A is an $n \times n$ matrix

The kernel of A is the set of solutions of $A\vec{x} = \vec{0}$. You can think of the kernel as all of the inputs that are sent to $\vec{0}$.

The image of A is the set of all $\vec{b}$ such that $A\vec{x} = \vec{b}$ is consistent. You can think of the image as all of the possible outputs.

↳ in general, $\operatorname{im}(A) = \operatorname{span}(\text{columns of } A)$ *note that we need to make sure the vectors are linearly independent*

An intuitive definition of linear independence: vectors $(\vec{v}_1, \ldots, \vec{v}_n)$ are linearly dependent if one of them is in the span of the others | is a linear combination of the others.

Otherwise, they are linearly independent. We'll give a more formal definition of this later.

When we solve for the kernel of a matrix, that tells us about the image too.

Example: Find the image and kernel of matrix A.

Steps: row reduce A to solve for the kernel.

use the kernel to solve for the image.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 4 & 0 \\ 3 & 8 & 2 \end{bmatrix} \begin{matrix} \\ -2\mathrm{I} \\ -3\mathrm{I} \end{matrix} \quad \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 5 & 5 \end{bmatrix} \begin{matrix} \\ \div 2 \\ \div 5 \end{matrix} \quad \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} -\mathrm{II} \\ \\ -\mathrm{II} \end{matrix} \quad \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \operatorname{Ker}(A) = \operatorname{span}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right)$$

Since the kernel of A is the same as solving $A\vec{x} = \vec{0}$, and the columns of A are vectors $(\vec{v}_1, \ldots, \vec{v}_n)$, this tells us $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$.

So $\vec{v}_3$ is in the span of $\vec{v}_1$ and $\vec{v}_2$. Therefore $\operatorname{im}(A) = \operatorname{span}(\text{columns of } A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} \right)$.

Note that the columns with leading ones are included in the image, and the columns with free variables are not.

More on this later in the course.