As a recap, λ is an eigenvalue of A if $A\vec{v} = \lambda\vec{v}$ for some vector $\vec{v}$.

As of now, however, we don't know how to find the eigenvectors or eigenvalues of A unless we are given them.

Let's work through how to find both.

If λ is an eigenvalue of A, then $A\vec{v} = \lambda\vec{v}$ for some vector $\vec{v}$.

Then $A\vec{v} - \lambda\vec{v} = \vec{0} \Leftrightarrow A\vec{v} - \lambda I_n\vec{v} = \vec{0} \Leftrightarrow (A - \lambda I_n)\vec{v} = \vec{0}$

So the set of all vectors with eigenvalue λ, $E_\lambda = \ker(A - \lambda I_n)$. This is called the λ eigenspace, where λ is the specific eigenvalue.

Working from this, λ is not an eigenvalue of A if and only if:

$E_\lambda = \{\vec{0}\}$ since $\vec{0}$ can't be the only thing in the eigenspace, since $\vec{0}$ is not an eigenvector

$\ker(A - \lambda I_n) = \{\vec{0}\}$

$\text{rref}\{A - \lambda I_n\} = I_n$

$A - \lambda I_n$ is invertible

$\det(A - \lambda I_n) \neq 0$

So λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$

$\det(A - \lambda I_n)$ is the characteristic polynomial of A. It is a degree n polynomial in λ.

Now we can find out our eigenvalues and their associated eigenvectors.

1) Use the characteristic polynomial to find all eigenvalues. $\det(A - \lambda I_n) = 0$

2) Use each eigenvalue to find each eigenspace. $E_\lambda = \ker(A - \lambda I_n)$

Some last definitions and facts. If λ is an eigenvalue of A,

- the geometric multiplicity of λ is the $\dim(E_\lambda)$, or the number of linearly independent eigenvectors with eigenvalue λ.
- the algebraic multiplicity of λ is the number of times it appears algebraically as a solution to the characteristic polynomial.
- $1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity

A has an eigenbasis if and only if the algebraic multiplicity is equal to the geometric multiplicity for each eigenvalue.

- This is the same as the geometric multiplicities summing to n (A is an $n \times n$ matrix).
- If A has n distinct eigenvalues, it has an eigenbasis (without us even knowing the eigenvectors), since eigenvectors with different eigenvalues are always linearly independent.

For a 2×2 matrix, the characteristic polynomial is $\lambda^2 - (\text{tr } A)\lambda + (\det A)$

- $\text{tr } A$ is the trace of A, which is the sum of the diagonal.