Mathematical Economics

ECON2050: Tutorial 8

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Introduction

In this tutorial, we will encounter production functions and optimization problems where the domain is not open. This is a practical issue often faced in real-world economic models. In such cases, the domain being restricted can impact the results, and some theorems may not directly apply. We will explore whether we can still solve such problems and maximize firm profits despite these constraints.

Topics Covered

- Production functions
- Optimization of inputs (capital and labor)
- Profit maximization in firms
- Open and closed domain functions in economics

Objectives

- Apply production functions to determine the optimal quantities of capital and labor.
- Understand how to maximize firm profits given constraints on inputs.
- Discuss the importance of domains in economic optimization problems.

Expected Outcomes

By the end of this tutorial, students should be able to:

- Analyze and solve for optimal inputs in production functions.
- Maximize firm profits using given input prices and product prices.
- Understand the role of domains in applied economics problems.

Problem 1: Profit Maximization Using a Production Function

Relevance in Economics

Production functions are essential in modeling how firms convert inputs (capital and labor) into outputs. The goal of firms is typically to maximize their profits by choosing the right amount of inputs based on their costs and the revenue generated by selling the outputs.

Applications in Economics

- **Firm Profit Maximization:** Firms optimize their use of inputs to maximize profits.
- **Cost-Benefit Analysis:** Understanding how changes in input costs affect profit margins.

Problem Description

The production function $Q : \mathbb{R}^2_+ \to \mathbb{R}_+$ is given by:

$$Q(K, L) \mapsto 8K^{\frac{1}{4}}L^{\frac{1}{2}}$$

where K represents capital and L represents labor. The good produced by the firm can be sold for \$4 per unit, and the prices of a unit of capital and a unit of labor are $8and4$, respectively. The objective is to calculate the levels of capital and labor that maximize the firm's profits and to find the corresponding volume of production.

Solution

We define the profit function $\pi : \mathbb{R}^2_+ \to \mathbb{R}$ as follows:

$$\pi(K, L) = 4Q(K, L) - 8K - 4L = 32K^{\frac{1}{4}}L^{\frac{1}{2}} - 8K - 4L$$

First-Order Conditions

The first-order conditions are obtained by differentiating π with respect to K and L and setting them equal to zero:

$$\frac{\partial \pi}{\partial K} = \frac{1}{4} \cdot 32K^{-\frac{3}{4}}L^{\frac{1}{2}} - 8 = 0 \quad \text{and} \quad \frac{\partial \pi}{\partial L} = \frac{1}{2} \cdot 32K^{\frac{1}{4}}L^{-\frac{1}{2}} - 4 = 0$$

These simplify to:

$$K^{-\frac{3}{4}}L^{\frac{1}{2}} = 1 \quad \text{and} \quad K^{\frac{1}{4}}L^{-\frac{1}{2}} = 1$$

From these equations, we deduce that:

$$K^{-\frac{3}{4}}L^{\frac{1}{2}} = 4K^{\frac{1}{4}}L^{-\frac{1}{2}} \quad \text{or} \quad 4K = L$$

Substituting $L = 4K$ into $K^{-\frac{3}{4}}L^{\frac{1}{2}} = 1$ gives $K = 16$. Therefore, $L = 64$.

Hessian Matrix and Second-Order Conditions

Next, we compute the Hessian of π to verify whether the critical point is a maximum. The Hessian matrix is given by:

$$H_\pi(K, L) = \begin{pmatrix} -6K^{-\frac{3}{4}}L^{\frac{1}{2}} & 4K^{-\frac{3}{4}}L^{-\frac{1}{2}} \\ 4K^{-\frac{3}{4}}L^{-\frac{1}{2}} & -8K^{\frac{1}{4}}L^{-\frac{3}{2}} \end{pmatrix}$$

The leading principal minors of $H_\pi(K, L)$ are:

$$D_1 = -6K^{-\frac{3}{4}}L^{\frac{1}{2}}, \quad D_2 = 32K^{-1}L^{-1}$$

Since $D_1 < 0$ and $D_2 > 0$ for all $(K, L) \in \mathbb{R}^2_+$, $H_\pi(K, L)$ is negative definite, and thus $\pi(K, L)$ is concave. Therefore, $(K, L) = (16, 64)$ is a global maximum of π on $\mathbb{R}^2_+$.

Conclusion

The optimal level of capital and labor is $K = 16$ units of capital and $L = 64$ units of labor. The corresponding volume of production is:

$$Q(16, 64) = 8(16)^{\frac{1}{4}}(64)^{\frac{1}{2}} = 8 \cdot 2 \cdot 8 = 128$$

Thus, the firm produces 128 units, and the maximum profit is:

$$\pi(16, 64) = 4 \times 128 - 8 \times 16 - 4 \times 64 = 512 - 128 - 256 = 128$$

Problem 2: Maximizing Artisan's Profits

Relevance in Economics

This problem demonstrates how artisans or small-scale producers can maximize their profits by determining the optimal amounts of labor and material inputs. Similar to large firms, small producers must balance input costs to achieve profit maximization.

Applications in Economics

- **Profit Maximization:** This is a classic problem in microeconomics where a producer seeks to optimize profits by choosing input quantities.
- **Cost Management:** Understanding how changes in input factors like labor and material affect overall profitability.

The Problem

An artisan uses x_h units of labor and x_m units of a unique material to produce a product. Her profits, employing $(x_h, x_m) \geq 0$ units of the factors, are given by:

$$\pi(x_h, x_m) = 20x_h + 26x_m + 4x_h x_m - 4x_h^2 - 3x_m^2$$

Find the values of x_h and x_m that maximize the artisan's profits.

Simple Explanation

We are given a profit formula for an artisan that tells us how much profit she can make depending on how much labor and material she uses. The goal is to figure out the best amounts of labor and material to maximize her profits.

Key Ideas

We need to:

- Write the profit function based on the inputs (labor and material).
- Take derivatives with respect to both inputs to find the critical points where profit might be maximized.
- Analyze the second derivatives to confirm whether these points are maxima.

Solution Strategy

We first write the partial derivatives of the profit function with respect to both x_h and x_m:

$$\frac{\partial \pi(x_h, x_m)}{\partial x_h} = 20 + 4x_m - 8x_h = 0$$

$$\frac{\partial \pi(x_h, x_m)}{\partial x_m} = 26 + 4x_h - 6x_m = 0$$

Solving these two equations will give the critical points for x_h and x_m.

Formal Solution

The system of equations is:

$$20 + 4x_m - 8x_h = 0 \quad (1)$$
$$26 + 4x_h - 6x_m = 0 \quad (2)$$

From (1), we can rearrange:

$$4x_m = 8x_h - 20 \implies x_m = 2x_h - 5$$

Substitute this into (2):

$$26 + 4x_h - 6(2x_h - 5) = 0 \implies 26 + 4x_h - 12x_h + 30 = 0 \implies -8x_h + 56 = 0 \implies x_h = 7$$

Now, substitute $x_h = 7$ into $x_m = 2x_h - 5$:

$$x_m = 2(7) - 5 = 9$$

Thus, the critical point is $(x_h^*, x_m^*) = (7, 9)$.

Second Derivatives and Concavity

Next, we compute the second partial derivatives to form the Hessian matrix:

$$\frac{\partial^2 \pi}{\partial x_h^2} = -8, \quad \frac{\partial^2 \pi}{\partial x_m^2} = -6, \quad \frac{\partial^2 \pi}{\partial x_h \partial x_m} = 4$$

The Hessian matrix is:

$$H_\pi(x_h, x_m) = \begin{pmatrix} -8 & 4 \\ 4 & -6 \end{pmatrix}$$

The leading principal minors are:

$$D_1 = -8 < 0, \quad D_2 = \det(H) = (-8)(-6) - 4^2 = 48 - 16 = 32 > 0$$

Since the leading principal minors have alternating signs, the Hessian is negative definite, meaning the function is concave. Therefore, $(x_h^*, x_m^*) = (7, 9)$ is a global maximum of the profit function.

Conclusion

The artisan maximizes her profits by using $x_h^* = 7$ units of labor and $x_m^* = 9$ units of material. This combination of inputs yields the highest profit.

Problem 3: Monopolist's Profit Maximization

Relevance in Economics

The study of monopolistic markets is central to understanding how firms with pricing power optimize their production levels and maximize profits. The inverse demand function and cost function provide key insights into the quantities to produce and the prices to set.

Applications in Economics

- **Monopoly Pricing:** Monopolists have the power to influence the market price through the quantities they produce.
- **Cost Management:** Firms analyze cost structures to ensure that their production choices lead to profit maximization.

The Problem

Suppose that the inverse demand functions for x and y are given by:

$$p(x) = 16 - x^2 \quad \text{and} \quad q(y) = 9 - y^2,$$

and that the cost function of a monopolist is given by:

$$C(x, y) = x^2 + 3y^2.$$

Determine the quantities x and y, and the prices p and q, that maximize the profits π of the monopolist, and calculate the maximum profits.

Layman's Explanation

We are given the demand functions for two goods and the cost function that tells us how much it costs the monopolist to produce different quantities of these goods. The goal is to find the production levels of each good that will lead to the highest possible profits for the monopolist.

Key Idea in Simple Terms

We need to:

- Set up the profit function as the revenue from selling x and y, minus the costs of producing them.
- Differentiate the profit function with respect to both x and y to find the critical points.
- Use second-order derivatives to check if these critical points represent a maximum.

Solution Strategy

The monopolist's profit function is:

$$\pi(x, y) = p(x)x + q(y)y - C(x, y).$$

Substituting the given functions for $p(x)$, $q(y)$, and $C(x, y)$, we have:

$$\pi(x, y) = (16 - x^2)x + (9 - y^2)y - (x^2 + 3y^2),$$

which simplifies to:

$$\pi(x, y) = 16x - x^3 + 9y - y^3 - x^2 - 3y^2.$$

Formal Solution

The first-order partial derivatives of the profit function with respect to x and y are:

$$\frac{\partial \pi(x,y)}{\partial x} = 16 - 3x^2 - 2x,$$

$$\frac{\partial \pi(x,y)}{\partial y} = 9 - 3y^2 - 6y.$$

Setting these partial derivatives equal to zero, we solve for x and y:

$$16 - 3x^2 - 2x = 0 \implies (x-2)(3x+8) = 0 \implies x = 2 \quad \text{(since } x = -\frac{8}{3} \text{ is not meaningful).}$$

$$9 - 3y^2 - 6y = 0 \implies (y-1)(3y+3) = 0 \implies y = 1 \quad \text{(since } y = -1 \text{ is not meaningful).}$$

Thus, the critical point is $(x^*, y^*) = (2, 1)$.

Second Derivatives and Concavity

Next, we compute the second partial derivatives to form the Hessian matrix:

$$\frac{\partial^2 \pi}{\partial x^2} = -6x - 2, \quad \frac{\partial^2 \pi}{\partial y^2} = -6y - 6, \quad \frac{\partial^2 \pi}{\partial x \partial y} = 0.$$

At the critical point $(x^*, y^*) = (2, 1)$, the Hessian matrix is:

$$H_\pi(x,y) = \begin{pmatrix} -6x-2 & 0 \\ 0 & -6y-6 \end{pmatrix} = \begin{pmatrix} -6(2)-2 & 0 \\ 0 & -6(1)-6 \end{pmatrix} = \begin{pmatrix} -14 & 0 \\ 0 & -12 \end{pmatrix}.$$

The leading principal minors are:

$$D_1 = -14 < 0, \quad D_2 = (-14)(-12) = 168 > 0.$$

Since the leading principal minors alternate in sign, the Hessian is negative definite, and the function is concave. Therefore, $(x^*, y^*) = (2, 1)$ is a global maximum of the profit function.

Conclusion

The monopolist should produce $x^* = 2$ units of x and $y^* = 1$ unit of y, and sell them at prices $p(x^*) = 16-(2)^2 = 12$ and $q(y^*) = 9 - (1)^2 = 8$. The cost function is $C(x^*, y^*) = (2)^2 + 3(1)^2 = 7$, and the maximum profit is:

$$\pi(x^*, y^*) = p(x^*)x^* + q(y^*)y^* - C(x^*, y^*) = 12 \cdot 2 + 8 \cdot 1 - 7 = 25.$$

Thus, the monopolist's maximum profit is \$25.

Problem 4 (Extra Practice Problem): Extrema and Saddle Points of a Logarithmic Function

Relevance in Economics

The analysis of extrema and saddle points is essential in understanding optimization in economic models. These points help in finding maximum profits, optimal production levels, and equilibrium points in various contexts.

Applications in Economics

- **Maximizing Utility:** Extrema are used to find points where utility is maximized.
- **Optimization in Production:** Saddle points and extrema are used to determine the best production levels that maximize profit or minimize cost.

The Problem

Calculate the extrema and saddle points of:

$$f : D \to \mathbb{R}, \quad (x, y) \mapsto \ln(x^2 + 2y^2 - 4x - 8y),$$

where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + 2y^2 - 4x - 8y > 0\}$. Determine whether the extrema are local or global.

Simple Explanation

We are given a function that represents a natural logarithm of a quadratic expression. The goal is to find the points where this function reaches its maximum or minimum values, or whether there are any saddle points where the function changes direction.

Key Ideas

We analyze the behavior of the underlying quadratic function to locate the points where it is maximized or minimized. Since the logarithmic function is strictly increasing, the extrema of the logarithmic function occur at the same points as the underlying quadratic function.

Solution Strategy

Since the logarithmic function $\ln(x)$ is strictly increasing, the extrema of the logarithmic function coincide with the extrema of the quadratic function $g(x, y) = x^2 + 2y^2 - 4x - 8y$. We will differentiate $g(x, y)$ and find its critical points.

Formal Solution

The gradient of $g(x, y)$ is:

$$\frac{\partial g(x,y)}{\partial x} = 2x - 4, \quad \frac{\partial g(x,y)}{\partial y} = 4y - 8.$$

Setting these partial derivatives equal to zero:

$$2x - 4 = 0 \implies x = 2,$$

$$4y - 8 = 0 \implies y = 2.$$

Thus, the critical point is $(x^*, y^*) = (2, 2)$. However, note that this point is not in the domain D, as substituting $(x, y) = (2, 2)$ into $g(x, y)$ gives:

$$g(2, 2) = 2^2 + 2(2^2) - 4(2) - 8(2) = 4 + 8 - 8 - 16 = -12,$$

which is less than zero, violating the condition $g(x, y) > 0$.

Conclusion

Since the quadratic function $g(x, y)$ has no critical points in the domain D, the logarithmic function $f(x, y)$ also has no critical points. Thus, $f(x, y)$ has no extrema and no saddle points.

Problem 5 (Extra Practice Problem): Production in Two Plants

Relevance in Economics

Production optimization is essential for firms that operate across multiple production plants. By analyzing the cost functions and determining the optimal production quantities, firms can maximize their profits.

Applications in Economics

- **Cost Minimization:** Firms aim to minimize the cost of production by optimizing output in each plant.
- **Profit Maximization:** By determining the quantities to produce in each plant, firms can ensure maximum profit.

The Problem

A firm produces the same telephones in two separate production plants. The cost of producing $x \geq 0$ units in production plant 1 is:

$$C_1(x) = 0.02x^2 + 4x + 500,$$

and the cost of producing $y \geq 0$ units in production plant 2 is:

$$C_2(y) = 0.05y^2 + 4y + 275.$$

The telephones are sold for \$15 each. Find the quantity of telephones that should be produced at each plant in order to maximize the firm's profits.

Simple Explanation

The firm wants to figure out how many telephones to produce at each of its two plants. The goal is to find the production levels that maximize the firm's total profit.

Key Ideas

We need to:

- Set up the firm's profit function as the revenue from selling the telephones, minus the costs of production in both plants.
- Differentiate the profit function with respect to x and y to find the critical points that maximize the profit.

Solution Strategy

The firm's profit function is given by:

$$\pi(x, y) = 15(x + y) - C_1(x) - C_2(y),$$

where $C_1(x)$ and $C_2(y)$ are the cost functions of the two plants. Substituting the cost functions, we get:

$$\pi(x, y) = 15(x + y) - (0.02x^2 + 4x + 500) - (0.05y^2 + 4y + 275),$$

which simplifies to:

$$\pi(x, y) = -0.02x^2 - 0.05y^2 + 11x + 11y - 775.$$

Formal Solution

The first-order partial derivatives of the profit function are:

$$\frac{\partial \pi(x, y)}{\partial x} = -0.04x + 11, \quad \frac{\partial \pi(x, y)}{\partial y} = -0.10y + 11.$$

Setting these partial derivatives equal to zero:

$$-0.04x + 11 = 0 \implies x = 275,$$

$$-0.10y + 11 = 0 \implies y = 110.$$

Thus, the firm should produce 275 units in plant 1 and 110 units in plant 2.

Conclusion

The firm maximizes its profit by producing $x = 275$ units in production plant 1 and $y = 110$ units in production plant 2.

Problem 6: Maximizing Profit for Electronic Items (Extra Practice Problem)

Relevance in Economics

Profit maximization is one of the primary goals of firms. This problem involves finding the production levels that maximize the firm's profits. This has significant applications in real-world settings where companies need to determine the optimal quantity of goods to produce to maximize their earnings.

Applications in Economics

- **Production Decision-Making:** Firms decide how much of each product to produce to maximize profits, taking costs and revenues into account.
- **Cost Efficiency:** By determining the optimal production levels, firms minimize waste and enhance profitability.

The Problem

A manufacturer of electronic items has determined that the profit $\pi(x, y)$ from producing x DVD players and y DVD recorders is given by:

$$\pi(x, y) = 8x + 10y - (0.001)(x^2 + xy + y^2) - 10000$$

Find the level of production that provides the maximum profit. What is the maximum profit?

Key Ideas

To find the critical points that maximize profit, we take the partial derivatives of the profit function with respect to x and y, set them equal to zero, and solve for the critical points. The second derivatives (Hessian matrix) help determine whether the critical points are local maxima or minima.

Solution Strategy

1. Compute the first-order partial derivatives $\frac{\partial \pi}{\partial x}$ and $\frac{\partial \pi}{\partial y}$.
2. Solve for x and y by setting these derivatives equal to zero to find the critical points.
3. Use the Hessian matrix to verify the nature of the critical points.

Solution

First-order conditions:

$$\frac{\partial \pi(x, y)}{\partial x} = 8 - 0.002x - 0.001y = 0 \quad \text{and} \quad \frac{\partial \pi(x, y)}{\partial y} = 10 - 0.001x - 0.002y = 0$$

Solving these gives the critical point $(x^*, y^*) = (2000, 4000)$.
Second-order conditions (Hessian matrix): The Hessian matrix $H_\pi(x, y)$ is:

$$H_\pi(x, y) = \begin{pmatrix} -0.002 & -0.001 \\ -0.001 & -0.002 \end{pmatrix}$$

Since the Hessian matrix is negative definite, the profit function is concave, and the critical point $(2000, 4000)$ is a global maximum.

Conclusion

The optimal production levels are $x^* = 2000$ DVD players and $y^* = 4000$ DVD recorders. The maximum profit is:

$$\pi(2000, 4000) = 8 \times 2000 + 10 \times 4000 - 0.001(2000^2 + 2000 \times 4000 + 4000^2) - 10000 = 18000$$

Thus, the maximum profit is \$18,000.

Problem 7: Maximizing Revenue for House Slippers (Extra Practice Problem)

Relevance in Economics

Revenue maximization is crucial for firms to sustain growth. By optimizing production strategies, companies ensure they meet consumer demand while maximizing the potential for earnings.

Applications in Economics

- **Revenue Maximization:** Firms seek to maximize revenue by adjusting production levels of goods to meet demand while optimizing prices.
- **Strategic Production Planning:** In industries with multiple product lines, optimizing the production mix ensures that the firm can generate the highest possible revenue.

The Problem

A firm produces two types of house slippers. The total revenue obtained from x_1 units of type 1 slippers and x_2 units of type 2 slippers is given by:

$$R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$$

where x_1 and x_2 are given in thousands of units. Find the values x_1 and x_2 that maximize the revenue.

Key Ideas

We will find the values of x_1 and x_2 that maximize revenue by computing the partial derivatives of the revenue function with respect to x_1 and x_2, setting them equal to zero, and solving for the critical points.

Solution Strategy

1. Compute the first-order partial derivatives $\frac{\partial R}{\partial x_1}$ and $\frac{\partial R}{\partial x_2}$.
2. Solve for the critical points by setting the partial derivatives equal to zero.
3. Use the Hessian matrix to classify the critical points as local maxima or minima.

Solution

First-order conditions:

$$\frac{\partial R(x_1, x_2)}{\partial x_1} = -10x_1 - 2x_2 + 42 = 0$$
$$\frac{\partial R(x_1, x_2)}{\partial x_2} = -16x_2 - 2x_1 + 102 = 0$$

Solving this system of equations yields the critical point $(x_1^*, x_2^*) = (3, 6)$.
Second-order conditions (Hessian matrix): The Hessian matrix $H_R(x_1, x_2)$ is:

$$H_R(x_1, x_2) = \begin{pmatrix} -10 & -2 \\ -2 & -16 \end{pmatrix}$$

The determinant of the Hessian is:

$$\det(H_R) = (-10)(-16) - (-2)(-2) = 160 - 4 = 156$$

Since $\det(H_R) > 0$ and $H_{11} < 0$, the function has a local maximum at $(x_1^*, x_2^*) = (3, 6)$.

Conclusion

The firm should produce 4,000 units of type 1 slippers and 5,000 units of type 2 slippers to maximize revenue. The maximum revenue is:

$$R(4, 5) = -5(4^2) - 8(5^2) - 2(4)(5) + 42(4) + 102(5) = 168$$

Thus, the maximum revenue is \$168,000.

Problem 8: Additional Practice on Convexity (Extra Practice Problem)

Relevance in Economics

Convexity plays a central role in optimization problems, ensuring that local optima are also global optima. Understanding convex sets and functions is crucial in economic analysis, particularly in consumer and production theory, where firms and individuals seek to optimize outcomes under constraints.

Applications in Economics

- **Optimization Theory:** Convex sets ensure that any weighted combination of two points within the set will also lie within the set, simplifying optimization problems.
- **Production Theory:** Convex production sets imply that firms can efficiently mix input combinations to achieve a desired output.

The Problem

Show that if $a \in \left[-\frac{2}{3}, \frac{2}{3}\right]$ and $b \in \mathbb{R}$, then the set:

$$M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - 3axy \leq 0, x + by + z \geq 0\}$$

is convex.

Key Ideas

We will break the set M into two parts, M_1 and M_2, and prove that each part is convex. If both parts are convex, then their intersection M will also be convex. Convexity is determined by checking the Hessian matrix of the function that defines the set.

Solution Strategy

1. Define the sets M_1 and M_2.
2. Show that $M_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - 3axy \leq 0\}$ is convex by analyzing the Hessian matrix of the defining function.
3. Show that $M_2 = \{(x, y, z) \in \mathbb{R}^3 : x + by + z \geq 0\}$ is convex because it represents a half-space.

Solution

Step 1: Show that M_1 is convex.

The set M_1 is defined by:

$$M_1 \equiv \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - 3axy \leq 0\}$$

We recognize that M_1 is the lower contour set of the function $f(x, y, z) = x^2 + y^2 - 3axy$. To check convexity, we need to examine the Hessian matrix of f.

The Hessian matrix $H_f(x, y, z)$ is:

$$H_f(x, y, z) = \begin{pmatrix} 2 & -3a & 0 \\ -3a & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To check whether f is convex, we need to verify if $H_f(x, y, z)$ is positive semidefinite. The principal minors of $H_f(x, y, z)$ are:

$$D_3 = 0, \quad \|H_f\|_2 = 0, \quad D_2 = \begin{vmatrix} 2 & -3a \\ -3a & 2 \end{vmatrix} = 4 - 9a^2$$

All principal minors are non-negative if and only if $4 - 9a^2 \geq 0$, or equivalently:

$$-\frac{2}{3} \leq a \leq \frac{2}{3}$$

Thus, M_1 is convex if $a \in \left[-\frac{2}{3}, \frac{2}{3}\right]$.

Step 2: Show that M_2 is convex.

The set M_2 is defined by:

$$M_2 \equiv \{(x, y, z) \in \mathbb{R}^3 : x + by + z \geq 0\}$$

This is the lower contour set of the linear function $f(x, y, z) = x + by + z$, which is convex (and concave) because linear functions are always convex. Therefore, M_2 is convex for any value of b.

Conclusion:

Since both M_1 and M_2 are convex, their intersection $M = M_1 \cap M_2$ is convex. Thus, M is convex if $a \in \left[-\frac{2}{3}, \frac{2}{3}\right]$ and $b \in \mathbb{R}$.