Mathematical Economics
ECON2050: Tutorial 6

Harley Blackwood 47473159

h.blackwood@uq.edu.au

Tutorial Overview:

Introduction

This tutorial focuses on the applications of convexity in economic theory, particularly in the context of optimization problems. Understanding convexity is essential for analyzing economic models that assume the existence of unique solutions.

Topics Covered

- **Problem 1: Convex Sets Defined by Convex Functions**
  Demonstrates how a convex function defines a convex set, introducing the concept of lower contour sets and their significance in economics.
- **Problem 2: Convexity of Specific Sets**
  Explores the convexity of various sets using the results from Problem 1, emphasizing the intersection of convex sets in multi-dimensional spaces.
- **Problem 3: Strict Convexity**
  Establishes the strict convexity of the quadratic function $f(x) = x^2$ by direct application of the definition, without relying on existing theorems about convex functions.
- **Problem 4: Convexity of Complex Sets**
  Examines more complex convex sets that involve multiple constraints, applying the concepts learned in previous problems to demonstrate their convexity.

Objectives

The primary objectives of this tutorial are:

- To understand and prove the convexity of sets defined by convex functions.
- To apply convexity concepts to specific sets and analyze their properties.
- To develop a deeper understanding of strict convexity and its implications in economic models.

Expected Outcomes

By the end of this tutorial, students should be able to:

- Prove that certain sets are convex based on the properties of convex functions.
- Understand the significance of convex and strictly convex functions in optimization.
- Apply these concepts to more complex economic models and optimization problems.

Problem 1: Convex Set Defined by a Convex Function

Relevance in Economics

The concept of convex sets and functions is crucial in optimization problems, where many economic models assume convexity to guarantee optimal solutions. Convexity ensures that any weighted average of two points within the set or function lies within the set or is at least as good as the value of the function at those points.

Applications

- **Utility Theory:** Convex utility functions indicate preferences where consumers prefer diversified bundles of goods.
- **Production Theory:** Convex production sets imply that firms can efficiently mix input combinations to achieve output levels.

Lower Contour Set

A **lower contour set** is a concept used in the study of functions, particularly in economics and optimization theory.

Definition

For a given function $g : \mathbb{R}^n \to \mathbb{R}$ and a real number b, the **lower contour set** (or sub-level set) corresponding to b is defined as the set of all points in the domain where the function's value is less than or equal to b. Formally, the lower contour set is given by:

$$S_b = \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq b\}$$

Interpretation

- **Graphically:** In the context of functions from $\mathbb{R}^2$ to $\mathbb{R}$, the lower contour set S_b represents the region in the plane where the function $g(x, y)$ has values that do not exceed b. Imagine a topographical map where each contour line represents a different elevation level. The lower contour set would be like all the areas at or below a certain elevation.
- **Economically:** In utility theory, if g represents a utility function and b represents a certain level of utility, then the lower contour set contains all the combinations of goods (or strategies, etc.) that provide a utility level of b or less. It can be used to understand the preferences of a consumer, as the lower contour set includes all the bundles of goods that are "less preferred" compared to a certain utility level b.

Properties

- **Convexity:** If the function g is convex, then its lower contour set S_b is a convex set. This is because, for any two points within the set S_b, the convex combination of these points will also be in S_b. This property is useful in optimization, where convex sets have desirable properties like the existence of a unique minimum.
- **Level Sets vs. Lower Contour Sets:** A level set of a function is where $g(\mathbf{x}) = b$, whereas the lower contour set includes all points where $g(\mathbf{x}) \leq b$. Therefore, the lower contour set includes the level set as well as all points where the function value is strictly less than b.

The Problem

Demonstrate that if $g : \mathbb{R}^2 \to \mathbb{R}$ is a convex function and $b \in \mathbb{R}$, then the set $S = \{(x, y) \in \mathbb{R}^2 : g(x, y) \leq b\}$ is a convex set. To prove that the set S is convex, we need to show that for any two points (x_1, y_1) and (x_2, y_2) in S, the line segment joining them also lies within S. This follows from the definition of a convex function.

Solution Strategy

1. Assume $(x_1, y_1) \in S$ and $(x_2, y_2) \in S$, so $g(x_1, y_1) \leq b$ and $g(x_2, y_2) \leq b$.
2. Consider a point (x, y) on the line segment joining (x_1, y_1) and (x_2, y_2). This can be expressed as $(x, y) = \lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)$, where $\lambda \in [0, 1]$.
3. Use the convexity of g to show that $g(\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)) \leq \lambda g(x_1, y_1) + (1 - \lambda)g(x_2, y_2) \leq b$.

Formal Proof

Want to Show

We want to show that the set $S = \{(x, y) \in \mathbb{R}^2 : g(x, y) \leq b\}$ is convex. Specifically, this means that if $(x_1, y_1) \in S$ and $(x_2, y_2) \in S$, then for any $\lambda \in [0, 1]$, the convex combination $\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)$ is also in S.

What We Have (Assumptions)

It is given that:

- $g : \mathbb{R}^2 \to \mathbb{R}$ is a convex function, meaning that for any $\lambda \in [0, 1]$ and any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,

$$g(\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)) \leq \lambda g(x_1, y_1) + (1 - \lambda)g(x_2, y_2).$$

- $(x_1, y_1) \in S$ and $(x_2, y_2) \in S$, meaning $g(x_1, y_1) \leq b$ and $g(x_2, y_2) \leq b$.

Working

Consider a point (x, y) on the line segment joining (x_1, y_1) and (x_2, y_2). This point can be expressed as:

$$(x, y) = \lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2),$$

where $\lambda \in [0, 1]$. By the convexity of g, we have:

$$g(\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)) \leq \lambda g(x_1, y_1) + (1 - \lambda)g(x_2, y_2).$$

Since $(x_1, y_1) \in S$ and $(x_2, y_2) \in S$, we know:

$$g(x_1, y_1) \leq b \quad \text{and} \quad g(x_2, y_2) \leq b.$$

Therefore:

$$\lambda g(x_1, y_1) + (1 - \lambda)g(x_2, y_2) \leq \lambda b + (1 - \lambda)b = b.$$

Formal Proof

We need to show that if $(x_1, y_1) \in S$ and $(x_2, y_2) \in S$, then $\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \in S$ for all $\lambda \in [0, 1]$. Given:

$$g(x_1, y_1) \leq b \implies \lambda g(x_1, y_1) \leq \lambda b, \quad (1) \tag{1}$$
$$g(x_2, y_2) \leq b \implies (1 - \lambda)g(x_2, y_2) \leq (1 - \lambda)b, \quad (2) \tag{2}$$

By the convexity of g and combining inequalities (1) and (2), we get:

$$g(\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)) \leq \lambda g(x_1, y_1) + (1 - \lambda)g(x_2, y_2) \leq \lambda b + (1 - \lambda)b = b.$$

Thus:

$$g(\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)) \leq b,$$

implying $\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \in S$. Therefore, S is convex.

Problem 2: Convexity of Specific Sets

Relevance in Economics

Understanding which sets are convex is vital in various economic models, such as feasible regions in optimization problems, and determining the shapes of production possibility frontiers.

Applications

- **Feasible Set in Optimization:** Ensuring the feasible set is convex guarantees that local optima are also global optima.
- **Risk Assessment:** In finance, convex sets are used to describe acceptable risk-return profiles.

The Problem

Apply the result from Problem 1 and the properties of convex sets to demonstrate that the following sets are convex:

1. $S = \{(x, y) \in \mathbb{R}^2 : y \geq x, x^2 + y^2 \leq 4\}$
2. $S = \{(x, y) \in \mathbb{R}^2 : y \geq x^2, y \leq 4\}$

Intersection of Convex Sets

Statement:
Let C_1 and C_2 be two convex sets in a vector space $\mathbb{R}^n$. The intersection of C_1 and C_2, denoted by $C_1 \cap C_2$, is also a convex set.
Proof:
Assume C_1 and C_2 are convex sets. By definition, a set C in $\mathbb{R}^n$ is convex if for any two points $x_1, x_2 \in C$ and any $\lambda \in [0, 1]$, the point $\lambda x_1 + (1 - \lambda)x_2$ also belongs to C.
Let $x_1, x_2 \in C_1 \cap C_2$. This implies that $x_1 \in C_1$ and $x_1 \in C_2$, and similarly $x_2 \in C_1$ and $x_2 \in C_2$.
Now consider any convex combination of x_1 and x_2, which is given by

$$\lambda x_1 + (1 - \lambda)x_2 \quad \text{for some } \lambda \in [0, 1].$$

Since C_1 is convex, it follows that

$$\lambda x_1 + (1 - \lambda)x_2 \in C_1.$$

Similarly, since C_2 is convex, it follows that

$$\lambda x_1 + (1 - \lambda)x_2 \in C_2.$$

Therefore, $\lambda x_1 + (1 - \lambda)x_2 \in C_1 \cap C_2$. This shows that any convex combination of points in $C_1 \cap C_2$ also lies in $C_1 \cap C_2$, hence $C_1 \cap C_2$ is convex.

Breakdown

We'll analyze each set individually using the results from Problem 1 and properties of convex sets, such as the intersection of convex sets being convex.

Solution Strategy

1. For S_1:
   - $y \geq x$ represents a half-space, which is convex.
   - $x^2 + y^2 \leq 4$ represents a disk, which is also convex.
   - The intersection of two convex sets is convex.
2. For S_2:
   - $y \geq x^2$ is the area above a parabola, which is convex.
   - $y \leq 4$ is a horizontal line, representing a convex half-space.
   - The intersection of these convex sets is also convex.

Formal Solution

For S_1:

- The set $\{(x, y) : y \geq x\}$ is convex because it is a half-space.
- The set $\{(x, y) : x^2 + y^2 \leq 4\}$ is convex because it is a disk.
- The intersection $S = \{(x, y) : y \geq x, x^2 + y^2 \leq 4\}$ is convex because the intersection of two convex sets is convex.

For S_2:

- The set $\{(x, y) : y \geq x^2\}$ is convex because the parabola $y = x^2$ defines a convex region.
- The set $\{(x, y) : y \leq 4\}$ is convex because it is a half-space.
- The intersection $S = \{(x, y) : y \geq x^2, y \leq 4\}$ is convex.

Problem 3: Strict Convexity of a Function

Relevance in Economics

Strict convexity ensures uniqueness in optimization solutions, making it a powerful property in economics, particularly in utility maximization and cost minimization.

Applications

- **Consumer Theory:** Strictly convex utility functions ensure that consumers prefer diversified bundles of goods, leading to unique consumption choices.
- **Production Theory:** Strictly convex cost functions guarantee that firms have a unique optimal input combination.

The Problem

Prove from the definition of a strictly convex function and without using any theorem about convex functions that $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is strictly convex.

Breakdown

We need to show that for any two distinct points x_1 and x_2, and $\lambda \in (0, 1)$, the function satisfies the inequality:

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda)f(x_2)$$

Solution Strategy

1. Define $f(x) = x^2$ and compute $f(\lambda x_1 + (1-\lambda)x_2)$.
2. Show that this is less than $\lambda x_1^2 + (1-\lambda)x_2^2$.

Theorem: Convex and Strictly Convex Functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function.

1. **Convexity**: The function f is said to be *convex* if for all $x, y \in \mathbb{R}^n$ and for all $\lambda \in [0, 1]$, the following inequality holds:
$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

2. **Strict Convexity**: The function f is said to be *strictly convex* if for all $x, y \in \mathbb{R}^n$ with $x \neq y$ and for all $\lambda \in (0, 1)$, the following strict inequality holds:
$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y).$$

Proof that $f(x) = x^2$ is strictly convex

We will prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is strictly convex by directly applying the definition of strict convexity.

1. Statement

We need to prove that for any $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$ and any $\lambda \in (0, 1)$, the following inequality holds:

$$f(\lambda x_1 + (1 - \lambda)x_2) < \lambda f(x_1) + (1 - \lambda)f(x_2).$$

2. Assumptions

Let $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$ and let $\lambda \in (0, 1)$.

3. Working

The function $f(x) = x^2$ at the point $\lambda x_1 + (1 - \lambda)x_2$ is:

$$f(\lambda x_1 + (1 - \lambda)x_2) = (\lambda x_1 + (1 - \lambda)x_2)^2.$$

Expanding this, we get:
$$(\lambda x_1 + (1 - \lambda)x_2)^2 = \lambda^2 x_1^2 + 2\lambda(1 - \lambda)x_1 x_2 + (1 - \lambda)^2 x_2^2.$$
Now consider the weighted sum:
$$\lambda f(x_1) + (1 - \lambda)f(x_2) = \lambda x_1^2 + (1 - \lambda)x_2^2.$$

4. Proof

To prove strict convexity, we need to show that:

$$(\lambda x_1 + (1 - \lambda)x_2)^2 < \lambda x_1^2 + (1 - \lambda)x_2^2.$$

Subtract the right-hand side from the left-hand side:

$$\lambda^2 x_1^2 + 2\lambda(1 - \lambda)x_1 x_2 + (1 - \lambda)^2 x_2^2 - \lambda x_1^2 - (1 - \lambda)x_2^2.$$

Simplify the expression:
$$(\lambda^2 - \lambda)x_1^2 + 2\lambda(1 - \lambda)x_1 x_2 + ((1 - \lambda)^2 - (1 - \lambda))x_2^2.$$
This simplifies further to:
$$\lambda(1 - \lambda)(x_1^2 - 2x_1 x_2 + x_2^2).$$
Recognizing that $(x_1 - x_2)^2 = x_1^2 - 2x_1 x_2 + x_2^2$, we have:

$$\lambda(1 - \lambda)(x_1 - x_2)^2.$$

Since $\lambda \in (0, 1)$ and $x_1 \neq x_2$, it follows that $\lambda(1 - \lambda)(x_1 - x_2)^2 > 0$, proving the strict inequality:

$$f(\lambda x_1 + (1 - \lambda)x_2) < \lambda f(x_1) + (1 - \lambda)f(x_2).$$

Therefore, $f(x) = x^2$ is strictly convex.

Solution

Statement: f is strictly convex if for all $x, x' \in \mathbb{R}$ such that $x \neq x'$, and all $\lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)x') < \lambda f(x) + (1-\lambda) f(x').$$

Proof: So let $x, x' \in \mathbb{R}$ be such that $x \neq x'$, and $\lambda \in (0,1)$. Then

$$\begin{aligned} f(\lambda x + (1-\lambda)x') &= (\lambda x + (1-\lambda)x')^2 \\ &= \lambda^2 x^2 + 2\lambda(1-\lambda)xx' + (1-\lambda)^2(x')^2 \qquad (3) \\ &< \lambda x^2 + (1-\lambda)(x')^2 \qquad (4) \\ &= \lambda f(x) + (1-\lambda) f(x') \end{aligned}$$

is equivalent to (as the following equals $0 < (4) - (3)$)

$$\begin{aligned} 0 &< \lambda(1-\lambda)x^2 - 2\lambda(1-\lambda)xx' + (1-\lambda)[1-(1-\lambda)](x')^2 \\ &= \lambda(1-\lambda)(x-x')^2 \end{aligned}$$

which is satisfied since $x \neq x'$ and $\lambda \in (0,1)$. Hence

$$f(\lambda x + (1-\lambda)x') < \lambda f(x) + (1-\lambda) f(x')$$

and f is strictly convex.

Problem 4: Convexity of Complex Sets

Relevance in Economics

Identifying convex sets in complex spaces is essential in understanding feasible regions for multivariate optimization problems, such as in resource allocation and production.

Applications

- **Feasibility in Production:** Convexity helps in determining whether a combination of inputs and outputs is feasible in production.
- **Risk Management:** Convex regions define acceptable combinations of risk and return in portfolio optimization.

The Problem

Apply the result from Problem 1 and the properties of convex sets to demonstrate that the following sets are convex.

1. $S = \{(x, y) \in \mathbb{R}^2 : y \geq x^2, x^2 + y^2 \leq 4, x \geq 0, y \geq 0\}$
2. $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 25, x + y \geq 5, y \geq 2, x \geq 2\}$

Breakdown

We need to examine the convexity of these sets by considering each constraint individually and then applying the intersection property of convex sets.

Solution Strategy

1. For S_1:
   - $y \geq x^2$ is convex.
   - $x^2 + y^2 \leq 4$ is convex.
   - $x \geq 0$ and $y \geq 0$ are convex half-spaces.
   - The intersection of these convex sets is convex.
2. For S_2:
   - $x^2 + y^2 \leq 25$ is convex.
   - $x + y \geq 5$, $y \geq 2$, and $x \geq 2$ are convex half-spaces.
   - The intersection of these convex sets is convex.

Formal Solution

For Set 1:

- The parabola constraint $y \geq x^2$ is convex.
- The disk $x^2 + y^2 \leq 4$ is convex.
- The half-space constraints $x \geq 0$ and $y \geq 0$ are convex.
- Therefore, the intersection $S = \{(x, y) : y \geq x^2, x^2 + y^2 \leq 4, x \geq 0, y \geq 0\}$ is convex.

For Set 2:

- The disk $x^2 + y^2 \leq 25$ is convex.
- The half-space $x + y \geq 5$ is convex.
- The half-space constraints $y \geq 2$ and $x \geq 2$ are convex.
- Therefore, the intersection $S = \{(x, y) : x^2 + y^2 \leq 25, x + y \geq 5, y \geq 2, x \geq 2\}$ is convex.