Self-similar Filament Calculations

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This document provides additional contexts to Appendix 1 of the article arXiv/astro-ph:2403.00912 “Entrainment of Hot Gas into Cold Streams: The Origin of Excessive Star-formation Rates at Cosmic Noon”, and include a minor correction to arXiv/astro-ph:2306.03966 “The Structure and Dynamics of Massive High-z Cosmic-Web Filaments: Three Radial Zones in Filament Cross-Sections”.

1 VIRIAL FILAMENT

To rederive the virialized quantities of filaments, we will start with the virial equilibriun condition given by eq. 9 of Lu et al. (2024),

$$2\mathcal{K}_{\rm x} = G\Lambda_{\rm tot}\Lambda_{\rm x}, \tag{1}$$

where $\mathcal{K}$ is the kinetic energy, and $\Lambda_{\rm tot}$ is the total line-mass or mass per unit length, and x is either for gas or for all particles. If x is tot, we get eq 9 back. From eq. 5 of Lu et al. (2024), the kinetic energy is given by

$$\mathcal{K}_{\rm x} = \frac{3}{2}\frac{k_B T\Lambda_{\rm x}}{\mu m_p}. \tag{2}$$

Thus, to solve for virial temperature of the filament,

$$k_B T = \frac{G\Lambda_{\rm tot}\mu m_p}{3}. \tag{3}$$

This differs from eq 65 of Lu et al. (2024) by a factor of f_b, and the correct eq 65 is

$$T_{\rm v,fil} = 1.2\times10^6{\rm K}\, M_{12}^{0.77}(1+z)_5^2 f_{s,3}\mathcal{M}_v^{-1} f_b. \tag{4}$$

2 SELF-SIMILAR DENSITY PROFILES OF FILAMENTS

The self-similar filament profiles are derived by combining elements of the collisionless cylindrical collapse model of Fillmore & Goldreich (1984) and the collisional spherical collapse model of Bertschinger (1985b). Following Fillmore & Goldreich (1984), the matter will break away from the expanding background and collapse at a turnaround radius if the gravitational pull due to the enclosed density is large enough to overcome the Hubble flow. Following Bertschinger (1985b), we do not assume virialization after the infalling mass shell reaches a certain radius, but rather consider shell crossing for dark matter and the formation of an accretion shock for gas. We assume an Einstein-de Sitter universe with $\Omega_m = 1$, $a \propto t^{2/3}$ and a background matter density $\rho_b = 1/6\pi G t^2$, with t the cosmic time. The model also assumes that the filament is infinite along its axis. Each mass shell around the overdense cylindrical region reaches its turnaround radius at some time $t_{\rm ita}$, the time of initial turnaround, and then proceeds to fall towards the filament axis. This sets the initial condition for the mass shell, where the turnaround radius is $r_{\rm ta} \equiv r(t_{\rm ita})$, the enclosed line-mass at turnaround is $\Lambda_{\rm ta} \equiv \Lambda(t_{\rm ita})$, and the initial velocity is $v_{\rm ta} \equiv v(t_{\rm ita}) = 0$. The filament line-mass enclosed within the turnaround radius at time t increases as a function of time as[1] $\Lambda(r_{\rm ta}) \propto a^{s-1} \propto t^{2(s-1)/3}$. Accordingly, the turnaround radius grows as $r_{\rm ta}(t) \propto \sqrt{\Lambda/\rho_b} = a^{3\delta/2}$, where $\delta = 2(1+s/2)/3$. The dark matter mass shell then follows the equation of motion (Fillmore & Goldreich 1984)

$$\frac{{\rm d}^2 r}{{\rm d}t^2} = \frac{d(Hr)}{dt} - \frac{2G(\Lambda-\Lambda_b)}{r} = \frac{2\pi G\rho_b r}{3} - \frac{2G\Lambda}{r}, \tag{5}$$

with $H = a^{-1}da/dt$ the Hubble constant at time t, and $\Lambda_b = \pi r^2\rho_b$.

The self-similar model assumes that the filament profile is universal for a given mass accretion rate, and we can, therefore, remove the time dependence when all parameters are normalised by appropriate quantities (see Table 1). The equation of motion can then be expressed in dimensionless form as

$$\frac{{\rm d}^2\lambda}{{\rm d}\xi^2} + (2\delta-1)\frac{{\rm d}\lambda}{{\rm d}\xi} + \delta(\delta-1)\lambda = \frac{\lambda}{9} - \frac{M}{3\lambda}, \tag{6}$$

where the normalized time, $\xi = \ln(t/t_{\rm ita})$.

We solve eq. (6) iteratively as follows. We begin by assuming a power-law mass profile, and then numerically integrate the equation with the outer boundary condition $M(\lambda = 1) = M_{\rm ta}$ and ${\rm d}\lambda/{\rm d}\xi(\lambda = 1) = -\delta$. Once we obtain the solution for the trajectory of the mass

$x(r,t)$	r	ρ	Λ	p	v
$X(\lambda)$	$\lambda(r,t)$	D	M	P	V
$\chi(t)$	$r_{\rm ta}$	ρ_b	$\rho_b\pi r_{\rm ta}^2$	$\rho_b(r_{\rm ta}/t)^2$	$r_{\rm ta}/t$

Table 1. The normalization assumed under the self-similar model. The top row indicates physical quantity $x(r,t)$. The middle indicates the corresponding dimensionless quantity $X(\lambda)$, which is a function of dimensionless position only. The bottom row shows the normalized dimensions $\chi = x/X$, which is a function of time only.

[1] Self-similar models for halo mass growth assume $m(r_{\rm ta}) \propto a^s$ (Fillmore & Goldreich 1984; Shi 2016). For filaments, the filament axis is expanding due to the expansion of the Universe, which decreases the line-mass and leads to the -1 in the power law exponent.

shell, we update the total enclosed line-mass at radius λ according to Bertschinger (1985a,b)

$$M(\lambda) = M_{\rm ta} \sum_{i=1}^{N(\lambda)} (-1)^{i-1} \exp(-2(s-1)\xi_i/3), \tag{7}$$

where the index i runs over all $N(\lambda)$ mass shells that are currently at radius λ, and ξ_i are the times with respect to each mass shell's turnaround time. $M_{\rm ta}$ is the normalised line mass inside the current turnaround radius. Thus, $M_{\rm ta}\exp[-2(s-1)\xi_i/3] = M_{\rm ta}(a_{\rm ita}/a_i)^{s-1} = \Lambda_{\rm ita}/(\pi r_{\rm ta}^2 \rho_{\rm b})$ is the enclosed line-mass when the shell was at the turnaround normalised by the current density and the turnaround radius, so that the overall normalisation of the mass profile is uniform among all shells. The alternating signs $(-1)^{i-1}$ account for the fact that the shells at the radius λ are alternating whether they are flowing in or out. The first shell is on its way in along the first infall, the second has fallen in and is on its way back out towards the first splashback, the third is on its way in along the second infall, etc. We then insert the updated mass profile obtained with eq. (7) back into eq. (6) and solve it again to obtain a new mass profile. We repeat this process until the mass profiles have converged to within $< 3\%$.

The collisional gas, on the other hand, follows the continuity equations expressed as

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\frac{\rho}{r}\frac{\partial}{\partial r}(rv), \\
\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{2\pi G}{3}\rho_b r - \frac{2G\Lambda}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r}, \\
\frac{\mathrm{d}k}{\mathrm{d}t} &= 0, \\
\frac{\partial \Lambda}{\partial r} &= 2\pi r \rho,
\end{aligned} \tag{8}$$

with the entropy $k \equiv p\rho^{-\gamma}$, and the Lagrangian derivative $\mathrm{d}f/\mathrm{d}t \equiv \partial f/\partial t + \mathbf{v}\cdot\nabla f = \partial f/\partial t + v\,\partial f/\partial r$. The gas is assumed to be pressureless outside the shock and infalls similarly to dark matter until it is shock-heated at a radius $r_{\rm sh}$. The post-shock properties are given by

$$\begin{aligned}
v_2 &= \frac{\gamma-1}{\gamma+1}\left[v_1 - v_{\rm sh}\right] + v_{\rm sh}, \\
\rho_2 &= \frac{\gamma+1}{\gamma-1}\rho_1, \\
p_2 &= \frac{2}{\gamma+1}\rho_1\left[v_1 - v_{\rm sh}\right]^2, \\
\Lambda_2 &= \Lambda_1,
\end{aligned} \tag{9}$$

assuming infinite Mach number due to pressureless pre-shock conditions. $v_{\rm sh}$ is the speed at which the accretion shock propagates, given by differentiating $r_{\rm sh}(t) = \lambda_{\rm sh} r_{\rm ta}(t)$, where $\lambda_{\rm sh}$ is constant with time due to the assumption of self-similarity. The continuity equations and the shock jump conditions can be rewritten as follows.

$$\begin{aligned}
-2D + (V - \delta\lambda)\,D' &= -\frac{D}{\lambda}(\lambda V)', \\
V(\delta - 1) + (V - \lambda\delta)V' &= \left[\frac{\lambda}{9} - \frac{M}{3\lambda}\right] - \frac{P'}{D}, \\
(V - \lambda\delta)\frac{(PD^{-\gamma})'}{PD^{-\gamma}} &= 2(1-\gamma) + 2(1-\delta), \\
M' &= 2\lambda D,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
V_2 &= \frac{\gamma-1}{\gamma+1}\left[V_1 - \lambda_{\rm sh}\delta\right] + \lambda_{\rm sh}\delta, \\
D_2 &= \frac{\gamma+1}{\gamma-1}D_1, \\
P_2 &= \frac{2}{\gamma+1}D_1\left[V_1 - \lambda_{\rm sh}\delta\right]^2, \\
M_2 &= M_1,
\end{aligned} \tag{11}$$

where $\lambda_{\rm sh}$ is the normalized shock radius. These equations are solved using the same initial conditions at the turnaround radius as for dark matter, and the shock radius is set such that the solution ensures the inner boundary condition $V = 0$.

3 CALCULATIONS

Here are some quick shortcuts for calculations:

$$\begin{aligned}
\frac{\mathrm{d}\rho_b}{\mathrm{d}t} &= -2\frac{\rho_b}{t}, \\
\frac{\mathrm{d}r_{\rm ta}}{\mathrm{d}t} &= \delta\frac{r_{\rm ta}}{t},
\end{aligned}$$

Additionally, partial time derivative of all dimensionless quantities are not zero, but should be

$$\begin{aligned}
\frac{\partial X}{\partial t} &= \frac{\partial X}{\partial \lambda}\frac{\partial \lambda}{\partial t}, \\
&= X' r \frac{\partial 1/r_{\rm ta}}{\partial t}, \\
&= -X'\frac{r}{r_{\rm ta}^2}\frac{\partial r_{\rm ta}}{\partial t}, \\
&= -X'\frac{\lambda}{r_{\rm ta}}\delta\frac{r_{\rm ta}}{t}, \\
&= -\frac{\delta\lambda X'}{t},
\end{aligned} \tag{12}$$

for any dimensionless X. Partial time derivative calculates the change of the variable at a constant point r. The dimensionless quantities are self-similar and does not change with time at constant λ. Thus, the dimensionless quantities at constant point r change with time because the normalization $r_{\rm ta}$ is growing with time.

For full time derivative,

$$\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= \frac{\mathrm{d}X}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}, \\
&= X'\left(\frac{1}{r_{\mathrm{ta}}}\frac{\mathrm{d}r}{\mathrm{d}t} + r\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r_{\mathrm{ta}}}\right), \\
&= X'\left(\frac{v}{r_{\mathrm{ta}}} + -\frac{r}{r_{\mathrm{ta}}^2}\frac{\mathrm{d}r_{\mathrm{ta}}}{\mathrm{d}t}\right), \\
&= X'\left(\frac{V}{t} + -\frac{r}{r_{\mathrm{ta}}^2}\delta\frac{r_{\mathrm{ta}}}{t}\right), \\
&= \frac{X'}{t}\left(V - \delta\lambda\right). \qquad (13)
\end{aligned}$$

For partial spatial derivative, we assume at constant time so we can take all time-dependent quantities, i.e. the quantities used to normalise, out of derivative. For example

$$\frac{\partial x}{\partial r} = \frac{\chi}{r_{\mathrm{ta}}}\frac{\partial X}{\partial \lambda} = \frac{\chi}{r_{\mathrm{ta}}}X' \qquad (14)$$

We substitute the variables in Table 1 in LHS of the continuity equation of Equation (9),

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= D\frac{\mathrm{d}\rho_b}{\mathrm{d}t} + \rho_b\frac{\mathrm{d}D}{\mathrm{d}t}, \\
&= -2D\frac{\rho_b}{t} + \rho_b\frac{D'}{t}\left(V - \delta\lambda\right), \\
&= \frac{\rho_b}{t}\left[-2D + D'(V - \delta\lambda)\right],
\end{aligned}$$

where in the second step, we use eq. (13).

Alternatively,

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\partial\rho}{\partial t} + v\frac{\partial\rho}{\partial r}, \\
&= D\frac{\partial\rho_b}{\partial t} + \rho_b\frac{\partial D}{\partial t} + \frac{v\rho_b}{r_{\mathrm{ta}}}\frac{\partial D}{\partial\lambda}, \\
&= -2D\frac{\rho_b}{t} - \rho_b\frac{\delta\lambda D'}{t} + \frac{V r_{\mathrm{ta}}\rho_b}{r_{\mathrm{ta}}t}\frac{\partial D}{\partial\lambda}, \\
&= \frac{\rho_b}{t}\left[-2D - \delta\lambda D' + V D'\right],
\end{aligned}$$

where we use eq. (12) and eq. (14) for second and third term respectively.

RHS of the continuity equation can be simplified to given expression using eq. (14).

For LHS of the momentum equation of Equation (9),

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{\mathrm{d}V r_{\mathrm{ta}}/t}{\mathrm{d}t}, \\
&= \frac{r_{\mathrm{ta}}}{t}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V}{t}\frac{\mathrm{d}r_{\mathrm{ta}}}{\mathrm{d}t} + V r_{\mathrm{ta}}\frac{\mathrm{d}1/t}{\mathrm{d}t}, \\
&= \frac{r_{\mathrm{ta}}}{t}\frac{V'}{t}(V - \delta\lambda) + \frac{V}{t}\frac{\delta r_{\mathrm{ta}}}{t} - \frac{V r_{\mathrm{ta}}}{t^2}, \\
&= \frac{r_{\mathrm{ta}}}{t^2}\left[V'(V - \delta\lambda) + V(\delta - 1)\right],
\end{aligned}$$

where leftmost term is simplified using eq. (13).

For the entropy equation of Equation (9),

$$\begin{aligned}
\frac{\mathrm{d}p\rho^{-\gamma}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}PD^{-\gamma}\frac{\rho_b r_{\mathrm{ta}}^2}{t^2\rho_b^\gamma} \\
&= PD^{-\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\rho_b r_{\mathrm{ta}}^2}{t^2\rho_b^\gamma} + \frac{\rho_b r_{\mathrm{ta}}^2}{t^2\rho_b^\gamma}\frac{\mathrm{d}PD^{-\gamma}}{\mathrm{d}t} \\
0 &= PD^{-\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\rho_b^{1-\gamma}r_{\mathrm{ta}}^2}{t^2} + \frac{\rho_b^{1-\gamma}r_{\mathrm{ta}}^2}{t^2}\frac{\mathrm{d}PD^{-\gamma}}{\mathrm{d}t} \\
&= -[2(1-\gamma) + 2(2-\delta)]PD^{-\gamma} + (PD^{-\gamma})'(V - \delta\lambda)
\end{aligned}$$

where

$$\frac{\rho_b^{1-\gamma}r_{\mathrm{ta}}^2}{t^2} \propto t^{-2(1-\gamma)+2\delta-2} \propto t^{-[2(1-\gamma)+2(2-\delta)]} \qquad (15)$$

REFERENCES

Bertschinger E., 1985a, ApJS, 58, 1

Bertschinger E., 1985b, ApJS, 58, 39

Fillmore J. A., Goldreich P., 1984, ApJ, 281, 1

Lu Y. S., Mandelker N., Oh S. P., Dekel A., van den Bosch F. C., Springel V., Nagai D., van de Voort F., 2024, MNRAS, 527, 11256

Shi X., 2016, MNRAS, 461, 1804