Equations of time evolution for sequential learning

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Here I will write the solutions that I have for the system so far

Let's assume for a moment that we have a pulse in o. That is

$$o_i = \delta_{i,k}$$

Where the pulse happens at mini-column k. If we remember that the equations for the traces z are of the following form:

$$\tau_z \frac{dz_i}{dt} = o_{i,k} - z_i \tag{1}$$

Where we have to remember that we actually have two τ_z. One for the pre-synaptic dynamics and other for the post-synaptic dynamics.

Subjected to an impulse function and using simple separation of variable to solve the ODE we arrive at a solution for the case when z is in the mini-column of the impulse function and another for the z outside of it.

$$z(t) = \begin{cases} 1 - (1 - z(0))e^{-\frac{t}{\tau_z}}, & \text{if } i = k \\ z_0 e^{-\frac{t}{\tau_z}} & \text{otherwise} \end{cases} \tag{2}$$

Now with this mind we wonder how this gets propagated to the equations that trace the probabilities

$$\tau_p \frac{dp_i}{dt} = z_i(t) - p_i(t) \tag{3}$$

$$\tau_p \frac{dp_{ij}}{dt} = z_i(t) z_j(t) - p_{ij}(t) \tag{4}$$

First we will solve (3) for the case $i = k$. This is a non-homogeneous EDO which we can solve by finding the general solution for the homogeneous part and then finding a particular solution fro the non-homogeneous case. The equation in particular is the following:

$$\tau_p \frac{dp}{dt} + p(t) = 1 - (1 - z(0))e^{-\frac{t}{\tau_z}}$$

The solution for the homogeneous part is $p(t) = ce^{-\frac{t}{\tau_p}}$. Using the method of undetermined coefficients we arrive at a particular solution for the non-homogeneous part that is $p(t) = 1 + \frac{\tau_z}{\tau_p - \tau_z}(1 - z(0))e^{-\frac{t}{\tau_z}}$. With this information in our hand we can build the general solution for equation (3) as:

$$p(t) = ce^{-\frac{t}{\tau_p}} + 1 + \frac{\tau_z}{\tau_p - \tau_z}(1 - z(0))e^{-\frac{t}{\tau_z}}$$

And using the initial condition we end up with a general solution for $p(t)$

$$p(t) = 1 - (1 - p(0))e^{-\frac{t}{\tau_p}} - \frac{\tau_z}{\tau_p - \tau_z}(1 - z(0))(e^{-\frac{t}{\tau_p}} - e^{-\frac{t}{\tau_z}})$$

Using a very similar procedure we will solve equation (4). We will use the fact that the solution to linear equation can be find by superposition to build a solution for the non-homogeneous part of this equation. Fully expressed equation (4) becomes:

$$\begin{aligned}
\tau_p \frac{dp_{ij}}{dt} + p_{ij}(t) = z_i(t)z_j(t) \\
&= (1 - (1 - z_i(0))e^{-\frac{t}{\tau_{z_i}}})(1 - (1 - z_j(0))e^{-\frac{t}{\tau_{z_j}}}) \\
&= 1 - Ae^{-\frac{t}{\tau_{z_i}}} - Be^{-\frac{t}{\tau_{z_j}}} + ABe^{-\frac{t}{\tau_{s_{ij}}}}
\end{aligned}$$

Where $A = (1 - z_i(0)$, $B = (1 - z_j(0)$ and we define the special time constant by $\tau_{s_{ij}} = \frac{\tau_{z_i}\tau_{z_j}}{\tau_{z_i} + \tau_{z_i}}$. We can identify those terms from the solution to the independent solution. The general solution therefore is:

$$\begin{aligned}
p_{ij}(t) =& ce^{-\frac{t}{\tau_p}} + 1 + \frac{\tau_{z_i}}{\tau_p - \tau_{z_i}}(1 - z_i(0))e^{-\frac{t}{\tau_{z_i}}} + \frac{\tau_{z_j}}{\tau_p - \tau_{z_j}}(1 - z_j(0))e^{-\frac{t}{\tau_{z_j}}} \\
&- \frac{\tau_{s_{ij}}}{\tau_p - \tau_{s_{ij}}}(1 - z_i(0))(1 - z_j(0))e^{-\frac{t}{\tau_{s_{ij}}}}
\end{aligned}$$

Which we can also write:

$$p_{ij}(t) = ce^{-\frac{t}{\tau_p}} + 1 + a_1 e^{-\frac{t}{\tau_{z_i}}} + a_2 e^{-\frac{t}{\tau_{z_j}}} - a_3 e^{-\frac{t}{\tau_{s_{ij}}}}$$

If we use the initial conditions we can get the value of c, in this case $c = -(1 - p_{ij}(0)) - a_1 - a_2 + a_3$. With this the final solution for the join equation is:

$$\begin{aligned}
p_{ij}(t) &= 1 - (1 - p_{ij}(0))e^{-\frac{t}{\tau_p}} \\
&- \frac{\tau_{z_i}}{\tau_p - \tau_{z_i}}(1 - z_i(0))(e^{-\frac{t}{\tau_p}} - e^{-\frac{t}{\tau_{z_i}}}) \\
&- \frac{\tau_{z_j}}{\tau_p - \tau_{z_j}}(1 - z_j(0))(e^{-\frac{t}{\tau_p}} - e^{-\frac{t}{\tau_{z_j}}}) \\
&+ \frac{\tau_{s_{ij}}}{\tau_p - \tau_{s_{ij}}}(1 - z_i(0))(1 - z_j(0))(e^{-\frac{t}{\tau_p}} - e^{-\frac{t}{\tau_{s_{ij}}}})
\end{aligned}$$