A linear transformation is a function T from $\mathbb{R}^m \to \mathbb{R}^n$ that

1) preserves addition $T(\vec{x}) + T(\vec{y}) = T(\vec{x}+\vec{y})$ for all $\vec{x}, \vec{y}$ in $\mathbb{R}^m$

2) preserves scalar multiplication $T(k\vec{x}) = kT(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^m$ and all scalars k

$\quad \hookrightarrow T(\vec{0}) = \vec{0}$ note that this is just the case when $k=0$, but is a good first step to check

If we have $T: \mathbb{R}^4 \to \mathbb{R}^3$, then the domain (the world our inputs live in) is $\mathbb{R}^4$. The codomain is $\mathbb{R}^3$ (the world our outputs live in).

So you can expect inputs to have 4 entries and outputs to have 3 entries.

Some common examples of linear transformations are scaling and rotations. We'll learn more examples in a few classes.

To prove something is a linear transformation, you must show that the 2 properties are BOTH upheld. You must use general vectors $\vec{x}$ and $\vec{y}$ and scalar k, or else you'll have only shown 1 specific case works.

To prove something is NOT a linear transformation, you only have to show that a property is NOT upheld. You can use a specific case if it's easier.

A general framework for these types of problems:

First check if $T(\vec{0}) = \vec{0}$. If this isn't true, you're done, this is NOT a linear transformation.

Otherwise, prove (or disprove) the 2 properties hold up.

1) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$? $\qquad$ 2) $T(k\vec{x}) = kT(\vec{x})$?

Example: $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \cdot x_2 \\ x_3 - x_1 \end{bmatrix}$. Is T a linear transformation?

1) preserves addition: $T(\vec{x}) + T(\vec{y}) = T(\vec{x}+\vec{y})$?

$$\begin{bmatrix} x_1 \cdot x_2 \\ x_3 - x_1 \end{bmatrix} + \begin{bmatrix} y_1 \cdot y_2 \\ y_3 - y_1 \end{bmatrix} = \begin{bmatrix} (x_1+y_1)\cdot(x_2+y_2) \\ (x_3+y_3)-(x_1+y_1) \end{bmatrix} ?$$

$$\begin{bmatrix} x_1 x_2 + y_1 y_2 \\ \cdots \end{bmatrix} \neq \begin{bmatrix} x_1 x_2 + x_1 y_2 + x_2 y_1 + y_1 y_2 \\ \cdots \end{bmatrix}$$

here we can stop since we can see these two statements are not equal

This is NOT a linear transformation.

Now we can use our knowledge of linear transformations to our advantage to figure out what happens to other vectors beyond what we are given.

Example 2: Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and that $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

1) What is $T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$? We can use our properties for this! Remember that $T(k\vec{x}) = kT(\vec{x})$, and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is a scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$T\left(2\cdot\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \to T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} \to T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

2) What is $T\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right)$? Again we can use our properties; $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) \to T\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \to T\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$