Coordinates give us another way to solve for the matrix of a linear transformation, which is arguably easier and less time-consuming than previous strategies.

Let $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$ be a basis of $\mathbb{R}^n$ and $\vec{x}$ be an arbitrary vector in $\mathbb{R}^n$.

this means $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$ can be written in exactly one way

We call $[\vec{x}]_{\mathfrak{B}}$ the $\mathfrak{B}$-coordinates of $\vec{x}$

If the matrix $S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$, then $[\vec{x}]_{\mathfrak{B}} = S^{-1}\vec{x}$. S is the "change of basis" matrix.

$$\begin{tikzcd} \vec{x} \arrow[r, bend left, "S^{-1}"] & {[\vec{x}]_{\mathfrak{B}}} \arrow[l, bend left, "S"] \end{tikzcd}$$

Why does this work? Since $[\vec{x}]_{\mathfrak{B}}$ tells the c's in $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{x}$, and the columns of S are $\vec{v}_1, \dots, \vec{v}_n$ then $S[\vec{x}]_{\mathfrak{B}} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$ which is just $\vec{x}$. In the other direction $S^{-1}\vec{x}$ gives us $[\vec{x}]_{\mathfrak{B}}$. Note that this is only possible because S is invertible, which we know since $\vec{v}_1, \dots, \vec{v}_n$ is a basis.

We can translate our vector into the world of our new basis $\mathfrak{B}$ using S^{-1}, but we need to describe the transformation in this new world. We can describe the matrix C, which tells us what the transformation does to our basis vectors in terms of the basis.

$C = \begin{bmatrix} [T(\vec{v}_1)]_{\mathfrak{B}} & \cdots & [T(\vec{v}_n)]_{\mathfrak{B}} \end{bmatrix}$ each column of C is what the transformation does to each column in S, put into terms of the basis.

Now that we can translate our vectors into the world of $\mathfrak{B}$ and describe the transformation in that world, we need to convert back to the standard world. Recall that S does this for us. Now we can write $A = SCS^{-1}$.

Summary: We want to find A, the matrix of the transformation T, but want a better way than what we've previously used. We use coordinates, to describe our vector $\vec{x}$ in terms of a new basis, describe our transformation in the new basis, and then translate back into the standard basis. We get $A = SCS^{-1}$ from this.

translate your vector $\vec{x}$ into the $\mathfrak{B}$-world

$$\begin{tikzcd} \vec{x} \arrow[r, "S^{-1}"] \arrow[d, "A?"'] & {[\vec{x}]_{\mathfrak{B}}} \arrow[d, "C"] \\ T(\vec{x}) & {[T(\vec{x})]_{\mathfrak{B}}} \arrow[l, "S"'] \end{tikzcd}$$

we want A, but we don't know it

do the transformation in the $\mathfrak{B}$-world

translate your vector back into the standard world

Example: Find the matrix of the reflection over the line $y = 2x$ in $\mathbb{R}^2$.

1) Find our basis $\mathfrak{B}$. $\mathfrak{B} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right)$

↳ this basis is picked since we can find $T(\vec{v}_1)$ and $T(\vec{v}_2)$ easily

2) Write S and S^{-1}. $S = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, $S^{-1} = -\frac{1}{5}\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$

↳ S is just the matrix with our basis as columns

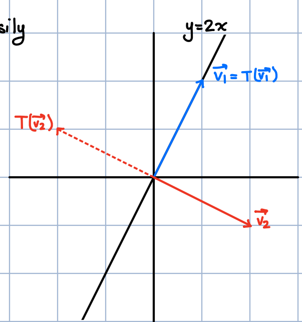
3) Find C. $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

↳ when transformed $\vec{v}_1$ stays the same, so it stays $\vec{v}_1$.

↳ $[T(\vec{v}_1)]_{\mathfrak{B}} = [\vec{v}_1]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ you need $1\vec{v}_1 + 0\vec{v}_2$ to make $\vec{v}_1$

↳ when transformed $\vec{v}_2$ flips over the line, so it becomes $-\vec{v}_2$

↳ $[T(\vec{v}_2)]_{\mathfrak{B}} = [-\vec{v}_2]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ you need $0\vec{v}_1 - 1\vec{v}_2$ to make $-\vec{v}_2$



4) Write down your answer A and simplify if asked.

↳ $A = SCS^{-1}$ $A = \frac{-1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$