Vectors $(\vec{v}_1, \ldots, \vec{v}_K)$ are linearly independent if there exists a non-trivial linear relation among them.

A trivial linear relation is $c_1\vec{v}_1 + \ldots + c_K\vec{v}_K = \vec{0}$ where all c_i's are 0. If any of the coefficients are non-zero, then we have a non-trivial linear relation.

Example: $2\vec{v}_1 - 4\vec{v}_3 + 5\vec{v}_4 = \vec{0}$ is a non-trivial linear relation. $0\vec{v}_1 + \ldots + 0\vec{v}_K = \vec{0}$ is a trivial linear relation.

We can now update our definition of a basis to include this idea. A basis of $\mathbb{R}^n$ is a collection of linearly independent vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ that span $\mathbb{R}^n$.

Note that with this definition, we don't specify that we have n vectors, since linear independence implies we will always only have n vectors.

A subspace of $\mathbb{R}^n$ is a set V in $\mathbb{R}^n$ such that

1) the sum of any two vectors in V is still in V. V is closed under addition

   If $\vec{v}_1, \vec{v}_2$ are in V, then $\vec{v}_1 + \vec{v}_2$ is in V

2) every scalar multiple of a vector in V is still in V. V is closed under scalar multiplication

   If $\vec{v}_1$ is in V and K is a scalar, then $K\vec{v}$ is in V

   this property implies that $\vec{0}$ is always in every subspace, and is always a good first thing to check

To show a subset V of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, you must show the subset is closed under addition and scalar multiplication.

1) check if $\vec{0}$ is in V. If it isn't, V isn't a subspace.

2) show V is closed under addition.

   This usually involves taking the properties of $\vec{v}_1$ and $\vec{v}_2$ and adding the properties together

3) show V is closed under scalar multiplication.

   This usually involves taking the properties of $\vec{v}$ and rewriting/distributing the scalar

Example 2: V is a plane consisting of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $2x - 4y + 3z = 0$. Show V is a subspace and find a basis of V.

1) $\vec{0}$ is in V ✓

2) let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in V$ so $2x_1 - 4y_1 + 3z_1 = 0$ and $2x_2 - 4y_2 + 3z_2 = 0$

   is $\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$ in V? $2x_1 - 4y_1 + 3z_1 + 2x_2 - 4y_2 + 3z_2 = 0 \longrightarrow 2x_1 + 2x_2 - 4y_1 - 4y_2 + 3z_1 + 3z_2 = 0$

   $2(x_1 + x_2) - 4(y_1 - y_2) + 3(z_1 - z_2) = 0$ this is exactly the requirement to be in V; closed under addition

3) let $\vec{v} \in V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $K \in \mathbb{R}$ so $2x - 4y + 3z = 0$

   is $K\vec{v} = \begin{bmatrix} Kx \\ Ky \\ Kz \end{bmatrix}$ in V? $K(2x - 4y + 3z) = 0 \longrightarrow 2Kx - 4Ky + 3Kz = 0$

   $2(Kx) - 4(Ky) + 3(Kz) = 0$ this is exactly the requirement to be in V; closed under scalar multiplication

Therefore V is a subspace. A basis of V is $\left(\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \right)$. Since V is a plane, to find the basis just pick two non-parallel vectors in the plane.