A Discrete Dynamical System (DDS) models some system so that you can express it as $\vec{x}(t) = A^t \vec{x}(0)$

Where $\vec{x}(0)$ is your inital state. However this isn't a closed form solution. At $t=1000$, it would be difficult to figure out what happens.

The point is to figure out what happens in the system in the long term given some initial state, in a closed form.

If A is a matrix, and $\vec{v}$ is an eigenvector * of A with eigenvalue λ, then $A\vec{v} = \lambda\vec{v}$.

* $\vec{0}$ cannot be an eigenvector

An eigenbasis of $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$ made up of eigenvectors

Using our knowledge, here is how we can solve a DDS.

1) Find our eigenvectors * we don't know how to do this yet, but they are given to you for now

2) Find the associated eigenvalues for each eigenvector $(\lambda_1, \dots, \lambda_n)$

3) Express your initial state $\vec{x}(0)$ as a linear combination of your eigenvectors

$\vec{x}(0) = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$ we can do this since we have a basis

$\vec{x}(t) = A^t(c_1\vec{v}_1 + \dots + c_n\vec{v}_n)$ since $\vec{x}(t) = A^t\vec{x}(0)$

$\vec{x}(t) = c_1 A^t\vec{v}_1 + \dots + c_n A^t \vec{v}_n$

$\vec{x}(t) = c_1(\lambda_1)^t\vec{v}_1 + \dots + c_n(\lambda_n)^t\vec{v}_n$ since $\vec{v}_k$ is an eigenvector with eigenvalue λ_k and $A\vec{v}_k = \lambda_k\vec{v}_k$, so $A^t\vec{v}_k = \lambda_k \vec{v}_k$

We now have a closed form general solution.

4) Do analysis on the system. Ask yourself these questions:

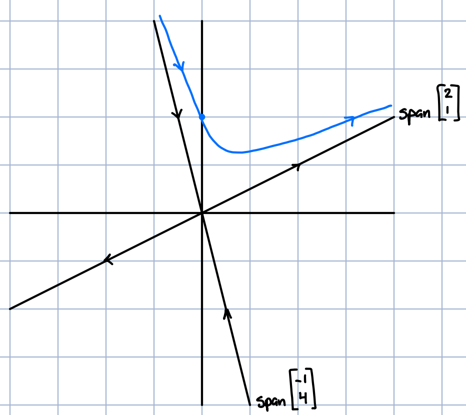
What happens along the span of the eigenvectors as $t \to \infty$?

What happens if your initial state is not on the span of the eigenvectors? Which span does it get closer to?

Example: Suppose you have a system where your eigenvectors and eigenvalues were

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with $\lambda = 1.2$ and $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ with $\lambda = 0.8$, and your initial condition $\vec{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Draw the solution trajectories.



$\vec{x}(t) = c_1(1.2)^t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2(0.8)^t\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ general solution

solving for c_1, c_2 gives

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 4 & 2 \end{array}\right] \to \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$

$\vec{x}(t) = \frac{1}{3}(1.2)^t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{3}(0.8)^t\begin{bmatrix} -1 \\ 4 \end{bmatrix}$

as $t \to \infty$, $\vec{x}(t)$ will be close to span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

as $t \to -\infty$, $\vec{x}(t)$ will be close to span of $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$

along the span of the eigenvectors, as $t \to \infty$

if $\vec{x}(0)$ is a multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{x}(t) = c(1.2)^t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and the trajectory goes away from the origin

if $\vec{x}(0)$ is a multiple of $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $\vec{x}(t) = c(0.8)^t\begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and the trajectory goes in towards the origin

With this info, we can make our sketch