Analytical Geometry and Linear Algebra. Lecture 4.

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Review. Lecture 4

- Part 1. Change of basis and coordinates
- Part 2. Matrix inverse

Objectives for today

1. To be able to change basis of a vector space
2. To be able to apply formula for changing coordinates
3. To be understand what an inverse matrix is (aka A^{-1})

Change of basis and coordinates

Here is the link to the core material in moodle.

https://moodle.innopolis.university/pluginfile.php/206799/mod_resource/content/1/AGLA1__Lecture_3-2-2.pdf

Break 5 min.

Matrix inverse

Matrix B is called inverse of a square matrix A if

$$AB = BA = I$$

Notation

$$B = A^{-1}$$

Simple view

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Notation

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$$AA^{-1} = A^{-1}A = I$$

Example of inverse matrix

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}$$

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$$= \frac{1}{6} \begin{bmatrix} 4*2+1*(-2) & 4*(-1)+1*(4) \\ 2*2+2*(-2) & 2*(-1)+2*4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = I$$

What if matrix A is **nonsquare**?

Left and Right inverse

Left inverse

Consider an $m \times n$ matrix A and $n \times m$ matrix B.
If $BA = I$, then we say B is the **left inverse** of A.

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If $AC = I$, then we say C is the **right inverse** of A.

Let A be a square matrix. Show that its left and right inverses are the same.
Hint: use associative property of matrix multiplication.

If A has an inverse, then A is ***invertible***

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Are all matrices invertible?

Provide a simple counter-example of noninvertible 3×3 matrix.

Important property

If A and B are invertible and AB is invertible, then

$$(AB)^{-1} = B^{-1}A^{-1}$$

Prove it, using pen and paper.

Hint: multiply $(B^{-1}A^{-1})$ by (AB).

How to find an inverse of 2×2 matrix A?

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A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

Step 0: Find determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = **ad-bc.** If $det(A) = 0$, then A^{-1} **does not exist**.

Step 1: Swap **main** diagonal elements:

$$\begin{bmatrix} \mathbf{d} & b \\ c & \mathbf{a} \end{bmatrix},$$

Step 2: Multiply off-diagonal elements by -1:

$$\begin{bmatrix} d & \mathbf{-b} \\ \mathbf{-c} & a \end{bmatrix}$$

Step 3: Divide by $det(A)$. So, A^{-1} = $\frac{1}{det(A)} \begin{bmatrix} d & \mathbf{-b} \\ \mathbf{-c} & a \end{bmatrix}$

Exercise

A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; A^{-1} = $\frac{1}{det(A)}\begin{bmatrix} d & \textbf{-b} \\ \textbf{-c} & a \end{bmatrix}$

Check with pen and paper

$$A^{-1}A = \dots$$

Important case: Orthogonal matrix

$$A^{-1} = A^{\top}$$

For orthogonal matrix:

$$AA^{\top} = A^{\top}A = I$$

Example

Rotation matrix is an example of an orthogonal matrix.

Rotation matrix

$$R = \begin{bmatrix} \cos(\theta) & \text{-}\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix};\ R^\top = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \text{-}\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Assignment:

- Find R^{-1}
- Show that $det(R) = 1$ (this is true for any rotation matrix)
- Given that $\theta = \frac{\pi}{2}$ Find R^2, R^3, R^4?

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- The rank of the matrix A is n
- $A^\top$ is invertible
- The rows of matrix A form a basis for $\mathbb{R}^n$
- $A\mathbf{x} = \mathbf{b}$ has exactly one solution ($\mathbf{x} = A^{-1}\mathbf{b}$)
- $A\mathbf{x} = \mathbf{0}$ has only a *trivial* solution ($\mathbf{x} = \mathbf{0}$, zero vector)

Matrix rank

Consider the following matrices ($a \neq b \neq 0$)

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; B = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} a & 0 & a \\ 0 & b & b \end{bmatrix}$$

$$D = \begin{bmatrix} a & 0 & a & -2a \\ 0 & b & b & -2b \end{bmatrix} E = \begin{bmatrix} a & 0 & a & -2a & 3a \\ 0 & b & b & -2b & 2b \end{bmatrix}$$

1) What can you say about columns-vectors inside each matrix?

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1) What can you say about columns-vectors inside each matrix?
2) Which matrices contain basis for $\mathbb{R}^2$?
3) Which matrices contain 'redundant' information about space spanned by column-vectors?

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Theorem

For ANY $m \times n$ matrix the column rank equals to row rank.

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Theorem

For ANY $m \times n$ matrix the column rank equals to row rank.

So, there is only **one** matrix rank. $rank(A) = rank(A^\top)$

Examples. Calculate rank of a matrix and its transpose

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} a & 0 & a \\ 0 & b & b \end{bmatrix}, rank(C) = rank(C^\top) =?$$

$$D = \begin{bmatrix} a & 0 & a & -2a \\ 0 & b & b & -2b \end{bmatrix},$$

More examples

For the following $m \times m$ matrices, which value of λ would give each matrix rank $m-1$?

$$A = \begin{bmatrix} 1 & 3 \\ 1 & \lambda \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & \lambda \\ 1 & 0 & 3 \end{bmatrix}$$

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- $rank(AB) \leq \min(rank(A), rank(B))$
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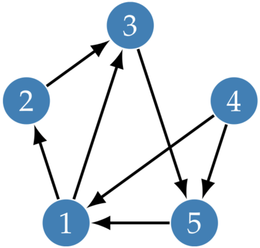
What about $rank(\lambda A)$?
$\lambda \in \mathbb{R}$

Break. 5 min.

Applications

Graphs and Matrices

Given a graph you can define its **adjacency** matrix, A



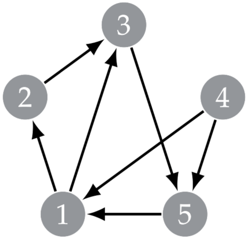
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix represents paths of length 1 (e.g. one ‘hop’ between 4 and 1)

Graphs and Matrices: Powers of A

Given an adjacency matrix, A you can find its power ($A^2 = AA$)

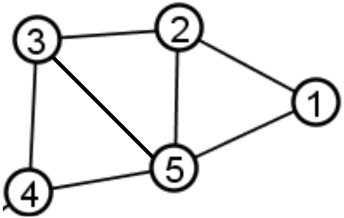
$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$



Matrix represents paths of length 2 (e.g. two 'hops' to reach 5 from 1)

Graphs and Matrices: Example

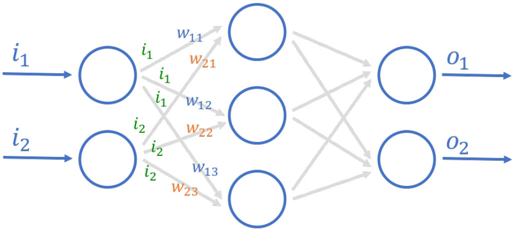
Given a graph G



Build its adjacency matrix, A
Find A^3.
Find the trace of A^3, $Tr(A^3)$

How can you interpret it?

Neural Networks and Matrices

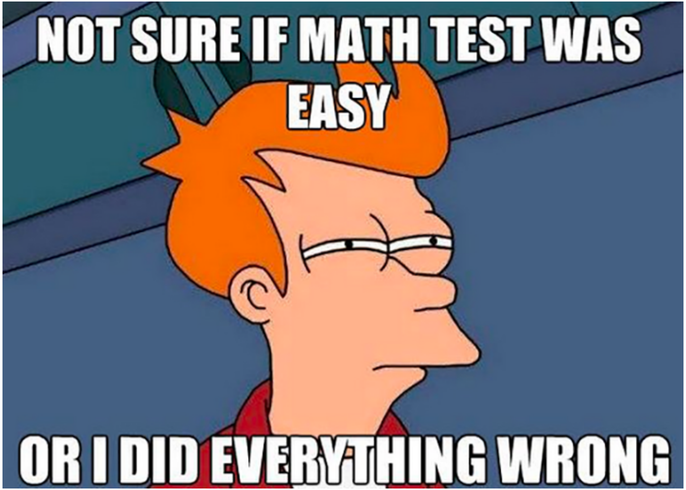


$$\begin{bmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \\ w_{13} & w_{23} \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} (w_{11} \times i_1) + (w_{21} \times i_2) \\ (w_{12} \times i_1) + (w_{22} \times i_2) \\ (w_{13} \times i_1) + (w_{23} \times i_2) \end{bmatrix}$$

+ Non-linear transformation of result !

Source: `https://sausheong.github.io/posts/`

End of Lecture #5



Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila