The Gram-Schmidt Process is a way to turn a basis of a subspace into an orthonormal basis of the subspace.

This process makes it possible for us to use the handy projection formula we learned before.

Given a subspace V of $\mathbb{R}^n$ and a basis $(\vec{v_1}, \ldots, \vec{v_m})$ of V, Find an orthonormal basis of V

1) Find $\vec{u_1}$ so that $(\vec{u_1})$ is an orthormal basis of $\text{Span}(\vec{v_1})$

$$\vec{u_1} = \frac{1}{\|\vec{v_1}\|}\vec{v_1}$$

2) Find $\vec{u_2}$ so that $(\vec{u_1}, \vec{u_2})$ is an orthormal basis of $\text{Span}(\vec{v_1}, \vec{v_2})$

$$\vec{v_2}^{*} = \vec{v_2} - \text{proj}_{\text{span }\vec{v_1}}(\vec{v_2})$$ Since $(\vec{u_1})$ is an orthonormal basis of $\text{span}(\vec{v_1})$

$$\vec{v_2}^{*} = \vec{v_2} - (\vec{v_2}\cdot\vec{u_1})\vec{u_1}$$

$$\vec{u_2} = \frac{1}{\|\vec{v_2}^{*}\|}\vec{v_2}^{*}$$

3) Repeat the process to Find $\vec{u_3}$ up to $\vec{u_m}$.

$$\vec{v_3}^{*} = \vec{v_3} - (\vec{v_3}\cdot\vec{u_1})\vec{u_1} - (\vec{v_3}\cdot\vec{u_2})\vec{u_1}$$

$$\vec{u_3} = \frac{1}{\|\vec{v_3}^{*}\|}\vec{v_3}^{*}$$

$$\vec{v_m}^{*} = \vec{v_m} - (\vec{v_m}\cdot\vec{u_1})\vec{u_1} - \ldots - (\vec{v_m}\cdot\vec{u_{m-1}})\vec{u_{m-1}}$$

$$\vec{u_m} = \frac{1}{\|\vec{v_m}^{*}\|}\vec{v_m}^{*}$$

Now you have your orthornomal basis $(\vec{u_1}, \ldots, \vec{u_m})$

Switching gears a bit, if we have V, a subspace of $\mathbb{R}^n$, then $V^{\perp}$ is all vectors in $\mathbb{R}^n$ orthogonal to V.

We call this the orthogonal complement. If we have a projection onto V, then we get some insights into V and $V^{\perp}$.

$\dim(\text{im proj}_V) + \dim(\text{Ker proj}_V) = \dim(V) + \dim(V^{\perp}) = n$. So the dimensions of V and $V^{\perp}$ must sum to n.

Finally, if we have matrix A, the transpose of A, called A^T, is the matrix where the i-th column of A is the i-th row of A^T

Similar to inverses, $(AB)^T = B^T A^T$.

Example: $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ the columns and rows "switched".

We can see a really interesting relation between the orthogonal complement and the transpose.

$(\text{Im } A)^{\perp} = \text{Ker}(A^T)$