Mathematical Economics
ECON2050: Tutorial 4

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Tutorial Overview:

This tutorial delves into the essential mathematical tools used in economic analysis, focusing on the application of calculus in understanding and solving economic problems. The problems presented cover a range of topics including the computation of partial derivatives using the implicit function theorem, finding tangent lines to level curves, second-order Taylor approximations, and calculating the Hessian matrix. Additionally, the tutorial explores the concepts of total derivatives and their significance in economic modeling. By working through these exercises, students will gain a deeper understanding of how these mathematical concepts are applied in optimization problems, comparative statics, and other areas of economic analysis, providing a strong foundation for further study and practical application in economics.

Problem 1: Partial Derivatives Using Implicit Function Theorem

Relevance in Economics

Partial Derivatives: Partial derivatives are crucial in economics as they measure how one variable affects another, holding all other variables constant. This is fundamental in understanding marginal changes in economics, such as marginal cost, marginal revenue, and marginal utility.
Implicit Function Theorem: The implicit function theorem is used in economics to study equilibrium analysis where certain relationships are defined implicitly rather than explicitly. For instance, in a market with multiple goods, prices and quantities may be related implicitly through demand and supply functions.

Applications

Utility Maximization: Determining how a consumer's utility changes with respect to changes in goods while keeping their budget constraint satisfied.
Cost Functions: Understanding how a firm's cost changes with respect to output levels while satisfying production constraints.

The Problem

Suppose that it follows from the implicit function theorem that, in a neighborhood of (x_0, y_0, z_0), z is implicitly defined by the equation $ze^x + e^y - ye^z = 0$ as a function of x and y. Compute the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point (x_0, y_0).

Implicit Function Theorem

Theorem (Implicit Function Theorem). Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a continuously differentiable function, and let $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$ be such that $F(\mathbf{a}, \mathbf{b}) = \mathbf{0}$. Suppose that the Jacobian matrix of partial derivatives of F with respect to the y-variables (the $\mathbb{R}^m$ component), denoted by $D_y F(\mathbf{a}, \mathbf{b})$, is invertible. Then there exists an open set $U \subseteq \mathbb{R}^n$ containing $\mathbf{a}$, an open set $V \subseteq \mathbb{R}^m$ containing $\mathbf{b}$, and a unique continuously differentiable function $\varphi : U \to V$ such that for all $\mathbf{x} \in U$,

$$F(\mathbf{x}, \varphi(\mathbf{x})) = \mathbf{0}.$$

Moreover, the derivative of φ at $\mathbf{a}$ is given by

$$D\varphi(\mathbf{a}) = -\left[D_y F(\mathbf{a}, \mathbf{b})\right]^{-1} D_x F(\mathbf{a}, \mathbf{b}),$$

where $D_x F(\mathbf{a}, \mathbf{b})$ is the Jacobian matrix of partial derivatives of F with respect to the x-variables (the $\mathbb{R}^n$ component).

Breakdown

This problem asks us to find how the variable z, defined implicitly by the given equation, changes with respect to x and y. These changes are measured by the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Essentially, we're trying to understand how z behaves as x or y varies slightly, while still satisfying the original equation.
The implicit function theorem helps us differentiate functions where one variable is not explicitly solved in terms of the others. Here, z is defined implicitly by the equation $ze^x + e^y - ye^z = 0$. We use this theorem to find the derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, which tell us how z changes with x and y.

Solution Strategy

The key steps involve differentiating the given equation with respect to x and y while treating z as a function of x and y. This differentiation yields expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Formal Solution

1. Differentiation with Respect to x:

$$\frac{\partial f(x_0, y_0, z_0)}{\partial x}\left(ze^x + e^y - ye^z\right) = z_0 e^{x_0}$$

2. Differentiation with Respect to y:

$$\frac{\partial f(x_0, y_0, z_0)}{\partial y}\left(ze^x + e^y - ye^z\right) = e^{y_0} - e^{z_0}$$

3. Differentiation with Respect to z:

$$\frac{\partial f(x_0, y_0, z_0)}{\partial z}\left(ze^x + e^y - ye^z\right) = e^{x_0} - y_0 e^{z_0}$$

We must have $\frac{\partial f(x_0,y_0,z_0)}{\partial z} \neq 0$ otherwise the implicit function would not apply. Using the implicit function thereom

$$\frac{\partial z(x_0, y_0)}{\partial x} = -\frac{\frac{\partial f(x_0,y_0,z_0)}{\partial x}}{\frac{\partial f(x_0,y_0,z_0)}{\partial z}} = -\frac{z_0 e^{x_0}}{e^{x_0} - y_0 e^{z_0}}$$

$$\frac{\partial z(x_0, y_0)}{\partial y} = -\frac{\frac{\partial f(x_0,y_0,z_0)}{\partial y}}{\frac{\partial f(x_0,y_0,z_0)}{\partial z}} = -\frac{e^{z_0} - e^{y_0}}{e^{x_0} - y_0 e^{z_0}}$$

Conclusion

Using the implicit function theorem, we found that:

$$\frac{\partial z}{\partial x} = -\frac{z_0 e^{x_0}}{e^{x_0} - y_0 e^{z_0}}, \frac{\partial z}{\partial y} = -\frac{e^{z_0} - e^{y_0}}{e^{x_0} - y_0 e^{z_0}}$$

These expressions represent how z changes as x and y vary, given that $ze^x + e^y - ye^z = 0$ holds.

Problem 2: Tangent Lines to Level Curves

Relevance in Economics

Tangent Lines: In economics, the tangent line to a level curve (such as an indifference curve or isoquant) provides important information about the rate of substitution between goods or inputs. The slope of the tangent line is often interpreted as the marginal rate of substitution (MRS) or the marginal rate of technical substitution (MRTS).

Applications

Indifference Curves: Analyzing how consumers make choices between two goods while maintaining the same level of utility.

Production Theory: Determining how firms can substitute between labor and capital while maintaining the same level of output.

The Problem

Obtain an equation of the tangent line to the level curve of $f : \mathbb{R}^2 \to \mathbb{R}$ at the points:

1. $(1, 1)$ if $f(x, y) = xe^{xy}$,
2. $(-1, 2)$ if $f(x, y) = y - x^2$,
3. $(1, 2)$ if $f(x, y) = yx$.

Breakdown

This problem asks us to find the equations of tangent lines to specific level curves of different functions at given points. A tangent line to a curve at a point is the straight line that just touches the curve at that point, matching its slope. The level curve is a set of points where the function has the same value, and we want to know the equation of the line that is tangent to this curve at the specified point.
To find the tangent line, we need to calculate the gradient of the function $f(x, y)$ at the given point. The gradient vector points in the direction of the steepest increase of the function and is perpendicular to the level curve at the point of interest. The equation of the tangent line can then be derived using this gradient.

Solution Strategy

For each part of the problem, we will:

1. Compute the gradient $\nabla f(x, y)$ of the function.
2. Evaluate the gradient at the given point.
3. Use the point-slope form of the equation of a line to derive the tangent line.

Formal Solution

1. **Part (a):** $f(x,y) = xe^{xy}$ **at** $(1,1)$
   Compute the gradient:

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial x} = e^{xy} + xye^{xy}, \quad \frac{\partial f}{\partial y} = x^2 e^{xy}$$

Evaluate at $(1,1)$:

$$\nabla f(1,1) = (e+e, e) = (2e, e)$$

Tangent line equation:

$$2e(x-1) + e(y-1) = 0 \quad \Rightarrow \quad 2ex + ey = 3e$$

2. **Part (b):** $f(x,y) = y - x^2$ **at** $(-1,2)$
   Compute the gradient:

$$\frac{\partial f}{\partial x} = -2x, \quad \frac{\partial f}{\partial y} = 1$$

Evaluate at $(-1,2)$:

$$\nabla f(-1,2) = (2,1)$$

Tangent line equation:

$$2(x+1) + (y-2) = 0 \quad \Rightarrow \quad 2x + y = 0$$

3. **Part (c):** $f(x,y) = yx$ **at** $(1,2)$
   Compute the gradient:

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x$$

Evaluate at $(1,2)$:

$$\nabla f(1,2) = (2,1)$$

Tangent line equation:

$$2(x-1) + 1(y-2) = 0 \quad \Rightarrow \quad 2x + y = 4$$

Rearrange to make y the subject of the equation.

Conclusion

For each function, we found the gradient vector and used it to derive the equation of the tangent line at the specified point. The equations of the tangent lines are:

1. $2ex + ey = 3e$ for $f(x,y) = xe^{xy}$ at $(1,1)$.
2. $2x + y = 0$ for $f(x,y) = y - x^2$ at $(-1,2)$.
3. $2x + y = 4$ for $f(x,y) = yx$ at $(1,2)$.

Problem 3: Second-Order Taylor Approximation

Relevance in Economics

Taylor Approximation: Taylor approximations are used to simplify complex functions, making them easier to analyze. In economics, second-order Taylor expansions are used to approximate functions like utility, production, or cost functions around a given point, allowing for easier computation of marginal values and elasticity.

Applications

Optimization Problems: Simplifying objective functions in optimization problems, such as profit maximization or cost minimization, to make them easier to solve analytically.
Macroeconomic Models: Approximating dynamic equations in macroeconomic models to analyze stability and predict future states.

The Problem

Determine the second-order Taylor approximation of $f : \mathbb{R}^2_{++} \to \mathbb{R}$, $(x, y) \mapsto \ln(3x + 2y)$ around the point $a = (2, 1)$ in polynomial form.

Breakdown

This problem asks us to approximate the function $f(x, y) = \ln(3x+2y)$ near the point $(2, 1)$ using a quadratic polynomial. The Taylor series provides a way to approximate a function around a specific point by summing the derivatives of the function at that point. The second-order Taylor approximation uses up to the second derivatives, resulting in a quadratic polynomial.
The Taylor approximation is a method used to approximate the value of a function near a given point. The second-order approximation involves the function's value, its first derivatives, and its second derivatives at the point. This allows us to create a quadratic polynomial that approximates the function near the point $(2, 1)$.

Why Use Taylor Approximation?

Taylor approximation is particularly useful because it allows us to approximate complex functions using polynomials, which are easier to work with. Polynomials are simple to differentiate and integrate, and their behavior near the point of expansion closely mirrors that of the original function. This approximation is especially valuable in optimization problems, numerical analysis, and when making predictions based on local behavior. By using the second-order Taylor approximation, we can capture not just the slope (first derivative) but also the curvature (second derivative) of the function, providing a more accurate approximation near the point of interest.

Solution Strategy

To find the second-order Taylor approximation:

1. Compute the first and second partial derivatives of $f(x, y)$.
2. Evaluate these derivatives at the point $(2, 1)$.
3. Use the Taylor series formula to write the quadratic approximation.

Formal Solution

1. Compute the First-Order Partial Derivatives:

$$f(x, y) = \ln(3x + 2y)$$

The first-order partial derivatives are:

$$\frac{\partial f}{\partial x} = \frac{3}{3x + 2y}, \quad \frac{\partial f}{\partial y} = \frac{2}{3x + 2y}$$

Evaluating at the point $(2, 1)$:

$$\frac{\partial f}{\partial x}(2, 1) = \frac{3}{8}, \quad \frac{\partial f}{\partial y}(2, 1) = \frac{2}{8} = \frac{1}{4}$$

2. Compute the Second-Order Partial Derivatives: The second-order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = -\frac{9}{(3x + 2y)^2}, \quad \frac{\partial^2 f}{\partial y^2} = -\frac{4}{(3x + 2y)^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = -\frac{6}{(3x + 2y)^2}$$

Evaluating at the point $(2, 1)$:

$$\frac{\partial^2 f}{\partial x^2}(2, 1) = -\frac{9}{64}, \quad \frac{\partial^2 f}{\partial y^2}(2, 1) = -\frac{4}{64} = -\frac{1}{16}, \quad \frac{\partial^2 f}{\partial x \partial y}(2, 1) = -\frac{6}{64} = -\frac{3}{32}$$

3. Construct the Second-Order Taylor Polynomial: The second-order Taylor series expansion around the point $(2, 1)$ using the Hessian matrix is:

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a}) H f(\mathbf{a})(\mathbf{x} - \mathbf{a})^T$$

$$f(x, y) \approx f(2, 1) + \nabla f(2, 1) \cdot \begin{pmatrix} x - 2 \\ y - 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x - 2 \\ y - 1 \end{pmatrix} H(f(2, 1)) \begin{pmatrix} x - 2 \\ y - 1 \end{pmatrix}$$

Substituting the values:

$$f(x, y) \approx \ln(8) + \frac{3}{8}(x - 2) + \frac{1}{4}(y - 1) + \frac{1}{2} \begin{pmatrix} x - 2 \\ y - 1 \end{pmatrix} \begin{pmatrix} -\frac{9}{64} & -\frac{3}{32} \\ -\frac{3}{32} & -\frac{1}{16} \end{pmatrix} \begin{pmatrix} x - 2, y - 1 \end{pmatrix}$$

$$= \ln(8) + \frac{3}{8}(x - 2) + \frac{1}{4}(y - 1) + \frac{1}{2} \cdot \left(\frac{-9}{64}(x - 2)^2 + 2 \cdot \frac{-6}{64}(y - 1)(x - 2) + \frac{4}{64}(y - 1)^2 \right)$$

$$= \frac{-9}{128}x^2 - \frac{3}{32}xy - \frac{1}{32}y^2 + \frac{3}{4}x + \frac{1}{2}y + \ln(8) - \frac{3}{2}$$

Conclusion

The second-order Taylor approximation of the function $f(x, y) = \ln(3x + 2y)$ around the point $(2, 1)$ is:

$$f(x, y) \approx \frac{-9}{128}x^2 - \frac{3}{32}xy - \frac{1}{32}y^2 + \frac{3}{4}x + \frac{1}{2}y + \ln(8) - \frac{3}{2}$$

This quadratic polynomial provides a good approximation of the function near the point $(2, 1)$, capturing both the slope and curvature of the function at that point.

Problem 4: Total Derivative Using Implicit Function Theorem

Relevance in Economics

Total Derivative: The total derivative is important in economics when changes in one variable indirectly cause changes in another through an implicit relationship. For example, changes in income could lead to changes in consumption patterns, even if the relationship isn't explicit.

Applications

Comparative Statics: In comparative statics, economists use total derivatives to understand how an economic equilibrium changes in response to changes in parameters like prices, taxes, or technology.
Sensitivity Analysis: Evaluating how sensitive an economic outcome is to changes in exogenous factors, which is important for policy analysis and forecasting.

The Problem

Suppose that it follows from the implicit function theorem that, in a neighborhood of (x_0, y_0, z_0), z is implicitly defined by the equation $x^2 + y^2 + z^2 = a^2$ as a function of x and y. Compute the total derivative $Dz(x_0, y_0)$.

Breakdown

This problem asks us to find the total derivative of the variable z, which is implicitly defined by the equation $x^2+y^2+z^2 = a^2$. The total derivative tells us how z changes with respect to both x and y combined. This derivative is crucial for understanding the overall behavior of z when both x and y vary simultaneously.
The total derivative combines the effects of changes in x and y on z. Here, z is not explicitly solved in terms of x and y, but we can still find out how z changes when x and y change by using the implicit function theorem. The theorem provides a way to differentiate the equation $x^2 + y^2 + z^2 = a^2$ with respect to x and y to find the total derivative of z.

Solution Strategy

To find the total derivative:

1. Differentiate the given equation implicitly with respect to x and y.
2. Solve for dz in terms of dx and dy to find the total derivative $Dz(x_0, y_0)$.

Formal Solution

1. Given Equation:

$$x^2 + y^2 + z^2 = a^2$$

Here, z is implicitly defined as a function of x and y. We need to differentiate this equation to find how z changes with x and y.

2. Differentiate with Respect to x and y: Differentiating both sides with respect to x and y:

$$\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + \frac{\partial}{\partial y}(x^2 + y^2 + z^2) = 0$$

Applying the chain rule:

$$2x\,dx + 2y\,dy + 2z\,dz = 0$$

3. Solve for dz: Rearrange the terms to solve for dz:

$$2z\,dz = -2x\,dx - 2y\,dy$$

$$dz = -\frac{x\,dx + y\,dy}{z}$$

This equation represents the total derivative $Dz(x_0, y_0)$.

Conclusion

The total derivative $Dz(x_0, y_0)$ is given by:

$$Dz(x_0, y_0) = \left(\frac{-x_0}{z_0}, \frac{-y_0}{z_0}\right)$$

This derivative tells us how the variable z changes when both x and y change, considering the constraint $x^2 + y^2 + z^2 = a^2$.

Problem 5: Hessian Matrix Calculation

Relevance in Economics

Hessian Matrix: The Hessian matrix is used in determining the nature of critical points (i.e., maxima, minima, or saddle points) in multivariable functions. This is vital for optimization problems in economics, such as maximizing profit or utility, or minimizing costs.

Applications

Optimization: The Hessian is used in optimization problems to confirm whether a solution provides a maximum, minimum, or saddle point. For example, firms might use this to verify whether their current production levels are cost-efficient.

Econometric Analysis: In econometrics, the Hessian is used to estimate the curvature of likelihood functions, which is important for understanding the confidence intervals of estimated parameters.

The Problem

Calculate the Hessian matrix of each of the following functions:

1. $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto xe^{xy}$
2. $f : \mathbb{R}^2_{++} \to \mathbb{R}$, $(x, y) \mapsto \ln(xy + y^2 + x^2)$
3. $f : \mathbb{R}^3 \setminus \{(x, y, z) \in \mathbb{R}^3 : x = 0 \text{ or } y = 0\} \to \mathbb{R}$, $(x, y, z) \mapsto \frac{1+xz}{yx^2}$
4. $f : \{(x, y) \in \mathbb{R}^2_+ : xy \leq 3\} \to \mathbb{R}$, $(x, y) \mapsto \sqrt{3x - x^2y}$

Breakdown

This problem requires us to compute the Hessian matrix for a series of functions. The Hessian matrix is a square matrix consisting of all the second-order partial derivatives of a function. It is a powerful tool in multivariable calculus, especially in optimization, as it provides insight into the curvature of the function. The entries in the Hessian matrix tell us how the slope of the function changes in different directions.

The Hessian matrix gives us a detailed picture of how a function behaves around a certain point. Each element of the matrix represents how the function's slope changes with respect to changes in the variables. By calculating the Hessian, we can understand whether a function is concave, convex, or has any saddle points at that location.

Solution Strategy

To find the Hessian matrix for each function:

1. Compute the first-order partial derivatives.
2. Compute the second-order partial derivatives for each pair of variables.
3. Arrange these second-order derivatives into the Hessian matrix.

Formal Solution

1. Part (a): $f(x,y) = xe^{xy}$

First-order partial derivatives:

$$\frac{\partial f}{\partial x} = e^{xy} + xye^{xy}, \quad \frac{\partial f}{\partial y} = x^2 e^{xy}$$

Second-order partial derivatives:

Hessian matrix:

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial x \partial y} \end{pmatrix} = e^{xy} \begin{pmatrix} 2y + xy^2 & 2x + x^2 y \\ 2x + x^2 y & x^3 \end{pmatrix}$$

2. Part (b): $f(x,y) = \ln(xy + y^2 + x^2)$

First-order partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{2x + y}{xy + y^2 + x^2}, \quad \frac{\partial f}{\partial y} = \frac{2y + x}{xy + y^2 + x^2}$$

Second-order partial derivatives: Hessian matrix:

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

3. **Part (c):** $f(x, y, z) = \frac{1+xz}{yx^2}$

First-order partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{-2 - xz}{x^3 y}, \quad \frac{\partial f}{\partial y} = \frac{-1 - xz}{y^2 x^2}, \quad \frac{\partial f}{\partial z} = \frac{1}{xy}$$

Second-order partial derivatives: Hessian matrix:

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

4. **Part (d):** $f(x, y) = \sqrt{3x - x^2 y}$
   First-order partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{3 - 2xy}{2\sqrt{3x - x^2 y}}, \quad \frac{\partial f}{\partial y} = \frac{-x^2}{2\sqrt{3x - x^2 y}}$$

Second-order partial derivatives: Hessian matrix:

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Conclusion

The Hessian matrix for each function has been calculated as shown above. These matrices give insight into the curvature of the functions at any given point and are essential in understanding the nature of the critical points, such as whether they are minima, maxima, or saddle points.

Problem 6: Gradient, Directional Derivative, and Hessian

Relevance in Economics

Gradient: The gradient vector is used to find the direction of the steepest ascent or descent in a function. In economics, this is relevant in optimization problems where agents want to maximize utility or profit.
Directional Derivative: This helps in understanding how a function changes in a particular direction, which is crucial in constrained optimization problems where direction matters (e.g., budget constraints).
Hessian Matrix: The Hessian matrix is particularly important in assessing the convexity or concavity of functions, helping determine whether a solution is a maximum, minimum, or saddle point.

Applications

Utility Maximization: Understanding how a consumer's utility changes as they adjust consumption of goods, given a certain budget.
Cost Minimization: Firms can use the Hessian to evaluate cost functions and make decisions about input levels that minimize costs.
Market Equilibrium Analysis: Determining the stability of equilibria in markets by analyzing the curvature of excess demand functions.

The Problem

Consider $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$, $(x, y) \mapsto \frac{x^2 y}{x^2+y^2}$.

1. Calculate the gradient of f.
2. Calculate the directional derivative of f at the point $(1, 1)$ in the direction of the vector $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.
3. Calculate the Hessian matrix of f.

Breakdown

This problem involves finding the gradient, directional derivative, and Hessian matrix for a function of two variables. The gradient is a vector that points in the direction of the steepest increase of the function. The directional derivative measures the rate of change of the function in a specific direction. The Hessian matrix, which consists of second-order partial derivatives, provides insight into the curvature of the function.

1. Gradient: The gradient tells us how the function changes most rapidly and in which direction. It is composed of the partial derivatives with respect to each variable.
2. Directional Derivative: This is a measure of how the function changes as we move in a specific direction, which is not necessarily aligned with the gradient.
3. Hessian Matrix: The Hessian provides a deeper understanding of the function's behavior by showing how the gradient itself changes as the variables change, indicating concavity, convexity, or saddle points.

Solution Strategy

To solve each part of the problem:

1. Calculate the partial derivatives to find the gradient.
2. Use the gradient to compute the directional derivative in the given direction.
3. Compute the second-order partial derivatives to construct the Hessian matrix.

Formal Solution

1. Part (a): Calculate the Gradient of $f(x,y) = \frac{x^2y}{x^2+y^2}$

- First, compute the partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{2xy(x^2+y^2) - x^2y(2x)}{(x^2+y^2)^2} = \frac{2xy(x^2+y^2-x^2)}{(x^2+y^2)^2} = \frac{2xyy^2}{(x^2+y^2)^2} = \frac{2xy^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^2(x^2+y^2) - x^2y(2y)}{(x^2+y^2)^2} = \frac{x^2(x^2+y^2-2y^2)}{(x^2+y^2)^2} = \frac{x^2(x^2-y^2)}{(x^2+y^2)^2}$$

- The gradient is then:

$$\nabla f(x,y) = \left(\frac{2xy^3}{(x^2+y^2)^2}, \frac{x^2(x^2-y^2)}{(x^2+y^2)^2}\right)$$

2. Part (b): Calculate the Directional Derivative at $(1,1)$ in the Direction $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

- First, evaluate the gradient at $(1,1)$:

$$\nabla f(1,1) = \left(\frac{2(1)(1^3)}{(1^2+1^2)^2}, \frac{1^2(1^2-1^2)}{(1^2+1^2)^2}\right) = \left(\frac{2}{4}, 0\right) = \left(\frac{1}{2}, 0\right)$$

- The directional derivative is given by:

$$D_{\mathbf{u}}f(1,1) = \nabla f(1,1) \cdot \mathbf{u} = \left(\frac{1}{2}, 0\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}$$

3. Part (c): Calculate the Hessian Matrix of $f(x,y) = \frac{x^2y}{x^2+y^2}$

- Compute the second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2xy^3}{(x^2+y^2)^2}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x^2(x^2-y^2)}{(x^2+y^2)^2}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{2xy^3}{(x^2+y^2)^2}\right)$$

- The Hessian matrix $H(f)$ is:

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Conclusion

For the function $f(x,y) = \frac{x^2y}{x^2+y^2}$, we have calculated:

1. The gradient $\nabla f(x,y) = \left(\frac{2xy^3}{(x^2+y^2)^2}, \frac{x^2(x^2-y^2)}{(x^2+y^2)^2}\right)$.
2. The directional derivative at $(1,1)$ in the direction $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is $\frac{1}{2\sqrt{2}}$.
3. The Hessian matrix $H(f)$ consists of the second-order partial derivatives, offering insights into the function's curvature and critical points.

Conclusion

The mathematical concepts explored in these problems are not just abstract tools but have practical applications in economics. Understanding partial derivatives, Taylor approximations, the total derivative, and the Hessian matrix enables economists to analyze, model, and solve complex economic problems, thereby aiding in decision-making processes that impact everything from individual consumer choices to global economic policies.