If $T:\mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then there is an $n \times m$ matrix A such that $A\vec{x} = T(\vec{x})$, where $A = \left[T(\vec{e_1})\ T(\vec{e_2})\ \dots\ T(\vec{e_m})\right]$

A is called the matrix of T. Note that in $\mathbb{R}^n$, we write $\vec{e_1} = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$ $\vec{e_2} = \begin{bmatrix}0\\1\\\vdots\end{bmatrix}$ $\vec{e_m} = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$

Linear combination: a vector $\vec{x}$ is a linear combination of vectors $\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}$ if $\vec{x} = c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_k\vec{v_k}$ for some scalars $c_1, c_2, \dots, c_k$

span: the span of $\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}$ is the set of all linear combinations of $\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}$. Intuitively this is all the vectors you could potentially make from adding multipes of $\vec{v_1}, \dots, \vec{v_k}$

basis: a basis of $\mathbb{R}^m$ is m vectors whose span is $\mathbb{R}^m$. $(\vec{e_1}, \vec{e_2}, \dots, \vec{e_m})$ is the standard basis of $\mathbb{R}^m$ * we will revisit this definition later

If $T:\mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and we know what T does to a basis of $\mathbb{R}^m$, then we know what T does to every vector in $\mathbb{R}^m$!

Key idea: A linear transformation is determined by what it does to a basis of its domain

Example: Is $\vec{v_3}$ a linear combination of $\vec{v_1}$ and $\vec{v_2}$? (Same question as is $\vec{v_3}$ in the span of $\vec{v_1}$ and $\vec{v_2}$?)

$$\vec{v_1} = \begin{bmatrix}1\\3\\-2\\1\end{bmatrix} \quad \vec{v_2} = \begin{bmatrix}3\\2\\-1\\1\end{bmatrix} \quad \vec{v_3} = \begin{bmatrix}6\\4\\-1\\2\end{bmatrix} \quad c_1\begin{bmatrix}1\\3\\-2\\1\end{bmatrix} + c_2\begin{bmatrix}3\\2\\-1\\1\end{bmatrix} = \begin{bmatrix}6\\4\\-1\\2\end{bmatrix} \to \begin{bmatrix}1&3\\3&2\\-2&-1\\1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}6\\4\\-1\\2\end{bmatrix} \to \left[\begin{array}{cc|c}1&3&6\\3&2&4\\-2&-1&-1\\1&1&2\end{array}\right]$$

Always use Gauss-Jordan, this guarantees a result.

$$\left[\begin{array}{cc|c}1&3&6\\3&2&4\\-2&-1&-1\\1&1&2\end{array}\right]\begin{array}{l}-(IV)\\-3(IV)\\+2(IV)\\ \\\end{array} \to \left[\begin{array}{cc|c}0&2&4\\0&-1&-2\\0&1&3\\1&1&2\end{array}\right]+(III) \to \left[\begin{array}{cc|c}0&2&4\\0&0&1\\0&1&3\\1&1&2\end{array}\right]$$ inconsistent, so $\vec{v_3}$ is NOT a linear combination of $\vec{v_1}$ and $\vec{v_2}$

This process can tell us if we can determine what happens to some vectors in linear transformations. If we only know what $T(\vec{v_1})$ and $T(\vec{v_2})$ equalled, we would not be able to figure out what $T(\vec{v_3})$ is with that info alone.

Example 2: $T:\mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation, where $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}0\\-1\\2\end{bmatrix}$. Find the matrix of T.

To find the matrix of T, we need to know what it does to an arbitrary vector $\vec{x}$. We can express $\vec{x}$ as a linear combination of what we already know.

$$c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \quad \left[\begin{array}{cc|c}1&3&x_1\\2&4&x_2\end{array}\right]\div 2 \to \left[\begin{array}{cc|c}1&3&x_1\\1&2&\frac{1}{2}x_2\end{array}\right]-I \to \left[\begin{array}{cc|c}1&3&x_1\\0&-1&-x_1+\frac{1}{2}x_2\end{array}\right]\begin{array}{l}+3II\\ \times -1\end{array} \to \left[\begin{array}{cc|c}1&0&-2x_1+\frac{3}{2}x_2\\0&1&x_1-\frac{1}{2}x_2\end{array}\right] \to \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}-2x_1+\frac{3}{2}x_2\\x_1-\frac{1}{2}x_2\end{bmatrix}$$

now by our properties of a linear transformation

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = T\left(c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}3\\4\end{bmatrix}\right) = T\left(c_1\begin{bmatrix}1\\2\end{bmatrix}\right) + T\left(c_2\begin{bmatrix}3\\4\end{bmatrix}\right) = c_1\left(T\begin{bmatrix}1\\2\end{bmatrix}\right) + c_2\left(T\begin{bmatrix}3\\4\end{bmatrix}\right) = c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}0\\-1\\2\end{bmatrix} = (-2x_1+\tfrac{3}{2}x_2)\begin{bmatrix}1\\2\\3\end{bmatrix} + (x_1-\tfrac{1}{2}x_2)\begin{bmatrix}0\\-1\\2\end{bmatrix}$$

$$\begin{bmatrix}-2x_1+\frac{3}{2}x_2\\-4x_1+3x_2-x_1+\frac{1}{2}x_2\\-6x_1+\frac{9}{2}x_2+2x_1-x_2\end{bmatrix} = \begin{bmatrix}-2x_1+\frac{3}{2}x_2\\-5x_1+\frac{7}{2}x_2\\-4x_1+\frac{7}{2}x_2\end{bmatrix} \to T(\vec{x}) = \begin{bmatrix}-2&\frac{3}{2}\\-5&\frac{7}{2}\\-4&\frac{7}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

so we have found the matrix of T

Check your matrix is correct by multiplying A by your original vectors and see if you get the correct result