2 The Greeks

2.1 Definitions

Mathematical definitions of Greeks

$$\Delta\,(\text{delta}) = \frac{\partial f}{\partial S_t} \qquad \Gamma\,(\text{gamma}) = \frac{\partial^2 f}{\partial S_t^2} = \frac{\partial \Delta}{\partial S_t} \qquad \Theta\,(\text{theta}) = \frac{\partial f}{\partial t}$$

$$V\,(\text{vega}) = \frac{\partial f}{\partial \sigma} \qquad \rho\,(\text{rho}) = \frac{\partial f}{\partial r} \qquad \lambda\,(\text{lambda}) = \frac{\partial f}{\partial q}$$

- The **Greeks** measure the **sensitivity** of the **price of a derivative** (or, in the case of gamma, the sensitivity of its delta) to **small changes** in the parameters.
- These sensitivities are important to measure for **risk management** purposes.
- Mathematically, they are defined as **partial derivatives** (second-order for gamma, but first-order for the others) of the **theoretical price** with respect to the appropriate **parameter**.
- This definition assumes that the **other parameters** remain **unchanged**. In reality, changes in one parameter may have a **knock-on effect** on the others (*eg* a change in the dividend rate q is likely to affect the share price S_t). So the individual Greeks do not give the full picture.
- The Greeks depend on the particular **pricing model** used (which need not be **Black-Scholes**). So the **numerical values** of the Greeks will have (slightly) different values according to the model assumed.
- We are not interested in the sensitivity to small changes in the **strike price** K, because this is (usually) a fixed quantity. So there is no corresponding Greek. (However, we may still be interested in comparing the prices of options with different strike prices.)
- **Vega** is an **invented** Greek letter (whose name was chosen to match the **V** in **volatility**). Some people insist on using a genuine Greek letter (*eg* kappa) instead.
- The three Greeks that appear in the Black-Scholes PDE are sometimes called the **major Greeks** (**delta**, **gamma** and **theta**). The other three (**vega**, **rho** and **lambda**) are the **minor Greeks**.

2.2 Greeks based on the Black-Scholes model

Lemma

The following lemma (= useful result) is needed to **simplify the formulae** for the **Greeks** when using the **Garman-Kohlhagen model**.

Lemma for simplifying Greeks based on Garman-Kohlhagen model

$$S_t e^{-q(T-t)}\phi(d_1) = Ke^{-r(T-t)}\phi(d_2)$$

- A useful **mnemonic** is that the two terms in the lemma correspond to the two terms in the **Garman-Kohlhagen call option formula** (which is on page 47 of the Tables), but with the **capital** phis (Φ) replaced with **small** phis (ϕ).

Proof of lemma

The lemma is proved in Section 1 of Unit 12 of the Core Reading. The **steps** are:

1. Start with the **ratio** $\dfrac{\phi(d_1)}{\phi(d_2)}$.

2. **Substitute** the formulae for the PDFs for the **standard normal distribution** – see page 9 of the Tables – and cancel the $\sqrt{2\pi}$'s.

3. **Substitute** $d_2 \to d_1 - \sigma\sqrt{T-t}$.

4. **Multiply out** the square and **cancel** the $\frac{1}{2}d_1^2$ terms.

5. **Substitute** $d_1\sigma\sqrt{T-t} \to \log\dfrac{S_t}{K} + (r - q + ½\sigma^2)(T-t)$ from the definition of d_1.

6. **Simplify** using $\exp\left(-\log\dfrac{S_t}{K}\right) = 1\Big/\exp\left(\log\dfrac{S_t}{K}\right) = 1\Big/\left(\dfrac{S_t}{K}\right) = \dfrac{K}{S_t}$.

7. **Rearrange** to prove the lemma.

Formulae for the Greeks

By differentiating and applying the lemma, we can derive the following formulae for the **Greeks** for **call / put options** based on the **Garman-Kohlhagen formulae**.

Greek	Formula	Range
Delta	$\Delta_{call} = e^{-qu}\Phi(d_1)$ $\Delta_{put} = -e^{-qu}\Phi(-d_1) = \Delta_{call} - e^{-qu}$	$0 < \Delta_{call} < 1$ $-1 < \Delta_{put} < 0$
Gamma	$\Gamma_{call} = \Gamma_{put} = \dfrac{e^{-qu}\phi(d_1)}{S_t\sigma\sqrt{u}}$	$\Gamma_{call} = \Gamma_{put} > 0$
Theta	$\Theta_{call} = qS_te^{-qu}\Phi(d_1) - rKe^{-ru}\Phi(d_2) - S_te^{-qu}\phi(d_1)\dfrac{\sigma}{2\sqrt{u}}$ $\Theta_{put} = rKe^{-ru}\Phi(-d_2) - qS_te^{-qu}\Phi(-d_1) - S_te^{-qu}\phi(d_1)\dfrac{\sigma}{2\sqrt{u}}$	Θ_{call}, Θ_{put} are usually both < 0
Vega	$V_{call} = V_{put} = S_te^{-qu}\phi(d_1)\sqrt{u}$	$V_{call} = V_{put} > 0$
Rho	$\rho_{call} = uKe^{-ru}\Phi(d_2)$ $\rho_{put} = -uKe^{-ru}\Phi(-d_2)$	$\rho_{call} > 0$ $\rho_{put} < 0$
Lambda	$\lambda_{call} = -uS_te^{-qu}\Phi(d_1)$ $\lambda_{put} = uS_te^{-qu}\Phi(-d_1)$	$\lambda_{call} < 0$ $\lambda_{put} > 0$

- $T-t$, the **remaining term** of the options, has been abbreviated as u (as in the Core Reading).

- For the **basic Black-Scholes model**, replace q with **zero**.

- All formulae involve a single "clump" except for **theta**, which has **three** separate **terms**. (This is a consequence of the Black-Scholes PDE which relates theta to delta, gamma and the option price.)

- **Theta** is usually **negative**, but for extreme values of the parameters, it can be positive. (There is a numerical example of this in the next chapter.)

Derivation of Greeks

The derivations are included in Section 1 of Unit 12 of the Core Reading. Some of these derivations can be made easier using the following useful tip:

Useful tip when deriving the Greeks

When deriving the formulae for the Greeks, try to apply the lemma **as soon as possible**.

This often makes the algebra a lot easier because it allows you to work with the difference of the d_1 and d_2 terms (which is just $\sigma\sqrt{T-t}$), and avoids you having to differentiate each one separately.

For example, **delta** for a **call option** can be derived most efficiently as follows:

$$\begin{aligned}
\Delta_{call} &= \frac{\partial}{\partial S_t}\left[S_t e^{-q(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)\right] \\
&= e^{-q(T-t)}\Phi(d_1) + S_t e^{-q(T-t)}\phi(d_1)\frac{\partial d_1}{\partial S_t} - Ke^{-r(T-t)}\phi(d_2)\frac{\partial d_2}{\partial S_t} \\
&= e^{-q(T-t)}\Phi(d_1) + S_t e^{-q(T-t)}\phi(d_1)\left[\frac{\partial d_1}{\partial S_t} - \frac{\partial d_2}{\partial S_t}\right] \qquad \text{(from lemma)} \\
&= e^{-q(T-t)}\Phi(d_1) + S_t e^{-q(T-t)}\phi(d_1)\frac{\partial}{\partial S_t}(d_1 - d_2) \\
&= e^{-q(T-t)}\Phi(d_1) \qquad \text{(since } d_1 - d_2 = \sigma\sqrt{T-t}\text{)}
\end{aligned}$$

Mnemonics

Mnemonics for Garman-Kohlhagen model

Delta, rho and lambda

You get the correct formulae for delta, rho and lambda (for both calls and puts) if you naively differentiate the Garman-Kohlhagen formulae treating d_1 and d_2 as constants!

Calls / puts

The Garman-Kohlhagen **pricing formula** and the formulae for the **Greeks** for a **put** option can be obtained from the corresponding formulae for a **call** option by replacing all expressions of the form $\Phi(x)$ with $-\Phi(-x)$, *ie* insert a **minus** sign **inside and outside** of every **capital** Φ function.

- Note that **small** phis are not affected by this rule. As a result, gamma and vega have the same formulae (and values) for calls and puts.

Other results

If derivatives of d_1 or d_2 are required, they can easily be found by rewriting the definitions of d_1 and d_2 as shown in the table below.

Parameter being varied	$d_1, d_2 = \dfrac{\log \frac{S_t}{K} + (r - q \pm \frac{1}{2}\sigma^2)u}{\sigma\sqrt{u}}$	Derivatives of d_1 and d_2
S_t	$\dfrac{\log S_t + \cdots}{\sigma\sqrt{u}}$	$\dfrac{\partial d_1}{\partial S_t} = \dfrac{\partial d_2}{\partial S_t} = \dfrac{1}{S_t \sigma\sqrt{u}}$
K		(not needed for Greeks)
t	$\left(\dfrac{1}{\sigma}\log\dfrac{S_t}{K}\right)u^{-\frac{1}{2}} + \dfrac{(r - q \pm \frac{1}{2}\sigma^2)}{\sigma}u^{\frac{1}{2}}$	$\dfrac{\partial d_1}{\partial t} = \dfrac{\log\frac{S_t}{K} - (r - q + \frac{1}{2}\sigma^2)u}{2\sigma u\sqrt{u}}$ $\dfrac{\partial d_2}{\partial t} = \dfrac{\log\frac{S_t}{K} - (r - q - \frac{1}{2}\sigma^2)u}{2\sigma u\sqrt{u}}$
σ	$\dfrac{\log\frac{S_t}{K} + (r - q)u}{\sigma\sqrt{u}} \pm \frac{1}{2}\sigma\sqrt{u}$	$\dfrac{\partial d_1}{\partial \sigma} = -\dfrac{d_2}{\sigma}, \quad \dfrac{\partial d_2}{\partial \sigma} = -\dfrac{d_1}{\sigma}$
r	$\dfrac{\cdots + r\sqrt{u} + \cdots}{\sigma}$	$\dfrac{\partial d_1}{\partial r} = \dfrac{\partial d_2}{\partial r} = \dfrac{\sqrt{u}}{\sigma}$
q	$\dfrac{\cdots - q\sqrt{u} + \cdots}{\sigma}$	$\dfrac{\partial d_1}{\partial q} = \dfrac{\partial d_2}{\partial q} = -\dfrac{\sqrt{u}}{\sigma}$

- $T - t$, the **remaining term** of the options has been abbreviated as u.
- "..." indicates "constant" terms that disappear when differentiated.
- The formulae for d_1 and d_2 themselves differ only in the sign of the "½ term" (which is + for d_1 and – for d_2).
- If you use the lemma described above, you can avoid having to use these results when deriving formulae for the Greeks for **vanilla** options, except for **gamma**. However, they are required for some of the **exotic** options we will look at in Chapter 9 of the ActEd Notes.

