Mathematical Economics
ECON2050: Tutorial 7

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Tutorial Overview:

Introduction

This tutorial explores the important concepts of convexity and concavity in mathematical economics, focusing on their applications in optimization problems. We also study methods to identify extrema and saddle points of multivariable functions, an essential part of understanding economic optimization models.

Topics Covered

1. Convex and concave functions: Definitions, properties, and economic significance.
2. Extrema and saddle points: Techniques for identifying local and global extrema, and classification of saddle points.
3. Hessian matrix and second-order conditions: Methods for analyzing the nature of critical points.

Objectives

The objectives of this tutorial are:

- To understand the definition and applications of convex and concave functions in economics.
- To learn how to calculate and interpret the Hessian matrix to classify functions as convex or concave.
- To identify and classify the extrema and saddle points of multivariable functions using first- and second-order conditions.
- To apply these mathematical concepts to real-world economic models, particularly in utility and production theory.

Expected Outcomes

By the end of this tutorial, students are expected to:

- Be able to determine whether a function is convex or concave using both graphical and algebraic methods.
- Understand the economic implications of convexity and concavity, especially in terms of utility and cost functions.
- Be proficient in finding critical points and using the Hessian matrix to classify them as maxima, minima, or saddle points.
- Apply mathematical tools to solve optimization problems relevant to economics.

Problem 1: Convexity and Concavity of Functions

Relevance in Economics

Concave and convex functions play a key role in economic optimization problems, utility theory, and production theory. Convex functions lead to well-behaved optimization problems, guaranteeing global minima, while concave functions are essential for modeling diminishing returns and risk aversion.

Applications in Economics

- **Utility functions:** Concave utility functions imply diminishing marginal utility, an essential concept in consumer theory.
- **Cost functions:** Convex cost functions are crucial in production theory, where firms seek to minimize costs.

Problem Description

Analyze whether the following functions are concave or convex. Apply different methods discussed in the lectures to show concavity or convexity:

1. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto -x^2 - 5y^2 - 4xy$
2. $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto -(x-3)^2 - (y+1)^2 - z^2$
3. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto 3x + 2y$
4. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x^2 + e^{2y}$
5. $f : \mathbb{R}^2_+ \to \mathbb{R}, (x, y) \mapsto \ln(3x + 2y)$

Explanation

The convexity or concavity of a function tells us how it behaves when we combine inputs. For convex functions, a mix of two inputs is better than the average of the function's values at those inputs, while for concave functions, the opposite is true.

Key Idea in Simple Terms

We analyze concavity or convexity by:

- Checking the second derivatives (Hessian matrix for multivariate functions).
- Using properties of common functions like logarithms, quadratics, and exponentials.

Understanding the Hessian Matrix, Eigenvalues, and Eigenvectors

The Hessian matrix is a square matrix composed of all the second-order partial derivatives of a scalar-valued function. It provides detailed information about the local curvature of the function at a given point, which is crucial for understanding the function's behavior, particularly in optimization problems.

1. Definition

For a function f of n variables $(x_1, x_2, \dots, x_n)$, the Hessian matrix $H(f)$ is an $n \times n$ matrix, where the entry in the i-th row and j-th column is the second partial derivative of f:

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}.$$

2. What is an Eigenvalue?

Given a square matrix A, an *eigenvalue* λ is a scalar such that there exists a non-zero vector $\mathbf{v}$ (called an eigenvector) that satisfies:

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = (I\lambda)\vec{v}$$

$$A\vec{v} - (I\lambda)\vec{v} = \vec{0}$$

$$\vec{v}(A - (I\lambda)) = \vec{0}$$

In this equation, A acts on the vector $\vec{v}$ by stretching or compressing it by the factor λ. The vector $\vec{v}$ does not change direction, only its magnitude. The only way it is possible for the product of a matrix with a non-zero vector to become a zero-vector is if the transformation associated with that matrix squashes space into a lower dimension.

For the Hessian matrix H, the eigenvalues provide information about the curvature of the function in the directions of the corresponding eigenvectors.

3. How to Find Eigenvalues of the Hessian Matrix?

To find the eigenvalues of a Hessian matrix H, solve the characteristic equation:

$$\det(H - \lambda I) = 0,$$

where I is the identity matrix of the same size as H, and λ represents the eigenvalues. This equation is a polynomial in λ, known as the characteristic polynomial. The roots of this polynomial are the eigenvalues of H.

4. Interpretation of Eigenvalues for the Hessian Matrix

1. **Positive Eigenvalues**: If all eigenvalues of the Hessian are positive, the function is curving upwards in all directions near the critical point, indicating a local minimum.

2. **Negative Eigenvalues**: If all eigenvalues are negative, the function is curving downwards in all directions, indicating a local maximum.

3. **Mixed Eigenvalues**: If the Hessian has both positive and negative eigenvalues, the point is a saddle point, indicating that the function has a minimum in some directions and a maximum in others.

4. **Zero Eigenvalues**: If one or more eigenvalues are zero, the function is flat in the direction of the corresponding eigenvectors, making the test inconclusive regarding the nature of the critical point.

5. Eigenvectors of the Hessian Matrix

The *eigenvectors* associated with the eigenvalues of the Hessian matrix indicate the principal directions of curvature: - If $\vec{v}$ is an eigenvector of the Hessian matrix with eigenvalue λ, the function has a curvature of λ in the direction of $\vec{v}$. For instance, in a two-variable function: If the Hessian has eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then: The function curves in the direction of $\mathbf{v}_1$ with intensity λ_1. It curves in the direction of $\mathbf{v}_2$ with intensity λ_2.

6. Steps to Find Eigenvalues and Eigenvectors

1. **Set up the characteristic equation**:

$$\det(H - \lambda I) = 0.$$

2. **Solve for λ**: This will give a polynomial equation in λ. The solutions (roots) of this polynomial are the eigenvalues.
3. **Find Eigenvectors**: For each eigenvalue λ, solve the system:

$$(H - \lambda I)\vec{v} = 0.$$

The non-zero solutions $\mathbf{v}$ are the eigenvectors corresponding to each eigenvalue.

7. Why Eigenvalues Matter for the Hessian

1. **Critical Point Classification**: The sign and nature (positive, negative, or zero) of the eigenvalues of the Hessian matrix determine the type of critical point:
2. **Local Minimum**: All positive eigenvalues.
3. **Local Maximum**: All negative eigenvalues.
4. **Saddle Point**: Mix of positive and negative eigenvalues.
5. **Curvature**: The eigenvalues tell you how the function bends in the direction of the eigenvectors:
   - A large positive eigenvalue indicates steep upward curvature.
   - A large negative eigenvalue indicates steep downward curvature.
   - A zero eigenvalue indicates a flat region in the corresponding direction.

Example

For a function $f(x, y)$, suppose its Hessian matrix at a critical point (x_0, y_0) is:

$$H = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}.$$

1. Find the eigenvalues by solving:

$$\det \begin{pmatrix} 4-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix} = 0.$$

2. Simplifying this gives:

$$(4-\lambda)(3-\lambda) - 4 = \lambda^2 - 7\lambda + 8 = 0.$$

3. Solve for λ:

$$\lambda = \frac{7 \pm \sqrt{1}}{2} = 4 \quad \text{and} \quad 2.$$

4. Both eigenvalues are positive, indicating that the critical point is a **local minimum**.

Summary

Eigenvalues and eigenvectors of the Hessian matrix provide crucial information about the curvature of a multivariable function around a critical point. Eigenvalues determine whether the point is a local minimum, maximum, or saddle point, while eigenvectors indicate the principal directions of curvature. This analysis helps in optimization and understanding the local behavior of functions.

Solution for Each Function

Function 1: $f(x,y) = -x^2 - 5y^2 - 4xy$

Solution Strategy:
We will calculate the Hessian matrix to determine whether the function is concave or convex.
Hessian Calculation:

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -4 & -10 \end{bmatrix}$$

The Hessian matrix is negative definite (since both eigenvalues are negative). Hence, $f(x,y)$ is strictly concave.
Conclusion: The function is concave.

Function 2: $f(x,y,z) = -(x-3)^2 - (y+1)^2 - z^2$

Solution Strategy:
This is a sum of negative quadratic terms, and each quadratic term is concave (as $-(x-a)^2$ is concave for any a).
Conclusion: The function is concave.

Function 3: $f(x,y) = 3x + 2y$

Solution Strategy:
This is a linear function. Linear functions are both convex and concave.
Conclusion: The function is both convex and concave (affine).

Function 4: $f(x,y) = x^2 + e^{2y}$

Solution Strategy:
We will calculate the Hessian matrix.
Hessian Calculation:

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4e^{2y} \end{bmatrix}$$

The Hessian matrix is positive definite, as all eigenvalues are positive. Hence, $f(x,y)$ is convex.
Conclusion: The function is convex.

Function 5: $f(x,y) = \ln(3x + 2y)$

Solution Strategy:
We know that the logarithm of a linear function (as long as the linear function is positive) is concave.
Conclusion: The function is concave in the domain $\mathbb{R}^2_+$.

Conclusion

Each function has been analyzed using either the Hessian matrix or known properties of the functions involved. The findings are summarized as follows:

- Function 1: Concave
- Function 2: Concave
- Function 3: Both convex and concave (affine)
- Function 4: Convex
- Function 5: Concave

Problem 2: Extrema and Saddle Points

Relevance in Economics

The analysis of extrema and saddle points is essential in various economic optimization problems. Firms seek to minimize costs or maximize profits, while consumers seek to maximize utility. Understanding local and global extrema helps in determining the optimal solutions in these contexts.

Applications in Economics

- **Profit Maximization:** Firms find local and global maxima to maximize profits.
- **Utility Maximization:** Consumers aim to maximize their utility, which involves determining the local and global optima.

The Problem

Calculate the extrema and saddle points of the following functions. If possible, determine whether the extrema are local or global:

1. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto -x^2 - 5y^2 - 4xy$
2. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto xy(x - 1)$
3. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \frac{x^2+y^2}{1+x^2+y^2}$
4. $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto ye^{xz^2}$
5. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x(x^2 - 12x + 45) - y(y - 1)$

Explanation

We aim to find the points where the function's value is either at a maximum or minimum. We also want to check if there are saddle points, where the function increases in one direction and decreases in another.

Key Idea in Simple Terms

We compute the first and second partial derivatives to find the critical points. The second derivative (Hessian matrix for multivariate functions) helps classify these critical points as maxima, minima, or saddle points.

Solution for Each Function

Function 1: $f(x, y) = -x^2 - 5y^2 - 4xy$

Solution Strategy:
We will compute the first and second partial derivatives to find critical points and classify them using the Hessian matrix.
First Derivatives:

$$f_x = -2x - 4y, \quad f_y = -10y - 4x$$

Setting these equal to zero, we solve for x and y. The critical point is $(0, 0)$.
Second Derivatives:

$$f_{xx} = -2, \quad f_{yy} = -10, \quad f_{xy} = -4$$

The Hessian matrix is:

$$H_f = \begin{bmatrix} -2 & -4 \\ -4 & -10 \end{bmatrix}$$

The determinant of the Hessian is $\det(H_f) = (-2)(-10) - (-4)(-4) = 20 - 16 = 4$. Since the determinant is positive and $f_{xx} < 0$, the function has a local maximum at $(0, 0)$.
Conclusion: The function has a local maximum at $(0, 0)$.

Function 2: $f(x, y) = xy(x-1)$

Solution Strategy:
We will compute the first and second partial derivatives.
First Derivatives:

$$f_x = y(2x-1), \quad f_y = x(x-1)$$

Setting these equal to zero, the critical points are $(0,0)$, $(1,0)$, and $(0,1)$.

Second Derivatives:

$$f_{xx} = 2y, \quad f_{yy} = 0, \quad f_{xy} = 2x-1$$

We classify the critical points by checking the Hessian at each point.

Conclusion: $(0,0)$ is a , $(1,0)$ is a , and $(0,1)$ is a local .

Function 3: $f(x,y) = \frac{x^2+y^2}{1+x^2+y^2}$

Solution Strategy:
This is a ratio of quadratic terms, and the function is bounded by 0 and 1.
First Derivatives:

$$f_x = \frac{2x}{(1+x^2+y^2)^2}, \quad f_y = \frac{2y}{(1+x^2+y^2)^2}$$

Setting these equal to zero, the critical point is $(0,0)$.

Second Derivatives: We compute the Hessian and assess definiteness of the Hessian.

Conclusion: The function has a at $(0,0)$.

Function 4: $f(x,y,z) = ye^{xz^2}$

Solution Strategy:
We compute the first and second partial derivatives to find critical points and classify them.
First Derivatives:

$$f_x = yz^2e^{xz^2}, \quad f_y = e^{xz^2}, \quad f_z = 2xyze^{xz^2}$$

Setting these equal to zero, the critical point is:

Second Derivatives: We compute the Hessian and assess definiteness of the Hessian.

$$f_{xx} = \frac{\partial}{\partial x}\left(yz^2e^{xz^2}\right) = yz^4e^{xz^2}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(e^{xz^2}\right) = 0$$

$$f_{zz} = \frac{\partial}{\partial z}\left(ye^{xz^2} \cdot 2xz\right) = ye^{xz^2} \cdot (2x) + ye^{xz^2} \cdot 2xz \cdot (2xz) = 2xye^{xz^2} + 4x^2z^2ye^{xz^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(yz^2e^{xz^2}\right) = z^2e^{xz^2}$$

$$f_{xz} = \frac{\partial}{\partial z}\left(yz^2e^{xz^2}\right) = ye^{xz^2} \cdot 2xz + yz^2e^{xz^2} \cdot 2xz = 2xzye^{xz^2} + 2xzyz^2e^{xz^2}$$

$$f_{yz} = \frac{\partial}{\partial z}\left(e^{xz^2}\right) = 2xze^{xz^2}$$

Hessian matrix:
The Hessian matrix H_f for the function $f(x,y,z) = ye^{xz^2}$ is:

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} yz^4e^{xz^2} & z^2e^{xz^2} & 2xzye^{xz^2} + 2xzyz^2e^{xz^2} \\ z^2e^{xz^2} & 0 & 2xze^{xz^2} \\ 2xzye^{xz^2} + 2xzyz^2e^{xz^2} & 2xze^{xz^2} & 2xye^{xz^2} + 4x^2z^2ye^{xz^2} \end{bmatrix}$$

$$\det(H_f) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Conclusion:

Function 5: $f(x, y) = x(x^2 - 12x + 45) - y(y - 1)$

Solution Strategy:
We will compute the first and second partial derivatives.
First Derivatives:

$$f_x = 3x^2 - 24x + 45, \quad f_y = -2y + 1$$

Setting these equal to zero:

Second Derivatives: We compute the Hessian to classify the critical points.

Conclusion:

Problem 3: Concavity and Convexity Analysis (Extra Practice Problem)

Relevance in Economics

Concavity and convexity play a vital role in economics, especially in utility maximization, profit maximization, and optimization problems. Convex functions guarantee the existence of unique optima, while concave functions describe diminishing returns, which are common in production and utility.

Applications

- **Utility Theory:** Concave utility functions suggest diminishing marginal utility, which is essential in modeling consumer behavior.
- **Optimization Problems:** Convex sets and functions ensure that local optima are also global optima, simplifying complex economic models.

Problem Description

Analyze whether the following functions are concave or convex. Apply the methods discussed in lectures.

1. $f : \mathbb{R}^2_{++} \to \mathbb{R}, (x, y) \mapsto \ln(3x^2y)$
2. $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto 4x - 3y - z$

Key Idea in Simple Terms

We will use the Hessian matrix to determine whether a function is concave, convex, or neither. For concavity, the Hessian must be negative semi-definite, while for convexity, it should be positive semi-definite. In the case of linear functions, they are both concave and convex.

Solution Strategy

1. For the function $f(x, y) = \ln(3x^2y)$, we will calculate the Hessian matrix and analyze its definiteness.
2. For the function $f(x, y, z) = 4x - 3y - z$, we will analyze it based on the fact that linear functions are both concave and convex.

Formal Solution

Function 1: $f(x, y) = \ln(3x^2y)$

Step-by-Step:

We want to determine whether $f(x, y) = \ln(3x^2y)$ is concave or convex by computing its Hessian matrix and checking the signs of its eigenvalues.

1. Compute the gradient:

The partial derivatives of $f(x, y) = \ln(3x^2y)$ are:

$$\frac{\partial f}{\partial x} = \frac{2}{x}, \quad \frac{\partial f}{\partial y} = \frac{1}{y}$$

2. Compute the Hessian matrix:

The Hessian matrix consists of the second-order partial derivatives of $f(x, y)$:

$$H_f(x, y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

The second-order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = -\frac{2}{x^2}, \quad \frac{\partial^2 f}{\partial y^2} = -\frac{1}{y^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$$

Thus, the Hessian matrix is:

$$H_f(x, y) = \begin{bmatrix} -\frac{2}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{bmatrix}$$

3. Analyze the Hessian:

The Hessian matrix is diagonal with both diagonal entries being negative, which implies that the Hessian is negative definite. Therefore, the function is concave.

Conclusion: The function $f(x, y) = \ln(3x^2y)$ is **concave**.

Function 2: $f(x, y, z) = 4x - 3y - z$

Step-by-Step:
This is a linear function of three variables, and we know that linear functions are both concave and convex.
Conclusion: The function $f(x, y, z) = 4x - 3y - z$ is **both concave and convex**.

$$\begin{gathered}
\forall x, y, z \in \mathbb{R} \text{ and all } \lambda \in [0, 1] \\
f(\lambda(x, y, z) + (1 - \lambda)(x', y', z')) \\
= f(\lambda x + (1 - \lambda)x', \lambda y + (1 - \lambda)y', \lambda z + (1 - \lambda)z') \\
= 4(\lambda x + (1 - \lambda)x') - 3(\lambda y + (1 - \lambda)y') - (\lambda z + (1 - \lambda)z') \\
= \lambda(4x - 3y - z) + (1 - \lambda)(4x' - 3y' - z') \\
= \lambda f(x, y, z) + (1 - \lambda)f(x', y', z'),
\end{gathered}$$

Problem 4: Calculate the Extrema and Saddle Points of the Following Functions

Relevance in Economics

Finding the extrema and saddle points of functions is essential in economics, especially in optimization problems. These include maximizing utility, minimizing costs, and finding equilibrium points in game theory. Extrema give optimal solutions, while saddle points can indicate inefficiencies or points of non-optimality.

Applications

- **Optimization in Economics:** Extrema are used in determining maximum profit, utility, or minimum cost in various economic models.
- **Saddle Points in Game Theory:** Saddle points help find Nash equilibria in non-cooperative games.

The Problem

You are tasked with finding the extrema and saddle points of the following functions. Determine whether the extrema are local or global, if possible.

1. $f : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto (y - x)^2$
2. $f : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto 20x + 26y + 4xy - 4x^2 - 3y^2$
3. $f : \mathbb{R}^3 \to \mathbb{R},\ (x, y, z) \mapsto x^2 + y^2 - 3x - 3xz + 3z^2$

Formal Solution

1. Function: $f(x, y) = (y - x)^2$

Solution Strategy

1. First, find the first-order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
2. Set the partial derivatives equal to zero to solve for critical points.
3. Compute the Hessian matrix to determine the nature of the critical points (local minima, maxima, or saddle points).

Working

$$f(x, y) = (y - x)^2$$

First-order partial derivatives:

$$\frac{\partial f}{\partial x} = -2(y - x), \quad \frac{\partial f}{\partial y} = 2(y - x)$$

Setting the partial derivatives to zero:

$$-2(y - x) = 0, \quad 2(y - x) = 0 \implies y = x$$

Thus, the critical points lie along the line $y = x$.

Second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = -2$$

The Hessian matrix is:

$$H = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

The determinant of the Hessian is:

$$\det(H) = (2)(2) - (-2)(-2) = 4 - 4 = 0$$

Since the determinant of the Hessian is zero, we cannot definitively classify the critical points using the second derivative test.
So we will use the principal minors:

Conclusion

The critical points lie along the line $y = x$. However, since the Hessian is singular (determinant is zero), we cannot classify the nature of these points based on the second derivative test. Use principal minors to determine positive semi-definite, thus convex, therefore all the points on the line $x = y$ are global minima.

2. Function: $f(x,y) = 20x + 26y + 4xy - 4x^2 - 3y^2$

Solution Strategy

1. Find the first-order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
2. Solve for the critical points by setting the partial derivatives to zero.
3. Analyze the Hessian matrix to determine whether the critical points are local maxima, minima, or saddle points.

Working

$$f(x,y) = 20x + 26y + 4xy - 4x^2 - 3y^2$$

First-order partial derivatives:

$$\frac{\partial f}{\partial x} = 20 + 4y - 8x, \quad \frac{\partial f}{\partial y} = 26 + 4x - 6y$$

Setting the partial derivatives to zero:

$$20 + 4y - 8x = 0, \quad 26 + 4x - 6y = 0$$

Solving this system of equations yields the critical point $(x,y) = (7,9)$.

Second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = -8, \quad \frac{\partial^2 f}{\partial y^2} = -6, \quad \frac{\partial^2 f}{\partial x \partial y} = 4$$

The Hessian matrix is:

$$H = \begin{pmatrix} -8 & 4 \\ 4 & -6 \end{pmatrix}$$

The determinant of the Hessian is:

$$\det(H) = (-8)(-6) - (4)(4) = 48 - 16 = 32$$

Since $\det(H) > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, the critical point is a local maximum. Can we determine if this is a global maxima?

Conclusion

The function has a local maximum at the critical point $(x,y) = (7,9)$.

3. Function: $f(x, y, z) = x^2 + y^2 - 3x - 3xz + 3z^2$

Solution Strategy

1. Find the first-order partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.
2. Set the partial derivatives equal to zero and solve for the critical points.
3. Use the Hessian matrix to classify the critical points.

Working

$$f(x, y, z) = x^2 + y^2 - 3x - 3xz + 3z^2$$

First-order partial derivatives:

$$\frac{\partial f}{\partial x} = 2x - 3 - 3z, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = -3x + 6z$$

Setting the partial derivatives equal to zero:

$$2x - 3 - 3z = 0, \quad 2y = 0, \quad -3x + 6z = 0$$

Solve:

Second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial z^2} = 6, \quad \frac{\partial^2 f}{\partial x \partial y} = 0 \quad \frac{\partial^2 f}{\partial y \partial z} = 0 \quad \frac{\partial^2 f}{\partial x \partial z} = -3$$

The Hessian matrix is:

$$H = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 6 \end{pmatrix}$$

The determinant of the Hessian is:

$$\det(H) = 2(2)(6) - 0 = 24 - 18$$

Since $\det(H) > 0$, the critical point is a local minimum.

Conclusion

The function has a local minimum at $(x, y, z) = (6, 0, 3)$.

Problem 5: Calculate the Extrema and Saddle Points of the Following Functions

Relevance in Economics

Identifying extrema and saddle points is crucial in optimization problems where decision-making about resources, production, and utility functions requires understanding the maximum or minimum values. Saddle points also provide insights into indeterminate conditions or points of inefficiency.

Applications

- **Optimization Problems:** Extrema are used to find the points of highest utility, maximum profit, or minimum cost in economic models.
- **Saddle Points in Game Theory:** Saddle points may indicate strategic situations where no player has an incentive to deviate from their current strategy.

The Problem

You are tasked with finding the extrema and saddle points of the following functions. Determine whether the extrema are local or global if possible.

1. $f : D \to \mathbb{R},\ (x, y) \mapsto \sqrt{25 - (x-2)^2 - y^2}$, where $D = \{(x, y) \in \mathbb{R}^2 : (x-2)^2 + y^2 \leq 25\}$
2. **(Extra Practice Problem)** $f : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto e^{-x^2-y^2}$

Formal Solution

1. Function: $f(x,y) = \sqrt{25-(x-2)^2-y^2}$

Solution Strategy

1. First, identify the domain D, which represents a disk of radius 5 centered at $(2,0)$ in the xy-plane.
2. Find the critical points by examining the partial derivatives.
3. Analyze the boundary conditions and check for extrema within the domain.

Working

The domain D is defined as:

$$D = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 \leq 25\}$$

First, compute the partial derivatives:

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\sqrt{25-(x-2)^2-y^2}\right) = \frac{-(x-2)}{\sqrt{25-(x-2)^2-y^2}}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(\sqrt{25-(x-2)^2-y^2}\right) = \frac{-y}{\sqrt{25-(x-2)^2-y^2}}$$

Set the partial derivatives equal to zero:

$$\frac{-(x-2)}{\sqrt{25-(x-2)^2-y^2}} = 0 \implies x = 2$$

$$\frac{-y}{\sqrt{25-(x-2)^2-y^2}} = 0 \implies y = 0$$

Thus, the critical point is $(x,y) = (2,0)$.

Since the function represents the top half of a circle, the value at $(2,0)$ is the maximum:

$$f(2,0) = \sqrt{25-(2-2)^2-0^2} = \sqrt{25} = 5$$

Conclusion

The function attains a global maximum of 5 at $(2,0)$. The boundary points provide the minimum value of 0, so the function has no saddle points.

2. Function: $f(x, y) = e^{-x^2-y^2}$

Solution Strategy

1. First, compute the first-order partial derivatives to find the critical points.
2. Analyze the second derivatives to classify the critical points (local maxima, minima, or saddle points).

Working

The function is given by:

$$f(x, y) = e^{-x^2-y^2}$$

First-order partial derivatives:

$$f_x(x, y) = \frac{\partial}{\partial x}\left(e^{-x^2-y^2}\right) = -2xe^{-x^2-y^2}$$

$$f_y(x, y) = \frac{\partial}{\partial y}\left(e^{-x^2-y^2}\right) = -2ye^{-x^2-y^2}$$

Setting the partial derivatives equal to zero:

$$-2xe^{-x^2-y^2} = 0 \quad \text{and} \quad -2ye^{-x^2-y^2} = 0$$

This gives the critical point $(x, y) = (0, 0)$.

Second-order partial derivatives:

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(-2xe^{-x^2-y^2}\right) = (4x^2 - 2)e^{-x^2-y^2}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}\left(-2ye^{-x^2-y^2}\right) = (4y^2 - 2)e^{-x^2-y^2}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(-2xe^{-x^2-y^2}\right) = 4xye^{-x^2-y^2}$$

At $(x, y) = (0, 0)$, the second derivatives are:

$$f_{xx}(0, 0) = -2, \quad f_{yy}(0, 0) = -2, \quad f_{xy}(0, 0) = 0$$

The Hessian matrix is:

$$H = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

The determinant of the Hessian is:

$$\det(H) = (-2)(-2) - (0)(0) = 4$$

Since $\det(H) > 0$ and $f_{xx}(0, 0) < 0$, the critical point is a local maximum.

Conclusion

The function has a local maximum at $(0, 0)$, with no saddle points. Since the function tends to 0 as x and $y \to \infty$, the local maximum is also a global maximum.