Comparing Nonlinear Least Squares & Closed Form Solutions of the 3D Conformal Coordinate Transformation

Andrew Lassiter
Assistant professor, University of Florida

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GEO WEEK

SCHOOL OF FOREST, FISHERIES, AND GEOMATICS SCIENCES

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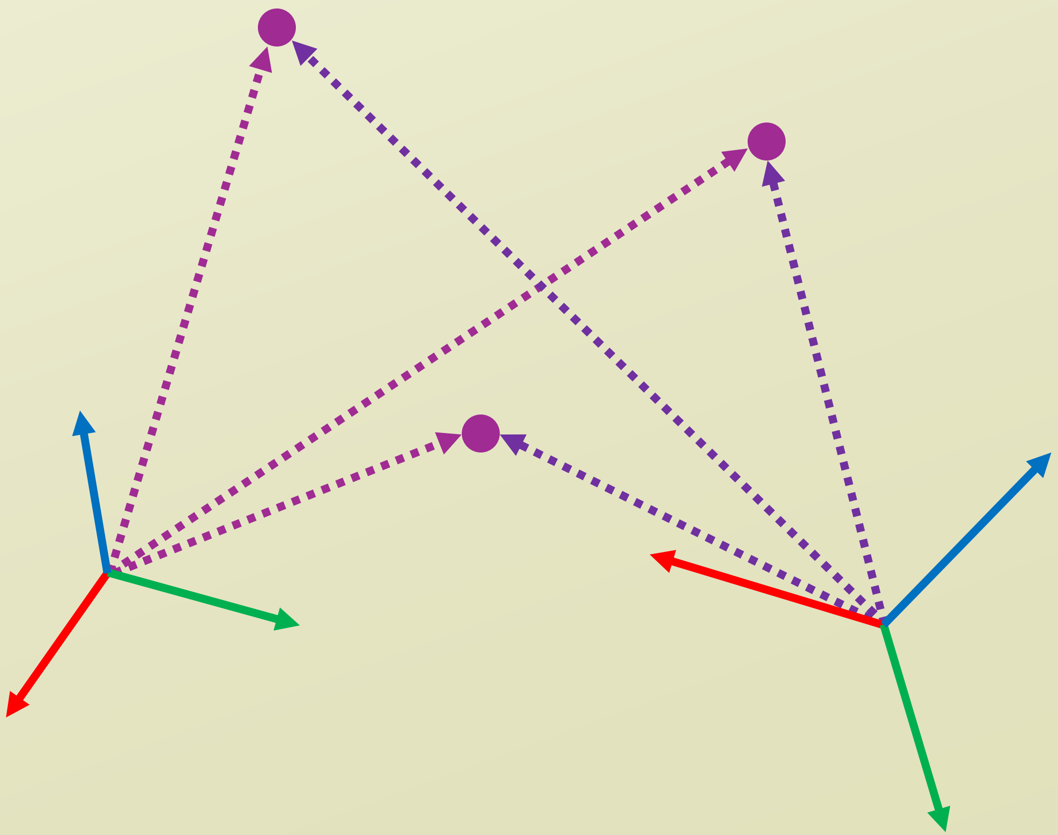
3D conformal coordinate transformation

- Also referred to as **absolute orientation** or the **seven-parameter transform**
  - 3 rotations, 3 translations, 1 scale
- Means of expressing the coordinates of a set of 3D points known in one coordinate system relative to another coordinate system
  - **Arbitrary** and **control** CS
- Applications in photogrammetry, lidar, robotics, geodesy



3D conformal transform in photogrammetry

Wolf, Dewitt, Wilkinson "Elements of Photogrammetry"

$$\mathbf{x}_c = sR^T\mathbf{x}_a + \mathbf{T}$$

- Elements of R contain trigonometric functions ∴ nonlinear least squares solution
  - Cannot treat elements of R as coefficients without unwieldy constraints to maintain orthogonality
- Nonlinear LS requires initial approximations
- Dewitt (1996; included in EOP text) offers a five-step method
  - Requires user selection of three points and use of azimuth-tilt-swing rotation system

3D conformal transform in CV and robotics

Horn BK "Closed-form solution of absolute orientation using unit quaternions"

- Refer to manuscript for derivation
- **Rotation** is solved as a unit quaternion (no trig)
- **Scale** is ratio of sum of vector norms of the control vs. arbitrary systems (each translated to respective origins)
- **Translation** is difference between centroid of the control system and the scaled and rotated centroid of arbitrary system
- **Closed-form**: Solved without iteration
  - Will still handle noisy data

Comparison of methods

Nonlinear Least Squares

- ✅ Can be taught to geomatics undergraduates via LSA
- ✅ Error propagation
- ✅ Outlier detection
- ❌ Initial approximations cumbersome
- ❌ Sensitive to local minima

Horn closed-form solution

- ✅ Faster
- ✅ Closed-form = more stable
- ✅ No initial approximations
- ❌ No error propagation
- ❌ No outlier detection
- ❌ Barrier to teaching at UG level

Easier initial approximations for NLS

$$\mathbf{x}_c = sR^T\mathbf{x}_a + \mathbf{T}$$

$$X = sr_{11}x + sr_{21}y + sr_{31}z + T_X = ax + by + cz + T_X$$

$$Y = sr_{12}x + sr_{22}y + sr_{32}z + T_X = dx + ey + fz + T_Y$$

$$Z = sr_{13}x + sr_{23}y + sr_{33}z + T_X = gx + hy + iz + T_Z$$

- Treat elements $a - i$ as coefficients and solve via linear LS
  - The result is not conformal because orthogonality not enforced
  - Still good enough for generating initial approximations?
- **Scale**: length of any column vector, e.g. $s' = ||\langle sr_{11}, sr_{12}, sr_{13}\rangle||$
- **Rotation**: ω', ϕ', κ' derived from R'^T

Test 1: DLT method

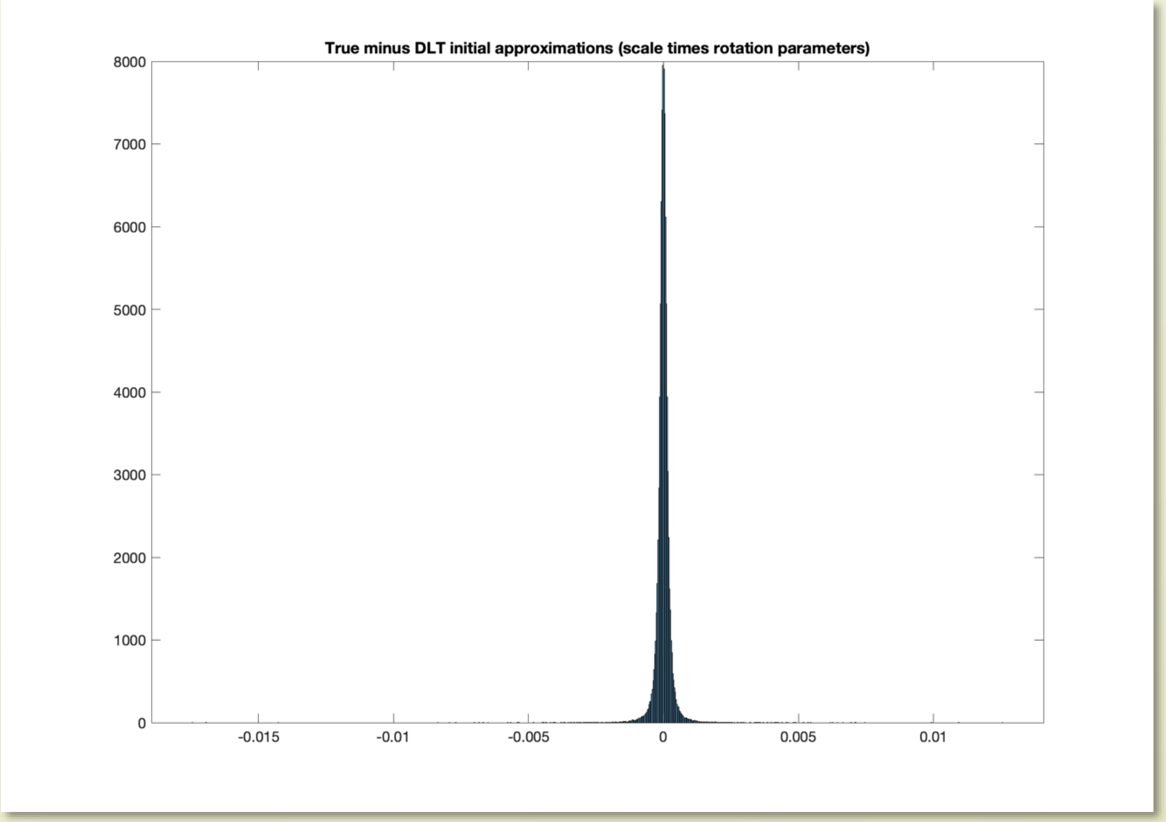
Objective

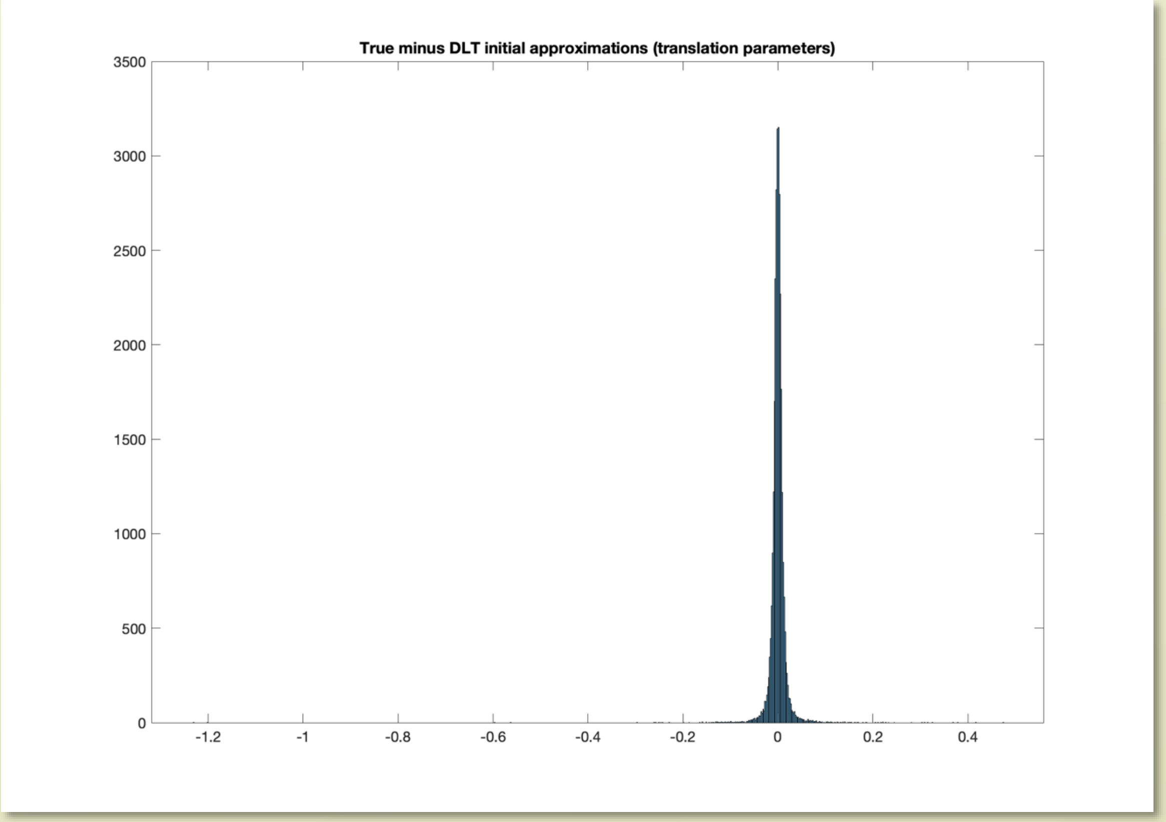
Test efficacy of direct linear transformation method of initial approximations

Method

1. Generate random set of “true” arbitrary points (n = 5)
2. Generate random “true” transformation parameters
3. Transform true arbitrary to generate true control
4. Add noise to arbitrary points
5. Find initial approximations via DLT using noisy arbitrary and true control
6. Compare true transformation matrix to DLT initial transformation matrix

DLT approximation results





Monte Carlo (n = 10 000)

Arbitrary, control coords on order of 10s, 100s respectively; random translations range (-1000, 1000)

No incidences of “failed” approximations

Test 2: Accuracy of NLS vs. Horn methods

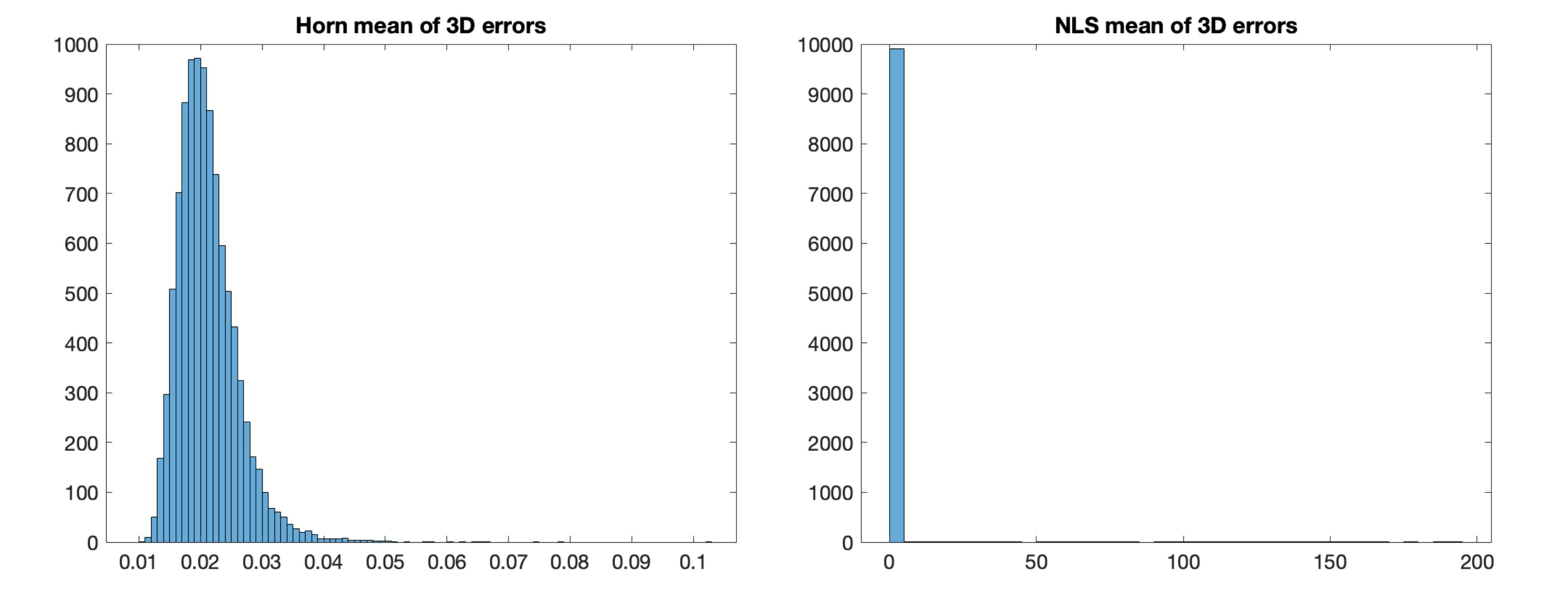
Objective

Compare accuracy of each 3D conformal method

Method

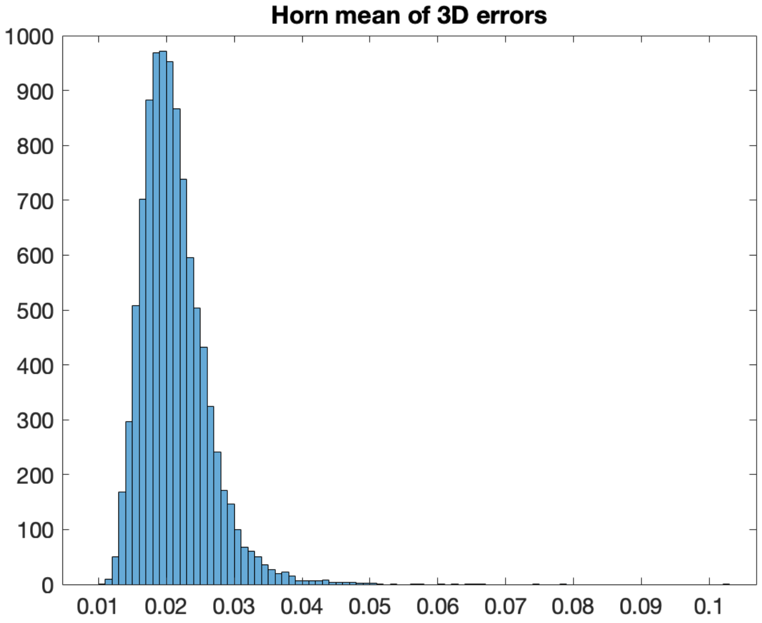
1. Generate random set of "true" arbitrary points (n = 25)
2. Generate random "true" transformation parameters
3. Transform true arbitrary to generate true control
4. Add noise to arbitrary points
5. Test NLS and Horn methods on solving between noisy arbitrary and true control (n = 5 for transformation, n = 20 check)
6. Test accuracy by comparing checkpoints to true control values (mean of norms)

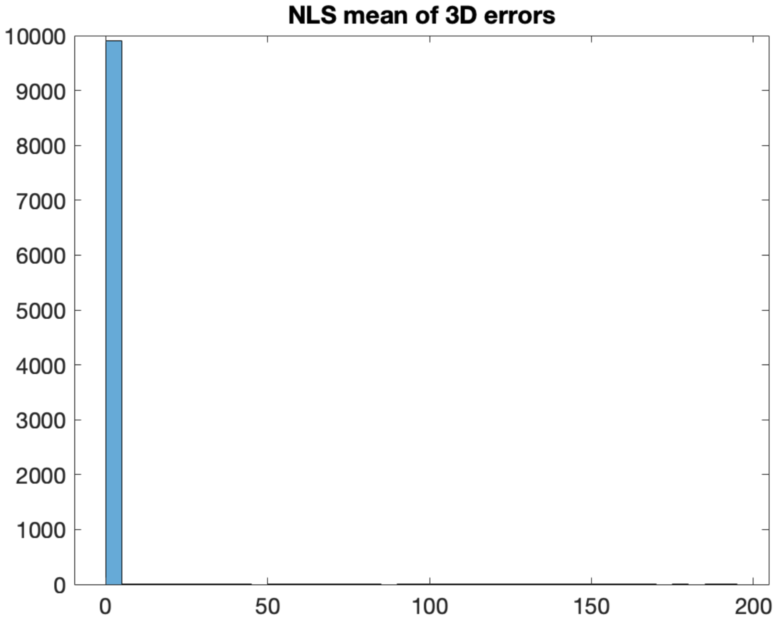
Accuracy comparison results

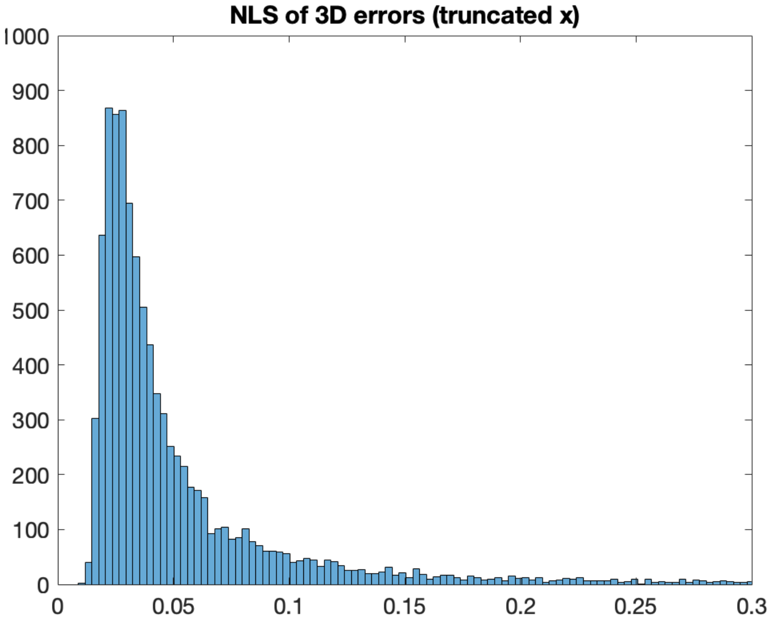


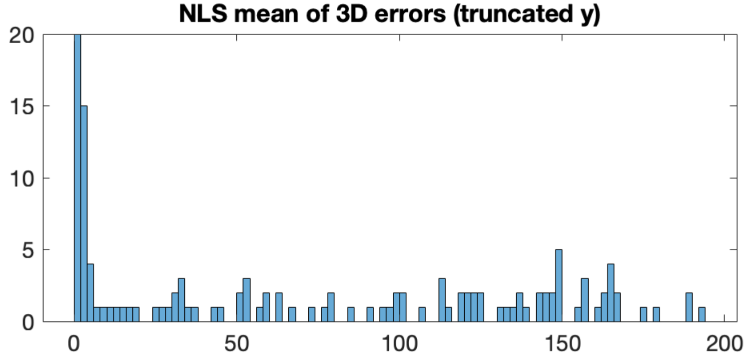
Monte Carlo (n = 10 000)
Large failures: 84/10000 gimbal lock
Other lesser failures in NLS

Accuracy comparison results









Monte Carlo (n = 10 000)
Large failures: 84/10000 gimbal lock
Other lesser failures in NLS

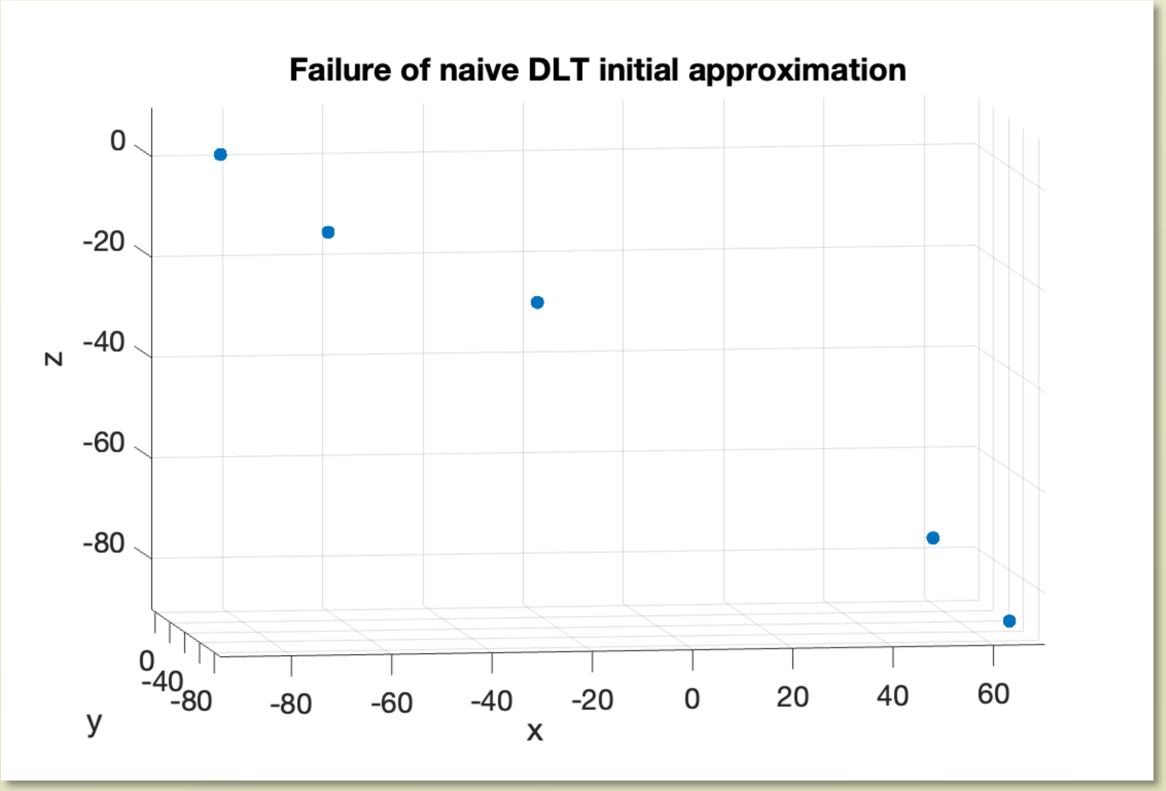
Failure in NLS

"Small" failures

- Nearly coplanar points chosen for finding initial approximations; mean of norms 5-10x larger than typical

"Large" failures

- Gimbal lock in initial approximations
- Despite method for handling gimbal lock (setting one of three rotations and solving for other two), NLS transformation yields mean of norms 100-1000x larger than typical
  - Could be my failure? **(Edit: It was! These large failures are gone.)**



Best algorithm?

- Some mixture of DLT, NLS, Horn methods should yield the optimal solution for performing the transformation
- Something will be lost

Example

1. **DLT**: Opportunity for outlier detection/data snooping
2. **Horn**: Accurate and efficient transformation

Disadvantage: No classical *a posteriori* error propagation on elements of transformation

(Enthralling stuff, right?)

Questions and comments!

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