Analytical Geometry and Linear Algebra. Lecture 1.

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Outline

- Part 1. About the course
- Part 2. Introduction. Vector spaces. Linear independence. Basis
- Part 3. Dot product

Main questions for today's lecture

- What is this course about?

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- How to get high grade in this course?

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- How to use this course in your projects?

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What is this course about?

Topics of the course

1. Vector spaces, Change of basis of the vector space
2. Matrices and transformations in 2D and 3D
3. Lines and planes
4. Conics or quadric curves
5. Quadratic surfaces
6. Polar and spherical coordinates

Goals of this course

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8. to solve applied problems with vectors/matrices

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7. to operate with quadric curves, such as ellipse, hyperbola and parabola
8. to solve applied problems with vectors/matrices
9. many more + some examples in Python :)

Main questions for today's lecture

How to get a high grade in this course?

Grading in the course

1. Test 1 12%
2. Midterm 35%
3. Test 2 18%
4. Final Exam 35%

In total, 100 %
No bonus points. :(

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No bonus points. :(
But you can have up to **10 points** of the course from your lab's instructors.

How to get the highest grade?

1. Attend classes
   1. Labs
   2. Tutorials
   3. Lectures
2. Solve assignments (also at home) on your own and in groups
3. Read books (check the list in moodle)
4. Come to office hours (info is in moodle)

Repeat :)

What is the exact process you can follow?

1. **Wednesday**
   1. attend lecture and tutorial
   2. review materials after classes
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4. **Tuesday**
    1. review last week, your questions
    2. do not forget about other courses

Wrong way to go is...

1. **Monday – Sunday**
   1. Dota, dota, DoTa...

Team of the course and Resources

- Vladimir Ivanov (PhD), Principal Instructor, Lectures
- Ivan Konyukhov (PhD), Tutorials
- Amer Albadr, Labs
- Oleg Bulichev, Labs
- Eugene Marchuk, Labs
- Egor Dmitriev, Labs
- A secret TA, Labs

Resources: Books, Assignments, Useful links, etc.

Please, check Moodle!

Main questions for today's lecture

Applications of Linear Algebra and Analytical Geometry

How to use this course in your projects?

Applications of AGLA in Computer Science and Engineering

Areas:

1. Computer Graphics and Computer Games
2. Machine Learning, Data Analysis
3. Natural Language Processing
4. Robotics
5. Computer Vision

6. and many, many other areas...
7. maybe, even in the backend...

Applications of AGLA

Computer Graphics and Computer Games

- 2D/3D graphics
- Projective geometry, Homogeneous coordinates
- Collision detection in games. Calculation of trajectories

Machine Learning, Data Analysis

- Linear Regression
- Eigenvalue decomposition
- Singular Value Decomposition
- Covariance matrix
- Linear Layers, Attention Mechanism in Neural Networks

Break 5 min.

Agenda: Week 1

Vectors. Linear Independence

- Points and Vectors
- Vector Addition. Scalar Vector Multiplication
- Properties of Vector Arithmetic
- Vector spaces, Subspaces
- Span, Linear Independence
- Vector Bases and Vector Coordinates in Basis

Notation

- We denote **points** by capital italic letters, e.g., $A, B, ..., Q, ...$
- We denote **scalars** (numbers) by Greek letters, e.g., $\alpha, \beta, ..., \lambda, \theta, ...$ and sometimes by Latin letters, $a, b, ..., v, u, x, ...$
- We denote **vectors** by **bold** letters, e.g., $\mathbf{a}, \mathbf{b}, ..., \mathbf{v}, \mathbf{u}, \mathbf{x}, ...,$
- and also we denote vectors by a letter with an arrow, e.g. $\vec{a}, \vec{b}, \vec{u}$
- and sometimes we denote vectors by end-points, e.g. $\overline{AB}, \overline{BC}, \overline{OA}$
- $\mathbb{R}$ is the set of **real** numbers
- $\mathbb{C}$ is the set of **complex** numbers

Introduction

Points and Vectors (**informally**). Direction

Vector. Geometrical point of view. Vectors as ‘arrows’ in plane or in 3D space

Let A and B be two points.

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Thus, a vector is a equivalence class of directed line segments with the same direction and length.

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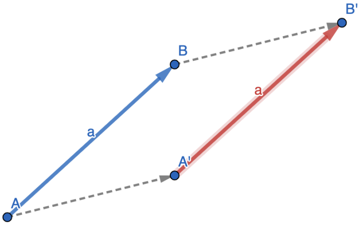
We define **a vector** as all directed line segments sharing the same direction and length.
Thus, a vector is a equivalence class of directed line segments with the same direction and length.

Thus, each vector can be associated with a notion of *direction*. In this case, we can think of a vector as an "arrow" in space.

If you move the line segment to another line segment with the same direction and length, they constitute **the same vector**.



Points and Vectors (**informally**). Magnitude

Length (or Magnitude) of a Vector

Also, often (**but not always!**) vector has a *length* (or a magnitude). The length of a vector is denoted by $\|\mathbf{v}\|$.

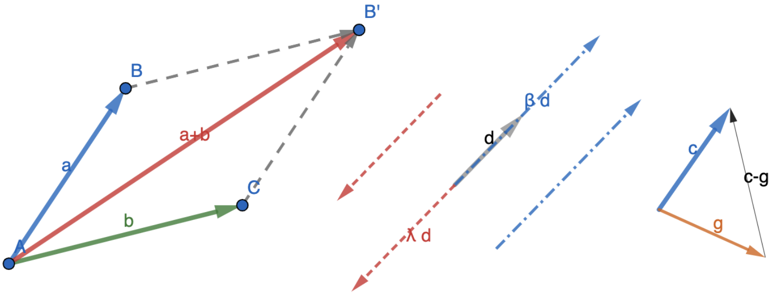
Unit vector

A *unit vector*, $\mathbf{u}$ is a vector with unit length (so $\|\mathbf{u}\|$=1).
We can derive a unit vector as $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$.

The length of a vector is closely related to the **dot product**, an operation which will be discussed in the next lecture. $\mathbf{v}/\|\mathbf{v}\|$ is called a normalized vector.

Examples: Points and Vectors (informally)

Note that vector $\lambda\mathbf{d}$ is either parallel ($\lambda > 0$) to or anti-parallel ($\lambda < 0$) to $\mathbf{d}$.



$$\lambda, \beta \in \mathbb{R}$$

In this figure: $\lambda > 0$?

What if $\lambda = 0$?

Vector spaces

Vector space definition

Vector space

A *vector space* V over $\mathbb{R}$ (or $\mathbb{C}$) is a collection of vectors, together with two operations:

- $\mathbf{a} + \mathbf{b}$, addition of two vectors and
- $\lambda\mathbf{a}$, multiplication by a scalar ($\lambda \in \mathbb{R}$)

A scalar is a number from $\mathbb{R}$ or $\mathbb{C}$, respectively.

Addition and multiplication SHOULD satisfy following axioms

Vector addition axioms

Vector addition $\mathbf{a} + \mathbf{b}$ is defined $\forall \mathbf{a}, \mathbf{b} \in V$

Vector addition has to satisfy the following axioms:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (associativity)
3. There is a vector $\mathbf{0}$ (zero vector) such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$. (identity)
4. For each vector $\mathbf{a}$, there exists a vector $(-\mathbf{a})$ such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ (inverse)

Axioms of multiplication (by a scalar)

$\lambda\mathbf{a}$ is defined $\forall\lambda \in \mathbb{R}, \forall\mathbf{a} \in V$

Scalar multiplication has to satisfy the following axioms:

1. $\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$.
2. $(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$.
3. $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a} = \mu(\lambda)\mathbf{a}$.
4. $1\mathbf{a} = \mathbf{a}$ (here $\lambda = 1$).

The scalar is called a *scalar*, because it **scales** a vector :)

Homework Assignment: Prove 2 facts using the axioms

Prove

The zero vector is unique.

Prove

The inverse vector $(-\mathbf{a})$ is unique for any vector $\mathbf{a}$.

Vectors as lists of numbers

Column vectors. Examples

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ — we will use **this notation**! We represent vectors as **columns**!

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Row vectors. Examples

$\begin{bmatrix} 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$, $\begin{bmatrix} x & y & z \end{bmatrix}$ Even though vectors can be represented as rows.

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$\begin{bmatrix}3 & 4\end{bmatrix} \neq \begin{bmatrix}3\\4\end{bmatrix}$ Contents is the same, but **shapes** of the vectors are not the same.

Transposition

Transposition

$$\begin{bmatrix} 3 & 4 \end{bmatrix}^\top = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix}^\top = \begin{bmatrix} 3 & 4 \end{bmatrix} \quad (2)$$

This operation transforms a row-vector to a column-vector and back

For any vector

$$(\mathbf{v}^\top)^\top = \mathbf{v}$$

Examples

Example (extra)

Vector space V consisting of all functions $f(x)$ that are continuous on $\mathbb{R}$

$$V = \{f(x), \texttt{such that} f(x) \texttt{is continuous on } \mathbb{R}\}$$

Linear combination and linear independence

Linear combination

Vector $\mathbf{w} \in V$ is a linear combination of vectors $\mathbf{v_1}, \ldots, \mathbf{v_n} \in V$ with coefficients $c_k \in \mathbb{R}; (k = 1..n)$
such that

$$\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \sum_{k=1}^{n} c_k\mathbf{v_k}$$

Span

Span

Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\} \subset V$.

$$span(S) \equiv \left\{ \mathbf{w} \in V : \mathbf{w} = \sum_{k=1}^{n} c_k \mathbf{v_k}, \quad \forall c_k \in \mathbb{R} \right\}$$

Basically, $W = span(S)$ is the set of all (possible) linear combinations of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$.

Subspace

Definition

W is a subspace of V if

a)	$W \subset V$	(subset)
b)	$\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$	(closure under addition)
c)	$\mathbf{u} \in W, \lambda \in \mathbb{R} \Rightarrow \lambda\mathbf{u} \in W$	(closure under scalar multiplication)

Examples

Linear independence in $\mathbb{R}^2$ and in $\mathbb{R}^3$

Linearly independent vectors in $\mathbb{R}^2$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *linearly independent*
if for $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1\mathbf{a} + \alpha_2\mathbf{b} = \mathbf{0}$ if and only if $\alpha_1 = \alpha_2 = 0$.

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Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are *linearly independent* if for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, $\alpha_1\mathbf{a} + \alpha_2\mathbf{b} + \alpha_3\mathbf{c} = \mathbf{0}$ if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

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Try to give a definition for Linearly independent vectors in $\mathbb{R}^n$

Basis

A **set** of vectors is a *basis* of a vector space if it spans a vector space and this set is **linearly independent**.

Basis in $\mathbb{R}^2$ and $\mathbb{R}^3$

Basis in $\mathbb{R}^2$

A **set** of vectors is a *basis* of $\mathbb{R}^2$ if it spans $\mathbb{R}^2$ and this set is **linearly independent.**

Standard basis in $\mathbb{R}^2$

$\{\hat{\mathbf{i}}, \hat{\mathbf{j}}\} = \{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$. They are the standard basis in $\mathbb{R}^2$.

Standard basis in $\mathbb{R}^3$

$\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\} = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$. They are the standard (canonical) basis in $\mathbb{R}^3$.

Examples

Representation of a Vector in Vector Space

Theorem

Let V be a vector space over $\mathbb{R}^n$ and let $\{\mathbf{e_1}, ..., \mathbf{e_n}\}$ be a basis.
Then each vector $\mathbf{u}$ can be identified with its coordinates $\{u_1, ..., u_n\}$ in the basis.

$$\mathbf{u} = \sum_{k=1}^{n} u_k \mathbf{e_k}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$$

Homework Assignment

Let P_3, be a set of all polynomials of degree 3 or less.

Show that P_3 is a vector space over $\mathbb{R}$.
Hint: check axioms of vector space.

What could be a basis of P_3?
Give examples of two bases in P_3.
Express the polynomial $x^3 - 2x^2 + 3$ in the basis.

End of Lecture 1.

Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila
- http://brilliant.com

Lecture 2.

Outline

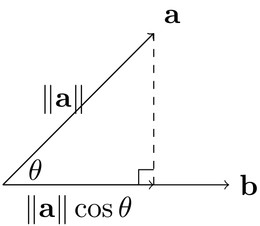
- Part 3. The Dot Product and its properties
  - Norm of a vector
  - Cauchy-Schwarz inequality
  - Triangle Inequality

Dot Product

Geometric view (in $\mathbb{R}^2$ and $\mathbb{R}^3$)

Scalar/dot product

$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.



Examples

Scalar projection

Scalar projection of vector $\mathbf{a}$ on vector $\mathbf{b}$ is **a scalar**: $a_b = \|\mathbf{a}\| \cos \theta$

Find the scalar projections $\mathbf{a_b}$ and $\mathbf{b_a}$.

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

Orthogonal projection

Orthogonal projection of vector $\mathbf{a}$ on vector $\mathbf{b}$ is **a vector**: $\mathbf{a_b} = \hat{\mathbf{b}}\|\mathbf{a}\| \cos \theta$

$\hat{\mathbf{b}}$ is the unit vector in the direction of $\mathbf{b}$

Dot Product. Algebraic view

Definition

Let V be a vector space over $\mathbb{R}$.
By a dot product on V we mean a real valued function $\mathbf{u} \cdot \mathbf{v}$ on $V \times V \to \mathbb{R}$ which satisfies the following axioms:

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- $\mathbf{u} \cdot (\mathbf{w} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{w} + \mathbf{u} \cdot \mathbf{v} \quad , \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

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- $\mathbf{u} \cdot \mathbf{u} = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$

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- $(\lambda \mathbf{u}) \cdot \mathbf{v} = \lambda (\mathbf{u} \cdot \mathbf{v}) \quad , \quad \forall \mathbf{u}, \mathbf{v} \in V, \lambda \in \mathbb{R}$
- $\mathbf{u} \cdot \mathbf{u} \geq 0 \quad , \quad \forall \mathbf{u} \in V$
- $\mathbf{u} \cdot \mathbf{u} = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$

Notation

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{u}, \mathbf{v}) = \mathbf{u}^\top \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle$$

Dot Product. Calculation

Dot product in $\mathbb{R}^n$

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

If $\mathbf{u}, \mathbf{v}$ are column vectors, then

$$\mathbf{u}^\top \mathbf{v} = u_1 v_1 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i = \mathbf{u} \cdot \mathbf{v}$$

Examples

Question. Find the angle between $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ -1 \\ -1 \end{bmatrix}$$

Hint

$$\| \mathbf{u} \| \equiv \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

Norm (or a length) of a vector

A norm on any vector space is defined as follows:

Definition

We say $\| \mathbf{u} \|$ is a norm on a vector space V if $\forall \mathbf{u}, \mathbf{v} \in V$ and $\alpha \in \mathbb{R}$,

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d) $\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \|$

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Check

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

Cauchy-Schwarz inequality

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For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|.$$

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Proof

Consider the expression $\|\mathbf{x} - \lambda\mathbf{y}\|^2$. We must have

$$\begin{aligned} \|\mathbf{x} - \lambda\mathbf{y}\|^2 &\geq 0 \\ (\mathbf{x} - \lambda\mathbf{y}) \cdot (\mathbf{x} - \lambda\mathbf{y}) &\geq 0 \\ \lambda^2\|\mathbf{y}\|^2 - \lambda(2\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{x}\|^2 &\geq 0. \end{aligned}$$

Cauchy-Schwarz inequality

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For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

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Consider the expression $\|\mathbf{x} - \lambda \mathbf{y}\|^2$. We must have $\lambda^2 \|\mathbf{y}\|^2 - \lambda(2\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{x}\|^2 \geq 0$.

Viewing this as a quadratic in λ, we see that the quadratic is non-negative. Thus, it cannot have 2 different real roots. The discriminant $\Delta = b^2 - 4ac \leq 0$. So

$$4(\mathbf{x} \cdot \mathbf{y})^2 \leq 4\|\mathbf{y}\|^2 \|\mathbf{x}\|^2$$
$$(\mathbf{x} \cdot \mathbf{y})^2 \leq \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$$
$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

Write some code

Here we open Google Colab...

... to check Cauchy-Schwarz inequality

https://colab.research.google.com/drive/1QKCs22fjRaLks5oSA2QjssqXYgBHMn1A?usp=sharing

Triangle inequality

Triangle inequality

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Triangle inequality

Triangle inequality

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Proof

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) \\
&= \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 \\
&\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\
&= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.
\end{aligned}$$

Orthogonality

Definition

Let V be vector space with a dot product.
Vectors $\mathbf{u}, \mathbf{v} \in V$ are said to be **orthogonal** if

$$\mathbf{u} \cdot \mathbf{v} = 0$$

Examples

Here we open the Geogebra :)

Homework

Show that the difference between a vector $\mathbf{a}$ and its orthogonal projection ($\mathbf{a}_{\mathbf{b}}$) on a vector $\mathbf{b}$ is orthogonal to the vector $\mathbf{b}$.

If

$$\mathbf{p} = \mathbf{a} - \mathbf{a}_{\mathbf{b}}$$

then

$$\mathbf{p} \cdot \mathbf{b} = 0$$

Homework

In the triangle ABC the median AD is divided into three equal segments: AE, EF and FD.

$$\overline{BA} \cdot \overline{CA} = 4$$

$$\overline{BF} \cdot \overline{CF} = -1$$

Find $\overline{BE} \cdot \overline{CE}$.

Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila
- http://brilliant.com