -0.122 -0.673
## [92,]  0.305  0.908
## [93,] -0.336 -0.356                                         0.654
## [94,]  0.132        0.223  0.251 -0.108                      0.698
## [95,]               0.170  0.324                             0.762
## [96,] -0.134              0.121                              0.731
## [97,] -0.144                    -0.909 -0.237
## [98,] -0.227 -0.110              -0.254  0.137
## [99,]  0.130                     0.912  0.211
## [100,] -0.184 -0.117            -0.345  0.136
## [101,]                          -0.674 -0.335
## [102,] -0.142                   -0.280
## [103,]                                 -0.210
## [104,] -0.227 -0.109            -0.254  0.137
## [105,]        -0.132                                               -0.807
## [106,] -0.158 -0.321                        0.111        0.777
## [107,]                                -0.182      -0.701  0.108  0.108
## [108,]         0.574                               -0.442 -0.116 -0.277
## [109,]  0.108  0.244                                       -0.156 -0.124
## [110,]         0.360                  -0.165        -0.888        -0.157
## [111,]                          -0.809 -0.255
## [112,]                                -0.118                      -0.127
## [113,]         0.214  0.190                                0.189
## [114,]        -0.116                 -0.709  0.115        -0.142
## [115,]  0.115  0.576        -0.115   -0.137 -0.184        0.178
## [116,]  0.349  0.265                -0.256 -0.304         0.146
## [117,] -0.195              0.106   -0.566 -0.309                    0.140
## [118,]                          -0.843 -0.262
## [119,]  0.201                                      -0.105  0.271  0.120
## [120,]                           0.163
## [121,]         0.115                  -0.411
## [122,] -0.145                   -0.238  0.130
##         [,11]  [,12]  [,13]  [,14]  [,15]  [,16]  [,17]  [,18]  [,19]
##   [1,]
##   [2,]                0.177 -0.114                        -0.156 -0.481
##   [3,]         0.106               0.223          0.124         0.345
##   [4,]                                           -0.136        -0.313
##   [5,]  0.171                     -0.268          0.167 -0.114
##   [6,] -0.154        0.151                               0.105
```

```
##    [7,] -0.114                    -0.115      0.213      -0.228
##    [8,]                                       0.269
##    [9,]                                       0.281
##   [10,]                            0.107     -0.107
##   [11,]                                 0.173
##   [12,]   0.131                          0.752
##   [13,]                                 0.112      -0.298
##   [14,]                  -0.194         0.154      -0.186
##   [15,] -0.505                     -0.140 -0.275  0.110  0.115 -0.197
##   [16,]                                            -0.377
##   [17,]                            0.143 -0.110 -0.143
##   [18,]                 0.298          -0.139          0.343
##   [19,]   0.174                          -0.264 -0.257 -0.121
##   [20,]                                            -0.268
##   [21,]                                      0.117 -0.106
##   [22,]                                      0.114
##   [23,]   0.129                          0.209      0.220 -0.121
##   [24,]          0.101
##   [25,]                 0.113          -0.139 -0.139
##   [26,]          0.452      -0.210
##   [27,]                                 0.102          -0.106
##   [28,]
##   [29,] -0.166          0.253          -0.154  0.186  0.112  0.258
##   [30,] -0.139                     0.125 -0.154        0.223  0.105
##   [31,]                 0.102      0.180 -0.110        0.126  0.181
##   [32,]                                      0.212  0.167 -0.166
##   [33,] -0.110          0.306      0.147                    0.222
##   [34,]                 -0.364         0.189
##   [35,]   0.204                                 0.101
##   [36,]                                      0.179      -0.130
##   [37,]   0.125                          -0.123          0.166
##   [38,]                                      0.118  0.163 -0.179
##   [39,]                      -0.112                       0.801
##   [40,]   0.101                     0.197      0.242      0.205
##   [41,]                            -0.163                 0.829
##   [42,]                      -0.111 -0.126                0.840
##   [43,] -0.118          0.194                      -0.198 -0.271
##   [44,]                            -0.112      -0.113      -0.659
##   [45,]                            -0.116                 -0.612
##   [46,]                            -0.102                 -0.562
##   [47,]          -0.118            -0.136      -0.115      -0.638
##   [48,]                 -0.857
##   [49,]                 -0.873
##   [50,]                                                  0.136
##   [51,]          0.100              0.245      0.122      0.461
##   [52,]
##   [53,]                                       0.837
##   [54,]                                       0.907      0.111
##   [55,]                                       0.879      0.152
##   [56,]                                       0.798      0.186
##   [57,]   0.139                          0.320
##   [58,]   0.123                          0.289
##   [59,]   0.119                          0.255
##   [60,]   0.111                          0.221
```

```
##  [61,]                    -0.103                              -0.124
##  [62,]
##  [63,] -0.117         0.186                              -0.221
##  [64,] -0.125         0.201                              -0.227 -0.149
##  [65,] -0.106         0.198                       -0.133 -0.210 -0.435
##  [66,]                0.119                  0.121 -0.128 -0.155 -0.422
##  [67,] -0.126         0.208 -0.149                        -0.196 -0.136
##  [68,] -0.140                                    -0.246         -0.627
##  [69,]                0.111       -0.115                            0.123
##  [70,]                                  0.111        0.142  0.122  0.561
##  [71,]                                            -0.149 -0.111 -0.409
##  [72,]                                                          0.138
##  [73,]   0.257              0.183 -0.270  0.121        -0.122 -0.221
##  [74,] -0.169              -0.133               -0.244         -0.631
##  [75,]                0.195       0.237                            0.210
##  [76,] -0.164         0.163 -0.140  0.408 -0.132 -0.114         -0.145
##  [77,] -0.197                    -0.321                         -0.155
##  [78,] -0.224         0.176             -0.236        0.214  0.313
##  [79,]                0.117       0.173 -0.189        0.138  0.140
##  [80,]
##  [81,]
##  [82,]
##  [83,]                                  0.165        -0.128
##  [84,]                                  0.125        -0.114
##  [85,]                            0.100  0.122        -0.111 -0.106
##  [86,]                                  0.130        -0.116 -0.116
##  [87,]                0.176 -0.260  0.112               -0.199  0.126
##  [88,]                      -0.140  0.246               -0.148
##  [89,]                             0.712
##  [90,]                                                          0.120
##  [91,]                                                          0.101
##  [92,]   0.106                           0.102
##  [93,]                                                        -0.109
##  [94,]                            0.255        -0.283         -0.302
##  [95,]                            0.104  0.118 -0.255
##  [96,] -0.147                                  -0.212
##  [97,]                                  -0.240
##  [98,]                                  -0.912
##  [99,]                                   0.209
## [100,]                                  -0.849
## [101,]   0.203  0.100       -0.390        -0.124
## [102,]         0.133       -0.483        -0.721                0.147
## [103,]         0.175       -0.822         0.117                0.123
## [104,]                                  -0.912
## [105,]         0.122                            -0.119
## [106,]                                                  -0.144
## [107,]                -0.119        0.153        -0.277  0.194
## [108,]                0.162       -0.115               0.347
## [109,]   0.498                    -0.189        0.117         0.161
## [110,]                                                          0.110
## [111,]                            -0.227
## [112,] -0.121  0.916               -0.110                      0.138
## [113,] -0.390              0.146 -0.232        -0.222  0.173
## [114,]
```

```
## [115,]   0.263                         0.215  0.191          -0.107  0.277
## [116,]   0.203                         0.398  0.144                  0.131
## [117,]                                       -0.470                  0.117
## [118,]                                       -0.307
## [119,] -0.106  0.175          -0.121          0.357
## [120,] -0.127          -0.103         -0.198          0.165
## [121,]                                        0.172          0.209
## [122,]                                       -0.911
##
##                  [,1]    [,2]  [,3]  [,4]  [,5]  [,6]  [,7]  [,8]  [,9] [,10]
## SS loadings     19.753 14.022 6.268 4.464 4.660 8.311 1.964 4.183 4.351 2.543
## Proportion Var   0.162  0.115 0.051 0.037 0.038 0.068 0.016 0.034 0.036 0.021
## Cumulative Var   0.162  0.277 0.328 0.365 0.403 0.471 0.487 0.522 0.557 0.578
##                 [,11] [,12] [,13] [,14] [,15] [,16] [,17] [,18] [,19]
## SS loadings     1.718 1.336 2.577 1.619 2.129 5.828 4.484 1.286 7.917
## Proportion Var  0.014 0.011 0.021 0.013 0.017 0.048 0.037 0.011 0.065
## Cumulative Var  0.592 0.603 0.624 0.637 0.655 0.703 0.739 0.750 0.815
```

The rotated matrix is more easily interpretable as it "encourages" large (in absolute) values and many near-zero values, this makes it more interpretable as it reduces the number of factors a given feature relies on. In the output above small values aren't printed, hence the empty values. Let's try and interpret some of the factors, starting with factor 1 we can see that multiple features have large values for this factor, let's find the features with the largest absolute values:

```
colnames(crimeData)[which(abs(rotated_loadings[,1]) > 0.9)]
```

```
## [1] "x.V25" "x.V26" "x.V27" "x.V85" "x.V86" "x.V87" "x.V89" "x.V90"
```

These attributes correspond to the median family income, per capita income, per capita income for caucasians, owner occupied housing lower quartile, median and upper quartile, median rent for rental housing and upper quartile rent for medium housing. This indicates that the first factor is some weighted average of most of the factors with a large emphasis on income and accommodation costs, perhaps it is trying to model the general economic state of the community.

```
colnames(crimeData)[which(abs(rotated_loadings[,2]) > 0.9)]
```

```
## [1] "x.V66" "x.V67" "x.V97"
```

Factor 2 has a particularly high weighting for the following features: percentage of kids born never to be married, total number of people known to be foreign born, percent of people foreign born. And so perhaps has more of a focus on demographics as opposed to economic data as in factor 1.

```
colnames(crimeData)[which(abs(rotated_loadings[,3]) > 0.8)]
```

```
## [1] "x.V15" "x.V22"
```

Factor 3 has particularly high weightings for the percentage of population that is 65 and over in age and percentage of households with social security income in 1989. This factor could perhaps be modeling something such as the level of vulnerability of the community.

Let's now let k=2 and compare our two-dimensional reduction of the data to our PCA in the previous portfolio on this same dataset:

```
fit2 <- fapa(R, numFactors = 2)
plot(fit2$loadings,type="n") # set up plot
text(fit2$loadings,labels=names(crimeData),cex=.7)
```
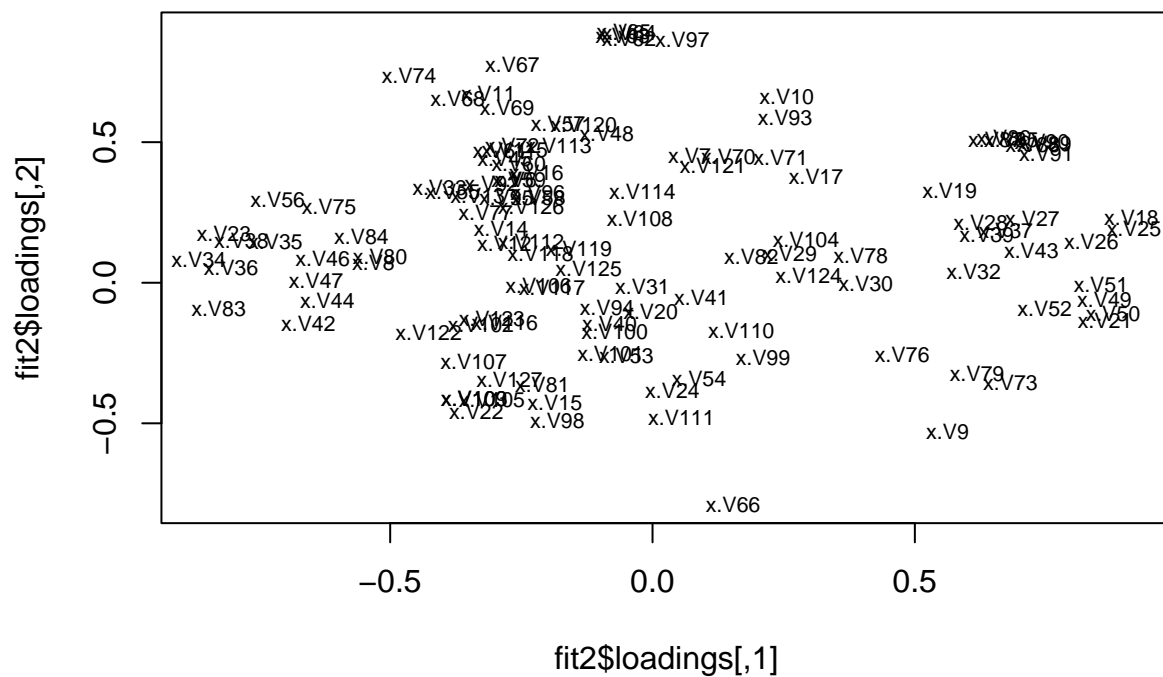
```
pc_crimeData <- prcomp(~ . -y , crimeData, scale. = TRUE, retx=TRUE)
autoplot(pc_crimeData, data=crimeData[,-ncol(crimeData)], loadings = TRUE, alpha = 0, loadings.colour =
        loadings.label.colour='black', loadings.label = TRUE, loadings.label.size = 4,
        loadings.label.repel=TRUE)
```
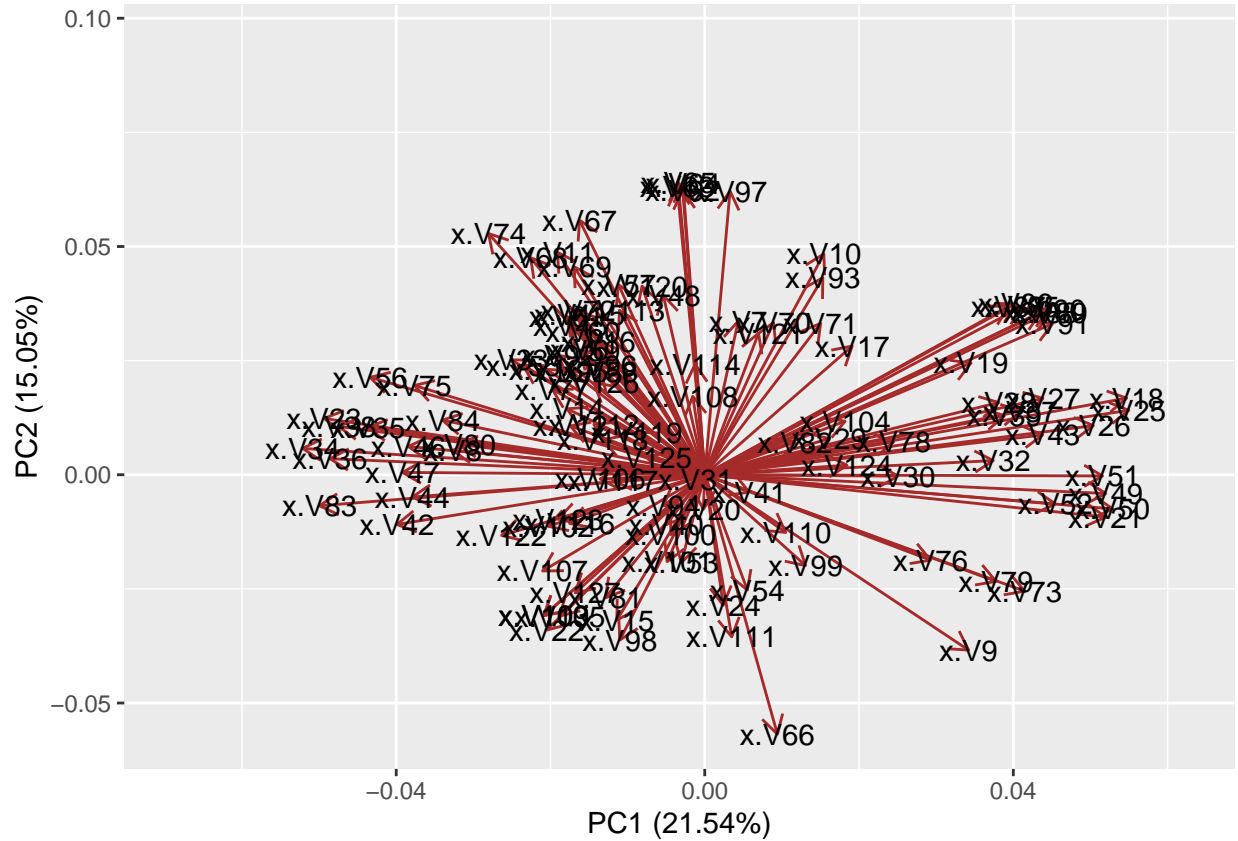
Comparing the two plots we see that the contributions of the features to the first two principal components matches up well with the contributions of the features to the factors, this means that the results of the PCA and FA are very similar.

In this scenario it may make more sense to use FA as opposed to PCA as we don't simply want a dimensionality reduction of the dataset but we are trying to create some interpretation of the dataset. By using FA we infer latent variables which can be thought of as "abstract concepts" inferred from the data. For this dataset for example we would like to infer what concepts such as a communities finances and demographics cause correlation in our data.

Task 2