Some review about the dot product:

if $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$

the dot product of $\vec{v}$ and $\vec{w}$ is $\vec{v} \cdot \vec{w} = v_1 w_1 + \ldots + v_n w_n$

the length/norm of $\vec{v}$ is $\|\vec{v_1}\| = \sqrt{\vec{v_1} \cdot \vec{v_1}}$

the angle θ between $\vec{v}$ and $\vec{w}$ is $\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\ }$

$\vec{v}$ and $\vec{w}$ are perpendicular/orthogonal $\iff \vec{v} \cdot \vec{w} = \vec{0}$

Vectors $(\vec{u_1}, \ldots, \vec{u_m})$ are orthonormal if

$\vec{u_i} \cdot \vec{u_j} = 1$ if $i = j$

$\vec{u_i} \cdot \vec{u_j} = 0$ if $i \neq j$

This is saying every vector is of unit length, and is orthogonal to every other vector in the set

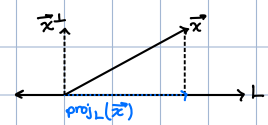
An orthonormal basis of $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$ where the vectors are orthonormal

If we have an orthonormal basis of a subspace V, we can leverage this to project vectors onto V efficiently without needing to solve for the matrix of the transformation!

Let V be a subspace of $\mathbb{R}^n$, $(\vec{u_1}, \ldots, \vec{u_m})$ be an orthonormal basis of V, and $\vec{x}$ be a vector in $\mathbb{R}^n$ we want to project onto V.

$\text{proj}_V(\vec{x}) = c_1\vec{u_1} + \ldots + c_m\vec{u_m}$ since $(\vec{u_1}, \ldots, \vec{u_m})$ is an orthonormal basis

$\vec{x} = (c_1\vec{u_1} + \ldots + c_m\vec{u_m}) + \vec{x}^{\perp}$ since $\vec{x} = \text{proj}_V(\vec{x}) + \vec{x}^{\perp}$ * visually here it is in $\mathbb{R}^2$, and you can generalize to $\mathbb{R}^n$ *



We can actually solve for $(c_1, \ldots, c_m)$ using the dot product

$\vec{x} \cdot \vec{u_k} = ((c_1\vec{u_1} + \ldots + c_m\vec{u_m}) + \vec{x}^{\perp}) \cdot \vec{u_k}$

$\vec{x} \cdot \vec{u_k} = (c_1\vec{u_1} + \ldots + c_m\vec{u_m}) \cdot \vec{u_k} + \vec{x}^{\perp} \cdot \vec{u_k}$

$\vec{x} \cdot \vec{u_k} = c_1\vec{u_1} \cdot \vec{u_k} + \ldots + c_k\vec{u_k} \cdot \vec{u_k} + \ldots + c_m\vec{u_m} \cdot \vec{u_k} + \vec{x}^{\perp} \cdot u_k$

Since $\vec{u_k}$ is orthogonal to all vectors except $\vec{u_k}$

$\vec{x} \cdot \vec{u_k} = c_k$

Now we have our formula for a projection onto V that is easy to compute!

$\text{proj}_V(\vec{x}) = (\vec{x} \cdot \vec{u_1})\vec{u_1} + \ldots + (\vec{x} \cdot \vec{u_m})\vec{u_m}$

However, this actually isn't as easy at it seems. This only works out easily if we have an orthonormal basis of V.

How do we actually find an orthonormal basis of V? Next class we'll learn the method.