On the Optimality of Treating Interference as Noise: General Message Sets Revisited

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***Abstract*—We study the optimality of power control and treating interference as noise (TIN) in the $M \times N$ X channel, from the generalized degrees-of-freedom (GDoF) and constant-gap capacity perspectives. A result by Geng, Sun and Jafar shows that if there exist $K = \min(M, N)$ transmitter-receiver pairs such that each direct link strength is no less than the sum of the strongest incoming and strongest outgoing cross link strengths (all in dB), then it is optimal to reduce the $M \times N$ X channel to a K-user interference channel and use TIN. The proof of this result relies on a deterministic approximation of the original Gaussian network, specifically for the case $M < N$. Here we present a simpler proof by working directly with the original Gaussian network. Our proof relies on a new "less noisy under interference" order exhibited by TIN-optimal $M \times N$ X channels, akin to the "less noisy" order in broadcast channels.**

I. Introduction

Geng et al. [1] showed that in a K-user Gaussian interference channel, if certain conditions on channel strength levels are satisfied, then power control at the transmitters and treating interference as noise at the receivers (TIN for short) is optimal from the generalized degrees-of-freedom (GDoF) and constant-gap capacity perspectives. These TIN conditions are described as follows: for each designated transmitter-receiver pair, the strength of the direct (i.e. desired) link must be no less than the sum of the strengths of the strongest incoming (from other transmitters) and the strongest outgoing (to other receivers) cross links, where all link strengths are in dB scale. This pioneering work by Geng et al. [1] has inspired a surge of interest in studying the optimality of TIN-based schemes in a variety of wireless network settings [2]–[12].

In this paper, we are primarily interested in the optimality of TIN in the X channel. Geng, Sun and Jafar [2] showed that even after enhancing a TIN-optimal K-user interference channel to an X channel, i.e. by expanding the message set to include an independent message from each transmitter to each receiver; it remains optimal from the sum-GDoF and constant-gap sum-capacity perspectives to operate the new channel as the original interference channel and to use TIN. The authors in [2] have also extended the result to more general $M \times N$ X channels, with distinct M and N. In this case, the TIN conditions are modified as follows: there must exist K transmitter-receiver pairs, where $K = \min(M, N)$, such that the TIN condition is satisfied by each designated pair against all other transmitters and receivers (see Theorem 1). In this case, it is optimal from the sum-GDoF and constant-gap sum-capacity perspectives to operate the $M \times N$ X channel as a K-user interference channel and use TIN.

The main technical difficulty in [2] is driving a tight information-theoretic outer bound which matches the TIN achievable inner bound in the sum-GDoF and constant-gap sum-capacity sense. This turns out to be particularly challenging in the case with more receivers than transmitters (i.e. $M < N$). This challenge is circumvented in [2] through a deterministic approximation of the original Gaussian network, inspired by the Avestimehr-Diggavi-Tse approach in [13]. The capacity of the original Gaussian channel is shown to be upper bounded by the capacity of the corresponding deterministic channel, up to a constant gap. Furthermore, the deterministic model explicitly marks out the combinatorial structure arising from the interaction between different signal strength levels, which is exploited in the outer bound proof in [2].

While deterministic approximations have contributed invaluable insights into the capacity of Gaussian networks, including the TIN-optimality result for the $M \times N$ X channel [2] of interest to us here; one may wonder whether it is possible to prove this result while working directly with the original Gaussian model. We answer this question in the affirmative—working directly with the Gaussian $M \times N$ X channel (with $M < N$) in the TIN regime, we derive an outer bound for the sum-capacity which is tight up to constant gap (hence tight in the sum-GDoF sense). Our proof is surprisingly simple, and avoids the intricacies of translating the Gaussian network into a counterpart deterministic network and then proving the faithfulness of such translation—see [2]. A key ingredient of our proof is the utilization of a new "less noisy under interference" order [10], akin to the "less noisy" order in broadcast channels (see Lemma 1). It turns out that in $M \times N$ X channels satisfying the TIN conditions, and with respect to any transmitter and its set of N messages, users are partially ordered such that the corresponding designated receiver is (approximately) less noisy than all other receivers.

Notation: For positive integers z_1 and z_2 where $z_1 \leq z_2$, the sets $\{1, 2, \ldots, z_1\}$ and $\{z_1, z_1 + 1, \ldots, z_2\}$ are denoted by $\langle z_1 \rangle$ and $\langle z_1 : z_2 \rangle$, respectively. Bold symbols denote tuples, e.g. $\boldsymbol{a} = (a_1, \ldots, a_z)$, also expressed as $(a_i : i \in \langle z \rangle)$, and

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$\boldsymbol{A} = (A_1, \ldots, A_z)$; and calligraphic symbols denote sets, e.g. $\mathcal{A} = \{a_1, \ldots, a_z\}$. $\mathcal{A}^{\mathrm{c}}$ denotes the complement of set $\mathcal{A}$.

II. Problem Setting and Preliminaries

A. Network model

We consider a wireless network with M transmitters and N receivers. For GDoF studies [1], [2], [10], the input-output relationship at the t-th channel use is described by

$$Y_k(t) = \sum_{i=1}^{M} \bar{P}^{\alpha_{ki}} e^{\mathrm{j}\theta_{ki}} X_i(t) + Z_k(t) \tag{1}$$

where $X_i(t), Y_k(t), Z_k(t) \in \mathbb{C}$ respectively denote the symbol transmitted by transmitter i, the symbol received by receiver k, and the zero-mean unit-variance additive white Gaussian noise (AWGN) at receiver k. The signal of transmitter i is subject to the unit average power constraint $\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\left[\left|X_i(t)\right|^2\right] \leq 1$, where T is the communication duration in channel uses. $\bar{P}^{\alpha_{ki}}$ and θ_{ki} are the magnitude and phase, respectively, of the channel between transmitter i and receiver k. For GDoF purposes, we define $\bar{P} \triangleq \sqrt{P}$, where $P > 1$ is a nominal power parameter that approaches infinity in the GDoF limit. The exponent $\alpha_{ki} > 0$ is known as the channel strength parameter, or channel strength level, of the link between transmitter i and receiver k.

When operating the above $M \times N$ network as an X channel, each of the M transmitters has a set of N independent messages, one for each receiver. The message from transmitter m to receiver n, where $m \in \langle M \rangle$ and $n \in \langle N \rangle$, is denoted by W_{nm}, and the corresponding rate is given by R_{nm}. Rates, capacity and GDoF are all defined in a standard Shannon-theoretic fashion, see, e.g., [1], [2], [10]. For fixed P, the capacity region, denoted by $\mathcal{C}(P)$, is the closure of the set of all achievable rate tuples $\boldsymbol{R} = \big(R_{nm} : m \in \langle M \rangle, n \in \langle N \rangle\big)$. A GDoF tuple is denoted by $\boldsymbol{d} = \big(d_{nm} : m \in \langle M \rangle, n \in \langle N \rangle\big)$, and the GDoF region is defined as

$$\mathcal{D} \triangleq \Big\{\boldsymbol{d} : d_{nm} = \lim_{P \to \infty} \frac{R_{nm}}{\log(P)}, \forall m \in \langle M \rangle, n \in \langle N \rangle, \boldsymbol{R} \in \mathcal{C}(P)\Big\}.$$

Here we are primarily interested in the sum-GDoF, obtained from the GDoF region as

$$d_{\Sigma} \triangleq \max_{\boldsymbol{d} \in \mathcal{D}} \sum_{m=1}^{M} \sum_{n=1}^{N} d_{nm}. \tag{2}$$

We now introduce some shorthand notations for message sets, and their corresponding rates and GDoF. The message set originating from transmitter m, where $m \in \langle M \rangle$, is denoted by $\mathcal{W}_m \triangleq \{W_{1m}, \ldots, W_{Nm}\}$. The sum-rate of $\mathcal{W}_m$ is given by $R_m \triangleq \sum_{n=1}^{N} R_{nm}$, and the corresponding sum-GDoF is denoted by $d_m \triangleq \sum_{n=1}^{N} d_{nm}$. On the other hand, the message set intended to user n, where $n \in \langle N \rangle$, is given by $\mathcal{W}'_n \triangleq \{W_{n1}, \ldots, W_{nM}\}$. The corresponding sum-rate and sum GDoF are given by $R'_n \triangleq \sum_{m=1}^{M} R_{nm}$ and $d'_n \triangleq \sum_{m=1}^{M} d_{nm}$, respectively. The following simple identities hold: $\bigcup_{m=1}^{M} \mathcal{W}_m = \bigcup_{n=1}^{N} \mathcal{W}'_n$, $\sum_{m=1}^{M} R_m = \sum_{n=1}^{N} R'_n$, and $\sum_{m=1}^{M} d_m = \sum_{n=1}^{N} d'_n$.

B. Power control and treating interference as noise (TIN)

Define $K \triangleq \min(M, N)$, and let $\pi : \langle M \rangle \to \langle M \rangle$ and $\pi' : \langle N \rangle \to \langle N \rangle$ be arbitrary permutations over the sets of transmitter and receiver indices, respectively. The $M \times N$ X channel can be operated as a regular K-user interference channel by eliminating all messages except for a set of K messages given by $\big\{W_{\pi'(k)\pi(k)} : k \in \langle K \rangle\big\}$, for some π and π'. Without loss of generality, we may assume that $\pi(m) = m$ and $\pi'(n) = n$, for all $m \in \langle M \rangle$ and $n \in \langle N \rangle$, by relabelling transmitters and receivers if necessary. In this case, the K-user interference channel is given by the first K transmitter-receiver pairs of the underlying $M \times N$ X channel.

By further restricting operation in the K-user interference channel to schemes based on power control and treating interference as noise (i.e. TIN), a sum-GDoF denoted by $d_{\Sigma}^{\mathrm{TIN}}$ is achievable, which is a lower bound for d_{Σ} in (2). $d_{\Sigma}^{\mathrm{TIN}}$ is characterized through the following linear program [1], [2]

$$d_{\Sigma}^{\mathrm{TIN}} \triangleq \max_{(d_1, \ldots, d_K) \in \mathcal{D}^{\mathrm{TIN}}} \sum_{k=1}^{K} d_k \tag{3}$$

where the region $\mathcal{D}^{\mathrm{TIN}}$ is defined as the set of all non-negative tuples $(d_1, \ldots, d_K) \in \mathbb{R}_{+}^{K}$ that satisfy

$$d_k \leq \alpha_{kk}, \ \forall k \in \langle K \rangle \tag{4}$$

$$\sum_{k \in \{\sigma\}} d_k \leq \sum_{s=1}^{|\sigma|} \Big(\alpha_{\sigma(s)\sigma(s)} - \alpha_{\sigma(s+1)\sigma(s)}\Big), \ \forall \sigma \in \Sigma_K. \tag{5}$$

In (5), σ is a cyclic sequence (or cycle) of users in $\langle K \rangle$ of length $|\sigma|$, and Σ_K is the set of all such cycles of any length[1] $|\sigma| \in \langle 2 : K \rangle$. A cycle σ of length $|\sigma| = S$ is written as

$$\sigma = \big(\sigma(1) \to \sigma(2) \to \cdots \to \sigma(S)\big) \tag{6}$$

where $\{\sigma\} \triangleq \big\{\sigma(1), \sigma(2), \ldots, \sigma(S)\big\} \subseteq \langle K \rangle$ is the set of users traversed by σ. For any cycle of length S, indices are interpreted modulo S, i.e. $\sigma(s+S) = \sigma(s)$, for all integers s.

The TIN achievable sum-GDoF in (3) is optimal for the K-user interference channel given that channel strength parameters satisfy the following TIN optimality condition [1]

$$\alpha_{kk} \geq \alpha_{jk} + \alpha_{kl}, \ \forall k, j, l \in \langle K \rangle, \ j, l \neq k. \tag{7}$$

As it turns out, this result generalizes to the $M \times N$ X channel [2]. This is reviewed next.

C. Optimality of TIN with general message sets

The main results of [2] are stated in the following theorem.

Theorem 1. [2, Th. 2 and Th. 3] *Suppose that there exists two permutations π and π' for the transmitter and receiver indices, respectively, such that*

$$\alpha_{\pi'(k)\pi(k)} \geq \alpha_{\pi'(n)\pi(k)} + \alpha_{\pi'(k)\pi(m)},$$
$$\forall k \in \langle K \rangle, \ m \in \langle M \rangle, \ n \in \langle N \rangle, \ m, n \neq k. \tag{8}$$

[1]Trivial cycles of length 1 are excluded from the set Σ_K, as their corresponding GDoF bounds take on a different form, see e.g. (4).

Then it is optimal from the sum-GDoF and constant-gap sum-capacity perspectives to operate the $M \times N$ X channel as a K-user interference channel in which transmitter $\pi(k)$ communicates an independent message to receiver $\pi'(k)$ only, where $k \in \langle K \rangle$, and to use TIN.

As noted in the previous subsection, we may assume without loss of generality that the permutations in Theorem 1 are given by $\pi(m) = m$ and $\pi'(n) = n$, for all $m \in \langle M \rangle$ and $n \in \langle N \rangle$. This is assumed to be the case henceforth, with no loss of generality. The condition in (8) becomes

$$\alpha_{kk} \geq \alpha_{nk} + \alpha_{km}, \quad \forall k \in \langle K \rangle,\ m \in \langle M \rangle,\ n \in \langle N \rangle,\ m, n \neq k. \tag{9}$$

Whenever the condition in (9) holds, we have

$$d_{\Sigma} = d_{\Sigma}^{\mathrm{TIN}} \tag{10}$$

where d_{Σ} is the sum-GDoF of the X channel defined in (2), while $d_{\Sigma}^{\mathrm{TIN}}$ is the TIN achievable sum-GDoF defined in (3). Under the TIN condition in (9), and for $M \geq N = K$, Theorem 1 implies that transmitters indexed by $\langle K+1 : M \rangle$ are redundant from the sum-GDoF standpoint, and hence can be omitted. On the other hand, for $K = M < N$, receivers indexed by $\langle K+1 : N \rangle$ are redundant, and hence can be omitted in this case. The main challenge in proving Theorem 1 is the converse, especially for the case $M < N$. A new proof for this result, based on the notion of *less noisiness under interference*, is presented in the following section.

III. A New Proof for Theorem 1

We have the following outer bound.

Theorem 2. *For $K = M \leq N$, and under the TIN condition in* (9)*, the rate tuple $(R_1, \ldots, R_K)$ of transmitter message sets is included in the region specified by*

$$0 \leq R_k \leq \alpha_{kk} \log(P) + O(1),\ \forall k \in \langle K \rangle \tag{11}$$

$$\sum_{k \in \{\sigma\}} R_k \leq \sum_{s=1}^{|\sigma|} \Big(\alpha_{\sigma(s)\sigma(s)} - \alpha_{\sigma(s+1)\sigma(s)} \Big) \log(P) + O(1),\ \forall \sigma \in \Sigma_K. \tag{12}$$

In the GDoF sense, the rate outer bound in Theorem 2 is identical to the TIN region $\mathcal{D}^{\mathrm{TIN}}$ described in (4) and (5), which implies that $d_{\Sigma} \leq d_{\Sigma}^{\mathrm{TIN}}$. Since $d_{\Sigma}^{\mathrm{TIN}}$ is also achievable, it follows that $d_{\Sigma} = d_{\Sigma}^{\mathrm{TIN}}$ under the TIN condition in (9). The remainder of this section is dedicated to proving Theorem 2.

Before we proceed, it is worthwhile highlighting that the single transmitter bounds in (11) follow directly from the capacity of the degraded Gaussian BC, see, e.g. [14]. Therefore, we focus on the multi-transmitter cyclic bounds in (12). We start by reviewing a key lemma from [10].

A. Less noisy under interference

Let us focus on a 2-transmitter, 2-receiver sub-network of the $M \times N$ network. We select transmitters k and m, and receivers k and i, where $k \neq m \neq i$, and eliminate all other nodes. The signal model for this sub-network is given by

$$Y_k(t) = \bar{P}^{\alpha_{kk}} e^{\mathrm{j}\theta_{kk}} X_k(t) + \bar{P}^{\alpha_{km}} e^{\mathrm{j}\theta_{km}} X_m(t) + Z_k(t) \tag{13}$$
$$Y_i(t) = \bar{P}^{\alpha_{ik}} e^{\mathrm{j}\theta_{ik}} X_k(t) + \bar{P}^{\alpha_{im}} e^{\mathrm{j}\theta_{im}} X_m(t) + Z_i(t). \tag{14}$$

We further define an arbitrary random variable W, independent of $\boldsymbol{X}_m$, $\boldsymbol{Z}_k$ and $\boldsymbol{Z}_i$, and which forms a Markov chain given by $W \to \boldsymbol{X}_k \to (\boldsymbol{Y}_k, \boldsymbol{Y}_i)$. The following result holds.

Lemma 1. (Less noisy under interference [10, Lemma 2]) *Suppose that $\alpha_{kk} - \alpha_{km} \geq \alpha_{ik}$. Then for any $\boldsymbol{X}_k$, $\boldsymbol{X}_m$ and W as defined above, we have*

$$I\big(W; \boldsymbol{Y}_i \mid \boldsymbol{X}_m\big) \leq I\big(W; \boldsymbol{Y}_k\big) + T. \tag{15}$$

It is worthwhile highlighting that in [10, Lemma 2], a weaker version of (15) is presented, given by

$$I\big(W; \boldsymbol{Y}_i\big) \leq I\big(W; \boldsymbol{Y}_k\big) + T \tag{16}$$

which is implied by (15) due to the independence of W and $\boldsymbol{X}_m$ (and hence $I\big(W; \boldsymbol{Y}_i \mid \boldsymbol{X}_m\big) \geq I\big(W; \boldsymbol{Y}_i\big)$). The stronger inequality in (15) follows from the proof of [10, Lemma 2], presented in [10, Appendix C]. This means that even after removing interference from the signal $\boldsymbol{Y}_i$, user i remains more noisy than user k, up to a constant of 1 bit per channel use. Next, we employ Lemma 1 to prove Theorem 2.

B. Proof of Theorem 2: Simple example

To gain some insights into the proof of Theorem 2, we start with a simple example. We consider a 2×4 X channel in which each user n wishes to decode $\mathcal{W}'_n \triangleq \{W_{n1}, W_{n2}\}$, with W_{nm} originating from transmitter m, where $n \in \langle 4 \rangle$ and $m \in \langle 2 \rangle$. For this 2×4 network, Theorem 2 gives rise to only one multi-transmitter bound given by

$$R_1 + R_2 \leq (\alpha_{11} - \alpha_{21} + \alpha_{22} - \alpha_{12}) \log(P) + O(1). \tag{17}$$

Next, we show that this is a valid outer bound. Starting from Fano's inequality, the sum-rate $R'_1 \triangleq R_{11} + R_{12}$ of user 1 is bounded above as

$$\begin{aligned} T\big(R'_1 - \epsilon_T\big) &\leq I\big(\mathcal{W}'_1; \boldsymbol{Y}_1\big) \\ &= I\big(W_{11}; \boldsymbol{Y}_1\big) + I\big(W_{12}; \boldsymbol{Y}_1 \mid W_{11}\big) \\ &\leq I\big(W_{11}; \boldsymbol{Y}_1\big) + I\big(W_{12}; \boldsymbol{Y}_1 \mid W_{11}, W_{21}, W_{31}, W_{41}\big) && (18) \\ &= I\big(W_{11}; \boldsymbol{Y}_1\big) + I\big(W_{12}; \boldsymbol{Y}_1 \mid \mathcal{W}_1, \boldsymbol{X}_1\big). && (19) \end{aligned}$$

In the above, and henceforth, we use ϵ_T to denote terms that approach zero as T goes to infinity. The inequality in (18) holds due to the independence of messages; while (19) holds since $\boldsymbol{X}_1$ is fully determined by the message set $\mathcal{W}_1$ of transmitter 1. Next, we invoke Lemma 1 to bound the mutual information term $I\big(W_{12}; \boldsymbol{Y}_1 \mid \mathcal{W}_1, \boldsymbol{X}_1\big)$. We have

$$\begin{aligned} I\big(W_{12}; \boldsymbol{Y}_1 \mid \mathcal{W}_1, \boldsymbol{X}_1\big) &= I\big(W_{12}; \bar{P}^{\alpha_{12}} e^{\mathrm{j}\theta_{12}} \boldsymbol{X}_2 + \boldsymbol{Z}_1\big) \\ &\leq I\big(W_{12}; \boldsymbol{Y}_2\big) + T && (20) \end{aligned}$$

which holds due to the TIN condition $\alpha_{22} - \alpha_{21} \geq \alpha_{12}$. Combining (20) and (19), we obtain

$$T\big(R_1' - \epsilon_T\big) \leq I\big(W_{11}; \boldsymbol{Y}_1\big) + I\big(W_{12}; \boldsymbol{Y}_2\big) + T. \tag{21}$$

In a similar fashion, we obtain a sum-rate bound for user 2 as

$$T\big(R_2' - \epsilon_T\big) \leq I\big(W_{21}; \boldsymbol{Y}_1\big) + I\big(W_{22}; \boldsymbol{Y}_2\big) + T. \tag{22}$$

We now consider the remaining users. For user 3, we have

$$\begin{aligned} T\big(R_3' - \epsilon_T\big) &\leq I\big(W_{31}, W_{32}; \boldsymbol{Y}_3\big) \\ &\leq I\big(W_{31}; \boldsymbol{Y}_3 \mid W_{32}\big) + I\big(W_{32}; \boldsymbol{Y}_3 \mid W_{31}\big) \\ &\leq I\big(W_{31}; \boldsymbol{Y}_3 \mid \mathcal{W}_2, \boldsymbol{X}_2\big) + I\big(W_{32}; \boldsymbol{Y}_3 \mid \mathcal{W}_1, \boldsymbol{X}_1\big). \end{aligned} \tag{23}$$

Here we apply Lemma 1 to both mutual information terms in (23), from which we obtain

$$T\big(R_3' - \epsilon_T\big) \leq I\big(W_{31}; \boldsymbol{Y}_1\big) + I\big(W_{32}; \boldsymbol{Y}_2\big) + 2T \tag{24}$$

which holds due to the TIN conditions $\alpha_{11} - \alpha_{12} \geq \alpha_{31}$ and $\alpha_{22} - \alpha_{21} \geq \alpha_{32}$. Through similar steps, we obtain a sum-rate bound for user 4 as

$$T\big(R_4' - \epsilon_T\big) \leq I\big(W_{41}; \boldsymbol{Y}_2\big) + I\big(W_{42}; \boldsymbol{Y}_2\big) + 2T. \tag{25}$$

From the above bounds for all 4 users, we bound the sum-rate of all messages as

$$\begin{aligned} T\sum_{n=1}^{4}\big(R_n' - \epsilon_T\big) &\leq \sum_{k=1}^{2}\sum_{n=1}^{4} I\big(W_{nk}; \boldsymbol{Y}_k\big) + 6T \\ &\leq \sum_{k=1}^{2}\sum_{n=1}^{4} I\big(W_{nk}; \boldsymbol{Y}_k \mid W_{1k}, \ldots, W_{(n-1)k}\big) + 6T \\ &= I\big(\mathcal{W}_1; \boldsymbol{Y}_1\big) + I\big(\mathcal{W}_2; \boldsymbol{Y}_2\big) + 6T \\ &\leq I\big(\boldsymbol{X}_1; \boldsymbol{Y}_1\big) + I\big(\boldsymbol{X}_2; \boldsymbol{Y}_2\big) + 6T. \end{aligned} \tag{26}$$

Recall that $\mathcal{W}_m \triangleq \{W_{1m}, W_{2m}, W_{3m}, W_{4m}\}$ is the message set of transmitter m. After normalizing by T and omitting the constant additive factor, it becomes evident that the right-hand-side of (26) bounds the sum-rate of a regular 2-user interference channel, in which transmitter 1 communicates $\boldsymbol{X}_1$ to user 1 and transmitter 2 communicates $\boldsymbol{X}_2$ to user 2. Therefore, we invoke the 2-user interference channel genie-aided outer bound of Etkin et al. [15], from which we obtain

$$\begin{aligned} I\big(\boldsymbol{X}_1; \boldsymbol{Y}_1\big) + I\big(\boldsymbol{X}_2; \boldsymbol{Y}_2\big) &\leq T\log\left(1 + P^{\alpha_{12}} + \frac{P^{\alpha_{11}}}{1 + P^{\alpha_{21}}}\right) \\ &\quad + T\log\left(1 + P^{\alpha_{21}} + \frac{P^{\alpha_{22}}}{1 + P^{\alpha_{12}}}\right) & (27) \\ &\leq T\log\left(3P^{\alpha_{11} - \alpha_{21}}\right) + T\log\left(3P^{\alpha_{22} - \alpha_{12}}\right) & (28) \end{aligned}$$

where (27) follows from [15, Th. 1], while (28) holds due to the TIN condition and $P > 1$. Combining (28) and (26), we obtain the desired sum-rate outer bound

$$R_1 + R_2 \leq (\alpha_{11} - \alpha_{21} + \alpha_{22} - \alpha_{12})\log(P) + 6 + 2\log(3)$$

which translates to $d_1 + d_2 \leq \alpha_{11} - \alpha_{21} + \alpha_{22} - \alpha_{12}$.

C. Proof of Theorem 2: General case

We now extend the proof presented in the previous part to the $M \times N$ X channel of interest. Focusing on an arbitrary cycle $\sigma \in \Sigma_K$ of length $|\sigma| = S$, we show that the corresponding cyclic bound in (12) is a valid outer bound. To this end, we start by eliminating all non-participating transmitters and their corresponding message sets, i.e. $\big\{\mathcal{W}_m : m \in \{\sigma\}^{\mathrm{c}}\big\}$. This step does not decrease the rates of the remaining message sets, i.e. $\big\{\mathcal{W}_k : k \in \{\sigma\}\big\}$, and effectively reduces the $M \times N$ channel to an $S \times N$ channel. Note that the message set of each user n, where $n \in \langle N \rangle$, reduces to $\mathcal{W}_n' \triangleq \big\{W_{nm} : m \in \{\sigma\}\big\}$ here. We focus on the cycle $\sigma = (1 \to 2 \to \cdots \to S)$ for ease of exposition, for which we have $\{\sigma\} = \langle S \rangle$.

Starting from Fano's inequality, the sum-rate R_1' of user 1 is bounded above as

$$\begin{aligned} T\big(R_1' - \epsilon_T\big) &\leq I\big(\mathcal{W}_1'; \boldsymbol{Y}_1\big) \\ &= I\big(W_{11}; \boldsymbol{Y}_1\big) + \sum_{m=2}^{S} I\big(W_{1m}; \boldsymbol{Y}_1 \mid W_{11}, \ldots, W_{1(m-1)}\big) \\ &\leq I\big(W_{11}; \boldsymbol{Y}_1\big) + \sum_{m=2}^{S} I\big(W_{1m}; \boldsymbol{Y}_1 \mid \{\mathcal{W}_m\}^{\mathrm{c}}, \{\boldsymbol{X}_m\}^{\mathrm{c}}\big) & (29) \\ &\leq I\big(W_{11}; \boldsymbol{Y}_1\big) + \sum_{m=2}^{S}\Big[I\big(W_{1m}; \boldsymbol{Y}_m \mid \{\boldsymbol{X}_{m-1}, \boldsymbol{X}_m\}^{\mathrm{c}}\big) + T\Big] & (30) \\ &\leq (S-1)T + \sum_{m=1}^{S} I\big(W_{1m}; \boldsymbol{Y}_m \mid \{\boldsymbol{X}_{m-1}, \boldsymbol{X}_m\}^{\mathrm{c}}\big). & (31) \end{aligned}$$

In the above, $\{\mathcal{W}_m\}^{\mathrm{c}}$ comprises all message sets except for $\mathcal{W}_m$; and $\{\boldsymbol{X}_m\}^{\mathrm{c}}$ comprises all transmitted signals except for $\boldsymbol{X}_m$. The inequality in (29) holds due to the independence of messages; the fact that $\{\mathcal{W}_m\}^{\mathrm{c}}$ includes $\{W_{11}, \ldots, W_{1(m-1)}\}$; and since $\{\boldsymbol{X}_m\}^{\mathrm{c}}$ is fully determined by $\{\mathcal{W}_m\}^{\mathrm{c}}$. Going from (29) to (30) is accomplished through the following steps:

$$\begin{aligned} I\big(W_{1m}; \boldsymbol{Y}_1 \mid &\{\mathcal{W}_m\}^{\mathrm{c}}, \{\boldsymbol{X}_m\}^{\mathrm{c}}\big) \\ &= I\big(W_{1m}; \bar{P}^{\alpha_{1m}} e^{\mathrm{j}\theta_{1m}} \boldsymbol{X}_m + \boldsymbol{Z}_1\big) & (32) \\ &\leq I\big(W_{1m}; \boldsymbol{Y}_m \mid \{\boldsymbol{X}_{m-1}, \boldsymbol{X}_m\}^{\mathrm{c}}\big) + T. & (33) \end{aligned}$$

In the above, (32) holds since conditioning on $\{\boldsymbol{X}_m\}^{\mathrm{c}}$ is equivalent to removing the contributions of all such signals from $\boldsymbol{Y}_1$, leaving only the contribution of $\boldsymbol{X}_m$ and noise which are independent of $\{\boldsymbol{X}_m\}^{\mathrm{c}}$. On the other hand, (33) follows from Lemma 1, which applies here due to the TIN condition, and by a similar argument to the one used in (32).

Finally, the inequality in (31) follows from

$$I\big(W_{11}; \boldsymbol{Y}_1\big) \leq I\big(W_{11}; \boldsymbol{Y}_1 \mid \{\boldsymbol{X}_S, \boldsymbol{X}_1\}^{\mathrm{c}}\big)$$

which in turn holds due to the independence of the message W_{11} and signals in $\{\boldsymbol{X}_S, \boldsymbol{X}_1\}^{\mathrm{c}}$. Note that the index m in (31) is interpreted modulo S.

The above steps leading to (31) for user 1 can be repeated for all users in the set $\{\sigma\} = \langle S \rangle$. This yields a bound on the sum-rate R_k' of each user k, where $k \in \langle S \rangle$, given by

$$T\big(R'_k - \epsilon_T\big) \leq$$
$$(S-1)T + \sum_{m=1}^{S} I\big(W_{km}; \boldsymbol{Y}_m \mid \{\boldsymbol{X}_{m-1}, \boldsymbol{X}_m\}^{\mathrm{c}}\big). \quad (34)$$

We now consider remaining users. The sum-rate R'_i of user i, where $i \in \langle S+1 : N \rangle$, is bounded as follows:

$$\begin{aligned} T\big(R'_i - \epsilon_T\big) &\leq I\big(\mathcal{W}'_i; \boldsymbol{Y}_i\big) \\ &= \sum_{m=1}^{S} I\big(W_{im}; \boldsymbol{Y}_i \mid W_{i1}, \ldots, W_{i(m-1)}\big) \\ &\leq \sum_{m=1}^{S} I\big(W_{im}; \boldsymbol{Y}_i \mid \{\mathcal{W}_m\}^{\mathrm{c}}, \{\boldsymbol{X}_m\}^{\mathrm{c}}\big) \\ &\leq ST + \sum_{m=1}^{S} I\big(W_{im}; \boldsymbol{Y}_m \mid \{\boldsymbol{X}_{m-1}, \boldsymbol{X}_m\}^{\mathrm{c}}\big) \end{aligned} \quad (35)$$

From (34) and (35), we obtain the bound on the sum-rate of all N users, given as follows

$$\begin{aligned} &- NST + T\sum_{n=1}^{N} \big(R'_n - \epsilon_T\big) \\ &\leq \sum_{n=1}^{N}\sum_{m=1}^{S} I\big(W_{nm}; \boldsymbol{Y}_m \mid \{\boldsymbol{X}_{m-1}, \boldsymbol{X}_m\}^{\mathrm{c}}\big) \\ &\leq \sum_{m=1}^{S}\sum_{n=1}^{N} I\big(W_{nm}; \boldsymbol{Y}_m \,|\, \{\boldsymbol{X}_{m-1}, \boldsymbol{X}_m\}^{\mathrm{c}}, W_{1m}, \ldots, W_{(n-1)m}\big) \\ &= \sum_{m=1}^{S} I\big(\mathcal{W}_m; \boldsymbol{Y}_m \mid \{\boldsymbol{X}_{m-1}, \boldsymbol{X}_m\}^{\mathrm{c}}\big) \\ &\leq \sum_{m=1}^{S} I\big(\boldsymbol{X}_m; \boldsymbol{Y}_m \mid \{\boldsymbol{X}_{m-1}, \boldsymbol{X}_m\}^{\mathrm{c}}\big). \end{aligned} \quad (36)$$

After normalizing by T, the right-hand-side of (36) bounds the sum-rate of an S-user *cyclic* interference channel in which each receiver m, where $m \in \langle S \rangle$, is connected to its designated transmitter m and receives interference from transmitter $m-1$ only. This is bounded above by invoking the result in [1, Th. 3], which extends the genie-aided outer bound of Etkin et al. [15] (see also [16] for a similar result). This leads to

$$\begin{aligned} &\sum_{m=1}^{S} I\big(\boldsymbol{X}_m; \boldsymbol{Y}_m \mid \{\boldsymbol{X}_{m-1}, \boldsymbol{X}_m\}^{\mathrm{c}}\big) \\ &\leq T\sum_{m=1}^{S} \log\left(1 + P^{\alpha_{m(m-1)}} + \frac{P^{\alpha_{mm}}}{1 + P^{\alpha_{(m+1)m}}}\right) \quad (37) \\ &\leq 2ST + T\sum_{s=1}^{S} \big(\alpha_{mm} - \alpha_{(m+1)m}\big)\log(P). \quad (38) \end{aligned}$$

By combining the bounds in (36) and (38), we obtain the desired cyclic bound given by

$$\sum_{k \in \langle S \rangle} R_k \leq (N+2)S + \sum_{m=1}^{S} \big(\alpha_{mm} - \alpha_{(m+1)m}\big)\log(P). \quad (39)$$

The same approach applies to all cycles in Σ_K, from which we obtain the bounds in (12). This concludes the proof.

IV. Conclusion

We presented a new converse proof for the TIN-optimality result in the $M \times N$ X channel ($M < N$), originally shown by Geng, Sun and Jafar in [2]. Our new proof deals directly with the Gaussian setting, and avoids the intricate step of translation into a deterministic model. The main ingredient of our proof is the utilization of a new "less noisy under interference" order, induced in the $M \times N$ X channel by the TIN conditions. This new order approximates any sub-network of the X channel by a regular interference channel, from which desired cyclic sum-rate upper bounds are obtained using previously known results. The TIN regime for the $M \times N$ X channel in [2], which we considered here, has been expanded in [7], albeit only for the $M \times 2$ special case (i.e. M transmitters and 2 receivers). It is of interest to explore whether the new outer bound techniques utilized here can help generalize the expanded regime in [7] to settings with arbitrary M and N.

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