Mathematical Economics

ECON2050: Tutorial 5

Harley Blackwood 47473159

h.blackwood@uq.edu.au

Tutorial Overview:
This tutorial explores key mathematical techniques applied to economic problems, focusing on Cobb-Douglas production functions, quadratic forms, convex sets, and Taylor expansions. These problems are crucial for understanding production functions, matrix analysis, the properties of convex sets, and approximations in economic contexts. The exercises enhance students' abilities to analyze and solve optimization problems, assess matrix definiteness, apply convexity in economic theory, and approximate functions around specific points using Taylor expansions.

Problem 1: Cobb-Douglas Production Function

Relevance in Economics

Production Functions: Cobb-Douglas production functions are fundamental in economics for modeling the output of firms based on their inputs. Understanding these functions helps economists analyze how changes in input factors affect production levels, and how marginal products can be optimized.

Applications

Firm Output Analysis: Economists and analysts use Cobb-Douglas functions to estimate the output of firms given certain levels of input. This is crucial in understanding production efficiency and the impact of scaling inputs.

The Problem

A firm that uses 3 input factors has the Cobb-Douglas production function given by $f(x, y, z) = 10x^{\frac{1}{3}}y^{\frac{1}{2}}z^{\frac{1}{6}}$. Currently, the firm uses the following combination of input factors: $x = 27$, $y = 16$, $z = 64$.

1. How many units of output does the firm currently produce?
2. What is the marginal product of each production factor?
3. How does the production change if x is increased by 0.1 units, y is decreased by 0.3 units, and the third factor is unchanged?

Breakdown

This problem involves analyzing a Cobb-Douglas production function to calculate current output, marginal products of inputs, and the impact of small changes in input levels on production. These concepts are key to understanding the responsiveness of output to changes in input factors.

Solution Strategy

To solve each part:

1. Calculate the output by substituting the input values into the Cobb-Douglas function.
2. Derive the marginal products by differentiating the production function with respect to each input.
3. Apply the marginal products to calculate the change in production given the changes in input levels.

Formal Solution

1. Calculate Output:

$$f(27, 16, 64) = 10 \times 27^{\frac{1}{3}} \times 16^{\frac{1}{2}} \times 64^{\frac{1}{6}} = 10 \times 3 \times 4 \times 2 = 240 \text{ units}$$

2. Marginal Products:

$$\frac{\partial f}{\partial x} = \frac{1}{3} \times 10x^{-\frac{2}{3}}y^{\frac{1}{2}}z^{\frac{1}{6}} = \frac{f(x,y,z)}{3x}, \quad \frac{\partial f}{\partial y} = \frac{1}{2} \times 10x^{\frac{1}{3}}y^{-\frac{1}{2}}z^{\frac{1}{6}} = \frac{f(x,y,z)}{2y}$$

$$\frac{\partial f}{\partial z} = \frac{1}{6} \times 10x^{\frac{1}{3}}y^{\frac{1}{2}}z^{-\frac{5}{6}} = \frac{f(x,y,z)}{6z}$$

3. Change in Production:

$$\Delta f \approx \frac{80}{27} \times 0.1 - \frac{15}{2} \times 0.3 + 0 = \frac{8}{27} - \frac{9}{4} \approx -1.95 \text{ units}$$

Conclusion

The firm currently produces 240 units. The marginal products and small changes in inputs suggest a production decrease of approximately 1.95 units with the given input adjustments.

Problem 2: Quadratic Forms and Matrix Definiteness

Relevance in Economics

Quadratic Forms: Understanding quadratic forms and matrix definiteness is essential for analyzing the curvature of functions, particularly in optimization problems in economics. The definiteness of a matrix can indicate whether a function is concave, convex, or has saddle points, which in turn affects optimization strategies.

Applications

Optimization Problems: In economics, quadratic forms are used to study functions related to costs, utilities, and profits. Determining the definiteness of matrices helps in identifying the nature of equilibrium points and ensuring optimization solutions are robust.

The Problem

Express the quadratic form defined by the following matrix A in polynomial form and classify A according to its definiteness.

$$A = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Breakdown

This problem involves determining the polynomial form of a quadratic function defined by matrix A and classifying the matrix based on its definiteness. Understanding the definiteness of a matrix is crucial in economic optimization problems as it indicates the nature of the function's critical points.

Solution Strategy

To solve:

1. Compute the quadratic form by multiplying the matrix A with the vector (x, y, z).
2. Determine the definiteness of A by calculating the leading principal minors.

Formal Solution

1. Quadratic Form:

$$q_A(x,y,z) = (x,y,z)\begin{pmatrix} -3 & 1 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -3x^2 - 4y^2 - 3z^2 + 2xy$$

2. Matrix Definiteness:

$$D_1 = -3 < 0, \quad D_2 = \begin{vmatrix} -3 & 1 \\ 1 & -4 \end{vmatrix} = 11 > 0, \quad D_3 = \begin{vmatrix} -3 & 1 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -33 < 0$$

Matrix A is negative definite.

Conclusion

The quadratic form is $q_A(x_1, x_2, x_3) = -3x_1^2 - 4x_2^2 - 3x_3^2 + 2x_1x_2$, and the matrix A is classified as negative definite.

Problem 3: Quadratic Forms and Definiteness of Matrices

Relevance in Economics

Quadratic Forms: Quadratic forms are used in economics to study various optimization problems, including utility maximization and cost minimization. They are particularly important in analyzing the curvature of functions, which helps in understanding whether a function is concave, convex, or has saddle points.
Definiteness of Matrices: The definiteness of a matrix, which is derived from its quadratic form, provides insight into the nature of critical points in optimization problems. For example, a positive definite matrix indicates a local minimum, while a negative definite matrix indicates a local maximum.

Applications

Utility Maximization: In utility maximization problems, the definiteness of the Hessian matrix, which is a quadratic form, determines whether the consumer's utility function is concave or convex.
Cost Minimization: Firms use quadratic forms to represent cost functions, and the definiteness of these forms helps determine whether the cost function has a minimum at a particular production point.

Problem Description

For each of the following matrices, express the quadratic form that it defines in polynomial form. Moreover, classify each matrix according to its definiteness.

$$B = \begin{pmatrix} -8 & 2 & -2 \\ 2 & 6 & 1 \\ -2 & 1 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Breakdown

This problem requires us to:

1. Compute the quadratic form associated with each matrix, which involves expressing the matrix-vector multiplication in polynomial form.
2. Analyze the leading principal minors of each matrix to classify them as positive definite, negative definite, or indefinite.

Solution Strategy

To solve the problem:

1. Express the quadratic form $q(x, y, z) = \mathbf{x}^T A\mathbf{x}$ for each matrix B and C.
2. Compute the leading principal minors of each matrix.
3. Use the conditions for definiteness (based on the signs of the leading principal minors) to classify each matrix.

Formal Solution

Matrix $B = \begin{pmatrix} -8 & 2 & -2 \\ 2 & 6 & 1 \\ -2 & 1 & -2 \end{pmatrix}$

1. **Quadratic Form:**

$$q_B(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} -8 & 2 & -2 \\ 2 & 6 & 1 \\ -2 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

   Expanding this gives:

$$q_B(x, y, z) = -8x^2 + 6y^2 - 2z^2 + 4xy - 4xz + 2yz$$

2. **Definiteness:** The leading principal minors of B are:

$$D_1 = -8 < 0$$

$$D_2 = \det \begin{pmatrix} -8 & 2 \\ 2 & 6 \end{pmatrix} = (-8)(6) - (2)(2) = -48 - 4 = -52 < 0$$

   Since $D_2 < 0$, matrix B is classified as **indefinite**.

Matrix $C = \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

1. **Quadratic Form:**

$$q_C(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

   Expanding this gives:

$$q_C(x, y, z) = 5x^2 + 5y^2 + 2z^2 + 8xy$$

2. **Definiteness:** The leading principal minors of C are:

$$D_1 = 5 > 0$$

$$D_2 = \det \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} = (5)(5) - (4)(4) = 25 - 16 = 9 > 0$$

$$D_3 = \det \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2 \cdot (25 - 16) = 18 > 0$$

   Since all leading principal minors are positive, matrix C is classified as **positive definite**.

Conclusion

For matrix B, the quadratic form is $q_B(x, y, z) = -8x^2 + 6y^2 - 2z^2 + 4xy - 4xz + 2yz$, and it is classified as indefinite. For matrix C, the quadratic form is $q_C(x, y, z) = 5x^2 + 5y^2 + 2z^2 + 8xy$, and it is classified as positive definite. The analysis of the leading principal minors helps determine the definiteness of the matrices, which is crucial in understanding the nature of the corresponding quadratic forms.

Problem 4: Convexity of Sets

Relevance in Economics

Convex Sets: Convexity is a fundamental concept in economics, particularly in optimization and game theory. Convex sets ensure that any weighted average of two points within the set also lies within the set, which is crucial for stability and equilibrium analysis.
Linear Programming and Optimization: Many economic problems can be formulated as linear programming problems, where the feasible region is a convex set. Convexity ensures that any local optimum is also a global optimum.

Applications

Optimization Problems: In optimization, convex sets are essential because they guarantee that the solutions obtained are optimal. For instance, in consumer theory, the budget set and preference sets are convex, ensuring that the consumer's utility maximization problem has a solution.
Game Theory: In game theory, convexity of strategy sets ensures the existence of mixed strategy equilibria.

Problem Description

Let A be an $m \times n$ matrix and b be an $m \times 1$ matrix (i.e., a column vector). We are asked to demonstrate that the following sets are convex by applying the definition of convexity:

1. $S = \{x \in \mathbb{R}^n : Ax = b\}$
2. $S = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$
3. $S = \{(x, y) \in \mathbb{R}^2 : x + y \leq 6, x - 2y \leq 2\}$

Breakdown

To show that each set is convex, we will:

1. Use the definition of convexity, which states that a set S is convex if for any $x, y \in S$ and any $\lambda \in [0, 1]$, the point $\lambda x + (1 - \lambda)y$ is also in S.
2. For each set, we will pick arbitrary points $x, y \in S$ and show that the point $\lambda x + (1 - \lambda)y$ satisfies the defining conditions of the set.

Solution Strategy

The strategy involves:

1. Substituting the points x and y into the set's defining conditions.
2. Showing that these conditions hold for the convex combination $\lambda x + (1 - \lambda)y$.

Formal Solution

1. Set $S = \{x \in \mathbb{R}^n : Ax = b\}$ is Convex:

1. **Let $x, y \in S$.** This means that x and y satisfy the equation $Ax = b$ and $Ay = b$, respectively.
$$Ax = b \quad \text{and} \quad Ay = b$$
2. **Consider any $\lambda \in [0, 1]$.** We want to show that the point $z = \lambda x + (1 - \lambda)y$ also lies in S. Substituting z into the condition defining S, we have:
$$Az = A[\lambda x + (1 - \lambda)y]$$
3. **Use the linearity of the matrix product A to expand:**
$$Az = \lambda Ax + (1 - \lambda)Ay$$
4. **Since $Ax = b$ and $Ay = b$, substitute these into the equation:**
$$Az = \lambda b + (1 - \lambda)b = [\lambda + (1 - \lambda)]b = b$$
5. **Thus, $z = \lambda x + (1 - \lambda)y$ satisfies $Az = b$, meaning $z \in S$.** Therefore, S is convex.

2. Set $S = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$ is Convex:

1. **Let $x, y \in S$.** This means that x and y satisfy the conditions $Ax \leq b$, $Ay \leq b$, and $x \geq 0$, $y \geq 0$.
$$Ax \leq b \quad \text{and} \quad Ay \leq b$$
$$x \geq 0 \quad \text{and} \quad y \geq 0$$
2. **Consider any $\lambda \in [0, 1]$.** We want to show that the point $z = \lambda x + (1 - \lambda)y$ also lies in S. First, substitute z into the condition $Ax \leq b$:
$$Az = A[\lambda x + (1 - \lambda)y]$$
3. **Use the linearity of the matrix product A to expand:**
$$Az = \lambda Ax + (1 - \lambda)Ay$$
4. **Since $Ax \leq b$ and $Ay \leq b$, substitute these into the equation:**
$$Az \leq \lambda b + (1 - \lambda)b = [\lambda + (1 - \lambda)]b = b$$
5. **Additionally, consider the non-negativity condition $x \geq 0$ and $y \geq 0$:**
$$z = \lambda x + (1 - \lambda)y \geq \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0$$
6. **Thus, $z = \lambda x + (1 - \lambda)y$ satisfies $Az \leq b$ and $z \geq 0$, meaning $z \in S$.** Therefore, S is convex.

3. Set $S = \{(x, y) \in \mathbb{R}^2 : x + y \leq 6, x - 2y \leq 2\}$ is Convex:

1. **Let $(x_1, y_1), (x_2, y_2) \in S$.** This means that $x_1 + y_1 \leq 6$ and $x_2 + y_2 \leq 6$, as well as $x_1 - 2y_1 \leq 2$ and $x_2 - 2y_2 \leq 2$.
$$x_1 + y_1 \leq 6 \quad \text{and} \quad x_2 + y_2 \leq 6$$
$$x_1 - 2y_1 \leq 2 \quad \text{and} \quad x_2 - 2y_2 \leq 2$$
2. **Consider any $\lambda \in [0, 1]$.** We want to show that the point $(x, y) = \lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)$ also lies in S.
$$(x, y) = (\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2)$$
3. **Check the condition $x + y \leq 6$:**
$$x + y = (\lambda x_1 + (1 - \lambda)x_2) + (\lambda y_1 + (1 - \lambda)y_2)$$
$$x + y = \lambda(x_1 + y_1) + (1 - \lambda)(x_2 + y_2) \leq \lambda \cdot 6 + (1 - \lambda) \cdot 6 = 6$$
4. **Check the condition $x - 2y \leq 2$:**
$$x - 2y = (\lambda x_1 + (1 - \lambda)x_2) - 2(\lambda y_1 + (1 - \lambda)y_2)$$
$$x - 2y = \lambda(x_1 - 2y_1) + (1 - \lambda)(x_2 - 2y_2) \leq \lambda \cdot 2 + (1 - \lambda) \cdot 2 = 2$$
5. **Thus, $(x, y) = \lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)$ satisfies both conditions, meaning $(x, y) \in S$.** Therefore, S is convex.

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