Mathematical Economics

ECON2050: Tutorial 9

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Introduction

In this tutorial, we will explore various optimization problems in economics, focusing on constrained maximization and minimization. These problems are highly relevant in real-world economic models, where firms and individuals often operate under constraints such as budgets, resource availability, or production limitations. We will study how to apply optimization techniques, particularly using Lagrange multipliers, to solve these problems and determine the optimal outcomes.

Topics Covered

- Constrained optimization problems
- Distance minimization and cost minimization
- Application of Lagrange multipliers in economic models
- Global maxima and minima under constraints

Objectives

- Apply optimization techniques to minimize production costs and distances.
- Solve constrained maximization problems in economic contexts.
- Understand how to use Lagrange multipliers to incorporate constraints into economic decision-making.
- Analyze the impact of constraints on solutions and determine whether global maxima or minima exist.

Expected Outcomes

By the end of this tutorial, students should be able to:

- Solve constrained optimization problems using Lagrange multipliers.
- Determine global maxima and minima under different constraints.
- Apply these techniques to practical economic problems, such as minimizing production costs and optimizing resource allocation.
- Understand the importance of convexity and the role of constraints in determining optimal solutions.

Problem 1: Maximizing $f(x, y) = x - y$

Relevance in Economics

Optimization problems like this are fundamental in economics, especially in scenarios involving profit maximization or cost minimization, where firms need to optimize their production inputs or outputs.

Applications in Economics

Such optimization is critical in fields like production theory, where firms aim to adjust inputs to achieve the best possible outcomes. It is also relevant in consumer theory for utility maximization given constraints like income or prices.

Problem Description

Let $f(x, y) = x - y$. Solve the following optimization problems graphically:

- (a) $\max_{x,y \in \mathbb{R}} f(x, y)$
- (b) $\max_{x,y \in \mathbb{R}} f(x, y) \quad \text{subject to} \quad x^2 + y^2 = 8$
- (c) $\max_{x \geq 0, y \geq 0} f(x, y) \quad \text{subject to} \quad x^2 + y^2 \leq 8$

Simple Explanation

The goal is to find the values of x and y that maximize the function $f(x, y) = x - y$. In (a), we are looking for the global maximum without any constraints. In (b), the problem is constrained by the condition that the values of x and y must lie on a circle with radius $\sqrt{8}$. In (c), the constraints are more specific: x and y must be non-negative and lie within or on the boundary of a circle with radius $\sqrt{8}$.

Key Idea

The key idea is to first understand the function $f(x, y) = x - y$ geometrically and then apply the constraints to determine where the maximum occurs. For constrained optimization (in parts (b) and (c)), we use techniques like Lagrange multipliers or graphical methods to solve.

Solution Strategy

(a) Global Maximization:

For $f(x, y) = x - y$, there is no bound on x and y in this case, so the function increases indefinitely as $x \to \infty$ and $y \to -\infty$. Therefore, the function has no maximum in the unconstrained case. Thus:

$$\text{No global maximum exists.}$$

(b) Maximization with Constraint $x^2 + y^2 = 8$:

The constraint $x^2 + y^2 = 8$ describes a circle centered at the origin with radius $\sqrt{8}$. To find the maximum of $f(x, y) = x - y$ subject to this constraint, we can apply Lagrange multipliers or analyze it graphically.

Using Lagrange multipliers, we set up the following system:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(x^2 + y^2 - 8)$$

Taking the partial derivatives:

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda x = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = -1 - 2\lambda y = 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 8 = 0$$

Solving these yields the points $(2, -2)$ and $(-2, 2)$.

Evaluating $f(x, y)$ at these points:

$$f(2, -2) = 2 - (-2) = 4, \quad f(-2, 2) = -2 - 2 = -4$$

Thus, the maximum is $f(2, -2) = 4$ and occurs at $(2, -2)$.

(c) Maximization with Constraints $x \geq 0, y \geq 0, x^2 + y^2 \leq 8$:

The constraint $x^2 + y^2 \leq 8$ describes a disk with radius $\sqrt{8}$, and the additional constraints $x \geq 0$ and $y \geq 0$ restrict the domain to the first quadrant.

We analyze the boundary of the disk $x^2 + y^2 = 8$ in the first quadrant. Points on this boundary that satisfy $x^2 + y^2 = 8$ and are in the first quadrant include:

$$(2\sqrt{2}, 0), \quad (0, 2\sqrt{2})$$

Evaluating $f(x, y)$ at these points:

$$f(2\sqrt{2}, 0) = 2\sqrt{2} - 0 = 2\sqrt{2}, \quad f(0, 2\sqrt{2}) = 0 - 2\sqrt{2} = -2\sqrt{2}$$

Thus, the maximum occurs at $(2\sqrt{2}, 0)$ and is $f(2\sqrt{2}, 0) = 2\sqrt{2}$.

Problem 2: Global Maxima and Minima under Constraints

Relevance in Economics

Determining global maxima and minima is crucial for understanding how firms, individuals, or other agents can optimize their choices subject to constraints like budgets, resource limitations, or market restrictions.

Applications in Economics

These problems are particularly useful in optimization tasks such as maximizing utility or profits under budget constraints, or minimizing costs while maintaining production levels. They are widely used in production theory, consumer theory, and general equilibrium analysis.

Problem Description

Determine all global maxima and minima of $f : \mathbb{R}^2 \to \mathbb{R}$ subject to the constraint $h(x, y) = c$.

(a) $f(x, y) = x - y$, $h(x, y) = x^2 + y^2$, and $c = 9$.

(b) $f(x, y) = 3x^2 + 4y^2 - xy$, $h(x, y) = 2x + y$, and $c = 21$.

Simple Explanation

We are asked to find where the function $f(x, y)$ reaches its highest and lowest values while satisfying a given constraint. For example, in part (a), the constraint forces x and y to lie on a circle of radius 3, and we are interested in finding the highest and lowest values of $f(x, y) = x - y$ on that circle.

Key Idea

The key idea is to use constrained optimization methods such as the method of Lagrange multipliers to solve for the points that maximize or minimize the function $f(x, y)$ while satisfying the constraint $h(x, y) = c$.

Solution Strategy

(a) Global Maxima and Minima for $f(x, y) = x - y$, $h(x, y) = x^2 + y^2 = 9$:

We apply the method of Lagrange multipliers. The Lagrangian function is:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(h(x, y) - 9)$$

$$\mathcal{L}(x, y, \lambda) = (x - y) - \lambda(x^2 + y^2 - 9)$$

Taking the partial derivatives with respect to x, y, and λ, we get:

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda x = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2x}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -1 - 2\lambda y = 0 \quad \Rightarrow \quad \lambda = -\frac{1}{2y}$$

Equating the two expressions for λ:

$$\frac{1}{2x} = -\frac{1}{2y} \quad \Rightarrow \quad x = -y$$

Substitute into the constraint $x^2 + y^2 = 9$:

$$x^2 + (-x)^2 = 9 \quad \Rightarrow \quad 2x^2 = 9 \quad \Rightarrow \quad x^2 = \frac{9}{2} \quad \Rightarrow \quad x = \pm\frac{3\sqrt{2}}{2}$$

Thus, $y = \mp\frac{3\sqrt{2}}{2}$.

Evaluating $f(x, y) = x - y$ at these points:

$$f\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} = 3\sqrt{2}$$

$$f\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} = -3\sqrt{2}$$

Thus, the global maximum is $3\sqrt{2}$, and the global minimum is $-3\sqrt{2}$.

(b) Global Maxima and Minima for $f(x, y) = 3x^2 + 4y^2 - xy$, $h(x, y) = 2x + y = 21$:
We apply the method of Lagrange multipliers to solve this constrained optimization problem.
The Lagrangian function is given by:

$$\mathcal{L}(x, y, \mu) = 3x^2 + 4y^2 - xy - \mu(2x + y - 21)$$

where μ is the Lagrange multiplier. The first-order conditions (FOCs) are obtained by taking the partial derivatives of $\mathcal{L}(x, y, \mu)$ with respect to x, y, and μ:

$$\frac{\partial \mathcal{L}(x, y, \mu)}{\partial x} = 6x - y - 2\mu = 0$$

$$\frac{\partial \mathcal{L}(x, y, \mu)}{\partial y} = 8y - x - \mu = 0$$

$$\frac{\partial \mathcal{L}(x, y, \mu)}{\partial \mu} = -(2x + y - 21) = 0$$

From the first two conditions, we can solve for μ as follows:

$$2\mu = 6x - y \quad \text{and} \quad \mu = 8y - x$$

Equating the two expressions for μ, we have:

$$6x - y = 2(8y - x) \quad \Rightarrow \quad 6x - y = 16y - 2x$$

$$8x = 17y \quad \Rightarrow \quad y = \frac{8}{17}x$$

Substitute $y = \frac{8}{17}x$ into the constraint $2x + y = 21$:

$$2x + \frac{8}{17}x = 21 \quad \Rightarrow \quad \frac{42}{17}x = 21 \quad \Rightarrow \quad x = \frac{17}{2}$$

Now, substitute $x = \frac{17}{2}$ back into $y = \frac{8}{17}x$ to find y:

$$y = \frac{8}{17} \cdot \frac{17}{2} = 4$$

Thus, the critical point is $(x^*, y^*) = \left(\frac{17}{2}, 4\right)$. To confirm whether this point is a minimum or maximum, we check the Hessian matrix of $f(x, y)$, which is:

$$H_f(x, y) = \begin{pmatrix} 6 & -1 \\ -1 & 8 \end{pmatrix}$$

Since the Hessian is positive definite for all $(x, y) \in \mathbb{R}^2$ (the determinant of the Hessian is positive and the leading principal minors are both positive), $f(x, y)$ is a strictly convex function. Additionally, because the constraint $h(x, y) = 2x+y = 21$ is affine, the constraint set is convex. Therefore, this is a convex optimization problem. Since the function is convex and the constraint is convex, the critical point $\left(\frac{17}{2}, 4\right)$ is the unique global (constrained) minimum.

$$\boxed{\text{The global constrained minimum is at } \left(\frac{17}{2}, 4\right).}$$

Problem 3: Constrained Maximization Problem

Relevance in Economics

Constrained optimization problems are fundamental in economic modeling, especially when resources or choices are restricted. They allow us to understand how individuals or firms can maximize their utility or profits given specific constraints.

Applications in Economics

This type of constrained optimization is used in production, consumer behavior analysis, and resource allocation, where agents aim to achieve optimal outcomes within the boundaries of their constraints. It's widely applied in the study of utility maximization, profit maximization, and cost minimization.

Problem Description

Consider the following constrained maximization problem:

$$\max_{(x,y)\in\mathbb{R}^2} x \quad \text{s.t.} \quad x^2 - y = 0, \quad -x^2 - y = 0$$

Solve this problem graphically and using Lagrange multipliers.

Simple Explanation

The goal here is to maximize the variable x, but we are given two constraints on the relationship between x and y. In simpler terms, we want to find the highest value of x while ensuring that both conditions hold.

Key Idea

The key idea in solving this problem is to handle the constraints effectively, both graphically and mathematically. We will use Lagrange multipliers to formalize the solution and illustrate the geometric interpretation of the constraints.

Solution Strategy

(a) Solve the Problem Graphically:
1. **Objective Function Level Curves**: We need to plot a few level curves of the objective function $f(x, y) = x$. The level curves of $f(x, y) = x$ are vertical lines since the function depends only on x and is independent of y.
2. **Constraints**: The two constraints are:

$$x^2 - y = 0 \quad \Rightarrow \quad y = x^2$$

$$-x^2 - y = 0 \quad \Rightarrow \quad y = -x^2$$

The first constraint represents a parabola that opens upwards, and the second constraint represents a parabola that opens downwards.
3. **Graphical Interpretation**: Both constraints are parabolas, and the feasible region is the intersection of these two parabolas. However, since the two curves do not intersect at any point in $\mathbb{R}^2$, there is no feasible solution. Therefore, the constrained maximization problem has no solution.

(b) Write the Constraint Set by Extension:
The constraint set is the intersection of the two constraints:

$$C = \{(x, y) \in \mathbb{R}^2 : x^2 - y = 0 \quad \text{and} \quad -x^2 - y = 0\}$$

However, upon inspecting the constraints, we see that there is no $(x, y) \in \mathbb{R}^2$ that satisfies both constraints simultaneously, meaning the constraint set is empty:

$$C = \emptyset$$

(c) Gradients of the Constraint and Objective Functions:
Let $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$ denote the objective function, and $h_1 : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 - y$ and $h_2 : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 + y$ denote the constraint functions (if you choose $h_2 : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto -x^2 - y$, you get equivalent results). The gradients are illustrated as:

$$\nabla f(0,0) = (1,0),$$
$$\nabla h_1(0,0) = (2x,-1)\big|_{(x,y)=(0,0)} = (0,-1),$$
$$\nabla h_2(0,0) = (2x,1)\big|_{(x,y)=(0,0)} = (0,1),$$

in the figure from part (a).
Since the constraint set is empty, there are no points (x, y) to evaluate the gradients. However, if points existed, evaluate and visualize these gradients at each point.

(d) Solve the Problem Using the Lagrange Theorem:

To solve the problem using Lagrange multipliers, we first write the Lagrangian function:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2) = x - \lambda_1(x^2 - y) - \lambda_2(-x^2 - y)$$

where λ_1 and λ_2 are the Lagrange multipliers for the two constraints.

The first-order conditions (FOCs) are obtained by taking the partial derivatives of $\mathcal{L}(x, y, \lambda_1, \lambda_2)$ with respect to x, y, λ_1, and λ_2:

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda_1 x + 2\lambda_2 x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = -(x^2 - y) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = -(-x^2 - y) = 0$$

From $\lambda_1 + \lambda_2 = 0$, we conclude $\lambda_1 = -\lambda_2$. Substituting this into the first equation gives:

$$1 - 2\lambda_1 x - 2\lambda_1 x = 0 \quad \Rightarrow \quad 1 = 4\lambda_1 x \quad \Rightarrow \quad \lambda_1 = \frac{1}{4x}$$

However, substituting this into the constraints leads to a contradiction, meaning the problem has no solution in $\mathbb{R}^2$. Thus, there is no feasible point that satisfies both constraints, and the problem has no solution.

$$\boxed{\text{The problem has no feasible solution.}}$$

Problem 4: Constrained Maximization Problem

Relevance in Economics

This type of constrained maximization is crucial in economics, especially when dealing with optimization problems where multiple constraints must be satisfied simultaneously. It applies in various fields, including production theory, resource allocation, and profit maximization.

Applications in Economics

Constrained optimization is widely applied in economic models where firms or individuals aim to maximize output, utility, or profit subject to limitations such as resource availability or specific technical requirements. In this problem, the constraints represent geometric and algebraic conditions that must hold for optimal decision-making.

Problem Description

Solve the constrained maximization problem:

$$\max_{(x,y,z)\in\mathbb{R}^3} yz + xz \quad \text{s.t.} \quad y^2 + z^2 = 1, \quad xz = 3.$$

(You may assume that this problem has a solution.)

Simple Explanation

We are trying to find the maximum value of $yz+xz$, but we need to satisfy two constraints: the sum of the squares of y and z must equal 1, and the product of x and z must equal 3. This setup mimics optimization problems where firms or individuals need to make the best choices while staying within certain limits, such as budgets or production capacities.

Key Idea

The key idea is to apply constrained optimization techniques, specifically the Lagrange multiplier method. This method allows us to maximize or minimize an objective function while considering multiple constraints, ensuring that all conditions are met.

Solution Strategy

The solution involves setting up the Lagrangian function, which incorporates the objective function $yz + xz$ and the constraints $y^2 + z^2 = 1$ and $xz = 3$, along with their corresponding Lagrange multipliers. From the Lagrangian, we derive the first-order conditions (FOCs) and solve for the variables x, y, z, and the multipliers.

Solution

Let h_1, h_2 be such that $h_1(x, y, z) = y^2 + z^2$ and $h_2(x, y, z) = xz$. Then:

$$\nabla h_1(x, y, z) = (0, 2y, 2z), \quad \nabla h_2(x, y, z) = (z, 0, x).$$

Every (x, y, z) in the constraint set satisfies $xz = 3$ and thus $z \neq 0$. Therefore, for every (x, y, z) in the constraint set, $\nabla h_1(x, y, z)$ and $\nabla h_2(x, y, z)$ are linearly independent:

$$c_1(0, 2y, 2z) + c_2(z, 0, x) = 0 \quad \text{and} \quad z \neq 0 \quad \Rightarrow \quad c_1 = c_2 = 0.$$

Thus every (x, y, z) that is in the constraint set satisfies the NDCQ. Form the Lagrangian:

$$\mathcal{L}(x, y, z, \mu_1, \mu_2) = yz + xz - \mu_1(y^2 + z^2 - 1) - \mu_2(xz - 3)$$

and derive the first-order conditions (FOCs):

$$\begin{aligned}
\frac{\partial \mathcal{L}(x, y, z, \mu_1, \mu_2)}{\partial x} &= z - \mu_2 z = 0 \\
\frac{\partial \mathcal{L}(x, y, z, \mu_1, \mu_2)}{\partial y} &= z - 2\mu_1 y = 0 \\
\frac{\partial \mathcal{L}(x, y, z, \mu_1, \mu_2)}{\partial z} &= y + x - 2\mu_1 z - \mu_2 x = 0 \\
\frac{\partial \mathcal{L}(x, y, z, \mu_1, \mu_2)}{\partial \mu_1} &= -(y^2 + z^2 - 1) = 0 \\
\frac{\partial \mathcal{L}(x, y, z, \mu_1, \mu_2)}{\partial \mu_2} &= -(xz - 3) = 0
\end{aligned}$$

The last equation implies $z \neq 0$. Therefore, the first equation implies $\mu_2 = 1$. The third equation simplifies to $y - 2\mu_1 z = 0$. Together with the second equation, this implies $y = 4\mu_1^2 y$. Note that the second equation, together with $z \neq 0$, implies $y \neq 0$. Therefore, $y = 4\mu_1^2 y$ further simplifies to $1 = 4\mu_1^2$ or $\mu_1 = \pm 0.5$. If $\mu_1 = 0.5$, then $z = y$, and if $\mu_1 = -0.5$, then $z = -y$. In either case, from the fourth equation, we get $2y^2 = 1$ or $y = \pm\frac{1}{\sqrt{2}}$. In all cases, $x = \frac{3}{z}$ by the fifth equation. In summary, we obtain four critical points of the constrained problem:

$$\begin{aligned}
(x_1^*, y_1^*, z_1^*, \mu_1^*, \mu_2^*) &= \left(3\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}, 1\right) \\
(x_2^*, y_2^*, z_2^*, \mu_1^*, \mu_2^*) &= \left(-3\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{2}, 1\right) \\
(x_3^*, y_3^*, z_3^*, \mu_1^*, \mu_2^*) &= \left(-3\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}, 1\right) \\
(x_4^*, y_4^*, z_4^*, \mu_1^*, \mu_2^*) &= \left(3\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{2}, 1\right)
\end{aligned}$$

Since a solution exists and the objective function is 3.5 if evaluated at (x_1^*, y_1^*, z_1^*) or (x_2^*, y_2^*, z_2^*) and 2.5 if evaluated at (x_3^*, y_3^*, z_3^*) or (x_4^*, y_4^*, z_4^*), we find that the solutions to the problem are (x_1^*, y_1^*, z_1^*) and (x_2^*, y_2^*, z_2^*).

Problem 6 (Extra Practice Problem): Global Maxima and Minima under Constraints

Relevance in Economics

Determining global maxima and minima under constraints is critical in various fields of economics. These optimization problems often arise in production, utility maximization, and cost minimization scenarios, where economic agents are subject to constraints like budgets, resources, or technological limits.

Applications in Economics

Constrained optimization is frequently used in economic modeling. Firms aim to maximize profit while adhering to production constraints, consumers maximize utility subject to budget constraints, and social planners optimize welfare under resource limitations. These types of problems form the basis of economic analysis and decision-making in markets, firms, and households.

Problem Description

For each of the following cases, determine all global maxima and all global minima of $f : \mathbb{R}^2 \to \mathbb{R}$ subject to the constraint $h(x, y) = c$:

- (a) $f(x, y) = x - y^2$, $h(x, y) = x + y$, and $c = 2$.
- (b) $f(x, y) = xy$, $h(x, y) = x^2 + y^2$, and $c = 4$.
- (c) $f(x, y) = x^2 + y^2 - 2xy$, $h(x, y) = x^2 + y^2$, and $c = 50$.
- (d) $f(x, y) = x^2 - 10$, $h(x, y) = x - y$, and $c = 18$.

Simple Explanation

In each case, we want to find the points where the function $f(x, y)$ achieves its highest or lowest values, while ensuring that the constraint $h(x, y) = c$ holds. These types of problems can be solved using mathematical techniques that handle optimization with constraints, such as the method of Lagrange multipliers.

Key Idea

The key idea is to use the method of Lagrange multipliers to determine where the function $f(x, y)$ is maximized or minimized, subject to the given constraints. By setting up a Lagrangian function that incorporates both the objective function and the constraint, we derive the first-order conditions (FOCs) and solve for the optimal points.

Solution Strategy

For each part, the strategy involves:

- Formulating the Lagrangian function, combining the objective function $f(x, y)$ and the constraint $h(x, y) = c$.
- Deriving the FOCs by taking partial derivatives of the Lagrangian with respect to x, y, and the Lagrange multiplier μ.
- Solving the system of equations to identify the critical points.
- Determining whether the critical points correspond to global maxima or minima based on the behavior of the function and the constraint.

Solution

(a) The objective function is concave, as its Hessian matrix is negative semidefinite for all $(x, y) \in \mathbb{R}^2$. The constraint set is convex because the constraint function is linear. Therefore, we have a concave program. As $\nabla h(x, y) = (1, 1) \neq (0, 0)$ for all (x, y), the NDCQ is satisfied for all elements of the constraint set. From the Lagrangian defined by

$$\mathcal{L}(x, y, \mu) = x - y^2 - \mu(x + y - 2)$$

we derive the FOCs:

$$1 - \mu = 0$$

$$-2y - \mu = 0$$

$$x + y = 2$$

The unique critical point is $(x^*, y^*, \mu^*) = \left(\frac{5}{2}, \frac{-1}{2}, 1\right)$. The unique global maximizer is $(x^*, y^*) = \left(\frac{5}{2}, \frac{-1}{2}\right)$. There are no global minima.

(b) The objective function is continuous. The constraint set is compact. Therefore, the Extreme Value Theorem implies that the objective function has a global maximum and a global minimum in the constraint set. The NDCQ holds for all points in the constraint set, as $\nabla h(x, y) = (2x, 2y) \neq (0, 0)$ for all (x, y) such that $x^2 + y^2 = 4$. The Lagrangian is given by

$$\mathcal{L}(x, y, \mu) = xy - \mu(x^2 + y^2 - 4)$$

The FOCs are:

$$y - 2\mu x = 0$$

$$x - 2\mu y = 0$$

$$x^2 + y^2 = 4$$

We deduce that μ, x, y are all nonzero. Further calculations show that the critical points are:

$$(x_1^*, y_1^*, \mu_1^*) = \left(\sqrt{2}, \sqrt{2}, \frac{1}{2}\right)$$

$$(x_2^*, y_2^*, \mu_2^*) = \left(\sqrt{2}, -\sqrt{2}, -\frac{1}{2}\right)$$

$$(x_3^*, y_3^*, \mu_3^*) = \left(-\sqrt{2}, -\sqrt{2}, \frac{1}{2}\right)$$

$$(x_4^*, y_4^*, \mu_4^*) = \left(-\sqrt{2}, \sqrt{2}, -\frac{1}{2}\right)$$

Since $f(x_1^*, y_1^*) = f(x_3^*, y_3^*) = 2$ and $f(x_2^*, y_2^*) = f(x_4^*, y_4^*) = -2$, (x_1^*, y_1^*) and (x_3^*, y_3^*) are the global maxima, and (x_2^*, y_2^*) and (x_4^*, y_4^*) the global minima.

(c) The objective function is continuous. The constraint set is compact. Therefore, the Extreme Value Theorem implies that the objective function has a global maximum and a global minimum in the constraint set. The NDCQ holds for all points in the constraint set, as $\nabla h(x, y) = (2x, 2y) \neq (0, 0)$ for all (x, y) such that $x^2 + y^2 = 50$. The Lagrangian is given by

$$\mathcal{L}(x, y, \mu) = x^2 + y^2 - 2xy - \mu(x^2 + y^2 - 50)$$

The FOCs are:

$$2x - 2y - 2\mu x = 0$$
$$2y - 2x - 2\mu y = 0$$
$$x^2 + y^2 = 50$$

The critical points are:

$$(x_1^*, y_1^*, \mu_1^*) = (5, 5, 0)$$
$$(x_2^*, y_2^*, \mu_2^*) = (5, -5, 2)$$
$$(x_3^*, y_3^*, \mu_3^*) = (-5, -5, 0)$$
$$(x_4^*, y_4^*, \mu_4^*) = (-5, 5, 2)$$

Since $f(x_1^*, y_1^*) = f(x_3^*, y_3^*) = 0$ and $f(x_2^*, y_2^*) = f(x_4^*, y_4^*) = 100$, (x_1^*, y_1^*) and (x_3^*, y_3^*) are the global minima, and (x_2^*, y_2^*) and (x_4^*, y_4^*) the global maxima.

(d) The objective function is convex because its Hessian is positive semidefinite for all $(x, y) \in \mathbb{R}^2$. The constraint function is linear, implying that the constraint set is convex. The problem is thus a convex program. The NDCQ holds for all points in the constraint set, as $\nabla h(x, y) = (1, -1) \neq (0, 0)$ for all (x, y). The Lagrangian is given by

$$\mathcal{L}(x, y, \mu) = x^2 - 10 - \mu(x - y - 18)$$

The FOCs are:

$$2x - \mu = 0$$

$$\mu = 0$$

$$x - y = 18$$

The unique critical point is $(x^*, y^*, \mu^*) = (0, -18, 0)$. The unique global minimum is $(x^*, y^*) = (0, -18)$.

Problem 7 (Extra Practice Problem): Distance Minimization Problem

Relevance in Economics

Minimizing distance under constraints can be analogous to solving optimization problems in economics, such as minimizing costs or transportation distances under specific constraints. Understanding these types of geometric optimization problems is useful in resource allocation, supply chain optimization, and production processes where distance and constraints are key factors.

Applications in Economics

In economics, firms often seek to minimize transportation costs or optimize the placement of resources relative to distribution centers. This problem mirrors such scenarios, where the goal is to minimize distance while adhering to certain geometric constraints, such as available resources or spatial limitations.

Problem Description

Calculate the point or points on the circle with center $(0, 0)$ and radius 1 that are the closest to the point $(1, 1)$.

Layman's Explanation

We are trying to find the point(s) on a circle that are closest to a given point. The distance between any two points in space can be found using the distance formula, and here, we want to minimize this distance under the constraint that the point lies on a specific circle.

Key Idea

The key idea is to minimize the distance between a point on the circle and the point $(1, 1)$. Since the square root function is increasing, we can minimize the square of the distance instead. Using Lagrange multipliers, we can incorporate the constraint that the point lies on a circle, and then solve for the points that minimize the distance.

Solution Strategy

The solution involves:

- Formulating the distance as the objective function, minimizing $(x - 1)^2 + (y - 1)^2$ subject to the constraint that $x^2 + y^2 = 1$ (i.e., the point lies on the circle).
- Setting up the Lagrangian function that incorporates the distance function and the constraint.
- Deriving the first-order conditions (FOCs) and solving the resulting system of equations to find the points that minimize the distance.

Solution

(Sketch only). The distance between $(x, y) \in \mathbb{R}^2$ and $(1, 1)$ is:

$$\sqrt{(x-1)^2 + (y-1)^2}.$$

Since the square root is an increasing function, we can solve the question by solving the following minimization problem:

$$\min (x-1)^2 + (y-1)^2 \quad \text{s.t.} \quad x^2 + y^2 = 1.$$

The objective function is continuous. The constraint set is compact. Therefore, the Extreme Value Theorem implies that the objective function has a global minimum in the constraint set. The NDCQ is satisfied by all points in the constraint set.

The Lagrangian is given by:

$$\mathcal{L}(x, y, \mu) = (x-1)^2 + (y-1)^2 - \mu(x^2 + y^2 - 1).$$

The FOCs are:

$$2(x-1) - 2\mu x = 0$$
$$2(y-1) - 2\mu y = 0$$
$$x^2 + y^2 = 1.$$

The critical points are:

$$(x_1^*, y_1^*, \mu_1^*) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1 - \sqrt{2}\right),$$
$$(x_2^*, y_2^*, \mu_2^*) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1 + \sqrt{2}\right).$$

By evaluating the objective function at both critical points, we find that (x_1^*, y_1^*) is the global minimum and solution to the problem.

Problem 8 (Extra Practice Problem): Constrained Optimization Problem

Relevance in Economics

Constrained optimization problems like this arise frequently in economics, particularly in maximizing or minimizing costs, profits, or utility while adhering to certain limitations or resource constraints. Understanding these problems is essential for economic decision-making, where multiple conditions must be satisfied simultaneously.

Applications in Economics

This type of problem is applicable in fields like production theory, where firms aim to minimize costs or maximize profits while adhering to resource constraints, or consumer theory, where individuals maximize utility subject to budgetary limits. It is a standard technique for modeling real-world decision-making problems in economics.

Problem Description

Consider the constrained optimization problem:

$$\min f(x, y) = (x-1)^2 + (y-2)^2 \quad \text{s.t.} \quad x + y = 1.$$

(a) Solve this problem.
(b) What is the value of the objective function at point $(1, 2)$? Does the point $(1, 2)$ solve the problem?

Simple Explanation

We want to minimize the distance from the point (x, y) to $(1, 2)$, but the point must also lie on the line $x + y = 1$. The task is to find the point on this line that minimizes the distance.

Key Idea

The key idea is to minimize the given objective function subject to the constraint using Lagrange multipliers. By setting up the Lagrangian function, we incorporate both the objective function and the constraint, and then derive the first-order conditions (FOCs) to find the optimal point that solves the problem.

Solution

(Sketch only)

(a) The constraint set is convex because the constraint function is affine. The objective function f is convex because the Hessian matrix of f is:

$$Hf(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

which is positive definite for all (x, y). Thus, this is a convex program. The NDCQ is satisfied for all points in the constraint set.

From the Lagrangian:

$$\mathcal{L}(x,y,\mu) = (x-1)^2 + (y-2)^2 - \mu(x+y-1),$$

we derive the FOCs:

$$2(x-1) - \mu = 0,$$

$$2(y-2) - \mu = 0,$$

$$x + y = 1.$$

The unique critical point is $(x^*, y^*, \mu^*) = (0, 1, -2)$. The unique global minimum and solution is $(x^*, y^*) = (0, 1)$. Note that $f(x^*, y^*) = f(0, 1) = 2$.

(b) The value of the objective function at $(1, 2)$ is $f(1, 2) = 0$. The point $(x, y) = (1, 2)$ is the global (unconstrained) minimum of f, but this point is not a solution of the constrained problem because it does not satisfy the restriction $x + y = 1$.

Problem 9 (Extra Practice Problem): Production Cost Minimization

Relevance in Economics

Minimizing production costs is a key concern for firms seeking to maximize profits. This type of optimization problem is central to production theory, where firms must determine the most cost-effective allocation of resources to produce a given output.

Applications in Economics

This problem is applicable to production and operations management, where firms need to determine how much of each type of product to produce in order to minimize costs while satisfying production constraints. It is also relevant in supply chain management, resource allocation, and industrial planning.

Problem Description

A factory produces two types of machinery in quantities x and y. The production costs are given by the function:

$$f(x, y) = x^2 + y^2 - xy.$$

In order to minimize the production costs, how many machines of each type should be produced if in total exactly 8 machines must be manufactured?

Simple Explanation

The factory needs to produce a total of 8 machines, but they want to minimize the cost of production. The production cost is given as a function of the number of each type of machine produced, and the goal is to find the right combination of machines that results in the lowest cost.

Key Idea

The key idea is to minimize the cost function subject to the constraint that the total number of machines produced is 8. By using the method of Lagrange multipliers, we can incorporate the constraint into the problem and solve for the optimal production quantities.

Solution Strategy

The solution involves:

- Formulating the cost function $f(x, y)$ and the constraint $x + y = 8$.
- Setting up the Lagrangian function to combine the objective (minimizing cost) and the constraint (producing exactly 8 machines).
- Solving the first-order conditions (FOCs) to find the optimal quantities x and y.

Solution

(Sketch only) We solve the minimization problem:

$$\min f(x,y) = x^2 + y^2 - xy \quad \text{s.t.} \quad x + y = 8.$$

The constraint set is convex since the constraint function is affine. The objective function f is convex because the Hessian matrix of $f(x,y)$ is:

$$Hf(x,y) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

which is positive definite for all (x,y). Thus, this is a convex program.

From the Lagrangian:

$$\mathcal{L}(x,y,\mu) = x^2 + y^2 - xy - \mu(x + y - 8),$$

we find the FOCs:

$$2x - y - \mu = 0,$$

$$2y - x - \mu = 0,$$

$$x + y = 8.$$

Solving this system admits the unique critical point $(x^*, y^*, \mu^*) = (4, 4, 4)$. Thus, $(x^*, y^*) = (4, 4)$ is the global minimum. This point is also feasible in the economically relevant problem:

$$\min f(x,y) = x^2 + y^2 - xy \quad \text{s.t.} \quad x + y = 8, \quad x \geq 0, \quad y \geq 0,$$

so it solves this problem as well. Producing four machines of each type minimizes the cost.