Mathematical Economics

ECON2050: Tutorial 3

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Tutorial Overview:
This tutorial explores various fundamental concepts in mathematical economics, focusing on the application of calculus to economic problems. The exercises cover a range of topics including the uniqueness of derivatives, the computation and interpretation of directional derivatives, the analysis of multivariable functions, and the calculation of partial derivatives from first principles. Additionally, the tutorial delves into the practical applications of these concepts in economic contexts, such as production optimization, utility maximization, and cost minimization. Each problem is carefully structured to enhance the understanding of how mathematical tools can be applied to real-world economic scenarios, providing a solid foundation for further study and application in economic analysis.

Problem 1: Uniqueness of the Derivative

Relevance & Applications in Economics

In economics, the uniqueness of the derivative is crucial for analyzing functions that represent economic relationships such as utility, cost, or production functions. The uniqueness ensures that the marginal rate of change at any given point is well-defined, allowing for precise predictions about how small changes in inputs or other variables affect the outcome.

- Cost Minimization: Firms rely on the uniqueness of the marginal cost function to determine the optimal level of production that minimizes costs.
- Consumer Theory: The uniqueness of the marginal utility function is essential for understanding consumer behavior and predicting how consumption changes in response to variations in price or income.

The Problem Description

We are working with a function f that is defined on some set D and maps to the real numbers $\mathbb{R}$. The function f is said to be differentiable at a point x_0 if there exists a number a such that we can create a specific linear (affine) function $g(x)$ that closely approximates $f(x)$ near x_0. This affine function $g(x)$ is given by:

$$g(x) = f(x_0) + a(x - x_0)$$

Here, $g(x)$ is just a straight line that touches the curve of $f(x)$ at the point x_0 and has a slope of a. The function $g(x)$ is supposed to match $f(x)$ so closely that the difference between $f(x)$ and $g(x)$ divided by $(x - x_0)$ becomes almost zero as x gets closer to x_0. In mathematical terms, this means:

$$\lim_{x \to x_0} \frac{|f(x) - g(x)|}{|x - x_0|} = 0$$

This condition essentially means that $f(x)$ and $g(x)$ behave almost the same when x is very close to x_0, and the difference between them becomes negligible.

Explanation

Imagine you're looking at a curve and you're trying to draw the best straight line that touches the curve at a particular point. The slope of that line tells you how steep the curve is at that point.
If f is differentiable at x_0, it should have exactly one tangent line at that point, with a unique slope a.
The problem asks us to prove that there cannot be two different slopes a and $\hat{a}$ that both satisfy the definition of the derivative at x_0.
To prove this, we'll show that if there were two such slopes, it would lead to a contradiction. Now, the problem is asking: Can there be two different straight lines with different slopes that both perfectly touch the curve at the same point? The answer, intuitively, is no. There can only be one best line (with one unique slope) that matches the curve exactly at that point. If you had two different lines with different slopes, it would mean that the curve has two different steepnesses at the same point, which doesn't make sense.

Solution Strategy

The strategy is to use a proof by contradiction. We will assume that two different derivatives, a and $\hat{a}$, exist and satisfy the conditions for differentiability at $x_0 = 0$. By exploring the consequences of this assumption, we will demonstrate that it leads to a contradiction, thereby proving that the derivative must be unique.

Formal Proof: Uniqueness of the Derivative

Given:

We are given a function $f : D \to \mathbb{R}$ that is differentiable at $x_0 = 0$. This means there exists a number a such that the function $g(x) = f(0) + a(x - 0) = f(0) + ax$ satisfies:

$$\lim_{x \to 0} \frac{|f(x) - g(x)|}{|x - 0|} = 0$$

Similarly, if there were another number $\hat{a}$ such that the function $\hat{g}(x) = f(0) + \hat{a}x$ satisfies:

$$\lim_{x \to 0} \frac{|f(x) - \hat{g}(x)|}{|x - 0|} = 0$$

we want to prove that a must equal $\hat{a}$.

Proof by Contradiction

1. **Assume Two Different Derivatives Exist:**
   Suppose, for contradiction, that there are two different values a and $\hat{a}$ such that both $g(x) = f(0) + ax$ and $\hat{g}(x) = f(0) + \hat{a}x$ satisfy the conditions for differentiability:

   $$\lim_{x \to 0} \frac{|f(x) - g(x)|}{|x|} = 0 \quad \text{and} \quad \lim_{x \to 0} \frac{|f(x) - \hat{g}(x)|}{|x|} = 0$$

   Assume $a \neq \hat{a}$.

2. **Consider the Difference Between the Two Affine Functions:**
   Consider the difference between the two approximating affine functions:

   $$|g(x) - \hat{g}(x)| = |(f(0) + ax) - (f(0) + \hat{a}x)| = |ax - \hat{a}x| = |x(a - \hat{a})|$$

   Simplifying, we get:

   $$|g(x) - \hat{g}(x)| = |x||a - \hat{a}|$$

3. **Examine the Behavior as x Approaches 0:**
   Divide this expression by $|x|$ to get:

   $$\frac{|g(x) - \hat{g}(x)|}{|x|} = |a - \hat{a}|$$

   Notice that this expression does not depend on x; it is a constant.

4. **Analyze the Limits:**
   For both $g(x)$ and $\hat{g}(x)$ to be valid derivatives, they must satisfy:

   $$\lim_{x \to 0} \frac{|f(x) - g(x)|}{|x|} = 0 \quad \text{and} \quad \lim_{x \to 0} \frac{|f(x) - \hat{g}(x)|}{|x|} = 0$$

   However, from the previous step, we see that:

   $$\lim_{x \to 0} \frac{|g(x) - \hat{g}(x)|}{|x|} = |a - \hat{a}|$$

   Since $a \neq \hat{a}$, $|a - \hat{a}|$ is a non-zero constant, implying that the limit is not zero. This contradicts the requirement for differentiability that the difference between the function and the affine approximation must vanish as x approaches 0.

Conclusion

The assumption that a and $\hat{a}$ are different leads to a contradiction. Therefore, $a = \hat{a}$, meaning the derivative of f at $x_0 = 0$ is unique. This concludes the proof that a function can have only one derivative at any given point.

Problem 2: Directional Derivatives

Relevance & Applications in Economics

Directional derivatives are crucial in economics, particularly when analyzing how a function, such as a production or utility function, changes in a specific direction. They allow economists to determine the rate of change of an economic variable with respect to others, which is key in optimization problems.

- Optimization Problems: Directional derivatives are used in determining the optimal direction to adjust variables in order to maximize profits or minimize costs.
- Marginal Analysis: They help in understanding how changing one economic factor while keeping others constant will impact the overall outcome.

The Problem Description

Let the function $f : \mathbb{R}^3_{++} \to \mathbb{R}$ be defined as:

$$f(x_1, x_2, x_3) = (3x_1^2 + e^{x_2})2x_3^{-1}$$

We are to compute the directional derivatives at all points (x_1, x_2, x_3), denoted by $D_v f(x_1, x_2, x_3)$, for the following directions:

1. $v = (1, 0, 0)$
2. $v = (0, 1, 0)$
3. $v = (0, 0, 1)$

Additionally, determine $D_v f(1, 2, 3)$ for $v = (0, 0, 1)$.

Explanation

Introduction: Understanding the Notation

In calculus, when we want to know how a function changes as we move in a certain direction, we use something called the directional derivative. The directional derivative is found using a mathematical tool called the gradient.

- The gradient of a function $f(x_1, x_2, x_3)$, denoted by ∇f, is a vector that points in the direction of the steepest increase of the function. It consists of the partial derivatives of f with respect to each of its variables.
- If you think of the gradient as an arrow, it shows you the direction in which the function increases the most. The length of this arrow tells you how steep the function is in that direction.

The directional derivative in the direction of a vector $v = (v_1, v_2, v_3)$ tells you how fast the function $f(x_1, x_2, x_3)$ is changing as you move in the direction of v. Mathematically, it's the dot product of the gradient ∇f and the direction vector v.

Part (a): Understanding Directional Derivatives

Imagine you have a hill, and you're standing on it. The shape of the hill is described by a function $f(x_1, x_2, x_3)$, where x_1, x_2, and x_3 are like coordinates telling you where you are on the hill.

Now, you want to know how steep the hill is in different directions. The directional derivative tells you exactly that—it tells you how much the hill goes up or down if you start walking in a specific direction from your current position.

- **Direction** $v = (1, 0, 0)$: You're moving only in the x_1 direction (east-west direction).
- **Direction** $v = (0, 1, 0)$: You're moving only in the x_2 direction (north-south direction).
- **Direction** $v = (0, 0, 1)$: You're moving only in the x_3 direction (up-down direction).

For each direction, the directional derivative will tell you how steep the hill is in that specific direction.

Part (b): Specific Directional Derivative

After understanding the general idea, you're asked to compute the directional derivative specifically at the point $(1, 2, 3)$ when moving in the direction $v = (0, 0, 1)$. This will tell you how steep the hill is at the point $(1, 2, 3)$ when moving straight up or down.

Solution Strategy

To solve this problem:

1. 1. Compute the gradient $\nabla f(x_1, x_2, x_3)$ of the function.
2. For each direction vector v, calculate the directional derivative $D_v f(x_1, x_2, x_3)$ as the dot product of the gradient and v.
3. Evaluate the directional derivative at the specified points.

Formal Solution

Part (a): Directional Derivatives at All Points

1. Compute the Gradient $\nabla f(x_1, x_2, x_3)$:

$$\nabla f(x_1, x_2, x_3) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$$

The partial derivatives are:

$$\frac{\partial f}{\partial x_1} = 6x_1, \quad \frac{\partial f}{\partial x_2} = e^{x_2} \cdot 2x_3^{-1}, \quad \frac{\partial f}{\partial x_3} = -2e^{x_2} \cdot x_3^{-2}$$

2. Calculate the Directional Derivatives:

$$D_v f(x_1, x_2, x_3) = \nabla f(x_1, x_2, x_3) \cdot v$$

For each direction:

- $v = (1, 0, 0)$:
$$D_v f(x_1, x_2, x_3) = 6x_1$$
- $v = (0, 1, 0)$:
$$D_v f(x_1, x_2, x_3) = e^{x_2} \cdot 2x_3^{-1}$$
- $v = (0, 0, 1)$:
$$D_v f(x_1, x_2, x_3) = -2e^{x_2} \cdot x_3^{-2}$$

3. **Evaluate $D_v f(1, 2, 3)$ for $v = (0, 0, 1)$:**

$$D_v f(1, 2, 3) = -\frac{2e^2}{9}$$

Conclusion

The directional derivatives tell us the rate of change of the function $f(x_1, x_2, x_3)$ as we move in the specified directions. For instance, at the point $(1, 2, 3)$, the function decreases at a rate of $-\frac{2e^2}{9}$ in the direction of $v = (0, 0, 1)$. Understanding these derivatives helps in determining the optimal direction to adjust variables to achieve desired outcomes in economic models.

Problem 3: Analyzing the Function $f(x,y)$

Relevance & Applications in Economics

Understanding the behavior of multivariable functions is critical in economics, particularly when analyzing functions that describe production, utility, or cost. These functions often depend on multiple variables, and analyzing their gradients and directional derivatives can provide insights into how small changes in inputs affect outputs or other economic variables.

- Production Functions: Economists use functions like $f(x,y)$ to model the relationship between inputs (like labor and capital) and output. Analyzing the gradient helps in understanding how changes in each input affect the overall production.
- Utility Functions: Utility functions often depend on several goods. The gradient and directional derivatives can indicate how changes in consumption patterns affect overall utility.
- Cost Functions: Firms analyze cost functions to determine how changes in production levels or input prices impact total costs.

The Problem Description

Given the function $f : D \to \mathbb{R}, (x,y) \mapsto e^{\frac{x}{y}} + \ln(x^2+1)$, where D is the largest domain $D \subseteq \mathbb{R}^2$ on which f is well-defined, we need to solve the following:

- (a) Determine the domain D.
- (b) Calculate the gradient $\nabla f(x,y)$.
- (c) Show that all points of the form $(0,y)$, $y \neq 0$, are located on the same level curve.
- (d) Analyze whether f increases or decreases at the point $(0,1)$ when moving in the direction of vector $(1,1)$.
- (e) Calculate $D_v f(0,1)$ for $v = (1,1)$ and $v' = (2,2)$, and determine the slope of the curve in $\mathbb{R}^3$ at $(x,y) = (0,1)$.

Explanation

You are given a function $f(x,y) = e^{\frac{x}{y}} + \ln(x^2+1)$. This function takes two inputs, x and y, and gives you an output. The problem involves determining the domain, calculating gradients, analyzing level curves, and determining the behavior of the function in certain directions.

Solution Strategy

To approach this problem, we will:

- (a) Determine the domain D. Identify the set of all points (x,y) for which the function $f(x,y)$ is well-defined.
- (b) Calculate the gradient $\nabla f(x,y)$ at every point (x,y). Compute the gradient $\nabla f(x,y)$ to determine the direction of the steepest ascent.
- (c) Show that all points of the form $(0,y)$, $y \neq 0$, are located on the same level curve. Show that all points of the form $(0,y)$, $y \neq 0$, lie on the same level curve.
- (d) Analyze whether f increases or decreases at the point $(0,1)$ when we move in the direction of vector $(1,1)$. Analyze the behavior of $f(x,y)$ at $(0,1)$ when moving in the direction $(1,1)$.
- (e) Calculate $D_v f(0,1)$ for $v = (1,1)$ and $v' = (2,2)$, and determine the slope of the curve in $\mathbb{R}^3$ at $(x,y) = (0,1)$. Compute the directional derivatives and determine the slope of the curve in $\mathbb{R}^3$ at $(x,y) = (0,1)$.

Formal Solution

Part (a): Determine the Domain D

The function $f(x, y) = e^{\frac{x}{y}} + \ln(x^2 + 1)$ is well-defined if both components $e^{\frac{x}{y}}$ and $\ln(x^2 + 1)$ are well-defined.
The exponential function $e^{\frac{x}{y}}$ is defined for all real numbers x and y, provided that $y \neq 0$.
The logarithm $\ln(x^2 + 1)$ is defined when $x^2 + 1 > 0$, which is always true since $x^2 + 1 \geq 1$.
Thus, the largest domain $D \subseteq \mathbb{R}^2$ is $D = \mathbb{R}^2 \setminus \{(x, y) \mid y = 0\}$.

Part (b): Calculate the Gradient

The gradient $\nabla f(x, y)$ of $f(x, y)$ is given by:

$$\nabla f(x, y) = \left(\frac{\partial f(x, y)}{\partial x}, \frac{\partial f(x, y)}{\partial y} \right)$$

Calculate the partial derivatives:

$$\frac{\partial f(x, y)}{\partial x} = \frac{1}{y} e^{\frac{x}{y}} + \frac{2x}{x^2 + 1}$$

$$\frac{\partial f(x, y)}{\partial y} = -\frac{x}{y^2} e^{\frac{x}{y}}$$

Thus, the gradient is:

$$\nabla f(x, y) = \left(\frac{1}{y} e^{\frac{x}{y}} + \frac{2x}{x^2 + 1}, -\frac{x}{y^2} e^{\frac{x}{y}} \right)$$

Part (c): Level Curves for $(0, y)$, $y \neq 0$

At any point of the form $(0, y)$, the function simplifies to:

$$f(0, y) = e^0 + \ln(0^2 + 1) = 1 + \ln(1) = 1$$

Since $f(0, y) = 1$ for any $y \neq 0$, all such points lie on the same level curve where $f(x, y) = 1$.

Part (d): Analyzing Increase or Decrease at $(0, 1)$ in the Direction $(1, 1)$

To analyze whether the function $f(x, y) = e^{\frac{x}{y}} + \ln(x^2 + 1)$ increases or decreases at the point $(0, 1)$ when moving in the direction $(1, 1)$, we need to compute the directional derivative at that point. The directional derivative provides a measure of how the function changes as we move in a specific direction, which is represented by a vector. In this case, the direction vector $v = (1, 1)$ implies that we are moving equally in both the x and y directions.

First, we recall that the directional derivative $D_v f(x, y)$ in the direction of a vector $v = (v_1, v_2)$ is given by the dot product of the gradient $\nabla f(x, y)$ and the direction vector v:

$$D_v f(x, y) = \nabla f(x, y) \cdot v$$

At the point $(0, 1)$, the gradient $\nabla f(x, y)$ simplifies as follows:

$$\nabla f(0, 1) = \left(\frac{\partial f(0, 1)}{\partial x}, \frac{\partial f(0, 1)}{\partial y} \right) = \left(\frac{1}{1} e^0 + \frac{2 \cdot 0}{0^2 + 1}, -\frac{0}{1^2} e^0 \right) = (1, 0)$$

We then calculate the directional derivative in the direction of $v = (1, 1)$:

$$D_v f(0, 1) = \nabla f(0, 1) \cdot v = (1, 0) \cdot (1, 1) = 1 \cdot 1 + 0 \cdot 1 = 1$$

The positive value of the directional derivative indicates that the function $f(x, y)$ is increasing as we move from the point $(0, 1)$ in the direction $(1, 1)$. This means that if we take a small step in this direction, the value of the function will increase.

The result is significant because it tells us the behavior of the function in a particular direction from a given point. Knowing whether a function increases or decreases in a certain direction can be crucial in optimization problems where the goal is to find the maximum or minimum value of the function.

Part (e): Directional Derivatives and Slope of the Curve

Determine $D_v f(0,1)$ for $v = (1,1)$ and for $v' = (2,2)$. Consider the curve in $\mathbb{R}^3$ consisting of all the points in $(x,y,z) \in \mathbb{R}^3 : y = x+1, z = f(x,y)$. What is the slope of this curve at $(x,y) = (0,1)$?

In this part, we extend the analysis of the directional derivatives by comparing the rates of change in different directions and using this information to determine the slope of the curve in $\mathbb{R}^3$ at the point $(x,y) = (0,1)$.
First, we calculate the directional derivatives at $(0,1)$ in the directions $v = (1,1)$ and $v' = (2,2)$:

1. **Directional Derivative in the Direction $v = (1,1)$:**

$$D_v f(0,1) = \nabla f(0,1) \cdot v = (1,0) \cdot (1,1) = 1$$

2. **Directional Derivative in the Direction $v' = (2,2)$:**

$$D_{v'} f(0,1) = \nabla f(0,1) \cdot v' = (1,0) \cdot (2,2) = 2$$

The larger directional derivative in the direction $v' = (2,2)$ compared to $v = (1,1)$ indicates that the function increases more rapidly in the direction v'.

Next, we consider the curve in $\mathbb{R}^3$ defined by the equation $z = f(x,y)$ and the constraint $y = x+1$. The slope of this curve at $(x,y) = (0,1)$ is found by taking the derivative of $z = f(x,y)$ with respect to x along the curve, which involves considering the directional derivative in the direction tangent to the curve.
Since the constraint $y = x+1$ gives us a direction vector $(1,1)$ in the x-y plane, the slope $\frac{dz}{dx}$ is calculated as:

$$\text{slope} = D_{(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}})} f(0,1) = \frac{D_{(1,1)} f(0,1)}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}$$

This slope represents the rate at which the function $f(x,y)$ changes as we move along the line $y = x+1$ from the point $(0,1)$. Understanding the slope helps in visualizing the steepness of the curve in three-dimensional space and is essential for analyzing the behavior of multivariable functions, particularly in economic models where such curves might represent trade-offs between different variables.
The curve described inthe question is the curve in $\mathbb{R}^3$ that runs "on" the graph of f and above the line $y = x+1$ in the $x - y$ plane. We can obtain its slope by computing the directional derivative of f at $(0,1)$ "in the direction of the line $y = x+1$."

Conclusion

In summary, the directional derivative analysis shows that the function $f(x,y)$ increases when moving from the point $(0,1)$ in the direction $(1,1)$. Additionally, the slope calculation provides insight into how the function behaves along a specific curve in $\mathbb{R}^3$. These results are particularly useful in economic contexts where the goal is to understand how changes in variables impact outcomes, allowing for better decision-making in areas like production, cost, and utility optimization.

Problem 4: Calculating a Partial Derivative from First Principles

Relevance & Applications in Economics

Partial derivatives play a crucial role in economics by allowing us to understand how a small change in one variable affects the outcome while keeping other variables constant. This concept is particularly useful in various economic analyses, including:

- Marginal Analysis: Economists use partial derivatives to calculate marginal cost, marginal utility, and marginal revenue, which are essential for firms to determine optimal production levels and pricing strategies.
- Optimization Problems: Partial derivatives help in identifying the rate of change in utility, profit, or cost with respect to one variable, aiding in optimization of resources and decision-making.
- Economic Modelling: In economic models, partial derivatives are used to assess the sensitivity of economic outcomes to changes in parameters, such as prices, income, or technology.

The Problem Description

We are given the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \frac{xy}{x^2+y^2+1}$. The goal is to determine the partial derivative $\frac{\partial f(1,1)}{\partial x}$ directly from the definition of the partial derivative, which involves calculating the appropriate limit.

Explanation

In this problem, we deal with a function $f(x, y) = \frac{xy}{x^2+y^2+1}$ that depends on two inputs, x and y. The task is to find the partial derivative of this function with respect to x at the point $(1, 1)$. A partial derivative measures how the function changes as x changes slightly, keeping y constant. This type of analysis is essential for understanding marginal effects in economics, where small changes in inputs or variables can have significant impacts on outputs or outcomes.

Solution Strategy

To solve this problem, we will:

1. Step 1: Calculate the function value at the point $(1, 1)$ to establish a base reference for our calculations.
2. Step 2: Calculate the function value at the point $(1 + h, 1)$, where h is a small perturbation in x.
3. Step 3: Subtract the base value from the perturbed value to determine the change in the function due to the change in x.
4. Step 4: Divide this difference by h to form the difference quotient, representing the average rate of change.
5. Step 5: Take the limit as h approaches zero to obtain the exact value of the partial derivative.

Formal Solution

Calculating $\frac{\partial f(1,1)}{\partial x}$ from the Definition

The partial derivative of $f(x, y)$ with respect to x at the point $(1, 1)$ is given by the following limit:

$$\frac{\partial f(1,1)}{\partial x} = \lim_{h \to 0} \frac{f(1+h,1) - f(1,1)}{h}$$

Step 1: Calculate $f(1, 1)$:
First, calculate the value of the function at the point $(1, 1)$:

$$f(1,1) = \frac{1 \cdot 1}{1^2 + 1^2 + 1} = \frac{1}{3}$$

Step 2: Calculate $f(1 + h, 1)$:
Next, calculate the value of the function at the point $(1 + h, 1)$:

$$f(1+h,1) = \frac{(1+h) \cdot 1}{(1+h)^2 + 1^2 + 1} = \frac{1+h}{(1+h)^2 + 2}$$

Expand and simplify the denominator:

$$(1+h)^2 + 2 = 1 + 2h + h^2 + 2 = h^2 + 2h + 3$$

So:

$$f(1+h,1) = \frac{1+h}{h^2 + 2h + 3}$$

Step 3: Compute the Difference:
Now, compute the difference between the function values:

$$f(1+h,1) - f(1,1) = \frac{1+h}{h^2 + 2h + 3} - \frac{1}{3}$$

To combine these into a single fraction:

$$f(1+h,1) - f(1,1) = \frac{3(1+h) - (h^2 + 2h + 3)}{3(h^2 + 2h + 3)}$$

Simplify the numerator:

$$3(1+h) - (h^2 + 2h + 3) = 3 + 3h - h^2 - 2h - 3 = h - h^2$$

So:

$$f(1+h,1) - f(1,1) = \frac{h - h^2}{3(h^2 + 2h + 3)}$$

Step 4: Divide by h:
Now, divide the difference by h:

$$\frac{f(1+h,1) - f(1,1)}{h} = \frac{h - h^2}{3h(h^2 + 2h + 3)} = \frac{1 - h}{3(h^2 + 2h + 3)}$$

Step 5: Take the Limit as h Approaches 0:
Finally, take the limit as h approaches 0 to find the partial derivative:

$$\lim_{h \to 0} \frac{1 - h}{3(h^2 + 2h + 3)} = \frac{1}{9}$$

Therefore, the partial derivative is:

$$\frac{\partial f(1,1)}{\partial x} = \frac{1}{9}$$

This concludes the formal proof of the partial derivative $\frac{\partial f(1,1)}{\partial x}$. ■

Problem 5: Analyzing the Production Function

Relevance & Applications in Economics

Production functions are fundamental in economics as they describe the relationship between inputs (such as capital and labor) and output. Understanding how changes in these inputs affect output is crucial for decision-making in firms and for economic policy. The concepts applied here are relevant in the following areas:

- Cost-Benefit Analysis: Firms use production functions to determine the most efficient combination of inputs to maximize output or minimize costs.
- Marginal Productivity: By analyzing partial derivatives, firms can determine the marginal productivity of each input, guiding decisions on resource allocation.
- Economic Growth Models: Production functions are also used in macroeconomics to model economic growth, where capital and labor are key inputs determining the overall output of an economy.

The Problem Description

We are given the production function $f : \mathbb{R}^2_+ \to \mathbb{R}$, defined by $f(K, L) = 4K^{3/4}L^{1/4}$, where K represents capital, and L represents labor. The firm currently uses 10,000 units of capital ($K = 10,000$) and employs 625 workers ($L = 625$). The task is to determine how the output $f(K, L)$ changes when both the capital and labor increase by one unit.

Explanation

The production function $f(K, L) = 4K^{3/4}L^{1/4}$ describes how output depends on two inputs: capital K and labor L. The firm's management is interested in understanding how a small increase in both inputs affects the output. To answer this, we calculate the partial derivatives of the function with respect to K and L. These derivatives represent the marginal increase in output when either capital or labor is increased by one unit while holding the other input constant.

Solution Strategy

To analyze the change in output, we will:

1. Step 1: Calculate the partial derivatives of the production function with respect to both K and L.
2. Step 2: Evaluate these partial derivatives at the given point $(K, L) = (10,000, 625)$ to find the marginal products of capital and labor.
3. Step 3: Add the contributions from both inputs to approximate the total change in output when both capital and labor increase by one unit.

Formal Solution

Calculating the Partial Derivatives

The production function is given by:

$$f(K, L) = 4K^{3/4}L^{1/4}$$

Step 1: Partial Derivative with Respect to K

$$\frac{\partial f(K, L)}{\partial K} = 4 \cdot \frac{3}{4}K^{3/4-1}L^{1/4} = 3K^{-1/4}L^{1/4}$$

Step 2: Partial Derivative with Respect to L

$$\frac{\partial f(K, L)}{\partial L} = 4 \cdot \frac{1}{4}K^{3/4}L^{1/4-1} = K^{3/4}L^{-3/4}$$

Evaluating the Partial Derivatives at $(K, L) = (10, 000, 625)$

Next, evaluate these partial derivatives at the given point:

$$\frac{\partial f(10, 000, 625)}{\partial K} = 3 \cdot (10, 000)^{-1/4} \cdot (625)^{1/4}$$

$$\frac{\partial f(10, 000, 625)}{\partial L} = (10, 000)^{3/4} \cdot (625)^{-3/4}$$

Now, simplify these expressions:

$$10, 000^{-1/4} = \left(10^4\right)^{-1/4} = 10^{-1} = 0.1$$

$$625^{1/4} = \left(5^4\right)^{1/4} = 5$$

Thus:

$$\frac{\partial f(10, 000, 625)}{\partial K} = 3 \cdot 0.1 \cdot 5 = 1.5$$

For the partial derivative with respect to L:

$$10, 000^{3/4} = \left(10^4\right)^{3/4} = 10^3 = 1, 000$$

$$625^{-3/4} = \left(5^4\right)^{-3/4} = 5^{-3} = \frac{1}{125}$$

Thus:

$$\frac{\partial f(10, 000, 625)}{\partial L} = 1, 000 \cdot \frac{1}{125} = 8$$

Total Change in Production Δf

The total change in production Δf when both capital and labor increase by one unit can be approximated by:

$$\Delta f \approx \frac{\partial f(10, 000, 625)}{\partial K} \cdot \Delta K + \frac{\partial f(10, 000, 625)}{\partial L} \cdot \Delta L$$

Since $\Delta K = 1$ and $\Delta L = 1$:

$$\Delta f \approx 1.5 \cdot 1 + 8 \cdot 1 = 1.5 + 8 = 9.5$$

$\therefore$ When both capital and labor increase by one unit, the output is expected to increase by approximately 9.5 units.

Conclusion

The analysis of the production function reveals that small increases in both capital and labor result in a significant increase in output. This insight is valuable for the firm's management in making informed decisions regarding resource allocation and investment in capital and labor. By understanding the marginal contributions of each input, the firm can optimize its production process to achieve desired output levels efficiently.

Problem 6: Calculating the Direction of Maximum Increase

Relevance & Applications in Economics

In economics, understanding the direction of maximum increase in functions is essential for optimization problems. For example, firms may want to know the direction in which increasing certain inputs will lead to the greatest increase in output or profit. The gradient and its unit vector help in determining the most efficient direction for resource allocation, pricing strategies, and other economic decisions.

- Optimization in Production: By calculating the gradient of a production function, firms can determine how to adjust inputs like labor and capital to maximize output efficiently.
- Utility Maximization: In consumer theory, gradients help in identifying the direction in which a consumer can increase utility by reallocating their consumption bundle.
- Cost Minimization: Firms use gradients to find the direction in which they can decrease costs most rapidly, given changes in input prices or quantities.

The Problem Description

We are given two functions:

(a) $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \ln(x^2 + y^2 + 1)$

(b) $f : \mathbb{R}^2_{++} \to \mathbb{R}, (x, y) \mapsto \sqrt{xy + y^3}$

The task is to find the direction v with $\|v\| = 1$ in which the function increases most rapidly at the point $(1, 1)$. This involves calculating the gradient and then normalizing it to obtain the unit vector that represents the direction of maximum increase.

Explanation

In this problem, we are given two functions and asked to find the direction in which the function increases most rapidly at a specific point. The direction in which a function increases most rapidly is given by the gradient of the function at that point.
Gradient and Direction of Maximum Increase: The gradient of a function is a vector that points in the direction of the steepest increase. If you're standing on a hill, the gradient would tell you which way to walk to go uphill the fastest. The length of the gradient vector tells you how steep the hill is in that direction.

Norm and Unit Vector:

Norm: The norm of a vector is a measure of its length or magnitude. It's similar to the absolute value that you might be familiar with in the context of single numbers. The absolute value of a number gives you the distance of that number from zero. For instance: The absolute value of 3 is $|3| = 3$, and the absolute value of -3 is $|-3| = 3$. Both numbers are 3 units away from zero. When we move from single numbers to vectors (which have multiple components, like coordinates), the concept of distance from zero extends to the idea of a norm:
For a vector $\mathbf{v} = (v_1, v_2)$ in two dimensions, its norm (or length) is calculated using the formula $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$. This is similar to how the distance from the origin to a point (v_1, v_2) is found using the Pythagorean theorem.
Unit Vector: The problem also asks that the direction vector v has a norm (length) of 1, which means it's a unit vector. We will calculate the gradient at the given point and then find the unit vector that points in the same direction as the gradient.
Normalized Vector: When we talk about a "unit vector," we mean a vector that has been scaled to have a norm of 1. This is called normalizing the vector. The direction of the vector remains the same, but its length becomes exactly 1. To normalize a vector $\mathbf{v}$, we divide each component of the vector by its norm:

$$\mathbf{v}_{\text{unit}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

This new vector $\mathbf{v}_{\text{unit}}$ has the same direction as the original vector $\mathbf{v}$, but with a length of 1. In this problem, we are asked to find the gradient of the function at a specific point and then determine the unit vector that points in the direction of the gradient. This unit vector will tell us the direction in which the function increases most rapidly.

Solution Strategy

To solve this problem, we will:

1. Step 1: Calculate the gradient $\nabla f(x, y)$ of each function at the point $(1, 1)$.
2. Step 2: Normalize the gradient to obtain the unit vector v that points in the direction of maximum increase.
3. Step 3: Verify that the unit vector v satisfies the condition $\|v\| = 1$, ensuring it is a direction vector with a norm of 1.

Formal Solution

Part (a): $f(x, y) = \ln(x^2 + y^2 + 1)$

First, compute the gradient $\nabla f(x, y)$ of $f(x, y)$:

$$\text{Gradient: } \nabla f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}\right)$$

At the point $(1, 1)$:

$$\nabla f(1, 1) = \left(\frac{2 \cdot 1}{1^2 + 1^2 + 1}, \frac{2 \cdot 1}{1^2 + 1^2 + 1}\right) = \left(\frac{2}{3}, \frac{2}{3}\right)$$

Next, normalize the gradient to get the unit vector v:

$$\text{Norm of the Gradient: } \|\nabla f(1, 1)\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \frac{2\sqrt{2}}{3}$$

$$\text{Unit Vector: } v = \frac{\nabla f(1, 1)}{\|\nabla f(1, 1)\|} = \frac{\left(\frac{2}{3}, \frac{2}{3}\right)}{\frac{2\sqrt{2}}{3}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Thus, the direction v of maximum increase at $(1, 1)$ is:

$$v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Part (b): $f(x, y) = \sqrt{xy + y^3}$

First, compute the gradient $\nabla f(x, y)$ of $f(x, y)$:

$$\text{Gradient: } \nabla f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{y}{2\sqrt{xy + y^3}}, \frac{x + 3y^2}{2\sqrt{xy + y^3}}\right)$$

At the point $(1, 1)$:

$$\nabla f(1, 1) = \left(\frac{1}{2\sqrt{1 + 1}}, \frac{1 + 3 \cdot 1^2}{2\sqrt{1 + 1}}\right) = \left(\frac{1}{2\sqrt{2}}, \frac{4}{2\sqrt{2}}\right) = \left(\frac{1}{2\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$$

Next, normalize the gradient to get the unit vector v:

$$\text{Norm of the Gradient: } \|\nabla f(1, 1)\| = \sqrt{\left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{2}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{8} + 2} = \sqrt{\frac{17}{8}}$$

$$\text{Unit Vector: } v = \frac{\nabla f(1, 1)}{\|\nabla f(1, 1)\|} = \frac{\left(\frac{1}{2\sqrt{2}}, \frac{2}{\sqrt{2}}\right)}{\sqrt{\frac{17}{8}}} = \left(\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$$

Thus, the direction v of maximum increase at $(1, 1)$ is:

$$v = \left(\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$$

Conclusion

In conclusion, the direction of maximum increase for the functions at the given point was determined by calculating the gradient and normalizing it to obtain a unit vector. The results provide the direction in which the function's value increases most rapidly from the point $(1, 1)$. This method is particularly useful in economics for finding optimal directions for resource allocation, maximizing production, or minimizing costs.