Mathematical Economics

ECON2050: Tutorial 10

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Introduction

This tutorial covers a series of optimization problems that are fundamental to economic theory. We will apply mathematical techniques such as Lagrange multipliers and the Kuhn-Tucker theorem to solve constrained optimization problems. These methods are commonly used in economics to maximize utility, minimize costs, and optimize resource allocation under given constraints.

Topics Covered

- Logarithmic utility functions and their applications
- Quadratic optimization problems
- Nonlinear programming and the Kuhn-Tucker theorem
- Utility maximization with budget constraints

Objectives

- Apply Lagrange multipliers to solve constrained maximization and minimization problems.
- Understand the role of the Kuhn-Tucker theorem in nonlinear optimization.
- Explore how small changes in constraints affect the optimal value in optimization problems.

Expected Outcomes

By the end of this tutorial, students should be able to:

- Solve constrained optimization problems using Lagrange multipliers.
- Apply the Kuhn-Tucker conditions to solve nonlinear programming problems.
- Evaluate the sensitivity of optimal solutions to changes in constraints.

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Problem 1: Maximizing $f(x, y) = \ln x + \ln y$

Relevance in Economics

Optimization problems involving logarithmic utility functions are essential in economics, particularly in consumer theory and production functions. In cases where agents need to maximize their utility or firms aim to optimize production under resource constraints, these functions model diminishing returns to scale effectively.

Applications in Economics

Logarithmic utility functions are often used in models of consumer behavior, where agents derive satisfaction (utility) from consuming goods under budget constraints. These functions are also applied in Cobb-Douglas production functions to represent how firms allocate resources between inputs like labor and capital to maximize output while facing constraints like production capacity.

Problem Description

Solve the following constrained optimization problem:

$$\max_{x,y\in\mathbb{R}_{++}} \ln x + \ln y \quad \text{subject to} \quad 3x + 2y = 72.$$

Without solving a new problem, calculate the approximate change in the maximum value if the independent term of the constraint is changed to 72.5.

Simple Explanation

The objective is to maximize the function $f(x, y) = \ln x + \ln y$, which represents the total utility derived from consuming two goods x and y. The constraint $3x + 2y = 72$ implies that there is a resource limit on how much of these goods can be consumed. The goal is to find the values of x and y that maximize utility, given the resource constraint.

Key Idea

The key idea is to use Lagrange multipliers to solve the constrained maximization problem. The logarithmic form of the utility function implies diminishing returns to consumption, while the constraint enforces a budget-like limitation. After solving the problem, we will explore how a small change in the constraint affects the maximum utility.

Solution Strategy

(a) Constrained Maximization:
To solve the constrained maximization problem, we first set up the Lagrangian:

$$\mathcal{L}(x, y, \mu) = \ln x + \ln y - \mu(3x + 2y - 72).$$

The first-order conditions (FOCs) are derived as:

$$\frac{\partial \mathcal{L}(x, y, \mu)}{\partial x} = \frac{1}{x} - 3\mu = 0,$$

$$\frac{\partial \mathcal{L}(x, y, \mu)}{\partial y} = \frac{1}{y} - 2\mu = 0,$$

$$\frac{\partial \mathcal{L}(x, y, \mu)}{\partial \mu} = 3x + 2y - 72 = 0.$$

From the first two equations, we have:

$$\mu = \frac{1}{3x} = \frac{1}{2y}.$$

This implies $3x = 2y$. Using this in the constraint equation $3x + 2y = 72$, we substitute $2y = 3x$ to find:

$$3x + 3x = 72 \quad \Rightarrow \quad 6x = 72 \quad \Rightarrow \quad x = 12.$$

Substitute $x = 12$ into $2y = 3x$ to get:

$$2y = 36 \quad \Rightarrow \quad y = 18.$$

Thus, the critical point is $(x^*, y^*) = (12, 18)$ and the Lagrange multiplier is $\mu^* = \frac{1}{36}$.

(b) Impact of Changing the Constraint:
We are asked to compute the approximate change in the maximum value of the objective function if the independent term of the constraint is changed from 72 to 72.5. The sensitivity of the optimal value to changes in the constraint is given by the Lagrange multiplier μ^*.
The approximate change in the maximum value is:

$$\Delta f \approx \mu^*(72.5 - 72) = \frac{1}{36} \times 0.5 = \frac{1}{72}.$$

Thus, the maximum value of the function will increase by approximately $\frac{1}{72}$ when the constraint is relaxed slightly.
Conclusion: The optimal point is $(12, 18)$, and a small increase in the constraint (from 72 to 72.5) increases the maximum value of the function by approximately $\frac{1}{72}$.

Problem 2: Minimizing and Maximizing Quadratic Functions

Relevance in Economics

Minimization and maximization problems involving quadratic functions are important in economics, especially in cost minimization or utility maximization. These problems often arise when firms need to allocate resources optimally while facing certain constraints.

Applications in Economics

Quadratic optimization is crucial in production theory, where firms aim to minimize costs subject to input constraints or maximize profits given limited resources. It is also used in portfolio optimization, where investors need to maximize returns or minimize risk while adhering to budget or risk limits.

Problem Description

Solve the following optimization problems:

- (a) $\min 4x^2 + 5y^2 - 6y \quad \text{subject to} \quad x + 2y \geq 18$
- (b) $\max 16x^2 + 12y - 2x^2 - 3y^2 - 6 \quad \text{subject to} \quad x + y \leq 11$

Simple Explanation

The goal is to optimize quadratic functions (minimize in (a) and maximize in (b)) while considering the constraints. These problems involve quadratic forms, and the constraints place limits on the possible values of x and y.

Key Idea

The key idea is to use the method of Lagrange multipliers to solve these constrained optimization problems. By setting up the Lagrangian function for each problem, we can incorporate the constraints and find the critical points that optimize the functions.

Solution Strategy

(a) Minimization:
We are asked to minimize the function $f(x, y) = 4x^2 + 5y^2 - 6y$ subject to the constraint $x + 2y \geq 18$. To solve this, we can instead maximize:

$$\max -4x^2 - 5y^2 + 6y \quad \text{subject to} \quad -x - 2y \leq -18.$$

The NDCQ is satisfied at all points of the constraint set, as the gradient $(-1, -2)$ of the constraint function is not equal to $(0, 0)$ for all (x, y). The Lagrangian function is:

$$\mathcal{L}(x, y, \lambda) = -4x^2 - 5y^2 + 6y - \lambda(-x - 2y + 18).$$

The first-order conditions (FOCs) are:

$$\frac{\partial \mathcal{L}(x, y, \lambda)}{\partial x} = -8x + \lambda = 0,$$
$$\frac{\partial \mathcal{L}(x, y, \lambda)}{\partial y} = -10y + 6 + 2\lambda = 0,$$
$$\frac{\partial \mathcal{L}(x, y, \lambda)}{\partial \lambda} = x + 2y - 18 = 0.$$

From these equations, we find $\lambda = 8x$ and $\lambda = 5y - 3$. By equating the two, we obtain:

$$8x = 5y - 3.$$

Solving this system with the constraint $x + 2y = 18$, we find the critical solution $(x^*, y^*) = (4, 7)$.
The objective function $f(x, y) = 4x^2 + 5y^2 - 6y$ is strictly convex, meaning that the critical point is a global minimum. Thus, the optimal solution is $(x^*, y^*) = (4, 7)$ and the optimal value is:

$$f(4, 7) = 267.$$

(b) Maximization:
We are asked to maximize $f(x, y) = 16x^2 + 12y - 2x^2 - 3y^2 - 6$ subject to the constraint $x + y \leq 11$. The NDCQ is satisfied at all points of the constraint set, as the gradient $(1, 1)$ of the constraint function is not equal to $(0, 0)$ for all (x, y). The Lagrangian function is:

$$\mathcal{L}(x, y, \lambda) = 16x^2 + 12y - 2x^2 - 3y^2 - 6 - \lambda(x + y - 11).$$

The first-order conditions (FOCs) are:

$$\frac{\partial \mathcal{L}(x, y, \lambda)}{\partial x} = 16 - 4x - \lambda = 0,$$
$$\frac{\partial \mathcal{L}(x, y, \lambda)}{\partial y} = 12 - 6y - \lambda = 0,$$
$$\frac{\partial \mathcal{L}(x, y, \lambda)}{\partial \lambda} = x + y - 11 = 0.$$

From these equations, if $\lambda > 0$, then $x + y = 11$ by the last equation. Also:

$$16 - 4x = \lambda, \quad 12 - 6y = \lambda.$$

Substituting λ from one equation into the other, we solve for $(x^*, y^*) = (4, 7)$.
Since the objective function $f(x, y)$ is strictly concave (the Hessian matrix is negative definite) and the constraint set is convex, the unique global maximizer is $(x^*, y^*) = (4, 7)$, and the optimal value is:

$$f(4, 7) = 267.$$

Problem 3: Nonlinear Programming and the Kuhn-Tucker Theorem

Relevance in Economics

Nonlinear programming is a common tool used in economics for solving optimization problems that involve non-linear objective functions. Such problems appear frequently in production optimization, where firms maximize profits or minimize costs subject to nonlinear constraints.

Applications in Economics

This type of nonlinear programming is often applied in economic models where resources or production processes follow non-linear patterns. Nonlinear constraints also emerge in utility maximization and environmental economics when firms or individuals operate under natural or technological limits.

Problem Description

Consider the following problem of nonlinear programming:

$$\max 2x - 2y^2 \quad \text{s.t.} \quad 2x - y \leq 5, \quad y \leq 4, \quad x \geq 0, \quad y \geq 0.$$

Solve the following:

- (a) Try to solve the problem graphically.
- (b) Solve the problem by applying the Kuhn-Tucker theorem.

Simple Explanation

The objective is to maximize the function $2x - 2y^2$, subject to several constraints on x and y, including inequality constraints $2x - y \leq 5$, $y \leq 4$, and non-negativity constraints $x \geq 0$ and $y \geq 0$. Graphically, this problem involves finding the point on the feasible region where the objective function achieves its highest value.

Key Idea

The key idea is to first attempt a graphical solution by plotting the feasible region and identifying the point where the objective function attains its maximum value. Then, we apply the Kuhn-Tucker theorem, which provides necessary conditions for an optimum in constrained optimization problems like this one.

Solution Strategy

(a) Graphical Solution:
The level curve of the objective function of level C is defined by the equation:

$$x = y^2 + \frac{C}{2}.$$

You have to draw quite precisely to see the exact solution, which is $(x^*, y^*) = (2.625, 0.25)$.

(b) Kuhn-Tucker Theorem:
The NDCQ is satisfied at all relevant points since:

- If exactly one constraint binds, the gradient of the constraint function of this binding constraint is not equal to $(0, 0)$.
- If exactly two constraints bind, the corresponding Jacobian has rank two, ensuring linear independence of the gradients.
- At most two constraints bind simultaneously.

We set up the Kuhn-Tucker Lagrangian:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2) = 2x - 2y^2 + \lambda_1(2x - y - 5) + \lambda_2(4 - y).$$

From the Kuhn-Tucker conditions, we derive the FOCs:

$$\frac{\partial \mathcal{L}(x, y, \lambda_1, \lambda_2)}{\partial x} = 2 - 2\lambda_1 = 0 \quad (1),$$
$$\frac{\partial \mathcal{L}(x, y, \lambda_1, \lambda_2)}{\partial y} = -4y - \lambda_1 + \lambda_2 = 0 \quad (2),$$
$$\frac{\partial \mathcal{L}(x, y, \lambda_1, \lambda_2)}{\partial \lambda_1} = (2x - y - 5) \geq 0 \quad (3),$$
$$\frac{\partial \mathcal{L}(x, y, \lambda_1, \lambda_2)}{\partial \lambda_2} = (y - 4) \geq 0 \quad (4),$$
$$\lambda_1(2x - y - 5) = 0 \quad (5),$$
$$\lambda_2(y - 4) = 0 \quad (6).$$

From (1), we have $\lambda_1 = 1$. From (6), we see that $\lambda_2 > 0$, so $y = 4$. Substituting into (2), we obtain:

$$-4(4) - 1 + \lambda_2 = 0 \quad \Rightarrow \quad \lambda_2 = 15.$$

Using (5), we get:

$$2x - 4 - 5 = 0 \quad \Rightarrow \quad x = 2.625.$$

Thus, the optimal solution is $(x^*, y^*) = (2.625, 0.25)$.
The only point satisfying the FOCs is $(x^*, y^*, \lambda_1^*, \lambda_2^*) = (2.625, 0.25, 1, 0)$. Since the objective function is continuous and the constraint set is compact, the Extreme Value Theorem applies, and we conclude that $(x^*, y^*) = (2.625, 0.25)$ is the global constrained maximum.

Problem 4: Maximizing a Nonlinear Objective Function

Relevance in Economics

Maximizing nonlinear objective functions subject to constraints is a common problem in economics. These types of problems arise in various optimization scenarios, such as utility maximization, profit maximization, or cost minimization, where economic agents must operate under certain limitations.

Applications in Economics

Nonlinear optimization with constraints can be found in production theory, where firms aim to maximize output given resource constraints, or in consumer theory, where consumers maximize utility given a budget. These types of problems also appear in environmental economics and other fields where non-linear systems are subject to natural or technological constraints.

Problem Description

Consider maximizing an objective function $f : \mathbb{R}^2 \to \mathbb{R}$ subject to the constraints:

$$x^2 + y^2 - 8y \geq 0, \quad y \geq (x-1)^2 + 1, \quad x \geq 0, \quad y \geq 0.$$

Solve the following:

- (a) Draw the constraint set in a figure.
- (b) Determine all points that are candidates to be a solution to this problem because they violate the NDCQ.

(Useful fact: The only solution (x, y) with $y \geq 0$ to the system of equations:

$$x^2 + y^2 - 8y = 0, \quad y = (x-1)^2 + 1$$

is $(x, y) = (2, 2)$.)

Simple Explanation

The goal is to maximize the objective function $f(x, y)$ subject to the given nonlinear constraints. In part (a), we will graph the constraints to identify the feasible region. In part (b), we will determine the points that violate the necessary conditions for a solution, as defined by the Nonlinear Domain Condition Qualification (NDCQ).

Key Idea

The key idea is to graph the constraint set to understand the feasible region where the solution might lie. We then check the candidates for solutions based on the violation of the NDCQ, focusing on points where the Jacobian matrix of the constraints does not have full rank.

Solution Strategy

(a) Drawing the Constraint Set: The constraint set is defined by the following inequalities:

$$x^2 + y^2 - 8y \geq 0 \quad \text{and} \quad y \geq (x-1)^2 + 1.$$

By plotting these two curves, we obtain the region of feasible points C, as shown in the diagram. The region C is bounded by the parabola $y = (x-1)^2 + 1$ and the circle $x^2 + y^2 - 8y = 0$. **(b) Candidates for Solutions**

Violating the NDCQ: The constraint functions and their gradients are:

$$g_1(x, y) = x^2 + y^2 - 8y, \quad \nabla g_1(x, y) = (2x, 2y - 8),$$

$$g_2(x, y) = (x-1)^2 + 1 - y, \quad \nabla g_2(x, y) = (2(x-1), -1),$$

$$g_3(x, y) = x, \quad \nabla g_3(x, y) = (1, 0),$$

$$g_4(x, y) = y, \quad \nabla g_4(x, y) = (0, 1).$$

A point will be a candidate solution if the Jacobian matrix of the constraints that bind at this point does not have full rank. We check the NDCQ at all points where at least some constraints bind. The NDCQ is violated if the gradients of the binding constraints are linearly dependent. We distinguish the following cases:

- **Exactly one constraint binds:** If $g_1(x, y) = 0$ or $g_2(x, y) = 0$ and the other constraints do not bind, the NDCQ is violated at the origin $(0, 0)$.
- **Exactly two constraints bind:** If two constraints bind, the corresponding Jacobian must have rank 1, meaning the NDCQ is violated. At $(x, y) = (2, 2)$, both $g_1(x, y)$ and $g_2(x, y)$ bind, violating the NDCQ. Additionally, we find that:

  $$\nabla g_1(2, 2) = (4, -4), \quad \nabla g_2(2, 2) = (2, -1),$$

  which are linearly dependent, implying that the NDCQ is violated at $(x, y) = (2, 2)$.
- **All four constraints bind:** This case is not possible since $x \geq 0$ and $y \geq 0$ never bind at the same time as g_1 and g_2.

In summary, an external point (x, y) of the constraint set is a candidate solution because it violates the NDCQ if and only if $x^2 + y^2 - 8y = 0$ and $y = (x-1)^2 + 1$. The point $(2, 2)$ is a candidate solution because it violates the NDCQ, and $(x, y) = (2, 2)$ is the unique solution to this problem.

Problem 5: Minimization of a Quadratic Function

Relevance in Economics

Quadratic minimization problems arise in various economic contexts, particularly in cost minimization and portfolio optimization. These types of optimization problems are fundamental for firms aiming to reduce production costs or for investors seeking to minimize risk under certain constraints.

Applications in Economics

This type of quadratic minimization is particularly useful in production economics, where firms seek to minimize costs while meeting production constraints. Similarly, it applies to financial economics, where an investor may want to minimize risk while ensuring that the sum of investment weights equals a specified target.

Problem Description

Solve the following minimization problem:

$$\min_{x,y\in\mathbb{R}} x^2 + 2y \quad \text{s.t.} \quad x + y = 1,$$

and estimate the optimal value of f when the constraint is substituted with $4y = 3 - 4x$.

Simple Explanation

The objective is to minimize the quadratic function $f(x, y) = x^2 + 2y$, subject to the constraint $x + y = 1$. Once we solve this, we will explore how the optimal value of f changes when the constraint is modified to $4y = 3 - 4x$.

Key Idea

The key idea is to use the method of Lagrange multipliers to solve this constrained optimization problem. By setting up the Lagrangian function, we can incorporate the constraint into the problem and solve for the values of x and y that minimize $f(x, y)$. We will then analyze how changing the constraint impacts the minimum value of the function.

Solution Strategy

We begin by setting up the Lagrangian:

$$\mathcal{L}(x, y, \mu) = x^2 + 2y - \mu(x + y - 1).$$

The first-order conditions (FOCs) are:

$$\frac{\partial \mathcal{L}(x, y, \mu)}{\partial x} = 2x - \mu = 0,$$

$$\frac{\partial \mathcal{L}(x, y, \mu)}{\partial y} = 2 - \mu = 0,$$

$$\frac{\partial \mathcal{L}(x, y, \mu)}{\partial \mu} = x + y - 1 = 0.$$

From the second equation, we find that $\mu = 2$. Substituting this into the first equation, we get:

$$2x - 2 = 0 \quad \Rightarrow \quad x = 1.$$

Using the constraint $x + y = 1$, we substitute $x = 1$ to find:

$$1 + y = 1 \quad \Rightarrow \quad y = 0.$$

Thus, the critical point is $(x^*, y^*, \mu^*) = (1, 0, 2)$.
This problem is a convex program, as the constraint $x+y = 1$ is linear and the objective function $f(x, y) = x^2+2y$ is convex. The Hessian matrix of $f(x, y)$ is:

$$Hf(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

which is positive semidefinite for all $(x, y) \in \mathbb{R}^2$. Hence, $(x^*, y^*) = (1, 0)$ is the global constrained minimum, and the optimal value of f is:

$$f(x^*, y^*) = 1^2 + 2(0) = 1.$$

Impact of Changing the Constraint:
Now, we estimate the optimal value of f when the constraint is replaced with $4y = 3 - 4x$, which simplifies to $x + y = 0.75$.
The change in the independent term of the constraint from 1 to 0.75 corresponds to $\Delta b = 0.75 - 1 = -0.25$. The approximate new value of f is given by:

$$f(x^*, y^*) + \mu^* \Delta b = 1 + 2(-0.25) = 0.5.$$

Thus, the approximate new value of f is 0.5 when the constraint is modified to $4y = 3 - 4x$.

Problem 6: Maximizing Utility with a Budget Constraint

Relevance in Economics

Utility maximization is a fundamental concept in microeconomics, where consumers seek to maximize their satisfaction (utility) given their budget constraints. This problem provides an example of how consumers allocate their resources to purchase different goods while maximizing utility.

Applications in Economics

The utility maximization problem is widely applicable in consumer theory. It allows economists to understand how consumers make decisions about spending their income on various goods and services. Firms can also use such optimization models to determine how to price goods based on consumer preferences and constraints.

Problem Description

Let $u : \mathbb{R}^2_+ \to \mathbb{R}, (x, y) \mapsto \sqrt{xy}$ be a consumer's utility function, where x and y represent the quantities of two goods available to the consumer. The per unit prices of the goods are \$5 (for X) and \$10 (for Y), respectively, and the consumer has \$500 available. Formulate and solve the optimization problem of the consumer.

Simple Explanation

The consumer aims to maximize their utility, represented by $u(x, y) = \sqrt{xy}$, while ensuring that their total spending does not exceed their budget of \$500. The price of X is \$5, and the price of Y is \$10. The goal is to determine the optimal quantities x and y that the consumer should purchase to maximize their utility while adhering to the budget constraint.

Key Idea

The key idea is to set up and solve the optimization problem using Lagrange multipliers. We will first express the problem in terms of a more manageable objective function, then solve the constrained maximization problem. By finding the critical points, we will determine the quantities x and y that maximize the consumer's utility.

Solution Strategy

The consumer's problem is:

$$\max_{(x,y)\in\mathbb{R}^2_+} \sqrt{xy} \quad \text{s.t.} \quad 5x+10y \leq 500, \quad x \geq 0, \quad y \geq 0.$$

Since $\sqrt{\cdot} : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, the solutions to the consumer's problem coincide with those of:

$$\max_{(x,y)\in\mathbb{R}^2_+} xy \quad \text{s.t.} \quad 5x+10y \leq 500, \quad x \geq 0, \quad y \geq 0.$$

To solve this, we use the Lagrangian function:

$$\mathcal{L}(x,y,\lambda) = xy - \lambda(5x+10y-500).$$

The first-order conditions (FOCs) are:

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x} = y - 5\lambda = 0,$$
$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} = x - 10\lambda = 0,$$
$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda} = -(5x+10y-500) = 0.$$

From the first two equations, we solve for λ:

$$\lambda = \frac{y}{5} \quad \text{and} \quad \lambda = \frac{x}{10}.$$

Equating these two, we get:

$$\frac{y}{5} = \frac{x}{10} \quad \Rightarrow \quad 2y = x.$$

Substituting this into the budget constraint $5x+10y=500$, we find:

$$5x + 10\left(\frac{x}{2}\right) = 500 \quad \Rightarrow \quad 10x = 500 \quad \Rightarrow \quad x = 50.$$

Using $x = 50$, we find y from $2y = x$:

$$2y = 50 \quad \Rightarrow \quad y = 25.$$

Thus, the optimal solution is $(x^*, y^*) = (50, 25)$.

Conclusion

Since the constraint set (the budget set) is compact and the objective function $u(x,y) = \sqrt{xy}$ is continuous, the Extreme Value Theorem applies. From $x^* \cdot y^* = 1250$, we conclude that $(x^*, y^*) = (50, 25)$ solves the consumer's problem.